Introduction to the Beilinson conjectures

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One of the most beautiful formulas in classical algebraic number theory is the analytic class number formula: The Dedekind zeta function of an algebraic number field has a simple pole at $s = 1$ and its residue is given in terms of the class number and the unit regulator of that field. Remarkable about this result is the fact that on the one hand side the zeta function, for $Re(s) > 1$, is defined as a convergent Euler product given completely in terms of the local arithmetic of the field and then is shown to have a meromorphic continuation to the whole complex plane. On the other hand, the ideal class group and the group of units are genuine invariants of the global arithmetic of the field. Therefore the class number formula is some kind of a highly nontrivial local-to-global principle, or said differently: The zeta function transforms the relatively simple local information it is built out of into a rather deep knowledge about the global arithmetic.

Over the past decades it became increasingly clear that this is not a singular phenomenon but a manifestation of a general principle in arithmetic. The values or better the leading coefficients at integral arguments of the L-functions of algebraic varieties over number fields seem to be closely related to the global arithmetical geometry of these varieties (e.g., the conjecture of Birch and Swinnerton-Dyer). Beilinson, in [Bei 1], has developed a completely general conjectural formalism which connects the "transcendental" parts of that leading coefficients to so-called regulators. These regulators are sophisticated generalizations of the classical unit regulator defined by purely algebraic and geometrical means.

The purpose of these Notes is twofold. First and mainly we want to present Beilinson's extremely fascinating conjectures in a way which leads the reader to their statement as directly as possible. So we will explain most of the necessary formalism only up to the point which is needed for the statement of the conjectures. (The reader will find detailed treatments in the subsequent Chapters.) The exception to this is the theory of Chern classes. A basic ingredient in the definition of the Beilinson regulators is the construction of Chern class maps from the higher algebraic K-theory into the Deligne cohomology. Since this type of construction most probably will turn out to be important in similar situations also (e.g., p-adic L-functions) our second purpose is to explain the theory of Chern classes from higher K-theory into any reasonable cohomology theory in a rather formal and detailed way. Furthermore, a basic but for the understanding of the conjectures important result concerns the behaviour of the Chern classes with respect to the γ-filtration on K-theory. Since this, although well-known to the experts, seems not to be contained in the literature we will include a proof.

It need not to be emphasized, of course, that nothing in these Notes is original. They are the poor result of the author's attempt to understand these beautiful conjectures. Finally I want to thank the members of the Arbeitsgemeinschaft Heidelberg-Mannheim; in a common effort we managed to go through Beilinson's paper. I am also grateful to S. Kosarew for pointing out some stupidities in a first version of these Notes.

§1 Complex L-functions

Let us start by briefly looking at the "simplest" example of an L-function: The Riemann zeta function

$$\zeta(s) := \prod_p \frac{1}{1-p^{-s}}$$

is absolutely convergent (and nonzero) for $Re(s) > 1$. It has a meromorphic continuation and $\zeta(s) \cdot \Gamma(s/2) \cdot \pi^{-s/2}$ has a functional equation with respect to $s \mapsto 1-s$. Concerning the values at integral arguments we know that

$$\zeta(n) = \begin{cases} (2\pi\sqrt{-1})^n \bmod \mathbb{Q}^\times & \text{if } n > 1 \text{ is even ,} \\ ? & \text{if } n > 1 \text{ is odd} \end{cases}$$

and therefore, by the functional equation, that

$$\zeta(n) \quad \begin{cases} \in \mathbb{Q}^\times & \text{if } n < 0 \text{ is odd,} \\ \text{simple zero,} & \text{if } n < 0 \text{ is even.} \end{cases}$$

We make the following observations:

1) The zeros of $\zeta(s)$ in the region $\mathrm{Re}(s) < 0$ are completely determined by the Γ-factor in the functional equation.

2) The value $(2\pi\sqrt{-1})^n (\bmod \mathbb{Q}^\times)$ at an even $n > 1$ is a quite elementary example of a period. On the other hand, the unknown values $\zeta(n)$ for odd $n > 1$ are related in an obvious way to the leading coefficients of $\zeta(s)$ at $s = 1-n$. To compute them mod $\mathbb{Q}^\times$ therefore requires probably a much more sophisticated "period"-construction (usually called "regulator"). (In case of $\zeta(s)$ this was done by Borel in [Bor].)

3) In order to determine the multiplicity of $\zeta(s)$ at $s = 0$ and $s = 1$ one needs additional information (in this case that we have a simple pole at $s = 1$).

In the following we will see that these observations most likely reflect general principles in the theory of motivic complex L-functions. Since there is an excellent reference ([Ser] and also [Del]) for the construction and the expected analytic properties of these L-functions we only will give a short review here. In addition, we are going to work always over the rational numbers $\mathbb{Q}$ as base field. Because of the fact that the restriction of scalars does not change the L-functions this is not really a loss of generality but it simplifies the notation a lot. Our main aim in this Paragraph is to work out from the expected functional equation quite explicitly what our first observation above becomes in the general case. Although this will consist in elementary computations it is a useful exercise for becoming acquainted with some basic facts which any conjecture has to take into account; it also serves as a piece of motivation for the introduction of the Deligne cohomology in the next Paragraph.

Let $X_{/\mathbb{Q}}$ be a projective smooth variety over $\mathbb{Q}$. We fix an algebraic closure $\overline{\mathbb{Q}}/\mathbb{Q}$ and put $\overline{X} := X \underset{\mathbb{Q}}{\times} \overline{\mathbb{Q}}$. We also fix an integer i between 0 and $2\dim X$ and we denote by M (for motive) the family of all ith cohomology groups of X as there are

- the ℓ-adic cohomology $H^i(\overline{X}, \mathbb{Q}_\ell)$, for each prime number ℓ, viewed as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module,
- the de Rham cohomology $H^i_{DR}(X(\mathbb{C}))$ of the complex manifold $X(\mathbb{C})$, and
- the singular cohomology $H^i(X(\mathbb{C}), \mathbb{Q})$.

For each prime number p we choose a prime $\overline{p}$ of $\overline{\mathbb{Q}}$ above p and we let $D_{\overline{p}}$, resp. $I_{\overline{p}}$, denote the corresponding decomposition, resp. inertia, subgroup in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In the factor group $D_{\overline{p}}/I_{\overline{p}}$ we have the arithmetic Frobenius $\phi_{\overline{p}}$ as a distinguished element so that we can consider, for each $\ell \neq p$, the characteristic polynomial

$$P_p(T) := \det(1 - \phi_{\overline{p}}^{-1}T; H^i(\overline{X}, \mathbb{Q}_\ell)^{I_{\overline{p}}})$$

of the geometric Frobenius $\phi_{\overline{p}}^{-1}$ on the subspace of $I_{\overline{p}}$ -invariant elements in $H^i(\overline{X}, \mathbb{Q}_\ell)$. Obviously, this polynomial does not depend on the particular choice of $\overline{p}$ above p. Furthermore, it is conjectured ([Ser]) that the following hypothesis always is fulfilled.

Hypothesis:

(I) $P_p(T)$ has coefficients in $\mathbb{Z}$ which are independent of ℓ.

By Deligne's proof of the Weil conjectures this is known to be true if X has good reduction at p (which is the case for almost all p). Assuming (I) we then define the complex L-function of M by the Euler product

$$L(M, s) := \prod_p P_p(p^{-s})^{-1}$$

which converges absolutely for $\mathrm{Re}(s) >> 0$. Here is a list of some of the expected analytic properties of $L(M, s)$.

Hypothesis:

(II) The above Euler product converges absolutely for $\mathrm{Re}(s) > \frac{i}{2} + 1$ (and therefore does not vanish in this region);

(III) $L(M, s)$ has a meromorphic continuation to the whole complex plane; the only possible pole occurs at $s = \frac{i}{2} + 1$ for even i;

(IV) $L(M, \frac{i}{2}+1) \neq 0$;

(V) $L(M,s) \cdot L_\infty(M,s)\cdot$(exponential factor) has a functional equation with respect to $s \mapsto i+1-s$ (the precise definition of the archimedean Euler factor $L_\infty(M,s)$ is given below).

If $\Gamma(s)$ denotes the usual Γ-function we put

$$\Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \cdot \Gamma(s/2) \quad \text{and} \quad \Gamma_{\mathbb{C}}(s) := 2 \cdot (2\pi)^{-s} \cdot \Gamma(s) \ .$$

The Euler factor $L_\infty(M,s)$ is built out of these Γ-factors by a rule which involves the Hodge structure on the singular cohomology $H^i(X(\mathbb{C}),\mathbb{C})$. This Hodge structure consists in the Hodge decomposition (compare [GH])

$$H^i(X(\mathbb{C}),\mathbb{C}) = \bigoplus_{\substack{p+q=i \\ p,q\geq 0}} H^{pq}$$

together with the $\mathbb{C}$-linear involution F_∞ on $H^i(X(\mathbb{C}),\mathbb{C})$ induced by the complex conjugation on the manifold $X(\mathbb{C})$; one has

$$F_\infty(H^{pq}) = H^{qp} \ .$$

We put

$$h^{pq} := \dim_{\mathbb{C}} H^{pq} \quad \text{and} \quad h^{p\pm} := \dim_{\mathbb{C}} H^{p,\pm(-1)^p}$$

where $H^{pp} = H^{p+} \oplus H^{p-}$ is the decomposition into eigenspaces with respect to F_∞. Then $L_\infty(M,s)$ is defined to be

$$\prod_{\substack{p<q \\ p+q=i}} \Gamma_{\mathbb{C}}(s-p)^{h^{pq}} \qquad \text{if i is odd ,}$$

$$\text{-}\,"\,\text{-} \quad \cdot\ \Gamma_{\mathbb{R}}(s-\tfrac{i}{2})^{h^{\frac{i}{2}+}} \ \cdot\ \Gamma_{\mathbb{R}}(s-\tfrac{i}{2}+1)^{h^{\frac{i}{2}-}} \qquad \text{if i is even .}$$

In the following we always assume that the above Hypotheses I-V are fulfilled. Then it is clear that the location and multiplicity of the zeros of $L(M,s)$ in the region $\mathrm{Re}(s) < \frac{i}{2}$ are completely determined by the poles of the Euler factor $L_\infty(M,s)$. Since the Γ-function has (simple) poles precisely at the nonpositive integers $0,-1,-2,...$ those zero multiplicities must depend in an elementary way on the Hodge structure of M. In order to state the result in a convenient form we first recall that there is a canonical isomorphism

$$H^i(X(\mathbb{C}),\mathbb{C}) = H^i_{DR}(X(\mathbb{C}))$$

between singular and de Rham cohomology (compare [GH]) and that the de Rham filtration $F^\cdot H^i_{DR}$ on the right hand side is related to the Hodge decomposition on the left hand side by

$$F^p H^i_{DR}(X(\mathbb{C})) = \bigoplus_{p'\geq p} H^{p'q} \ .$$

Proposition:

The only poles of $L_\infty(M,s)$ occur at integer points $s = m \le \frac{i}{2}$ with multiplicity equal to

$$\dim_{\mathbb{C}} H^i(X(\mathbb{C}),\mathbb{C})^{(-1)^{n-1}} - \dim_{\mathbb{C}} F^n H^i_{DR}(X(\mathbb{C}))$$

where $n := i+1-m$ and the exponent (as always in the following) denotes the corresponding eigenspace with respect to F_∞.

Proof: The first assertion is obvious. Now, the multiplicity at $s = m \le \frac{i}{2}$ by definition is equal to

$$\sum_{m \le p < q} h^{pq} (+h^{\frac{i}{2},(-1)^{m-\frac{i}{2}}}) \quad \text{if i is odd (even)} \ .$$

On the other hand, because of $F_\infty(H^{pq}) = H^{qp}$, we have

$$\dim_{\mathbb{C}} H^i(X(\mathbb{C}),\mathbb{C})^{(-1)^{n-1}} = \sum_{p<q} h^{pq} (+h^{\frac{i}{2},(-1)^{\frac{i}{2}-m}})$$

and

$$\dim_{\mathbb{C}} F^n H^i_{DR}(X(\mathbb{C})) = \sum_{p \ge i+1-m} h^{pq} = \sum_{q \ge i+1-m} h^{pq} = \sum_{p \le m-1} h^{pq} \ .$$

Since $m-1 < \frac{i}{2}$ the relations $p \le m-1$ and $p+q = i$ imply $p < \frac{i}{2} < q$. We therefore get

$$\dim_{\mathbb{C}} F^n H^i_{DR}(X(\mathbb{C})) = \sum_{\substack{p \le m-1 \\ p<q}} h^{pq} \ .$$

If we put

$$\operatorname{ord}_{s=m} L(M,s) := \text{ multiplicity of } L(M,s) \text{ at } s = m$$

(poles are counted negatively) then our Hypotheses allow to reformulate the Proposition in the following way.

Corollary:

For any integer $m \le \frac{i}{2}$ and $n := i+1-m$ we have

$$\begin{aligned} &\dim_{\mathbb{C}} H^i(X(\mathbb{C}),\mathbb{C})^{(-1)^{n-1}} - \dim_{\mathbb{C}} F^n H^i_{DR}(X(\mathbb{C})) \\ &= \begin{cases} \operatorname{ord}_{s=m} L(M,s) & \text{if } m < \frac{i}{2} \ , \\ \operatorname{ord}_{s=m} L(M,s) - \operatorname{ord}_{s=m+1} L(M,s) & \text{if } m = \frac{i}{2} \ . \end{cases} \end{aligned}$$

Corollary:

For any integer $m \le \max(0, i - \dim X)$ such that $m \ne \frac{i}{2}$ we have

$$\operatorname{ord}_{s=m} L(M,s) = \dim_{\mathbb{C}} H^i(X(\mathbb{C}),\mathbb{C})^{(-1)^m} \quad ;$$

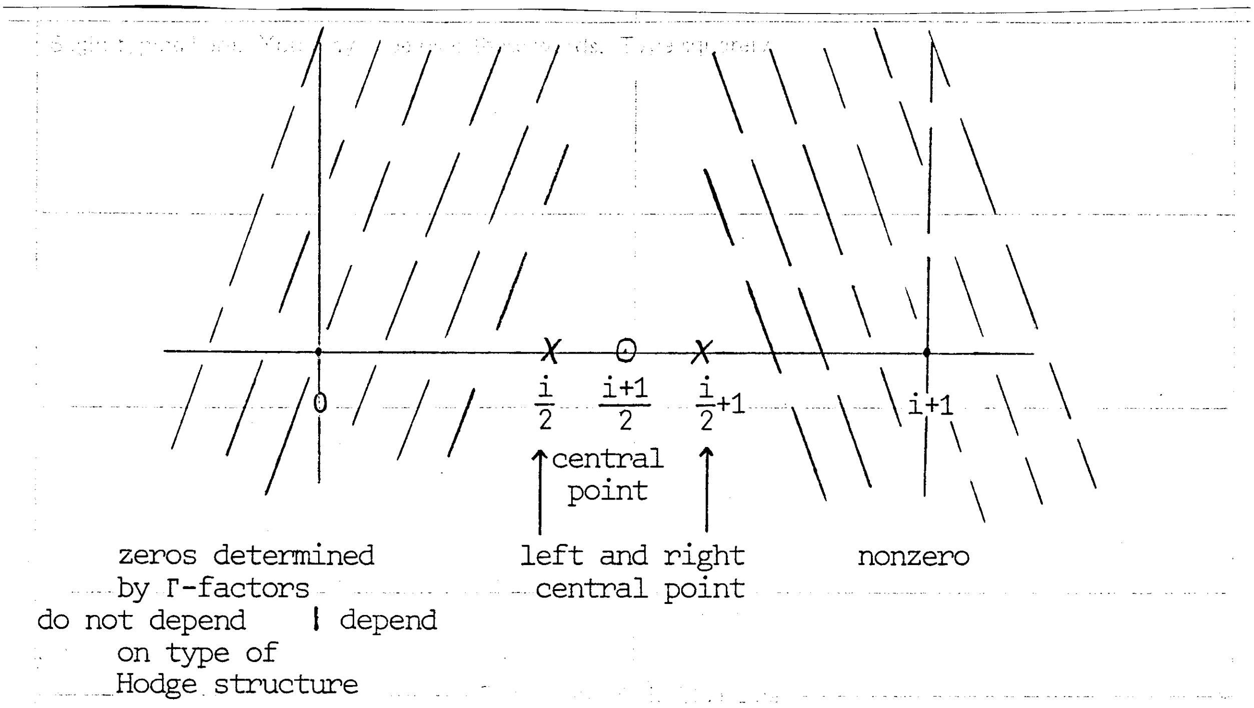
furthermore, for odd i the right hand side is equal to $\frac{1}{2} \cdot \dim_{\mathbb{C}} H^i(X(\mathbb{C}), \mathbb{C})$ and therefore is even independent of m.

Proof: By assumption we have $m < \frac{i}{2}$ and $n := i + 1 - m > \min(i, \dim X)$ so that

$$F^n H^i_{DR}(X(\mathbb{C})) = \bigoplus_{\substack{p \geq n \\ p+q=i}} H^q(X(\mathbb{C}), \Omega^p) = 0 \quad .$$

The above Corollary then implies

$$\begin{aligned} \operatorname{ord}_{s=m} L(M,s) &= \dim_{\mathbb{C}} H^i(X(\mathbb{C}), \mathbb{C})^{(-1)^{i-m}} \\ &= \dim_{\mathbb{C}} H^i(X(\mathbb{C}), \mathbb{C})^{(-1)^m} \quad . \end{aligned}$$



In the next Paragraph we will see that there is a more "natural" formula for $\operatorname{ord}_{s=m} L(M,s)$ and any integer $m < \frac{i}{2}$ which, like in the last Corollary, computes this multiplicity as the dimension of a certain cohomology group.

§2 Deligne cohomology

The de Rham cohomology $H^*_{DR}(X(\mathbb{C}))$, by definition, is the cohomology of the complex of sheaves of holomorphic differential forms

$$\Omega^\cdot : \mathcal{O}_{X(\mathbb{C})} \longrightarrow \Omega^1 \longrightarrow \Omega^2 \longrightarrow \ldots$$

on $X(\mathbb{C})$ and the de Rham filtration $F^\cdot H^*_{DR}$ is induced by the naive filtration of this complex:

$$F^p H^i_{DR}(X(\mathbb{C})) = \mathrm{image}(H^i(\Omega^\cdot_{\geq p}) \longrightarrow H^i(\Omega^\cdot)) \quad ;$$

here the map on the right hand side is derived from the first arrow in the short exact sequence of complexes

$$\begin{array}{lcccccccccccccc}
\Omega^\cdot_{\geq p} & : & & & & & & & 0 & \to & \Omega^p & \to & \Omega^{p+1} & \to & \ldots \\
\downarrow & & & & & & & & \downarrow & & \| & & \| & & \\
\Omega^\cdot & : & \mathcal{O}_{X(\mathbb{C})} & \to & \Omega^1 & \to & \ldots & \to & \Omega^{p-1} & \to & \Omega^p & \to & \Omega^{p+1} & \to & \ldots \\
\downarrow & & \| & & \| & & & & \| & & \downarrow & & & & \\
\Omega^\cdot_{<p} & : & \mathcal{O}_{X(\mathbb{C})} & \to & \Omega^1 & \to & \ldots & \to & \Omega^{p-1} & \to & 0 & & & . &
\end{array}$$

In addition, we conclude from the existence of the Hodge decomposition that the hypercohomology spectral sequence $H^j(X(\mathbb{C}), \Omega^i) \Longrightarrow H^{i+j}_{DR}(X(\mathbb{C}))$ degenerates. Consequently, the map

$$H^i(\Omega^\cdot_{\geq p}) \longrightarrow H^i(\Omega^\cdot)$$

is injective and we get

$$H^i(\Omega^\cdot_{<p}) = H^i_{DR}(X(\mathbb{C}))/F^p \quad .$$

The real <u>Deligne cohomology</u> $H^i_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p))$ of $X_{/\mathbb{C}}$ is defined to be the cohomology of the complex

$$\mathbb{R}(p)_{\mathcal{D}} : \mathbb{R}(p) \to \mathcal{O}_{X(\mathbb{C})} \to \Omega^1 \to \ldots \to \Omega^{p-1} \to 0$$

where the first arrow simply is the inclusion of the "twisted" constants $\mathbb{R}(p) := (2\pi\sqrt{-1})^p\mathbb{R} \subseteq \mathbb{C} \subseteq \mathcal{O}_{X(\mathbb{C})}$. We apparently have a short exact sequence of complexes

$$0 \longrightarrow \Omega^\cdot_{<p}[-1] \longrightarrow \mathbb{R}(p)_{\mathcal{D}} \longrightarrow \mathbb{R}(p) \longrightarrow 0$$

from which we derive the long exact cohomology sequence

$$\begin{aligned}
&\to H^i(X(\mathbb{C}), \mathbb{R}(p)) \to H^i_{DR}(X(\mathbb{C}))/F^p \to H^{i+1}_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p)) \to \\
&\to H^{i+1}(X(\mathbb{C}), \mathbb{R}(p)) \to \ldots
\end{aligned}$$

Remark:

Replacing $\mathbb{R}$ by any subring $A \subseteq \mathbb{R}$ we get corresponding cohomology groups $H^i_{\mathcal{D}}(X_{/\mathbb{C}}, A(p))$. There is a long exact cohomology sequence

$$\to H^i_{\mathcal{D}}(X_{/\mathbb{C}}, A(p)) \to H^i_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p)) \to H^i(X(\mathbb{C}), \mathbb{R}/A) \to \ldots$$

Examples:

1) $\mathbb{Z}(1)_{\mathcal{D}} \xrightarrow{\sim} \mathcal{O}/\mathbb{Z}(1)[-1] \xrightarrow{\sim} \mathcal{O}^\times[-1] \quad ;$

2) For $p > \dim X$ we have

$$\mathbb{Z}(p)_{\mathcal{D}} \xrightarrow{\sim} [0 \to \mathcal{O}/\mathbb{Z}(p) \to \Omega^1 \to \ldots \to \Omega^{\dim X}] \xleftarrow{\sim} \mathbb{C}/\mathbb{Z}(p)[-1]$$

and consequently $H^{i+1}_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p)) = H^i(X(\mathbb{C}), \mathbb{R}(p-1))$.

The above long exact cohomology sequence already indicates that the real Deligne cohomology in some sense measures how the natural real structure on the singular cohomology is behaved with respect to the de Rham filtration. And we have to explore this a bit further. Let us first recall that this real structure

$$H^i(X(\mathbb{C}), \mathbb{R}) \underset{\mathbb{R}}{\otimes} \mathbb{C} = H^i(X(\mathbb{C}), \mathbb{C})$$

on the singular cohomology is given by the $\mathbb{R}$-linear involution $^-$ on the right hand side which is induced by the complex conjugation on the coefficients. On the other hand, by GAGA the algebraic de Rham cohomology $H^i_{DR}(X_{/\mathbb{R}})$ of $X_{/\mathbb{R}}$ defines a real structure

$$H^i_{DR}(X_{/\mathbb{R}}) \underset{\mathbb{R}}{\otimes} \mathbb{C} = H^i_{DR}(X(\mathbb{C}))$$

on the analytic de Rham cohomology. The corresponding $\mathbb{R}$-linear involution on the right hand side which we simply call the DR-conjugation is induced by the obvious complex conjugation on the pair $(X(\mathbb{C}), \Omega^\cdot)$. We have

- $\overline{H^{pq}} = H^{qp}$.

- The de Rham filtration already is defined over $\mathbb{R}$.

- Under the canonical identification $H^i_{DR}(X(\mathbb{C})) = H^i(X(\mathbb{C}), \mathbb{C})$ (which is induced by the obvious quasi-isomorphism of complexes $\mathbb{C} \to \Omega^\cdot$) the DR-conjugation on the left hand side corresponds to $\overline{F}_\infty$ on the right hand side ([Del] Prop.1.4.). For the following it is useful to define the real Deligne cohomology of $X_{/\mathbb{R}}$

$$H^i_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(p)) := H^i_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p))^{\mathrm{DR-conjugation}}$$

to be the subspace of elements invariant with respect to the DR-conjugation. We now are prepared to analyze the above long exact cohomology sequence more closely.

Lemma:

For $i < 2p$ the natural map $H^i(X(\mathbb{C}), \mathbb{R}(p)) \to H^i_{DR}(X(\mathbb{C}))/F^p$ is injective.

Proof: The involution $^-$ acts on the left hand side by multiplication by $(-1)^p$. But we have

$$F^p \cap \overline{F^p} = (\bigoplus_{p' \geq p} H^{p'q}) \cap (\bigoplus_{q \geq p} H^{p'q}) = \bigoplus_{p', q \geq p} H^{p'q}$$

which is zero because of $2p > i$.

For $i < 2p$ our long exact cohomology sequence therefore becomes a short exact sequence

$$0 \to H^{i-1}(X(\mathbb{C}), \mathbb{R}(p)) \to H^{i-1}_{DR}(X(\mathbb{C}))/F^p \to H^i_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p)) \to 0 \quad .$$

Using the decomposition $\mathbb{C} = \mathbb{R}(p) \oplus \mathbb{R}(p-1)$ we can rewrite this as a short exact sequence

$$0 \to F^p H^{i-1}_{DR}(X(\mathbb{C})) \to H^{i-1}(X(\mathbb{C}), \mathbb{R}(p-1)) \to H^i_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p)) \to 0 \quad .$$

Remark:

For $i > 2p$ the groups $H^i_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p))$ should be considered pathological. In fact, in [Bei 2] Beilinson defines "absolute Hodge" cohomology groups which coincide with the Deligne cohomology in the range $i \leq 2p$ but vanish for $i > 2p$. In the last Paragraph we will say something about the very interesting groups $H^{2p}_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{Z}(p))$.

For our purposes it is convenient to change the notation a little bit. We fix once and for all an integer

$$\boxed{m < \frac{i+1}{2}}$$

and we put $n := i+1-m$. Passing to invariants with respect to the DR-conjugation $(= \overline{F}_\infty)$ in the above short exact sequence we derive our basic exact sequence

$$(*) \qquad \begin{aligned} 0 \to F^n H^i_{DR}(X_{/\mathbb{R}}) \to H^i(X(\mathbb{C}), \mathbb{R}(n-1))^{(-1)^{n-1}} \\ \to H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)) \to 0 \ . \end{aligned}$$

We immediately realize that our first Corollary in the last Paragraph can now be reformulated in the following more "natural" way.

Proposition:

$$\dim_{\mathbb{R}} H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)) = \begin{cases} \operatorname{ord}_{s=m} L(M,s) & \text{if } \ m < \frac{i}{2} \ , \\ \operatorname{ord}_{s=m} L(M,s) - \operatorname{ord}_{s=m+1} L(M,s) & \text{if } \ m = \frac{i}{2} \ . \end{cases}$$

But we also want to predict (up to a rational number) the leading coefficient (in a Taylor series expansion) of $L(M,s)$ at $s = m$. Let us first look at the case where we have a "honest" value. For $m < \frac{i}{2}$ the following conditions are equivalent:

i. $L_\infty(M,s)$ has no poles at $s = m$ and $s = n$,

ii. $L_\infty(M,s)$ has no pole at $s = m$,

iii. $L(M,m) \neq 0$,

iv. $H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)) = 0$.

According to Deligne such an integer m is called critical. Our sequence (*) in that case becomes an isomorphism

$$F^n H^i_{DR}(X_{/\mathbb{R}}) \xrightarrow{\cong} H^i(X(\mathbb{C}), \mathbb{R}(n-1))^{(-1)^{n-1}} \ .$$

We now observe that both sides carry natural $\mathbb{Q}$-structures: The left hand side by the algebraic de Rham cohomology of $X_{/\mathbb{Q}}$ and the right hand side by the singular cohomology with coefficients $\mathbb{Q}(n-1)$. Therefore, the determinant of this isomorphism calculated in $\mathbb{Q}$-rational bases defines a number

$$c_M(m) \in \mathbb{R}^\times / \mathbb{Q}^\times$$

called the Deligne period of the twisted motive $M(m)$.

Conjecture: (Deligne in [Del])

If $m < \frac{i}{2}$ is critical we have $L(M,m) \equiv c_M(m) \bmod \mathbb{Q}^\times$.

Remarks:

1) Deligne actually defines his period in a slightly different way. The computation which shows that Deligne's and Beilinson's definitions lead to the same period is given in a subsequent Chapter of this book.

2) If $m < \frac{i}{2}$ is critical with $m \leq \max(0, i - \dim X)$ then our second Corollary above says that $c_M(m) = 1$. Deligne's conjecture in this case amounts to the assertion that $L(M, m)$ is $\mathbb{Q}$-rational.

For general m the exact sequence (*) still provides an isomorphism

$$\begin{aligned} &\wedge^{\max} F^n H^i_{DR}(X_{/\mathbb{R}}) \underset{\mathbb{R}}{\otimes} \wedge^{\max} H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)) \\ &\qquad \xrightarrow{\cong} \wedge^{\max} H^i(X(\mathbb{C}), \mathbb{R}(n-1))^{(-1)^{n-1}} \end{aligned}$$

between the maximal exterior powers of the respective $\mathbb{R}$-vector spaces. Now, Beilinson's idea how to proceed is the following:

- Show that the Deligne cohomology $H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n))$ carries a natural $\mathbb{Q}$-structure, too.

- Define the regulator $c_M(m) \in \mathbb{R}^\times / \mathbb{Q}^\times$ to be the above isomorphism calculated in $\mathbb{Q}$-rational bases.

- Conjecture that $c_M(m)$ is the leading coefficient of $L(M, s)$ at $s = m$ up to a rational multiple.

Furthermore, his hope is that the Chern class maps from higher algebraic K-theory into Deligne cohomology will provide the required $\mathbb{Q}$-structure. In the next two Paragraphs we will discuss in some detail the construction of these Chern classes. In order to understand why this theory has to come in it is useful to observe the following two facts:

1) There is no "easy" way of getting a $\mathbb{Q}$-structure. In fact, our considerations which led to the exact sequence (*) also imply that the natural map

$$H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{Q}(n)) \longrightarrow H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n))$$

is surjective.

2) Of course, there also are Chern class maps from higher K-theory into de Rham and singular cohomology. But a weight argument shows that apart from the cycle map on K_0 they are trivial. In contrary to these two cohomology theories the Deligne cohomology is of a transcendental nature so that we should expect highly interesting Chern class maps.

§3 Absolute cohomology

The higher algebraic K-groups of a scheme in a natural way break up into pieces which behave very much like a usual cohomology theory. From this point of view the theory of Chern classes then appears as a technique to construct natural transformations from this "absolute" cohomology into any other reasonable cohomology theory.

In the framework of the +-construction one can define a family of natural operations $\{\psi^k\}_{k\geq 1}$ called the Adams operations on the K-groups $K_i(A)$ of any affine scheme Spec(A). Their most important property is that they induce a decomposition

$$K_i(A) \otimes \mathbb{Q} = \bigoplus_{j\geq 0} K_i^{(j)}(A) \qquad \text{with}$$
$$K_i^{(j)}(A) := \{x \in K_i(A) \otimes \mathbb{Q} : \psi^k(x) = k^j x \quad \text{for all} \quad k \geq 1\} \ .$$

See [Hil] or [Kra] or the corresponding Chapter in this book.

Remark:
$K_i^{(0)}(A) = 0$ for $i \geq 1$, and $K_0(A) \xrightarrow{\text{rank}} H^0(\text{Spec}(A), \mathbb{Z})$ induces an isomorphism $K_0^{(0)}(A) \cong H^0(\text{Spec}(A), \mathbb{Z})$.

In order to get Adams operations on the K-groups of our variety X we use the following facts:

1) There is a torsor $p : W \to X$ for a vector bundle on X which is an affine scheme ([Jou] p.297).

2) (Homotopy property) If $f : Y \to Z$ is a faithfully flat morphism whose fibres are affine spaces then $f^* : K'_*(Z) \xrightarrow{\cong} K'_*(Y)$ is an isomorphism ([Qui] p.120).

3) For any regular scheme Y we have $K_*(Y) = K'_*(Y)$ ([Qui] p.116).

We consequently have an isomorphism

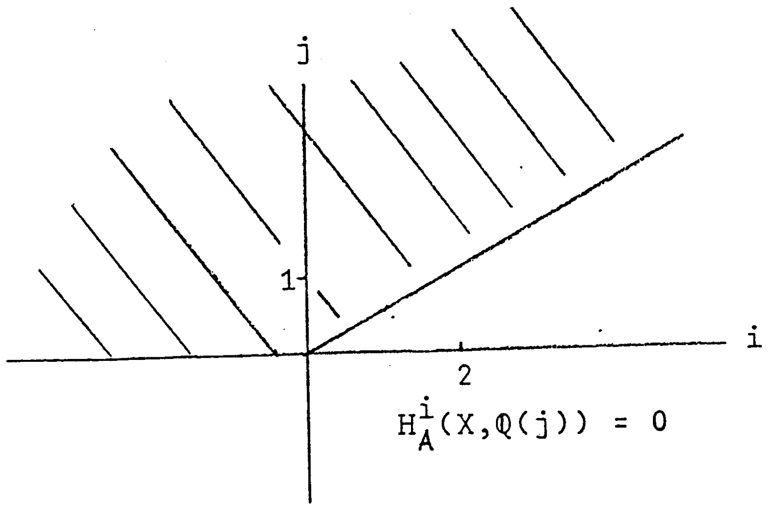
$$p^* : K_*(X) \xrightarrow{\cong} K_*(W)$$

which we use to transfer the Adams operations from the right to the left hand side. The result is independent of the particular choice of W: Namely, if $W' \to X$ is a second such torsor then all morphisms in the cartesian diagram

$$\begin{array}{ccc} W'' & \longrightarrow & W \\ \downarrow & & \downarrow \\ W' & \longrightarrow & X \end{array}$$

induce isomorphisms in K-theory and W'' is affine, too. The <u>absolute cohomology groups</u> of X now are defined by

$$H^i_{\mathcal{A}}(X, \mathbb{Q}(j)) := K^{(j)}_{2j-i}(X) \quad .$$



Problem: Is $H^i_{\mathcal{A}} = 0$ for $i < 0$?

Beilinson hopes that these groups form some kind of universal cohomology theory for X. In the same spirit he expects that the projections

$$ch_{\mathcal{A}} : K_i(X) \longrightarrow \bigoplus_{j \geq 0} H^{2j-i}_{\mathcal{A}}(X, \mathbb{Q}(j))$$

define an universal Chern character.

Remarks:

1) As motivation one should have in mind that, for the complex K-theory of a compact topological space Y, the classical Chern character induces isomorphisms ([Kar])

$$K_0^{(j)}(Y) \cong H^{2j}(Y, \mathbb{Q}) \quad \text{and} \quad K_1^{(j)}(Y) \cong H^{2j-1}(Y, \mathbb{Q}) \ .$$

2) The groups $H^i_{\mathcal{A}}(Y, \mathbb{Q}(j))$ can be defined for any scheme Y which is quasi-projective over a regular scheme; they form a cohomology theory in the sense of Bloch/Ogus (see [Sou 2] or the Chapter on Riemann-Roch in this book).

As a first step towards the wanted $\mathbb{Q}$-structures we will construct in the next Paragraph natural maps

$$H^i_{\mathcal{A}}(X, \mathbb{Q}(j)) \longrightarrow H^i_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(j)) \ .$$

Bloch/Grayson ([BG]) discovered that for certain elliptic curves $X_{/\mathbb{Q}}$ the map $H^2_{\mathcal{A}}(X, \mathbb{Q}(2)) \otimes \mathbb{R} \to H^2_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(2))$ is not injective. Therefore Beilinson proposes the following modification of the absolute cohomology.

Conjecture: (Beilinson)

If $\mathcal{X}_{/\mathbf{Z}}$ is a proper flat model of $X_{/\mathbb{Q}}$ (it always exists!) then

$$\text{image}(K'_*(\mathcal{X}) \otimes \mathbb{Q} \longrightarrow K_*(X) \otimes \mathbb{Q})$$

is independent of the choice of $\mathcal{X}_{/\mathbf{Z}}$ and is compatible with the Adams operations and the formation of inverse images with respect to X.

Remark:

image$(K'_*(\mathcal{X}) \to K_*(X))$ *is independent of the choice of a regular proper model $\mathcal{X}_{/\mathbf{Z}}$ of X (if it exists).*

Proof: Let $\mathcal{X}_{/\mathbf{Z}}$ and $\mathcal{X}'_{/\mathbf{Z}}$ be two regular proper models of X. The Zariski closure $\mathcal{X}''_{/\mathbf{Z}}$ of the diagonal of $X \underset{\mathbb{Q}}{\times} X$ in $\mathcal{X} \underset{\mathbf{Z}}{\times} \mathcal{X}'$ again is a proper model of X; the canonical morphisms

$$\mathcal{X} \underset{\pi}{\longleftarrow} \mathcal{X}'' \underset{\pi'}{\longrightarrow} \mathcal{X}'$$

are proper. Consider now the commutative diagram

$$\begin{tikzcd}
K_i(\mathcal{X}) \arrow[r, "\pi^*"] \arrow[dr] & K_i(\mathcal{X}'') \arrow[r] \arrow[d] & K'_i(\mathcal{X}'') \arrow[r, "\pi'_*"] \arrow[dl] & K'_i(\mathcal{X}') \arrow[dll] \arrow[r, equal] & K_i(\mathcal{X}') \\
& K_i(X) & & &
\end{tikzcd}$$

and its counterpart where the roles of $\mathcal{X}$ and $\mathcal{X}'$ are exchanged. (The construction of π_* for an arbitrary proper morphism π is carried out in [Gil 2] §4.)

Assuming that the above Conjecture holds true we define

$$H^i_{\mathcal{A}}(X, \mathbb{Q}(j))_{\mathbf{Z}} := \text{image}(K'_{2j-i}(\mathcal{X}) \otimes \mathbb{Q} \to K_{2j-i}(X) \otimes \mathbb{Q} \to H^i_{\mathcal{A}}(X, \mathbb{Q}(j)))$$

where $\mathcal{X}_{/\mathbf{Z}}$ is any proper flat model of X. Certain conjectures about the K-theory in characteristic > 0 would imply, via the localization sequence, the following statements about the relation between $H^i_{\mathcal{A}}(X, \mathbb{Q}(j))_{\mathbf{Z}}$ and $H^i_{\mathcal{A}}(X, \mathbb{Q}(j))$.

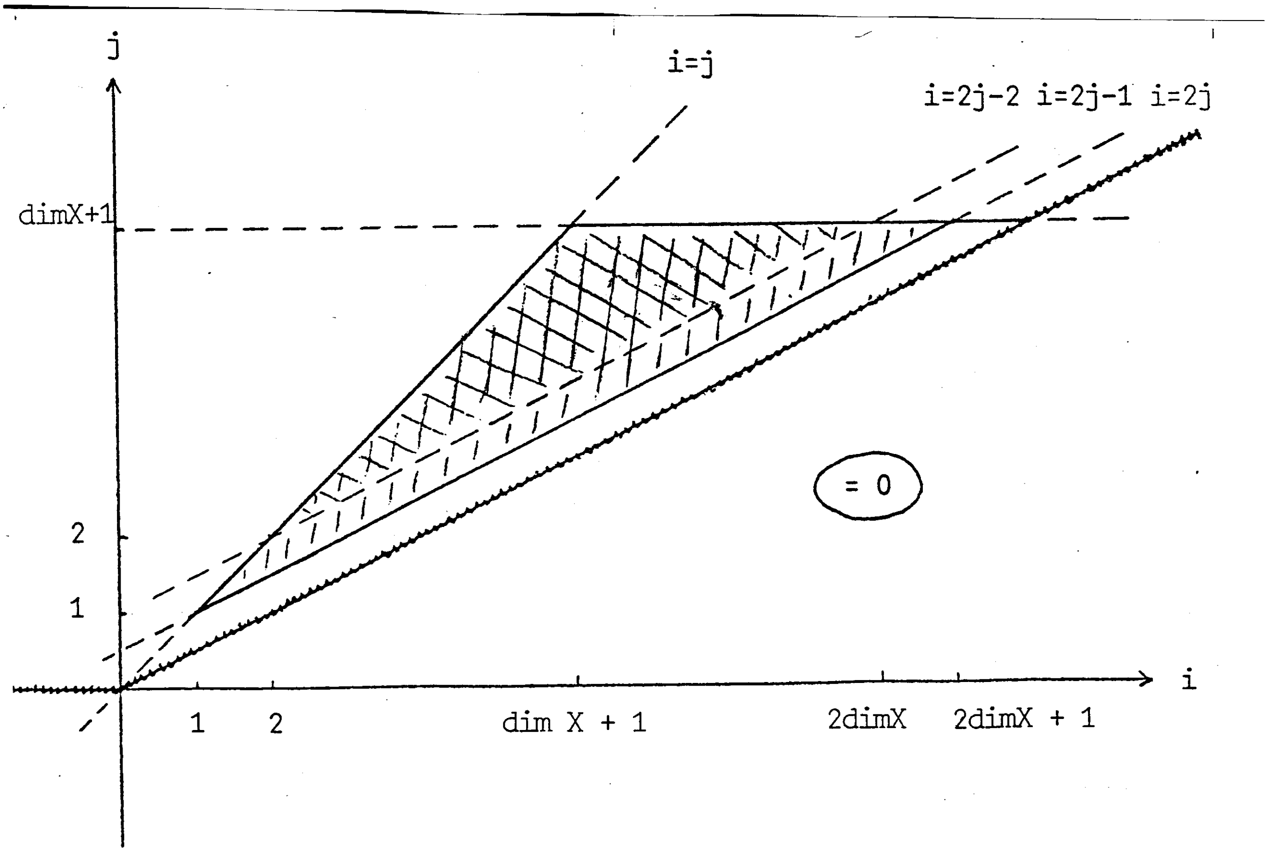
Conjecture:

a) $H^i_{\mathcal{A}}(X, \mathbb{Q}(j))_{\mathbf{Z}} = H^i_{\mathcal{A}}(X, \mathbb{Q}(j))$ except for (i,j) with $j \le i \le 2j-1$ and $j \le \dim X + 1$;

b) $H^i_{\mathcal{A}}(X, \mathbb{Q}(j))/H^i_{\mathcal{A}}(X, \mathbb{Q}(j))_{\mathbf{Z}}$, for $i \le 2j-2$, only depends on the bad fibres of $\mathcal{X}_{/\mathbf{Z}}$;

c) (reformulation in the indices $0 \le i \le 2\dim X$, $m \le \frac{i}{2}$, and $n = i+1-m$) $H^{i+1}_{\mathcal{A}}(X, \mathbb{Q}(n))_{\mathbf{Z}} = H^{i+1}_{\mathcal{A}}(X, \mathbb{Q}(n))$ except if $m \ge \max(0, i - \dim X)$; for $m < \frac{i}{2}$ the difference only depends on the bad fibres of $\mathcal{X}_{/\mathbf{Z}}$.

The following picture might illustrate these statements. The large shadowed triangle is the range where difference can occur; in the smaller one the difference only depends on the bad fibres.



A trivial example is

$$H^1_{\mathcal{A}}(\mathrm{Spec}(\mathbb{Q}),\mathbb{Q}(1))_{\mathbb{Z}} = 0 \underset{\neq}{\subset} \mathbb{Q}^\times \underset{\mathbb{Z}}{\otimes} \mathbb{Q} = H^1_{\mathcal{A}}(\mathrm{Spec}(\mathbb{Q}),\mathbb{Q}(1)) \quad .$$

§4 Chern classes

To a large part this theory is completely formal and relies on some manipulations in the context of simplicial algebra. Let $\mathcal{V}$ be the category of smooth quasi-projective schemes over some fixed base field equipped with the Zariski topology. Any scheme Y in $\mathcal{V}$ represents a sheaf of sets $\underline{\underline{Y}}$ on $\mathcal{V}$. Let $\mathbb{Z}\underline{\underline{Y}}$ denote the sheafification of the presheaf of free $\mathbb{Z}$-modules over $\underline{\underline{Y}}$. For any sheaf $\mathcal{F}$ on $\mathcal{V}$ we then have $\mathcal{F}(Y) = \mathrm{Hom}_{\mathcal{V}}(\mathbb{Z}\underline{\underline{Y}},\mathcal{F})$ and therefore

$$H^*(Y,\mathcal{F}) = \mathrm{Ext}^*_{\mathcal{V}}(\mathbb{Z}\underline{\underline{Y}},\mathcal{F}) \quad .$$

We need a generalization of this identity for the cohomology of any simplicial scheme $Y.$ in $\mathcal{V}$. Similarly as before $Y.$ represents a simplicial sheaf of sets $\underline{\underline{Y}}.$ on $\mathcal{V}$. In an obvious way we get the associated simplicial sheaf $\mathbb{Z}\underline{\underline{Y}}.$ of $\mathbb{Z}$-modules which gives rise to a (negative cohomological) complex of sheaves

$$N\mathbb{Z}\underline{\underline{Y}}. : \ldots \to \mathbb{Z}\underline{\underline{Y}}_{-k} \xrightarrow[\sum_{\nu=0}^{-k}(-1)^\nu d_\nu]{} \mathbb{Z}\underline{\underline{Y}}_{-k+1} \to \ldots$$

with $\mathbb{Z}\underline{\underline{Y}}_{-k}$ in degree k. Let $D(\mathcal{V})$ (resp. $D^+(\mathcal{V})$) be the derived category of complexes of abelian sheaves on $\mathcal{V}$ (which are bounded below).

Lemma:

$H^*(Y.,\mathcal{F}^\cdot) = \mathrm{Hom}_{D(\mathcal{V})}(N\mathbb{Z}\underline{\underline{Y}}.,\mathcal{F}^\cdot[*])$ *for any* $\mathcal{F}^\cdot \in D^+(\mathcal{V})$.

Proof: Without loss of generality we can assume that $\mathcal{F}^\cdot$ is a complex of injective sheaves on $\mathcal{V}$. An examination of the proof of Prop.2.4 in [Fri] then shows that the cohomology $H^*(Y.,\mathcal{F}^\cdot)$ can be computed from the double complex

$$(\mathrm{Hom}_{Y_k}(\mathbb{Z},\mathcal{F}^\ell))_{k,\ell} \quad .$$

But this double complex is equal to the double complex

$$(\mathrm{Hom}_{\mathcal{V}}(\mathbb{Z}\underline{\underline{Y}}_k,\mathcal{F}^\ell))_{k,\ell}$$

which, by Yoneda (compare [Har] I.6.4), computes the groups

$$\mathrm{Ext}^*_{D(\mathcal{V})}(N\mathbb{Z}\underline{\underline{Y}}.,\mathcal{F}^\cdot) = \mathrm{Hom}_{D(\mathcal{V})}(N\mathbb{Z}\underline{\underline{Y}}.,\mathcal{F}^\cdot[*]) \quad .$$

We apply this to the "classifying" simplicial scheme

$$B.GL_n \; : \; \mathrm{Spec}\binom{\text{base}}{\text{field}} \; \begin{array}{c} \longleftarrow \\ \longleftarrow \end{array} \; GL_n \; \begin{array}{c} \xleftarrow{id\times 1} \\ \xleftarrow{\mu} \\ \xleftarrow{1\times id} \end{array} \; GL_n \times GL_n \; \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow \end{array}$$

(where μ denotes the multiplication map for the group scheme GL_n) and get natural identifications

$$H^*(B.GL_n,\mathcal{F}^\cdot) = \mathrm{Hom}_{D(\mathcal{V})}(N\mathbb{Z}\underline{\underline{B.GL}}_n,\mathcal{F}^\cdot[*])$$

for any $\mathcal{F}^\cdot \in D^+(\mathcal{V})$. Next we want to pass to the limit with respect to n in that identity. This requires a stability result and is therefore, of course, not a formal matter. Put $\underline{\underline{B.GL}} := \varinjlim \underline{\underline{B.GL}}_n$.

Proposition:

The natural homomorphism of complexes of sheaves

$$N\mathbb{Z}\underline{\underline{B.GL}}_n \longrightarrow N\mathbb{Z}\underline{\underline{B.GL}}$$

is a quasi-isomorphism in degree $\geq -\frac{n-1}{2}$.

Proof: Since we can check our assertion stalkwise we have to show (use [Mil] II.2.9(d) and 3.20(a)) that

$$N\mathbb{Z}B.GL_n(A) \longrightarrow N\mathbb{Z}B.GL(A)$$

is a quasi-isomorphism in degree $\geq -\frac{n-1}{2}$ for any local ring A. Now we observe that these complexes are the standard bar resolutions which compute the group homology of $GL_n(A)$ and $GL(A)$, respectively (compare [Mac]). What we have to prove therefore amounts to the statement that the natural homomorphisms

$$H_k(GL_n(A),\mathbb{Z}) \longrightarrow H_k(GL(A),\mathbb{Z})$$

are bijective for $k \leq \frac{n-1}{2}$ and any local ring A. This stability result is established in [Kal] Th.4.11 or [Sus 1] Cor.8.3.

Corollary:

For any $\mathcal{F}^{\cdot} \in D^+(\mathcal{V})$ there is a $n' \geq 1$ such that

$$\mathrm{Hom}_{D(\mathcal{V})}(N\mathbb{Z}\underline{\underline{B.GL_n}}, \mathcal{F}^{\cdot}) = \mathrm{Hom}_{D(\mathcal{V})}(N\mathbb{Z}\underline{\underline{B.GL}}, \mathcal{F}^{\cdot}) \quad \textit{for} \quad n \geq n' \ .$$

We now fix a complex $\mathcal{F}^{\cdot} \in D^+(\mathcal{V})$ and we assume that, for $j \geq 0$ and $n >> 0$, there are given classes $c_j^{(n)} \in H^{2j}(B.GL_n, \mathcal{F}^{\cdot})$ which are compatible with respect to $GL_n \hookrightarrow GL_{n+1}$. We have seen that these classes induce a homomorphism

$$c_j : N\mathbb{Z}\underline{\underline{B.GL}} \longrightarrow \mathcal{F}^{\cdot}[2j]$$

in $D(\mathcal{V})$. If $U = \mathrm{Spec}(A)$ is an affine scheme in $\mathcal{V}$ passing to global sections in U then gives a homomorphism of complexes

$$c_j : N\mathbb{Z}B.GL(A) \to (N\mathbb{Z}\underline{\underline{B.GL}})(U) \to \mathcal{F}^{\cdot}[2j](U)$$

(where, for a moment, we again think of $\mathcal{F}^{\cdot}$ being a complex of injective sheaves) which induces homomorphisms

$$c_{i,j} : H_i(GL(A), \mathbb{Z}) \longrightarrow H^{2j-i}(U, \mathcal{F}^{\cdot})$$

in cohomology (remember that $N\mathbb{Z}B.GL(A)$ is the standard bar resolution for the group $GL(A)$). In order to get maps on K-theory we use the Hurewicz map and the fact that the $+$-construction leaves homology invariant. We put

$$\begin{array}{rcl} c_{i,j} : K_i(A) = \pi_i(BGL(A)^+) & \underset{\text{Hurewicz}}{\longrightarrow} & H_i(BGL(A)^+, \mathbb{Z}) \\ & & \| \\ & & H_i(BGL(A), \mathbb{Z}) \\ & & \| \\ & & H_i(GL(A), \mathbb{Z}) \\ & & c_{i,j} \downarrow \\ & & H^{2j-i}(U, \mathcal{F}^{\cdot}) \end{array}$$

for $i \geq 1$.

Result:

Any compatible family of classes $c_j^{(n)} \in H^{2j}(B.GL_n, \mathcal{F}^{\cdot})$ induces natural homomorphisms

$$c_{i,j} : K_i(A) \longrightarrow H^{2j-i}(U, \mathcal{F}^{\cdot})$$

for $i \geq 1$ and any affine scheme $U = \mathrm{Spec}(A)$ in $\mathcal{V}$.

Remarks:

1) Since we hope that, in an appropriate setting, the Chern classes on higher K-theory provide $\mathbb{Q}$-structures on cohomology it seems worth to note that the Hurewicz map $K_i(A) \otimes \mathbb{Q} \to H_i(GL(A), \mathbb{Q})$ is injective (see [MM] App.). So the above construction does not cause any a priori loss of information about the ranks of the K-groups.

2) The above construction works in any "reasonable" topology (e.g., the etale topology) instead of the Zariski topology. This can be seen either by examining the arguments or (at least if the new topology is finer than the Zariski topology) by applying the "Zariski" construction to the total direct image on the Zariski site of the respective complex of sheaves on the finer site.

In order to proceed we assume that the cohomology of our fixed complex $\mathcal{F}^{\cdot}$ has four particular properties (which are familiar from classical or etale cohomology).

(I) **(Homotopy property)** The natural map $\mathbf{A}^1_Y \to Y$, for any scheme Y in $\mathcal{V}$, induces a cohomology isomorphism $H^*(Y,\mathcal{F}^{\cdot}) \xrightarrow{\cong} H^*(\mathbf{A}^1_Y,\mathcal{F}^{\cdot})$.

This property will enable us to extend the definition of the maps $c_{i,j}$ to any scheme in $\mathcal{V}$.

Lemma:

Any morphism $Y \to Z$ in $\mathcal{V}$ which (Zariski) locally is of the form $\mathbf{A}^n_U \to U$ for some $n \geq 0$ (e.g., if Y is a torsor for a vector bundle on Z) induces a cohomology isomorphism $H^*(Z,\mathcal{F}^{\cdot}) \xrightarrow{\cong} H^*(Y,\mathcal{F}^{\cdot})$.

Proof: If $f : \mathbf{A}^n \to \mathrm{Spec}(\text{base field})$ denotes the structure morphism then the homotopy property (I) is equivalent to the assertion that the canonical homomorphism

$$\mathcal{F}^{\cdot} \longrightarrow Rf_*(f^*\mathcal{F}^{\cdot})$$

in $D^+(\mathcal{V})$ is an isomorphism. But this statement obviously is of a local nature so that it holds true for any morphism in $\mathcal{V}$ which locally is of the form $\mathbf{A}^n_U \to U$.

Let now Y be any scheme in $\mathcal{V}$. Jouanolou's lemma ([Jou] p.297) tells us that there is a torsor $p : W \to Y$ for a vector bundle on Y which is an affine scheme. Because of the homotopy property we therefore can define maps $c_{i,j}$ for Y and $i \geq 1$ by the commutative diagram

$$\begin{tikzcd} K_i(W) \arrow[r, "c_{i,j}"] & H^{2j-i}(W,\mathcal{F}^{\cdot}) \\ K_i(Y) \arrow[u, "p^*"', "\cong"] \arrow[r, dashed, "c_{i,j}"'] & H^{2j-i}(Y,\mathcal{F}^{\cdot}) \arrow[u, "\cong"', "p^*"] \end{tikzcd} .$$

The same argument as in the last Paragraph shows that this definition does not depend on the particular choice of W.

Remark:

If one is prepared to use more complicated techniques from homotopical algebra one can avoid to require the homotopy property (I). The idea, of course, is to "sheafify" the above construction for affine schemes. Since the $+$-construction $BGL(A)^+$ is natural in A only up to homotopy it is first necessary to replace it by the homotopy equivalent and truly functorial Bousfield/Kan completion $\mathbb{Z}_\infty B.GL(A)$. This gives a simplicial presheaf of sets on $\mathcal{V}$ and the K-theory turns out to be equal to the generalized sheaf cohomology

$$K_i(Y) = H^{-i}(Y, \mathbb{Z}_\infty \underline{\underline{B.GL}}), \quad \text{for} \quad i \geq 1 \ ,$$

of the associated (pointed) simplicial sheaf $\mathbb{Z}_\infty \underline{\underline{B.GL}}$. The Hurewicz map in this context simply becomes the natural map of simplicial sheaves

$$\mathbb{Z}_\infty \underline{\underline{B.GL}} \longrightarrow \mathbb{Z}\mathbb{Z}_\infty \underline{\underline{B.GL}} \ .$$

On the other hand, by the acyclicity of the +-construction, we know that the natural map $\mathbb{Z}\underline{\underline{B.GL}} \to \mathbb{Z}\mathbb{Z}_\infty\underline{\underline{B.GL}}$ is a weak equivalence. We therefore get homomorphisms

$$K_i(Y) \longrightarrow H^{-i}(Y, \mathbb{Z}\underline{\underline{B.GL}}) \quad .$$

Finally, we observe that the generalized sheaf cohomology of a simplicial abelian sheaf is equal to the usual hypercohomology of the corresponding complex of abelian sheaves:

$$H^{-i}(Y, \mathbb{Z}\underline{\underline{B.GL}}) = H^{-i}(Y, N\mathbb{Z}\underline{\underline{B.GL}}) \text{ for } i \geq 0 \quad .$$

See [Gil 1] and [BrG] .

So far our classes $c_j^{(n)} \in H^{2j}(B.GL_n, \mathcal{F}^\cdot)$ were completely arbitrary. The properties (II) - (IV) below will enable us to make a specific choice of those classes. They simply axiomatize the usual procedure for defining Chern classes of vector bundles so that we also will get maps (not homomorphisms!) on $K_0(Y)$.

Commentary:

The technique in the above Remark actually gives maps

$$K_0(Y) \longrightarrow H^0(Y, \mathbb{Z}_\infty\underline{\underline{B.GL}}) \longrightarrow H^{2j}(Y, \mathcal{F}^\cdot)$$

for any family of classes $c_j^{(n)}$. On the other hand, the classes $c_j^{(n)}$ are determined by those maps: $c_j^{(n)}$ is the image of the class of the universal rank n vector bundle on $B.GL_n$ under the map $K_0(B.GL_n) \to H^{2j}(B.GL_n, \mathcal{F}^\cdot)$. So the correct order seems first to define the maps on K_0 which then give classes $c_j^{(n)}$ and therefore maps on the higher K-groups.

(II) (**Product structure**) There are homomorphisms

$$\cup : \mathcal{F}^\cdot \overset{L}{\underset{\mathbb{Z}}{\otimes}} \mathcal{F}^\cdot \longrightarrow \mathcal{F}^\cdot \text{ and } e : \mathbb{Z} \longrightarrow \mathcal{F}^\cdot$$

in $D^+(\mathcal{V})$ such that $\cup$ is associative and (graded) commutative with unit e.

Explanations:

1) For principal ideal domains like $\mathbb{Z}$ the derived tensor product $\overset{L}{\underset{\mathbb{Z}}{\otimes}}$ exists on $D(\mathcal{V})$ and respects $D^+(\mathcal{V})$ (see [God] Th.I.5.5.2 and [Har] II §4).

2) The homomorphism $\cup$ induces cup-product pairings

$$\begin{array}{ccccc} H^k(Y., \mathcal{F}^\cdot) & \times & H^\ell(Y., \mathcal{F}^\cdot) & \longrightarrow & H^{k+\ell}(Y., \mathcal{F}^\cdot) \\ & (x,y) & & \longmapsto & x \cup y \end{array}$$

on the cohomology of any simplicial scheme $Y.$ in $\mathcal{V}$.

If we interprete cohomology classes as homomorphisms in $D(\mathcal{V})$ then $x \cup y$ is given by the commutative diagram

$$\begin{array}{ccc}
N\mathbb{Z}\underline{\underline{Y}}_{\cdot} \overset{L}{\otimes} N\mathbb{Z}\underline{\underline{Y}}_{\cdot} & \overset{x\otimes y}{\longrightarrow} & \mathcal{F}^{\cdot}[k] \overset{L}{\otimes} \mathcal{F}^{\cdot}[\ell] \\
EZ \downarrow \sim & & \\
N\mathbb{Z}(\underline{\underline{Y_{\cdot} \times Y_{\cdot}}}) & & \Big\downarrow \cup \\
\text{diagonal} \uparrow & & \\
N\mathbb{Z}\underline{\underline{Y}}_{\cdot} & \overset{x\cup y}{\longrightarrow} & \mathcal{F}^{\cdot}[k+\ell]
\end{array}$$

Here EZ denotes the homotopy equivalence given by the theorem of Eilenberg-Zilber.

(III) **(Cohomology of projective space)** There is a homomorphism

$$\tilde{c} : \mathbf{G}_m[-1] \longrightarrow \mathcal{F}^{\cdot}$$

in $D^+(\mathcal{V})$ such that, for any scheme Y in $\mathcal{V}$ and any $n \geq 0$, the map

$$\sum_{k=0}^{n} \pi^*(\ ) \cup \xi^k : \bigoplus_{k=0}^{n} H^{*-2k}(Y, \mathcal{F}^{\cdot}) \xrightarrow{\cong} H^*(\mathbf{P}^n_Y, \mathcal{F}^{\cdot})$$

is an isomorphism where $\pi : \mathbf{P}^n_Y \to Y$ is the structure morphism and ξ is the image of the canonical line bundle, i.e.,

$$\begin{array}{ccc}
H^1(\mathbf{P}^n_Y, \mathcal{O}^\times) & \overset{\tilde{c}}{\longrightarrow} & H^2(\mathbf{P}^n_Y, \mathcal{F}^{\cdot}) \\
\mathcal{O}(1) & \longmapsto & \xi
\end{array} .$$

Like the homotopy property this property (III) is of a local nature. Therefore it generalizes to arbitrary projective bundles.

Proposition:

Let E be a rank n vector bundle on a simplicial scheme $Y_{\cdot}$ in $\mathcal{V}$. Then the map

$$\sum_{k=0}^{n-1} \pi^*(\ ) \cup \xi_E^k : \bigoplus_{k=0}^{n-1} H^{*-2k}(Y_{\cdot}, \mathcal{F}^{\cdot}) \xrightarrow{\cong} H^*(\mathbf{P}(E), \mathcal{F}^{\cdot})$$

is an isomorphism; here $\pi : \mathbf{P}(E) \to Y_{\cdot}$ is the structure morphism of the associated projective bundle and $\xi_E \in H^2(\mathbf{P}(E), \mathcal{F}^{\cdot})$ is the image under $\tilde{c}$ of the canonical line bundle on $\mathbf{P}(E)$.

Proof: See [Gil 1] Lemma 2.4 (for the definition of a vector bundle on a simplicial scheme consult [Gil 3] Ex. 1.1).

This result, in particular, gives an identity

$$\xi_E^n + \pi^*(c_1(E)) \cup \xi_E^{n-1} + \ldots + \pi^*(c_n(E)) = 0$$

in $H^{2n}(\mathbf{P}(E), \mathcal{F}^\cdot)$ with uniquely determined classes

$$c_j(E) \in H^{2j}(Y_\cdot, \mathcal{F}^\cdot)$$

(put $c_0(E) = 1$ and $c_j(E) = 0$ for $j > n$); they are called the Chern classes of the vector bundle E.

Remark:

For a line bundle E viewed as an element of $H^1(Y_\cdot, \mathcal{O}^\times)$ we have $c_1(E) = \tilde{c}(E)$.

In order to get the usual properties for these Chern classes we need a very weak version of the formalism of Gysin maps.

(IV) **(Weak Gysin property)** Let $\iota : Z \hookrightarrow Y$ be a closed immersion of pure codimension 1 in $\mathcal{V}$ and let $[Z] \in H^1(Y, \mathcal{O}^\times)$ be the class of the divisor Z on Y; for any $x \in H^{2*}(Y, \mathcal{F}^\cdot)$ such that $\iota^* x = 0$ we have

$$x \cup \tilde{c}([Z]) = 0 \quad .$$

The behaviour of our Chern classes with respect to short exact sequences, tensor products, and exterior powers of vector bundles can now most conveniently be described in the following way. Those operations on vector bundles give $K_0(Y_\cdot)$ the structure of an augmented $H^0(Y_\cdot, \mathbb{Z}) - \lambda$-algebra (SGA 6 exp. VI Th. 3.3). On the other hand we put

$$Ch(Y_\cdot) := H^0(Y_\cdot, \mathbb{Z}) \times \{(x_j) \in \prod_{j \geq 0} H^{2j}(Y_\cdot, \mathcal{F}^\cdot) : x_0 = 1\}$$

which obviously forms an abelian group with respect to the cup-product as addition (it is suggestive to think of elements in the second factor as being power series in one variable with constant coefficient 1). Furthermore, using certain universal polynomials, $Ch(Y_\cdot)$ in a natural way can be made into an augmented $H^0(Y_\cdot, \mathbb{Z}) - \lambda$-algebra, too. The interested reader should consult SGA 6 exp. 0 App. I §3 or exp. V §6 for the details. The only fact about $Ch(Y_\cdot)$ we need to know in the following is that the action of the Adams operations ψ^k, for $k \geq 1$, on it can be determined explicitly.

Lemma:

For $x = (r, 1, (x_j)_{j \geq 1}) \in Ch(Y_\cdot)$ and $k \geq 1$ we have

$$\psi^k x = (r, 1, (k^j x_j)_{j \geq 1}) \quad .$$

Proof: We freely use the notations of SGA 6 exp. V. The same argument as in the proof of loc.cit. (6.6.1) shows that we have

$$\psi^k(\ell, 1, \ldots, x_\ell, 0, \ldots) = (\ell, 1, \ldots, k^\ell x_\ell, 0, \ldots) \text{ for all } k, \ell \geq 1 \ ;$$

one only has to observe that loc.cit. (6.2.1) implies

$$\psi^k(1, 1+T_i) = (1, 1+kT_i) \quad .$$

Since obviously $\psi^k(\ell, 1, 0, ...) = (\ell, 1, 0, ...)$ we then also get

$$\psi^k(0, 1, ..., x_\ell, 0, ...) = (0, 1, ..., k^\ell x_\ell, 0, ...) \text{ for all } \ell \geq 1 \ .$$

But $(Ch(Y.))_{\ell+1}$ is a λ-ideal according to loc.cit. (6.6.3). Consequently

$$\begin{aligned} \psi^k(0, 1, (x_j)_{j\geq 1}) &\equiv \psi^k(0, 1, ..., x_\ell, 0, ...) \\ &= (0, 1, ..., k^\ell x_\ell, 0, ...) \\ &\equiv (0, 1, (k^j x_j)_{j\geq 1}) \bmod (Ch(Y.))_{\ell+1} \end{aligned}$$

holds true for all $\ell \geq 1$ which proves the assertion.

All the important properties of Chern classes now can be expressed by the following statement.

Proposition:

The map $[E] \mapsto (\text{rank } E, c_0(E), c_1(E), ...)$ *induces a natural homomorphism*

$$c : K_0(Y.) \longrightarrow Ch(Y.)$$

of augmented $H^0(Y., \mathbb{Z}) - \lambda$*-algebras. Furthermore, the family of these homomorphisms (for all (simplicial) schemes in* $\mathcal{V}$*) is uniquely characterized by the fact that*

$$c([E]) = (1, 1, \tilde{c}(E), 0, ...) \textit{ for line bundles } E \quad .$$

Proof: See [Gro 1] §3. The reader will realize that the purpose of the weak Gysin property is to ensure the validity of the corollary on p. 142 of loc.cit.

In particular, we get natural maps

$$\begin{array}{cccccc} c_{0,j} & : & K_0(Y.) & \longrightarrow & H^{2j}(Y., \mathcal{F}^\cdot) & \text{for} \quad j \geq 0 \quad . \\ & & [E] & \longmapsto & c_j(E) & \end{array}$$

It remains to explain which choice of classes $c_j^{(n)} \in H^{2j}(B.GL_n, \mathcal{F}^\cdot)$ we are going to make. Since those classes should be universal in the sense that their origin does not depend on the particular cohomology theory $H^*(., \mathcal{F}^\cdot)$ we are dealing with, the obvious idea is to use the map

$$c : \varprojlim K_0(B.GL_n) \longrightarrow \varprojlim Ch(B.GL_n)$$

which is provided by the above Proposition. Indeed, if E^n, resp. $\underline{1}^n$, denotes the universal, resp. trivial, rank n vector bundle on $B.GL_n$ (compare [Gil 1] p. 218) then we have the element

$$u := \{[E^n] - [\underline{1}^n]\}_n \in \varprojlim K_0(B.GL_n) \quad .$$

We define the <u>universal Chern classes</u> $c_j^{(n)} \in H^{2j}(B.GL_n, \mathcal{F}^\cdot)$ by

$$c(u) = \{(0, c_0^{(n)}, c_1^{(n)}, ...)\}_n \quad .$$

The homomorphisms $c_{i,j}$ on higher K-groups constructed from these particular classes $c_j^{(n)}$ are called <u>Chern class maps</u>.

Remark:

The structure of the λ-ring $K_0(B.GL_n)$ is known explicitly: Let $R(GL_n)$ denote the Grothendieck ring of rational linear (over the base field) representations of the group scheme GL_n. This is a λ-ring (SGA 6 exp. 0 App. I §2). Furthermore, we have the homomorphism of λ-rings

$$\begin{array}{ccc} R(GL_n) & \longrightarrow & K_0(B.GL_n) \\ [\rho : GL_n \to GL_m] & \longmapsto & (B.\rho)^*[E^m] \end{array}$$

which, in fact, is an isomorphism: A vector bundle V on $B.GL_n$ (up to isomorphism) is completely determined by the following data (compare [Gil 3] p. 7/8):

- a trivial vector bundle $\mathcal{O}^m$ on $B_1GL_n = GL_n$, and
- an automorphism ρ of $\mathcal{O}^m$ such that $d_2^*\rho \circ d_0^*\rho = d_1^*\rho$ (on B_2GL_n).

Obviously, ρ defines a homomorphism of group schemes $\rho : GL_n \to GL_m$ such that $[V] = (B.\rho)^*[E^m]$. Now, let $id_n : GL_n \to GL_n$ be the identity representation. We then have (SGA 6 exp. 0 App. I §2)

$$R(GL_n) = \mathbb{Z}[\lambda^1[id_n], \dots, \lambda^n[id_n], \lambda^n[id_n]^{-1}] \quad .$$

We now have achieved the construction of Chern class maps

$$c_{i,j} : K_i(Y) \longrightarrow H^{2j-i}(Y, \mathcal{F}^{\cdot})$$

for $i, j \geq 0$ and any scheme Y in $\mathcal{V}$. By definition, they are homomorphisms in case $i \geq 1$. The above Proposition on the other hand gives a rather complete description of their properties in case $i = 0$. We therefore still have the task to determine their behaviour with respect to the Adams operations and the product on higher K-theory in case $i \geq 1$. For that purpose it is necessary to consider all maps which arise from classes in the image of c simultaneously: We fix an affine scheme $U = \mathrm{Spec}(A)$ in $\mathcal{V}$. For any $v \in K_0(B.GL_n)$ and $i \geq 1, j \geq 0$ let

$$\begin{array}{ccc} c_{i,j}(v) : \pi_i(BGL_n(A)^+) & \underset{\text{Hurewicz}}{\longrightarrow} & H_i(GL_n(A), \mathbb{Z}) \\ & & \downarrow {\scriptstyle H_i(c_j(v))} \\ & & H^{2j-i}(U, \mathcal{F}^{\cdot}) \end{array}$$

denote the homomorphism constructed from the class $c_j(v) \in H^{2j}(B.GL_n, \mathcal{F}^{\cdot})$ given by $c(v) = (\text{rank } v, c_0(v), c_1(v), \dots)$.

Lemma:

For $v, w \in K_0(B.GL_n)$ we have $c_{i,j}(v + w) = c_{i,j}(v) + c_{i,j}(w)$.

Proof: (Compare [Gil 1] 2.25) From the commutative diagram

$$\begin{array}{ccc} K_0(B.GL_n) & \xrightarrow{c} & Ch(B.GL_n) \\ \downarrow {\scriptstyle \text{unit section}} & & \downarrow \\ K_0(\substack{\text{base} \\ \text{field}}) = \mathbb{Z} & \xrightarrow{c} & Ch(\substack{\text{base} \\ \text{field}}) \end{array}$$

we see that, for $j > 0$, the composed homomorphism

$$\mathbb{Z} \underset{\text{section}}{\overset{\text{unit}}{\longrightarrow}} N\mathbb{Z}\underline{\underline{B.GL}}_n \overset{c_j(v)}{\longrightarrow} \mathcal{F}^{\cdot}[2j]$$

is the zero map. Passing to cohomology groups this implies that we have commutative diagrams

$$\begin{tikzcd}
 & H_i(GL_n(A) \times GL_n(A), \mathbb{Z}) \arrow[d] \\
H_i(GL_n(A), \mathbb{Z}) \arrow[ur, "\text{diagonal}"] \arrow[r, "\text{diagonal}"] \arrow[dr, "H_i(c_{j'}(v) \cup c_{j''}(w))"'] & H_i(BGL_n(A)^+ \times BGL_n(A)^+, BGL_n(A)^+ \vee BGL_n(A)^+; \mathbb{Z}) \arrow[d] \\
 & H^{2(j'+j'')-i}(U, \mathcal{F}^{\cdot})
\end{tikzcd}$$

for all positive integers $i, j', j'' > 0$ (compare [Dol] VI. 12.8). We now make use of the following two facts from topology:

- For any pointed CW-complex (T, P) there is a commutative diagram

$$\begin{tikzcd}
 & H_*(T \times T, T \vee T; \mathbb{Z}) \\
H_*(T, P; \mathbb{Z}) \arrow[ur, "\text{diagonal}"] \arrow[dr, "\text{diagonal}"'] & \\
 & H_*(T \wedge T, P; \mathbb{Z}) \arrow[uu]
\end{tikzcd}$$

  (see [Dol] V.4.4).

- For any topological space T and any $i \geq 1$ the composed map

$$\pi_i(T) \underset{\text{Hurewicz}}{\longrightarrow} H_i(T, \mathbb{Z}) \underset{\text{diagonal}}{\longrightarrow} H_i(T \wedge T, \mathbb{Z})$$

  is the zero map (since $H_i(S^i \wedge S^i, \mathbb{Z}) = H_i(S^{2i}, \mathbb{Z}) = 0$).

If we combine these facts with the above diagram we end up with a commutative diagram

$$\begin{tikzcd}
\pi_i(BGL_n(A)^+) \arrow[r, "\text{Hurewicz}"] \arrow[dr, "0"'] & H_i(GL_n(A), \mathbb{Z}) \arrow[r, "H_i(c_{j'}(v) \cup c_{j''}(w))"] \arrow[d, "\text{diagonal}"] & H^{2(j'+j'')-i}(U, \mathcal{F}^{\cdot}) \\
 & H_i(BGL_n(A)^+ \wedge BGL_n(A)^+, \mathbb{Z}) \arrow[ur] &
\end{tikzcd}$$

which shows that

$$H_i(c_{j'}(v) \cup c_{j''}(w)) \circ \text{Hurewicz} = 0 \quad \text{for} \quad i, j', j'' > 0 \ .$$

Consequently we have

$$\begin{aligned} c_{i,j}(v+w) &= H_i(c_j(v+w)) \circ \text{Hurewicz} \\ &= H_i(\sum_{\substack{j'+j''=j \\ j',j''\geq 0}} c_{j'}(v) \cup c_{j''}(w)) \circ \text{Hurewicz} \\ &= H_i(c_j(v)) \circ \text{Hurewicz} \ + H_i(c_j(w)) \circ \text{Hurewicz} \\ &= c_{i,j}(v) + c_{i,j}(w) \quad . \quad \text{q.e.d.} \end{aligned}$$

Next we will see that all the maps $c_{i,j}(v)$ can actually be computed in terms of the Chern class maps $c_{i,j}$. First we recall from the above Remark that we have a canonical isomorphism

$$R(GL_n) \xrightarrow{\cong} K_0(B.GL_n)$$

which we view from now on as an identification. In particular, we will write $c_j(\rho)$ and $c_{i,j}(\rho)$ for $\rho \in R(GL_n)$. According to [Kra] Cor. 3.2 there is a natural homomorphism of groups

$$R(GL_n) \longrightarrow [BGL_n(A)^+, BGL(A)^+]$$

where the group structure on the right hand side comes from the H-space structure on $BGL(A)^+$. The image of (the class of) a representation $\rho : GL_n \to GL_m$ under this homomorphism is the (pointed) homotopy class of the map

$$BGL_n(A)^+ \xrightarrow{B\rho(A)^+} BGL_m(A)^+ \longrightarrow BGL(A)^+ \quad .$$

Via this homomorphism any $\rho \in R(GL_n)$ induces natural maps

$$\pi_i(\rho) : \pi_i(BGL_n(A)^+) \longrightarrow K_i(A) \quad \text{for} \quad i \geq 1$$

on homotopy groups with the property that

$$\pi_i(\rho + \rho') = \pi_i(\rho) + \pi_i(\rho') \quad .$$

Lemma:

For $\rho \in R(GL_n)$ we have $c_{i,j}(\rho) = c_{i,j} \circ \pi_i(\rho)$.

Proof: Let ρ first be the class of a "true" representation $\rho : GL_n \to GL_m$. We then have the commutative diagram

$$\begin{array}{ccc} \pi_i(BGL_n(A)^+) & \xrightarrow{\pi_i(\rho)} & K_i(A) \\ \downarrow{\scriptstyle \text{Hurewicz}} & & \downarrow \\ H_i(GL_n(A), \mathbb{Z}) & \xrightarrow{H_i(\rho(A))} & H_i(GL(A), \mathbb{Z}) \\ {\scriptstyle H_i(c_j(\rho))} \searrow & & \swarrow {\scriptstyle H_i(c_j(u))} \\ & H^{2j-i}(U, \mathcal{F}^{\cdot}) & \end{array}$$

for any $i \geq 1$ and $j \geq 0$. The commutativity of the square, resp. triangle, is obvious, resp. follows from the naturality of the Chern classes on K_0. Since any class in $R(GL_n)$ can be written as a difference of classes of "true" representations the previous Lemma implies that, for arbitrary ρ, we still have a commutative diagram

$$\begin{tikzcd}
\pi_i(BGL_n(A)^+) \arrow[rr, "\pi_i(\rho)"] \arrow[dr, "c_{i,j}(\rho)"'] & & K_i(A) \arrow[dl, "c_{i,j}"] \\
& H^{2j-i}(U,\mathcal{F}^{\cdot}) &
\end{tikzcd} \quad .$$

Proposition:

For $i,k \geq 1$ and $j \geq 0$ we have $c_{i,j} \circ \psi^k = k^j \cdot c_{i,j}$ where ψ^k denotes the k-th Adams operation on K-theory.

Proof: By the homotopy property it suffices to prove the assertion for an affine scheme $U = \mathrm{Spec}(A)$ in $\mathcal{V}$. According to [Kra] §5 we have

$$\psi^k \text{ on } K_i(A) = \varinjlim \pi_i(\psi^k([id_n] - [\underline{1}_n]))$$

where id_n, resp. $\underline{1}_n$, denotes the identity, resp. trivial, n-dimensional representation of GL_n. Using the last Lemma and our Lemma about the Adams operations on $Ch(.)$ we compute

$$\begin{aligned}
c_{i,j} \circ \psi^k &= \varinjlim c_{i,j}(\psi^k([id_n] - [\underline{1}_n])) \\
&= \varinjlim H_i(c_j(\psi^k([id_n] - [\underline{1}_n]))) \circ \text{Hurewicz} \\
&= \varinjlim H_i(k^j c_j([id_n] - [\underline{1}_n])) \circ \text{Hurewicz} \\
&= k^j \cdot \varinjlim H_i(c_j^{(n)}) \circ \text{Hurewicz} \\
&= k^j \cdot c_{i,j} \quad . \qquad \text{q.e.d.}
\end{aligned}$$

In the affine case the product in K-theory can be defined in the following way ([Lod]): The tensor product representations $id_r \otimes id_s$, $id_r \otimes \underline{1}_s$, and $\underline{1}_r \otimes id_s$ define continuous maps

$$id_r \otimes id_s, \ldots : BGL_r(A)^+ \times BGL_s(A)^+ \longrightarrow BGL_{rs}(A)^+$$

and using the H-space structure of $BGL(A)^+$ we obtain the homotopy class of maps

$$id_r \otimes id_s - id_r \otimes \underline{1}_s - \underline{1}_r \otimes id_s : BGL_r(A)^+ \times BGL_s(A)^+ \to BGL(A)^+ \; .$$

This homotopy class factorizes through a homotopy class of maps

$$BGL_r(A)^+ \wedge BGL_s(A)^+ \xrightarrow{\mu_{r,s}} BGL(A)^+$$

and those $\mu_{r,s}$ are compatible with respect to varying r and s and define in the limit a weak homotopy class of maps

$$BGL(A)^+ \wedge BGL(A)^+ \xrightarrow{\mu} BGL(A)^+ \; .$$

If we fix $i_1, i_2 \geq 1$ and put $i := i_1 + i_2$ the product is given by the composed homomorphism

$$\cdot : K_{i_1}(A) \times K_{i_2}(A) \to \pi_i(BGL(A)^+ \wedge BGL(A)^+) \stackrel{\pi_i(\mu)}{\longrightarrow} K_i(A)\ .$$

First we have to see how this product is behaved with respect to the Hurewicz map. Let $\otimes_{r,s}$ denote the composed homomorphism

$$\begin{aligned}\otimes_{r,s} : H_{i_1}(GL_r(A), \mathbb{Z}) \otimes H_{i_2}(GL_s(A), \mathbb{Z}) &\longrightarrow H_i(GL_r(A) \times GL_s(A), \mathbb{Z}) \\ &\stackrel{H_i(id_r \otimes id_s)}{\longrightarrow} H_i(GL_{rs}(A), \mathbb{Z}) \longrightarrow H_i(GL(A), \mathbb{Z})\ .\end{aligned}$$

Similarly the direct sum representation $id_r \oplus id_s$ induces a homomorphism

$$\begin{aligned}\oplus_{r,s} : H_{i_1}(GL_r(A), \mathbb{Z}) \otimes H_{i_2}(GL_s(A), \mathbb{Z}) &\longrightarrow H_i(GL_r(A) \times GL_s(A), \mathbb{Z}) \\ &\stackrel{H_i(id_r \oplus id_s)}{\longrightarrow} H_i(GL_{r+s}(A), \mathbb{Z}) \longrightarrow H_i(GL(A), \mathbb{Z})\ .\end{aligned}$$

Lemma:

The diagram

$$\begin{array}{ccc} \pi_{i_1}(BGL_r(A)^+) \otimes \pi_{i_2}(BGL_s(A)^+) & \stackrel{\cdot}{\longrightarrow} & K_i(A) \\ \text{Hurewicz} \big\downarrow \otimes \text{Hurewicz} & & \big\downarrow \text{Hurewicz} \\ H_{i_1}(GL_r(A), \mathbb{Z}) \otimes H_{i_2}(GL_s(A), \mathbb{Z}) & \stackrel{\otimes_{r,s} - rs \cdot \oplus_{r,s}}{\longrightarrow} & H_i(GL(A), \mathbb{Z}) \end{array}$$

is commutative.

Proof: [Sus 2] (4.2).

Recall that $u = \{[E^n] - [\underline{1}^n]\}_n \in \varprojlim K_0(B.GL_n)$.

Lemma:

The diagram

$$\begin{array}{ccc} H_{i_1}(GL_r(A), \mathbb{Z}) \otimes H_{i_2}(GL_s(A), \mathbb{Z}) & \stackrel{\oplus_{r,s}}{\longrightarrow} & H_i(GL(A), \mathbb{Z}) \\ \sum H_{i_1}(c_{j_1}(u)) \otimes \big\downarrow H_{i_2}(c_{j_2}(u)) & & H_i(c_j(u)) \big\downarrow \\ \bigoplus_{j_1 + j_2 = j} H^{2j_1 - i_1}(U, \mathcal{F}^\cdot) \otimes H^{2j_2 - i_2}(U, \mathcal{F}^\cdot) & \stackrel{\cup}{\longrightarrow} & H^{2j-i}(U, \mathcal{F}^\cdot) \end{array}$$

is commutative.

Proof: This follows from the fact that the preimage of the universal vector bundle E^{r+s} under the morphism

$$B.GL_r \times B.GL_s \stackrel{id_r \oplus id_s}{\longrightarrow} B.GL_{r+s}$$

is isomorphic to the direct sum bundle $pr_1^* E^r \oplus pr_2^* E^s$.

Lemma:

The diagram

$$\begin{array}{ccc}
\pi_{i_1}(BGL_r(A)^+)\otimes\pi_{i_2}(BGL_s(A)^+) & \xrightarrow[\text{Hurewicz}]{\text{Hurewicz}\otimes} & H_{i_1}(GL_r(A),\mathbb{Z})\otimes H_{i_2}(GL_s(A),\mathbb{Z}) \\
 & & \otimes_{r,s}\downarrow \\
\sum\left(rs-\frac{(j-1)!}{(j_1-1)!(j_2-1)!}\right)\Big\downarrow c_{i_1,j_1}\otimes c_{i_1,j_2} & & H_i(GL(A),\mathbb{Z}) \\
 & & H_i(c_j(u))\downarrow \\
\bigoplus_{j_1+j_2=j} H^{2j_1-i_1}(U,\mathcal{F}^{\cdot})\otimes H^{2j_2-i_2}(U,\mathcal{F}^{\cdot}) & \xrightarrow{\cup} & H^{2j-i}(U,\mathcal{F}^{\cdot})
\end{array}$$

is commutative.

Proof: The preimage of the universal vector bundle E^{rs} under the morphism

$$B.GL_r\times B.GL_s \xrightarrow{id_r\otimes id_s} B.GL_{rs}$$

is isomorphic to the tensor product bundle $pr_1^*E^r\otimes pr_2^*E^s$. Therefore we have the commutative diagram

$$\begin{array}{ccccc}
H_i(GL_r(A)\times GL_s(A),\mathbb{Z}) & \xrightarrow{H_i(id_r\otimes id_s)} & H_i(GL_{rs}(A),\mathbb{Z}) & \longrightarrow & H_i(GL(A),\mathbb{Z}) \\
 & {\scriptstyle H_i(c_j(pr_1^*E^r\otimes pr_2^*E^s))}\searrow & & \swarrow{\scriptstyle H_i(c_j(u))} & \\
 & & H^{2j-i}(U,\mathcal{F}^{\cdot}) & &
\end{array}$$

.

By the theory of the Chern ring $Ch(.)$ the Chern classes of a tensor product bundle can be expressed as a polynomial in the Chern classes of the two factors. This leads to a commutative diagram

$$\begin{array}{ccc}
H_{i_1}(GL_r(A),\mathbb{Z})\otimes H_{i_2}(GL_s(A),\mathbb{Z}) & \longrightarrow & H_i(GL_r(A)\times GL_s(A),\mathbb{Z}) \\
q\downarrow & & H_i(c_j\downarrow(pr_1^*E^r\otimes pr_2^*E^s)) \\
\bigoplus_{j_1+j_2=j} H^{2j_1-i_1}(U,\mathcal{F}^{\cdot})\otimes H^{2j_2-i_2}(U,\mathcal{F}^{\cdot}) & \xrightarrow{\cup} & H^{2j-i}(U,\mathcal{F}^{\cdot})
\end{array}$$

where the homomorphism q is of the form

$$q=\sum_{\nu} a_\nu H_{i_1}(M_\nu(c_1(u),...,c_j(u)))\otimes H_{i_2}(N_\nu(c_1(u),...,c_j(u)))$$

with certain universal monomials $M_\nu(X_1,...,X_j)$ and $N_\nu(Y_1,...,Y_j)$ and certain universal integers a_ν both depending only on r, s, and j. But we already know that

$$H_{*+1}(M(c_1(u),...,c_j(u)))\circ\text{Hurewicz}=0$$

for any monomial M of degree > 1. On the other hand an explicit calculation in the Chern ring (SGA 6 exp. 0 App. I §3) shows that the monomial $X_{j_1}Y_{j_2}$ occurs in the above expression with the coefficient

$$rs - \frac{(j-1)!}{(j_1-1)!(j_2-1)!} \quad .$$

Proposition:

For $i_1, i_2 \geq 1, i := i_1 + i_2, j \geq 0$, and $x \in K_{i_1}, y \in K_{i_2}$ we have

$$c_{i,j}(x \cdot y) = \sum_{j_1+j_2=j} \frac{-(j-1)!}{(j_1-1)!(j_2-1)!} c_{i_1,j_1}(x) \cup c_{i_2,j_2}(y) \quad .$$

Proof: Again by the homotopy property it suffices to treat the affine case. But here one only has to combine the three Lemmata above. (For different proofs compare [Sou 1] p. 262-265 and [Gil 1] Prop. 2.35.)

In order to bring these results in a particularly nice form we now assume that the cohomology groups $H^*(.,\mathcal{F}^\cdot)$ of our complex $\mathcal{F}^\cdot$ are $\mathbb{Q}$-vector spaces. We then define the Chern character

$$ch : K_i(.) \longrightarrow \bigoplus_{j\geq 0} H^{2j-i}(.,\mathcal{F}^\cdot)$$

by

$$ch := \begin{cases} \sum_{j\geq 1} \frac{(-1)^{j-1}}{(j-1)!} c_{i,j} & \text{if} \quad i \geq 1 \quad , \\ ch_{0,0} + \sum_{j\geq 1} \frac{(-1)^{j-1}}{(j-1)!} \tilde{c}_{0,j} & \text{if} \quad i = 0 \end{cases}$$

where

$$ch_{0,0} : K_0(.) \xrightarrow{\text{rank}} H^0(.,\mathbb{Z}) \xrightarrow{e} H^0(.,\mathcal{F}^\cdot)$$

and

$$\sum_{j\geq 1} \tilde{c}_{0,j} t^j = \log(1 + \sum_{j\geq 1} c_{0,j} t^j)$$

(compare SGA 6 exp. V §6.3).

Corollary:

i. For $i, j \geq 0$ we have $ch(K_i^{(j)}(.)) \subseteq H^{2j-i}(.,\mathcal{F}^\cdot)$;
ii. for $i_1, i_2 \geq 0$ and $x \in K_{i_1}, y \in K_{i_2}$ we have

$$ch(x \cdot y) = ch(x) \cup ch(y) \quad .$$

Taking into account that Adams operations and product on K-theory are compatible ([Hil] or [Kra]) we get a "natural transformation"

$$R : H^*_{\mathcal{A}}(.,\mathbb{Q}(*)) \longrightarrow H^*(.,\mathcal{F}^\cdot)$$

which respects products and which satisfies the relation

$$ch = R \circ ch_{\mathcal{A}} \quad .$$

In particular, this justifies Beilinson's point of view that $H^*_{\mathcal{A}}$ and $ch_{\mathcal{A}}$ are some kind of universal objects.

Remark:

In the applications $\mathcal{F}^{\cdot}$ often is a graded complex $\mathcal{F}^{\cdot} = \bigoplus_{j\geq 0} \mathcal{F}^{\cdot}(j)$. The product structure then should be given by homomorphisms

$$e : \mathbb{Z} \to \mathcal{F}^{\cdot}(0) \quad \text{and} \quad \cup : \mathcal{F}^{\cdot}(j) \overset{L}{\underset{\mathbb{Z}}{\otimes}} \mathcal{F}^{\cdot}(j') \to \mathcal{F}^{\cdot}(j+j') \quad ;$$

furthermore, the homomorphism $\tilde{c}$ should be of the form $\tilde{c} : \mathbf{G}_m[-1] \to \mathcal{F}^{\cdot}(1)$. The universal Chern classes then lie in $H^{2j}(B.GL_n, \mathcal{F}^{\cdot}(j))$ and the Chern class maps consequently are of the form

$$c_{i,j} : K_i(.) \longrightarrow H^{2j-i}(., \mathcal{F}^{\cdot}(j)) \quad .$$

Here is a list of the most important examples of complexes $\mathcal{F}^{\cdot}$ which have the properties (I)-(IV) and therefore give rise to corresponding Chern class maps:

- $\mathcal{F}^{\cdot} := \Omega^{\cdot}_Y$ the algebraic de Rham complex (in the Zariski topology); here the base field has characteristic 0; - [Hart].
- $\mathcal{F}^{\cdot}(j) := \mu_m^{\otimes j}$ the j-th tensor power of the sheaf of m-th roots of unity in the etale topology; her m is prime to the characteristic of the base field; - [Mil].
- $\mathcal{F}^{\cdot}(j) := W.\Omega^j_{Y,\log}[-j]$ the "logarithmic part" (in the etale topology) of the de Rham - Witt complex; here the base field is perfect of characteristic > 0; - [Gros]. The homotopy property does not hold! Similarly, $\mathcal{F}^{\cdot} := W.\Omega^{\cdot}_Y$ gives rise to the crystalline Chern class maps.
- $\mathcal{F}^{\cdot}(j) := \mathcal{K}_j[-j]$ the sheafification (in the Zariski topology) of Quillen's K-groups; - [Gil 1], [She], [Sch].
- $\mathcal{F}^{\cdot}(j) := z^j[-2j]$ the complex (in the Zariski topology) which computes Bloch's higher Chow groups (it is expected but not known, at present, to be bounded below); - [Blo 3]. Bloch proves the very remarkable fact that the "natural transformation" R in this case induces isomorphisms $H^*_{\mathcal{A}}(., \mathbb{Q}(j)) \xrightarrow{\cong} H^*(., z^j[-2j]) \otimes \mathbb{Q}$.

The example of a complex $\mathcal{F}^{\cdot}$ we are especially interested in in this paper is the Deligne complex $\bigoplus_{j\geq 0} \mathbb{Z}(j)_{\mathcal{D}}$. Here, $\mathcal{V}$ is the category of smooth quasi-projective schemes over the field $\mathbb{C}$ equipped with the analytic topology. In §2 we defined the complexes $\mathbb{Z}(j)_{\mathcal{D}}$ on projective schemes in $\mathcal{V}$. If we take the same definition on any scheme $\mathcal{V}$ then these complexes seem to have the properties (II)-(IV) (not the homotopy property) which would suffice for the construction of Chern class maps. But we do not pursue this here since there is a second and much more important way to extend the definition of the complexes $\mathbb{Z}(j)_{\mathcal{D}}$ to all schemes in $\mathcal{V}$. It involves the theory of smooth compactifications by divisors with normal crossings and the theory of holomorphic forms with logarithmic singularities and is explained in [Bei 1] §1 or in the Chapter on Deligne cohomology in this book. In this case the properties (I)-(IV) are established in [Bei 1] §1. Consequently we have the Chern class maps

$$c_{i,j} : K_i(.) \longrightarrow H^{2j-i}_{\mathcal{D}}(., \mathbb{Z}(j))$$

which induce (as described above) a "natural transformation"

$$R : H^*_{\mathcal{A}}(., \mathbb{Q}(*)) \longrightarrow H^*_{\mathcal{D}}(., \mathbb{Q}(*)) \ .$$

For any smooth projective variety X over $\mathbb{Q}$ we now define the regulator map to be

$$\begin{array}{l} \mathrm{reg} : H^*_{\mathcal{A}}(X, \mathbb{Q}(*)) \to H^*_{\mathcal{A}}(X_{/\mathbb{R}}, \mathbb{Q}(*)) \xrightarrow{R} H^*_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(*))^{\mathrm{DR-conjugation}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad H^*_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(*)) \ . \end{array}$$

§5 The conjectures

Now, let $X_{/\mathbb{Q}}$ be again a projective smooth variety over $\mathbb{Q}$. As before we let M denote the family of all i-th cohomology groups of X for some fixed integer i between 0 and $2\dim X$. We assume that the Hypotheses (I)-(V) in §1 are fulfilled so that we have the complex L-function $L(M,s)$ of M with all its expected analytic properties. Our interest lies in the numbers

$$\mathrm{ord}_{s=m} L(M,s) := \text{multiplicity of } L(M,s) \text{ at } s = m$$

and

$$L^*(M,m) := \text{leading coefficient of } L(M,s) \text{ in a Taylor series expansion at } s = m$$

where m is an integer $\leq \frac{i}{2}+1$ (in the following we exclude the central point $m = \frac{i+1}{2}$ since it is somewhat of a different nature - but see the last Paragraph). In §2 we have seen that

$$\mathrm{ord}_{s=m} L(M,s) = \dim_{\mathbb{R}} H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n))(+\,\mathrm{ord}_{s=n} L(M,s))$$

holds true if $m < \frac{i}{2} (m = \frac{i}{2})$; here again we always put $n := i+1-m$. Furthermore we have constructed a canonical isomorphism

$$\begin{array}{l} \wedge^{\max} F^n H^i_{DR}(X_{/\mathbb{R}}) \underset{\mathbb{R}}{\otimes} \wedge^{\max} H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)) \xrightarrow{\cong} \\ \wedge^{\max} H^i(X(\mathbb{C}), \mathbb{R}(n-1))^{(-1)^{n-1}} \end{array}$$

for any $m < \frac{i+1}{2}$ and have discussed already that the first and the third term carry an obvious $\mathbb{Q}$-structure. The first conjecture says that the regulator map

$$\mathrm{reg} : H^*_{\mathcal{A}}(X, \mathbb{Q}(*)) \longrightarrow H^*_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(*))$$

constructed in §4 leads to a $\mathbb{Q}$-structure on the second term.

Conjecture I:

For $m < \frac{i}{2}$, the regulator map induces an isomorphism

$$H^{i+1}_{\mathcal{A}}(X, \mathbb{Q}(n))_{\mathbb{Z}} \otimes \mathbb{R} \xrightarrow{\cong} H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)) \ .$$

We now define the regulator $c_M(m) \in \mathbb{R}^\times/\mathbb{Q}^\times$ for $m < \frac{i}{2}$ to be the above isomorphism calculated in $\mathbb{Q}$-rational bases.

Conjecture II:

For $m < \frac{i}{2}$, we have $L^*(M,m) \equiv c_M(m) \bmod \mathbb{Q}^\times$.

In case $m = \frac{i}{2}$ the regulator map alone is not sufficient to induce a $\mathbb{Q}$-structure. It is easy to see that in addition the group

$$\begin{aligned} N^m(X) := {} & m\text{-codimensional cycles on } X_{/\mathbb{Q}} \\ & \text{modulo homological equivalence (over } \overline{\mathbb{Q}}) \end{aligned}$$

in a natural way is contained in the corresponding Deligne cohomology group: Let

$$z : N^m(X) \longrightarrow H^{2m}_{DR}(X_{/\mathbb{R}}) \subseteq H^{2m}_{DR}(X_{/\mathbb{C}}) = H^{2m}(X(\mathbb{C}), \mathbb{C})$$

be the cycle map into the de Rham cohomology. It is well-known that we have

$$z(N^m(X)) \subseteq H^{m,m} \cap H^{2m}(X(\mathbb{C}), \mathbb{Q}(m))$$

(compare [Gro 2] (6.14)). Consequently

$$z(N^m(X)) \subseteq H^{2m}(X(\mathbb{C}), \mathbb{R}(m))^{(-1)^m}$$

and

$$z(N^m(X)) \cap F^{m+1} H^{2m}_{DR}(X_{/\mathbb{R}}) = 0$$

hold true. We therefore see from the exact sequence $(*)$ in §2 that the composed map

$$\tilde{z} : N^m(X) \xrightarrow{z} H^{2m}(X(\mathbb{C}), \mathbb{R}(m))^{(-1)^m} \longrightarrow H^{2m+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(m+1))$$

is injective. (Warning: $\tilde{z}$ is not the cycle map into the Deligne cohomology.)

Conjecture III:

For $m = \frac{i}{2}$ and $n = \frac{i}{2} + 1$ we have:

a. The maps reg *and $\tilde{z}$ together induce an isomorphism*

$$(H^{i+1}_{\mathcal{A}}(X, \mathbb{Q}(n))_{\mathbb{Z}} \otimes \mathbb{R}) \oplus (N^m(X) \otimes \mathbb{R}) \xrightarrow{\cong} H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)) \quad ;$$

b. $\mathrm{ord}_{s=m} L(M,s) = \dim_{\mathbb{Q}} H^{i+1}_{\mathcal{A}}(X, \mathbb{Q}(n))_{\mathbb{Z}}$*;*

c. ([Tat 1]) $\mathrm{ord}_{s=n} L(M,s) = -\mathrm{rank}\, N^m(X)$*;*

d. if $c_M(m) \in \mathbb{R}^\times/\mathbb{Q}^\times$ denotes the regulator defined as before by using part a. then $L^*(M,m) \equiv c_M(m) \bmod \mathbb{Q}^\times$.

§6 Further hints

In the last Paragraph we did not discuss the center $m = \frac{i+1}{2}$ of the functional equation. In that case the exact sequence (*) in §2 has to be replaced by the exact sequence

$$0 \longrightarrow F^m H^i_{DR}(X_{/\mathbb{R}}) \longrightarrow H^i(X(\mathbb{C}), \mathbb{R}(m-1))^{(-1)^{m-1}} \longrightarrow H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(m))$$
$$\longrightarrow H^{2m}(X(\mathbb{C}), \mathbb{R}(m))^{(-1)^m} \cap H^{m,m} \longrightarrow 0 \ .$$

Since the first two terms obviously have the same $\mathbb{R}$-dimension this sequence breaks up into two isomorphisms of which the first one

$$F^m H^i_{DR}(X_{/\mathbb{R}}) \xrightarrow{\cong} H^i(X(\mathbb{C}), \mathbb{R}(m-1))^{(-1)^{m-1}}$$

can be used, as in §2, to define the Deligne period $c_M(m) \in \mathbb{R}^\times/\mathbb{Q}^\times$ (= the determinant calculated in the obvious $\mathbb{Q}$-rational bases). And, indeed, Deligne conjectures ([Del]§1) that

$$L(M,m) \cdot c_M(m)^{-1} \in \mathbb{Q} \ .$$

But, of course, $L(M,s)$ often will vanish at $s = m = \frac{i+1}{2}$; the functional equation only can detect the parity of the vanishing order. In the following we will very briefly indicate a refined conjecture which was proposed by Beilinson ([Bei 1,3]) and Bloch ([Blo 1,2]). We fix an odd i, put $m = \frac{i+1}{2}$, and define

$$Ch^m(X)^0 := (m\text{-codimensional cycles on } X \text{ cohomologous to 0 (over } \overline{\mathbb{Q}})$$
$$\text{modulo rational equivalence}) \otimes \mathbb{Q}.$$

Conjecture:

a. $Ch^m(X)^0$ has finite dimension;

b. there exists a natural nondegenerate "height pairing"

$$< \ , \ >_m : CH^m(X)^0 \times CH^{\dim X - m + 1}(X)^0 \longrightarrow \mathbb{R} \ ;$$

c. $\mathrm{ord}_{s=m} L(M,s) = \dim_{\mathbb{Q}} CH^m(X)^0$ and

$$L^*(M,m) \equiv c_M(m) \cdot \det \ < \ , \ >_m \bmod \mathbb{Q}^\times \ .$$

If X is an abelian variety and $i = m = 1$ then part a. of the above Conjecture is the theorem of Mordell-Weil, part b. is the theory of the Néron-Tate height, and part c. is part of the conjecture of Birch and Swinnerton-Dyer. In [Tat 2] the reader may find a discussion of this case in which the conjectural picture is even more precise insofar as $L^*(M,m)$ itself (not only $\bmod \mathbb{Q}^\times$) is predicted in terms of arithmetic invariants of X. The general conjecture certainly is modeled on this case. Beilinson ([Bei 1,3]), Bloch ([Blo 2]), and Gillet/Soulé ([GS]) construct - all three by different techniques - a natural height pairing for any X which has certain geometric properties (conjecturally it always should have those). At least the archimedean component of this pairing is defined for any X independent of additional assumptions; we should indicate that the reason for this lies in the fact, which we have seen above, that the canonical map $H^{2m}_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(m)) \to H^{2m}(X(\mathbb{C}), \mathbb{R}(m))$ is injective.

I also have to keep the promise to say something about the groups $H^{2p}_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{Z}(p))$ which turn out to be very important but did not play a role in the previous Paragraphs. It is straightforward that the exact sequence $(*)$ in §2 in this context becomes an exact sequence

$$\begin{aligned} 0 \longrightarrow H^{2p-1}(X(\mathbb{C}), \mathbb{Z}(p)) \backslash H^{2p-1}_{DR}(X(\mathbb{C}))/F^p \longrightarrow H^{2p}_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{Z}(p)) \\ \longrightarrow \text{preimage of } H^{p,p} \text{ in } H^{2p}(X(\mathbb{C}), \mathbb{Z}(p)) \longrightarrow 0 \ . \end{aligned}$$

The middle term appears as an extension of the group of Hodge p-cycles by the p-th intermediate Jacobian of Griffiths. Furthermore, the Chern character into the middle term combines the usual cycle map and Griffiths' Abel-Jacobi map (see [Bei 1] §1 or the Chapter on Deligne cohomology in this book).

Finally I cannot refrain from mentioning the following extremely fascinating line of thought due to Deligne and Beilinson. In [Bei 2] it is shown that, for $i < 2j$, the Deligne cohomology $H^i_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(j))$ can be interpreted as the Yoneda group $\mathrm{Ext}^1(\mathbb{R}, H^{i-1}(X(\mathbb{C}), \mathbb{R}(j)))$ in the category of mixed $\mathbb{R}$-Hodge structures over $\mathbb{R}$. One may speculate whether the absolute cohomology $H^i_{\mathcal{A}}(X, \mathbb{Q}(j))$ has a similar interpretation, for $i < 2j$, as a Yoneda group $\mathrm{Ext}^1(\mathbb{Q}, H^{i-1}(X)(j))$ in a not yet existing category of $\mathbb{Q}$-linear mixed motives over $\mathbb{Q}$. In this light the regulator map should simply be induced by the functor which associates with each mixed motive its realization as a mixed Hodge structure. The reader will find more about this in the final Chapter by Jannsen in this book.

For additional hints the reader is advised to read Soulé's Bourbaki article [Sou 3].

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