The Beilinson Conjecture for Algebraic Number Fields

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Preface

In this exposition we consider a conjecture of B. Gross on the Artin L-series $L(M,s)$ of a representation M of the galois group $\Gamma = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, which may be viewed as the zero dimensional case of Beilinson's general conjecture on motivic L-series. The conjecture gives a K-theoretic interpretation of the transcendental nature of the values of $L(M,s)$ at the integral places $s = n \in \mathbb{Z}$. Included are those places where the L-series vanishes. In this case the "value" is meant to be the first non-vanishing coefficient in the Taylor expansion and is denoted by $L(M,n)^*$.

For example, let F be a finite algebraic number field and let $X = \mathrm{Spec}\, F$. Then X gives rise to a representation $H(X)$ of Γ, namely $H^0(X \otimes \bar{\mathbb{Q}}, \mathbb{Q}) = \mathbb{Q}^{\mathrm{Hom}(F,\bar{\mathbb{Q}})}$. The associated L-series $L(H(X),s)$ is in this case the Dedekind zeta function $\zeta_F(s)$ of F. We consider the following two groups, which we denote as cohomology groups:

$$H^1_{\mathcal{A}}(X, \mathbb{Q}(n)) = K_{2n-1}(F) \otimes \mathbb{Q} ,$$

$$H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)) = [\prod_{\alpha} (2\pi i)^{n-1}\mathbb{R}]^+ .$$

Here α runs through $\mathrm{Hom}(F,\mathbb{C})$ and $[\ \ ]^+$ means the fixed module under complex conjugation acting on $\mathrm{Hom}(F,\mathbb{C})$ and $(2\pi i)^{n-1}\mathbb{R}$. By a theorem of Borel we have a canonical isomorphism

$$r_{\mathcal{D}} : H^1_{\mathcal{A}}(X,\mathbb{Q}(n)) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)).$$

Both $\mathbb{R}$-vectorspaces are equipped with canonical "$\mathbb{Q}$-structures", the left side with the $\mathbb{Q}$-subspace $V_{\mathbb{Q}} = H^1_{\mathcal{A}}(X,\mathbb{Q}(n))$ and the right side with the $\mathbb{Q}$-subspace $W_{\mathbb{Q}} = [\prod_{\alpha}(2\pi i)^{n-1}\mathbb{Q}]^+$. The n-th regulator $c_X(1-n)$ of F is defined to be the determinant of the linear map $r_{\mathcal{D}}$ with respect to a $\mathbb{Q}$-basis of $V_{\mathbb{Q}}$ and of $W_{\mathbb{Q}}$. It is determined up to a rational number, i.e.

$$c_X(1-n) \in \mathbb{R}^*/\mathbb{Q}^* .$$

The Gross conjecture for the representation $H(X)$ (which in this case has been proven by Borel) then says that for $n \geq 1$,

$$c_X(1-n) = \zeta_F(1-n)^* \mod \mathbb{Q}^* .$$

For an arbitrary Artin L-series $L(M,s)$ the conjecture is formulated in quite the same way, just that $X = \mathrm{Spec}(F)$ has to be replaced by an "Artin motive", which produces the representation M. The main purpose of this exposition is, to give a presentation of Beilinson's proof of the Gross conjecture in the case of Dirichlet L-series.

The exposition is divided into two parts. The first part is rather independent of the general Beilinson theory and may serve as an elementary introduction into the general set up. It comprises the explanation of the absolute cohomology, the Deligne cohomology and the higher regulators of Artin motives, and it describes the link of these concepts with the values of Artin L-series. The main result of this part is the verification of the Gross conjecture for Dirichlet L-series. This verification is obtained by elementary arguments up to a theorem on the explicit description of the regulator map $r_{\mathcal{D}}$, the proof of which is subject to the second part. I have tried to write the first part in such a way that it may serve as a basis for a student's seminar.

The second part is devoted to a presentation of Beilinson's proof of the theorem mentioned above, which describes the higher regulator map $r_{\mathcal{D}}$ by means of the polylogarithm function. The case n = 2 has been proven before by S. Bloch, by means of the dilogarithm. This proof was an initial point for the emergence of the general conjecture. The proof for n > 2 is very difficult and makes use of the full generality of Beilinson's theory. Based on a first version, which was incomplete and partially also incorrect, this part has been rewritten by M. Rapoport and P. Schneider. So the presentation of the proof is essentially due to them. Unfortunately, we were not able to understand the claim (7.0.2) in Beilinson's paper [3] (see the "crucial lemma" (2.4)), so that there remains a serious gap in the proof.

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Part I

Regulators and Values of Artin L-series

§1. Regulators for Algebraic Number Fields

The theory which we shall develop is based on a canonical homomorphism

$$r: K_{2n-1}(\mathbb{C}) \to \mathbb{R}(n-1)$$

which is given for each $n \geq 1$, where $\mathbb{R}(n-1) = (2\pi i)^{n-1}\mathbb{R}$. When $n = 1$, then $K_1(\mathbb{C}) = \mathbb{C}^*$, and r is the homomorphism

$$\mathbb{C}^* \to \mathbb{R} \quad , \quad z \mapsto \log|z| \ .$$

In the general case the definition of the map r relies on the following three facts (see [20]).

1). We have a canonical homomorphism

$$K_q(\mathbb{C}) = \pi_q(BGL^+(\mathbb{C})) \to H_q(BGL^+(\mathbb{C})) = H_q(GL(\mathbb{C}),\mathbb{Z}),$$

the Hurewicz-map.

2). We have a canonical pairing

$$H^q(GL(\mathbb{C}),\mathbb{R}(n-1)) \times H_q(GL(\mathbb{C}),\mathbb{Z}) \xrightarrow{<,>} \mathbb{R}(n-1) \ .$$

3). In the continuous cohomology H_c^* of the topological group $GL(\mathbb{C})$ with coefficients in $\mathbb{R}(n-1)$ we have canonically constructed "Borel regulator elements"

$$b_{2n-1} = \gamma((2\pi i)^n u_{2n-1}) \in H_c^{2n-1}(GL(\mathbb{C}),\mathbb{R}(n-1)) \quad ^{1)} \ .$$

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1) These elements yield an identification

$$H_c^*(GL(\mathbb{C}),\mathbb{R}) = \Lambda_{\mathbb{R}}^r(v_1,v_3,v_5,\dots)$$

of the continuous cohomology with the free exterior algebra generated by the cohomology classes $v_{2n-1} = \frac{i}{(2\pi i)^n} b_{2n-1}$ of degree $2n-1$ (see [12] and [6]).

which are invariant under the involution induced by complex conjugation on $GL(\mathbb{C})$ and $\mathbb{R}(n-1)$. For the definition of u_{2n-1}, γ and b_{2n-1} we refer the reader to Rapoport [20], §1. We denote the image of b_{2n-1} under the canonical map

$$H_c^{2n-1}(GL(\mathbb{C}),\mathbb{R}(n-1)) \to H^{2n-1}(GL(\mathbb{C}),\mathbb{R}(n-1))$$

also by b_{2n-1} and obtain our homomorphism r now as the composite of

$$K_{2n-1}(\mathbb{C}) \to H_{2n-1}(GL(\mathbb{C}),\mathbb{Z}) \xrightarrow{<b_{2n-1},->} \mathbb{R}(n-1).$$

It is called the "Borel regulator map".

Now let F be a finite algebraic number field, let $\mathcal{O}_F$ be the ring of integers in F and let

$$X = \mathrm{Spec}(F)\ ,\ X_{\mathbb{Z}} = \mathrm{Spec}(\mathcal{O}_F)\ ,\ X(\mathbb{C}) = \mathrm{Hom}(F,\mathbb{C})\ .$$

Any complex imbedding $\alpha: F \to \mathbb{C}$ induces a map $\alpha_*: K_{2n-1}(F) \to K_{2n-1}(\mathbb{C})$ by functoriality and we obtain a homomorphism

$$K_{2n-1}(F) \to K_{2n-1}(\mathbb{C})^{X(\mathbb{C})}\ ,\ a \mapsto (\ldots,\alpha_*(a),\ldots)_{\alpha\in X(\mathbb{C})}\ ,$$

which is functorial in F. We consider the composite homomorphism

$$K_{2n-1}(F) \to K_{2n-1}(\mathbb{C})^{X(\mathbb{C})} \xrightarrow{r} \mathbb{R}(n-1)^{X(\mathbb{C})}\ .$$

The complex conjugation acts on $X(\mathbb{C})$, on $K_{2n-1}(\mathbb{C})$ and on $\mathbb{R}(n-1)$, and thus on the middle and the right group. r is compatible with this action. Indicating the fixed module by $[\ \ ]^+$, we obtain a canonical homomorphism

$$r_{\mathcal{D}} : K_{2n-1}(F) \to [\mathbb{R}(n-1)^{X(\mathbb{C})}]^+\ ,$$

and, as composite with $K_{2n-1}(\mathcal{O}_F) \to K_{2n-1}(F)$,

$$r_{\mathcal{D}} : K_{2n-1}(\mathcal{O}_F) \to [\mathbb{R}(n-1)^{X(\mathbb{C})}]^+\ .$$

This homomorphism is called the n-th <u>regulator map</u>. We have two fundamental theorems about these maps:

<u>(1.1) Theorem</u> (Dirichlet's unit theorem): For $n = 1$, the map $r_{\mathcal{D}}$ together with the diagonal map $\mathbb{Z} \to \mathbb{R}^{X(\mathbb{C})}$ induces an isomorphism

$$(K_1(\mathcal{O}_F) \oplus \mathbb{Z}) \otimes \mathbb{R} \xrightarrow{\sim} [\mathbb{R}^{X(\mathbb{C})}]^+ .$$

<u>(1.2) Theorem</u> (Borel): For $n > 1$ the map $r_{\mathcal{D}}$ induces an isomorphism

$$K_{2n-1}(F) \otimes \mathbb{R} \xrightarrow{\sim} [\mathbb{R}(n-1)^{X(\mathbb{C})}]^{+\ 2)} .$$

For the proof of theorem (1.2) we refer the reader to [5] and [6].

We now reformulate these results in the language of Beilinson. We set

$$H^1_A(X_{\mathbb{Z}}, \mathbb{Q}(n)) := K_{2n-1}(\mathcal{O}_F) \otimes \mathbb{Q} \quad , \quad H^1_A(X, \mathbb{Q}(n)) := K_{2n-1}(F) \otimes \mathbb{Q} .$$

For the reader who is familiar with the general Beilinson theory we remark, that this designation is justified by the fact that the Adams operator ψ^k acts on $K_{2n-1}(\mathcal{O}_F) \otimes \mathbb{Q}$ as multiplication by k^n because of $r \circ \psi^k = k^n r$ and (1.2). From this and from $K_{2n}(\mathcal{O}_F) \otimes \mathbb{Q} = 0$ follows also $H^i_A(X_{\mathbb{Z}}, \mathbb{Q}(n)) = 0$ for $i \neq 1$. We set on the other hand

$$\begin{aligned} H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)) &:= H^0(X(\mathbb{C}), \mathbb{C}/\mathbb{R}(n))^+ \\ &= H^0(X(\mathbb{C}), \mathbb{R}(n-1))^+ = [\mathbb{R}(n-1)^{X(\mathbb{C})}]^+ , \end{aligned}$$

regarding that $\mathbb{C} = \mathbb{R}(n-1) \oplus \mathbb{R}(n)$. We then obtain a regulator map

$$r_{\mathcal{D}}: H^1_A(X_{\mathbb{Z}}, \mathbb{Q}(n)) \to H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n))$$

between the "absolute cohomology" H^1_A and the

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2) We remark that $K_i(\mathcal{O}_F) \otimes \mathbb{Q} = K_i(F) \otimes \mathbb{Q}$ for $i > 1$. This is a consequence of the localization sequence and of the fact that the K-groups of the finite residue class fields of $\mathcal{O}_F$ are finite groups.

"Deligne cohomology" $H^1_{\mathcal{D}}$. In [20] (see also the appendix to §2 of [2]) it is shown that this map coincides with the regulator map that has been defined by Beilinson quite generally for arbitrary Grothendieck motives over number fields. From Dirichlet's and Borel's theorem we obtain

(1.3) Theorem: The regulator map $r_{\mathcal{D}}$ induces isomorphisms

$$(H^1_A(X_{\mathbb{Z}},\mathbb{Q}(1)) \oplus \mathbb{Q}) \otimes \mathbb{R} \xrightarrow{\sim} H^1_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(1)) \qquad \text{for } n = 1,$$
$$(H^1_A(X,\mathbb{Q}(n)) \otimes \mathbb{R} \xrightarrow{\sim} H^1_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(n)) \qquad \text{for } n > 1.$$

We have formulated the above situation for schemes $X = \mathrm{Spec}(F)$, but it clearly extends to arbitrary zero-dimensional varieties X over $\mathbb{Q}$, i.e. to finite disjoint unions of schemes $\mathrm{Spec}(F)$. The regulator map $r_{\mathcal{D}}$ is functorial in a twofold sense. Namely, let $H(X)$ denote one of the groups $H^1_A(X_{\mathbb{Z}},\mathbb{Q}(n))$, $H^1_A(X,\mathbb{Q}(n))$, $H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n))$. A morphism

$$p: X \to Y$$

of 0-dimensional varieties over $\mathbb{Q}$ induces then two homomorphisms

$$H(X) \underset{p^*}{\overset{p_*}{\rightleftarrows}} H(Y) ,$$

where $p^* = H(p)$. In the case $H = H^1_A$ the map p_* is induced by the usual transfer

$$tr : K_{2n-1}(X) \to K_{2n-1}(Y)$$

of K-theory. In the case $H = H^1_{\mathcal{D}}$ the map p_* is induced by the homomorphism

$$p_* : \mathbb{R}(n)^{X(\mathbb{C})} \to \mathbb{R}(n)^{Y(\mathbb{C})} ,$$

which associates to a function $f: X(\mathbb{C}) \to \mathbb{R}(n)$ the function

$$(p_*f)(y) = \sum_{x\in p^{-1}(y)} f(x)$$

on $Y(\mathbb{C})$. If, in particular, $p: X \to X$ is an automorphism, then p_* and p^* are mutually inverse automorphisms of $H(X)$. Quite generally the maps p_* and p^* commute with the regulator map.

The central result in this exposition is an explicit description of the regulator map $r_{\mathcal{D}}$ for $n > 1$, in the case that F is the field $\mathbb{Q}(\mu_N)$ of the N-th roots of unity and $X = \mathrm{Spec}(F)$. This description is obtained by the <u>poly-logarithm function</u> which is defined for all complex s and z with $|z| < 1$ by the convergent power series

$$\mathcal{L}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} .$$

It extends to a function which is defined, and holomorphic in both variables, for all s and z in the region $\mathbb{C}-[1,\infty)$ (cf [18]). For $\zeta \in \mu_N-\{1\}\subset F$ and $n > 1$ we set

$$\mathcal{L}_n(\zeta) = (\ldots,\mathcal{L}_n(\alpha\zeta),\ldots)_{\alpha:F\to\mathbb{C}}$$

Viewing this as an element of $(\mathbb{C}/\mathbb{R}(n))^{X(\mathbb{C})}$, we have

<u>(1.4) Theorem:</u> For each $n > 1$ we have a map of G-sets

$$\varepsilon_n : \mu_N-\{1\} \to H^1_{\mathcal{A}}(X,\mathbb{Q}(n))$$

such that for $\zeta \in \mu_N-\{1\}$,

$$r_{\mathcal{D}}(\varepsilon_n(\zeta)) = \mathcal{L}_n(\zeta) .$$

We shall see in part II, §1, that this theorem yields a complete and explicit description of the regulator map for $X = \mathrm{Spec}(\mathbb{Q}(\mu_N))$.

The proof of theorem (1.4) is difficult and uses the full generality of Beilinson's theory of absolute cohomology, Deligne cohomology and regulator map. We shall give a presentation of it in part II of this exposition. The consequences for the values of Dirichlet L-series, however, to which this part is devoted, will be obtained by elementary methods.

§2. Regulators for Artin Motives

Let k be a finite algebraic number field, $\bar{k}$ an algebraic closure and $\Gamma = \mathrm{Gal}(\bar{k}|k)$ the absolute galois group of k. Let X be a zero-dimensional variety over k, i.e. a finite disjoint union of schemes Spec(F), where F is a finite field extension. Then X gives rise to a representation of Γ, namely

$$H(X) = H^{o}(\bar{X},\mathbb{Q}) = \mathbb{Q}^{\pi_o(\bar{X})} \quad ,$$

where $\bar{X} = X \times_k \bar{k}$ and $\pi_o(\bar{X})$ denotes the set of connected components of $\bar{X}$ [3]). To this representation belongs an Artin L-series, which is nothing but the zeta function of F. It is our purpose to study arbitrary Artin L-series and relate them to regulators. For this reason we view the situation of §1 as attached to the representation H(X) and refine it now by replacing H(X) by a representation which is associated to an arbitrary <u>Artin motive</u>. By this we mean the following.

Let E be a finite algebraic number field. An <u>Artin motive</u> over k with coefficients in E is a pair

$$M = (X,p) \quad ,$$

where X is a zero-dimensional variety over k and p is a Γ-homomorphism

$$p: H(X) \otimes E \to H(X) \otimes E$$

with $p^2 = p$ of the E-vector space $H(X) \otimes E = E^{\pi_o(\bar{X})}$. We set in particular

$$EX = (X,\mathrm{id})$$

To each Artin motive $M = (X,p)$ we associate the E-vector space

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3) Each connected component of $\bar{X}$ consists of one point only. If X = Spec(F), then we obtain a bijection

$$X(\bar{k}) = \mathrm{Hom}_k(F,\bar{k}) \xrightarrow{\sim} \pi_o(\bar{X}) ,$$

which associates to a $\bar{k}$-rational point of X the connected component of the associated point of the scheme $\bar{X}$. Note however, that Γ acts on $X(\bar{k})$ from the left and on $\pi_o(\bar{X})$ from the right.

$$H(M) := p(H(X)\otimes E) = pE^{\pi_0(\bar{X})},$$

which is a representation of Γ with coefficients in E. We have in particular

$$H(EX) = H(X)\otimes E = E^{\pi_0(\bar{X})}.$$

A <u>morphism</u> between two Artin motives with coefficients in F,

$$f: (X,p) \to (Y,q)$$

is an E-linear Γ-homomorphism

$$\varphi: H(EY) \to H(EX)$$

with $p\circ\varphi = \varphi\circ q$. In this way the Artin motives over k with coefficients in E form an abelian category

$$\mathcal{M} = \mathcal{M}_k^o(E) .$$

Note, that we have let φ and f go into opposite directions. In this way the functor

$$V_k^o \to \mathcal{M} \quad , \quad X \mapsto EX ,$$

from the category V_k^o of zero-dimensional varieties over k to the category $\mathcal{M}$ is <u>covariant</u>, thus giving the motives the character of "spaces". This convention coincides with that of Beilinson [3] and Jannsen [14]. It is however opposite to that of [9] . We remark, that

$$\mathrm{Hom}_{\mathcal{M}}(EX,EY) = \mathrm{Hom}_{E,\Gamma}(H(EY),H(EX)) \cong CH^o(X\times Y) \otimes_{\mathbb{Z}} E ,$$

where

$$CH^o(X\times Y) = \mathbb{Z}^{\pi_0(X\times Y)}$$

is the free abelian group over the set $\pi_0(X\times Y)$ of connected components of $X\times Y$, i.e. the group of (zero-codimensional) cycles on $X\times Y$. In fact, we have

$$CH^o(X\times Y) \otimes E = (\mathbb{Z}^{\pi_0(\overline{X\times Y})})^{\Gamma} \otimes E \cong \mathrm{Hom}_{E,\Gamma}(H(EY),H(EX))$$

by associating to a function $f: \pi_0(\overline{X\times Y}) = \pi_0(\bar{X}) \times \pi_0(\bar{Y}) \to \mathbb{Z}$ the map

$$E^{\pi_0(\bar{Y})} \to E^{\pi_0(\bar{X})} ,$$

$$(g:\pi_0(\bar{Y})\to E) \mapsto (\tilde{g}:\pi_0(\bar{x})\to E), \quad \tilde{g}(\bar{x}) = \sum_{\bar{y}\in\pi_0(\bar{Y})} f(\bar{x},\bar{y})g(\bar{y}) .$$

Since $H(EX) = E^{\pi_0(\bar{X})}$ has the canonical basis $\bar{x}_*$, $\bar{x} \in \bar{X}$, we have canonically

$$\mathrm{Hom}_E(H(EX),E) = H(EX) .$$

Therefore to each Artin motive $M = (X,p)$, there is associated the dual Artin motive

$$\check{M} = (X,\check{p}) ,$$

$\check{p}$ being the dual map of $p: H(EX) \to H(EX)$.

The functor $M \mapsto H(M)$ is contravariant and yields an equivalence

$$H: \mathcal{M} \tilde{\to} \mathrm{Rep}_E(\Gamma)^{\circ}$$

between the category of Artin motives over k with coefficients in E and the opposite category of the category $\mathrm{Rep}_E(\Gamma)$ of finite dimensional representations of $\Gamma = \mathrm{Gal}(\bar{k}|k)$ over E . One should however not identify $\mathcal{M}$ with $\mathrm{Rep}_E(\Gamma)^{\circ}$; an Artin motive is not simply a representation of Γ , but it is determined by the selection of a direct summand of a representation $H(X) \otimes E$.

We are now going to associate to every Artin motive cohomology groups $H^1_{\mathcal{A}}$ and $H^1_{\mathcal{D}}$ as follows. Let

$$\mathcal{H}: V_k^{\circ} \to \mathrm{Vec}_{\mathbb{Q}}$$

be any contravariant functor on the category V_k° of 0-dimensional varieties over k into the category of $\mathbb{Q}$-vector spaces, which sends sums into products and associates to a morphism $f: X \to Y$ functorially two homomorphisms

$$\mathcal{H}(Y) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{H}(X) .$$

where $f^* = \mathcal{H}(f)$ and $f^* \circ f_* = 1$ if f is an isomorphism. Let

$$\mathcal{H}(EX) := \mathcal{H}(X) \otimes E .$$

To every morphism

$$f \in \mathrm{Hom}_{\mathcal{M}}(EX,EY) = CH^{\circ}(X \times Y) \otimes E$$

we associate two homomorphisms

$$\mathcal{H}(EY) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{H}(EX)$$

as follows. The group

$$CH^{\circ}(X \times Y) \otimes E = E^{\pi_{\circ}(X \times Y)} = \prod_{Z \in \pi_{\circ}(X \times Y)} E$$

acts on

$$\mathcal{H}(E(X\times Y)) = \prod_{Z\in\pi_0(X\times Y)} \mathcal{H}(EZ)$$

componentwise by scalar multiplication, so that f yields an endomorphism fU of $\mathcal{H}(E(X\times Y))$. The maps f^* and f_* are then defined as the composites of

$$\mathcal{H}(EY) \underset{pr_{2*}\otimes 1}{\overset{pr_2^*\otimes 1}{\rightleftarrows}} \mathcal{H}(E(X\times Y)) \underset{fU}{\overset{fU}{\rightleftarrows}} \mathcal{H}(E(X\times Y)) \underset{pr_1^*\otimes 1}{\overset{pr_{1*}\otimes 1}{\rightleftarrows}} \mathcal{H}(EX) \quad .$$

If $\varphi: X \to Y$ is a morphism in V_k^o and $f = \mathrm{graph}(\varphi) \otimes 1 \in CH^o(X\times Y) \otimes E$, then it is easy to see that

$$f^* = \varphi^* \otimes 1 \quad , \quad f_* = \varphi_* \otimes 1 \ .$$

We may now extend the functor $\mathcal{H}$ from the category V_k^o to the category $\mathcal{M}$ of Artin motives $M = (X,p)$, by setting

$$\mathcal{H}(M) := p^*\mathcal{H}(EX)$$

We apply this to the functors $\mathcal{H}(X) = H^1(X,\mathbb{Q}(n))$ and $\mathcal{H}(X) = H^1_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(n))$:

(2.1) Definition: Let $M = (X,p)$ be an Artin motive over k with coefficients in E. Then p is an idempotent in the ring $\mathrm{End}_{\mathcal{M}}(EX)$, and we set

$$H^1_{\mathcal{A}}(M,\mathbb{Q}(n)) = p^*H^1_{\mathcal{A}}(EX,\mathbb{Q}(n))$$

$$H^1_{\mathcal{D}}(M_{\mathbb{R}},\mathbb{Q}(n)) = p^*H^1_{\mathcal{D}}(EX_{\mathbb{R}},\mathbb{R}(n)).$$

The first group is an E-vector space and the second a free $\mathbb{R}\otimes E$-module. The E-vectorspaces $H^1_{\mathcal{A}}(M_{\mathbb{Z}},\mathbb{Q}(n))$ are defined in the same way starting from the functor $\mathcal{H}(X) = H^1_{\mathcal{A}}(X_{\mathbb{Z}},\mathbb{Q}(n))$.

As an immediate consequence of theorem (1.3) we obtain the first part of Beilinson's conjectures for arbitrary Artin motives over k:

(2.2) Theorem: If $M = (X,p)$ is an Artin motive over k with coefficients in E, then the regulator map $r_{\mathcal{D}}$ for X induces isomorphisms

$$r_{\mathcal{D}} : (H^1_{\mathcal{A}}(M_{\mathbb{Z}},\mathbb{Q}(1)) \oplus H(M)^{\Gamma})\otimes\mathbb{R} \xrightarrow{\sim} H^1_{\mathcal{D}}(M_{\mathbb{R}},\mathbb{R}(1)) \text{ for } n = 1 \ .$$

$$r_{\mathcal{D}} : H^1_{\mathcal{A}}(M, \mathbb{Q}(n)) \otimes \mathbb{R} \xrightarrow{\sim} H^1_{\mathcal{D}}(M_{\mathbb{R}},\mathbb{R}(n)) \text{ for } n > 1 \ .$$

From the regulator maps $r_{\mathcal{D}}$ we obtain regulators of the motive M as follows. If E is a ring, A an E-algebra and V an A-module, then an E-structure (or E-lattice) of V is a free E-submodule V_E of finite rank such that

$$V = V_E \underset{E}{\otimes} A .$$

Now to any isomorphism

$$f : V \to V'$$

of A-modules V, V' which are endowed with E-structures V_E and V'_E we associate the regulator, which is the element

$$R(f) \in A^*/E^*$$

given by the determinant of the matrix associated to f after the choice of an E-basis of V_E and V'_E .

We apply this to the maps $r_{\mathcal{D}}$ of theorem (2.2), which are isomorphisms of free $\mathbb{R} \otimes E$-modules. Both sides are equipped with a canonical E-structure, the left side with

$H^1_{\mathcal{A}}(M_{\mathbb{Z}}, \mathbb{Q}(1)) \oplus H(M)^{\Gamma}$ for $n=1$ and $H^1_{\mathcal{A}}(M, \mathbb{Q}(n))$ for $n > 1$,

and the right side with

$$H^0(M_{\mathbb{R}}, \mathbb{Q}(n-1)) := p^*(H^0(X_{\mathbb{R}}, \mathbb{Q}(n-1)) \otimes E),$$

regarding that

$H^0(X_{\mathbb{R}}, \mathbb{Q}(n-1)) = [\mathbb{Q}(n-1)^{X(\mathbb{C})}]^+ \subset [\mathbb{R}(n-1)^{X(\mathbb{C})}]^+ = H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n))$.

(2.3) Definition: We denote the regulator of the isomorphism $r_{\mathcal{D}}$ with respect to these two E-structures by

$$c_M(1-n) \in (\mathbb{R} \otimes E)^*/E^*$$

and call it the n-th regulator of the motive M .

§3. L-series of Artin Motives

We are now going to explain the second and third part of Beilinson's conjectures, concerning the values of L-series, here for the case of algebraic number fields. The third conjecture is identical with a conjecture, previously posed by

B. Gross [12] (which, for s=0 , is in turn Stark's conjecture (see [25])).

We consider an Artin motive $M = (X,p)$ over k with coefficients in the finite algebraic number field E; M yields a representation

$$\rho : \Gamma \to \mathrm{Aut}_E(H(M))$$

of the absolute galois group $\Gamma = \mathrm{Gal}(\bar{k}|k)$ on the finite dimensional E-vector space $H(M)$. For every $\sigma \in \mathrm{Hom}(E,\mathbb{C})$ we obtain a complex representation

$$\rho^\sigma : \Gamma \to \mathrm{Aut}_{\mathbb{C}}(H(M)^\sigma) \quad , \quad H(M)^\sigma = H(M) \underset{E,\sigma}{\otimes} \mathbb{C} ,$$

to which we associate the L-series

$$L(M^\sigma,s) = \prod_{\mathfrak{p}\nmid\infty} \det(1-\rho^\sigma(\varphi_{\mathfrak{p}}^{-1})\mathfrak{N}(\mathfrak{p})^{-s};(H(M)^\sigma)^{I_{\mathfrak{p}}}) .$$

Here $\mathfrak{p}$ runs over the finite primes of k and $\varphi_{\mathfrak{p}}$ is an element in a decomposition group $\Gamma_{\mathfrak{p}} \subset \Gamma$ over $\mathfrak{p}$, that is mapped onto the Frobenius automorphism in $\Gamma_{\mathfrak{p}}/I_{\mathfrak{p}}$, $I_{\mathfrak{p}}$ being the inertia group in $\Gamma_{\mathfrak{p}}$ [4]).

To the Artin motive M we associate the <u>L-series of</u> M by setting

$$L(M,s) = (\ldots,L(M^\sigma,s),\ldots)_{\sigma\in\mathrm{Hom}(E,\mathbb{C})} .$$

This is a meromorphic function on $\mathbb{C}$ with values in $\mathbb{C} \underset{\mathbb{Q}}{\otimes} E$. To be precise, every $\sigma \in \mathrm{Hom}(E,\mathbb{C})$ induces a homomorphism

$$\sigma_* : \mathbb{C} \otimes E \to \mathbb{C} \quad , \quad z \otimes a \mapsto z\cdot(\sigma a) .$$

We obtain in this way a decomposition $\mathbb{C} \otimes E = \prod_\sigma \mathbb{C}$ and the L-series $L(M,s) \in \mathbb{C} \otimes E$ is defined by

$$\sigma_* L(M,s) = L(M^\sigma,s) .$$

Our aim is to study the values of $L(M,s)$ at the integral points $s = m \leq 0$. The order $d_m(M)$ of $L(M^\sigma,s)$ at $s = m$ is independent of σ and is given by

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[4]) We stress that we have put $\varphi_{\mathfrak{p}}^{-1}$ and not $\varphi_{\mathfrak{p}}$ in the definition of $L(M^\sigma,s)$. So $L(M^\sigma,s)$ is not the usual Artin L-series of $H(M)^\sigma$ but of the dual of $H(M)^\sigma$.

$$d_m(M) = \begin{cases} \sum\limits_{\mathfrak{p}|\infty} \dim_E H(M)^{\Gamma_{\mathfrak{p}}} - \dim_E H(M)^{\Gamma} & \text{for } m=0 \\ \sum\limits_{\mathfrak{p}|\infty} \dim_E H(M)^{\Gamma_{\mathfrak{p}}} & \text{for } m<0 \text{ even} \\ \sum\limits_{\substack{\mathfrak{p}|\infty \\ \text{complex}}} \dim_E H(M)^{\Gamma_{\mathfrak{p}}} + \sum\limits_{\substack{\mathfrak{p}|\infty \\ \text{real}}} \dim_E (H(M)/H(M)^{\Gamma_{\mathfrak{p}}}) & \text{for } m<0 \text{ odd}. \end{cases}$$

This is well known (see [25], CH O, §6) and follows from the functional equation of $L(M^{\sigma},s)$, which shows that only the Γ-factors contribute to the order.

On the other hand, we have the following explicit description of the cohomology group $H^1_{\mathcal{D}}(M_{\mathbb{R}},\mathbb{R}(n))$:

<u>(3.1) Proposition</u>: Let $M = (X,p)$ be an Artin motive with coefficients in E. Then, canonically

$$H^1_{\mathcal{D}}(M_{\mathbb{R}},\mathbb{R}(n)) \cong \begin{cases} (\bigoplus\limits_{\mathfrak{p}|\infty} H(M)^{\Gamma_{\mathfrak{p}}}) \otimes \mathbb{R} & \text{for } n \text{ odd,} \\ (\bigoplus\limits_{\substack{\mathfrak{p}|\infty \\ \text{complex}}} H(M)^{\Gamma_{\mathfrak{p}}} \oplus \bigoplus\limits_{\substack{\mathfrak{p}|\infty \\ \text{real}}} H(M)/H(M)^{\Gamma_{\mathfrak{p}}})\otimes\mathbb{R} & \text{for } n \text{ even,} \end{cases}$$

where $\mathfrak{p}$ runs over the infinite primes of k and $\Gamma_{\mathfrak{p}} \subset \Gamma = \mathrm{Gal}(\bar{k}|k)$ is a decomposition group over $\mathfrak{p}$.

Proof: Let $S = \mathrm{Spec}(k)$. The morphism $X \to S$ yields a map

$$X(\mathbb{C}) \to S(\mathbb{C})$$

and we let $X(\mathbb{C})_s$ be the fibre over $s \in S(\mathbb{C}) = \mathrm{Hom}(k,\mathbb{C})$. We fix an extension $\tilde{s}: \bar{k} \to \mathbb{C}$ of $s: k \to \mathbb{C}$. Then $\tilde{s}$ induces a bijection

$$X(\bar{k}) \xrightarrow{\sim} X(\mathbb{C})_s \quad , \quad x \mapsto \tilde{s}\circ x .$$

The idempotent p induces endomorphisms p^* of $H(EX):=E^{X(\bar{k})}$ and $\tilde{H}(EX):= E^{X(\mathbb{C})_s}$ as described in §2. By functoriality we have a commutative diagram

$$\begin{array}{ccc} E^{X(\mathbb{C})_s} & \xrightarrow{\tilde{s}^*} & E^{X(\bar{k})} \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle p^*} \\ E^{X(\mathbb{C})_s} & \xrightarrow{\tilde{s}^*} & E^{X(\bar{k})} \end{array} ,$$

where $(\tilde{s}^*f)(x) = f(\tilde{s}\circ x)$.

Let now $\mathfrak{p}$ be an infinite prime of k and let $s_{\mathfrak{p}}: k \to \mathbb{C}$ be an imbedding defining $\mathfrak{p}$. Then

$$H^1_{\mathcal{D}}(M_{\mathbb{R}}, \mathbb{R}(n)) = p^*([\mathbb{R}(n-1)^{X(\mathbb{C})}]^+ \otimes E) = p^*[E^{X(\mathbb{C})} \otimes \mathbb{R}(n-1)]^+ =$$

$$\bigoplus_{\substack{\mathfrak{p}|\infty \\ \text{real}}} p^*[E^{X(\mathbb{C})_{s_{\mathfrak{p}}}} \otimes \mathbb{R}(n-1)]^+ \oplus \bigoplus_{\substack{\mathfrak{p}|\infty \\ \text{complex}}} p^*[(E^{X(\mathbb{C})_{s_{\mathfrak{p}}}} \oplus E^{X(\mathbb{C})_{\bar{s}_{\mathfrak{p}}}}) \otimes \mathbb{R}(n-1)]^+ .$$

Let $\mathfrak{p}$ be real and let $s = s_{\mathfrak{p}}$ and consider the commutative diagram

$$\begin{array}{ccc} E^{X(\mathbb{C})_s} \otimes \mathbb{R}(n-1) & \xrightarrow{\tilde{s}^* \otimes 1/(2\pi i)^{n-1}} & E^{X(\bar{k})} \otimes \mathbb{R} \\ \downarrow p^* & & \downarrow p^* \\ E^{X(\mathbb{C})_s} \otimes \mathbb{R}(n-1) & \xrightarrow{\tilde{s}^* \otimes 1/(2\pi i)^{n-1}} & E^{X(\bar{k})} \otimes \mathbb{R} . \end{array}$$

Since complex conjugation on $X(\mathbb{C})_s$ corresponds to the action of the non-trivial element $\varphi_{\mathfrak{p}}$ in $\Gamma_{\mathfrak{p}}$ on $X(\bar{k})$ under the map $\tilde{s}_o : X(\bar{k}) \to X(\mathbb{C})_s$, we see that the complex conjugation on the left side of the diagram corresponds to the action of $(-1)^{n-1}\varphi_{\mathfrak{p}}$ on the right side. Regarding that the two maps p^* are compatible with these actions, and that $p^*E^{X(\bar{k})} = H(M)$ as $\Gamma_{\mathfrak{p}}$-modules we obtain

$$p^*[E^{X(\mathbb{C})_s} \otimes \mathbb{R}(n-1)]^+ = (p^*E^{X(\bar{k})})^{(-1)^{n-1}} \otimes \mathbb{R} = \begin{cases} H(M)^{\Gamma_{\mathfrak{p}}} \otimes \mathbb{R}, & n \text{ odd} \\ H(M)/H(M)^{\Gamma_{\mathfrak{p}}} \otimes \mathbb{R}, & n \text{ even} \end{cases}$$

where the exponent $(-1)^{n-1}$ denotes the eigenspace of $\varphi_{\mathfrak{p}}$ with eigenvalue $(-1)^{n-1}$.

Let now $\mathfrak{p}$ be a complex prime and let $s = s_{\mathfrak{p}}$, $\bar{s}: k \to \mathbb{C}$ be the pair of conjugate imbeddings defining $\mathfrak{p}$. Consider the homomorphism

$$(E^{X(\mathbb{C})_s} \oplus E^{X(\mathbb{C})_{\bar{s}}}) \otimes \mathbb{R}(n-1) \xrightarrow{s^* \otimes 1/(2\pi i)^{n-1}} E^{X(\bar{k})} \otimes \mathbb{R}$$

given by

$$s^*(f + g) = \tfrac{1}{2}(\tilde{s}^* f + (-1)^{n-1}\tilde{\bar{s}}^* g) .$$

The fixed module of complex conjugation on the left side is given by the elements $\frac{1}{2}(f + \bar{f})$, where $f \in E^{X(\mathbb{C})_s}$ and $\bar{f}(x) = f(\bar{x})$. Since $s^*(\frac{1}{2}(f + \bar{f})) = \tilde{s}^* f = f \circ \tilde{s}$, we see that the restriction of the map $s^* \otimes 1/(2\pi i)^{n-1}$ to this fixed module yields an isomorphism

$$[(E^{X(\mathbb{C})_s} \oplus E^{X(\mathbb{C})_{\bar{s}}}) \otimes \mathbb{R}(n-1)]^+ \cong E^{X(\bar{k})} \otimes \mathbb{R}.$$

Applying as above the maps p^* we obtain

$$p^*[(E^{X(\mathbb{C})_s} \oplus E^{X(\mathbb{C})_{\bar{s}}}) \otimes \mathbb{R}(n-1)]^+ \cong (p^*E^{X(\bar{k})}) \otimes \mathbb{R} = H(M) \otimes \mathbb{R}.$$

Inserting these results into the above direct composition of $H^1_{\mathcal{D}}(M_{\mathbb{R}}, \mathbb{R}(n))$ we obtain the assertion of the proposition. □

Comparing proposition (3.1) with the results on the pole order of the L-function $L(M,s)$ at $s = 1-n$, and applying theorem (2.2) we obtain the second part of Beilinson's conjectures for arbitrary Artin motives:

<u>(3.2) Theorem:</u> For every $n \geq 1$ we have

$$\mathrm{ord}_{s=1-n} L(M,s) = \dim_E H^1_A(M_{\mathbb{Z}}, \mathbb{Q}(n)).$$

We now consider the first non-vanishing coefficient a_o in the Taylor series expansion

$$L(M,s) = a_o(s - (1-n))^{d_{n-1}(M)} + a_1(s - (1-n))^{d_{n-1}(M)+1} + \ldots$$

at the point $s = 1-n$. This coefficient is an element of $(\mathbb{C} \otimes E)^* = \prod_\sigma \mathbb{C}^*$, where $\sigma \in \mathrm{Hom}(E,\mathbb{C})$. We denote it by

$$L(M,1-n)^* = \lim_{s\to 1-n} \frac{L(M,s)}{(s-(1-n))^{d_{n-1}(M)}}$$

and call it briefly the "value" of $L(M,s)$ at $s = 1-n$. For this value we now have, as a special case of the last part of Beilinson's general conjecture:

<u>Conjecture</u> (Gross): For every $n \geq 1$ we have in $(\mathbb{C} \otimes E)^*/E^*$ 5) ,

$$c_{\check{M}}(1-n) = L(M,1-n)^* \bmod E^*$$

where $\check{M}$ is the dual motive of M .

For the <u>trivial</u> motive, i.e. for the Dedekind zeta function $\zeta(k,s)$, this conjecture has been proven by A. Borel (see [5]).

When $H^1_{\mathcal{D}}(M_{\mathbb{R}}, \mathbb{R}(n)) = 0$, then $L(M,1-n) \neq 0$ by theorem (3.2) and the conjecture is a special case of Deligne's conjecture [8] on the "critical" values of motivic L-series. Here it says that

$$L(M,1-n) \in E^* \subset (\mathbb{C} \otimes E)^* ,$$

and this is in fact true by the results of Siegel [22]. For these algebraic values we have further going conjectures and results. For example, let $G = \mathrm{Gal}(\mathbb{Q}(\mu_\ell)|\mathbb{Q})$ be the galois group of the ℓ-th cyclotomic field. Let $\theta : G \xrightarrow{\sim} (\mathbb{Z}/\ell)^*$ be the canonical isomorphism and let $M^{(i)}$ be the Artin motive with character

$$\theta^i : G \to (\mathbb{Z}/\ell)^* \hookrightarrow \mathbb{Q}(\mu_{\ell-1}) \subset \mathbb{Q}_\ell , \text{ i even.}$$

For the values $L(M^{(i)},1-n)$ with $n \equiv i \bmod (\ell-1)$ we then obtain an interpretation as multiplicative Euler characteristics in ℓ-adic cohomology (cf [1])

$$|L(M^{(i)},1-n)|_\ell = \prod_j \# H^j_!(M^{(i)}, \mathbb{Z}_\ell(n))^{(-1)^{j+1}} .$$

S. Lichtenbaum has proposed conjectures that predict formulas

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5) We remark, that the imbedding $E \xrightarrow{1 \otimes id} \mathbb{C} \otimes E$ becomes the map $a \mapsto (\ldots, \sigma a, \ldots)_{\sigma \in \mathrm{Hom}(E,\mathbb{C})}$ after the identification $\mathbb{C} \otimes E = \prod_\sigma \mathbb{C}$.

of this type in a most general frame (see [16]).

Of particular interest is the case $s = 0$, for which the conjecture goes back to H. Stark [25] . When M is the trivial motive $ESpec(k)$ then $L(M,s)$ is the Dedekind zeta function $\zeta(k,s)$ of k . At $s = 1$ it has a simple pole with residue

$$\zeta(k,1)^* = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d_k|}} \frac{hR}{e} ,$$

where h is the class number, e the number of roots of unity in k and R the <u>regulator</u> of k . This is the classical <u>class number formula</u>. Transforming this formula from $s = 1$ to $s = 0$ via the functional equation we get

$$\zeta(k,0)^* = -\frac{h}{e}R \equiv R \bmod \mathbb{Q}^* = R(r_{\mathcal{D}}).$$

So for the trivial motive and $s = 0$ our conjecture is a direct consequence of the class number formula. If the character of $H(M)$ has rational values, then the Stark conjecture has been proven by Tate (see [25]). There is no proof, however, for an arbitrary Artin motive.

The main purpose of this exposition is to present Beilinson's proof of the Gross conjecture for the Dirichlet L-series at the points $s = 1-n < 0$. The method does not apply to the case $s = 0$ for which we have fortunately a classical proof of Dirichlet (see [25]).

§4. Dirichlet L-series

Let $\chi: \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ be a Dirichlet character, and let f be the conductor, so that χ comes from a <u>primitive</u> character χ' mod f . The link between the Dirichlet L-series

$$L(\chi,s) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s}\ , \quad \mathrm{Re}(s) > 1\ ,$$

and the regulators is given by the polylogarithm function

$$L_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}\ ,$$

which we have introduced in §1. The function

$$\ell_s(x) = L_s(e^{2\pi i x})$$

extends for $s \neq 1$ from $(0,1)$ to a continous function on $\mathbb{R}/\mathbb{Z}$ (see [18] Lemma 7) and we set

$$\ell(\chi,s) = \sum_{a\in\mathbb{Z}/N} \chi(a)\ell_s(\tfrac{a}{N})\ .$$

For this function we have the following (see [15], CH. I. §2)

<u>(4.1) Lemma:</u> For all $s \in \mathbb{C}-\{1\}$ we have

$$N^{s-1}\ell(\chi,s) = \prod_{\substack{p\mid N\\ p\nmid f}} (1-\chi(p)p^{s-1}) f^{s-1}\ell(\chi',s)\ .$$

Proof: By a straightforward power series computation one proves, that for $m > 0$,

$$L_s(z) = m^{s-1} \sum_{w^m=z} L_s(w)\ .$$

For the function $\ell_s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ this implies

$$(*) \qquad \ell_s(x) = m^{s-1} \sum_{\nu=0}^{m-1} \ell_s(\tfrac{x+\nu}{m})\ .$$

(i.e. $\ell_s(x)$ is a distribution of weight $s-1$). If f and N have the same prime factors, then the function $\chi : \mathbb{Z}/N \to \mathbb{C}$ factors through the function $\chi' : \mathbb{Z}/f \to \mathbb{C}$ and our formula is a direct consequence of $(*)$.

It will now suffice to prove the formula for the case $N = pf$, $(p,f) = 1$. We have

$$N^{s-1}\ell(\chi,s) = N^{s-1} \sum_{a\in\mathbb{Z}/N} \chi(a)\ell_s(\tfrac{a}{N})$$

$$= N^{s-1} \sum_{b\in\mathbb{Z}/f} \sum_{\substack{x\in\mathbb{Z}/N\\ x\equiv b(f)}} \chi(x)\ell_s(\tfrac{x}{N})\ .$$

Now $\chi(x) = \chi'(b)$ for $x \equiv b \bmod f$ and $(x,p) = 1$, whereas $\chi(a) = 0$ if $a = pc$ for some c. Hence

$$N^{s-1}\ell(\chi,s) = \sum_{b\in\mathbb{Z}/f} \chi'(b) N^{s-1} \sum_{\substack{x\in\mathbb{Z}/N \\ x\equiv b(f)}} \ell_s(\tfrac{x}{N}) - N^{s-1} \sum_{\substack{a\in\mathbb{Z}/N \\ a=pc}} \chi'(a)\ell_s(\tfrac{a}{N}).$$

Applying (*) for $m = p$ to the first sum and making in the second the change of variables $a = pc$ where $c \in \mathbb{Z}/f$ we obtain

$$\begin{aligned} N^{s-1}\ell(\chi,s) &= \sum_{b\in\mathbb{Z}/f} \chi'(b) f^{s-1}\ell_s(\tfrac{b}{f}) - \chi'(p)p^{s-1}f^{s-1} \sum_{c\in\mathbb{Z}/f} \chi'(c)\ell_s(\tfrac{c}{f}) \\ &= (1 - \chi'(p)p^{s-1})f^{s-1}\ell(\chi',s) \quad , \quad \text{q.e.d.} \end{aligned}$$

□

The Dirichlet L-series may now be expressed by the polylogarithm as follows.

<u>(4.2) Proposition:</u> For all $s \in \mathbb{C}-\{1\}$ we have

$$L(\bar{\chi},s) = g(\chi,s)\ell(\chi,s) \quad ,$$

where

$$g(\chi,s) = \frac{1}{\tau(\chi)}\left(\frac{N}{f}\right)^{s-1} \prod_{\substack{p\mid N \\ p\nmid f}} \frac{1-\bar{\chi}'(p)p^{-s}}{1-\chi'(p)p^{s-1}} \quad ,$$

$\tau(\chi)$ being the Gauss sum

$$\tau(\chi) = \sum_{a=0}^{f-1} \chi(a)e^{2\pi i a/f} \quad .$$

Proof: Since both sides are holomorphic in s, it will suffice to consider the case $\mathrm{Re}(s) > 1$. We first assume that χ is primitive, i.e. $f = N$. Then, for $k \geq 1$,

$$\sum_{a\in\mathbb{Z}/N} \chi(a)e^{2\pi i ak/N} = \tau(\chi)\bar{\chi}(k).$$

If $(k,N) = 1$, this follows from $\chi(a) = \chi(ak)\bar{\chi}(k)$, and if $(k,N) \neq 1$, then both sides are zero. Now dividing both sides by k^s and summing over $k \geq 1$, we get

$$\sum_{a\in\mathbb{Z}/N} \chi(a) L_s(e^{2\pi i a/N}) = \tau(\chi)L(\bar{\chi},s).$$

Since $L_s(e^{2\pi i a/N}) = \ell_s(\frac{a}{N})$, this is the desired formula for the case that χ is primitive. In the general case we have

$$L(\bar{\chi},s) = \prod_{\substack{p|N \\ p\nmid f}} (1-\bar{\chi}'(p)p^{-s})L(\bar{\chi}',s)$$
$$= \frac{1}{\tau(\chi)} \prod_{\substack{p|N \\ p\nmid f}} (1-\bar{\chi}'(p)p^{-s})\ell(\chi',s).$$

The proposition follows now from (4.1). □

The generalized <u>Bernoulli numbers</u> $B_{n,\chi}$ are defined by the expansion

$$\sum_{a=1}^{f} \chi(a) \frac{ze^{az}}{e^{fz}-1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{z^n}{n!}.$$

They are connected with the ordinary Bernoulli numbers $B_n \in \mathbb{Q}$ by

$$B_{n,\chi} = \frac{1}{f} \sum_{a=1}^{f} \chi(a) f^n B_n(\tfrac{a-f}{f}),$$

where $B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i}$ is the n-th Bernoulli polynomial (cf [13]). For the values of the L-series $L(\chi,s)$ at the negative integers we now obtain:

<u>(4.3) Theorem:</u> For $n > 1$ we have:

(i) If $\chi(-1) \neq (-1)^{n-1}$, then

$$L(\chi,1-n) = -\frac{B_{n,\chi}}{n} \neq 0.$$

(ii) If $\chi(-1) = (-1)^{n-1}$, then $L(\chi,1-n) = 0$ and

$$L'(\chi,1-n) = \frac{a(\chi)}{(2\pi i)^{n-1}} \ell(\chi,n) \neq 0,$$

where

$$a(\chi) = \frac{(n-1)!}{2} N^{n-1} \prod_{\substack{p|N \\ p\nmid f}} \frac{1-\bar{\chi}'(p)p^{-n}}{1-\chi'(p)p^{n-1}}.$$

Proof: It is well known (cf [13]) that

$$L(\chi,1-n) = -\frac{B_{n,\chi}}{n}$$

for every $n > 1$, and $B_{n,\chi} \neq 0$ precisely, when $\chi(-1) \neq (-1)^{n-1}$. So assume $\chi(-1) = (-1)^{n-1}$. The functional equation for the Dirichlet L-series may be written

$$L(\chi,s) = \frac{\tau(\chi)}{2i^{\delta}} (\frac{2\pi}{f})^s \frac{L(\bar{\chi},1-s)}{\Gamma(s)\cos\frac{\pi}{2}(s-\delta)}$$

with $\delta = 0$ or 1 according $\chi(-1) = 1$ or -1 (cf [13], p. 104). Since $\chi(-1) = (-1)^{n-1}$, $n-\delta$ must be odd and therefore

$$\cos \pi \frac{1-n-\delta}{2} = (-1)^{(1-n-\delta)/2} = i^{1-n-\delta}.$$

$\Gamma(s)$ has a simple pole at $s = 1-n$ with residue $\frac{(-1)^{n-1}}{(n-1)!}$, and $L(\bar{\chi},n) \neq 0,\infty$ as is well known. Expanding now each term in the functional equation in a Laurent series at $s = 1-n$, we see that

$$L'(\chi,1-n) = \frac{\tau(\chi)}{2i^{\delta}} \left(\frac{2\pi}{f}\right)^{1-n} \frac{(n-1)!}{(-1)^{n-1} i^{1-n-\delta}} L(\bar{\chi},n)$$

$$= \frac{\tau(\chi)}{2} \frac{(n-1)!}{(2\pi i)^{n-1}} f^{n-1} L(\bar{\chi},n) \neq 0 .$$

Inserting the result of proposition (4.2) for $s = n$ we obtain the desired formula. □

We now consider a character $\chi: (\mathbb{Z}/N)^* \to E^*$ that takes values in the multiplicative group of a finite algebraic number field E and extend it by 0 to $\mathbb{Z}/N$. We associate to χ an Artin motive over $\mathbb{Q}$ with values in E as follows.

Let $F = \mathbb{Q}(\mu_N)$, μ_N the group of N-th roots of unity, and let $G = \mathrm{Gal}(F|\mathbb{Q})$. The right action of G on $X = \mathrm{Spec}(F)$ induces a left action on

$$H(X) = H^0(\bar{X},\mathbb{Q}) = \mathbb{Q}^{\pi_0(\bar{X})} \quad \text{and } H(EX) = H(X)\otimes E = E^{\pi_0(\bar{X})},$$

thus an identification

$$E[G] = \mathrm{End}_{E,\Gamma}(H(EX)).$$

Via the canonical isomorphism $G \cong (\mathbb{Z}/N)^*$ we may view χ as a character of G, and we consider in $E[G]$ the element

$$e_\chi = \sum_{\tau\in G} \chi^{-1}(\tau)\tau ,$$

which acts on $H(EX) = \{f: \pi_0(\bar{X}) \to E\}$ by

$$(e_\chi f)(\bar{x}) = \sum_{\tau\in G} \chi^{-1}(\tau) f(\bar{x}\tau) .$$

$p_\chi = \frac{1}{\# G} e_\chi$ is an idempotent in $\mathrm{End}_{E,\Gamma}(H(EX))$ and therefore defines an Artin motive

$$M = (X,p_\chi)$$

over $\mathbb{Q}$ with coefficients in E. We denote this motive by $[\chi]$ and call it a <u>Dirichlet motive</u>.

We assume $F \subseteq \bar{\mathbb{Q}}$. We then have a homomorphism $\Gamma = \mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q}) \overset{\pi}{\to} G$ and a distinguished point $\bar{x} \in \mathrm{Spec}(F \otimes_{\mathbb{Q}} \bar{\mathbb{Q}})$ (the kernel of $F \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}$, $a \otimes b \mapsto ab$) by which we get a bijection $G \tilde{\to} \pi_0(\bar{X})$, $\tau \mapsto \bar{x}\tau$, of right G-sets. Associating to a function $f: \pi_0(\bar{X}) \to E$ the element $\sum_\tau f(\bar{x}\tau^{-1})\tau \in E[G]$ we obtain an isomorphism

$$H(EX) = E^{\pi_0(\bar{X})} \cong E[G] .$$

This is an isomorphism of left Γ-modules, if we let $\sigma \in \Gamma$ act on $E[G]$ as multiplication by $\pi(\sigma)$ from the left. Therefore the endomorphism p_χ of $H(EX)$ becomes after the identification with $E[G]$ just multiplication of $E[G]$ by the element $\frac{1}{\#G} \sum_\tau \chi^{-1}(\tau)\tau = \frac{1}{\#G} e_\chi \in E[G]$. The associated representation of Γ is the 1-dimensional subspace

$$e_\chi E[G] = E e_\chi$$

of $E[G]$ on which Γ acts via the character χ. So the representation associated with the Dirichlet motive $[\chi]$ is $\Gamma \to G \overset{\chi}{\to} E^*$.

We now consider the L-series

$$L([\chi],s) = (\ldots, L(\chi^\sigma,s),\ldots)_{\sigma \in \mathrm{Hom}(E,\mathbb{C})}$$

of the Dirichlet motive. On the other hand we set

$$\ell([\chi],s) = \sum_{a \in \mathbb{Z}/N} \ell_s(\tfrac{a}{N}) \otimes \chi(a) \in \mathbb{C} \otimes E,$$

$$B_{n,[\chi]} = \frac{1}{f} \sum_{a=1}^{f} \chi(a) f^n B_n(\tfrac{a-f}{f}) \in E ,$$

$$a([\chi]) = \frac{(n-1)!}{2} N^{n-1} \prod_{\substack{p|N \\ p \nmid f}} \frac{1-\bar{\chi}'(p)^{-1}p^{-n}}{1-\chi'(p)p^{n-1}} \in E.$$

Here $B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i} \in \mathbb{Q}[x]$ is the n-th Bernoulli polynomial, f the conductor of χ, and χ' the associated primitive character.

The homomorphisms $\sigma_*: \mathbb{C} \otimes_{\mathbb{Q}} E \to \mathbb{C}$, $z \otimes \varepsilon \mapsto z\sigma\varepsilon$, $\sigma \in \mathrm{Hom}(E,\mathbb{C})$, yield the identification

$$\mathbb{C} \otimes E = \prod_\sigma \mathbb{C},$$

by which the subspace $E = 1 \otimes E \subseteq \mathbb{C} \otimes E$ becomes the image of $E \to \prod \mathbb{C}$, $\varepsilon \mapsto (\ldots,\sigma\varepsilon,\ldots)$. After this identification we have with the notions of theorem (4.3)

$$\ell([\chi],s) = (\ldots,\ell(\chi^{\sigma},s),\ldots)_{\sigma\in \mathrm{Hom}(E,\mathbb{C})} \ ,$$
$$B_{n,[\chi]} = (\ldots,B_{n,\chi^{\sigma}},\ldots\ldots)_{\sigma\in \mathrm{Hom}(E,\mathbb{C})} \ ,$$
$$a([\chi]) = (\ldots,a(\chi^{\sigma}),\ldots\ldots)_{\sigma\in \mathrm{Hom}(E,\mathbb{C})} \ .$$

We may therefore reformulate theorem (4.3) as follows

<u>(4.4) Theorem:</u> Let $n > 1$. Then in $\mathbb{C} \otimes E$ we have

(i) $L([\chi],1-n) = -\frac{B_{n,[\chi]}}{n} \neq 0$ if $\chi(-1) \neq (-1)^{n-1}$

(ii) If $\chi(-1) = (-1)^{n-1}$, then $L([\chi],1-n) = 0$ and

$$L'([\chi],1-n) = \frac{a([\chi])}{(2\pi i)^{n-1}}\ell([\chi],n) \neq 0$$

with $a([\chi]) \in E \subseteq E \otimes \mathbb{C}$.

§5. Regulators of Dirichlet Motives

Let again $F = \mathbb{Q}(\mu_N)$, where now $N > 1$. Let $G = \mathrm{Gal}(F|\mathbb{Q}), X = \mathrm{Spec}(F)$, $e_{\chi} = \sum_{\tau\in G} \chi^{-1}(\tau)\tau \in E[G]$ and

$$\ell([\chi],s) = \sum_{a\in \mathbb{Z}/N} \ell_s(\tfrac{a}{N}) \otimes \chi(a) \bmod E^* .$$

In the preceding paragraph we have explicitely computed the values of the L-series $L([\chi],s)$ at $s = 1-n$ of the Dirichlet motive $[\chi] = ([\chi],\frac{1}{\#G} e_{\chi})$. By means of theorem (1.4) we now determine the regulators $c_{[\chi]}(1-n)$.

<u>(5.1) Theorem:</u> The n-th regulator $c_{[\chi]}(1-n) \in (\mathbb{R} \otimes E)^*/E^*$, $n > 1$, of the Dirichlet motive $[\chi]$ is given by

(i) $\quad c_{[\chi]}(1-n) = 1 \qquad$ if $\chi(-1) \neq (-1)^{n-1}$

(ii) $\quad c_{[\chi]}(1-n) = \frac{1}{(2\pi i)^{n-1}} \ell([\chi^{-1}],n) \bmod E^*$, if $\chi(-1) = (-1)^{n-1}$.

Proof: We first determine explicitely the E-structure of

$$H^1_{\mathcal{D}}([\chi]_{\mathbb{R}}, \mathbb{R}(n)) = e^*_\chi (H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)) \otimes E).$$

Since G acts on X and $X(\mathbb{C})$ from the right, the elements $\tau \in G$ act on

$$H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)) \otimes E = [\mathbb{C}/\mathbb{R}(n)^{X(\mathbb{C})} \otimes E]^+ = [\mathbb{C}/\mathbb{R}(n) \otimes E^{X(\mathbb{C})}]^+$$

by τ^* from the left (see §1, p. 4). Directly from the definition (see §2) we see that the endomorphism

$$e^*_\chi : H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)) \otimes E \to H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)) \otimes E$$

is given by

$$e^*_\chi = \sum_{\tau \in G} \tau^* \otimes \chi^{-1}(\tau) .$$

We choose a primitive N-th root of unity ζ and we let

$$\alpha: F \to \mathbb{C}$$

be the imbedding determined by $\alpha\zeta = e^{2\pi i/N}$. We may then consider the bijection of G-sets $G \to X(\mathbb{C})$, $\tau \mapsto \alpha \circ \tau$, and the induced isomorphism of left $E[G]$ modules

$$\alpha^*: E^{X(\mathbb{C})} \xrightarrow{\sim} E[G] \quad , \quad (f: X(\mathbb{C}) \to E) \mapsto \sum_\tau f(\alpha\tau^{-1})\tau .$$

Complex conjugation on $X(\mathbb{C})$ corresponds then to multiplication with $c \in G$, $c: \zeta \to \zeta^{-1}$, the action of τ^* on $E^{X(\mathbb{C})}$ becomes multiplication by τ and the operator e^*_χ on $E^{X(\mathbb{C})}$ becomes multiplication by $e_\chi = \Sigma\chi^{-1}(\tau)\tau \in E[G]$. We have

$$e_\chi E[G] = E e_\chi$$

and

$$e_\chi \tau = \tau e_\chi = \chi(a) e_\chi$$

if $a \in (\mathbb{Z}/N)^*$ corresponds to $\tau \in G$ under $(\mathbb{Z}/N)^* \cong G$.
In particular

$$e_\chi c = c e_\chi = \chi(-1) e_\chi \quad .$$

By the above identifications we obtain a commutative diagram

$$\begin{array}{ccccc} H^1_A(X,\mathbb{Q}(n))\otimes E & \xrightarrow{r_{\mathcal{D}}} & H^1_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(n))\otimes E & \underset{\sim}{\xrightarrow{\alpha^*}} & [\mathbb{C}/\mathbb{R}(n)\otimes E[G]]^+ \\ \downarrow e^*_\chi & & \downarrow e^*_\chi & & \downarrow 1\otimes e_\chi \\ H^1_A([\chi],\mathbb{Q}(n)) & \xrightarrow{r_{\mathcal{D}}} & H^1_{\mathcal{D}}([\chi]_{\mathbb{R}},\mathbb{R}(n)) & \underset{\sim}{\xrightarrow{\alpha^*}} & [\mathbb{C}/\mathbb{R}(n)\otimes E\, e_\chi]^+ . \end{array}$$

The action of complex conjugation on $\mathbb{C}/\mathbb{R}(n)\otimes E\, e_\chi$ is multiplication by $(-1)^{n-1}\chi(-1)$, since it is multiplication by $(-1)^{n-1}$ on $\mathbb{C}/\mathbb{R}(n) \cong \mathbb{R}(n-1)$ and by $\chi(-1)$ on $E\, e_\chi$. Therefore $[\mathbb{C}/\mathbb{R}(n)\otimes E\, e_\chi]^+ = 0$ and thus

$$c_{[\chi]}(1-n) = 1 \quad , \quad \text{if} \quad \chi(-1) \neq (-1)^{n-1} .$$

Assume $\chi(-1) = (-1)^{n-1}$. Then the E-structure of

$$[\mathbb{C}/\mathbb{R}(n)\otimes E\, e_\chi]^+ = \mathbb{C}/\mathbb{R}(n)\otimes E\, e_\chi = \mathbb{R}(n-1)\otimes E\, e_\chi$$

is the 1-dimensional E-vectorspace $(2\pi i)^{n-1}\mathbb{Q}\otimes E\, e_\chi$ with basis $\omega_\chi = (2\pi i)^{n-1}\otimes e_\chi$. (Working with $\mathbb{C}/\mathbb{R}(n)$ in place of $\mathbb{R}(n-1)$ we have to interpret $2\pi i$ as $2\pi i \bmod \mathbb{R}(n)$). In order to determine the regulator $c_{[\chi]}(1-n)$ we must choose a basis element η_χ of the 1-dimensional E-structure $H^1_A([\chi],\mathbb{Q}(n))$ of $H^1_A([\chi],\mathbb{Q}(n))\otimes\mathbb{R}$. If after this choice

$$\alpha^* r_{\mathcal{D}}(\eta_\chi) = a\omega_\chi \quad , \quad a \in (\mathbb{R}\otimes E)^* ,$$

then $c_{[\chi]}(1-n) = a \bmod E^*$. Now theorem (1.4) yields the element $\varepsilon_n(\zeta) \in H^1_A(X,\mathbb{Q}(n))$ and we set

$$\eta_\chi = e^*_\chi(\varepsilon_n(\zeta)) \in H^1_A([\chi],\mathbb{Q}(n)) .$$

(Note that N is assumed to be >1 , so that $\zeta \in \mu_N - \{1\}$).
By (1.4) we have

$$r_{\mathcal{D}}(\varepsilon_n(\zeta)) = L_n(\zeta) = (\ \ldots, L_n(\iota\zeta),\ldots)_{\iota: F\to\mathbb{C}} \in (\mathbb{C}/\mathbb{R}(n))^{X(\mathbb{C})} .$$

From this we obtain the desired result

$$c_{[\chi]}(1-n) = \frac{1}{(2\pi i)^{n-1}}\, \ell([\chi^{-1}],n) \bmod E^* ,$$

once we have shown that $\eta_\chi \neq 0$. For this it suffices to show that

$$\ell([\chi^{-1}],n) = \sum_{a\in\mathbb{Z}/N} L_n(e^{2\pi i a})\otimes\chi^{-1}(a)$$

is $\neq 0$ considered as an element of $\mathbb{C}/\mathbb{R}(n) \otimes E$. By (4.4) we have, that $\ell([\chi^{-1}],n) \neq 0$ as an element in $\mathbb{C} \otimes E$. Therefore it suffices to show that $\ell([\chi^{-1}],n) \in \mathbb{R}(n-1) \otimes E \subseteq \mathbb{C} \otimes E$. Now for each summand $L_n(e^{2\pi ia/N}) \otimes \chi^{-1}(a)$ we have

$$L_n(e^{2\pi ia/N}) \otimes \chi^{-1}(a) + L_n(e^{2\pi i(-a)/N}) \otimes \chi^{-1}(-a) =$$

$$L_n(e^{2\pi ia/N}) \otimes \chi^{-1}(a) + \chi^{-1}(-1)L_n(e^{2\pi i(-a)/N}) \otimes \chi^{-1}(a) =$$

$$(L_n(e^{2\pi ia/N}) \oplus \chi^{-1}(-1)\overline{L_n(e^{2\pi ia/N})}) \otimes \chi^{-1}(a) .$$

For the term $L = L_n(e^{2\pi ia/N}) \oplus \chi^{-1}(-1)\overline{L_n(e^{2\pi ia/N})}$ we have $\bar{L} = \chi(-1)L = (-1)^{n-1}L$ and hence $L \subseteq \mathbb{R}(n-1) \subseteq \mathbb{C}$. From this follows $\ell([\chi^{-1}],n) \in \mathbb{R}(n-1) \otimes E \subseteq \mathbb{C} \otimes E$. □

Regarding that $[\chi^{-1}]$ is the dual motive $[\check{\chi}]$ of $[\chi]$, we obtain from the above theorem and from proposition (4.4) the result, that the Gross conjecture (which is the third part of Beilinson's conjectures) holds true for Dirichlet motives, if we take into account, that the case $n = 1$ and the case $\chi = 1$ has been established by Dirichlet and Borel:

<u>(5.2) Theorem</u>: For every Dirichlet motive $[\chi]$ over $\mathbb{Q}$ with coefficients in E and every $n \geq 1$ we have

$$c_{[\check{\chi}]}(1-n) = L([\chi],1-n)^* \bmod E^* .$$