On the Beilinson conjectures for elliptic curves with complex multiplication

Christopher Deninger and Kay Wingberg

Introduction

The aim of this paper is to give a coherent and detailed exposition of the Bloch-Beilinson results concerning the leading coefficient at zero of the L-function associated to an elliptic curve over $\mathbb{Q}$ with complex multiplication. According to the conjectures as formulated by Beilinson this value should be equal up to a rational factor to the determinant of a regulator map between certain K-groups and Deligne cohomology.

In contrast to the approach given by Bloch in [3] lect. 8,9 where he uses a definition of the regulator based upon relative cycles we start from Beilinson's definition as expounded in [2] . There he derives an expression for the regulator of an elliptic curve as a linear combination of Eisenstein-Kronecker-Lerch series. In order to complete his argument one has to refer to Bloch's computation of the L-series given in [3] lect. 9.

It seems to us however that Bloch considers only those elliptic curves with complex multiplication by $\mathcal{O}$ where the period lattice with respect to a real differential is generated as an $\mathcal{O}$-module by a real period.

Without this assumption a construction of his involving N-torsion points on the elliptic curve for a natural number N has to be generalized using ν-torsion points with $\nu \in \mathcal{O}$. We also note that the root of unity in Bloch's final result can be discarded.

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§1 A formula for the regulator of curves

An analytic space U_{an} over $\mathbb{R}$ can be given by a pair $(U_{an}(\mathbb{C}),F_\infty)$ consisting of a complex analytic space $U_{an}(\mathbb{C})$ and an antiholomorphic involution F_∞ on $U_{an}(\mathbb{C})$. A sheaf $\mathcal{F}$ on U_{an} is then a sheaf $\mathcal{F}_{\mathbb{C}}$ on $U_{an}(\mathbb{C})$ together with a morphism $\sigma\colon F_\infty^*\mathcal{F}_{\mathbb{C}} \to \mathcal{F}_{\mathbb{C}}$ such that $\sigma\circ F_\infty^*\sigma = \mathrm{id}$. In other words the pair (F_∞,σ) is an involution on $(U_{an}(\mathbb{C}),\mathcal{F}_{\mathbb{C}})$. For a $\mathbb{Q}$-sheaf one has essentially by definition

$$H^i(U_{an},\mathcal{F}) = H^i(U_{an}(\mathbb{C}),\mathcal{F}_{\mathbb{C}})^{(F_\infty,\sigma)} \quad \text{the } (F_\infty,\sigma)\text{-fixmodule}$$

of the analytic sheaf cohomology $H^i(U_{an}(\mathbb{C}),\mathcal{F}_{\mathbb{C}})$. See [6] 2.1. We will be concerned with sheaves $\mathcal{F} = A(n),\Omega^p_{U_{an}}$ given by $\mathcal{F}_{\mathbb{C}} = A(n),\Omega^p_{U_{an}(\mathbb{C})}$ together with the usual complex conjugation. Here $\mathbb{Q}\subset A\subset\mathbb{C}$ is a subgroup and $A(n) = (2\pi i)^n A$. For example an element of $H^0(U_{an},\Omega^1_{U_{an}})$ is given by a differential form ω on $U_{an}(\mathbb{C})$ such that $F_\infty^*\omega = \bar{\omega}$. We call ω a real holomorphic form.

Let X denote a smooth complete curve over $\mathbb{R}$ (i.e. a smooth, complete, geometrically irreducible, one-dimensional $\mathbb{R}$-scheme). Consider a closed nonempty subscheme $P\subset X$, $P \neq X$ and let $U = X\smallsetminus P$. Then U carries the structures of a smooth affine $\mathbb{R}$-scheme and that of an analytic space U_{an} over $\mathbb{R}$ given by $U_{an} = (U(\mathbb{C}),F_\infty)$. Here $U(\mathbb{C})$ is the manifold of complex points of U and F_∞ is induced by complex conjugation. From [6] (2.12) we know that

$$H^1_{\mathcal{D}}(U,\mathbb{R}(1)) = \{\varphi\in H^0(U_{an},\mathcal{O}_{U_{an}}/\mathbb{R}(1))\mid d\varphi\in F^1(U)\}$$

where $F^1(U) = H^0(X_{an},\Omega^1_{X_{an}}\langle P\rangle) = H^0(X(\mathbb{C}),\Omega^1_{X(\mathbb{C})}\langle P(\mathbb{C})\rangle)^{\bar{F}_\infty}$ consists of the real meromorphic forms on $X(\mathbb{C})$ which are holomorphic on $U(\mathbb{C})$ and have at most first order poles on $P(\mathbb{C})$. We need another description of this group. Clearly $\varepsilon = \mathrm{Re}\,\varphi$ is a well defined real valued F_∞-invariant C^∞-function on $U(\mathbb{C})$ (i.e. an element of $C^\infty(U_{an},\mathbb{R})$) which is integrable on $X(\mathbb{C})$ as it has only logarithmic poles. Moreover

$$2\partial\varepsilon = \partial\varphi = d\varphi \quad \text{is integrable as well and}$$

according to the generalized Cauchy integral formula ([7] 0.1) one has the equation of currents on $X(\mathbb{C})$:

$$(1.1) \qquad \frac{1}{\pi i}\bar{\partial}\partial\varepsilon = \sum_{x\in P(\mathbb{C})} \alpha_x \delta_x \quad ,$$

where δ_x is the Dirac distribution at the point x and $\alpha_x = \mathrm{Res}_x(d\varphi)$ is real and such that $\alpha_{F_\infty(x)} = \alpha_x$. We set

$$(1.1.1) \qquad \mathrm{div}\,\varepsilon = \mathrm{div}(2\partial\varepsilon) = \mathrm{div}(d\varphi) = \sum_{x\in P(\mathbb{C})} \alpha_x \cdot x \ .$$

Clearly $\mathrm{div}\,\varepsilon \in \mathbb{R}[P]^{0}$=(group of divisors of degree 0 on P)$\otimes\mathbb{R}$.

On the other hand a function ε on U_{an} with the above properties gives rise to an element φ in $H^1_{\mathcal{D}}(U,\mathbb{R}(1))$ such that $\varepsilon = \mathrm{Re}\,\varphi$ and hence we have [6] 2.17 ii

$$(1.2) \quad H^1_{\mathcal{D}}(U,\mathbb{R}(1)) = \{\varepsilon\in C^\infty(U_{an},\mathbb{R}) \mid \varepsilon\in L^1(U(\mathbb{C})), \frac{1}{\pi i}\bar{\partial}\partial\varepsilon = \sum_{x\in P(\mathbb{C})} \alpha_x\delta_x$$

$$\text{and } \alpha = \sum_{x\in P(\mathbb{C})} \alpha_x x \in \mathbb{R}[P]^{0}\} \ .$$

In the following we will use this description of Deligne cohomology.

The divisor mapping induces an exact (Gysin-) sequence

$$(1.3) \qquad 0 \to \mathbb{R} \to H^1_{\mathcal{D}}(U,\mathbb{R}(1)) \xrightarrow{\mathrm{div}} \mathbb{R}[P]^{0} \to 0 \ .$$

Exactness in the middle follows from Weyl's lemma: any distribution solution to the homogenous Poisson equation is a harmonic function on $X(\mathbb{C})$ hence a constant.

As F^2 of H^1 of curves is zero we have by [2](1.6)or[6]2.16.

$$H^2_{\mathcal{D}}(U,\mathbb{R}(2)) = H^1(U_{an},\mathbb{R}(1)) \subset H^1(U_{an},\mathbb{C}) \ .$$

Describing the latter group by closed $\bar{F}_\infty$-invariant C^∞ 1-forms on $U(\mathbb{C})$ modulo exact forms the cup product

$$\Lambda^2 H^1_{\mathcal{D}}(U,\mathbb{R}(1)) \overset{\cup}{\to} H^2_{\mathcal{D}}(U,\mathbb{R}(2))$$

is given by the formula $\varepsilon \cup \varepsilon' = 2(\varepsilon\pi_1(\partial\varepsilon') - \varepsilon'\pi_1(\partial\varepsilon))$. Here $\pi_1 : \mathbb{C} \to \mathbb{R}(1)$ is the canonical projection (see [2] (1.2.5) and [6] 3.12).

Let $F^1(U(\mathbb{C})) = H^0(X(\mathbb{C}),\Omega^1_{X(\mathbb{C})}\langle P(\mathbb{C})\rangle)$ be the F^1-term of the Hodge filtration on $H^1(U(\mathbb{C}),\mathbb{C})$. We have [5]
$H^1(X(\mathbb{C}),\mathbb{C}) = (F^1(U(\mathbb{C})) \cap H^1(X(\mathbb{C}),\mathbb{C})) \oplus (\overline{F^1(U(\mathbb{C}))} \cap H^1(X(\mathbb{C}),\mathbb{C}))$
and since $Gr^W_2 H^1(U(\mathbb{C}),\mathbb{C})$ is purely of type (1,1)
$H^1(U(\mathbb{C}),\mathbb{C}) = F^1(U(\mathbb{C})) + H^1(X(\mathbb{C}),\mathbb{C}) = \overline{F^1(U(\mathbb{C}))} + H^1(X(\mathbb{C}),\mathbb{C})$.
It follows that the canonical inclusions induce an isomorphism

(1.4) $\quad H^1(X(\mathbb{C}),\mathbb{C}) \oplus (F^1(U(\mathbb{C})) \cap \overline{F^1(U(\mathbb{C}))}) \overset{\sim}{\to} H^1(U(\mathbb{C}),\mathbb{C})$.

Taking the fixed modules under $\bar{F}_\infty$ we obtain the decomposition

(1.5) $\quad H^1(U_{an},\mathbb{C}) = H^1(X_{an},\mathbb{C}) \oplus (F^1(U) \cap \overline{F^1(U)})$

which finally gives

(1.6) $\quad H^1(U_{an},\mathbb{R}(1)) = H^1(X_{an},\mathbb{R}(1)) \oplus (H^1(U_{an},\mathbb{R}(1)) \cap F^1(U))$.

Let $pr_{\mathcal{D}}$ be the projection onto $H^1(X_{an},\mathbb{R}(1))$ associated with the decomposition (1.6) . Putting everything together we find a map $[\,,\,]_{\mathcal{D}}$ making the following diagram with exact columnes commute

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
\mathbb{R}\otimes H^1_{\mathcal{D}}(U,\mathbb{R}(1)) \oplus H^1_{\mathcal{D}}(U,\mathbb{R}(1))\otimes\mathbb{R} & \xrightarrow{\cup} & H^1(U_{an},\mathbb{R}(1))\cap F^1(U) \\
\downarrow & & \downarrow \\
\Lambda^2 H^1_{\mathcal{D}}(U,\mathbb{R}(1)) & \xrightarrow{\cup} & H^2_{\mathcal{D}}(U,\mathbb{R}(2)) = H^1(U_{an},\mathbb{R}(1)) \\
\downarrow \Lambda^2\mathrm{div} & & \downarrow pr_{\mathcal{D}} \\
\Lambda^2\mathbb{R}[P]^0 & \xrightarrow{[\,,\,]_{\mathcal{D}}} & H^2_{\mathcal{D}}(X,\mathbb{R}(2)) = H^1(X_{an},\mathbb{R}(1)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array} \qquad (1.7)$$

.

Using as an intermediate the Deligne cohomology of non-complete curves and its cup product structure we have thus associated to every pair of cycles on P an element in $H^2_{\mathcal{D}}(X,\mathbb{R}(2))$. An analogous construction for the absolute cohomology will be described later.

In order to identify the element $pr_{\mathcal{D}}(\varepsilon \cup \varepsilon')$ in $H^2_{\mathcal{D}}(X,\mathbb{R}(2))$ we introduce the pairing

(1.8) $$<\xi,\eta> = \frac{1}{2\pi i} \int_{X(\mathbb{C})} \xi \wedge \eta$$

defined for C^∞ 1-forms ξ,η on $X(\mathbb{C})$. Representing cohomology classes by closed forms we get an isomorphism

(1.9) $$H^1(X_{an},\mathbb{R}(1)) \xrightarrow{\sim} \mathrm{Hom}(F^1(X),\mathbb{R}) ,$$

where $F^1(X) = H^0(X(\mathbb{C}),\Omega^1)^{\bar{F}_\infty}$. A simple calculation based on Stokes's formula gives the following result ([2] (4.2)).

<u>(1.10) Lemma</u> Let $\alpha,\beta \in \mathbb{R}[P]^0$ and choose $\varepsilon_\alpha,\varepsilon_\beta \in H^1_{\mathcal{D}}(U,\mathbb{R}(1))$ such that $\mathrm{div}\ \varepsilon_\alpha = \alpha$, $\mathrm{div}\ \varepsilon_\beta = \beta$. Then for any real holomorphic form $\omega \in F^1(X)$ we have

$$\frac{1}{2} <\omega,[\alpha,\beta]_{\mathcal{D}}> = \frac{1}{2\pi i} \int_{X(\mathbb{C})} \varepsilon_\alpha(\bar{\partial}\varepsilon_\beta) \wedge \omega .$$

Observe that the integral exists since $\frac{1}{|z|} \log |z|$ is integrable.

§2 A weakened version of the Beilinson conjecture for elliptic curves

For an elliptic curve $X_{/\mathbb{Q}}$ defined over $\mathbb{Q}$ let $L(X,s) = L(H^1(X),s)$ be its L-series converging for $\operatorname{Re} s > \frac{3}{2}$. It is conjectured that $L(X,s)$ has an analytic continuation to the whole s-plane and that it satisfies a functional equation of the form

(2.1) $$\Lambda(X,s) = w\Lambda(X,2-s)$$

where $\Lambda(X,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(X,s)$ with $w = \pm 1$ and $N \in \mathbb{N}$ denoting the conductor of $X_{/\mathbb{Q}}$. This is proved if X is a modular curve and (in particular) if it has complex multiplication. Assuming (2.1) the L-function has a first order zero at $s=0$ and its leading coefficient is given by

$$L'(X,0) = \frac{wN}{(2\pi)^2} L(X,2) \ .$$

On the other hand $H^2_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(2)) = H^1(X_{an},\mathbb{R}(1))$ is one-dimensional and we have a regulator map

$$r_{\mathcal{D}}: H^2_{\mathcal{A}}(X_{\mathbb{Z}},\mathbb{Q}(2)) \to H^2_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(2)) \ .$$

Here $H^2_{\mathcal{A}}(X_{\mathbb{Z}},\mathbb{Q}(2))$ is the image of $K_2(\mathcal{X}) \otimes \mathbb{Q}$ in $H^2_{\mathcal{A}}(X,\mathbb{Q}(2))$ where $\mathcal{X}$ denotes a proper flat regular model of X over $\mathbb{Z}$. According to [2] 2.4.2 the group $H^2_{\mathcal{A}}(X_{\mathbb{Z}},\mathbb{Q}(2))$ is well defined. Bloch and Beilinson conjecture that $r_{\mathcal{D}} \otimes \mathbb{R}$ is an isomorphism and that

$$\operatorname{Im} r_{\mathcal{D}} = L'(X,0)\cdot H^1(X_{an},\mathbb{Q}(1)) \quad \text{in} \quad H^2_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(2)) \ .$$

Clearly this determines $L'(X,0)$ up to a rational multiple. As it is not even known if $K_2(X)$ is finitely generated we consider the following weaker version of the conjecture.

<u>(2.2) Conjecture</u> Let $X_{/\mathbb{Q}}$ be an elliptic curve. Then there exist $\Psi \in H^2_{\mathcal{A}}(X_{\mathbb{Z}},\mathbb{Q}(2))$ and $0 \neq \Phi \in H^1(X_{an},\mathbb{Q}(1))$ such that

$$r_{\mathcal{D}}(\Psi) = \frac{N}{4\pi^2} L(X,2)\Phi \quad \text{in} \quad H^2_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(2)) \ .$$

Observe that for modular elliptic curves this is equivalent to

$$r_{\mathcal{D}}(\Psi) = L'(X,0)\Phi \ .$$

In the rest of this paper we give the proof of the following result due to Bloch and Beilinson:

(2.3) Theorem Let $X_{/\mathbb{Q}}$ be an elliptic curve with complex multiplication by the ring of integers $\mathcal{O}$ in an imaginary quadratic extension $K/\mathbb{Q}$. Then conjecture (2.2) holds true.

Using the pairing (1.8) we obtain a commutative diagram

$$\begin{array}{ccccc} H^1(X_{an},\mathbb{Q}(1)) & \xrightarrow{<,>} & \mathrm{Hom}(H^1(X_{an},\mathbb{Q}),\mathbb{Q}) & \cong & H_1(X_{an},\mathbb{Q}) \\ \downarrow & & & & \downarrow \text{Integration} \\ H^1(X_{an},\mathbb{R}(1)) & & \xrightarrow{<,>} & & \mathrm{Hom}(F^1(X_{\mathbb{R}}),\mathbb{R}) \end{array} .$$

As $H_1(X_{an},\mathbb{Q}) = \langle X(\mathbb{R})^0 \rangle$ we find that (2.3) is equivalent to

(2.4) There exist $\psi \in H^2_{\mathcal{A}}(X_{\mathbb{Z}},\mathbb{Q}(2))$ and $\omega_{\mathbb{Q}} \in F^1(X_{\mathbb{R}})$ such that $\int_{X(\mathbb{R})^0} \omega \in \mathbb{Q}^\times$ and $<\omega_{\mathbb{Q}}, r_{\mathcal{D}}(\psi)> \equiv L'(X,0) \bmod \mathbb{Q}^\times$.

Later the element $r_{\mathcal{D}}(\psi)$ will have the form $r_{\mathcal{D}}(\psi) = [\alpha,\beta]_{\mathcal{D}}$ for cycles $\alpha,\beta \in \mathbb{R}[P]^0$ and hence we have to calculate the integral in (1.10) for elliptic curves. This is dealt with in the next section.

§3 Calculating $\langle\omega,[\alpha,\beta]_{\mathcal{D}}\rangle$ for elliptic curves over $\mathbb{R}$

In this section X denotes an elliptic curve over $\mathbb{R}$. Let $P \subset X$ be a finite closed subscheme of X and set $U = X \setminus P$. We choose a real holomorphic differential $\omega \in F^1(X)$ such that

(3.1) $$\frac{i}{2\pi} \int_{X(\mathbb{C})} \omega \wedge \bar{\omega} = 1 \;.$$

Its period lattice $\Gamma \subset \mathbb{C}$ is invariant under complex conjugation $\bar{\Gamma} = \Gamma$. The analytic isomorphism

$$J: X(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/\Gamma$$
$$x \mapsto \int_0^x \omega \bmod \Gamma$$

is $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant. It identifies $(X(\mathbb{C}),\omega,F_\infty)$ with $(\mathbb{C}/\Gamma, dz, -)$ where $-$ denotes complex conjugation. In particular $\pi = \frac{i}{2}\int_{\mathbb{C}/\Gamma} dz\wedge d\bar{z} = \mathrm{Vol}(\mathbb{C}/\Gamma) = \mathrm{Vol}(\Gamma)$.

The pairing $(,):\mathbb{C}/\Gamma \otimes \Gamma \to U(1) \subset \mathbb{C}^*$

$$(z,\gamma) = \exp(z\bar{\gamma} - \gamma\bar{z})$$

identifies $\mathbb{C}/\Gamma$ and Γ as Pontrjagin duals of each other.

For $\alpha \in \mathbb{R}[P]^0$ we construct $\varepsilon_\alpha \in H^1_{\mathcal{D}}(U,\mathbb{R}(1))$ with $\operatorname{div} \varepsilon_\alpha = \alpha$ as follows (compare (1.2)):

Define $f_\alpha : \Gamma \to \mathbb{C}$ by setting

$$f_\alpha(\gamma) = \frac{-1}{2|\gamma|^2} \sum_{x\in P(\mathbb{C})} \alpha_x (x,\gamma) \quad \text{for } \gamma \neq 0 \;,\; f_\alpha(0) = 0$$

and let $\hat{f}_\alpha$ be its Fourier transform in the sense of distributions.

Consider the series

$$\varepsilon_\alpha(z) = \sum_{\gamma\in\Gamma} f_\alpha(\gamma)\overline{(z,\gamma)} = -\frac{1}{2} \sum_{\substack{\gamma\in\Gamma \\ x\in P(\mathbb{C})}}{}' \frac{\alpha_x (x-z,\gamma)}{|\gamma|^2} \quad \text{for } z\in U(\mathbb{C})$$

which is given a sense by Eisenstein or Kronecker summation (c f. [14]). The dash indicates ommission of $\gamma = 0$ in the sum. The function $\varepsilon_\alpha(z)$ has logarithmic singularities as z approaches the point $x \in P(\mathbb{C})$ and hence it is integrable on $X(\mathbb{C})$. It can be shown that as distributions $\varepsilon_\alpha = \hat{f}_\alpha$. Differentiating in the sense of currents we get

$$\frac{1}{\pi i}\bar{\partial}\partial\varepsilon_\alpha = -\frac{1}{2\pi i}\sum_{\substack{\gamma\in\Gamma \\ x\in P(\mathbb{C})}}{}' \alpha_x(x-z,\gamma)\,dz\wedge d\bar{z}$$

$$= \sum_{x\in P(\mathbb{C})} \alpha_x \sum_{\gamma\in\Gamma}(z-x,\gamma)\,d\mu$$

where $d\mu = \frac{i}{2\pi}\,dz\wedge d\bar{z}$ is the normalized Haar measure on $\mathbb{C}/\Gamma$. Here we have changed γ into $-\gamma$ and used that $\alpha \in \mathbb{R}[P]^{o}$. It is well known that

$$\sum_{\gamma\in\Gamma}(z-x,\gamma)\,d\mu = \delta_x$$

and hence $\frac{1}{\pi i}\bar{\partial}\partial\varepsilon_\alpha = \sum_{x\in P(\mathbb{C})} \alpha_x\delta_x$. By Weyl's lemma ε_α is harmonic on $U(\mathbb{C})$ and in particular $\varepsilon_\alpha \in C^\infty(U_{an},\mathbb{R})$. Hence $\varepsilon_\alpha \in H^1_{\mathcal{D}}(U,\mathbb{R}(1))$ and $\operatorname{div}\varepsilon_\alpha = \alpha$.

We can now use lemma (1.10) to calculate $\langle\omega,[\alpha,\beta]_{\mathcal{D}}\rangle$

$$\frac{1}{2}\langle\omega,[\alpha,\beta]_{\mathcal{D}}\rangle = \frac{1}{2\pi i}\int_{X(\mathbb{C})}(\varepsilon_\alpha\bar{\partial}\varepsilon_\beta)\wedge\omega$$

$$= \int_{\mathbb{C}/\Gamma}\left(\varepsilon_\alpha\frac{\partial\varepsilon_\beta}{\partial\bar{z}}\right)d\mu$$

$$= \widehat{\varepsilon_\alpha\cdot\frac{\partial\varepsilon_\beta}{\partial\bar{z}}}(0)$$

$$= \hat{\varepsilon}_\alpha * \widehat{\frac{\partial\varepsilon_\beta}{\partial\bar{z}}}(0)$$

where $\hat{\varepsilon}$ is the Fouriertransform of ε, i.e. $\hat{\varepsilon}(\gamma)$ is the γ'th Fouriercoefficient of ε, and where $*$ denotes convolution on Γ. Using Fourier-inversion in the sense of distributions we find that

$$\hat{\varepsilon}_\alpha(\gamma) = \hat{\hat{f}}_\alpha(\gamma) = f_\alpha(-\gamma) \quad\text{and}$$

$$\widehat{\frac{\partial\varepsilon_\beta}{\partial z}}(\gamma) = \widehat{\widehat{\gamma f}}_\beta = -\gamma f_\beta(-\gamma) \quad\text{. Thus}$$

$$\frac{1}{2}\langle\omega,[\alpha,\beta]_{\mathcal{D}}\rangle = \sum_{\gamma\in\Gamma} f_\alpha(-\gamma)\gamma f_\beta(\gamma)$$

$$= -\sum_{\gamma\in\Gamma} f_\alpha(\bar{\gamma})\bar{\gamma} f_\beta(-\bar{\gamma}).$$

For a completely rigorous argument see [17].

In conclusion we obtain

<u>(3.2) Lemma</u> Let X be an elliptic curve over $\mathbb{R}$ and let $\omega \in F^1(X)$ be normalized by (3.1). For $\alpha,\beta \in \mathbb{R}[P]^{\circ}$ we have

$$\langle\omega,[\alpha,\beta]_{\mathcal{D}}\rangle = -\frac{1}{2} \sum_{\substack{\gamma\in\Gamma \\ x,y\in P(\mathbb{C})}}' \frac{\bar{\gamma}}{|\gamma|^4} \cdot \alpha_x \beta_y (y-x,\gamma) .$$

In addition we observe the following consequence of (3.2):

<u>(3.3) Remark</u> Assume that $P(\mathbb{C})$ is a subgroup of $X(\mathbb{C})$ and consider the cycle $\alpha = -(|P(\mathbb{C})|-1)\cdot 0 + \sum_{\substack{x\in P(\mathbb{C}) \\ x\neq 0}} x \in \mathbb{Z}[P]^{\circ}$

Then for any $\beta \in \mathbb{R}[P]^{\circ}$ we have the formula

$$\langle\omega,[\alpha,\beta]_{\mathcal{D}}\rangle = \frac{|P(\mathbb{C})|}{2} \sum_{\substack{\gamma\in\Gamma \\ y\in P(\mathbb{C})}}' \beta_y \frac{\bar{\gamma}}{|\gamma|^4} (y,\gamma) .$$

<u>Proof:</u> We have

$$- \sum_{x,y\in P(\mathbb{C})} \alpha_x \beta_y (y-x,\gamma) = - \sum_{x,z} \alpha_x \beta_{x+z} (z,\gamma)$$

$$= \sum_z \Big(- \sum_x \alpha_x \beta_{x+z}\Big)(z,\gamma)$$

and

$$- \sum_x \alpha_x \beta_{x+z} = -\alpha_0 \beta_z - \sum_{x\neq 0} \alpha_x \beta_{x+z}$$

$$= (|P(\mathbb{C})|-1)\beta_z - \sum_{x\neq 0} \beta_{x+z}$$

$$= |P(\mathbb{C})|\beta_z - \sum_x \beta_{x+z}$$

$$= |P(\mathbb{C})|\beta_z$$

since β has degree zero.

In the next section we will approach the conjectured equality in (2.4) from the point of view of L-series.

§4 Relations between the L-function of an elliptic curve over $\mathbb{Q}$ with complex multiplication and Eisenstein-Kronecker-Lerch series

In this section we consider an elliptic curve X defined over $\mathbb{Q}$ with complex multiplication by the ring of integers $\mathcal{O}$ in an imaginary quadratic field K. Observe that K has class number one since the values of the j-invariant of its ideal classes are in the field of definition of X i.e. in $\mathbb{Q}$. We consider K as a subfield of $\mathbb{C}$ such that the Hecke character ψ of $X_K = X \times_{\mathbb{Q}} K$ has the form

$$(4.1) \qquad \psi((\alpha)) = \chi(\alpha)\bar{\alpha}$$

on ideals (α) prime to the conductor (f) of ψ; here χ factors

$$\chi: (\mathcal{O}/(f))^* \to \mu_K \subset \mathbb{C}^* .$$

Then (f) is the conductor of χ as well and as usual we set

$$\chi(\alpha) = 0 \quad \text{for} \quad \alpha \in \mathcal{O} \quad \text{with} \quad (\alpha,(f)) \neq 1 .$$

Since X is defined over $\mathbb{Q}$ we have $\bar{\psi}((\alpha)) = \psi((\bar{\alpha}))$ for all $\alpha \in \mathcal{O}$ [8] Th. 10.1.3 and hence $(\bar{f}) = (f)$ and $\bar{\chi}(\alpha) = \chi(\bar{\alpha})$. In the following we choose a fixed generator $f \in \mathcal{O}$ of the conductor (f). Clearly $\bar{f} = \varepsilon f$ for some $\varepsilon \in \mu_K$.

Let $\theta: \mathcal{O} \xrightarrow{\sim} \mathrm{End}(X(\mathbb{C})) \cong \mathrm{End}(X_K)$ be a normalized isomorphism i.e. $\theta(\alpha)^*\eta = \alpha\eta$ for all $\alpha \in \mathcal{O}$ and $\eta \in H^0(X(\mathbb{C}),\Omega^1)$. We choose $\omega \in F^1(X_{\mathbb{R}})$ as in (3.1) and let $\Gamma \subset \mathbb{C}$ denote its period lattice. Via θ the analytic isomorphism $J: X(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/\Gamma$ considered earlier is $\mathcal{O}$-invariant thus we may further identify $X(\mathbb{C})$ with $\mathbb{C}/\Gamma$.

As K has class number one there is an $\Omega \in \mathbb{C}^*$ unique up to an element of μ_K such that as subsets of $\mathbb{C}$

$$\Gamma = \Omega\mathcal{O} .$$

If $\Omega_{\mathbb{R}} = \int_{X(\mathbb{R})^0} \omega$ denotes the real period of ω there is a number $h \in \mathcal{O}$ such that $\Omega_{\mathbb{R}} = \Omega h$. As $\bar{\Gamma} = \Gamma$ we have $(\bar{h}) = (h)$ for the $\mathcal{O}$-ideal (h), and setting

(4.2) $\nu = f\bar{f}h$ we have $(\bar{\nu}) = (\nu)$ as well.

By descent theory there is a closed subscheme P of $X_{/\mathbb{Q}}$ such that $P \times_{\mathbb{Q}} K = {}_{\nu}(X_K)$ the group scheme of ν-torsion points on X_K. Concerning h we refer to a conjecture of Gross [9]§5.

According to Deuring we have

$$(4.3) \qquad L(X,s) = L(\psi,s) = \sum_{(\mathfrak{a},f)=1} \frac{\psi(\mathfrak{a})}{N\mathfrak{a}^s}$$

$$= \frac{1}{|\mu_K|} \sum_{\substack{\alpha\in\mathcal{O} \\ (\alpha,f)=1}} \frac{\chi(\alpha)\bar{\alpha}}{|\alpha|^{2s}}$$

In order to get rid of the restriction $(\alpha,f)=1$ we introduce Gauss-sums. Define a perfect pairing

$$(4.4) \qquad <,> : \mathcal{O}/\nu \times \mathcal{O}/\nu \to U(1) \subset \mathbb{C}^*$$

by setting $<a,b> = (\frac{\Omega}{\bar{\nu}}a, \Omega b)$ where $(,): \mathbb{C}/\Gamma\otimes\Gamma \to U(1)$ is the duality pairing introduced in §3. Observe that $\frac{\Omega}{\bar{\nu}}a \in {}_{\nu}X(\mathbb{C})$ and $\Omega b \in \Gamma/\nu\Gamma$ because of (4.2).

Explicitly $<a,b> = \exp(|\Omega|^2(\frac{a\bar{b}}{\bar{\nu}} - \frac{\bar{a}b}{\nu}))$ and thus

$$<a,bc> = <a\bar{c},b> \quad \text{for} \quad a,b,c \in \mathcal{O}/\nu .$$

On $\mathcal{O}/f$ we get a perfect pairing

$$<,>_f : \mathcal{O}/f \times \mathcal{O}/f \to U(1)$$

by defining $<a,b>_f = <a,bg> = (\frac{\Omega}{f}a, \Omega b)$ where we have set $g = \bar{f}h$ such that $\nu = fg$.

The Gauss sum for χ is then given by

$$G(\chi,x) = \sum_{y\in\mathcal{O}/f} \chi(y)<y,x>_f \quad \text{for} \quad x \in \mathcal{O}/f .$$

It has the following properties:

<u>(4.5) Lemma</u>

i) $|G(\chi,1)| = |f|$

ii) $\chi(\alpha) = \frac{G(\chi,\alpha)}{G(\chi,1)}$ for <u>all</u> $\alpha \in \mathcal{O}$.

iii) $\overline{G(\chi,1)} = \bar{\varepsilon}\, G(\chi,1)$ where $\varepsilon \in \mu_K$ was defined by $\bar{f} = \varepsilon f$.

iv) For $g = \bar{f}h$, $\nu = fg$ we have

$$\sum_{z\in \mathcal{O}/\nu} \chi(z)\langle z,x\rangle = \begin{cases} 0 & \text{if } x \not\equiv 0 \bmod g \\ g\bar{g}\, G(\chi,\frac{x}{g}) & \text{if } x \equiv 0 \bmod g \end{cases}$$

Proof: For the standard properties i) and ii) we refer to [11] 22, §1.

iii) $$\overline{G(\chi,1)} = \sum_{y\in \mathcal{O}/f} \overline{\chi(y)}\ \overline{(\frac{\Omega}{\bar{f}}\, y,\Omega)}$$

$$= \sum_{y\in \mathcal{O}/f} \chi(\bar{y})\ (\frac{\Omega}{\bar{f}}\, \varepsilon\bar{y},\Omega)$$

$$= \sum_{y\in \mathcal{O}/f} \chi(\bar{y})\langle \varepsilon\bar{y},1\rangle_f$$

$$= \chi(\varepsilon)^{-1}\, G(\chi,1) = \bar{\varepsilon}\, G(\chi,1)$$

iv) Observe that

$$\sum_{y'\in \mathcal{O}/g} \langle y',fx\rangle = \begin{cases} 0 & \text{if } fx \not\equiv 0 \bmod \nu \\ g\bar{g} & \text{if } fx \equiv 0 \bmod \nu\ . \end{cases}$$

Hence decomposing $z \in \mathcal{O}/\nu$ as $z = y + \bar{f}y'$ with $y \in \mathcal{O}/f$ and $y' \in \mathcal{O}/g$ we get

$$\sum_{z\in \mathcal{O}/\nu} \chi(z)\langle z,x\rangle = \sum_{y\in \mathcal{O}/f} \chi(y)\langle y,x\rangle \sum_{y'\in \mathcal{O}/g} \langle y',fx\rangle$$

$$= g\bar{g} \sum_{y\in \mathcal{O}/f} \chi(y)\langle y,x\rangle \begin{cases} 0 \text{ if } x \not\equiv 0 \bmod g \\ 1 \text{ if } x \equiv 0 \bmod g \end{cases}$$

We can now return to the L-series and to (4.3).

$$L(X,s) \overset{ii)}{=} \frac{1}{|\mu_K|G(\chi,1)} \sum_{\alpha\in \mathcal{O}}{}' G(\chi,\alpha)\ \frac{\bar{\alpha}}{|\alpha|^{2s}}$$

$$\overset{iv)}{=} \frac{1}{|\mu_K|G(\chi,1)g\bar{g}} \sum_{\alpha\in \mathcal{O}}{}' \sum_{z\in \mathcal{O}/\nu} \chi(z)\langle z,g\alpha\rangle\ \frac{\bar{\alpha}}{|\alpha|^{2s}}$$

$$= \frac{|g|^{2s-2}}{|\mu_K|G(\chi,1)\bar{g}} \sum_{\alpha\in \mathcal{O}}{}' \sum_{z\in \mathcal{O}/\nu} \chi(z)\langle z,g\alpha\rangle\ \frac{\overline{g\alpha}}{|g\alpha|^{2s}}$$

$$\overset{iv)}{=} \frac{|g|^{2s-2}}{|\mu_K|G(\chi,1)\bar{g}} \sum_{\alpha\in \mathcal{O}}{}' \sum_{z\in \mathcal{O}/\nu} \chi(z)\langle z,\alpha\rangle\ \frac{\bar{\alpha}}{|\alpha|^{2s}}$$

According to (4.5) i), iii) the number $G(\chi,1)/\bar{f}$ is real of absolute value equal to one, hence

$$L(X,s) = \pm \frac{|g|^{2s-2}}{|\mu_K|\bar{f}\bar{g}} \frac{|\Omega|^{2s}}{\bar{\Omega}} \sum_{\gamma\in\Gamma}{}' \sum_{z\in\mathcal{O}/\nu} \chi(z)\langle z,\tfrac{\gamma}{\Omega}\rangle \frac{\bar{\gamma}}{|\gamma|^{2s}}$$

$$(4.6) \qquad = \pm \frac{|g|^{2s-2}}{|\mu_K|\,|f|^2} \frac{|\Omega|^{2s}}{\Omega_{\mathbb{R}}} \sum_{\gamma\in\Gamma}{}' \sum_{x\in P(\mathbb{C})} \chi(x)(x,\gamma) \frac{\bar{\gamma}}{|\gamma|^{2s}}$$

where we have used that $\bar{f}\bar{g}\bar{\Omega} = \bar{f}f\bar{h}\bar{\Omega} = \bar{f}f\Omega_{\mathbb{R}}$ and where for $x \in P(\mathbb{C}) = {}_{\nu}X(\mathbb{C})$ we have set

(4.7) $\chi(x) = \chi(\frac{\bar{\nu}}{\bar{\Omega}}x)$. Observe that since $\frac{\bar{\nu}}{\bar{\Omega}} = \frac{|f|^2|h|^2}{\Omega_{\mathbb{R}}}$ is real we have $\bar{\chi}(x) = \chi(\bar{x})$ for all $x \in P(\mathbb{C})$.

From $\Gamma = \Omega\mathcal{O}$ it follows that $|\Omega|^2\sqrt{|d_K|} = 2\,\mathrm{Vol}(\Gamma) = 2\pi$. Specializing to $s = 2$ in (4.6) and using that by the functional equation

$$L'(X,0) = \pm \frac{N}{4\pi^2} L(X,2)$$

where N is the conductor of $X_{/\mathbb{Q}}$ we get

$$L'(X,0) = \pm \frac{|\nu|^2}{|\mu_K|\,|f|^2\Omega_{\mathbb{R}}} \sum_{\gamma\in\Gamma}{}' \sum_{x\in P(\mathbb{C})} \chi(x)(x,\gamma) \frac{\bar{\gamma}}{|\gamma|^4} .$$

Here we have also taken into account that $N=|d_K|f\bar{f}$ (c f. [8] 10.3.2). Since $|P(\mathbb{C})| = |\nu|^2$ and since μ_K operates on Γ we get

$$(4.8) \quad L'(X,0) = \pm \frac{1}{|f|^2\Omega_{\mathbb{R}}} \frac{|P(\mathbb{C})|}{|\mu_K|} \sum_{\gamma\in\Gamma}{}' \sum_{x\in P(\mathbb{C})} (\bar{\chi}(x)x,\gamma) \frac{\bar{\gamma}}{|\gamma|^4} .$$

This is already rather close to the formula in (3.3). Observe that the theory developed in §3 applies to $X_{\mathbb{R}}, P_{\mathbb{R}}$.

Let $\alpha \in \mathbb{Z}[P_{/\mathbb{Q}}]^0$ be the cycle defined by

$$\alpha = -(|P(\mathbb{C})|-1)\cdot 0 + \sum_{\substack{x\in P(\mathbb{C})\\ x\neq 0}} x$$

and consider the cycle

$$\beta = \sum_{x\in P(\mathbb{C})/\mu_K} \beta(\bar{\chi}(x)x) \in \mathbb{Z}[P(\mathbb{C})]^0$$

where for $z \in P(\mathbb{C})$ we have set

$$\beta(z) = -0 + z \in \mathbb{Z}[P(\mathbb{C})]^0 .$$

Since $\beta^{F_\infty} = \sum_{x \in P(\mathbb{C})/\mu_K} (F_\infty(\bar{\chi}(x)x))$,

$$= \sum_{x \in P(\mathbb{C})/\mu_K} \beta(\chi(x)\bar{x})$$

$$= \sum_{x \in P(\mathbb{C})/\mu_K} \beta(\chi(\bar{x})x) = \beta \quad (\text{use } \chi(\bar{x}) = \bar{\chi}(x) \text{ by } (4.7))$$

we have that $\beta \in \mathbb{Z}[P_{\mathbb{R}}]^0$ and thus (3.3) applies

$$(4.9) \quad \langle \omega, [\alpha,\beta]_{\mathcal{D}} \rangle = \frac{|P(\mathbb{C})|}{8|\mu_K|} \sum_{\gamma \in \Gamma}' \sum_{x \in P(\mathbb{C})} (\bar{\chi}(x)x,\gamma) \frac{\bar{\gamma}}{|\gamma|^4} .$$

Here we have used that $\sum_{\gamma \in \Gamma}' \frac{\bar{\gamma}}{|\gamma|^4} = 0$ since $-\Gamma = \Gamma$.
A comparison of (4.8) and (4.9) shows that

$$\langle \omega, [\alpha,\beta]_{\mathcal{D}} \rangle = \pm \frac{1}{2} |f|^2 \Omega_{\mathbb{R}} L'(X,0)$$

$$= \pm \frac{N}{|d_K| 2} \Omega_{\mathbb{R}} L'(X,0)$$

since $N = |d_K| f\bar{f}$. In conclusion:

<u>(4.10) Theorem</u> Let X be an elliptic curve over $\mathbb{Q}$ with complex multiplication by the ring of integers $\mathcal{O}$ in an imaginary quadratic field K with discriminant d_K . Determine $\nu \in \mathcal{O}$ as in (4.2) and let P be the $\mathbb{Q}$-subscheme of X such that $P \times_{\mathbb{Q}} K = {}_{\nu}X_K$. Consider the following cycles on X with support in P .

$$\alpha = -(|P(\mathbb{C})|-1)\cdot 0 + \sum_{\substack{x \in P(\mathbb{C}) \\ x \neq 0}} x \in \mathbb{Z}[P_{/\mathbb{Q}}]^0$$

$$\beta = \sum_{x \in P(\mathbb{C})/\mu_K} \beta(\bar{\chi}(x)x) \in \mathbb{Z}[P_{/\mathbb{Q}}]^0$$

where for $z \in P(\mathbb{C})$ we have set $\beta(z) = -0+z \in \mathbb{Z}[P(\mathbb{C})]^0$ and χ is defined by (4.1) and (4.7) . Denote by $\omega_{\mathbb{Q}} \in F^1(X_{\mathbb{R}})$ the real differential such that $\int_{X(\mathbb{R})^0} \omega_{\mathbb{Q}} = 1$. Then we have

$$\langle \omega_{\mathbb{Q}}, [\alpha,\beta]_{\mathcal{D}} \rangle = \pm \frac{N}{2|d_K|} L'(X,0)$$

where N is the conductor of $X_{/\mathbb{Q}}$.

Proof: We have shown everything except the rationality of β. As we have already seen that β is real it suffices to show that β is defined over K. For any $\sigma \in \mathrm{Gal}(K({}_\nu X(\mathbb{C}))/K)$ there is an $a \in (\mathcal{O}/\nu)^*/\mu_K$ such that the action of σ on $x \in {}_\nu X(\mathbb{C}) \cong \frac{1}{\nu}\Gamma/\Gamma$ is given by $x^\sigma = a^{-1}\chi(a)x = a^{-1}\bar{\chi}(a^{-1})x$. Hence

$$\beta^\sigma = \sum_{x \in P(\mathbb{C})/\mu_K} \beta(\bar{\chi}(x)x)^\sigma = \sum_{x \in P(\mathbb{C})/\mu_K} \beta(a^{-1}\bar{\chi}(a^{-1})\bar{\chi}(x)x)$$

$$= \sum_{x \in P(\mathbb{C})/\mu_K} \beta(\bar{\chi}(a^{-1}x)(a^{-1}x)) = \beta .$$

Clearly the equivalent version (2.4) of theorem (2.3) will follow from this result once we have established that $[\alpha,\beta]_{\mathcal{D}} = r_{\mathcal{D}}(\psi)$ for some $\psi \in H^2_A(X_{\mathbb{Z}}, \mathbb{Q}(2))$. This is done in the next section.

Remarks (a) A closer look at the root numbers in the functional equation for the L-series of Hecke characters reveals that $w = G(\chi,1)/\bar{f}$ and therefore the sign in the final formula of (4.10) is in fact +1.

(b) One of the authors (C.D.) has extended (4.10) to the case of L-series for certain Hecke characters of imaginary quadratic fields at all negative integers. This is done in the context of the Beilinson conjectures for motives with coefficients in a number field [16] , [17].

§5 The absolute cohomology

In order to prove that $[\alpha,\beta]_{\mathcal{D}}$ lies in the image of the regulator map

$$r_{\mathcal{D}}\colon H^2_{\mathcal{A}}(X,\mathbb{Q}(2)) \to H^2_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(2))$$

we use the following theorem.

Let $X_{/k}$ be a smooth, projective curve over a field k and let $P \subset X$ be a finite closed subscheme over k with open complement $U = X \setminus P \overset{j}{\hookrightarrow} X$. By $\mathbb{Z}[P]$ and $\mathbb{Z}[P]^o$ we denote the group of divisors on X with support in P and its subgroup of divisors of degree zero respectively. Tensoring with $\mathbb{Q}$ we obtain $\mathbb{Q}[P]$ and $\mathbb{Q}[P]^o$.

(5.1) Theorem Let k be a number field contained in $\mathbb{R}$. If the image of $\mathbb{Z}[P]^o$ in the Jacobian of X is a torsion group, then there exists a pairing

$$[\,,\,]_{\mathcal{A}} : \Lambda^2\mathbb{Q}[P]^o \to H^2_{\mathcal{A}}(X,\mathbb{Q}(2))$$

such that the following diagram commutes

$$\begin{array}{ccc} \Lambda^2\mathbb{Q}[P]^o & \xrightarrow{[\,,\,]_{\mathcal{A}}} & H^2_{\mathcal{A}}(X,\mathbb{Q}(2)) \\ & \searrow^{[\,,\,]_{\mathcal{D}}} & \downarrow r_{\mathcal{D}} \\ & & H^2_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(2)) \end{array} .$$

We prepare the proof of this theorem by the following lemma. Let k' be a finite galois extension of k and write P', U', X' for the base changes of P, U, X with k'/k. Assume that all points of P' are k'-rational and let $\varphi\colon U' \to U$ denote the canonical covering.

(5.2) Lemma If the elements of $\mathbb{Z}[P]^o$ are of finite order in the Jacobian of X then we have

$$H^2_{\mathcal{A}}(U,\mathbb{Q}(2)) = H^2_{\mathcal{A}}(X,\mathbb{Q}(2)) + \varphi_*\{\mathcal{O}^*(U'),k'^*\} \otimes \mathbb{Q}$$

$$H^2_{\mathcal{A}}(\mathrm{Spec}\, k,\mathbb{Q}(2)) = H^2_{\mathcal{A}}(X,\mathbb{Q}(2)) \cap \varphi_*\{\mathcal{O}^*(U'),k'^*\} \otimes \mathbb{Q} .$$

Here $\{\mathcal{O}^*(U'),k'^*\}$ is the subgroup of $H^2_{\mathcal{A}}(U',\mathbb{Q}(2)) \subset K_2(k'(X')) \otimes \mathbb{Q}$ generated by the symbols $\{f,a\}$ with $f \in \mathcal{O}^*(U')$ and $a \in k'^*$.

<u>Proof:</u> The Gysin sequence for absolute cohomology ([12]Th.8,9)

$$H^0_A(P,\mathbb{Q}(1)) \to H^2_A(X,\mathbb{Q}(2)) \overset{j^*}{\to} H^2_A(U,\mathbb{Q}(2)) \to H^1_A(P,\mathbb{Q}(1))$$

shows that j^* is injective: Writing $P = \coprod \operatorname{Spec} k_i$ we get $H^0_A(P,\mathbb{Q}(1)) = \oplus K_2^{(1)}(k_i)$, but as $K_2(k_i)$ is generated by symbols we have $K_2(k_i) \otimes \mathbb{Q} = K_2^{(2)}(k_i)$ ([12] Th. 2) and hence $H^0_A(P,\mathbb{Q}(1)) = 0$.
In the commutative and exact diagram

$$\begin{array}{ccccccc}
0 \to & H^2_A(X,\mathbb{Q}(2)) & \to & H^2_A(U,\mathbb{Q}(2)) & \to & \operatorname{coker} j^* & \to 0 \\
 & \uparrow & & \uparrow \varphi_* & & & \\
0 \to & H^2_A(X',\mathbb{Q}(2)) & \to & H^2_A(U',\mathbb{Q}(2)) & \overset{\text{tame}}{\to} & \mathbb{Q}[P']^0 \otimes k'^* \subset H^1_A(P',\mathbb{Q}(1)) & \\
 & & & \uparrow \{,\} & \nearrow & -\operatorname{div} \otimes \operatorname{id} & \\
 & & & \mathcal{O}^*(U') \otimes k'^* \otimes \mathbb{Q} & & &
\end{array}$$

the image of the tame symbol coincides with $\mathbb{Q}[P']^0 \otimes k'^*$ since $\operatorname{div} \otimes \operatorname{id}$ is surjective by our assumption.
On the other hand the relation $\varphi_*\varphi^* = [k':k]$ implies that the map φ_* in the diagram is surjective and hence induces a surjection $\mathbb{Q}[P']^0 \otimes k'^* \twoheadrightarrow \operatorname{coker} j^*$. This proves the first assertion of (5.2).
The sequence

$$0 \to \mathcal{O}^*(X') \otimes k'^* \otimes \mathbb{Q} \to \mathcal{O}^*(U') \otimes k'^* \otimes \mathbb{Q} \to \operatorname{div} \mathcal{O}^*(U') \otimes k'^* \otimes \mathbb{Q} \to 0$$

being exact we obtain a commutative and exact diagram

$$\begin{array}{ccccc}
0 \to \{\mathcal{O}^*(X'),k'^*\} \otimes \mathbb{Q} & \to & \{\mathcal{O}^*(U'),k'^*\} \otimes \mathbb{Q} & \overset{\text{tame}}{\to} & H^1_A(P',\mathbb{Q}(1)) \\
\downarrow & & \downarrow & & \| \\
0 \to H^2_A(X',\mathbb{Q}(2)) & \to & H^2_A(U',\mathbb{Q}(2)) & \to & H^1_A(P',\mathbb{Q}(1)) .
\end{array}$$

Observing that $\mathcal{O}^*(X') = k'^*$ and that $H^2_A(\operatorname{Spec} k',\mathbb{Q}(2)) = K_2(k') \otimes \mathbb{Q}$ is generated by symbols we find

$$H^2_A(\operatorname{Spec} k',\mathbb{Q}(2)) = \{\mathcal{O}^*(X'),k'^*\} \otimes \mathbb{Q} = H^2(X',\mathbb{Q}(2)) \cap \{\mathcal{O}^*(U'),k'^*\} \otimes \mathbb{Q} .$$

Applying the surjective map φ_* gives the second assertion of (5.2) since $\varphi^*\varphi_*$ corresponds to taking $\mathrm{Gal}(k'/k)$-invariants and φ^* is injective.

Proof of (5.1): Since k is assumed to be a number field the group $H^2_A(\mathrm{Spec}\, k, \mathbb{Q}(2)) = K_2(k) \otimes \mathbb{Q}$ is zero. Let pr_A denote the projection of $H^2_A(U,\mathbb{Q}(2))$ onto $H^2_A(X,\mathbb{Q}(2))$ associated with the decomposition in (5.2). Since $k \subset \mathbb{R}$ we can use (1.6) to obtain an exact diagram

$$\begin{array}{ccc}
\varphi_*\{\mathcal{O}^*(U'),k'^*\}\otimes\mathbb{Q} & \dashrightarrow & H^1(U_{an},\mathbb{R}(1)) \cap F^1(U_{\mathbb{R}}) \\
\downarrow & & \downarrow \\
H^2_A(U,\mathbb{Q}(2)) & \xrightarrow{r_{\mathcal{D}}} & H^2_{\mathcal{D}}(U_{\mathbb{R}},\mathbb{R}(2)) = H^1(U_{an},\mathbb{R}(1)) \\
j^* \uparrow \downarrow pr_A & & j^* \uparrow \quad \downarrow pr_{\mathcal{D}} \\
H^2_A(X,\mathbb{Q}(2)) & \xrightarrow{r_{\mathcal{D}}} & H^2_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(2)) = H^1(X_{an},\mathbb{R}(1))
\end{array}$$

which commutes because $r_{\mathcal{D}} j^* = j^* r_{\mathcal{D}}$ and because $r_{\mathcal{D}}(\varphi_*\{\mathcal{O}^*(U'),k'^*\} \otimes \mathbb{Q})$ is contained in $F^1(U_{\mathbb{R}})$. Indeed, consider the commutative diagram

$$\begin{array}{ccc}
H^2_A(U',\mathbb{Q}(2)) & \xrightarrow{ch_{2,2}} & H^2_{\mathcal{D}}(U_{\mathbb{C}},\mathbb{R}(2)) \supset F^1(U(\mathbb{C})) \\
\uparrow \varphi^* & & \uparrow \\
H^2_A(U,\mathbb{Q}(2)) & \xrightarrow{r_{\mathcal{D}}} & H^2_{\mathcal{D}}(U_{\mathbb{R}},\mathbb{R}(2)) \supset F^1(U_{\mathbb{R}})\ .
\end{array}$$

Since $H^1(U_{an},\mathbb{R}(1)) \cap F^1(U_{\mathbb{R}}) = H^1(U_{an},\mathbb{R}(1)) \cap F^1(U(\mathbb{C}))$ we have to show that

$$ch_{2,2}(\varphi^*\varphi_*\{\mathcal{O}^*(U'),k'^*\}) \subset F^1(U(\mathbb{C}))\,.$$

But $ch_{2,2}$ is a ring homomorphism ([2] 2.3.1) and using that $\varphi^*\varphi_* = \sum_{\sigma\in \mathrm{Gal}(k'/k)} \sigma$ we get

$$\begin{aligned}
ch_{2,2}(\varphi^*\varphi_*\{f,a\}) &= \sum_\sigma ch_{2,2}\{f^\sigma,a^\sigma\} = \sum_\sigma(\log|f^\sigma| \cup \log|a^\sigma|) \\
&= -\sum_\sigma \log|a^\sigma|\,\partial \log|f^\sigma| \in F^1(U(\mathbb{C}))
\end{aligned}$$

for all $f \in \mathcal{O}^*(U')$ and $a \in k'^*$. In conclusion we have established that $pr_{\mathcal{D}} \circ r_{\mathcal{D}} = r_{\mathcal{D}} \circ pr_A$.

The identity $\varphi_* \varphi^* = [k':k]$ implies that for $f \in \mathcal{O}^*(U)$, $a \in k^*$ we have $[k':k]\{f,a\} = \varphi_*\{\varphi^*(f), \varphi^*(a)\}$ and hence that

$$\mathcal{O}^*(U) \otimes k^* \otimes \mathbb{Q} \subset \varphi_*\{\mathcal{O}^*(U'), k'^*\} \otimes \mathbb{Q} .$$

Hence there is a map $[\,,]_A$ completing the diagram

$$\begin{array}{ccc} \Lambda^2(\mathcal{O}^*(U) \otimes \mathbb{Q}) & \xrightarrow{\{,\}} & H^2_A(U, \mathbb{Q}(2)) \\ \Big\downarrow \Lambda^2 \mathrm{div} & & \Big\downarrow pr_A \\ \Lambda^2 \mathbb{Q}[P]^{\circ} & \xrightarrow{[,]_A} & H^2_A(X, \mathbb{Q}(2)) \end{array} .$$

Observe that $\mathrm{Ker\ div} = \mathcal{O}^*(X) = k^*$ and that because of $\{f,-f\} = 1$ the symbol becomes an alternating function on $(\mathcal{O}^*(U) \otimes \mathbb{Q}) \otimes (\mathcal{O}^*(U) \otimes \mathbb{Q})$. For $f,g \in \mathcal{O}^*(U)$ we now obtain

$$\begin{aligned} r_{\mathcal{D}}[\mathrm{div}\, f, \mathrm{div}\, g]_A &= r_{\mathcal{D}} pr_A\{f,g\} = pr_{\mathcal{D}} r_{\mathcal{D}}\{f,g\} \\ &= pr_{\mathcal{D}}(\log|f| \cup \log|g|) \\ &= [\mathrm{div}(\log|f|), \mathrm{div}(\log|g|)]_{\mathcal{D}} \text{ by (1.7).} \end{aligned}$$

But according to (1.1.1) we have

$$\mathrm{div}(\log|f|) = \mathrm{div}(2\partial \log|f|) = \mathrm{div}\left(\frac{df}{f}\right) = \sum_x \mathrm{Res}_x\left(\frac{df}{f}\right) x = \mathrm{div}\, f$$

and hence the proof of theorem (5.1) is complete.

Now let X be again an elliptic curve over $\mathbb{Q}$. It remains to prove that for the divisors α, β of theorem (4.10) the element $[\alpha,\beta]_A \in H^2_A(X, \mathbb{Q}(2))$ belongs to the "integral" subspace $H^2_A(X_{\mathbb{Z}}, \mathbb{Q}(2))$. In fact the following stronger result holds true:

<u>(5.3) Lemma</u> Let X be an elliptic curve over $\mathbb{Q}$ with potential good reduction at all finite places. Then

$$H^2_A(X_{\mathbb{Z}}, \mathbb{Q}(2)) = H^2_A(X, \mathbb{Q}(2)) .$$

Proof: Let $\mathcal{X}$ be the regular minimal model of X over Spec($\mathbb{Z}$).
By assumption it has either good or additive reduction $\mathcal{X}_p$ at the primes p of $\mathbb{Z}$. According to the localization exact sequence

$$K_2(\mathcal{X}) \to K_2(X) \to \bigoplus_p K_1'(\mathcal{X}_p)$$

it will be sufficient to show that $K_1'(\mathcal{X}_p) \otimes \mathbb{Q} = 0$ for all p. If Y is a smooth proper curve over a finite field F it follows immediately from the localization sequence

$$\bigoplus_y \kappa(y)^* \to K_1(Y) \to K_1(F(Y)) \xrightarrow{\text{div}} \bigoplus_y \mathbb{Z}$$

that $K_1(Y)$ is torsion. Here y runs over the closed points of Y and κ denotes the residue field.
We may thus assume that $\mathcal{X}_p$ is singular and also reduced because $K_n'(\mathcal{X}_p) = K_n'(\mathcal{X}_p^{red})$. By the Kodaira-Néron classification $\mathcal{X}_p^{red}$ is the disjoint union of a copy of $\mathbb{P}^1$ with copies of $\mathbb{A}^1$ (open in $\mathcal{X}_p^{red}$). Using the exact sequence

$$K_1'(\mathcal{X}_p^{red} \setminus \mathbb{A}^1) \to K_1'(\mathcal{X}_p^{red}) \to K_1(\mathbb{A}^1)$$

and the fact that $K_1(\mathbb{A}^1) \otimes \mathbb{Q} = 0$ one is reduced to $K_1(\mathbb{P}^1)$. But this group is torsion as well whence $K_1'(\mathcal{X}_p) \otimes \mathbb{Q} = 0$.

Remark: Bloch and Grayson [4] considered modular elliptic curves X *without* complex multiplication. With the aid of a computer program they found in this case as well rational relations between the value at two of the L-function of X and special values of Kronecker-Eisenstein-Lerch series.

These relations have the form

$$C\, L(C,2) + \sum_{\alpha \in X(\mathbb{Q})_{tor}} c_\alpha M(X,\alpha) = 0 \quad , \quad c_\alpha, C \in \mathbb{Z}$$

where $M(X,\alpha) = (\mathrm{Im}\,\tau)^2 \sum_{\gamma \in \Gamma} \overline{(\alpha,\gamma)} \dfrac{\bar{\gamma}}{|\gamma|^4}$

$(\alpha,\gamma) = \exp A^{-1}(\alpha\bar{\gamma} - \bar{\alpha}\gamma)$ and $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}$

is the period lattice of a real differential having volume $\mathrm{Vol}(\Gamma) = \pi A$.

For example the curve

$$y^2 + xy + y = x^3 - x^2 - 3x + 7$$

has a point of order 7 and the computer suggests the only relations

$$26\ L(X,2) + 28\ M(X,\tfrac{2}{7}) + 28\ M(X,\tfrac{3}{7}) = 0$$
$$5\ M(X,\tfrac{1}{7}) + 10\ M(X,\tfrac{2}{7}) + 8\ M(X,\tfrac{3}{7}) = 0\ .$$

This points towards

$$\mathrm{rank}\ K_2(X) = 2 \quad (>1!)$$

which is surprising at first glance. But X has <u>multiplicative</u> reduction at 2 (and good reduction at all other primes). Considering the localization sequence and observing that $K_1'(X_p)$ has rank 1 if the reduction X_p is of multiplicative type then implies

$$\mathrm{rank}\ K_2(X_{\mathbb{Z}}) = 1$$

as it should be by Beilinson's conjecture (Actually these computations had led to a revision of the original conjectures by taking into account the integral model). For further results on $H^2_A(X_{\mathbb{Z}},\mathbb{Q}(2))$ of elliptic curves we refer to [10], [13].

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After this paper was written we learned of related work by Rohrlich who computes the regulators for various choices of n-torsion points. See:

15) D. Rohrlich. Elliptic curves and values of L-functions. To appear in: Proc. Montreal Summer School of Can. Math. Soc. 1985.

Generalizations to L-series at negative integers are contained in

16) C. Deninger. Higher regulators of elliptic curves with complex multiplication. Preprint 1987.

17) C. Deninger. Higher regulators and Hecke L-series of imaginary quadratic fields. Preprint 1987.

For p-adic analogues we refer to [13] and

18) R. Coleman, E. de Shalit. p-adic regulators on curves and special values of p-adic L-functions. Preprint 1987.

Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
D-8400 Regensburg

Mathematisches Institut
Universität Erlangen
Bismarckstr. 1 1/2
D-8520 Erlangen