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AUTOMORPHISMS OF MANIFOLDS AND ALGEBRAIC K-THEORY, PART II

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We relate the L-groups of a ring with involution R to its higher algebraic K-groups. As a by-product, we obtain long exact 'higher' Rothenberg sequences. These involve a particular $K_n(R)$, with $n \geq 0$, in the same way the traditional Rothenberg sequences involve $K_0(R)$ and $K_1(R)$, respectively.

Introduction

In part I of this work (see [47]) we elaborated an idea of Hatcher [14] concerning the connection between the space of automorphisms of a manifold M and the concordance theory of M. Here in part II we take up an older suggestion of Cappell and Shaneson (see [32]) concerning the connection between the L-theory and the K-theory of a ring with involution R. The two ideas will eventually be unified, probably in part IV.

Let $\underline{\underline{L}}p_{\blacktriangledown}(R)$ be the projective quadratic L-theory spectrum of R, so that $\pi_n(\underline{\underline{L}}p_{\blacktriangledown}(R)) \cong \pi_{n+4}(\underline{\underline{L}}p_{\blacktriangledown}(R))$ for $n \geq 0$, and $\pi_n(\underline{\underline{L}}p_{\blacktriangledown}(R)) = 0$ for $n < 0$. This can be defined using the methods of Ranicki [25, 26]; see Sections 0 and 4 below for details. Let $\underline{\underline{K}}p(R)$ be the K-theory spectrum of R. It can be obtained e.g. by the method of Segal [30] from the symmetric monoidal category of finitely generated projective left R-modules. Replacing an isomorphism $P_1 \to P_2$ in this category by its adjoint inverse $P_1^* \to P_2^*$ defines (almost) an involution, first on the category, and then on $\underline{\underline{K}}p(R)$. Using this involution define the homotopy orbit spectrum

$$H_{\blacktriangledown}(Z_2; \underline{\underline{K}}p(R)) := (EZ_2)_+ \wedge_{Z_2} \underline{\underline{K}}p(R),$$

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and the homotopy fixed point spectrum,

$$\mathbb{H}^{\bullet}(Z_2; \underline{K}p(R)) := \underline{\underline{\mathrm{map}}}_{Z_2}((EZ_2)_+, \underline{K}p(R));$$

see 2.1. There is a norm map of spectra, due to Dwyer,

$$\mathbb{H}_{\bullet}(Z_2; \underline{K}p(R)) \to \mathbb{H}^{\bullet}(Z_2; \underline{K}p(R))$$

(see Section 2) whose cofibre we denote by $\hat{\mathbb{H}}^{\bullet}(Z_2; \underline{K}p(R))$.

Construction D. We construct a map of spectra

$$\Xi^p : \underline{L}p_{\bullet}(R) \to \hat{\mathbb{H}}^{\bullet}(Z_2; \underline{K}p(R)).$$

This appears to be what Cappell and Shaneson were looking for, and we shall use it in part III to construct the ∞-supersimple L-groups whose existence they also predicted. (The idea is sketched below, towards the end of Section 0.)

Now let Y be a pointed connected CW-space, and let γ be a spherical fibration on Y, trivialized over the base point. Write $\pi = \pi_1(Y)$, and let $w : \pi \to Z_2$ be the first Stiefel–Whitney class of γ. Let $\underline{A}p(Y)$ be the K-theory of finitely dominated retractive spaces over Y, as defined by Waldhausen [41]. Following Vogell [39] in most respects, we use the spherical fibration γ to define an involution on $\underline{A}p(Y)$. (Unlike Vogell's, our involution does not change if γ is stabilized. Nevertheless, the difference is not substantial. This will be explained in part III.)

Construction E. We construct a factorization

$$\begin{tikzcd} & \hat{\mathbb{H}}^{\bullet}(Z_2; \underline{A}p(Y)) \arrow[d, "\text{linearization}"] \\ \underline{L}p_{\bullet}(\mathbb{Z}\pi) \arrow[ur, dashed] \arrow[r, "\Xi^p"] & \hat{\mathbb{H}}^{\bullet}(Z_2; \underline{K}p(\mathbb{Z}\pi)). \end{tikzcd}$$

(As might be expected, the ring $Z\pi$ carries the w-twisted involution; the linearization map from $\underline{A}p(Y)$ to $\underline{K}p(\mathbb{Z}\pi)$ is compatible with the involutions defined previously.) The new map $\underline{L}p_{\bullet}(\mathbb{Z}\pi) \to \hat{\mathbb{H}}^{\bullet}(Z_2; \underline{A}p(Y))$ will also be written Ξ^p.

There are versions of constructions D and E where the letter p is replaced by h or s; for example, $\underline{L}s(\mathbb{Z}\pi)$ is the spectrum whose homotopy groups are the groups $L_n^s(\mathbb{Z}\pi)$, and $\underline{K}s(\mathbb{Z}\pi)$ is the K-theory of free based $\mathbb{Z}\pi$-modules and their simple isomorphisms. Similarly, $\underline{A}s(Y)$ is the K-theory of finite retractive spaces over Y and their homotopy equivalences over Y having torsion zero. The s-version is usually closer to geometry.

The methods we use here in Part II are algebraic, in the sense of algebraic K- and L-theory. To create a little suspense, let us note that there is also a geometric way of getting from L-theory to K-theory. Namely, if M is a compact smooth or topological manifold, use Sullivan–Wall theory to get from the L-theory of $\mathbb{Z}\pi_1(M)$ to the space of block automorphisms of M; use the map Φ^s of Part I to get from there to the concordance theory of M; and use Waldhausen theory to get from the concordance theory of M to the K-theory of $\mathbb{Z}\pi_1(M)$, or (preferably) to the K-theory of retractive spaces over M.

It is not a good idea to read this paper without being acquainted with Part I of [25] and the first two parts of [41]. There is no need to be put off by the length of [25], because it is written in a leisurely style.

0. Outline

Let R be a ring with involution (= involutory anti-automorphism). Recall that the quadratic and symmetric L-theory spectra $\underline{\underline{L}}p_\bullet(R)$ and $\underline{\underline{L}}p^\bullet(R)$ have a standard description in terms of chain complexes over R, due to Ranicki [25, 26]; see also [27] and [46] for surveys. Constructing the map Ξ^p in the introduction therefore requires a description of $\underline{\underline{K}}p(R)$ in terms of chain complexes also.

Let $\mathscr{D}$ be the category of chain complexes of projective left R-modules, graded over the integers and homotopy equivalent to finitely generated projective ones. (Call such a chain complex C f.g. projective if the direct sum of all C_r is.) The morphisms in $\mathscr{D}$ are the chain maps. Call such a morphism a weak equivalence if it is a homotopy equivalence, and call it a cofibration if it is split injective in each dimension $r \in \mathbb{Z}$. This makes $\mathscr{D}$ into a category with cofibrations and weak equivalences in the sense of Waldhausen [41]; its K-theory $\underline{\underline{K}}\mathscr{D}$, in the sense of Waldhausen [41], is homotopy equivalent to $\underline{\underline{K}}p(R)$. This was proved by Brinkmann [5] and Gillet [11], but it is also clear from general theorems in [41].

Now let po$[n]$ be the set of (nonempty) faces of the standard simplex Δ^n. Inclusion defines a partial ordering on po$[n]$. Let $\varrho_n\mathscr{D}$ be the category of covariant functors $C: \text{po}[n] \to \mathscr{D}$. This is again a category with cofibrations and weak equivalences: a natural transformation $g: B \to C$ in $\varrho_n\mathscr{D}$ is a cofibration (or a weak equivalence) if $g_s: B(s) \to C(s)$ is a cofibration (or a weak equivalence) for each s in po$[n]$. Note that $\varrho_0\mathscr{D} \cong \mathscr{D}$, that the rule $[n] \mapsto \varrho_n\mathscr{D}$ defines a simplicial category $\varrho_\bullet\mathscr{D}$, and that the rule $[n] \mapsto \underline{\underline{K}}\varrho_n\mathscr{D}$ defines a simplicial spectrum.

A suitable notion of duality makes each spectrum $\underline{\underline{K}}\varrho_n\mathscr{D}$ into a spectrum with Z_2-action. We shall discuss this first when $n=0$. To begin with, we shall have to say what it means for two objects B and C of $\mathscr{D} \cong \varrho_0\mathscr{D}$ to be dual to one another. To this end we introduce pairings and nondegenerate pairings. A pairing between B and C is simply a zero-dimensional cycle in the chain complex $B^t \otimes_R C$. Equivalently, it is a chain map $\mathbb{Z} \to B^t \otimes_R C$, where $\mathbb{Z}$ is regarded as a chain complex concentrated in dimension zero. (Recall from [25] that B^t is obtained from B by shifting the R-action from left to right using the involution on R.) Such a pairing induces homomorphisms from $H^{-n}(C; R)$ to $H_n(B)$ for all $n \in \mathbb{Z}$ (by a slant product), where $H^{-n}(C; R) = H_n(\mathrm{Hom}_R(C, R))$. See Section 3 below. It these homomorphisms are isomorphisms for all $n \in \mathbb{Z}$, then we call the pairing nondegenerate. Note how necessary it is here to allow chain complexes graded over the integers, not just over the positive integers. At any rate, if a nondegenerate pairing between B and C exists, then we may think of B and C as dual to one another. (Of course, the idea goes back to Spanier and Whitehead who used it in stable homotopy theory.)

We now introduce another idea, due to Vogell [39]. The method of Vogell, adapted to our circumstances, is to inflate Waldhausen's definition of $\underline{K}\mathscr{D}$ by substituting for each chain complex C in sight a triple (B, C, η). Here $\eta : \mathbb{Z} \to B^{\mathrm{t}} \otimes_R C$ is a nondegenerate pairing between the chain complexes B and C in $\mathscr{D}$. This inflation has no effect on the homotopy type of $\underline{K}\mathscr{D}$, because every chain complex C in $\mathscr{D}$ occurs in an essentially unique nondegenerate pairing. However, the new description of $\underline{K}\mathscr{D}$ makes it clear that $\underline{K}\mathscr{D}$ has an involution: interchange the objects B and C in all nondegenerate pairings $\eta : \mathbb{Z} \to B^{\mathrm{t}} \otimes_R C$. See Section 4 for more details.

In much the same way, we can understand duality in the categories $\varrho_n \mathscr{D}$, for arbitrary $n \geq 0$, by talking about pairings and nondegenerate pairings. A pairing between objects B and C in $\varrho_n \mathscr{D}$ is a natural chain map

$$\mathrm{cl}(s) \to B(s)^{\mathrm{t}} \otimes_R C(s)$$

where s ranges over the objects of $\mathrm{po}[n]$, or over the faces of Δ^n, and where $\mathrm{cl}(s)$ is the cellular chain complex of the face $s \subset \Delta^n$. Note that $\mathrm{cl}(s)$ has a standard basis with one generator for each face r contained in s. Again, such a pairing is nondegenerate if certain homological conditions are fulfilled; see 3.5 below. (Topologists should find it easy to guess what these conditions are by meditating on the Poincaré duality properties of a compact manifold V^{k+n} modelled on $\mathbb{R}^k \times \Delta^n$. See 3.13.) As before, the fact that each object in $\varrho_n \mathscr{D}$ occurs in an essentially unique nondegenerate pairing can be used to construct an involution on $\underline{K}\varrho_n \mathscr{D}$. We conclude that the rule $[n] \mapsto \underline{K}\varrho_n \mathscr{D}$ is a simplicial Z_2-spectrum.

Next we recall the precise definition of $Q\underline{\underline{L}}p^{\blacktriangledown}(R)$, the zeroth infinite loop space of the symmetric L-theory spectrum $\underline{\underline{L}}p^{\blacktriangledown}(R)$. Let W be the usual projective resolution over $\mathbb{Z}[Z_2]$ of the trivial module $\mathbb{Z}$. Let C be an object of $\varrho_n \mathscr{D}$. Generalizing the standard definition for $n = 0$, we declare that a symmetric pairing of C with itself is a natural chain map

$$\varphi : W \otimes \mathrm{cl}(s) \to C(s)^{\mathrm{t}} \otimes_R C(s)$$

(with $s \in \mathrm{po}[n]$) which is compatible with Z_2-actions. (Z_2 acts on $C(s)^{\mathrm{t}} \otimes_R C(s)$ by switching the factors, with the usual sign changes.) Such a symmetric pairing gives rise to an ordinary pairing: choose a splitting of the augmentation map $W \to \mathbb{Z}$. Therefore we can speak of nondegenerate symmetric pairings. The nondegenerate symmetric pairings (C, φ) as above, with C in $\varrho_n \mathscr{D}$, form the n-simplices of a simplicial set. Its geometric realization is $Q\underline{\underline{L}}p^{\blacktriangledown}(R)$. See also 4.8.

Several people, e.g. Quinn and Thomason [36], have observed that the 0-simplices in $Q\underline{\underline{L}}p^{\blacktriangledown}(R)$ give rise to Z_2-equivariant maps

$$EZ_2 \to Q\underline{K}\mathscr{D} \simeq Q\underline{K}p(R)$$

(where again Q marks the zeroth infinite loop space associated to a spectrum, and $\underline{K}\mathscr{D} \cong \underline{K}\varrho_0\mathscr{D}$ carries the duality involution described above). This is certainly plausible; but it is also plausible and true that the n-simplices in $Q\underline{L}p^{\blacktriangledown}(R)$ give rise to Z_2-equivariant maps $EZ_2 \to Q\underline{K}\varrho_n\mathscr{D}$, for any $n \geq 0$. What we have here is a simplicial map from a simplicial set to a simplicial space. The simplicial set has geometric realization $Q\underline{L}p^{\blacktriangledown}(R)$; the simplicial space is given by the rule $[n] \mapsto \mathrm{map}_{Z_2}(EZ_2, Q\underline{K}\varrho_n\mathscr{D})$ or equivalently by $[n] \mapsto QH^{\blacktriangledown}(Z_2; \underline{K}\varrho_n\mathscr{D})$. Applying the geometric realization functor $|\ldots|$, we may write the map in the form

$$Q\underline{L}p^{\blacktriangledown}(R) \to |QH^{\blacktriangledown}(Z_2; \underline{K}\varrho_{\bullet}\mathscr{D})|.$$

It is not too difficult to see that this is an infinite loop map between infinite loop spaces; composing with the symmetrization map $Q\underline{L}p_{\blacktriangledown}(R) \to Q\underline{L}p^{\blacktriangledown}(R)$ from quadratic to symmetric L-theory, we then have an infinite loop map

$$\Xi^p : Q\underline{L}p_{\blacktriangledown}(R) \to |QH^{\blacktriangledown}(Z_2; \underline{K}\varrho_{\bullet}\mathscr{D})|.$$

The crunch is now to identify $|QH^{\blacktriangledown}(Z_2; \underline{K}\varrho_{\bullet}\mathscr{D})|$ with $Q\hat{H}^{\blacktriangledown}(Z_2; \underline{K}\mathscr{D})$.

At this point it is natural to ask whether the simplicial Z_2-spectrum $[n] \mapsto \underline{K}\varrho_n\mathscr{D}$ can be described directly in terms of the Z_2-spectrum $\underline{K}\mathscr{D} \cong \underline{K}\varrho_0\mathscr{D}$. The following observation is a step in that direction:

Each face s of Δ^n determines a functor from $\varrho_n\mathscr{D}$ to $\mathscr{D}$ which sends an object C in $\varrho_n\mathscr{D}$ to $C(s)$. The functor is exact and so induces a map from $\underline{K}\varrho_n\mathscr{D}$ to $\underline{K}\mathscr{D}$. Since Δ^n has $2^{n+1}-1$ faces, we obtain $2^{n+1}-1$ different maps from $\underline{K}\varrho_n\mathscr{D}$ to $\underline{K}\mathscr{D}$. Their sum is a map from $\underline{K}\varrho_n\mathscr{D}$ to a wedge of $2^{n+1}-1$ copies of $\underline{K}\mathscr{D}$. This map is a homotopy equivalence. The proof uses the additivity theorem of [41].

To explain in an abstract setting what is going on, we introduce the notion of an augmented resolution, as follows. Let T be a discrete group acting on a spectrum X; we also say that X is a T-spectrum. An augmented T-resolution of X is a simplicial T-spectrum $\mathfrak{X} : [n] \mapsto \mathfrak{X}[n]$ having various properties reminiscent of augmented projective resolutions in homological algebra; in particular, $\mathfrak{X}[0] = X$. It turns out that, for such an augmented resolution, one always has

$$|H^{\blacktriangledown}(T; \mathfrak{X}[-])| \simeq \hat{H}^{\blacktriangledown}(T; X)$$

if T is finite. (This time the vertical bars denote the geometric realization of a simplicial spectrum, which is again a spectrum.) If X is (-1)-connected, then it is also true that

$$|QH^{\blacktriangledown}(T; \mathfrak{X}[-])| \simeq Q\hat{H}^{\blacktriangledown}(T; X).$$

This is chiefly useful because the simplicial Z_2-spectrum $\underline{K}\varrho_{\bullet}\mathscr{D}$ is in fact an augmented Z_2-resolution of $\underline{K}\mathscr{D}$. (More or less, this is what the splitting of each $\underline{K}\varrho_n\mathscr{D}$ into copies of $\underline{K}\mathscr{D}$ means.) The infinite loop map Ξ^p which we just obtained can therefore be written in the form

$$Q\underline{L}p_{\blacktriangledown}(R) \to Q\hat{H}^{\blacktriangledown}(Z_2; \underline{K}\mathscr{D}),$$

or as a map of spectra

$$\underline{\underline{L}}p_{\blacktriangledown}(R) \to \hat{H}^{\blacktriangledown}(Z_2; \underline{\underline{K}}\mathcal{D}).$$

This completes the sketch of construction D.

Let λ_i be the homotopy fibre of the composite map of spectra

$$\underline{\underline{L}}p_{\blacktriangledown}(R) \xrightarrow{\Xi^p} \hat{H}^{\blacktriangledown}(Z_2; \underline{\underline{K}}p(R)) \longrightarrow \hat{H}^{\blacktriangledown}(Z_2; \beta^i \underline{\underline{K}}p(R))$$

where β^i is the ith Postnikov base (obtained by killing homotopy groups in dimensions $\geq i$). Then λ_0 is $\underline{\underline{L}}p_{\blacktriangledown}(R)$, and λ_{i+1} maps to λ_i. So we may view the groups $\pi_n(\lambda_i)$ as refined versions of the quadratic (projective) L-groups $L_n(R) = \pi_n(\lambda_0)$, the degree of refinement depending on i. For any $i \geq 0$ there is a long exact 'Rothenberg' sequence

$$\cdots \to \pi_n(\lambda_{i+1}) \to \pi_n(\lambda_i) \to \hat{H}^{n-i}(Z_2; K_i(R)) \to \pi_{n-1}(\lambda_{i+1}) \to \cdots$$

with $n \in \mathbb{Z}$. (See 2.6 below.) For $i=0$ and $i=1$, these Rothenberg sequences are well known; Cappell and Shaneson conjectured their existence for larger i. The case $i=2$ is treated in the thesis of Kennedy [16].

We shall only say a few words about construction E in this outline. It is a nonlinear version of construction D. The nonlinearity comes from replacing chain complexes with an action of $\mathbb{Z}\pi$ by spectra with an action of a topological group G (with $\pi_0(G) = \pi$). The point is that K-theory is sensitive to the difference, but quadratic L-theory is not. See Proposition 6.2. We stress that the map in construction D factors through symmetric L-theory $\underline{\underline{L}}p^{\blacktriangledown}(R)$, but its nonlinear counterpart in construction E (with $R = \mathbb{Z}\pi$) does not do so for any obvious reason.

1. Simplicial spectra

Let Δ be the category with objects $[k] = \{0, 1, \ldots, k\}$ for $k \geq 0$ and with monotone maps as morphisms. If $\mathcal{A}$ is any category, then a simplicial $\mathcal{A}$-object is a contravariant functor from Δ to $\mathcal{A}$. For example, there are simplicial sets, simplicial spaces, simplicial spectra, etc. (So we do not use the expression "simplicial spectrum" in the same way as Kan and Whitehead [15].)

To avoid technical problems, let us agree that space means CW-space in this section; all maps between spaces are to be cellular. Similarly, spectrum means CW-spectrum in the sense of Boardman [3], see also [1], and maps between spectra are to be cellular.

The geometric realization of a simplicial space $\mathfrak{X}$ is the space

$$|\mathfrak{X}| = \coprod_{k \geq 0} \Delta^k \times \mathfrak{X}[k] \Big/ \sim,$$

where $\sim$ denotes the usual relations. If $\mathfrak{X}$ is a simplicial spectrum, its geometric realization is the spectrum

$$|\mathfrak{X}| = \bigvee_{k\geq 0} \Delta^k_+ \wedge \mathfrak{X}[k]\Big/\sim .$$

(See [30, Appendix].)

1.1. Question. Let $\mathfrak{X}$ be a simplicial spectrum. Form the simplicial space $Q(\mathfrak{X}) : [k] \mapsto Q(\mathfrak{X}[k])$. What can be said about the homotopy type of the geometric realization $|Q(\mathfrak{X})|$?

1.2. Definition. Let $\mathfrak{X}$ be as in 1.1. By generously killing homotopy groups in dimensions $\geq n$, it is easy to construct a simplicial spectrum $\beta^n\mathfrak{X}$ and a simplicial map $\mathfrak{X} \to \beta^n\mathfrak{X}$ such that for all k, the homomorphisms

$$\pi_j(\mathfrak{X}[k]) \to \pi_j(\beta^n\mathfrak{X}[k])$$

are isomorphisms for $j<n$ and $\pi_j(\beta^n\mathfrak{X}[k])=0$ for $j\geq n$. Define yet another simplicial spectrum $\varphi^n\mathfrak{X}$ by

$$\varphi^n\mathfrak{X}[k] = \Sigma^{-1}\text{ (cofibre of the map } \mathfrak{X}[k] \to \beta^n\mathfrak{X}[k]).$$

There is an evident map of simplicial spectra from $\varphi^n\mathfrak{X}$ to $\mathfrak{X}$, and we call $\varphi^n\mathfrak{X}$ the $(n-1)$-connected cover of $\mathfrak{X}$.

1.3. Lemma. *There are homotopy equivalences*

$$|Q(\mathfrak{X})| \xleftarrow{\simeq} |Q(\varphi^0\mathfrak{X})| \xrightarrow{\simeq} Q(|\varphi^0\mathfrak{X}|).$$

Proof. The map $\varphi^0\mathfrak{X} \to \mathfrak{X}$ induces a map of simplicial spaces $Q(\varphi^0\mathfrak{X}) \to Q(\mathfrak{X})$ which is a homotopy equivalence in each degree k; therefore the induced map of geometric realizations is a homotopy equivalence $|Q(\varphi^0\mathfrak{X})| \to |Q(\mathfrak{X})|$. (See again [30, Appendix].)

To prove that $|Q(\varphi^0\mathfrak{X})| \simeq Q(|\varphi^0\mathfrak{X}|)$ we assume, as we may, that $\varphi^0\mathfrak{X}$ is a simplicial Ω-spectrum. Then we see that

$$Q(|\varphi^0\mathfrak{X}|) = \operatorname*{holim}_{n\to\infty} \Omega^n(|Q(\Sigma^n\varphi^0\mathfrak{X})|),$$

where the homotopy direct limit is just a telescope. This provides us with a map

$$|Q(\varphi^0\mathfrak{X})| \to Q(|\varphi^0\mathfrak{X}|)$$

which is the inclusion of the beginning of the telescope. The next lemma shows that this map is a homotopy equivalence.

1.4. Lemma. *If $\mathfrak{Y}$ is a simplicial pointed connected space, then $\Omega|\mathfrak{Y}| \simeq |\Omega\mathfrak{Y}|$.*

Proof. This follows from [42, Lemma 5.2] on replacing all spaces in sight by their singular simplicial sets. $\square$

It is often useful to decompose a simplicial spectrum $\mathfrak{X}$ into 'minimal' building blocks. Note first that any subspectrum of the geometric realization $|\mathfrak{X}|$ has the form $|\mathfrak{Y}|$ for a uniquely determined simplicial subspectrum $\mathfrak{Y}\subset\mathfrak{X}$. If $|\mathfrak{Y}|$ has only finitely many cells, we call $\mathfrak{Y}$ finite. With this terminology $\mathfrak{X}$ is the union, even the direct limit, of its finite simplicial subspectra $\mathfrak{Y}$. Call a simplicial spectrum minimal if its geometric realization has exactly one cell (not counting the base point). It is an exercise to verify that a minimal simplicial spectrum $\mathfrak{Z}$ is determined up to isomorphism by two integers, viz., the integer k such that $|\mathfrak{Z}|$ is isomorphic to a k-fold suspension of the sphere spectrum $\underline{S}^0$, and the integer $m\geq 0$ such that $\mathfrak{Z}[m]\neq *$ and $\mathfrak{Z}[j]=*$ for all $j<m$. Every finite simplicial spectrum $\mathfrak{Y}$ has a filtration by simplicial subspectra

$$* = \mathrm{Filt}_0\,\mathfrak{Y} \subset \mathrm{Filt}_1\,\mathfrak{Y} \subset \cdots \subset \mathrm{Filt}_n\,\mathfrak{Y} = \mathfrak{Y}$$

such that $\mathrm{Filt}_{i+1}\,\mathfrak{Y}/\mathrm{Filt}_i\,\mathfrak{Y}$ is minimal whenever $0\leq i<n$.

Suppose now that E is any spectrum and $\mathfrak{X}$ is a simplicial spectrum. Then the rule $[k]\mapsto[E,\mathfrak{X}[k]]$ (where $[E,\mathfrak{X}[k]]$ is the set of homotopy classes of maps from E to $\mathfrak{X}[k]$) defines a simplicial abelian group. Therefore we can use known facts about simplicial abelian groups to obtain information about $\mathfrak{X}$. Specifically, we will use:

1.5. Fact (see Curtis [8, Section 5]). *Let J be a simplicial abelian group. For each $k\geq 0$, let*

$$NJ[k] = \bigcap_{i\neq 0} \ker(d_i : J[k]\to J[k-1])$$

where the d_i are the face operators in J and $i=1,\ldots,k$. Then

$$J[k] = \bigoplus_{p:[k]\twoheadrightarrow[m]} p^*(NJ[m])$$

where the direct sum ranges over all surjective monotone maps $p:[k]\to[m]$, with $m\geq 0$ arbitrary. (Each p^* is injective because p has a right inverse.)

By the remark preceding 1.5 we have a splitting of $[E,\mathfrak{X}[k]]$ into direct summands indexed by surjective monotone maps $p:[k]\to[m]$. Since the splitting is natural in E, we see that the homotopy type of $\mathfrak{X}[k]$ itself splits:

$$\mathfrak{X}[k]\simeq \bigvee_{p:[k]\twoheadrightarrow[m]} N\mathfrak{X}[m].$$

The wedge summands are only well defined up to homotopy equivalence, but this will be sufficient for our purposes.

1.6. Definition. We call $N\mathfrak{X}[k]$ the nondegenerate summand of $\mathfrak{X}[k]$.

1.7. Definition. The n-skeleton of $\mathfrak{X}$ is the smallest simplicial subspectrum $\mathfrak{Y}\subset\mathfrak{X}$ such that $\mathfrak{Y}[k]=\mathfrak{X}[k]$ for all $k\leq n$.

1.8. Observation (cf. Weibel [45, §3]). *The cofibre of the inclusion $|(n-1)$-skeleton of $\mathfrak{X}| \to |n$-skeleton of $\mathfrak{X}|$ is homotopy equivalent to $\Sigma^n N\mathfrak{X}[n]$.*

Proof. Denote the cofibre under consideration by $\mathrm{CF}_n\mathfrak{X}$. There is a natural homotopy class of maps

$$u:\Sigma^n N\mathfrak{X}[n]\to \mathrm{CF}_n\mathfrak{X}$$

given by the composition

$$\Sigma^n N\mathfrak{X}[n]\hookrightarrow \Sigma^n\mathfrak{X}[n]\cong(\Delta^n_+\wedge\mathfrak{X}[n])/(\partial\Delta^n_+\wedge\mathfrak{X}[n])\to\mathrm{CF}_n\mathfrak{X}.$$

It helps to regard u as a natural transformation between exact functors. They are exact in the following sense: If $\mathfrak{Y}$ is a simplicial subspectrum of $\mathfrak{X}$, then the homotopy groups of $\Sigma^n N\mathfrak{Y}[n]$, $\Sigma^n N\mathfrak{X}[n]$ and $\Sigma^n N(\mathfrak{X}/\mathfrak{Y})[n]$ can be arranged in a long exact sequence, and the homotopy groups of $\mathrm{CF}_n\mathfrak{Y}$, $\mathrm{CF}_n\mathfrak{X}$ and $\mathrm{CF}_n(\mathfrak{X}/\mathfrak{Y})$ can also be arranged in a long exact sequence. (This is obvious for the second of these functors, $\mathfrak{X}\mapsto\mathrm{CF}_n\mathfrak{X}$; for the first, $\mathfrak{X}\mapsto\Sigma^n N\mathfrak{X}[n]$, remember that $\Sigma^n N\mathfrak{X}[n]$ is a natural wedge summand of $\Sigma^n\mathfrak{X}[n]$.) It will therefore be sufficient to show that u is a homotopy equivalence when $\mathfrak{X}$ is minimal, and to proceed by induction on the number of cells of $\mathfrak{X}$ when $\mathfrak{X}$ is finite. (If $\mathfrak{X}$ is not finite, use a direct limit argument.) But the minimal case is obvious. □

Define a 'weak homotopy category of simplicial spectra' by formally inverting all morphisms $f:\mathfrak{X}\to\mathfrak{Y}$ in the category of simplicial spectra which are homotopy equivalences in each degree k. The rule which to each simplicial spectrum $\mathfrak{X}$ associates the filtered spectrum $|\mathfrak{X}|$, with filtration given by

$$\mathrm{Filt}_n|\mathfrak{X}|=|n\text{-skeleton of }\mathfrak{X}|$$

is a functor from the weak homotopy category of simplicial spectra to the homotopy category of filtered spectra. Conversely, suppose that

$$E=\bigcup_{n\geq 0}\mathrm{Filt}_n E$$

is a filtered spectrum. We can associate to it a simplicial spectrum E^Δ as follows. For each $k\geq 0$, regard the standard simplex Δ^k_+ (with added base point) as a filtered space such that

$$\mathrm{Filt}_n\Delta^k_+=n\text{-skeleton of }\Delta^k_+.$$

Let $E^\Delta[k]$ be the spectrum of filtration-preserving maps from Δ^k_+ to E. (Mapping spectra are defined in 2.2 below.) Then the rule $[k]\to E^\Delta[k]$ defines a simplicial spectrum.

1.9. Proposition (Kan–Dold Theorem for simplicial spectra). *The functors $\mathfrak{X}\mapsto|\mathfrak{X}|$ and $E\mapsto E^\Delta$ are mutually inverse equivalences of categories* (*between the weak homotopy category of simplicial spectra and the homotopy category of filtered spectra*).

Idea of proof. If the mapping spectra we used in defining E^Δ had some reasonable properties, then we could easily define natural transformations $\mathfrak{X} \to |\mathfrak{X}|^{\lrcorner}$ and $|E^{\lrcorner}| \to E$. (The reader is urged to try.) These should be weak equivalences and filtered homotopy equivalences, respectively, when $\mathfrak{X}$ is minimal and when E has just one cell, and therefore in general. The problem with this sketch proof is that mapping spectra do not have these reasonable properties. We defer the real proof of 1.9; see 2.12. The result will not be used anywhere.

Recall that the ordinary Kan–Dold theorem establishes an equivalence between the category of simplicial abelian groups and the category of chain complexes graded over the positive integers. In particular, if E is any spectrum and if $\mathfrak{X}$ is a simplicial spectrum, then the simplicial abelian group

$$[k] \mapsto [E, \mathfrak{X}[k]]$$

determines a chain complex of abelian groups

$$(*) \qquad [E, N\mathfrak{X}[0]] \leftarrow [E, N\mathfrak{X}[1]] \leftarrow [E, N\mathfrak{X}[2]] \leftarrow \cdots$$

whose differential is induced by the face operators

$$d_0 : N\mathfrak{X}[k] \to N\mathfrak{X}[k-1].$$

1.10. Definition. We call $\mathfrak{X}$ acyclic if the homology of the chain complex $(*)$ is zero for arbitrary E.

By 1.9, the notions of simplicial spectrum and filtered spectrum are essentially equivalent. One can therefore ask what 1.10 looks like in the world of filtered spectra. The following translation turns out to be correct: A filtered spectrum E is acyclic if the inclusion $\mathrm{Filt}_k E \hookrightarrow \mathrm{Filt}_{k+1} E$ is nullhomotopic for each $k \geq 0$. The spectrum E itself is then contractible, but the condition is stronger than that.

2. The norm map and augmented resolutions

There will be a few introductory words about spectra, smash products and mapping spectra. Our point of view is that the smash product of two spectra is not a spectrum, but a bispectrum in the sense of Kan and Whitehead [15]. So in this paper we do not rely on Boardman's ideas regarding smash products (see [3]), nor on those of Adams [1], May [20], Elmendorf [10], Robinson [28], Clapp and Puppe [7].

For the moment, space need not mean CW-space, and spectrum need not mean CW-spectrum.

Recall from Adams [1] that a spectrum X is a sequence of pointed spaces X_n, for $n \in \mathbb{Z}$, and pointed maps $\varepsilon_n : \Sigma X_n \to X_{n+1}$. If each X_n is a pointed CW-space and each ε_n is a CW-homeomorphism of ΣX_n with a CW-subspace of X_{n+1}, call X a CW-spectrum. A function from a spectrum X to a spectrum Y is a collection of maps $\{f_n : X_n \to Y_n \mid n \in \mathbb{Z}\}$ commuting with all the structure in sight.

If X is a CW-spectrum and $k \in \mathbb{Z}$, then there is an evident injection

$$\{(n+k)\text{-cells of } X_n\} \to \{(n+k+1)\text{-cells of } X_{n+1}\},$$

and we call

$$\lim_{n\to\infty} \{(n+k)\text{-cells of } X_n\}$$

the set of k-cells of X. A CW-subspectrum $X' \subset X$ is cofinal if the injection

$$\{\text{cells of } X'\} \to \{\text{cells of } X\}$$

is a bijection. A map from a CW-spectrum X to a spectrum Y is an equivalence class of functions

$$f : X' \to Y$$

where $X' \subset X$ is cofinal; two such functions, say $f : X' \to Y$ and $g : X'' \to Y$, are equivalent if they agree on the cofinal subspectrum $X' \cap X'' \subset X$. Note that Y need not be a CW-spectrum, and if it is, the map need not be cellular.

The singular homology group $H_k(X; \mathbb{Z})$ is

$$\lim_{n\to\infty} H_{k+n}(X_n, \{*\}; \mathbb{Z})$$

for any spectrum X. If X is a CW-spectrum, then the cellular chain complex $\mathrm{cl}(X)$ of X can be defined as

$$\lim_{n\to\infty} \Sigma^{-n}(\mathrm{cl}(X_n)/\mathrm{cl}(\{*\})),$$

so that $H_*(X; \mathbb{Z}) \cong H_*(\mathrm{cl}(X))$.

These superfluous reminders should make the following definitions more acceptable. A bispectrum U is a family of pointed spaces

$$\{U_{m,n} \mid m, n \in \mathbb{Z}\}$$

and pointed maps

$$\alpha_{m,n} : \Sigma U_{m,n} \to U_{m+1,n}, \qquad \omega_{m,n} : \Sigma U_{m,n} \to U_{m,n+1}$$

such that all squares

$$\begin{array}{ccc} \Sigma^2 U_{m,n} & \xrightarrow{\Sigma\omega} & \Sigma U_{m,n+1} \\ {\scriptstyle \Sigma\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \Sigma U_{m+1,n} & \xrightarrow{\omega} & U_{m+1,n+1} \end{array}$$

are strictly anticommutative. (Anticommutativity means here that the two composite maps differ by the automorphism of $\Sigma^2 U_{m,n}$ which interchanges the two suspension coordinates.) If each $U_{m,n}$ is a pointed CW-space, and if the maps $\alpha_{m,n}$ and $\omega_{m,n}$ are all CW-homeomorphisms of $\Sigma U_{m,n}$ with a CW-subspace of $U_{m+1,n}$ and $U_{m,n+1}$ respectively, then we speak of a CW-bispectrum. A function from a bispectrum U to a bispectrum V is a collection of maps $\{f_{m,n}: U_{m,n} \to V_{m,n} \mid m,n \in \mathbb{Z}\}$ commuting with all the structure in sight.

If U is a CW-bispectrum, then there are injections

$$\begin{tikzcd}[column sep=small] & \{(m+n+1+k)\text{-cells of } U_{m+1,n}\} \\ \{(m+n+k)\text{-cells of } U_{m,n}\} \arrow[ur] \arrow[dr] & \\ & \{(m+n+1+k)\text{-cells of } U_{m,n+1}\} \end{tikzcd}$$

and we call

$$\lim_{m,n\to\infty} \{(m+n+k)\text{-cells of } U_{m,n}\}$$

the set of k-cells of U. A CW-sub-bispectrum $U' \subset U$ is cofinal if the injection

$$\{\text{cells of } U'\} \to \{\text{cells of } U\}$$

is a bijection. A map from a CW-bispectrum U to a bispectrum V is an equivalence class of functions

$$f: U' \to V$$

where $U' \subset U$ is a cofinal CW-sub-bispectrum; two such are equivalent if they agree on the intersection of their domains.

The singular homology group $H_k(U;\mathbb{Z})$ of a bispectrum U is

$$\lim_{m,n\to\infty} H_{k+m+n}(U_{m,n},\{*\};\mathbb{Z}),$$

where the maps in the direct system are $(\alpha_{m,n})_*$ or $(-1)^m(\omega_{m,n})_*$ as appropriate.

For a CW-bispectrum U, define the cellular chain complex $\mathrm{cl}(U)$ as

$$\lim_{m,n\to\infty} \Sigma^{-m-n}(\mathrm{cl}(U_{m,n})/\mathrm{cl}(\{*\})).$$

(The chain maps in the direct system are the ones induced by α and ω, up to sign; the signs are chosen as above.) Then $H_*(U;\mathbb{Z}) \cong H_*(\mathrm{cl}(U))$.

2.1. Examples. If X and Y are spectra, then $X \wedge Y$ is a bispectrum, given by

$$(X \wedge Y)_{m,n} := X_m \wedge Y_n.$$

If X and Y are CW-spectra, then clearly $X \wedge Y$ is a CW-bispectrum, and there is an isomorphism of cellular chain complexes

$$\mathrm{cl}(X \wedge Y) \cong \mathrm{cl}(X) \otimes \mathrm{cl}(Y).$$

An important special case is the sphere bispectrum $\underset{=}{S}^0 \wedge \underset{=}{S}^0$. The suspension bispectrum of a pointed space X is defined to be the bispectrum obtained from $\underset{=}{S}^0 \wedge \underset{=}{S}^0$ by smashing each term with X (on the left, say). It is a CW-bispectrum if X is a pointed CW-space. In this situation we shall occasionally speak of pointed maps from X to some bispectrum U; these are just maps from the suspension bispectrum of X to U. (Similarly, a map from a pointed CW-space X to a spectrum Y is a map from $X \wedge \underset{=}{S}^0$ to Y.)

Suppose that we have a partition of $\mathbb{Z}$ into disjoint subsets B_1 and B_2 which are both unbounded from above. This determines a map $\lambda : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ such that $\lambda(0) = (0,0)$ and

$$\lambda(n) - \lambda(n-1) \begin{cases} (1,0) & \text{if } n \in B_1, \\ (0,1) & \text{if } n \in B_2. \end{cases}$$

The map λ gives rise to a functor e_λ from the category of CW-bispectra and their maps to the category of CW-spectra and their maps: Let $e_\lambda(U)_n = U_{\lambda(n)}$ etc., whenever U is a CW-bispectrum. The functor e_λ is an equivalence of categories. If $U = X \wedge Y$ as in 2.1, then $e_\lambda(Z \wedge Y)$ is what Adams calls a handicrafted smash product in [1].

By a combinatorial spectrum V we will mean a family of pointed simplicial sets $\{V_n \mid n \in \mathbb{Z}\}$ and injective simplicial maps $\Sigma V_n \to V_{n+1}$ for all n. (The suspension ΣV_n is a simplicial set such that e.g. $|\Sigma V_n|$ is CW-homeomorphic to $\Sigma |V_n|$; see [8].) The CW-spaces $|V_n|$ and the injections $\Sigma |V_n| \to V_{n+1}$ then form a CW-spectrum, which we also denote by V. We will also talk about combinatorial maps between combinatorial spectra. (As usual, maps are equivalence classes of functions.)

2.2. Notation. If X and Y are CW-spectra, let map(X, Y) be the geometric realization of the simplicial set of cellular maps from X to Y; so a k-simplex in map(X, Y) is a cellular map from $\Delta^k_+ \wedge X$ to Y. The mapping spectrum $\underline{\underline{\mathrm{map}}}(X, Y)$ is the Ω-spectrum whose ith term is the space map$(X, S^i \wedge Y)$. It is a CW-spectrum and even a combinatorial spectrum. (Do not confuse the pointed space S^i and the suspension spectrum $\underset{=}{S}^i$.) If X is merely a pointed CW-space, with suspension spectrum $\underset{=}{X}$, then we also write map(X, Y) and $\underline{\underline{\mathrm{map}}}(X, Y)$ to mean map$(\underset{=}{X}, Y)$ and $\underline{\underline{\mathrm{map}}}(\underset{=}{X}, Y)$, respectively.

2.3. Remarks. (i) There is a chain of natural homotopy equivalences connecting a CW-spectrum Y with the mapping spectrum $\underline{\underline{\mathrm{map}}}(S^0, Y)$. See 2.12 below. The distinction between Y and $\underline{\underline{\mathrm{map}}}(S^0, Y)$ may be pedantic, but here and in Section 6 it is useful.

(ii) Suppose that V is a combinatorial spectrum and $f: V \to \underline{\underline{\mathrm{map}}}(X, Y)$ is a combinatorial map. It is not hard to see that f is adjoint to, or worth as much as, a cellular map of CW-bispectra

$$V \wedge X \to \underset{=}{S}^0 \wedge Y.$$

(iii) For fixed CW-spectra X and Y, there is a natural bijection

$$[V, \underline{\underline{\mathrm{map}}}(X, Y)] \to [e_\lambda(V \wedge X), Y].$$

Here V can be any CW-spectrum; square brackets denote homotopy classes of maps, and $e_\lambda(V \wedge X)$ is a handicrafted smash product of V and X. (Proof: Reduce to the combinatorial setting; then use (ii) and the fact that $e_\lambda(\underline{S}^0 \wedge Y) \simeq Y$, which can be extracted from [1, Part III, Section 4].)

So much for preliminaries; we can move into deeper waters again. From now on until Section 5, "spectrum" will mean "CW-spectrum". Let T be a discrete group, and let X be a spectrum on which T acts by cellular maps; we also say that X is a T-spectrum. The homotopy fixed point spectrum of X is

$$H^{\blacktriangledown}(T; X) = \underline{\underline{\mathrm{map}}}_T((ET)_+, X),$$

the spectrum of T-maps from $(ET)_+$ to X. (Here ET can be any contractible CW-space on which T acts freely by cellular maps.) The homotopy orbit spectrum of X is

$$H_{\blacktriangledown}(T; X) = \frac{\underline{\underline{\mathrm{map}}}(S^0, (ET)_+ \wedge X)}{T}$$

where T acts diagonally on $(ET)_+ \wedge X$ and by the induced action (which is still free) on the mapping spectrum. This looks somewhat different from what was promised in the introduction. Our justification is that the target of the norm map to be constructed is (inevitably) a combinatorial spectrum, so its source had better be one, too. Notice that by 2.3(i) there is a chain of natural homotopy equivalences connecting $H_{\blacktriangledown}(T; X)$ and $(ET)_+ \wedge_T X$. (We pretend that it is clear what $(ET)_+ \wedge_T X$ means. It is not entirely clear, because T was supposed to act on X and on $(ET)_+ \wedge X$ by maps, not necessarily by functions. The induced action on $\underline{\underline{\mathrm{map}}}(S^0, (ET)_+ \wedge X)$ is by functions and causes no problems.)

The inclusion $\{1\} \to T$ induces a map from $X \simeq \underline{\underline{\mathrm{map}}}(S^0, X) = H_{\blacktriangledown}(\{1\}; X)$ to $H_{\blacktriangledown}(T; X)$. It also induces, by restriction, a map from $H^{\blacktriangledown}(T; X)$ to $H^{\blacktriangledown}(\{1\}; X) \simeq \underline{\underline{\mathrm{map}}}(S^0, X) \simeq X$.

2.4. Proposition (Dwyer). *If T is finite, then there is a norm map*

$$\mathcal{N} : H_{\blacktriangledown}(T; X) \to H^{\blacktriangledown}(T; X)$$

such that the composition

$$X \longrightarrow H_{\blacktriangledown}(T; X) \xrightarrow{\mathcal{N}} H^{\blacktriangledown}(T; X) \longrightarrow X$$

belongs to the homotopy class $\tau = \sum_{t \in T} t$. (Observe that each element of the group ring $\mathbb{Z}T$ determines a homotopy class of maps from X to X.) The norm map is natural in the variable X, for fixed T.

Let us agree on the following notation for the proof. Any spectrum Y can be smashed with a pointed space V, on the left or on the right; the resulting spectra $V\wedge Y$ and $Y\wedge V$ are such that $(V\wedge Y)_n = V\wedge Y_n$ and $(Y\wedge V)_n = Y_n\wedge V$. Clearly $V\wedge Y$ and $Y\wedge V$ are isomorphic. A spectrum Y can also be smashed with another spectrum Y', but the result will be a bispectrum, and one should note that $Y\wedge Y'$ and $Y'\wedge Y$ are not isomorphic bispectra in general. Then there are objects like

$$Y(1)\wedge Y(2)\wedge\cdots\wedge Y(r)$$

where each $Y(i)$ is either a spectrum or a pointed space. Such a smash product will be a pointed space if all the $Y(i)$ are; it will be a spectrum if just one of the $Y(i)$ is a spectrum, and a bispectrum if just two of the $Y(i)$ are spectra. (We will ensure that no more than two spectrum factors occur.)

Proof of Proposition 2.4. Suppose for the moment that we can find a map of spectra

$$f:(ET)_+\wedge \underline{S}^0\wedge (ET)_+\to T_+\wedge \underline{S}^0$$

having, briefly, the following properties.

(i) The homotopy class of f is

$$\tau\in \mathbb{Z}T\cong \pi_0(T_+\wedge \underline{S}^0).$$

(ii) The map f is a $T\times T$-map.

We pause to explain (i) and (ii). Since $(ET)_+\wedge \underline{S}^0\wedge(ET)_+$ is homotopy equivalent to $\underline{S}^0$, the map f represents an element in $\pi_0(T_+\wedge\underline{S}^0)$. Since $T_+\wedge\underline{S}^0$ is a wedge sum of copies of $\underline{S}^0$, with one summand for each element of T, we can identify $\pi_0(T_+\wedge\underline{S}^0)$ with $\mathbb{Z}T$. This explains (i). The group $T\times T$ acts on the spectrum $(ET)_+\wedge\underline{S}^0\wedge(ET)_+$ in the obvious way. It acts on the set T by

$$(a,b)\cdot g = agb^{-1}\quad \text{for } (a,b)\in T\times T \text{ and } g\in T;$$

smash with the trivial action on $\underline{S}^0$ to obtain an action on $T_+\wedge\underline{S}^0$. This explains (ii).

Given such an f, construct the norm map $\mathscr{N}$ as follows. Write $V=\underline{\underline{\mathrm{map}}}(\underline{S}^0,(ET)_+\wedge X)$ whenever that is convenient. The identity map from V to $\underline{\underline{\mathrm{map}}}(\underline{S}^0,(ET)_+\wedge X)$ is adjoint to a map of bispectra

$$e: V\wedge\underline{S}^0\to\underline{S}^0\wedge(ET)_+\wedge X,$$

by 2.3(ii). The composite map of bispectra

$$\begin{tikzcd}
V\wedge(ET)_+\wedge\underline{S}^0 \arrow[d, "\cong"'] & \\
(ET)_+\wedge(V\wedge\underline{S}^0) \arrow[r, "\mathrm{id}\wedge e"] & (ET)_+\wedge(\underline{S}^0\wedge(ET)_+\wedge X) \arrow[d, "f\wedge \mathrm{id}_X"] \\
\underline{S}^0\wedge X & T_+\wedge\underline{S}^0\wedge X \arrow[l, "p"']
\end{tikzcd}$$

where p sends $t \wedge s \wedge x$ to $s \wedge tx$, is adjoint to a map of spectra

$$\text{pre-}\mathcal{N}: V \to \underline{\mathrm{map}}((ET)_+ \wedge \underline{S}^0, X)$$

by 2.3(ii). Inspection shows that pre-$\mathcal{N}$ factors through the projection

$$V \to V/T = H_\bullet(T; X)$$

and through the inclusion

$$H^\bullet(T; X) = \mathrm{map}_T((ET)_+ \wedge S^0, X) \hookrightarrow \mathrm{map}((ET)_+ \wedge S^0, X).$$

(Use property (ii) of f.) So we obtain

$$\mathcal{N}: H_\bullet(T; X) \to H^\bullet(T; X)$$

from pre-$\mathcal{N}$, and property (i) of f implies that the composition

$$X \longrightarrow H_\bullet(T; X) \xrightarrow{\mathcal{N}} H^\bullet(T; X) \longrightarrow X$$

is $\tau \in \mathbb{Z}T$.

It remains to construct f. Since S^i is or can be identified with the geometric realization of a simplicial set for any $i \geq 0$ (with just two nondegenerate simplices), the product $(S^i)^T$ has a CW-structure coming from the simplicial product. Let $(\underline{S}^0)^T$ be the spectrum such that

$$((\underline{S}^0)^T)_i = (S^i)^T,$$

using this specific CW-structure; define the maps $\Sigma((\underline{S}^0)^T)_i \to ((\underline{S}^0)^T)_{i+1}$ in such a way that the various projections from $(\underline{S}^0)^T$ to $\underline{S}^0$ become maps of spectra. It is well known that products of this type are homotopy equivalent to wedge sums; in particular, the inclusion

$$T_+ \wedge \underline{S}^0 \to (\underline{S}^0)^T$$

is a homotopy equivalence. It induces a homotopy equivalence

$$z: \mathrm{map}(\underline{S}^0, T_+ \wedge \underline{S}^0) \to \mathrm{map}(\underline{S}^0, (\underline{S}^0)^T)$$

of mapping spaces (see 2.2). Further, z is a $T \times T$-map: the action of $T \times T$ on the set T specified earlier leads to actions of $T \times T$ on source and target of z. The diagonal embedding

$$\mu: \underline{S}^0 \to (\underline{S}^0)^T$$

is a 0-simplex in $\mathrm{map}(\underline{S}^0, (\underline{S}^0)^T)$ which is fixed under the $T \times T$-action. Writing Y for the space of $T \times T$-maps from $ET \times ET$ to $\mathrm{map}(\underline{S}^0, T_+ \wedge \underline{S}^0)$, and writing Y' for the space of $T \times T$-maps from $ET \times ET$ to $\mathrm{map}(\underline{S}^0, (\underline{S}^0)^T)$, we obtain a map

$$\bar{z}: Y \to Y'$$

given by composition with z. It is still a homotopy equivalence. (It is convenient to define Y and Y' as simplicial sets.) Let $\bar{\mu} \in Y'$ be the point or 0-simplex given by the constant map from $ET \times ET$ to $\mathrm{map}(\underline{S}^0, (\underline{S}^0)^T)$ with constant value μ. Choose a pair (f, w), where f is a 0-simplex in Y and w is a path (in the shape of a 1-simplex) connecting $\bar{z}(f)$ with $\bar{\mu}$. Since $\bar{z}$ is a homotopy equivalence, such a choice is possible; it is a contractible choice. The 0-simplex f is then a $T \times T$-map

$$ET \times ET \to \mathrm{map}(\underline{S}^0, T_+ \wedge \underline{S}^0)$$

or, in adjoint form, a $T \times T$-map of spectra

$$(ET)_+ \wedge (ET)_+ \wedge \underline{S}^0 \to T_+ \wedge \underline{S}^0.$$

The proof of 2.4 is complete. (It is remarkable that the sphere spectrum $\underline{S}^0$ was much more deeply involved than X.) □

2.5. Definition. A spectrum X with T-action is induced if there exists a spectrum Y (without T-action) and a map $T_+ \wedge Y \to X$ which is a homotopy equivalence of spectra and commutes with T-actions. (Compare the notion of an induced module in homological algebra; see [12].)

2.6. Observation. *If a T-spectrum X is induced, then the norm map $H_\bullet(T; X) \to H^\bullet(T; X)$ is a homotopy equivalence. (We assume that T is finite.)*

Proof. In this case there is, up to homotopy, just one possible way to write the map $\tau: X \to X$, where $\tau \in \mathbb{Z}T$ is the sum of the group elements, as a composition

$$X \longrightarrow H_\bullet(T; X) \dashrightarrow H^\bullet(T; X) \longrightarrow X.$$

(See the precise formulation of 2.4.) If $T_+ \wedge Y \to X$ is a homotopy equivalence as in 2.5, then $H_\bullet(T; X) \simeq Y$ and $H^\bullet(T; X) \simeq Y$, and the broken arrow corresponds to the identity $Y \to Y$. □

2.7. Definition. The mapping cone of the norm map $H_\bullet(T; X) \to H^\bullet(T; X)$ is denoted by $\hat{H}^\bullet(T; X)$.

2.8. Remark. The functors $X \mapsto H_\bullet(T; X)$, $X \mapsto H^\bullet(T; X)$ and $X \mapsto \hat{H}^\bullet(T; X)$ from T-spectra to spectra have a homotopy invariance property and an exactness property. That is, if $X \to Y$ is a T-map between T-spectra which is a homotopy equivalence of the underlying spectra, then the induced maps $H_\bullet(T; X) \to H_\bullet(T; Y)$, $H^\bullet(T; X) \to H^\bullet(T; Y)$ and $\hat{H}^\bullet(T; X) \to \hat{H}^\bullet(T; Y)$ are homotopy equivalences; and if $X \hookrightarrow Y$ is an inclusion map between T-spectra with quotient Y/X, then the diagrams

$$H_\bullet(T; X) \to H_\bullet(T; Y) \to H_\bullet(T; Y/X),$$

$$H^\bullet(T; X) \to H^\bullet(T; Y) \to H^\bullet(T; Y/X),$$

$$\hat{H}^\bullet(T; X) \to \hat{H}^\bullet(T; Y) \to \hat{H}^\bullet(T; Y/X)$$

are cofibrations up to homotopy (which means that they give rise to long exact sequences of homotopy groups). This is clear for $H_\bullet(T;-)$; it is also clear for $H^\bullet(T;-)$ if one remembers that cofibrations up to homotopy and fibrations up to homotopy are the same in the category of spectra; and it follows then for $\hat{H}^\bullet(T;-)$.

It may also be asked whether the functors $H_\bullet(T;-)$, $H^\bullet(T;-)$ and $\hat{H}^\bullet(T;-)$, applied to a simplicial T-spectrum, commute with geometric realization. This is false in general for $H^\bullet(T;-)$ and $\hat{H}^\bullet(T;-)$; see 2.10. But it is correct for $H_\bullet(T;-)$, so that

$$|H_\bullet(T;\mathfrak{X}[-])| \simeq H_\bullet(T;|\mathfrak{X}|)$$

for any simplicial T-spectrum $\mathfrak{X}$. (This would be obvious if we could use the naive definition of $H_\bullet(T;-)$, from the introduction.) Proof: Write $\mathfrak{X}^{(n)}$ for the n-skeleton of $\mathfrak{X}$, in the sense of 1.7; allow $n=\infty$. There are straightforward maps

$$\Delta_+^k \wedge H_\bullet(T;\mathfrak{X}^{(n)}[k]) \to H_\bullet(T;\Delta_+^k \wedge \mathfrak{X}^{(n)}[k]) \to H_\bullet(T;|\mathfrak{X}^{(n)}|)$$

for $n,k \geq 0$, giving rise to

$$v_n : |H_\bullet(T;\mathfrak{X}^{(n)}[-])| \to H_\bullet(T;|\mathfrak{X}^{(n)}|).$$

The maps v_n are homotopy equivalences for finite n (by induction on n, using the exactness property of $H_\bullet(T;-)$ and 1.8). By inspection, v_∞ is simply the union or direct limit of the v_n, so it is also a homotopy equivalence. (It is only the direct limit argument which fails if $H_\bullet(T;-)$ is replaced by $H^\bullet(T;-)$ or $\hat{H}^\bullet(T;-)$.)

2.9. Definition. Let X be a T-spectrum. An augmented T-resolution of X is a simplicial T-spectrum $\mathfrak{X}$ such that

(i) $\mathfrak{X}[0]=X$;

(ii) $\mathfrak{X}$ is acyclic in the sense of 1.10;

(iii) for each $k>0$, the cofibre of the degeneracy map $\mathfrak{X}[0]\to\mathfrak{X}[k]$ is an induced T-spectrum.

Any T-spectrum X admits an augmented T-resolution. Namely, define a filtered T-spectrum E by

$$\mathrm{Filt}_0 E = X,$$

$$\mathrm{Filt}_{i+1} = \text{mapping cone of the } T\text{-map } c\wedge \mathrm{id} : T_+ \wedge \mathrm{Filt}_i E \to S^0 \wedge \mathrm{Filt}_i E,$$

where $T_+\wedge \mathrm{Filt}_i E$ has the diagonal T-action and $c: T_+ \to S^0$ is the map of sets which sends only the base point to the base point. Then apply 1.9 to obtain an augmented T-resolution of X.

2.10. Proposition. *Adopt the notation of* 2.9, *and let* $H^\bullet(T;\mathfrak{X}[-])$ *be the simplicial* *spectrum obtained from $\mathfrak{X}$ by applying the functor $H^{\blacktriangledown}(T;-)$. Then, if T is finite,*

$$|H^{\blacktriangledown}(T;\mathfrak{X}[-])| \simeq \hat{H}^{\blacktriangledown}(T;X).$$

If X is (-1)-connected, then one also has

$$|QH^{\blacktriangledown}(T;\mathfrak{X}[-])| \simeq Q\hat{H}^{\blacktriangledown}(T;X).$$

Proof. First note that the map $|H^{\blacktriangledown}(T;\mathfrak{X}[-])| \to |\hat{H}^{\blacktriangledown}(T;\mathfrak{X}[-])|$ is a homotopy equivalence. Indeed its cofibre is

$$|\Sigma H_{\blacktriangledown}(T;\mathfrak{X}[-])| = \Sigma|H_{\blacktriangledown}(T;\mathfrak{X}[-])| \simeq \Sigma H_{\blacktriangledown}(T;|\mathfrak{X}|)$$

by 2.8, which is contractible since $|\mathfrak{X}|$ is contractible by 2.9(ii). Next, observe that

$$|\hat{H}^{\blacktriangledown}(T;\mathfrak{X}[-])| \simeq \hat{H}^{\blacktriangledown}(T;\mathfrak{X}[0]) = \hat{H}^{\blacktriangledown}(T;X)$$

because all simplicial operators $\hat{H}^{\blacktriangledown}(T;\mathfrak{X}[m]) \to \hat{H}^{\blacktriangledown}(T;\mathfrak{X}[k])$ are homotopy equivalences. (It is sufficient to check this with $m=0$, in which case it follows from 2.9 (iii), and 2.6 and the first part of 2.8.) This proves that $|H^{\blacktriangledown}(T;\mathfrak{X}[-])| \simeq \hat{H}^{\blacktriangledown}(T;X)$.

Much the same argument can be used to prove that $|QH^{\blacktriangledown}(T;\mathfrak{X}[-])| \simeq Q\hat{H}^{\blacktriangledown}(T;X)$ if $\mathfrak{X}[k]$ is (-1)-connected for all $k \geq 0$ (not just for $k=0$). In this case the diagram

$$|QH^{\blacktriangledown}(T;\mathfrak{X}[-])| \to |Q\hat{H}^{\blacktriangledown}(T;\mathfrak{X}[-])| \to |Q\Sigma H_{\blacktriangledown}(T;\mathfrak{X}[-])|$$

is a fibration up to homotopy by [4,2 Lemma 5.2]; here we use the connectedness assumption. Now 1.3 implies that $|Q\Sigma H_{\blacktriangledown}(T;\mathfrak{X}[-])| \simeq Q|\Sigma H_{\blacktriangledown}(T;\mathfrak{X}[-])|$, and we know already that $|\Sigma H_{\blacktriangledown}(T;\mathfrak{X}[-])|$ is contractible. Therefore,

$$|QH^{\blacktriangledown}(T;\mathfrak{X}[-])| \simeq |Q\hat{H}^{\blacktriangledown}(T;\mathfrak{X}[-])| \simeq Q\hat{H}^{\blacktriangledown}(T;X),$$

because all simplicial operators in $Q\hat{H}^{\blacktriangledown}(T;\mathfrak{X}[-])$ are homotopy equivalences.

Suppose finally that X is (-1)-connected, but that nothing of the sort is known about $\mathfrak{X}[k]$ for $k>0$. Proceeding as in 1.2, build a simplicial T-spectrum $\varphi^0\mathfrak{X}$ and a simplicial map $\varphi^0\mathfrak{X} \to \mathfrak{X}$ which is a (-1)-connected Postnikov cover in each degree k. Since $\varphi^0\mathfrak{X}$ is also an augmented T-resolution, we are in the special situation discussed before and

$$|QH^{\blacktriangledown}(T;\varphi^0\mathfrak{X}[-])| \simeq Q\hat{H}^{\blacktriangledown}(T;\varphi^0 X) \simeq Q\hat{H}^{\blacktriangledown}(T;X).$$

But the maps $QH^{\blacktriangledown}(T;\varphi^0\mathfrak{X}[k]) \to QH^{\blacktriangledown}(T;\mathfrak{X}[k])$ are homotopy equivalences for all $k \geq 0$, since the homotopy groups of $H^{\blacktriangledown}(T;\beta^0\mathfrak{X}[k])$ are trivial in dimensions ≥ 0 (see 1.2 and 2.8). Therefore,

$$|QH^{\blacktriangledown}(T;\mathfrak{X}[-])| \simeq |QH^{\blacktriangledown}(T;\varphi^0\mathfrak{X}[-])| \simeq Q\hat{H}^{\blacktriangledown}(T;X). \qquad \square$$

The norm map of 2.4 has a more simple-minded chain level analogue. Suppose that D is a chain complex of abelian groups (graded over the integers), equipped with an action of the group ring $\mathbb{Z}T$. Write D_T for $\mathbb{Z}\otimes_{\mathbb{Z}T} D$ and D^T for

$\mathrm{Hom}_{\mathbb{Z}T}(\mathbb{Z}, D)$, where $\mathbb{Z}$ is the trivial (left or right) $\mathbb{Z}T$-module. Let P be a left projective resolution of $\mathbb{Z}$ over the ring $\mathbb{Z}T$; since $\mathbb{Z}T$ has a standard involution sending the group elements to their inverses, we may also use P as a right projective resolution. The bottom row of the commutative diagram

$$\begin{tikzcd} & D \arrow[r, "\cdot\tau"] \arrow[d, two heads] & D & \\ P\otimes_{\mathbb{Z}T} D \arrow[r] & D_T \arrow[r, dashed] & D^T \arrow[u, hook] \arrow[r] & \mathrm{Hom}_{\mathbb{Z}T}(P, D), \end{tikzcd}$$

where $\tau \in \mathbb{Z}T$ is the sum of the group elements, defines an algebraic norm map. If D is the cellular chain complex of a T-spectrum X, and if P is the cellular chain complex of ET, then the cellular chain complex of $H_\bullet(T;X)$ maps to $P\otimes_{\mathbb{Z}T} D$, and the cellular chain complex of $H^\bullet(T;X)$ maps to $\mathrm{Hom}_{\mathbb{Z}T}(P, D)$.

2.11. Observation. The square of chain maps

$$\begin{tikzcd} \mathrm{cl}(H_\bullet(T;X)) \arrow[r] \arrow[d, dashed] & \mathrm{cl}(H^\bullet(T;X)) \arrow[d] \\ P\otimes_{\mathbb{Z}T} D \arrow[r] & \mathrm{Hom}_{\mathbb{Z}T}(P, D) \end{tikzcd}$$

is strictly commutative. (The upper horizontal arrow is the map of cellular chain complexes induced by the norm map of 2.4, and the lower one is the algebraic norm map.)

2.12. Postponed proofs. We return to the mapping spectra of 2.2 in order to prove 1.9 and to compare $\underline{\underline{\mathrm{map}}}(S^0, X)$ and X, for any spectrum X.

Choose a specific partition of $\mathbb{Z}$ into two disjoint sets unbounded from above, and write e_λ for the resulting functor from bispectra to spectra. From [1, Part III, Section 4] one can extract a chain of homotopy equivalences beginning with an arbitrary spectrum X and ending with $e_\lambda(\underline{S}^0 \wedge X)$, or with $e_\lambda(X \wedge \underline{S}^0)$ if preferred. Therefore

$$\begin{aligned} \underline{\underline{\mathrm{map}}}(S^0, Y) &\simeq e_\lambda(\underline{\underline{\mathrm{map}}}(S^0, Y) \wedge \underline{S}^0) && \text{(by Adams)} \\ &\simeq e_\lambda(\underline{S}^0 \wedge Y) && \text{(by evaluation)} \\ &\simeq Y && \text{(by Adams).} \end{aligned}$$

The second homotopy equivalence is induced by the evaluation

$$\underline{\underline{\mathrm{map}}}(S^0, Y) \wedge \underline{S}^0 \to \underline{S}^0 \wedge Y$$

(which is adjoint to the identity on $\underline{\underline{\mathrm{map}}}(S^0, Y)$, in the sense of 2.3(ii), and which is a homotopy equivalence of bispectra). Remember that $\underline{\underline{\mathrm{map}}}(S^0, Y) = \underline{\mathrm{map}}(S^0, Y)$. We conclude that

$$\underline{\underline{\mathrm{map}}}(S^0, Y) \simeq Y$$

by a natural chain of homotopy equivalences.

To prove 1.9, we ought to construct natural transformations $\mathfrak{X} \to |\mathfrak{X}|^{\Delta}$ and $|E^{\Delta}| \to E$, where $\mathfrak{X}$ denotes simplicial spectra and E denotes filtered spectra. But it will be easier, and quite sufficient, to construct natural transformations

$$v_1 : \underline{\underline{\mathrm{map}}}(S^0, \mathfrak{X}) \to |\mathfrak{X}|^{\Delta}, \qquad v_2 : |E^{\Delta}| \wedge S^0 \to \underline{S}^0 \wedge E.$$

(Here $\underline{\underline{\mathrm{map}}}(S^0, \mathfrak{X})$ is the simplicial spectrum given in degree k by $\underline{\underline{\mathrm{map}}}(S^0, \mathfrak{X}[k])$. The map v_2 is one of filtered bispectra.) The functors $E \to \underline{S}^0 \wedge E$ and $E \to E \wedge \underline{S}^0$ from filtered spectra to filtered bispectra are equivalences on the homotopy categories; the functor $\mathfrak{X} \mapsto \underline{\underline{\mathrm{map}}}(S^0, \mathfrak{X})$ is related to the identity functor $\mathfrak{X} \mapsto \mathfrak{X}$ by a chain of natural isomorphisms (in the weak homotopy category of simplicial spectra). Therefore v_1 and v_2 are indeed sufficient, provided they can be shown to be (weak) homotopy equivalences.

Define v_1 in degree k to be the composition

$$\begin{tikzcd}
\underline{\underline{\mathrm{map}}}(\underline{S}^0, \mathfrak{X}[k]) \arrow[r, "f"] & \mathrm{filt}\underline{\underline{\mathrm{map}}}(\Delta^k_+ \wedge \underline{S}^0, \Delta^k_+ \wedge \mathfrak{X}[k]) \arrow[d, "g"] \\
 & \mathrm{filt}\underline{\underline{\mathrm{map}}}(\Delta^k_+ \wedge \underline{S}^0, |\mathfrak{X}|)
\end{tikzcd}$$

where f is obtained by smashing all maps in sight with the identity $\Delta^k_+ \to \Delta^k_+$, and g is composition with the canonical map $\Delta^k_+ \wedge \mathfrak{X}[k] \to |\mathfrak{X}|$. We write

$$\mathrm{filt}\underline{\underline{\mathrm{map}}}(\cdots)$$

for the subspectrum of the appropriate mapping spectrum consisting of the filtration-preserving maps. Note that

$$\mathrm{filt}\underline{\underline{\mathrm{map}}}(\Delta^k_+ \wedge \underline{S}^0, |\mathfrak{X}|) = \mathrm{filt}\underline{\underline{\mathrm{map}}}(\Delta^k_+, |\mathfrak{X}|) = |\mathfrak{X}|^{\Delta}[k].$$

Next, define v_2 in such a way that the composition

$$\begin{tikzcd}
\mathrm{filt}\underline{\underline{\mathrm{map}}}(\Delta^k_+ \wedge \underline{S}^0, E) \wedge (\Delta^k_+ \wedge \underline{S}^0) \arrow[r, "="] & (E^{\Delta}[k] \wedge \Delta^k_+) \wedge \underline{S}^0 \arrow[d] \\
\underline{S}^0 \wedge E & |E^{\Delta}| \wedge \underline{S}^0 \arrow[l, "v_2"]
\end{tikzcd}$$

agrees with the evaluation map (adjoint to the identity on $\mathrm{filt}\underline{\underline{\mathrm{map}}}(\Delta^k_+ \wedge \underline{S}^0, E)$ in the sense of 2.3(ii)). Complete the proof of 1.9 by observing that v_1 is a weak homotopy equivalence when $\mathfrak{X}$ is minimal, and v_2 is a filtered homotopy equivalence when E has only one cell. □

3. Linear duality

Throughout this section we fix an integer $n \geq 0$ and investigate duality in the category $\varrho_n\mathscr{D}$. Recall that the objects of $\varrho_n\mathscr{D}$ are covariant functors $C: \mathrm{po}[n] \to \mathscr{D}$, where $\mathrm{po}[n]$ is the partially ordered set of faces of the simplex Δ^n. As explained in the outline, $\varrho_n\mathscr{D}$ inherits from $\mathscr{D}$ the structure of a category with cofibrations and weak equivalences. Occasionally it is useful to know that $\varrho_n\mathscr{D}$ also inherits from $\mathscr{D}$ a cylinder functor; see [41, Section 1.6]. (The cylinder functor in $\mathscr{D}$ is given by the usual construction which converts an arbitrary chain map into a cofibration, replacing its target by a mapping cylinder.)

A fundamental difficulty in dealing with $\varrho_n\mathscr{D}$ is that a weak equivalence $f: B \to C$ in $\varrho_n\mathscr{D}$ need not have a homotopy inverse. One way to avoid it is to impose conditions on B and C. Let $\widehat{\mathrm{po}}[n]$ be the partially ordered set of CW-subspaces of the standard simplex Δ^n, so that $\mathrm{po}[n] \subset \widehat{\mathrm{po}}[n]$. Call a functor $C: \mathrm{po}[n] \to \mathscr{D}$ well behaved if it extends to an intersection-preserving functor $\hat{C}: \widehat{\mathrm{po}}[n] \to \mathscr{D}$. The expression "intersection-preserving" means that, for any two CW-subspaces $X, Y \subset \Delta^n$, the square

$$\begin{array}{ccc} \hat{C}(X \cap Y) & \longrightarrow & \hat{C}(Y) \\ \downarrow & & \downarrow \\ \hat{C}(X) & \longrightarrow & \hat{C}(X \cup Y) \end{array}$$

consists of cofibrations and is a pushout square; it also means that $C(\emptyset) = 0$. Note that $\hat{C}$ is determined by C up to unique isomorphism if it exists. We shall therefore write $C(X)$ instead of $\hat{C}(X)$ whenever $\hat{C}$ does exist and X is a CW-subspace of Δ^n.

If C is a well-behaved object in $\varrho_n\mathscr{D}$, and D is any object, we let

$$\mathrm{Hom}_R(C, D) \subset \prod_{s \in \mathrm{po}[n]} \mathrm{Hom}_R(C(s), D(s))$$

be the chain subcomplex consisting of those collections $\{g_s \mid s \in \mathrm{po}[n]\}$ which are natural in s. (We explain this a little more: A j-chain in $\prod \mathrm{Hom}_R(C(s), D(s))$ is a collection $\{g_s \mid s \in \mathrm{po}[n]\}$, where each $g_s: \Sigma^j C(s) \to D(s)$ is a map of graded R-modules which need not commute with differentials. Such a collection $\{g_s\}$ is natural if the square

$$\begin{array}{ccc} \Sigma^j C(s) & \xrightarrow{g_s} & D(s) \\ \cap\downarrow & & \downarrow \\ \Sigma^j C(q) & \xrightarrow{g_q} & D(q) \end{array}$$

is commutative whenever s, q are elements of $\mathrm{po}[n]$ with $s \leq q$.) Observe that $H_j(\mathrm{Hom}_R(C, D))$ is the set of homotopy classes of natural chain maps $\Sigma^j C \to D$. (This is perhaps a good moment for saying that the sign conventions we use in defining Hom and $\otimes$ of chain complexes etc. are those of Dold [9].)

3.1. Lemma. *Let C, D and D' be objects of $\varrho_n\mathcal{D}$, and assume that C is well behaved. Let $f: D \to D'$ be a weak equivalence. Then the homomorphism of graded groups*

$$H_*(\mathrm{Hom}_R(C, D)) \to H_*(\mathrm{Hom}_R(C, D'))$$

induced by f is an isomorphism.

Further, suppose that $E \to E' \to E''$ is a short exact sequence in $\varrho_n\mathcal{D}$ (so that $E \to E'$ is a cofibration with quotient E''). Then the resulting sequence of chain complexes

$$\mathrm{Hom}_R(C, E) \to \mathrm{Hom}_R(C, E') \to \mathrm{Hom}_R(C, E'')$$

is also short exact, giving rise to a long exact sequence of homology groups.

Proof. We note that the chain complexes $\mathrm{Hom}_R(C, D)$ etc. have natural filtrations: Let $\mathrm{Filt}_i(\mathrm{Hom}_R(C, D))$ consist of those natural collections $\{g_s \mid s \in \mathrm{po}[n]\}$ which satisfy $g_s = 0$ whenever the dimension of s is $< n - i$. By inspection, the filtration quotients take the form

$$\mathrm{Filt}_i(\cdots)/\mathrm{Filt}_{i-1}(\cdots) \cong \prod_{\dim(s) = i} \mathrm{Hom}_R(C(s/\partial s), D(s))$$

where ∂s is the entire boundary of the face s and $C(s/\partial s)$ is an abbreviation for the quotient of the cofibration $C(\partial s) \to C(s)$. It follows by induction on i that the homomorphisms

$$H_*(\mathrm{Filt}_i(\mathrm{Hom}_R(C, D))) \to H_*(\mathrm{Filt}_i(\mathrm{Hom}_R(C, D')))$$

induced by the weak equivalence $f: D \to D'$ are isomorphisms, with $-1 \le i \le n$. For $i = n$ this is what we want. The exactness statement in 3.1 can be proved in the same way, using the natural filtrations of $\mathrm{Hom}_R(C, E)$ etc. and induction on i. □

A simple consequence of 3.1 is that a weak equivalence $f: C \to D$ is a natural homotopy equivalence provided C and D are well behaved. The homotopy inverse $g: D \to C$ can be found in the class $[g] \in H_0(\mathrm{Hom}_R(D, C))$ which maps to $[\mathrm{id}] \in H_0(\mathrm{Hom}_R(D, D))$ under f.

3.2. Lemma. *For any object C in $\varrho_n\mathcal{D}$ there exists a well-behaved object $\bar{C}$ in $\varrho_n\mathcal{D}$ and a weak equivalence $f: \bar{C} \to C$.*

Proof. We proceed inductively. If $s \in \mathrm{po}[n]$ and $\dim(s) = 0$, put $\bar{C}(s) = C(s)$ and $f_s =$ identity $: \bar{C}(s) \to C(s)$. Suppose for induction purposes that $\bar{C}(s)$ and $f_s: \bar{C}(s) \to C(s)$ have already been defined for all $s \in \mathrm{po}[n]$ such that $\dim(s) < k$. Here k is some integer with $1 \le k \le n$. Let $q \in \mathrm{po}[n]$ have dimension k. We are forced to define $\bar{C}(\partial q)$ as the direct limit of the $C(s)$ for $s < q$. Let $\bar{C}(q)$ be the mapping cylinder of the composite chain map

$$\bar{C}(\partial q) = \varinjlim \bar{C}(s) \xrightarrow{\{f_s\}} \varinjlim C(s) \longrightarrow C(q)$$

where the direct limits are taken over all s with $s<q$, and the arrow on the right uses the fact that C is a functor. Let $f_q: \bar{C}(q) \to C(q)$ be the projection of the mapping cylinder, and to the inclusion $\partial q \hookrightarrow q$ associate the evident inclusion of $\bar{C}(\partial q)$ in $\bar{C}(q)$. This completes the induction step. (A quicker way to describe $\bar{C}$ is to say that

$$\bar{C}(q) = \operatorname*{hocolim}_{s \leq q} C(s);$$

see [4].) $\square$

3.3. Remark. The rule $C \mapsto \bar{C}$ is a functor from $\varrho_n\mathscr{D}$ to itself, and the projection maps $f: \bar{C} \to C$ define a natural transformation from this functor to the identity.

3.4. Lemma. *For any object C in $\varrho_n\mathscr{D}$ there exists a finitely generated object B in $\varrho_n\mathscr{D}$ and a weak equivalence $f: B \to C$. (Call B finitely generated if the direct sum of all $B(s)_r$ is f.g., where $s \in \mathrm{po}[n]$ and $r \in \mathbb{Z}$.)*

Proof. By 3.2 we may assume that C is well behaved, and we will arrange B to be well behaved also. Suppose for induction purposes that $B(X)$ and $f_X: B(X) \to C(X)$ have already been defined for all CW-subspaces $X \subset \Delta^n$ of dimension $<k$, where k is some integer with $0 \leq k \leq n$. For any face $s \subset \Delta^n$ of dimension exactly k, choose a f.g. chain complex $B(s)$ in $\mathscr{D}$, a cofibration $B(\partial s) \to B(s)$ and a homotopy equivalence $f_s: B(s) \to C(s)$ such that the diagram

$$\begin{tikzcd} B(\partial s) \arrow[r, hook] \arrow[d, "f_{\partial s}"] & B(s) \arrow[d, "f_s"] \\ C(\partial s) \arrow[r, hook] & C(s) \end{tikzcd}$$

is strictly commutative. This is not difficult, since $B(\partial s)$ is f.g. and $C(s)$ is homotopy equivalent to a f.g. object of $\mathscr{D}$ by the very definition of $\mathscr{D}$. If $X \subset \Delta^n$ is a CW-subspace of dimension $<k+1$, we are forced to define $B(X)$ as the direct limit of the $B(s)$, where s ranges over the faces of Δ^n contained in X. This completes the induction step. $\square$

Suppose that U is any covariant functor from $\mathrm{po}[n]$ to the category of chain complexes of abelian groups (graded over the integers). For $j \in \mathbb{Z}$, define the total homology group $H_j(U)$ to be the abelian group of homotopy classes of natural chain maps

$$\Sigma^j \mathrm{cl} \to U,$$

where cl denotes the functor sending $s \in \mathrm{po}[n]$ to $\mathrm{cl}(s)$. (See the outline.) These total homology groups have a strong homotopy invariance property. Namely, if U' is another covariant functor from po$[n]$ to the category of chain complexes, and if $g: U \to U'$ is a natural chain map (i.e. a natural transformation) such that $g_s : U(s) \to U'(s)$ induces an isomorphism in homology for all $s \in$ po$[n]$, then the map $H_*(U) \to H_*(U')$ given by composition with g is also an isomorphism. To prove this, define a chain complex Hom(cl, U) in such a way that $H_*(\mathrm{Hom}(\mathrm{cl}, U))$ is the total homology of U; do the same for U', and argue as in the proof of 3.1. The point is that the functor cl from po$[n]$ to chain complexes is well behaved (over $\mathbb{Z}$), in the sense that it extends to an intersection-preserving functor on $\widehat{\mathrm{po}}[n]$.

Suppose next that B and C are objects in $\varrho_n \mathscr{D}$, and let $B^{\mathrm{t}} \otimes_R C$ be the covariant functor from po$[n]$ to the category of all chain complexes given by the rule $s \mapsto B(s)^{\mathrm{t}} \otimes_R C(s)$. Any $s \in$ po$[n]$ determines an injective monotone map $[m] \to [n]$ whose image consists of the vertices of s, with $m = \dim(s)$. Therefore s determines a restriction functor $\varrho_n \mathscr{D} \to \varrho_m \mathscr{D}$ whose value on an object B we denote by $B_{|s}$. We will define a cohomology slant product

$$H^j(C(s); R) \otimes H_k(B^{\mathrm{t}} \otimes_R C) \to H_{k-j}(B_{|s}),$$
$$f \otimes \eta \mapsto f \backslash \eta$$

where j, k are arbitrary integers, and $H^j(C(s); R)$ is the same as $H_{-j}(\mathrm{Hom}_R(C(s), R))$. (Compare [9, Chapter VII, 11.1].) The groups $H_k(B^{\mathrm{t}} \otimes_R C)$ and $H_{k-j}(B_{|s})$ are, of course, total homology groups as defined above.

Represent a typical element in $H^j(C(s); R)$ by a chain map

$$f: C(s) \to \Sigma^j R$$

where R is interpreted as a chain complex concentrated in degree zero. Represent a typical element in $H_k(B^{\mathrm{t}} \otimes_R C)$ by a natural chain map

$$\eta : \Sigma^k \mathrm{cl}(q) \to B(q)^{\mathrm{t}} \otimes_R C(q)$$

where q ranges over the objects of po$[n]$. For any $q \leq s$ we then have a chain map

$$\Sigma^{k-j} \mathrm{cl}(q) \to B(q),$$

or

$$\Sigma^k \mathrm{cl}(q) \to \Sigma^j B(q),$$

by composing

$$\Sigma^k \mathrm{cl}(q) \xrightarrow{\eta} B(q)^{\mathrm{t}} \otimes_R C(q) \xrightarrow{\mathrm{id} \otimes f \cdot e(q,s)} B(q)^{\mathrm{t}} \otimes_R \Sigma^j R.$$

Here $e(q,s)$ is the chain map from $C(q)$ to $C(s)$ induced by the inclusion $q \subset s$. Letting q vary (subject to the restriction $q \leq s$), we see that we have defined an element $f \backslash \eta$ in $H_{k-j}(B_{|s})$. (Remember that the definition of the total homology groups does not mention any R-module structures. It is sufficient to have an isomorphism of chain complexes of abelian groups

$$B(q)^{\mathrm{t}} \otimes_R \Sigma^j R \cong \Sigma^j B(q).)$$

3.5. Definition. A pairing between objects B and C in $\varrho_n\mathscr{D}$ is a natural chain map

$$\eta : \mathrm{cl}(s) \to B(s)^{\mathrm{t}} \otimes_R C(s)$$

where s ranges over the objects of po$[n]$. Such a pairing η is called nondegenerate if the homomorphisms

$$H^j(C(s); R) \to H_{-j}(B_{|s}); \qquad [f] \mapsto [f \backslash \eta]$$

are isomorphisms for any $j \in \mathbb{Z}$ and $s \in \mathrm{po}[n]$.

3.6. Proposition. *Every object B in $\varrho_n\mathscr{D}$ occurs in a nondegenerate pairing*

$$\eta : \mathrm{cl} \to B^{\mathrm{t}} \otimes_R C$$

with suitable C. (We have written cl *for the functor on* po$[n]$ *sending s to* cl(s).)

Proof. By 3.4 and the invariance property of total homology, we may assume that B is finitely generated. Write $B(q)^{-*}$ instead of $\mathrm{Hom}_R(B(q), R)$, for any $q \in \mathrm{po}[n]$. Let ∇B be the object of $\varrho_n\mathscr{D}$ given by

$$\nabla B(s) = \bigoplus_{q \le s} B(q)^{-*} \otimes \mathrm{cl}(q) \Big/ \sim .$$

(The relations $\sim$ are the usual ones: $e^*(f) \otimes v \sim f \otimes e_*(v)$ whenever $v \in \mathrm{cl}(q)$ and $f \in B(q')^{-*}$, with $q \le q' \le s$. We have written e for the inclusion $q \hookrightarrow q'$.)

For each $s \in \mathrm{po}[n]$ the obvious chain map from $B(s)^{-*} \otimes \mathrm{cl}(s)$ to $\nabla B(s)$ has an adjoint

$$\mathrm{cl}(s) \to B(s)^{\mathrm{t}} \otimes_R \nabla B(s)$$

which is natural in s. So we have a pairing η between B and ∇B. Observe that ∇B is well behaved!

We now prove that η is nondegenerate. Suppose for this purpose that E is any object in $\varrho_n\mathscr{D}$. There are homomorphisms

$$\backslash \eta : H_j(\mathrm{Hom}_R(\nabla B, E)) \to H_j(B^{\mathrm{t}} \otimes_R E)$$

for $j \in \mathbb{Z}$, given by the usual recipe. (A natural chain map $f : \Sigma^j \nabla B \to E$ gives rise to a composite map

$$\Sigma^j \mathrm{cl} \xrightarrow{\ \eta\ } B^{\mathrm{t}} \otimes_R \Sigma^j \nabla B \xrightarrow{\mathrm{id} \otimes f} B^{\mathrm{t}} \otimes_R E$$

which represents an element in $H_j(B^{\mathrm{t}} \otimes_R E)$.) These homomorphisms are isomorphisms. In fact, it is clear that the chain map $\mathrm{Hom}_R(\nabla B, E) \to \mathrm{Hom}(\mathrm{cl}, B^{\mathrm{t}} \otimes_R E)$ by which they are induced is an isomorphism. Now fix an element s in po$[n]$ and specify E as follows:

$$E(q) = \begin{cases} R & \text{if } q \le s, \\ 0 & \text{otherwise.} \end{cases}$$

(The maps $E(q)\to E(q')$ are to be identity maps $R\to R$ whenever $q\le q'\le s$.) Then $H_j(\mathrm{Hom}_R(\nabla B,E))=H^{-j}(B(s);R)$ and $H_j(B^{\mathrm{t}}\otimes_R E)=H_j(B|_s)$, and the homomorphisms $\backslash\eta$ just defined agree with those defined earlier in 3.5. Since they are isomorphisms, η is nondegenerate. So we can take $C=\nabla B$ in 3.6. □

3.7. Proposition. *Let $\eta:\mathrm{cl}\to B^{\mathrm{t}}\otimes_R C$ be a nondegenerate pairing as in 3.5, and assume that C is well behaved. Then for any object E in $\varrho_n\mathcal{D}$, the homomorphisms*

$$H_j(\mathrm{Hom}_R(C,E))\to H_j(B^{\mathrm{t}}\otimes_R E);\qquad f\to f\backslash\eta$$

are isomorphisms for arbitrary $j\in\mathbb{Z}$.

Taking $j=0$ for example, one obtains that up to natural homotopy every pairing between B and E is induced from η via a natural chain map $f: C\to E$, unique up to natural homotopy. To put it differently, nondegenerate pairings have a universal property.

Proof. Again, there is no harm in assuming B to be finitely generated. In this case we already know that the statement is true for the canonical nondegenerate pairing

$$\eta:\mathrm{cl}\to B^{\mathrm{t}}\otimes_R \nabla B.$$

If $\mu:\mathrm{cl}\to B^{\mathrm{t}}\otimes_R C$ is another nondegenerate pairing, let $f\colon \nabla B\to C$ be the unique natural chain map such that the composition

$$\mathrm{cl}\xrightarrow{\ \eta\ } B^{\mathrm{t}}\otimes_R \nabla B\xrightarrow{\mathrm{id}\otimes f} B^{\mathrm{t}}\otimes_R C$$

agrees with μ. The commutative diagram

$$\begin{array}{ccc} H^*(C(s);R) & \xrightarrow{\ f^*\ } & H^*(\nabla B(s);R)\\ \Big\downarrow{\scriptstyle \backslash\mu} & & \Big\downarrow{\scriptstyle \backslash\eta}\\ H_{-*}(B|_s) & \xrightarrow{\ =\ } & H_{-*}(B|_s)\end{array}$$

now shows that $f^*:H^*(C(s);R)\to H^*(\nabla B(s);R)$ is an isomorphism for every $s\in\mathrm{po}[n]$. Since $\nabla B(s)$ is f.g. and $C(s)$ is homotopy equivalent to a f.g. chain complex, it follows that $f_s\colon \nabla B(s)\to C(s)$ is a homotopy equivalence for all s. Therefore f is a weak equivalence. By 3.1 and subsequent remarks, f is a natural homotopy equivalence. It is then clear that μ must have the same universal property as η. □

3.8. Proposition. *Let $\eta:\mathrm{cl}\to B^{\mathrm{t}}\otimes_R C$ be a nondegenerate pairing between objects B and C in $\varrho_n\mathcal{D}$. Then the pairing obtained by composing η with the switching isomorphism,*

$$\mathrm{cl}\xrightarrow{\ \eta\ } B^{\mathrm{t}}\otimes_R C\longrightarrow C^{\mathrm{t}}\otimes_R B,$$

is also nondegenerate.

Proof. By the preceding discussion and by 3.4 we can assume that B is f.g., that $C=\nabla B$ and that η is the canonical pairing between B and ∇B. Fix $s\in \text{po}[n]$. We have to show that a certain homomorphism from $H^*(B(s);R)$ to $H_{-*}(\nabla B_{|s})$ is an isomorphism.

Observe that the functor $\nabla: B\to \nabla B$ is an exact functor from the full subcategory of f.g. objects in $\varrho_n\mathscr{D}$ to itself. (See [41].) Therefore the functor $B\mapsto H_{-*}(\nabla B_{|s})$ is a cohomology theory in the variable B. (That is, it associates isomorphisms to weak equivalences, and long exact sequences to short exact sequences $0\to B\to B'\to B''\to 0$ of f.g. objects in $\varrho_n\mathscr{D}$.) The natural homomorphism $H^*(B(s);R)\to H_{-*}(\nabla B_{|s})$ under consideration is a transformation of cohomology theories.

Next, observe that any object B in $\varrho_n\mathscr{D}$ has a canonical filtration by subobjects $\text{Filt}_i B$, where

$$(\text{Filt}_i B)(s)=\begin{cases} B(s) & \text{if } \dim(s)\geq n-i,\\ 0 & \text{otherwise}\end{cases}$$

whenever $-1\leq i\leq n$. The quotients $\text{Filt}_i B/\text{Filt}_{i-1} B$ split as direct sums of objects concentrated over a single face. (An object E of $\varrho_n\mathscr{D}$ is said to be concentrated over a single face q if $E(s)=0$ for $s\neq q$.) Further, a f.g. object in $\varrho_n\mathscr{D}$ which is concentrated over a single face has a canonical filtration whose quotients are concentrated in a single dimension $r\in\mathbb{Z}$. It is therefore sufficient (by the five lemma) to show that the homomorphism $H^*(B(s);R)\to H_{-*}(\nabla B_{|s})$ under consideration is an isomorphism if B is concentrated in a single dimension $r\in\mathbb{Z}$, and over a single face $q\in\text{po}[n]$. We leave this to the reader. □

3.9. Remark. A consequence of 3.7 and 3.8 is that maps can be dualized. Let $\eta:\text{cl}\to B^{\text{t}}\otimes_R C$ and $\mu:\text{cl}\to D^{\text{t}}\otimes_R E$ be nondegenerate pairings in $\varrho_n\mathscr{D}$. Suppose also that B and E are well behaved. Then there is a one-one correspondence ψ between homotopy classes of natural chain maps $B\to D$ and homotopy classes of natural chain maps $E\to C$. If we identify such homotopy classes with elements of $H_0(\text{Hom}_R(B,D))$ and $H_0(\text{Hom}_R(E,C))$, respectively, then ψ is characterized by the equation $f\setminus\eta=\psi(f)\setminus\mu$ which holds in the total homology group $H_0(D^{\text{t}}\otimes_R C)$.

Here is some notation which will help us to formulate the main result of the section. The category of all CW-spaces and their cellular maps acts on the category of all chain complexes and chain maps: if X is a CW-space and Y is a chain complex (graded over $\mathbb{Z}$), then we let

$$X\cdot Y=\text{cl}(X)\otimes Y$$

where $\text{cl}(X)$ is the cellular chain complex of X. In practice, X will be a simplex Δ^j for some j.

Now fix an object B in $\varrho_n\mathscr{D}$. We wish to state and prove, in a categorical way, that B occurs in an essentially unique nondegenerate pairing. (From 3.6 we know already that it occurs in some nondegenerate pairing.) Let $\mathscr{P}_B$ be the simplicial category whose objects in degree j are the nondegenerate pairings

$$\eta : \Delta^j \cdot \mathrm{cl} \to B^{\mathrm{t}} \otimes_R C$$

where C is an arbitrary object in $\varrho_n \mathscr{D}$ and where cl is the functor $s \mapsto \mathrm{cl}(s)$ on $\mathrm{po}[n]$. (We have taken the liberty to speak of nondegenerate pairings because the natural chain map

$$\Delta^j \cdot \mathrm{cl}(s) \to \mathrm{point} \cdot \mathrm{cl}(s) \cong \mathrm{cl}(s),$$

with $s \in \mathrm{po}[n]$, is a natural homotopy equivalence.) A morphism in degree j, say from $\eta : \Delta^j \cdot \mathrm{cl} \to B^{\mathrm{t}} \otimes_R C$ to $\eta' : \Delta^j \cdot \mathrm{cl} \to B^{\mathrm{t}} \otimes_R C'$, is a morphism $g : C \to C'$ in $\varrho_n \mathscr{D}$ such that $\eta' = (\mathrm{id}_B \otimes g) \cdot \eta$.

3.10. Proposition. *The nerve of $\mathscr{P}_B$ is contractible.*

Comment and proof. Nerves of categories are defined in [31]. The nerve of a category $\mathscr{A}$ is a simplicial class $\nu(\mathscr{A})$; it is not always a simplicial set, because we do not wish to assume that $\mathscr{A}$ is a small category. In fact, we will be considering simplicial categories $\mathscr{A}$ having the property that the connected components of the simplicial classes $\mathrm{Ob}(\mathscr{A})$ and $\mathrm{Mor}(\mathscr{A})$ are simplicial sets (which may be a relief to the reader). The nerve of a simplicial category $\mathscr{A}$ is a bisimplicial class $\nu(\mathscr{A})$ which we can interpret either as a contravariant functor

$$[j] \mapsto \nu(\mathscr{A}[j])$$

from Δ to simplicial classes, or as a contravariant functor

$$[r] \mapsto \nu_r(\mathscr{A})$$

from Δ to simplicial classes. Here $\nu_r(\mathscr{A})$ is the simplicial class whose class of j-simplices is the class of r-simplices of $\nu(\mathscr{A}[j])$.

A simplicial class will be called contractible if any simplicial map from a simplicial set to it can be factored through a contractible simplicial set. (A simplicial set will be called contractible if its geometric realization is.) There is a similar definition for bisimplicial classes, such as the nerve of $\mathscr{P}_B$.

For the proof of 3.10, choose a specific nondegenerate pairing

$$\mu : \mathrm{cl} \to B^{\mathrm{t}} \otimes_R D$$

such that D is a well behaved object in $\varrho_n \mathscr{D}$. This is possible by 3.6 and 3.2 (and by the invariance property of total homology). Let $\mathscr{A}$ be the simplicial category whose objects in degree j are the morphisms

$$f : \Delta^j \cdot D \to C$$

in $\varrho_n \mathscr{D}$, where C is arbitrary and f is a weak equivalence. A morphism in $\mathscr{A}[j]$, say from $f : \Delta^j \cdot D \to C$ to $g : \Delta^j \cdot D \to C'$, is a morphism $e : C \to C'$ in $\varrho_n \mathscr{D}$ such that $g = e \cdot f$. It is clear from this description that $\mathscr{A}[j]$ has an initial object for each $j \geq 0$;

therefore $v(\mathscr{A}[j])$ is contractible for all $j \geq 0$, and therefore $v(\mathscr{A})$ itself is contractible.

The pairing μ gives rise to a simplicial functor

$$\backslash\mu : \mathscr{A} \to \mathscr{P}_B$$

which sends an object $f: \Delta^j \cdot D \to C$ in $\mathscr{A}[j]$ to the object in $\mathscr{P}_B[j]$ obtained by composing the maps

$$\Delta^j \cdot \mathrm{cl} \xrightarrow{\Delta^j \cdot \mu} B^t \otimes_R (\Delta^j \cdot D) \xrightarrow{\mathrm{id} \otimes f} B^t \otimes_R C.$$

We now claim that the simplicial maps

$$v_r(\mathscr{A}) \to v_r(\mathscr{P}_B)$$

induced by $\backslash\mu$ are homotopy equivalences, for all $r \geq 0$. (This implies that $v(\mathscr{A})$, which is contractible, is homotopy equivalent to $v(\mathscr{P}_B)$, which is therefore also contractible.)

For $r=0$ we argue as follows. The simplicial class $v_0(\mathscr{A}) = \mathrm{Ob}(\mathscr{A})$ is a disjoint union of simplicial sets $X(C)$, where the j-simplices of $X(C)$ are the weak equivalences from $\Delta^j \cdot D$ to C. The disjoint union need only be taken over all C in $\varrho_n\mathscr{D}$ which are weakly equivalent to D. Each $X(C)$ is a Kan simplicial set whose homotopy groups/sets are, by inspection,

$$\pi_i(X(C)) \cong \begin{cases} H_i(\mathrm{Hom}_R(D,C)) \quad \text{if } i>0; \\ \text{the subset of } H_0(\mathrm{Hom}_R(D,C)) \text{ consisting of all} \\ \text{homotopy classes of weak equivalences from } D \\ \text{to } C, \text{ if } i=0. \end{cases}$$

Similarly, the simplicial class $v_0(\mathscr{P}_B) = \mathrm{Ob}(\mathscr{P}_B)$ is a disjoint union of simplicial sets $Y(C)$, where the j-simplices of $Y(C)$ are the nondegenerate pairings $\eta : \Delta^j \cdot \mathrm{cl} \to B^t \otimes_R C$. Here the disjoint union need only be taken over all C in $\varrho_n\mathscr{D}$ which occur in a nondegenerate pairing with B. Each $Y(C)$ is a Kan simplicial set whose homotopy groups/sets are

$$\pi_i(Y(C)) \cong \begin{cases} H_i(B^t \otimes_R C) \quad \text{if } i>0; \\ \text{the subset of } H_0(B^t \otimes_R C) \text{ consisting of the} \\ \text{homotopy classes of nondegenerate pairings} \\ \eta : \mathrm{cl} \to B^t \otimes_R C, \text{ if } i=0. \end{cases}$$

It is now clear from 3.7 that $\backslash\mu$ maps each $X(C)$ by a homotopy equivalence to $Y(C)$. That is, we have checked our claim for $r=0$.

The argument for $r>0$ is similar. Just note that

$$v_r(\mathscr{A}) = \coprod_C X(C) \times \alpha_r(C), \qquad v_r(\mathscr{P}_B) = \coprod_C Y(C) \times \alpha_r(C),$$

where C can be any object in $\varrho_n\mathscr{D}$ weakly equivalent to D, and where $\alpha_r(C)$ is the class of diagrams of the form $x_0 \to x_1 \to \cdots \to x_r$ in $w\varrho_n\mathscr{D}$ such that $x_0 = C$.

(The prefix w in $w\varrho_n\mathcal{D}$ denotes the subcategory of $\varrho_n\mathcal{D}$ consisting of the weak equivalences.) $\square$

3.11. Digression. We return to Definition 3.5 in order to discuss its geometric roots. Usually those objects in $\varrho_n\mathcal{D}$ which arise in geometric situations are well behaved. For a well-behaved object B in $\varrho_n\mathcal{D}$ there are isomorphisms expressing the total homology groups $H_j(B_{\cdot s})$ as ordinary homology groups:

$$H_j(B_{\cdot s}) \xrightarrow{\cong} H_{j+\dim(s)}(B(s/\partial s))$$

for any $s\in \mathrm{po}[n]$ and $j\in\mathbb{Z}$. As usual, $B(s/\partial s)$ abbreviates the quotient of the cofibration $B(\partial s)\to B(s)$. To obtain these isomorphisms, say γ_s, observe that any natural chain map $\Sigma^j \mathrm{cl}(q)\to B(q)$ defined for all $q\le s$ gives rise, by specialization, to a chain map $\Sigma^j\mathrm{cl}(s/\partial s)\to B(s/\partial s)$; this defines γ_s since $\mathrm{cl}(s/\partial s)\cong\Sigma^{\dim(s)}\mathbb{Z}$. For fixed s, the map γ_s is a natural transformation of homology theories on the category of well-behaved objects in $\varrho_n\mathcal{D}$. So we can prove that it is an isomorphism by decomposing the well-behaved objects into simpler (but well-behaved) pieces. (The same method was used in proving 3.8. We leave the details to the reader.) The conclusion is that the definition of nondegeneracy in 3.5 can be given a more familiar form if one of the participants in a pairing

$$\eta:\mathrm{cl}\to B^{\mathrm{t}}\otimes_R C,$$

say B, is well behaved. In fact, one is reminded of Poincaré duality for manifold pairs etc..

So let M be a compact manifold modelled on $\mathbb{R}^k\times\Delta^n$. This means the following. Write $\mathcal{G}$ for the pseudogroup consisting of all homeomorphisms

$$\psi:\omega_1\to\omega_2$$

where ω_1 and ω_2 are arbitrary open subsets of $\mathbb{R}^k\times\Delta^n$ and where

$$\psi(\omega_1\cap(\mathbb{R}^k\times d_i\Delta^n))=\omega_2\cap(\mathbb{R}^k\times d_i\Delta^n)$$

whenever $0\le i\le n$. (See [17, p.1].) A manifold modelled on $\mathbb{R}^k\times\Delta^n$ is a Hausdorff space M equipped with a complete atlas with charts in $\mathbb{R}^k\times\Delta^n$ and with changes of charts in $\mathcal{G}$. Given such an M, and given a CW-subspace $X\subset\Delta^n$, let $M(X)\subset M$ consist of those points which are taken to $\mathbb{R}^k\times X\subset\mathbb{R}^k\times\Delta^n$ by some chart. For example, if $s\subset\Delta^n$ is any face, then $M(s)$ is an ordinary manifold with boundary $M(\partial s)$. In particular, $M=M(\Delta^n)$ itself is a manifold with boundary $M(\partial\Delta^n)$. See also [6] for a less abstract definition (mock bundle over Δ^n), or [24].

Working over the ring $R=\mathbb{Z}$ for the moment, we can associate with M an object C in $\varrho_n\mathcal{D}$ by letting

$$C(s)=\text{singular chain complex of } M(s)$$

for $s\in\mathrm{po}[n]$. (We assume that M is compact.) Then C is well behaved; and for each CW-subspace $X\subset\Delta^n$ there is an inclusion of $C(X)$ into the singular chain complex of $M(X)$ which is a homotopy equivalence (though not an isomorphism in general). Using an Alexander–Whitney diagonal approximation we obtain a chain map

$$\mathrm{AW}: C(s) \to C(s) \otimes C(s)$$

which is natural in s. If M is also oriented, which we will assume, then the fundamental class in $H_{k+n}(M, \partial M; \mathbb{Z}) \cong H_{k+n}(C(\Delta^n/\partial\Delta^n))$ can be interpreted, as we have seen, as a total homology class in $H_k(C)$. Represent this by a natural chain map

$$u: \Sigma^k \mathrm{cl}(s) \to C(s)$$

with s in $\mathrm{po}[n]$. Then

$$\mathrm{AW} \cdot u: \Sigma^k \mathrm{cl} \to C \otimes C$$

or equivalently

$$\mathrm{AW} \cdot u: \mathrm{cl} \to C \otimes \Sigma^{-k} C$$

is a pairing of C with $\Sigma^{-k}C$. It is nondegenerate, as can be seen from the new definition of nondegeneracy given just above.

The construction can be refined in two respects. Firstly, there is no need to be content with the Alexander–Whitney diagonal approximation. A more powerful machine is the Eilenberg–Zilber diagonal approximation which, for any space Y with singular chain complex $\mathrm{sg}(Y)$, gives a Z_2-equivariant chain map

$$\mathrm{EZ}: W \otimes \mathrm{sg}(Y) \to \mathrm{sg}(Y) \otimes \mathrm{sg}(Y)$$

where W is the usual projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[Z_2]$. (Here Z_2 acts on the left-hand side by acting on W, and on the right-hand side by switching factors.) Using EZ instead of AW, and using the same notation as before, we obtain a natural chain map of the form

$$\mathrm{EZ} \cdot (\mathrm{id}_W \otimes u): W \otimes \Sigma^k \mathrm{cl} \to C \otimes C$$

which is Z_2-equivariant and nondegenerate. If $k=0$, this is a symmetric pairing of C with itself, as defined in the outline.

Secondly, the construction can be improved to give nondegenerate (symmetric) pairings over the ring $\mathbb{Z}\pi_1(M)$, if we assume that M is connected. There is then no need to assume that M be orientable; instead, the orientation behaviour of M can be encoded in the involution on $\mathbb{Z}\pi_1(M)$.

These refinements are discussed in much more detail by Mishchenko [21] and Ranicki [25], but only when n is 0 or 1.

3.12. Remark. Everything in this section, with the exception of 3.11, has a filtered version in the following sense. Let B be an object in $\varrho_n\mathcal{D}$ equipped with a filtration

$$0 = \mathrm{Filt}_0 B \subset \mathrm{Filt}_1 B \subset \cdots \subset \mathrm{Filt}_k B = B$$

such that the inclusions $\mathrm{Filt}_i B \hookrightarrow \mathrm{Filt}_{i+1} B$ are cofibrations in $\varrho_n\mathcal{D}$. Filtered objects of this type and their filtration-preserving natural chain maps (over R) form a category $\mathscr{S}_k \varrho_n \mathscr{D}$. It is a category with cofibrations and weak equivalences.

Given two objects B and C in $\mathscr{S}_k \varrho_n \mathscr{D}$, and given any $s \in \mathrm{po}[n]$, define $B(s)^t \odot_R C(s)$ to be the chain subcomplex of $B(s)^t \otimes_R C(s)$ generated by all chains $b \otimes c$ such that $b \in \mathrm{Filt}_i B$ and $c \in \mathrm{Filt}_j C$, with $i+j=k+1$. Write $B^t \odot_R C$ for the functor $s \mapsto B(s)^t \odot_R C(s)$, and call $B^t \odot_R C$ the filtered tensor product of B and C.

A filtered pairing between B and C is then, by analogy with 3.5, a natural chain map of the form

$$\mathrm{cl} \to B^t \odot_R C$$

where cl denotes the functor $s \mapsto \mathrm{cl}(s)$ on $\mathrm{po}[n]$. Such a filtered pairing induces several pairings in the sense of 3.5, namely,

$$\mathrm{cl} \to \mathrm{Filt}_i B^t \otimes_R \mathrm{Cofilt}_{k-i} C$$

where $\mathrm{Cofilt}_{k-i} C$ is the quotient of the cofibration $\mathrm{Filt}_{k-i} C \to C$, and where $0 \le i \le k$. If these are all nondegenerate, call the filtered pairing nondegenerate.

We are mostly interested in the filtered version of 3.10; this follows from the filtered version of 3.7, whose proof is almost identical with that of the unfiltered version.

Our use of the symbol $\mathscr{S}$ in $\mathscr{S}_k \varrho_n \mathscr{D}$ calls for an apology. Waldhausen, in [41], defines, for any category $\mathscr{C}$ with cofibrations and weak equivalences, the category $\mathscr{S}_k \mathscr{C}$ whose objects are diagrams in $\mathscr{C}$ of the form

$$\begin{array}{ccccccccc}
A_{0,0} & \to & A_{0,1} & \to & A_{0,2} & \to & \cdots & \to & A_{0,k} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
A_{1,0} & \to & A_{1,1} & \to & A_{1,2} & \to & \cdots & \to & A_{1,k} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
A_{2,0} & \to & A_{2,1} & \to & A_{2,2} & \to & \cdots & \to & A_{2,k} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
A_{k,0} & \to & A_{k,1} & \to & A_{k,2} & \to & \cdots & \to & A_{k,k}
\end{array}$$

in which any horizontal arrow is a cofibration, every square

$$\begin{array}{ccc}
A_{i,j} & \longrightarrow & A_{i,j+1} \\
\downarrow & & \downarrow \\
A_{i+1,j} & \longrightarrow & A_{i+1,j+1}
\end{array}$$

is a pushout, and $A_{i,i}$ is the zero object for all i. Such a diagram is, however, determined by its top row up to unique isomorphism. It is sometimes convenient to forget the other rows, as we have done in describing $\mathscr{S}_k \varrho_n \mathscr{D}$; but in other situations it is better to keep them, as Waldhausen explains [41, p. 329]. In any case the two definitions give rise to equivalent categories.

3.13. Remark. Let η be a pairing between objects B and C in $\varrho_n\mathcal{D}$, as in 3.5. We saw that such a pairing determines homomorphisms

$$\backslash\eta : H^*(C(s); R) \to H_{-*}(B_{|s})$$

for all $s \in \mathrm{po}[n]$. By the same procedure, one obtains homomorphisms

$$\backslash\eta : H^*(C(s); \mathcal{M}) \to H_{-*}(B_{|s}; \mathcal{M})$$

where $\mathcal{M}$ is any left R-module and

$$H^*(C(s); \mathcal{M}) = H_{-*}(\mathrm{Hom}_R(C(s), \mathcal{M})),$$

$$H_{-*}(B_{|s}; \mathcal{M}) = H_{-*}(B^t \otimes_R \mathcal{M}_{|s}).$$

Here $B^t \otimes_R \mathcal{M}$ is the functor on $\mathrm{po}[n]$ sending s to the chain complex $B(s)^t \otimes_R \mathcal{M}$. Note that $H_{-*}(B_{|s}; R)$ is just $H_{-*}(B_{|s})$. It turns out that

$$\backslash\eta : H^*(C(s); \mathcal{M}) \to H_{-*}(B_{|s}; \mathcal{M})$$

is an isomorphism for arbitrary $\mathcal{M}$ if it is an isomorphism for $\mathcal{M} = R$. In proving this we may assume that B is well behaved, and by 3.11 we may replace the total homology group $H_{-j}(B_{|s}; \mathcal{M})$ by an ordinary homology group $H_{-j}(B(s/\partial s); \mathcal{M})$. The assumption that

$$\backslash\eta : H^*(C(s); R) \to H_{-*}(B(s/\partial s); R)$$

is an isomorphism just means that the underlying chain map

$$\backslash\eta : \mathrm{Hom}_R(C(s), R) \to B(s/\partial s)^t$$

of right R-module chain complexes is a homotopy equivalence (since $C(s)$ and $B(s/\partial s)$ are homotopy equivalent to f.g. projective chain complexes). Therefore

$$\backslash\eta : H^*(C(s); \mathcal{M}) \to H_{-*}(B(s/\partial s); \mathcal{M})$$

is an isomorphism.

4. From L to K

The organization of this section corresponds roughly to that of the outline. We begin with a thorough description of the involution on $\underline{\underline{K}}\mathcal{D} \cong \underline{\underline{K}}\varrho_0\mathcal{D}$, the Waldhausen model of $\underline{\underline{K}}p(R)$.

Let $w\mathcal{D}$ be the subcategory of $\mathcal{D}$ having the same objects as $\mathcal{D}$, but only weak equivalences (= homotopy equivalences) as morphisms. Let $xw\mathcal{D}$ be the simplicial category whose objects in degree j are nondegenerate pairings of the form

$$\eta : \Delta^j \cdot \mathbb{Z} \to B^t \otimes_R C,$$

where B and C are arbitrary objects in $w\mathcal{D}$, and where $\mathbb{Z}$ is identified with the cellular chain complex of a one-point space. (Note that $\Delta^j \cdot \mathbb{Z} \cong \mathrm{cl}(\Delta^j)$.) A morphism in degree j, say from $(\eta : \Delta^j \cdot \mathbb{Z} \to B^t \otimes_R C)$ to $(\mu : \Delta^j \cdot \mathbb{Z} \to D^t \otimes_R E)$, shall be a pair $(f: B \to D, g: E \to C)$ of morphisms in $w\mathscr{D}$ such that

$$(f \otimes \mathrm{id}_C) \cdot \eta = (\mathrm{id}_D \otimes g) \cdot \mu : \Delta^j \cdot \mathbb{Z} \to D^t \otimes_R C.$$

Note the direction of g.

4.1. Lemma (Compare Vogell [39, 1.15]; beware different notation.) *The forgetful functor*

$$\lambda : xw\mathscr{D} \to w\mathscr{D}, \qquad (\eta : \Delta^j \cdot \mathbb{Z} \to B^t \otimes_R C) \mapsto B$$

induces a homotopy equivalence of nerves.

Comment and proof. We regard $w\mathscr{D}$ as a simplicial category in a trivial way, so that $w\mathscr{D}[j] = w\mathscr{D}[0] = w\mathscr{D}$ for all $j \geq 0$. Then λ is a simplicial functor.

The proof uses a version of Quillen's Theorem A [22] for simplicial categories. This is stated as Theorem A′ in [43, Section 4], together with an addendum which we also need.

For every object B in $w\mathscr{D}$, let λ/B be the simplicial category whose objects in degree j are pairs (η, f) where η is an object in $xw\mathscr{D}[j]$ and $f: \lambda(\eta) \to B$ is a morphism in $w\mathscr{D}$. (Waldhausen calls λ/B the left fibre of λ over B.) According to Theorem A′ and addendum we only have to prove that the nerve of λ/B is contractible for any B in $w\mathscr{D}$.

Those objects in λ/B which have the form (η, f) with $f = \mathrm{id}_B$ form a full subcategory of λ/B. It is the opposite category of $\mathscr{P}_B$ in 3.10. The inclusion $\mathscr{P}_B^{\mathrm{op}} \to \lambda/B$ has an adjoint functor $\lambda/B \to \mathscr{P}_B^{\mathrm{op}}$ which sends an object

$$(\eta : \Delta^j \cdot \mathbb{Z} \to C^t \otimes_R E, f: C \to B)$$

in λ/B to

$$((f \otimes \mathrm{id}_E) \cdot \eta : \Delta^j \cdot \mathbb{Z} \to B^t \otimes_R E)$$

in $\mathscr{P}_B^{\mathrm{op}}$. Therefore the nerve of λ/B is homotopy equivalent to that of $\mathscr{P}_B^{\mathrm{op}}$, which we know to be contractible from 3.10. See also [22, Section 1]. We stress that 3.10 has one interpretation for each $n \geq 0$; here we take $n = 0$. □

There is a filtered version of 4.1, as follows. Let $w\mathscr{S}_k\mathscr{D}$ be the subcategory of $\mathscr{S}_k\mathscr{D}$ (see 3.12) having the same objects as $\mathscr{S}_k\mathscr{D}$, but weak equivalences (= filtered homotopy equivalences) only as morphisms. Let $xw\mathscr{S}_k\mathscr{D}$ be the simplicial category whose objects in degree j are the nondegenerate pairings

$$\eta : \Delta^j \cdot \mathbb{Z} \to B^t \odot_R C,$$

where B and C are objects in $w\mathscr{S}_k\mathscr{D}$. (The definition of morphisms follows the pattern above.) Using the same ideas as before, we have

4.2. Lemma. *The forgetful functor*

$$\lambda : xw\mathscr{S}_k\mathscr{D} \to w\mathscr{S}_k\mathscr{D}, \qquad (\eta : \Delta^j \cdot \mathbb{Z} \to B^t \odot_R C) \mapsto B$$

induces a homotopy equivalence of nerves. □

Now recall that Waldhausen, in [41], makes the rule $[k] \mapsto w\mathcal{S}_k\mathcal{D}$ into something like a simplicial category, i.e. a contravariant functor from Δ to the category of categories. The face functors $d_i : w\mathcal{S}_k\mathcal{D} \to w\mathcal{S}_{k-1}\mathcal{D}$ are defined for $0 \le i \le k$ by

$$\mathrm{Filt}_j(d_i C) = \begin{cases} \mathrm{Filt}_j C & \text{if } i>0, i>j, \\ \mathrm{Filt}_{j+1} C & \text{if } i>0, i\le j, \\ \mathrm{Filt}_{j+1} C/\mathrm{Filt}_1 C & \text{if } i=0 \end{cases}$$

where C is in $w\mathcal{S}_k\mathcal{D}$ and where $0 \le j < k$. The degeneracy functors $s_i : w\mathcal{S}_k\mathcal{D} \to w\mathcal{S}_{k+1}\mathcal{D}$ are given for $0 \le i \le k$ by

$$\mathrm{Filt}_j(s_i C) = \begin{cases} \mathrm{Filt}_j C & \text{if } j \le i, \\ \mathrm{Filt}_{j-1} C & \text{if } j > i \end{cases}$$

where $0 \le j \le k+1$. (We say "something like a simplicial category" because the simplicial identities do not hold strictly, but only up to natural isomorphism. Waldhausen avoids the problem by using his luxurious definition of $\mathcal{S}_k(\cdots)$, the one we sketched at the end of 3.12. This results in an honest simplicial category $w\mathcal{S}_\bullet\mathcal{D}$. Such a modification will be understood in the sequel. Compare [33] and [35].) The rule $[k] \mapsto |v(w\mathcal{S}_k\mathcal{D})|$ is then a simplicial pointed space $\mathfrak{Y}$ (we use v for nerves, and vertical bars for geometric realization). The loop space of the geometric realization $|\mathfrak{Y}|$ is $\underline{QK}\mathcal{D}$, by definition. Since $\mathfrak{Y}[0]$ is a point and $\mathfrak{Y}[1] = |v(w\mathcal{D})|$, we have an obvious inclusion of $\Sigma|v(w\mathcal{D})|$ in $|\mathfrak{Y}|$; its adjoint is an inclusion $|v(w\mathcal{D})| \hookrightarrow \underline{QK}\mathcal{D}$. See also [41, bottom of p. 329].

It is straightforward to make the spaces $|v(xw\mathcal{S}_k\mathcal{D})|$, for $k \ge 0$, into a simplicial space also, such that the following holds:

(i) The forgetful maps $\lambda : |v(xw\mathcal{S}_k\mathcal{D})| \to |v(w\mathcal{S}_k\mathcal{D})|$, of 4.2 define a simplicial map between simplicial spaces as k varies.

(ii) The canonical involutions $|v(xw\mathcal{S}_k\mathcal{D})| \to |v(xw\mathcal{S}_k\mathcal{D})|$, obtained by interchanging the participants in all nondegenerate pairings, define an antisimplicial map between simplicial spaces as k varies.

Here the word antisimplicial means the following. The category Δ has an automorphism α sending a monotone map $f : [j] \to [k]$ to $\alpha(f) = r_k f r_j$, where r_j and r_k are the order-reversing bijections on $[j]$ and $[k]$. An antisimplicial map between simplicial objects X, Y in some category is a simplicial map from X to $Y \cdot \alpha$. An antisimplicial map between simplicial sets or simplicial spaces induces an honest map between their geometric realizations. For example, a contravariant functor from one category to another induces an antisimplicial map between the nerves. (Indeed, for fixed k the involution on $|v(xw\mathcal{S}_k\mathcal{D})|$ mentioned just above is induced by a contravariant functor; but that is another story.)

Summarizing, we can replace the categories $w\mathcal{S}_k\mathcal{D}$ by $xw\mathcal{S}_k\mathcal{D}$ in Waldhausen's definition of $\underline{QK}\mathcal{D}$. This will result in a new model for $\underline{QK}\mathcal{D}$, which we call the x-model because of the ubiquitous prefix x. It has the correct homotopy type by 4.2, but it also carries a Z_2-action. To be precise, we write $\underline{QK}\mathcal{D} := \Omega|\mathfrak{X}|$, where $\mathfrak{X}$ is the simplicial space $[k] \mapsto |v(xw\mathcal{S}_k\mathcal{D})|$. We have seen that the canonical contravariant involutions on $xw\mathcal{S}_k\mathcal{D}$ for $k\geq 0$ define an involution τ on the geometric realization $|\mathfrak{X}|$. We then obtain an involution on $\Omega|\mathfrak{X}|=Q\underline{K}\mathcal{D}$ by $\lambda\mapsto\tau\lambda r$, where $\lambda: S^1\to|\mathfrak{X}|$ is a pointed loop in $|\mathfrak{X}|$ and $r: S^1\to S^1$ is the group-theoretic inverse.

Using the x-model of $Q\underline{K}\mathcal{D}$, we have an evident inclusion $|v(xw\mathcal{D})|\hookrightarrow Q\underline{K}\mathcal{D}$ which is compatible with Z_2-actions. (Remember that the involution on the delooping $|\mathfrak{X}|$ of $Q\underline{K}\mathcal{D}$ was defined by means of an antisimplicial map, and see again [41, bottom of p. 329].)

Segal's machine shows once more that the involution we constructed on $Q\underline{K}\mathcal{D}$ is an infinite loop space involution, which is why we may regard it as an involution on the spectrum $\underline{K}\mathcal{D}$. See also 4.5 below.

We apologize for using the word space in a very liberal way. For example, it is not clear in what sense the geometric realization of a simplicial class is a space. It seems to be wise, then, to replace $\mathcal{D}$ by a full subcategory with cofibrations and weak equivalences $\mathcal{D}'\subset\mathcal{D}$ such that the inclusion $\mathcal{D}'\hookrightarrow\mathcal{D}$ satisfies the hypotheses of the approximation theorem 1.6.7 in [41], and such that $\mathcal{D}'$ is small. Then $Q\underline{K}\mathcal{D}'$ exists and has the correct homotopy type. We leave the choice of $\mathcal{D}'$ to the reader. We shall also continue to write $Q\underline{K}\mathcal{D}$ etc. when we should write $Q\underline{K}\mathcal{D}'$ etc.

4.3. Observation. *A formally* 0*-dimensional symmetric algebraic Poincaré complex* (C,φ) *gives rise to a* Z_2*-equivariant map* $EZ_2\to Q\underline{K}\mathcal{D}$.

Comment and proof. We use the x-model of $Q\underline{K}\mathcal{D}$ to make sense of the statement. Let us also agree that C can be any chain complex in $\mathcal{D}$, with possibly nontrivial homology in negative dimensions. The symbol φ denotes a chain map $W\to C^t\otimes_R C$ of chain complexes over $\mathbb{Z}[Z_2]$, where W is the usual free resolution of the trivial module $\mathbb{Z}$ over $\mathbb{Z}[Z_2]$. (See Section 0 and 3.11.) It is supposed to be nondegenerate.

If we use the standard (simplicial) model for EZ_2, which is the nerve of a category with two objects which are both initial, then W can be identified with the cellular chain complex $\mathrm{cl}(EZ_2)$. Therefore any j-simplex in EZ_2 gives rise to a chain map

$$\mathrm{cl}(\Delta^j)\to\mathrm{cl}(EZ_2)\cong W$$

which we may compose with φ to obtain a chain map

$$\mathrm{cl}(\Delta^j)\cong\Delta^j\cdot\mathbb{Z}\to C^t\otimes_R C.$$

But this is a j-simplex in the simplicial set or class $\mathrm{Ob}(xw\mathcal{D})$. Summarizing, we have constructed a Z_2-equivariant simplicial map from EZ_2 to $\mathrm{Ob}(xw\mathcal{D})$. Since there are Z_2-equivariant inclusions $|\mathrm{Ob}(xw\mathcal{D})|\hookrightarrow|v(xw\mathcal{D})|\hookrightarrow Q\underline{K}\mathcal{D}$, the proof is complete. □

So much for the involution on $\underline{K}\mathcal{D}$; using the full strength of 3.10, we can define similar involutions on $\underline{K}\varrho_n\mathcal{D}$ for all $n\geq 0$. Again we refer to the x-model of $\underline{K}\varrho_n\mathcal{D}$ or of $Q\underline{K}\varrho_n\mathcal{D}$.

4.4. Observation. *Any n-simplex in $Q\underline{\underline{L}}p^{\blacktriangledown}(R)$ gives rise to a Z_2-equivariant map $EZ_2 \to Q\underline{\underline{K}}\varrho_n\mathscr{D}$.*

Proof. Recall from Section 0 that an n-simplex in $Q\underline{\underline{L}}p^{\blacktriangledown}T(R)$ is a natural chain map of the form

$$\varphi : W \otimes \mathrm{cl} \to C^{\mathrm{t}} \otimes_R C$$

which is Z_2-equivariant and nondegenerate, with C in $\varrho_n\mathscr{D}$. The symbol cl denotes the usual functor on po$[n]$. With φ we associate a Z_2-equivariant simplicial map

$$EZ_2 \to \mathrm{Ob}(xw\varrho_n\mathscr{D})$$

as before. In short, the construction is a straightforward generalization of that in 4.3. □

As explained in Section 0, we obtain from 4.4 a map from $Q\underline{\underline{L}}p^{\blacktriangledown}(R)$ to $|QH^{\blacktriangledown}(Z_2; \underline{\underline{K}}\varrho_\bullet\mathscr{D})|$.

4.5. Observation. *This is an infinite loop map between infinite loop spaces.*

Proof. For $q \geq 0$ let X_q be the geometric realization of the simplicial set whose n-simplices are the n-simplices (C, φ) of $Q\underline{\underline{L}}p^{\blacktriangledown}(R)$, with C in $\varrho_n\mathscr{D}$, together with a splitting of C into q direct summands, and a compatible decomposition of φ: $W \otimes \mathrm{cl} \to C^{\mathrm{t}} \otimes_R C$ as $\varphi = \varphi_1 + \varphi_2 + \cdots + \varphi_q$. The collection $X = \{X_q \mid q \geq 0\}$ is then a Γ-space in the sense of Segal [30].

In the same spirit, let $\varrho_n\mathscr{D}_q$ be the category (with cofibrations and weak equivalences) whose objects are the objects of $\varrho_n\mathscr{D}$ together with a splitting into q direct summands. Using nondegenerate pairings etc., construct an involution on $\underline{\underline{K}}\varrho_n\mathscr{D}_q$ for each n and q and write $Y_q = |QH^{\blacktriangledown}(Z_2; \underline{\underline{K}}\varrho_\bullet\mathscr{D}_q)|$. The collection $Y = \{Y_q \mid q \geq 0\}$ is also a Γ-space. The map $X_1 \to Y_1$ in 4.5 extends to a map of Γ-spaces in the obvious way. (We will not use the categories $\mathscr{D}_q$ again, except for $q = 1$, in which case we write $\mathscr{D}$ as before.) □

4.6. Proposition. *The simplicial Z_2-spectrum $\underline{\underline{K}}\varrho_\bullet\mathscr{D}$ is an augmented Z_2-resolution.*

Proof. Fix $n > 0$. We first investigate the homotopy type of $\underline{\underline{K}}\varrho_n\mathscr{D}$, without any involution. (It is therefore convenient to use the original Waldhausen model of $\underline{\underline{K}}\varrho_n\mathscr{D}$, without the prefix x.) Let $\hat{\varrho}_n\mathscr{D} \subset \varrho_n\mathscr{D}$ be the full subcategory consisting of the well-behaved objects; it is a subcategory with cofibrations and weak equivalences. By 3.2 and 3.3 the inclusion $w\hat{\varrho}_n\mathscr{D} \subset w\varrho_n\mathscr{D}$ induces a homotopy equivalence of nerves, where the prefix w denotes weak equivalences. Similarly, the inclusions $w\mathscr{S}_k\hat{\varrho}_n\mathscr{D} \subset w\mathscr{S}_k\varrho_n\mathscr{D}$ induce homotopy equivalences of nerves. Consequently, inclusion defines a homotopy equivalence $\underline{\underline{K}}\hat{\varrho}_n\mathscr{D} \xrightarrow{\simeq} \underline{\underline{K}}\varrho_n\mathscr{D}$.

Next, let s be any face of Δ^n. Let $f_s : \hat{\varrho}_n\mathcal{D} \to \mathcal{D}$ be the functor sending an object C in $\hat{\varrho}_n\mathcal{D}$ to $C(s/\partial s)$, the quotient of the cofibration $C(\partial s) \to C(s)$. This is an exact functor, so it induces a map $\underline{\underline{K}}\hat{\varrho}_n\mathcal{D} \to \underline{\underline{K}}\mathcal{D}$ which we also write f_s.

In the opposite direction, define a functor $e_s : \mathcal{D} \to \hat{\varrho}_n\mathcal{D}$ by letting

$$(e_s B)(q) = \begin{cases} B & \text{if } q \geq s, \\ 0 & \text{otherwise} \end{cases}$$

for any B in $\mathcal{D}$ and any $q \in \mathrm{po}[n]$. (For $s \leq q \leq q'$, the structure maps $(e_s B)(q) \to (e_s B)(q')$ are to be identity maps $B \to B$.) Again, each e_s is an exact functor and induces a map $\underline{\underline{K}}\mathcal{D} \to \underline{\underline{K}}\hat{\varrho}_n\mathcal{D}$ which we also write e_s.

Now let

$$e = \bigvee_s e_s : \bigvee_s \underline{\underline{K}}\mathcal{D} \to \underline{\underline{K}}\hat{\varrho}_n\mathcal{D},$$

$$f = \prod_s f_s : \underline{\underline{K}}\hat{\varrho}_n\mathcal{D} \to \prod_s \underline{\underline{K}}\mathcal{D} \simeq \bigvee_s \underline{\underline{K}}\mathcal{D}$$

where in both cases s ranges over all faces of Δ^n. It is clear from the definitions that

$$f \cdot e \simeq \mathrm{id} : \bigvee_s \underline{\underline{K}}\mathcal{D} \to \bigvee_s \underline{\underline{K}}\mathcal{D}.$$

But it also follows from the additivity theorem in [41] that

$$e \cdot f \simeq \mathrm{id} : \underline{\underline{K}}\hat{\varrho}_n\mathcal{D} \to \underline{\underline{K}}\hat{\varrho}_n\mathcal{D}.$$

Namely, choose a filtration of Δ^n by CW-subspaces

$$\emptyset = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_{2^{n+1}-1} = \Delta^n$$

such that the X_i are all distinct (which means that $X_i - X_{i-1}$ must consist of exactly one cell if $0 < i \leq 2^{n+1} - 1$). Such a filtration of Δ^n induces a natural filtration on objects C of $\hat{\varrho}_n\mathcal{D}$, given by

$$\mathrm{Filt}_i\, C(s) = C(s \cap X_i) \quad \text{for } 0 < i \leq 2^{n+1} - 1$$

where s is any face of Δ^n. (Remember that C is well behaved.) By version (4) of the additivity theorem 1.3.2 in [41], the exact functor

$$C \mapsto \bigoplus_i (\mathrm{Filt}_i\, C / \mathrm{Filt}_{i-1}\, C)$$

from $\hat{\varrho}_n\mathcal{D}$ to itself, where $0 < i \leq 2^{n+1} - 1$, induces a self-map of $\underline{\underline{K}}\hat{\varrho}_n\mathcal{D}$ which is homotopic to the identity. On the other hand, this map is clearly also homotopic to $e \cdot f$ above, and we conclude that $e \cdot f \simeq \mathrm{id}$. Therefore

$$\underline{\underline{K}}\varrho_n\mathcal{D} \simeq \underline{\underline{K}}\hat{\varrho}_n\mathcal{D} \simeq \bigvee_s \underline{\underline{K}}\mathcal{D}.$$

(In Section 0 we specified a different map, say g, from $\underline{\underline{K}}\varrho_n\mathcal{D}$ to $\bigvee \underline{\underline{K}}\mathcal{D}$; but it is clear that the composition

$$\bigvee \underline{\underline{K}}\mathcal{D} \xrightarrow{e} \underline{\underline{K}}\hat{\varrho}_n\mathcal{D} \xrightarrow{\simeq} \underline{\underline{K}}\varrho_n\mathcal{D} \xrightarrow{g} \bigvee \underline{\underline{K}}\mathcal{D}$$

is a homotopy equivalence, and therefore g also is a homotopy equivalence.)

Next, observe that $\hat{\varrho}_\bullet\mathscr{D}$ is an incomplete simplicial category. That is, the face operators $d_i:\varrho_n\mathscr{D}\to\varrho_{n-1}\mathscr{D}$ restrict to face operators $d_i:\hat{\varrho}_n\mathscr{D}\to\hat{\varrho}_{n-1}\mathscr{D}$, but there are no reasonable degeneracy operators $\hat{\varrho}_n\mathscr{D}\to\hat{\varrho}_{n+1}\mathscr{D}$ (this is why we normally prefer to work with $\varrho_\bullet\mathscr{D}$). If we regard the splitting $f:\underline{K}\hat{\varrho}_n\mathscr{D}\simeq\bigvee\underline{K}\mathscr{D}$ above as a splitting of the contravariant functor

$$[-,\underline{K}\hat{\varrho}_n\mathscr{D}]\cong[-,\underline{K}\varrho_n\mathscr{D}]$$

on the homotopy category of spectra, then each face operator d_i becomes a projection

$$\bigoplus_{s\in \mathrm{po}[n]}[-,\underline{K}\mathscr{D}]\to\bigoplus_{q\in \mathrm{po}[n-1]}[-,\underline{K}\mathscr{D}]$$

which annihilates those summands whose label $s\in\mathrm{po}[n]$ contains $i\in[n]$. Consequently, in the splitting

$$\underline{K}\varrho_n\mathscr{D}\simeq\bigvee_s\underline{K}\mathscr{D},$$

the nondegenerate summand $N\underline{K}\varrho_n\mathscr{D}$ corresponds to the wedge summands labelled by $s=\Delta^n$ and $s=d_0\Delta^n$. (See 1.5 and its sequel.) Further, the face map

$$d_0:N\underline{K}\varrho_n\mathscr{D}\simeq\underline{K}\mathscr{D}\vee\underline{K}\mathscr{D}\to N\underline{K}\varrho_{n-1}\mathscr{D}\simeq\begin{cases}\underline{K}\mathscr{D}\vee\underline{K}\mathscr{D} & \text{if } n>1,\\ \underline{K}\mathscr{D} & \text{if } n=1\end{cases}$$

maps the second wedge summand $\underline{K}\mathscr{D}\subset N\underline{K}\varrho_n\mathscr{D}$ identically to the first wedge summand $\underline{K}\mathscr{D}\subset N\underline{K}\varrho_{n-1}\mathscr{D}$, and maps the first wedge summand $\underline{K}\mathscr{D}\subset N\underline{K}\varrho_n\mathscr{D}$ trivially. Therefore $\underline{K}\varrho_\bullet\mathscr{D}$ is acyclic.

In order to understand the involution on $\underline{K}\varrho_n\mathscr{D}$, or on $\underline{K}\hat{\varrho}_n\mathscr{D}$, we now introduce, for any $s\in\mathrm{po}[n]$, another exact functor $\bar{e}_s:\mathscr{D}\to\hat{\varrho}_n\mathscr{D}$. Let $s^\perp\subset\Delta^n$ be the union of all faces not containing the face s, and for any object C in $\mathscr{D}$ define $\bar{e}_sC$ by

$$(\bar{e}_sC)(q)=\mathrm{cl}(q/q\cap s^\perp)\otimes C$$

for $q\in\mathrm{po}[n]$. (As usual, $\mathrm{cl}(q/q\cap s^\perp)$ is the quotient of the cofibration $\mathrm{cl}(q\cap s^\perp)\to\mathrm{cl}(q)$; both q and $q\cap s^\perp$ are regarded as subspaces of Δ^n.) The structure maps $(\bar{e}_sC)(q)\to(\bar{e}_sC)(q')$ for $q\le q'$ are given by inclusion.

We also write $\bar{e}_s:\underline{K}\mathscr{D}\to\underline{K}\hat{\varrho}_n\mathscr{D}$ for the map of spectra induced by $\bar{e}_s$. It is useful because the square

$$(*)\qquad\begin{array}{ccc}\underline{K}\mathscr{D} & \xrightarrow{e_s} & \underline{K}\hat{\varrho}_n\mathscr{D}\\ \Big\downarrow{\scriptstyle\text{involution}} & & \Big\downarrow{\scriptstyle\text{involution}}\\ \underline{K}\mathscr{D} & \xrightarrow{\bar{e}_s} & \underline{K}\hat{\varrho}_n\mathscr{D}\end{array}$$

commutes up to homotopy. This can be seen as follows. Let $\eta:\mathbb{Z}\to B^t\otimes_R C$ be a nondegenerate pairing between objects of $\mathscr{D}$. Associate with η a nondegenerate pairing η^s between the objects $e_s B$ and $\bar{e}_s C$ in $\hat{\varrho}_n \mathscr{D}$ by observing that

$$(e_s B)(q)^{\mathrm{t}} \otimes_R (\bar{e}_s C)(q) \cong \mathrm{cl}(q/q \cap s^{\perp}) \otimes (B^{\mathrm{t}} \otimes_R C)$$

for any $q \in \mathrm{po}[n]$, and that $\mathrm{cl}(q) \cong \mathrm{cl}(q) \otimes \mathbb{Z}$ maps to $\mathrm{cl}(q/q \cap s^{\perp}) \otimes (B^{\mathrm{t}} \otimes_R C)$ by (projection$\otimes \eta$). Notice that η is an object of degree 0 in $xw\mathscr{D}$, and η^s is an object of degree 0 in $xw\hat{\varrho}_n \mathscr{D}$. We can proceed similarly with objects of degree $j \geq 0$ in $xw\mathscr{D}$, or with objects of degree $j \geq 0$ in $xw\mathscr{S}_k \mathscr{D}$, for any $k \geq 0$. In short, we obtain an explicit map from the x-model of $\underline{\underline{K}}\mathscr{D}$ to the x-model of $\underline{\underline{K}}\varrho_n \mathscr{D}$ or of $\underline{\underline{K}}\hat{\varrho}_n \mathscr{D}$, and the map clearly shows that the square $(*)$ commutes up to homotopy.

So the involution $\underline{\underline{K}}\hat{\varrho}_n \mathscr{D} \to \underline{\underline{K}}\hat{\varrho}_n \mathscr{D}$ can be described, as a homotopy class of self-maps of a $(2^{n+1}-1)$-fold wedge of copies of $\underline{\underline{K}}\mathscr{D}$, by a square matrix

$$(f_q \cdot \bar{e}_s \cdot \mathrm{inv})_{q,s \in \mathrm{po}[n]}$$

where $\mathrm{inv}: \underline{\underline{K}}\mathscr{D} \to \underline{\underline{K}}\mathscr{D}$ is the usual involution. It is easy to verify that $f_q \bar{e}_s$ is isomorphic, as an exact functor from $\mathscr{D}$ to itself, to an iterated suspension functor Σ^j for suitable j depending on s and q, or to the zero functor. As a map from $\underline{\underline{K}}\mathscr{D}$ to itself, it is therefore homotopic to plus or minus the identity, or to zero. (See [41, Proposition 1.6.2].) If we now restrict attention to those q and s which are equal to either Δ^n or $d_0 \Delta^n$, then we obtain a 2×2 submatrix

$$\begin{pmatrix} (-)^n \mathrm{inv} & (-)^n \mathrm{inv} \\ 0 & (-)^{n-1} \mathrm{inv} \end{pmatrix}$$

which describes the effect of the involution $\underline{\underline{K}}\varrho_n \mathscr{D} \to \underline{\underline{K}}\varrho_n \mathscr{D}$ on the nondegenerate summand $N\underline{\underline{K}}\varrho_n \mathscr{D} \simeq \underline{\underline{K}}\mathscr{D} \vee \underline{\underline{K}}\mathscr{D}$. We conclude that $N\underline{\underline{K}}\varrho_n \mathscr{D}$ is an induced Z_2-spectrum; since this is true for any $n > 0$, we have verified 2.9(iii) for the simplicial Z_2-spectrum $\underline{\underline{K}}\varrho_\bullet \mathscr{D}$. □

4.7. Remark. The splitting of $\underline{\underline{K}}\varrho_n \mathscr{D}$ into copies of $\underline{\underline{K}}\mathscr{D}$ is a special case of a splitting theorem for certain K-theories obtained by Lück [19]. Lück also has a more convincing characterization of well-behaved objects. (See p. 80/81 of tom Dieck's book [37]. We nevertheless prefer our own description because it remains meaningful when spaces are substituted for chain complexes.)

4.8. Remark. There are two slightly different descriptions of $Q\underline{\underline{L}}p^{\blacktriangledown}(R)$. In the first, which we used, an n-simplex of $Q\underline{\underline{L}}p^{\blacktriangledown}(R)$ is an object C in $\varrho_n \mathscr{D}$ with some extra structure; in the second description, one insists that C be well behaved. The first description gives a simplicial set, say X, which does not have the Kan property; the second gives an incomplete simplicial set (i.e. one without degeneracy operators), say X', which does have the Kan property. It is clear that $\pi_n(X')$ is the bordism group of formally n-dimensional symmetric algebraic Poincaré complexes. See [29] for information on incomplete simplicial sets.

We shall show that X' does indeed have the Kan property and that the inclusion $X' \hookrightarrow X$ is a homotopy equivalence. (For the time being we forget the degeneracy operators in X and regard the inclusion $X' \hookrightarrow X$ as one of incomplete simplicial sets. Incomplete simplicial sets have their own geometric realizations; moreover it makes no difference to the homotopy type whether or not we take care of the degeneracy operators in defining the geometric realization of X, or of any other complete simplicial set.) The reader is warned that we are up to something tedious.

For $0 \le i \le n$ let $\Lambda_{n,i} \subset \Delta^n$ be the union of all faces $d_j \Delta^n$ for $j \ne i$. A well-behaved object C of $\varrho_n \mathscr{D}$ will be called i-shallow if the cofibrations

$$C(d_i \Delta^n) \hookrightarrow C(\Delta^n), \qquad C(\Lambda_{n,i}) \hookrightarrow C(\Delta^n)$$

induced by inclusion are both homotopy equivalences. An arbitrary object C in $\varrho_n \mathscr{D}$ is i-shallow if its well-behaved approximation (of 3.2) is. The objects in $\varrho_n \mathscr{D}$ are the n-simplices of a simplicial set Y; and the well-behaved objects form an incomplete simplicial subset $Y' \subset Y$. There is a square of incomplete simplicial sets and maps

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ Y' & \longrightarrow & Y \end{array}$$

where the maps p are forgetful.

Now interpret Δ^n as an incomplete simplicial set (with one simplex for each $s \in \bar{\text{po}}[n]$) and interpret $\Lambda_{n,i}$ as an incomplete simplicial subset of Δ^n. Let $f: \Lambda_{n,i} \to X$ be a map and let $g: \Delta^n \to Y$ be an extension of $p \cdot f: \Lambda_{n,i} \to Y$. Can we find $\hat{f}: \Delta^n \to X$ extending f and lifting g?

Observation 1. Yes, if the n-simplex in Y determined by g is i-shallow.

It is not hard to see that any map $\Lambda_{n,i} \to Y'$ has an i-shallow extension $\Delta^n \to Y'$; together with the observation this implies that X' has the Kan property.

Let V be an arbitrary incomplete simplicial set. Write $[0,1]$ for the incomplete simplicial set generated by a single 1-simplex (with no relations), and write $V \otimes [0,1]$ for the geometric product of V and $[0,1]$. (See [29, Section 3] for details.) This is an incomplete simplicial set whose geometric realization is homeomorphic to the product of the geometric realizations of V and $[0,1]$. Given any map

$$e: V \to Y$$

we shall construct an extension

$$\hat{e}: V \otimes [0,1] \to Y$$

which maps $V \otimes \{1\}$ to $Y' \subset Y$ (and agrees with e on $V \otimes \{0\}$). We may assume that V is a standard simplex Δ^n; the general case can then be obtained by glueing. In this case the map $e: \Delta^r \to Y$ is worth as much as a simplex in Y, or a functor C from the category of faces of Δ^n to $\mathscr{D}$; we must try to extend it to a functor $\hat{C}$

on the category of faces (= simplices) of $\Delta^n\otimes[0,1]$. Let $\bar{C}$ be the well-behaved approximation of C constructed in 3.2. If w is a face (= simplex) of $\Delta^n\otimes[0,1]$, let

$$\hat{C}(w)=\begin{cases}\bar{C}(\mathrm{pr}_1(w)) & \text{if } w\subset\Delta^n\otimes\{1\},\\ C(\mathrm{pr}_1(w)) & \text{otherwise}\end{cases}$$

where $\mathrm{pr}_1(w)\subset\Delta^n$ is the image of w under the projection of geometric realizations $|\Delta^n\otimes[0,1]|\to|\Delta^n|$. (The dimension of $\mathrm{pr}_1(w)$ may be less than that of w.) The structure chain maps for $\hat{C}$ are obvious. This completes the construction of $\hat{e}$, in a special case and therefore in general.

The extension $\hat{e}$ has a remarkable property. Let x^n be an n-simplex in V. The inverse image of x^n in $|V\otimes[0,1]|$ under the projection map $|V\otimes[0,1]|\to|V|$ contains $n+1$ simplices of dimension $n+1$, which we label $z_0x^n, z_1x^n, \ldots, z_nx^n$. (Fix the order in such a way that the barycentre of z_ix^n maps to $(i+1)/(n+2)$ under the projection to $[0,1]$.) We now observe that $\hat{e}(z_ix^n)$ is an i-shallow simplex in Y, for any i between 0 and n.

Observation 2. Suppose that $e: V\to Y$ lifts to a map $q: V\to X$, so that $p\cdot q=e$. Then $\hat{e}$ also lifts to a map $\hat{q}: V\otimes[0,1]\to X$ which agrees with q on $V\otimes\{0\}$.

(Such a map q will automatically send $V\otimes\{1\}$ to X'.) Construct $\hat{q}$ as follows: $\hat{q}$ is already defined on $V\otimes\{0\}$. Use Observation 1 to extend it over the incomplete simplicial subset of $V\otimes[0,1]$ obtained from $V\otimes\{0\}$ by adjoining all simplices of the form z_0x^0, where x^0 can be any 0-simplex in V. Next, adjoin the simplices of the form z_0x^1, using Observation 1 again; next, those of the form z_1x^1; then those of the form z_0x^2; those of the form z_1x^2; and so on in anti-lexicographic order. The result is an extension over $V\otimes[0,1]$.

Taking q to be the identity on X in Observation 2, and looking at the restriction of $\hat{q}$ to $X\otimes\{1\}$, we see that X is a homotopy retract of X'. The corresponding retraction on $\pi_*(X')$ is the identity; this can be checked by hand since X' has the Kan property. Therefore the inclusion $X'\to X$ is a homotopy equivalence.

(We will not use incomplete simplicial sets again in this paper.)

4.9. Remark. Let C be an object of $\varrho_n\mathscr{D}$. We define a quadratic pairing of C with itself to be a natural chain map

$$\psi: \mathrm{cl}(s)\to W\otimes_{\mathbb{Z}[Z_2]}(C(s)^{\mathrm{t}}\otimes_R C(s))$$

with $s\in\mathrm{po}[n]$. Such a quadratic pairing gives rise to a symmetric pairing on composing with the algebraic norm map

$$W\otimes_{\mathbb{Z}[Z_2]}(C(s)^{\mathrm{t}}\otimes_R C(s))\to\mathrm{Hom}_{\mathbb{Z}[Z_2]}(W, C(s)^{\mathrm{t}}\otimes_R C(s));$$

therefore we can speak of nondegeneracy. (See the prelude to 2.11.) The nondegenerate quadratic pairings (C,ψ) as above, with C in $\varrho_n\mathscr{D}$, are the n-simplices of a simplicial set whose geometric realization we call $QL\underline{\underline{p}}_\blacktriangledown(R)$. Arguing as in 4.8 one can see that $\pi_n(QL\underline{\underline{p}}_\blacktriangledown(R))$ is the bordism group of formally n-dimensional quadratic algebraic Poincaré complexes.

The definitions of $Q\underline{\underline{L}}p^{\blacktriangledown}(R)$ and $Q\underline{\underline{L}}p_{\blacktriangledown}(R)$ are due to Ranicki. A sketch is given in [26]; a more detailed version appears in [18]. (The detailed version is not quite correct since Ranicki habitually works with homology classes of symmetric or quadratic pairings where we use explicit symmetric or quadratic pairings. This spoils the Kan condition despite claims made in [18].) The homotopy groups of $Q\underline{\underline{L}}p^{\blacktriangledown}(R)$ go back to Mishchenko [21]; we should also mention Quinn [23, 24] for introducing simplicial methods to the subject. See also Chapters 9 and 17A of Wall's book [44].

Technicalities apart, our definitions differ slightly from those of Ranicki in that we allow chain complexes (in $\mathscr{D}$) having no particular connectivity properties. This does not affect the groups $L_n(R):=\pi_n(Q\underline{\underline{L}}p_{\blacktriangledown}(R))$, because of the possibility of doing algebraic surgery below the middle dimension. But it does affect $L^n(R):=\pi_n(Q\underline{\underline{L}}p^{\blacktriangledown}(R))$. As a result the double skew suspension homomorphism $\bar{S}^2: L^n(R)\to L^{n+4}(R)$ is an isomorphism for any $n\geq 0$ in our version (it has an obvious inverse), but not in Ranicki's version. Our version can be obtained from Ranicki's by stabilizing with respect to the suspension.

5. Nonlinear duality

Having completed the linear part of our mission, we turn to nonlinear studies: spectra will replace chain complexes, and smash products of spectra will replace tensor products of chain complexes (as in 3.5, say). Our approach to smash products is that of 2.1. (In an earlier version, we made some incorrect assumptions about smash products; these were pointed out by Waldhausen and Vogt.) In this section and the next, spectrum need not mean CW-spectrum, and space need not mean CW-space.

Let X be a pointed space. By a filtration of X is meant a family of pointed subspaces $\mathrm{Filt}_i X$, with $i\in\mathbb{Z}$, such that $\mathrm{Filt}_i X\subseteq \mathrm{Filt}_{i+1} X$ for all i, and X is the union of the $\mathrm{Filt}_i X$. Call the filtration hemicellular if the pair $(X, \mathrm{Filt}_i X)$ is i-connected for all i, and $\mathrm{Filt}_{-1} X=\{*\}$. (See [34, Definition 3.12].) A filtration of a spectrum X by subspectra $\mathrm{Filt}_i X$ will be called hemicellular if the induced filtrations on the X_n are, after a suitable shift: i.e. the pair $(X_n, (\mathrm{Filt}_i X)_n)$ is $(i+n)$-connected for all n and all i, and $(\mathrm{Filt}_{-1-n} X)_n=\{*\}$. Hemicellular filtrations of bispectra can be defined in the same way. A map $f: X\to Y$ between pointed spaces (or a function between spectra, or a function between bispectra) equipped with hemicellular filtrations is called hemicellular if it respects the filtrations. If X is CW then the usual cellular filtration is hemicellular; with regard to this filtration any map f from X to some Y with hemicellular filtration is homotopic to a hemicellular map, and the homotopy can be arranged to fix a pointed CW-subspace (or a CW-subspectrum, or a CW-sub-bispectrum) $X'\subset X$ provided f is already hemicellular on X'.

5.1. Definition. The hemicellular chain complex $\mathrm{hcl}(Y)$ of a pointed space (or a spectrum, or a bispectrum) with hemicellular filtration is given in degree k by

$$H_k(\mathrm{Filt}_k\, Y, \mathrm{Filt}_{k-1}\, Y; \mathbb{Z})$$

with the obvious differential. (Compare [9, Chapter V, Definitions 1.1, 1.2].)

In the case of a pointed CW-space with the cellular filtration, $\mathrm{hcl}(Y)$ is just the reduced cellular chain complex; for a CW-spectrum or CW-bispectrum Y with the cellular filtration, $\mathrm{hcl}(Y)$ agrees with the cellular chain complex $\mathrm{cl}(Y)$.

Let G be a topological group. We shall not assume that G has a CW-structure. A G-space is a topological space X together with a continuous action $G\times X\to X$. A tame G-CW-space is a G-space X equipped with a base point fixed under the G-action, and with a filtration satisfying conditions (i)–(iii) below.

(i) $\mathrm{Filt}_{-1}\, X=\{*\}$.

(ii) For any $i\geq 0$, there exists a G-homeomorphism

$$\mathrm{Filt}_i\, X \cong \mathrm{Filt}_{i-1} X \amalg_f (G\times D^i\times w(i))$$

(relative to $\mathrm{Filt}_{i-1} X$), where $w(i)$ is a set, D^i is the standard disk, and $f: G\times S^{i-1}\times w(i)\to \mathrm{Filt}_{i-1} X$ is a G-map.

(iii) $X=\bigcup \mathrm{Filt}_i\, X$; a subset of X is closed if and only if its intersection with $\mathrm{Filt}_i\, X$ is closed for all $i\geq -1$.

Then X/G has the structure of a pointed CW-space, and we will say that X is a finite tame G-CW-space if X/G is a finite (=compact) CW-space. (For discrete G, our definition of a tame G-CW-space agrees with that of Ranicki [25], except that Ranicki does not use the word tame.)

A tame G-CW-space X is finitely dominated if there exists a finite tame G-CW-space Y and G-maps $i: X\to Y$, $r: Y\to X$ such that ri is G-homotopic to the identity on X.

The filtration of a tame G-CW-space X is hemicellular. The hemicellular chain complex $\mathrm{hcl}(X)$ is a chain complex of free modules over the group ring $\mathbb{Z}\pi$, where $\pi=\pi_0(G)$.

If X is a tame G-CW-space, then so is ΣX, with $(\Sigma X)/G\cong\Sigma(X/G)$. A tame G-CW-spectrum is a family of tame G-CW-spaces $\{X_i \mid i\in\mathbb{Z}\}$, together with G-maps $\varepsilon_i:\Sigma X_i\to X_{i+1}$; it is required that each ε_i be an isomorphism of ΣX_i with a tame G-CW-subspace of X_{i+1}. (For $G=\{1\}$ this is still Boardman's definition; for discrete G it is due to Ranicki [25].) There are notions of finite and finitely dominated for tame G-CW-spectra. G-maps between tame G-CW-spectra are defined in the expected way, as equivalence classes of G-functions (recall the definitions for $G=\{1\}$). Any tame G-CW-spectrum has a canonical hemicellular filtration. The associated hemicellular chain complex is one of free modules over $\mathbb{Z}\pi$.

Let $\mathscr{U}$ be the category of finitely dominated tame G-CW-spectra and hemicellular G-maps. The rule

$$X\mapsto \mathrm{hcl}(X)$$

is then a functor from $\mathscr{U}$ to $\mathscr{D}$, where $\mathscr{D}$ is the category introduced in Section 0, with $R=\mathbb{Z}\pi$ as ground ring.

5.2. Proposition. *The following conditions on a morphism $f: X \to Y$ in $\mathcal{U}$ are equivalent*:

(i) *f is a G-homotopy equivalence*;

(ii) *the chain map* $\mathrm{hcl}(X) \to \mathrm{hcl}(Y)$ *induced by f is a homotopy equivalence over* $\mathbb{Z}\pi$;

(iii) *the graded homomorphism*

$$H_*(\mathrm{hcl}(X)) \to H_*(\mathrm{hcl}(Y))$$

induced by f is an isomorphism.

Proof. For $n \in \mathbb{Z}$, let $\pi_n(X)$ be the group of homotopy classes of maps from the CW-spectrum $\Sigma^n \underline{S}^0$ to X. If V is any object in $\mathcal{U}$, write $[V, X]$ for the abelian group of G-homotopy classes of G-maps from V to X. Observe that

$$\pi_n(X) \cong [\Sigma^n(G_+ \wedge \underline{S}^0), X].$$

Saying that $f: X \to Y$ is a G-homotopy equivalence is equivalent to saying that $f_*: \pi_*(X) \to \pi_*(Y)$ is an isomorphism. For if the homomorphism $[V, X] \to [V, Y]$ given by composition with f is an isomorphism whenever V has the form $\Sigma^n(G_+ \wedge \underline{S}^0)$ for some n, then it will be an isomorphism for any finite tame G-CW-spectrum V (use induction on the number of cells of V/G). It will also be an isomorphism for finitely dominated V, and in particular for $V = X$ and $V = Y$.

Assuming 5.2(ii), we shall therefore prove that

$$f_*: \pi_*(X) \to \pi_*(Y)$$

is an isomorphism. The hemicellular filtration of X gives rise to a spectral sequence converging to $\pi_*(X)$. Its E^2-term is, by inspection,

$$E^2_{p,r} \cong H_p(\pi_r(G_+ \wedge \underline{S}^0) \otimes_{\mathbb{Z}\pi} \mathrm{hcl}(X)).$$

There is a similar spectral sequence converging to $\pi_*(Y)$. If we assume 5.2(ii), then f will induce an isomorphism between the E^2-terms of the respective spectral sequences, and therefore an isomorphism from $\pi_*(X)$ to $\pi_*(Y)$. The implication (iii) $\Rightarrow$ (ii) is well known; note that our chain complexes are homotopy equivalent over $\mathbb{Z}\pi$ to finitely generated ones. $\square$

We now describe duality in $\mathcal{U}$, first in an untwisted setting. Given two objects X and Y in $\mathcal{U}$, form the bispectrum $X \wedge_G Y$ such that

$$(X \wedge_G Y)_{m,n} = X_m \wedge_G Y_n := (X_m \wedge Y_n)/G,$$

where G acts simultaneously on X_m and Y_n. Equip $X \wedge_G Y$ with a suitable 'product filtration', using the intrinsic hemicellular filtrations of X and Y. This will again be hemicellular. One finds that

$$\mathrm{hcl}(X \wedge_G Y) \cong \mathrm{hcl}(X)^t \otimes_{\mathbb{Z}\pi} \mathrm{hcl}(Y),$$

where we use the untwisted involution on $\mathbb{Z}\pi$ (given by $g \mapsto g^{-1}$ for $g \in \pi$) to make sense of the superscript t.

5.3. Definition (untwisted version). A pairing of X and Y is a hemicellular map

$$\eta : S^0 \to X \wedge_G Y$$

from the pointed CW-space S^0 to the bispectrum $X \wedge_G Y$. (See the end of 2.1.)

Such a map η will induce a chain map

$$\mathbb{Z} \to \mathrm{hcl}(X \wedge_G Y) \cong \mathrm{hcl}(X)^t \otimes_{\mathbb{Z}\pi} \mathrm{hcl}(Y),$$

since the reduced cellular chain complex of S^0 is $\mathbb{Z}$. This is a pairing between the objects $\mathrm{hcl}(X)$ and $\mathrm{hcl}(Y)$ in $\mathscr{D}$; if it is nondegenerate, call η nondegenerate.

The twisted setting is as follows. The twist itself comes in the shape of a well-pointed G-space J together with an element in $\pi_q(J)$, for some $q \geq 0$, represented by a weak homotopy equivalence $S^q \to J$. ("Well-pointed" means that the inclusion $\{*\} \hookrightarrow J$ has the homotopy extension property; it is understood that G fixes the base point.) Since $\pi = \pi_0(G)$ acts on $H_q(J, \{*\}; \mathbb{Z}) \cong \mathbb{Z}$, we obtain a homomorphism

$$w : \pi \to Z_2 \cong \mathrm{Aut}(\mathbb{Z}).$$

Given two objects X and Y in $\mathscr{U}$, form the bispectrum $X \wedge_{G,J} Y$ such that

$$(X \wedge_{G,J} Y)_{m,n} = (X_m \wedge J \wedge Y_n)/G$$

where G acts on the three factors simultaneously. Now X and Y have intrinsic hemicellular filtrations, and J has a hemicellular filtration given by

$$\mathrm{Filt}_i\, J = \begin{cases} J & \text{if } i \geq q, \\ \{*\} & \text{if } i < q. \end{cases}$$

So there is a product filtration on $X \wedge_{G,J} Y$. Using the well-pointedness of J one finds that it is hemicellular, and

$$\mathrm{hcl}(X \wedge_{G,J} Y) \cong \Sigma^q(\mathrm{hcl}(X)^t \otimes_{\mathbb{Z}\pi} \mathrm{hcl}(Y)).$$

This time we use the w-twisted involution on $\mathbb{Z}\pi$, given by $g \mapsto w(g) \cdot g^{-1}$ for $g \in \pi$.

5.4. Definition (twisted version). A pairing between X and Y is a hemicellular map

$$\eta : S^q \to X \wedge_{G,J} Y$$

from the pointed CW-space S^q to the bispectrum $X \wedge_{G,J} Y$.

Again, such a pairing induces a chain map

$$\Sigma^q \mathbb{Z} \to \Sigma^q(\mathrm{hcl}(X)^t \otimes_{\mathbb{Z}\pi} \mathrm{hcl}(Y))$$

which is in effect a pairing between $\mathrm{hcl}(X)$ and $\mathrm{hcl}(Y)$. If it is nondegenerate, call η nondegenerate.

Taking $J=S^0$ shows that 5.4 includes 5.3. See Section 6 for other choices of J.

Fix G and J for the rest of this section, and fix a positive integer n as in Section 3. Let $\varrho_n\mathscr{U}$ be the category of covariant functors from $\mathrm{po}[n]$ to $\mathscr{U}$. Since $\mathscr{U}$ is a category with cofibrations and weak equivalences, so is $\varrho_n\mathscr{U}$. (A morphism $f:X\to Y$ in $\mathscr{U}$ is a cofibration if it is a G-CW-isomorphism of X with a subobject of Y. It is a weak equivalence if it is a G-homotopy equivalence. A morphism $f:X\to Y$ in $\varrho_n\mathscr{U}$ is a cofibration if each $f(s):X(s)\to Y(s)$ is a cofibration, for $s\in\mathrm{po}[n]$; it is a weak equivalence if each $f(s)$ is a weak equivalence.)

A pairing between objects X and Y in $\varrho_n\mathscr{U}$ is a natural map

$$\eta:\Sigma^q(s_+)\to X(s)\wedge_{G,J}Y(s)$$

where s ranges over the faces of Δ^n. (See the end of Definition 2.1; remember also that q occured in the description of J.) This will be considered nondegenerate if the induced pairing of objects $\mathrm{hcl}(X)$, $\mathrm{hcl}(Y)$ in $\varrho_n\mathscr{D}$ is nondegenerate. We want to show, in analogy with 3.6, that any object X in $\varrho_n\mathscr{U}$ occurs in a nondegenerate pairing, and that such a nondegenerate pairing is essentially unique (in analogy with 3.10). It seems best to reformulate the results of Section 3 in the nonlinear setting, one by one.

The nonlinear version of 3.1 is as follows. Let X and Y be objects in $\varrho_n\mathscr{U}$; assume that X is well behaved. (The definition of "well behaved" is literally the same as in the linear setting.) Let $[X,Y]_k$ be the group of homotopy classes of G-maps from Σ^kX to Y. If Y' is another object in $\varrho_n\mathscr{U}$, then the homomorphisms

$$[X,Y]_k\to[X,Y']_k$$

induced by f are isomorphisms.

The proof can be modelled on that of 3.1. (It is possible to define a mapping spectrum $\underline{\underline{\mathrm{map}}}_G(X,Y)$ whose homotopy groups are the groups $[X,Y]_k$; it has a natural filtration, and the proof consists in inspecting the homotopy groups of the filtration quotients.) There is also an exactness statement, corresponding to the second part of 3.1, whose formulation we leave to the reader.

There are obvious nonlinear versions of 3.2 and 3.3. The nonlinear version of 3.4 is simply false; even if $n=0$ there are objects in $\varrho_n\mathscr{U}\cong\mathscr{U}$ which are not G-homotopy equivalent to finite ones. (There is a finiteness obstruction in $\tilde{K}_0(\mathbb{Z}\pi)$.) Note that 3.4 was used only in proving 3.6. The nonlinear version of 3.5 has already been stated, as 5.4. We postpone the nonlinear version of 3.6 and turn to that of 3.7.

Suppose then that η is a nondegenerate pairing of objects X and Y in $\varrho_n\mathscr{U}$; assume that Y is well behaved. Denote by $\pi_k^{\mathrm{nat}}(X\wedge_{G,J}Y)$ the group of homotopy classes of natural maps

$$\Sigma^k(s_+)\to X(s)\wedge_{G,J}Y(s)$$

with $s \in \mathrm{po}[n]$. There are homomorphisms

$$\backslash\eta : [Y, Y']_k \to \pi^{\mathrm{nat}}_{k+q}(X \wedge_{G,J} Y')$$

for any Y' in $\varrho_n \mathscr{U}$, given by sending $f : \Sigma^k Y \to Y'$ to the composition

$$\Sigma^{q+k}(\ldots_+) \xrightarrow{\Sigma^k \eta} X \wedge_{G,J} \Sigma^k Y \xrightarrow{\mathrm{id} \wedge f} X \wedge_{G,J} Y'$$

(where ... stands for the identity functor $s \mapsto s$ on $\mathrm{po}[n]$). These homomorphisms are isomorphisms, for arbitrary Y'.

Proof: The statement remains true when Y' is any functor from $\mathrm{po}[n]$ to the category of tame G-CW-spectra (no finiteness conditions), and in this form it is easier to prove.

Step 1. Fix $r \in \mathrm{po}[n]$, and let Y' be such that the maps $Y'(s) \to Y'(r)$ are isomorphisms if $s \le r$, and $Y'(s) = \{*\}$ if s is not contained in r. Suppose also that $\pi_j(Y'(r)) = 0$ if $j \ne 0$: write $\pi_0(Y'(r)) = \mathscr{M}$. In this case

$$[Y, Y']_k \cong H^{-k}(\mathrm{hcl}(Y(r)); \mathscr{M})$$

and

$$\pi^{\mathrm{nat}}_{k+q}(X \wedge_{G,J} Y') \cong H_k(\mathrm{hcl}(X)_{!r}; \mathscr{M}),$$

which can be seen as follows. We may assume that $Y'(r)/G$ has no cells in negative dimensions, so that there is an 'augmentation'

$$\mathrm{hcl}(Y'(r)) \to \mathscr{M} \cong H_0(\mathrm{hcl}(Y'(r))) \cong \pi_0(Y'(r)).$$

Using this augmentation, define homomorphisms

$$[Y, Y']_k \to H^{-k}(\mathrm{hcl}(Y(r)); \mathscr{M})$$

and

$$\pi^{\mathrm{nat}}_{k+q}(X \wedge_{G,J} Y') \to H_k(\mathrm{hcl}(X)_{!r}; \mathscr{M})$$

for arbitrary k, which are natural in Y and X. These homomorphisms are isomorphisms, by inspection.

So, with our very special assumptions on Y', the homomorphisms

$$\backslash\eta : [Y, Y']_k \to \pi^{\mathrm{nat}}_{q+k}(X \wedge_{G,J} Y')$$

can be identified with

$$\backslash\eta : H^{-k}(\mathrm{hcl}(Y)(r); \mathscr{M}) \to H_k(\mathrm{hcl}(X)_{!r}; \mathscr{M}).$$

They are isomorphisms by 3.13, since η is nondegenerate.

Step 2. Fix $r \in \mathrm{po}[n]$, and let Y' be such that the maps $Y'(s) \to Y'(r)$ are isomorphisms for $s \le r$, and $Y'(s) = \{*\}$ if s is not contained in r. Now $Y'(r)$ has a Postnikov filtration. In other words, it is possible to construct tame G-CW-spectra $\varphi^i Y'(r)$, for $i \in \mathbb{Z}$, and G-maps

$$\varphi^i Y'(r) \to Y'(r)$$

inducing isomorphisms on homotopy groups in dimensions $\geq i$, whereas $\pi_j(\varphi^i Y'(r)) = 0$ for $j<i$. (This is straightforward; define first $\beta^i Y'(r)$ by killing homotopy groups in dimensions $\geq i$. Do so by attaching copies of $G_+ \wedge \mathrm{Disk}_+$. Compare 1.2.) Letting i tend to $-\infty$, one obtains the Postnikov filtration of $Y'(r)$. Because of our assumptions on Y', we can regard it as a filtration of Y' itself. As such it gives rise to two spectral sequences converging, respectively, to $[Y, Y']_*$ and to $\pi^{\mathrm{nat}}_{q+*}(X \wedge_{G,J} Y')$. Slant product with η defines a morphism of spectral sequences. It is an isomorphism on E^2-terms, by Step 1; so it is an isomorphism on E^∞-terms.

Step 3. Given an arbitrary Y', construct a weakly equivalent object with a finite filtration whose filtration quotients satisfy the restrictions imposed in Step 2 (for some r, which may vary). Then apply the five lemma (several times).

Next, there is 3.8. Here the proof of the nonlinear version is obvious, because nondegeneracy of a pairing in $\varrho_n \mathscr{U}$ simply means nondegeneracy of the induced pairing in $\varrho_n \mathscr{D}$. But the formulation, as opposed to the proof, requires care. Assume $n=0$ to simplify notation. For any bispectrum U, let U^{flip} be the bispectrum such that $(U^{\mathrm{flip}})_{m,n} = U_{n,m}$ etc., so that

$$U \mapsto U^{\mathrm{flip}}$$

is a functor from bispectra to bispectra. Let

$$\eta : S^q \to X \wedge_{G,J} Y$$

be a pairing in $\varrho_0 \mathscr{U} \cong \mathscr{U}$. Remember that this is shorthand notation for a map of bispectra

$$\eta : S^q \wedge \underline{S}^0 \wedge \underline{S}^0 \to (X \wedge J \wedge Y)/G.$$

The composite map of bispectra

$$\begin{tikzcd} S^q \wedge \underline{S}^0 \wedge \underline{S}^0 \arrow[d, "\cong"] & (Y \wedge J \wedge X)/G \\ (S^q \wedge \underline{S}^0 \wedge \underline{S}^0)^{\mathrm{flip}} \arrow[r, "\eta^{\mathrm{flip}}"] & (X \wedge J \wedge Y)^{\mathrm{flip}}/G \arrow[u, "\cong"'] \end{tikzcd}$$

where the vertical arrows are given in bidegree (j,k) by

$$\begin{tikzcd} S^q \wedge S^j \wedge S^k \arrow[d, "\cong"] \\ S^q \wedge S^k \wedge S^j \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y_j \wedge J \wedge X_k \\ X_k \wedge J \wedge Y_j \arrow[u, "\cong"'] \end{tikzcd}$$

defines another pairing. This is the switched pairing we want.

The nonlinear version of 3.9 is again obvious. But we still have to show that any X in $\varrho_n \mathscr{U}$, with arbitrary n, occurs in a nondegenerate pairing. (This corresponds to 3.6.) Here are two useful observations.

(i) If $f: X \to X'$ is a morphism in $\varrho_n \mathscr{U}$, and if both X and X' occur in a nondegenerate pairing (say, η and η'), then so does the mapping cone of f.

(ii) If X and Y are objects in $\varrho_n \mathscr{U}$ such that $X \vee Y$ occurs in a nondegenerate pairing, then so does X.

Proof of (i): We can assume that $f: X \to X'$ is a cofibration. Choose duals Y' and Y for X' and X, respectively, so that X and Y occur in a nondegenerate pairing η, and X', Y' occur in a nondegenerate pairing η'. Choose a map $g: Y' \to Y$ dual to f in the sense of 3.9 (nonlinear version). This means that $g \backslash \eta'$ and $f \backslash \eta$ represent the same class in $\pi_q^{\mathrm{nat}}(X' \wedge_{G,J} Y)$. Assume also that g is a cofibration and choose a specific homotopy from $g \backslash \eta'$ to $f \backslash \eta$. Passing to cofibres Y/Y' and X'/X, one finds that the homotopy projects to a nondegenerate pairing between X'/X and $\Sigma^{-1}(Y/Y')$.

Proof of (ii): Let η be a nondegenerate pairing of $X \vee Y$ with some U. Let $r: X \vee Y \to X$ be the projection and let $i: X \to X \vee Y$ be the inclusion. Let $p: U \to U$ be dual to $ir: X \vee Y \to X \vee Y$, in the sense of 3.9 (nonlinear version). Then $p \cdot p \simeq p$ because $ir \cdot ir \simeq ir$. Define U' to be the homotopy direct limit (in this case a telescope) of the diagram

$$U \xrightarrow{p} U \xrightarrow{p} U \xrightarrow{p} U \longrightarrow \cdots.$$

Then U' belongs to $\varrho_n \mathscr{U}$ (it is dominated by U). Write $e: U \to U'$ for the inclusion (which is really a projection from a homotopy-theoretic point of view). Go from $(X \vee Y) \wedge_{G,J} U$ to $X \wedge_{G,J} U'$ by $r \wedge e$. So the nondegenerate pairing η of $X \vee Y$ with U induces another pairing $(r \wedge e) \cdot \eta$ of X with U', which is also nondegenerate.

Now let X be an object in $\varrho_n \mathscr{U}$. The question is whether X occurs in a nondegenerate pairing. We may assume that X is well behaved. By (ii) above, we may also assume that $X(\Delta^n)$ is finite, not just finitely dominated. (If this is not yet the case, we can dominate X by a well-behaved object V with this property; the domination means that $V \simeq X \vee Y$ for suitable Y.) Using (i) and downward induction on the number of cells in the CW-spectrum $X(\Delta^n)/G$, reduce to the case where this number is one. In this case there exists an $r \in \mathrm{po}[n]$ such that

$$X(r) \cong G_+ \wedge \Sigma^k \underline{S}^0 \quad \text{for some } k \in \mathbb{Z},$$
$$X(s) = \{*\} \quad \text{if } r \not\leq s$$

and such that the map $X(r) \to X(s)$ is an isomorphism for all s containing r. (This is so because X is well behaved.) Define Y in $\varrho_n \mathscr{U}$ by

$$Y(r) = G_+ \wedge (r/\partial r) \wedge \Sigma^{-k} \underline{S}^0,$$
$$Y(s) = \{*\} \quad \text{for all } s \neq r.$$

It is not hard to find a nondegenerate pairing η of X with Y. (It is sufficient to specify

$$\eta(r): \Sigma^q(r_+) \to X(r) \wedge_{G,J} Y(r);$$

there is a canonical choice once a pointed map $S^q \to J$ in the distinguished homotopy class has been chosen.)

Finally, the uniqueness result 3.10 depends only on 3.6 and 3.7; since we have the nonlinear versions of 3.6 and 3.7, we have that of 3.10. The nonlinear formulation of 3.12 is obvious. There is some temptation in 3.11, too, but we must resist it until we get to Part III.

6. The π-π-theorem

Rewriting Section 4 in the nonlinear setting requires practically no further ideas, apart from the following definition: A skew-involution on a bispectrum U is a function $f: U \to U^{\text{flip}}$ (or a map $f: U \to U^{\text{flip}}$, if applicable) such that

$$f^{\text{flip}} \cdot f: U \to (U^{\text{flip}})^{\text{flip}} = U$$

is the identity. Recall that U^{flip} is the bispectrum such that $(U^{\text{flip}})_{m,n} = U_{n,m}$ etc., and that $U \mapsto U^{\text{flip}}$ is a functor.

Fix G and J from the previous section. A symmetric pairing in $\varrho_n \mathscr{U}$ is a natural map of the form

$$\varphi: \Sigma^q(\dots_+) \wedge (EZ_2)_+ \to X \wedge_{G,J} X$$

which is Z_2-equivariant.

Explanation: X is an object in $\varrho_n \mathscr{U}$. The dots ... denote the identity functor $s \mapsto s$ on po$[n]$. The map φ is a natural transformation between functors on po$[n]$. It is understood to be hemicellular. Recall that we use shorthand notation as in 2.1; so φ is really a natural map of bispectra

$$\Sigma^q(\dots_+) \wedge (EZ_2)_+ \wedge \underline{S}^0 \wedge \underline{S}^0 \to X \wedge_{G,J} X.$$

The source of φ has a skew-involution which interchanges the two copies of $\underline{S}^0$ and maps each point in EZ_2 to its antipode. The target of φ also has a skew-involution which interchanges the two copies of X. So φ can be required, and is required, to commute with the skew-involutions. (End of explanation.)

If the standard model of EZ_2 is used, then the cellular chain complex of EZ_2 is W (from the proof of 4.3 and 4.4). So a symmetric pairing φ in $\varrho_n \mathscr{U}$ as above gives rise to a symmetric pairing in $\varrho_n \mathscr{D}$, by means of the functor $X \mapsto \text{hcl}(X)$. If the latter is nondegenerate, then φ itself will be considered nondegenerate.

Let $Q\underline{\underline{L}}^{\bullet} \mathscr{U}$ be the geometric realization of the simplicial set whose n-simplices are the nondegenerate symmetric pairings in $\varrho_n \mathscr{U}$. (We usually write these in the form (X, φ), where X is in $\varrho_n \mathscr{U}$ and φ is a nondegenerate symmetric pairing of X with itself.) Arguing exactly as in Section 4, construct a map

$$Q\underline{\underline{L}}^{\bullet} \mathscr{U} \to Q\hat{H}^{\bullet}(Z_2; \underline{\underline{K}} \mathscr{U})$$

where $\underline{\underline{K}} \mathscr{U}$ is the K-theory spectrum of $\mathscr{U}$, in the sense of Waldhausen. This will be an infinite loop map between infinite loop spaces; it could also be written as a map between spectra,

$$\underline{\underline{L}}^{\bullet}\mathscr{U} \to \hat{H}^{\bullet}(Z_2; \underline{\underline{K}}\mathscr{U}).$$

Here $\underline{\underline{L}}^{\bullet}\mathscr{U}$ is the (-1)-connected spectrum associated with the infinite loop space $Q\underline{\underline{L}}^{\bullet}\mathscr{U}$. Note in passing that the homotopy group $\pi_n(\underline{\underline{L}}^{\bullet}\mathscr{U}) \cong \pi_n(Q\underline{\underline{L}}^{\bullet}\mathscr{U})$ is the bordism group of formally n-dimensional symmetric Poincaré objects in $\mathscr{U}$. (Argue as in 4.8.) It is reasonable to call $\underline{\underline{L}}^{\bullet}\mathscr{U}$ the symmetric L-theory spectrum of $\mathscr{U}$.

Symmetric L-theory is often more useful than interesting in itself, and so we should try to get to quadratic L-theory as quickly as possible. This brings us back to norm maps, because the symmetrization map from quadratic to symmetric L-theory involves norm technology. (See 4.9.)

6.1. Notation. Let V be a bispectrum with a hemicellular filtration and with a skew-involution z_V. Let U be a CW-bispectrum with a cellular skew-involution z_U. Form a mapping spectrum

$$\underline{\underline{\mathrm{map}}}(U, V)$$

using the same ideas as in 2.2. So the ith term of $\underline{\underline{\mathrm{map}}}(U, V)$ is the geometric realization of the simplicial set whose k-simplices are hemicellular maps of the form

$$f : \Delta^k_+ \wedge U \to S^i \wedge V,$$

for $i \geq 0$. (For $i < 0$ it can be taken to be a point.) The group Z_2 acts on $\underline{\underline{\mathrm{map}}}(U, V)$ by

$$f \mapsto z_V \cdot f \cdot z_U;$$

we let

$$\underline{\underline{\mathrm{map}}}_{Z_2}(U, V) \subset \underline{\underline{\mathrm{map}}}(U, V)$$

be the subspectrum consisting of those f which are fixed under the action, and we let

$$\frac{\underline{\underline{\mathrm{map}}}(U, V)}{Z_2}$$

be the quotient. If U is merely a pointed CW-space with a cellular involution, then we write

$$\underline{\underline{\mathrm{map}}}(U, V) := \underline{\underline{\mathrm{map}}}(U \wedge \underline{S}^0 \wedge \underline{S}, V),$$

$$\underline{\underline{\mathrm{map}}}_{Z_2}(U, V) := \underline{\underline{\mathrm{map}}}_{Z_2}(U \wedge \underline{S}^0 \wedge \underline{S}^0, V),$$

$$\frac{\underline{\underline{\mathrm{map}}}(U, V)}{Z_2} := \frac{\underline{\underline{\mathrm{map}}}(U \wedge \underline{S}^0 \wedge \underline{S}^0, V)}{Z_2}.$$

(Smash the involution on U with the skew-involution on $\underline{S}^0 \wedge \underline{S}^0$ which interchanges the factors; this results in a skew-involution on $U \wedge \underline{S}^0 \wedge \underline{S}^0$.) In particular, taking $U = (EZ_2)_+$, with the obvious involution, or taking $U = S^0$, we get

$$H^{\blacktriangledown}(Z_2;V) := \underline{\underline{\mathrm{map}}}_{Z_2}((EZ_2)_+, V)$$

and

$$H_{\blacktriangledown}(Z_2;V) := \frac{\underline{\underline{\mathrm{map}}}(S^0, (EZ_2)_+ \wedge V)}{Z_2}.$$

(Smash the involution on $(EZ_2)_+$ with the skew-involution on V to obtain a skew-involution on $(EZ_2)_+ \wedge V$.) There is a norm map

$$\mathcal{N}: H_{\blacktriangledown}(Z_2;V) \to H^{\blacktriangledown}(Z_2;V),$$

natural in V; see especially the last sentence of the proof of 2.4.

It is now possible, and necessary, to give another (more general) definition of symmetric pairings and of $Q\underline{\underline{L}}^{\blacktriangledown}\mathcal{U}$. A symmetric pairing in $\varrho_n\mathcal{U}$ is a natural (cellular) map

$$\varphi: \Sigma^q(s_+) \to H^{\blacktriangledown}(Z_2; X(s) \wedge_{G,J} X(s))$$

where X is an object in $\varrho_n\mathcal{U}$ and s ranges over po$[n]$. Such a φ induces a symmetric pairing in $\varrho_n\mathcal{D}$, of hcl(X) with itself. (See the prelude to 2.11.) Call φ nondegenerate if the induced symmetric pairing in $\varrho_n\mathcal{D}$ is nondegenerate. The nondegenerate symmetric pairings in $\varrho_n\mathcal{U}$ are the n-simplices of a simplicial set whose geometric realization we call $Q\underline{\underline{L}}^{\blacktriangledown}\mathcal{U}$. Careful inspection shows that the earlier version of $Q\underline{\underline{L}}^{\blacktriangledown}\mathcal{U}$ is contained in the new version. Both versions have the same homotopy groups (argue as in 4.8). So the inclusion is a homotopy equivalence, though not a homeomorphism.

The chief merit of the small version of $Q\underline{\underline{L}}^{\blacktriangledown}\mathcal{U}$ is that it maps directly to $Q\hat{H}^{\blacktriangledown}(Z_2; \underline{\underline{K}}\mathcal{U})$, which the large version apparently does not. (See 4.3 and 4.4.) The chief merit of the large version is that it receives a map from another infinite loop space $Q\underline{\underline{L}}_{\blacktriangledown}\mathcal{U}$ whose definition is as follows. A natural map of the form

$$\psi: \Sigma^q(s_+) \to H_{\blacktriangledown}(Z_2; X(s) \wedge_{G,J} X(s)),$$

where X belongs to $\varrho_n\mathcal{U}$ and s runs through po$[n]$, will be called a quadratic pairing in $\varrho_n\mathcal{U}$. It gives rise to a symmetric pairing on composing with the norm maps

$$H_{\blacktriangledown}(Z_2; X(s) \wedge_{G,J} X(s)) \to H^{\blacktriangledown}(Z_2; X(s) \wedge_{G,J} X(s)).$$

If the induced symmetric pairing is nondegenerate, then the quadratic pairing ψ itself will be considered nondegenerate. The nondegenerate quadratic pairings in $\varrho_n\mathcal{U}$ are the n-simplices of a simplicial set whose geometric realization we call $Q\underline{\underline{L}}_{\blacktriangledown}\mathcal{U}$. By construction, $Q\underline{\underline{L}}_{\blacktriangledown}\mathcal{U}$ comes equipped with a 'symmetrization' map to $Q\underline{\underline{L}}^{\blacktriangledown}\mathcal{U}$. This is again an infinite loop map between infinite loop spaces. Replacing infinite loop maps by maps of spectra throughout, we can summarize the constructions so far in a diagram:

$$\begin{tikzcd}
 & \underline{\underline{L}}^{\blacktriangledown}\mathcal{U} \arrow[r] \arrow[d, "\simeq"'] & \hat{H}^{\blacktriangledown}(Z_2;\underline{\underline{K}}\mathcal{U}) \arrow[dd, "\text{linearization}"] \\
\underline{\underline{L}}_{\blacktriangledown}\mathcal{U} \arrow[r, "\text{symmetrization}"] \arrow[d, "\text{linearization}"] & \underline{\underline{L}}^{\blacktriangledown}\mathcal{U} \arrow[d, "\text{linearization}"] & \\
\underline{\underline{L}}_{\blacktriangledown}\mathcal{D} \arrow[r, "\text{symmetrization}"] & \underline{\underline{L}}^{\blacktriangledown}\mathcal{D} \arrow[r] & \hat{H}^{\blacktriangledown}(Z_2;\underline{\underline{K}}\mathcal{D})
\end{tikzcd}$$

The vertical arrow $\underline{\underline{L}}^{\blacktriangledown}\mathcal{U}\xrightarrow{\simeq}\underline{\underline{L}}^{\blacktriangledown}\mathcal{U}$ just indicates that we used two different definitions of $\underline{\underline{L}}^{\blacktriangledown}\mathcal{U}$. The linearization maps are obtained by sending an object X in $\varrho_n\mathcal{U}$ to $\mathrm{hcl}(X)$ in $\varrho_n\mathcal{D}$. We have written $\underline{\underline{L}}_{\blacktriangledown}\mathcal{D}$, $\underline{\underline{L}}^{\blacktriangledown}\mathcal{D}$, $\underline{\underline{K}}\mathcal{D}$ to mean $\underline{\underline{L}}p_{\blacktriangledown}(\mathbb{Z}\pi)$, $\underline{\underline{L}}p^{\blacktriangledown}(\mathbb{Z}\pi)$, and $\underline{\underline{K}}p(\mathbb{Z}\pi)$. Note that $\underline{\underline{L}}_{\blacktriangledown}\mathcal{U}$, $\underline{\underline{L}}_{\blacktriangledown}\mathcal{D}$, $\underline{\underline{L}}^{\blacktriangledown}\mathcal{D}$ and $\underline{\underline{L}}^{\blacktriangledown}\mathcal{U}$ are (-1)-connected by definition. Deleting the symmetric L-theory from the diagram gives a square

$$\begin{tikzcd}
\underline{\underline{L}}_{\blacktriangledown}\mathcal{U} \arrow[r] \arrow[d, "\text{linearization}"] & \hat{H}^{\blacktriangledown}(Z_2;\underline{\underline{K}}\mathcal{U}) \arrow[d, "\text{linearization}"] \\
\underline{\underline{L}}_{\blacktriangledown}\mathcal{D} \arrow[r] & \hat{H}^{\blacktriangledown}(Z_2;\underline{\underline{K}}\mathcal{D})
\end{tikzcd}$$

which is commutative up to a preferred homotopy. If G is the loop group of some pointed connected CW-space Y, then $\underline{\underline{K}}\mathcal{U}$ is the A-theory of Y, in symbols $\underline{\underline{K}}\mathcal{U}=\underline{\underline{A}}p(Y)$. More details on $\underline{\underline{K}}\mathcal{U}$ are given below. We now unleash a version of the π-π-theorem to complete and explain construction E.

6.2. Proposition. *The linearization map in quadratic L-theory, from $\underline{\underline{L}}_{\blacktriangledown}\mathcal{U}$ to $\underline{\underline{L}}_{\blacktriangledown}\mathcal{D}$, is a homotopy equivalence.*

It appears to be impossible to give a proof which is both clean and illuminating. We have opted for a clean proof, but below in 6.4(i) we sketch an illuminating one. The key ingredient (in the clean proof) is a lemma for which we need some terminology. An object D in $\mathcal{D}$ is called $(k-1)$-connected if $H_m(D)=0$ for $m<k$, and it is called k-dimensional if $H^m(D;R)=0$ for $m>k$. (Take $R=\mathbb{Z}\pi$.) We will write £ for the linearization functor from $\mathcal{U}$ to $\mathcal{D}$ (instead of hcl, which can be confusing).

6.3. Lemma. *Let X be an object in $\mathcal{U}$, and let*

$$g:\pounds(X)\to D$$

be a morphism in $\mathcal{D}$ such that D is $(k-1)$-connected and the mapping cone of g is $(k+1)$-dimensional, for some integer k. Then g can be lifted to $\mathcal{U}$. In other words, there exists an object D' in $\mathcal{U}$, a morphism $g':X\to D'$ in $\mathcal{U}$, and a homotopy equivalence $\upsilon:\pounds(D')\to D$ such that $g=\upsilon\cdot\pounds(g')$. (We defer the proof.)

6.4. Lemma. *Let P be a $(k-1)$-connected object in $\mathscr{U}$. Then $P\wedge_{G,J}P$ is $(2k-1+q)$-connected, and so is $H_\blacktriangledown(Z_2, P\wedge_{G,J}P)$. The homomorphism*

$$\pi_m(P) \to H_m(\pounds(P))$$

is an isomorphism for $m \le k$ and a surjection for $m = k+1$. The homomorphisms

$$\pi_{m+q}(P\wedge_{G,J}P) \to H_m(\pounds(P)^{\mathrm{t}} \otimes_R \pounds(P))$$

and

$$\pi_{m+q}(H_\blacktriangledown(Z_2, P\wedge_{G,J}P)) \to H_m(W \otimes_{Z[Z_2]} (\pounds(P)^{\mathrm{t}} \otimes_R \pounds(P)))$$

(compare 5.3, 5.4 and 2.11) are isomorphisms for $m \le 2k$ and surjections for $m = 2k+1$.

Proof. The cellular filtration of P/G pulls back to a filtration of P. Similarly, the cellular filtration of $(P/G)\wedge(P/G)$ pulls back to a filtration of $P\wedge_{G,J}P$, and the cellular filtration of $H_\blacktriangledown(Z_2, (P/G)\wedge(P/G))$ pulls back to a filtration of $H_\blacktriangledown(Z_2, P\wedge_{G,J}P)$. These filtrations give rise to three spectral sequences converging to the homotopy of the objects in question. (We used the first to prove 5.2.) Inspection of these spectral sequences establishes 6.4. $\square$

We now prove 6.2 (using 6.3). For $n \ge 0$, the homotopy groups $\pi_n(\underline{\underline{L}}_\blacktriangledown \mathscr{U})$ and $\pi_n(\underline{\underline{L}}_\blacktriangledown \mathscr{D})$ are the bordism groups of formally n-dimensional quadratic Poincaré objects in $\mathscr{U}$ and in $\mathscr{D}$, respectively. (Argue as in 4.8.) We have to prove that the linearization homomorphism

$$\pi_n(\underline{\underline{L}}_\blacktriangledown \mathscr{U}) \to \pi_n(\underline{\underline{L}}_\blacktriangledown \mathscr{D})$$

is surjective and injective, for any $n \ge 0$.

For the surjectivity part, represent an element in $\pi_n(\underline{\underline{L}}_\blacktriangledown \mathscr{D})$ by a formally n-dimensional quadratic Poincaré object (D, ψ) in $\mathscr{D}$. Because of Ranicki's algebraic surgery (especially below the middle dimension), we may assume that D is $(k-1)$-connected, where $n = 2k$ or $n = 2k+1$. By Poincaré duality, D will also be $(k+1)$-dimensional if n is odd, and k-dimensional if n is even. In any case, taking $X = \{*\}$ in 6.3 shows that D can be lifted to $\mathscr{U}$. That is, there exists D' in $\mathscr{U}$ and a homotopy equivalence $\upsilon : \pounds(D') \to D$. By 6.4, the quadratic structure ψ on D, or at least its class

$$[\psi] \in H_n(W \otimes_{Z[Z_2]} (D^{\mathrm{t}} \otimes_R D))$$

can be lifted to a class

$$[\psi'] \in \pi_{n+q}(H_\blacktriangledown(Z_2, D'\wedge_{G,J}D')).$$

Then (D', ψ') is a quadratic Poincaré object of formal dimension n in $\mathscr{U}$ whose class in $\pi_n(\underline{\underline{L}}_\blacktriangledown \mathscr{U})$ maps to the class of (D, ψ) in $\pi_n(\underline{\underline{L}}_\blacktriangledown \mathscr{D})$. This proves surjectivity.

The proof of injectivity is similar but, of course, more relative. Start with a formally n-dimensional quadratic Poincaré object (X, θ) in $\mathscr{U}$, and assume that its class in $\pi_n(\underline{\underline{L}}_\blacktriangledown \mathscr{U})$ maps to zero in $\pi_n(\underline{\underline{L}}_\blacktriangledown \mathscr{D})$. Then there exists a quadratic Poincaré pair of formal dimension $n+1$ in $\mathscr{D}$, of the form

$$(\pounds(X) \xrightarrow{g} D, (\psi, \pounds(\theta))).$$

Here ψ is an $(n+1)$-chain in

$$W \otimes_{Z[Z_2]} (D^t \otimes_R D)$$

whose boundary agrees with the image of $\pounds(\theta)$ under g. (Certain nondegeneracy properties of ψ and θ are understood.) Because of Ranicki's algebraic surgery again, we may assume that D is $(k-1)$-connected, where $n+1=2k$ or $n+1=2k+1$. By Poincaré duality, the mapping cone of g will be $(k+1)$-dimensional. By 6.3, the map g can be lifted to $\mathscr{U}$. So we can find $g': X \to D'$ and a homotopy equivalence $v: \pounds(D') \to D$ such that $v \cdot \pounds(g') = g$. As in the first half of the proof, we can now use 6.4 to produce a lift ψ' of ψ, with a suitable interpretation of the word lift. Then

$$\left(X \xrightarrow{g'} D', (\psi', \theta)\right)$$

is a quadratic Poincaré pair of formal dimension $n+1$ in $\mathscr{U}$. This shows that (X, θ) is already nullbordant in $\mathscr{U}$ and completes the proof of injectivity.

It remains to prove 6.3. We begin with the observation that it is sufficient to produce D', g' and v such that g is homotopic to $v \cdot \pounds(g')$. Namely, let $h: v \cdot \pounds(g') \simeq g$ be a chain homotopy. We may assume that g' is a cofibration. Then so is $\pounds(g')$. Cofibrations have the homotopy extension property, so that h can be extended to a homotopy from v to something else, say v_1. Replacing v by v_1, we get strict equality: $v_1 \cdot \pounds(g') = g$.

Next, we prove 6.3 under the additional assumption that X is $(k-1)$-connected, like D. We may work with the homotopy categories $h\mathscr{U}$ and $h\mathscr{D}$. These are triangulated categories (see [2, 13, 38]). Embed g in a distinguished triangle

$$E \xrightarrow{f} \pounds(X) \xrightarrow{g} D \longrightarrow \Sigma E$$

in $h\mathscr{D}$. If f can be lifted to $\mathscr{U}$, then g can be lifted to $\mathscr{U}$. So we search for a lift of f. That is, we search for $f': E' \to X$ in $\mathscr{U}$ and $z: \pounds(E') \xrightarrow{\simeq} E$ in $h\mathscr{D}$ such that $z \cdot \pounds(f') = f$ in $h\mathscr{D}$. Now E is $(k-2)$-connected and k-dimensional. Therefore it is isomorphic in $h\mathscr{D}$ to a chain complex concentrated in dimensions $k-1$ and k whose chain modules are free (perhaps not f.g.). The existence of E' and z is then obvious. Further, f' is also easy to construct since X is $(k-1)$-connected and $\pi_k(X) = H_k(\pounds(X))$.

The general case is by induction on m, where m is the least positive integer such that X is $(k-m-1)$-connected. We have treated the case $m=0$. If $m>0$, choose a G-map e from a wedge of copies of $G_+ \wedge \Sigma^{k-m} \underline{S}^0$ to X which is surjective on π_{k-m}. Let $X^{\&}$ be the mapping cone of e. The composition $g \cdot \pounds(e)$ is nullhomotopic, since its target is $(k-1)$-connected and its source is $(k-m)$-dimensional. Therefore g has an extension

$$g^{\&}: \pounds(X^{\&}) \to D.$$

(We can identify $\pounds(X^{\&})$ with the mapping cone of $\pounds(e)$.) So we may concentrate on lifting $g^{\&}$ to $\mathscr{U}$. The mapping cone of $g^{\&}$ is still $(k+1)$-dimensional, and $X^{\&}$ is $(k-m)$-connected. That is, we have achieved the reduction from m to $m-1$. The proof is complete.

6.5. Remark. Here is an explanation of 6.2. Ranicki shows in [25] that any formally n-dimensional quadratic Poincaré object in $\mathscr{D}$ is bordant to a $(k-1)$-connected one, where $n=2k$ or $n=2k+1$. Quite simply, his arguments also work in $\mathscr{U}$. (The description of Ranicki's theory of algebraic surgery given in [46, Chapter 4] is more categorical than the original, so the reader might find that easier to generalize.) Working with highly connected representatives, one can then prove 6.2 by hand. This proof is more convincing than the one we gave, but it is longer.

6.6. Remark. The linearization map in symmetric L-theory, from $\underline{\underline{L}}^{\bullet}\mathscr{U}$ to $\underline{\underline{L}}^{\bullet}\mathscr{D}$, is usually not a homotopy equivalence.

6.7. Remark. Let G' be the geometric realization of the singular simplicial set of G. Let $\mathscr{U}'$ be the category of finitely dominated G'-CW-spectra. The map $\underline{\underline{K}}\mathscr{U}' \to \underline{\underline{K}}\mathscr{U}$ induced by $G' \to G$ is a homotopy equivalence by Waldhausen's approximation theorem in [41]. So there is no great loss of generality in assuming that G has the homotopy type of a CW-space. (By the same argument, we can assume that J has the homotopy type of a CW-space.) Assuming this for the moment, we claim that it is permitted to write

$$\underline{\underline{K}}\mathscr{U} = \underline{\underline{A}}p(BG).$$

In fact, Waldhausen has a definition of the A-theory of BG which is very similar, but still differs from the above in two minor respects. Firstly, Waldhausen works with unstable objects (certain spaces with a G-action) where we use stable objects (certain spectra with a G-action). By [41, Proposition 1.6.2], the corresponding K-theories have the same homotopy type. Secondly, Waldhausen works with finite objects where we work with finitely dominated ones. This does make a difference, which is why we distinguish between $\underline{\underline{A}}p(BG)$ and $\underline{\underline{A}}h(BG) = \underline{\underline{K}}\mathscr{U}^{\mathrm{f}}$, where $\mathscr{U}^{\mathrm{f}}$ is the full subcategory of finite objects in $\mathscr{U}$. The inclusion $\underline{\underline{A}}h(BG) \to \underline{\underline{A}}p(BG)$ induces an isomorphism on all homotopy groups except on π_0. The proof is not difficult, but we defer it to Part III. Nor is it hard to see that

$$\pi_0(\underline{\underline{A}}p(BG)) \cong K_0(\mathbb{Z}\pi), \quad \text{with } \pi = \pi_0(G),$$

whereas

$$\pi_0(\underline{\underline{A}}h(BG)) \cong \mathbb{Z}.$$

6.8. Remark. Assume as before that G and J are homotopy equivalent to CW-spaces. The universal principal G-bundle on BG determines an associated J-bundle on BG because J is a G-space. Since J has the homotopy type of a sphere, this is a spherical fibration. It has a distinguished section which picks the base point of J in each fibre.

Conversely, given a spherical fibration on BG with a distinguished section, pull it back to EG. The result is a spherical fibration over EG, with a distinguished section s. Write J for the mapping cone of s. Then J has the homotopy type of a sphere, and G acts on it by an action which fixes the base point.

What is the effect of suspension? We can either suspend J, or we can suspend (fibrewise) the spherical fibration on BG associated with J. It amounts to the same, and it has no effect at all. That is, J and ΣJ give rise to two slightly different models of $\underline{A}p(BG)$, both equipped with a strict involution. But it is clear from 5.4 that one of the two models is contained in the other, and the inclusion is a homotopy equivalence (apart from being Z_2-equivariant).

The moral is that we can associate an involution on $\underline{A}p(BG)$ with any spherical fibration γ on BG, even if γ has no distinguished section. (Subject γ to fibrewise suspension, once or several times. Then there will be a distinguished section.)

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