THE THEOREM OF RIEMANN-ROCH

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Introduction.

Let k be a fixed field, and $\mathcal{V}$ be the category of quasi-projective schemes over k. For every object X of $\mathcal{V}$, there are defined the groups $K_p(X)$, contravariant for arbitrary morphisms, and the groups $K'_p(X)$, covariant for proper morphisms in $\mathcal{V}$. The Riemann-Roch problem is to compute the map

$$f_* : K'_p(X) \to K'_p(Y)$$

for a proper morphism $f : X \to Y$; in case $p = 0$ this means to compute the Euler-Poincaré characteristic

$$f_* : K'_o(X) \to K'_o(Y)$$
$$[F] \mapsto \Sigma(-1)^q [R^q f_* (F)].$$

If both schemes X and Y are smooth, their K'-groups can be identified with the K-groups, and the push-forward can be read as a homomorphism $f_* : K_p(X) \to K_p(Y)$. The classical Grothendieck Riemann-Roch theorem [BS] is concerned with

$$f_* : K_o(X) \to K_o(Y)$$

and computes the Chern character of $f_*(x)$ for a given x, i.e. computes the image $f_*(x)$ modulo torsion. We recall, using the terminology of SGA 6:

For a smooth X, let $Gr^{\cdot}K_o(X)$ denote the associated graded ring of the Grothendieck filtration on $K_o(X)$, defined by the augmented λ-ring structure of $K_o(X)$ over $H^o(x,\mathbb{Z})$. Let $ch : K_o(X) \to Gr^{\cdot}K_o(X) \otimes \mathbb{Q}$ be the Chern character, and $Td(X) \in Gr^{\cdot}K_o(X) \otimes \mathbb{Q}$ be the Todd class of X, both defined by means of the universal Chern classes. Then the theorem asserts:

i) f_* induces a graded homomorphism

$$f_* : Gr^{\cdot}K_o(X) \otimes \mathbb{Q} \to Gr^{\cdot}K_o(Y) \otimes \mathbb{Q} ,$$

the Gysin homomorphism.

ii) The diagram

$$\begin{array}{ccc} K_o(X) & \xrightarrow{Td(X)ch} & Gr^{\cdot}K_o(X) \otimes \mathbb{Q} \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ K_o(Y) & \xrightarrow{Td(Y)ch} & Gr^{\cdot}K_o(Y) \otimes \mathbb{Q} \end{array}$$

commutes, i.e.

$$ch(f_*(x)) = Td(Y)^{-1} f_*(Td(X)ch(x))$$

for $x \in K_o(X)$.

The starting point for solving the general Riemann-Roch problem is given by two basic results, due to Quillen and due to Soulé ([S], and see §1 for more details):

1°. Let M be a smooth scheme, and $X \hookrightarrow M$ be an arbitrary closed subscheme. Then there is a canonical isomorphism

$$K'_p(X) \cong K_p^X(M),$$

where $K_p^X(M)$ denotes the K-groups of M with supports in X (Purity theorem).

2°. Let

$$K^X(M) = \bigoplus_{p \geqq 0} K_p^X(M).$$

Then $K^X(M)$ carries the structure of an augmented λ-algebra over the binomial λ-ring $H_X^0(M,\mathbb{Z})$. Let

$$ch : K^X(M) \to Gr^{\cdot} K^X(M) \otimes \mathbb{Q}$$

denote the associated Chern character.

These results have two aspects. The first one is the following: Let

$$\begin{array}{ccc} X & \hookrightarrow & M \\ \downarrow & & \downarrow f \\ Y & \hookrightarrow & N \end{array}$$

be a commutative diagram in $\mathcal{V}$, where the horizontal arrows are closed immersions into smooth schemes, and where $f : M \to N$ is proper. Then the push-forward for the K'-groups of X and Y induces via purity an additive homomorphism

$$f_* : K^X(M) \to K^Y(N).$$

For this homomorphism f_*, we prove as a main result and as a direct generalization of the classical Riemann-Roch the following Riemann-Roch theorem:

i) f_* induces a Gysin homomorphism

$$f_* : Gr^{\cdot} K^X(M) \otimes \mathbb{Q} \to Gr^{\cdot} K^Y(N) \otimes \mathbb{Q} .$$

ii) The diagram

$$\begin{array}{ccc} K^X(M) & \xrightarrow{Td(M)\,ch} & Gr^{\cdot} K^X(M) \otimes \mathbb{Q} \\ f_* \downarrow & & \downarrow f_* \\ K^Y(N) & \xrightarrow{Td(N)\,ch} & Gr^{\cdot} K^Y(N) \otimes \mathbb{Q} \end{array}$$

commutes.

The proof of this theorem is given in §3. The main step concerns the case in which $f : M \to N$ is a closed immersion. In this case, the theorem follows mainly from the Riemann-Roch theorem without denominators, proved in §2. It describes the effect of $f_* : K^X(M) \to K^Y(N)$ on the Adams operations, and more generally on arbitrary natural operations of augmentation 0.

The second aspect of the basic results above is the following ([BFM], [G], [S]):

Given a scheme X. We choose a closed immersion $X \hookrightarrow M$ of X into a smooth scheme M of pure dimension, say d. After identifying $K'(X) = \oplus K'_p(X)$ with $K^X(M)$ via purity, one defines a lower filtration on $K'(X) \otimes \mathbb{Q}$ by setting

$$F_n K'(X) \otimes \mathbb{Q} = F^{d-n} K^X(M) \otimes \mathbb{Q}$$

and a morphism

$$\tau : K'(X) \to Gr_{\bullet}K'(X) \otimes Q$$

into the associated graded group by the commutativity of

$$\begin{array}{ccc} K'(X) & \xrightarrow{\tau} & Gr_{\bullet}K'(X) \otimes Q \\ \| & & \| \\ K^X(M) & \xrightarrow{Td(M)ch} & Gr^{\bullet}K^X(M) \otimes Q \end{array} .$$

As more or less a corollary of the Riemann-Roch theorem above, we prove in §4 that the filtration on $K'(X) \otimes Q$ and the map $\tau : K'(X) \to Gr^{\bullet}K'(X) \otimes Q$ are well defined, and set up a singular Riemann-Roch theorem in the sense of Baum-Fulton-MacPherson, thus solving the original stated Riemann-Roch problem.

From the singular Riemann-Roch theorem we deduce in §5 that the absolute cohomology and homology on V, defined by

$$H^p(X,j) = Gr^j K_{2j-p}(X) \otimes Q$$
$$H_p(X,j) = Gr_j K'_{p-2j}(X) \otimes Q$$

satisfy the axioms of a twisted cohomology-homology theory with Poincaré duality in the sense of Bloch-Ogus [BO].

Needless to say that the main reference for this article is the beautiful paper [S].

§1. The λ-ring structure and the Chern character for K-theory with supports

In this paragraph we review the definition of algebraic K-theory with supports, its λ-ring structure, Chern classes and Chern character. We only give indications of proofs, if at all. We work with the category of quasi-projective schemes over a fixed field k; so all schemes under consideration are quasi-projective k-schemes and all morphisms of schemes are k-morphisms.

1. Definition and functorial behaviour of K-theory with supports

Let X be a scheme. Let $\mathcal{P}(X)$ denote the exact category of locally free $\mathcal{O}_X$-modules of finite rank. For $p \geq 0$ the p-th K-group of X is defined by

$$K_p(X) = \pi_{p+1}(B\,Q\,\mathcal{P}(X))$$

where $B\,Q\,\mathcal{P}(X)$ is the classifying space of the Quillen category $Q\,\mathcal{P}(X)$ associated to the exact category $\mathcal{P}(X)$ and the homotopy group is formed with respect to the zero object of $\mathcal{P}(X)$ as base point ([Q], §7).

Let $Y \hookrightarrow X$ be a closed subscheme of X. Then the restriction $\mathcal{P}(X) \to \mathcal{P}(X-Y)$ is an exact functor and induces a continuous map $B\,Q\,\mathcal{P}(X) \to B\,Q\,\mathcal{P}(X-Y)$ of pointed topological spaces.

Definition 1.1. For $p \geq 0$ the group

$$K_p^Y(X) = \pi_{p+1}(\text{Homotopy-fibre of } BQ\mathcal{P}(X) \to BQ\mathcal{P}(X-Y) \text{ over } 0)$$

is called the p-th K-group of X with supports in Y.

As the exact homotopy sequence of $BQ\mathcal{P}(X) \to BQ\mathcal{P}(X-Y)$ we get the long exact sequence

$$(1.2) \qquad \to K_p^Y(X) \to K_p(X) \to K_p(X-Y) \xrightarrow{\partial} K_{p-1}^Y(X) \to \quad .$$

Let

$$\begin{array}{ccc} Y' & \hookrightarrow & X' \\ \downarrow & & \downarrow f \\ Y & \hookrightarrow & X \end{array}$$

be a cartesian diagram of schemes, where the horizontal arrows are closed immersions. Then one gets the commutative diagram

$$\begin{array}{ccc} \mathcal{P}(X) & \longrightarrow & \mathcal{P}(X-Y) \\ \downarrow f^* & & \downarrow \\ \mathcal{P}(X') & \longrightarrow & \mathcal{P}(X'-Y') \end{array}$$

of exact functors and hence a map from the exact sequence of (X,Y) to the one for (X',Y'):

$$(1.3) \qquad \begin{array}{ccccccccc} \to K_p^Y(X) & \longrightarrow & K_p(X) & \longrightarrow & K_p(X-Y) & \xrightarrow{\partial} & K_{p-1}^Y(X) & \longrightarrow & \\ \downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \\ \to K_p^{Y'}(X') & \to & K_p(X') & \to & K_p(X'-Y') & \xrightarrow{\partial} & K_{p-1}^{Y'}(X') & \longrightarrow & . \end{array}$$

Let $Z \hookrightarrow Y \hookrightarrow X$ be closed subschemes. Then the exact functors

$$\mathcal{P}(X) \to \mathcal{P}(X-Z) \to \mathcal{P}(X-Y)$$

induce as the exact homotopy sequence of a composition of maps the long exact sequence

$$(1.4) \qquad \to K_p^Z(X) \to K_p^Y(X) \to K_p^{Y-Z}(X-Z) \xrightarrow{\partial} K_{p-1}^Z(X) \to \quad .$$

A morphism $f : X' \to X$ induces a map from the exact sequence of (X,Y,Z) to the one for $(X', f^{-1}(Y), f^{-1}(Z))$. The exact sequence of (X,X,Y) coincides with the exact sequence for the pair (X,Y).

2. The purity theorem

For a scheme X we denote by $\mathcal{M}(X)$ the abelian category of coherent $\mathcal{O}_X$-modules. For $p \geqq 0$ the p-th K'-group of X is defined by

$$K'_p(X) = \pi_{p+1}(B\,Q\,\mathcal{M}(X))$$

where again the homotopy group is formed with respect to the zero object of $\mathcal{M}(X)$ as base point ([Q], §7).
Let $Y \hookrightarrow X$ be a closed subscheme of X. The restriction $\mathcal{M}(X) \to \mathcal{M}(X-Y)$ is an exact functor, and we consider the homotopy group π_{p+1} of the homotopy fibre of $B\,Q\,\mathcal{M}(X) \to B\,Q\,\mathcal{M}(X-Y)$ over 0.
As well known, the restriction $\mathcal{M}(X) \to \mathcal{M}(X-Y)$ induces an equivalence of the category $\mathcal{M}(X-Y)$ with the quotient category of $\mathcal{M}(X)$ by the Serre subcategory $\mathcal{S}$ consisting of coherent $\mathcal{O}_X$-modules with support in Y. Hence by Quillen's localization theorem ([Q], §5, Th. 5), the considered homotopy group identifies with $\pi_{p+1}(B\,Q\,\mathcal{S})$. The devissage theorem ([Q], §5, Th. 4) implies that the direct image $\mathcal{M}(Y) \to \mathcal{S}$ induces a homotopy equivalence $B\,Q\,\mathcal{M}(Y) \to B\,Q\,\mathcal{S}$. So we get a canonical identification

(2.1) $\quad K'_p(Y) = \pi_{p+1}$ (Homotopy fibre of $BQM(X) \to BQM(X-Y)$ over 0).

Now the commutative diagram

$$\begin{array}{ccc} P(X) & \longrightarrow & P(X-Y) \\ \downarrow & & \downarrow \\ (X) & \longrightarrow & (X-Y) \end{array}$$

of exact functors induces a map of exact sequences

$$(2.2) \qquad \begin{array}{ccccccccc} \to & K_p^Y(X) & \to & K_p(X) & \to & K_p(X-Y) & \to & K_{p-1}^Y(X) & \to \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \to & K'_p(Y) & \to & K'_p(X) & \to & K'_p(X-Y) & \to & K'_{p-1}(Y) & \to \end{array}$$

More generally, for closed subschemes $Z \hookrightarrow Y \hookrightarrow X$ one gets a map from the exact sequence of (X,Y,Z) to the exact K'-sequence for (Y,Z).

Let

$$\begin{array}{ccc} Y' & \hookrightarrow & X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \hookrightarrow & X \end{array}$$

be a Cartesian diagram, where the horizontal arrows are closed immersions and the vertical arrows are <u>flat</u>. Then the diagram

$$\begin{array}{ccc} K_p^Y(X) & \xrightarrow{f^*} & K_p^{Y'}(X') \\ \downarrow & & \downarrow \\ K'_p(Y) & \xrightarrow{f'^*} & K'_p(Y') \end{array}$$

commutes, where the lower horizontal map is induced by the exact inverse image functor $f'^* : \mathcal{M}(Y) \to \mathcal{M}(Y')$.

For smooth schemes X the homomorophism $K_p(X) \to K'_p(X)$ is an isomorphism ([Q], §7), and it follows from (2.2):

<u>Theorem 2.4</u> (Purity for smooth schemes).
If X is a <u>smooth</u> scheme and $Y \hookrightarrow X$ a closed subscheme, then the canonical map

$$K_p^Y(X) \to K'_p(Y)$$

is an isomorphism.

From this purity theorem we get push-forward homomorphisms for the K-theory with supports as follows:
Given a commutative diagram

$$\begin{array}{ccc} Y' & \hookrightarrow & X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \hookrightarrow & X \end{array}$$

of schemes, where the horizontal arrows are closed immersions, and where $f : X' \to X$ is a proper morphism of smooth schemes.
Then we have a homomorphism

$$(2.5) \qquad f_* : K_p^{Y'}(X) \to K_p^Y(X)$$

uniquely determined by the commutativity of the diagram

$$\begin{array}{ccc} K_p^{Y'}(X') & \xrightarrow{f_*} & K_p^Y(X) \\ \downarrow \cong & & \downarrow \cong \\ K'_p(Y') & \xrightarrow{f'_*} & K'_p(Y) \end{array}$$

where the homomorphism f'_* between the K'-groups is induced by the proper morphism $f' : Y' \to Y$ (see [Q], §7, 2.7).

Let $\mathcal{V}_*$ denote the category of pairs (X,Y) with X a smooth scheme and $Y \hookrightarrow X$ a closed subscheme, in which a morphism $f : (X',Y') \to (X,Y)$ is a commutative diagram

$$\begin{array}{ccc} Y' & \hookrightarrow & X' \\ \downarrow & & \downarrow f \\ Y & \hookrightarrow & X \end{array}$$

with a proper morphism $f : X' \to X$. It is clear that the assignment $f \mapsto f_*$ is a covariant functor on $\mathcal{V}_*$. The smooth Riemann-Roch for K-theory with supports will be concerned with the push-forwards f_*.

3. <u>The λ-ring structure on $K^Y(X)$</u>

Let X be a scheme, and $Y \hookrightarrow X$ be a closed subscheme. Given E in $\mathcal{P}(X)$, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{P}(X) & \longrightarrow & \mathcal{P}(X-Y) \\ {\scriptstyle -\otimes E}\downarrow & & \downarrow {\scriptstyle -\otimes E/X-Y} \\ \mathcal{P}(X) & \longrightarrow & \mathcal{P}(X-Y) \end{array}$$

of exact functors which induces a homomorphism $(-\otimes E)_* : K_p^Y(X) \to K_p^Y(X)$. By the addition formula ([Q], §3, Cor. 1) we obtain a product

$$K_o(X) \times K_p^Y(X) \to K_p^Y(X) ,$$

and via $K_o^Y(X) \to K_o(X)$ hence a product

$$K_o^Y(X) \times K_p^Y(X) \to K_p^Y(X) .$$

In the following we put

$$K^Y(X) = \bigoplus_{p \geqq 0} K_p^Y(X) ,$$

and we define on $K^Y(X)$ the structure of a commutative ring (without identity in general) by linear extension of the products

$$\begin{cases} K_o^Y(X) \times K_p^Y(X) \to K_p^Y(X) \\ K_p^Y(X) \times K_q^Y(X) \to 0 \quad \text{for } pq \neq 0 . \end{cases}$$

The ring homomorphisms

$$\begin{cases} H_Y^o(X,\mathbb{Z}) \to K_o^Y(X) \to K^Y(X) \\ \varepsilon : K^Y(X) \to K_o^Y(X) \to H_Y^o(X,\mathbb{Z}) \end{cases}$$

give $K^Y(X)$ the structure of an augmented algebra over the ring $H_Y^o(X,\mathbb{Z}) = \mathrm{Ker}(H^o(X,\mathbb{Z}) \to H^o(X-Y,\mathbb{Z}))$; here the maps $H_Y^o(X,\mathbb{Z}) \to K_o^Y(X)$ and $K_o^Y(X) \longrightarrow H_Y^o(X,\mathbb{Z})$ are induced by the evident map $H^o(X,\mathbb{Z}) \to K_o(X)$ and the rank map $K_o(X) \to H^o(X,\mathbb{Z})$.

$H_Y^o(X,\mathbb{Z})$ is a binomial ring (without identity), and as such it carries a canonical λ-ring structure, defined by the binomial coefficients $\lambda^n(x) = \binom{x}{n}$ for $n \geqq 1$ ([SGA 6], V, 2.6). Taking the opportunity, we recall: A λ-ring (sometimes called a special λ-ring) is a commutative ring K, together with a family $(\lambda^n)_{n\in\mathbb{N}}$ of maps $\lambda^n : K \to K$, such that $\lambda^1(x) = x$ and the universal formulas

$$\lambda^n(x+y) = \lambda^n(x) + \lambda^{n-1}(x)\lambda^1(y) + \ldots + \lambda^n(y)$$
$$\lambda^n(xy) = P_n(\lambda^1(x),\ldots,\lambda^n(x);\lambda^1(y),\ldots,\lambda^n(y))$$
$$\lambda^m(\lambda^n(x)) = P_{m,n}(\lambda^1(x),\ldots,\lambda^{mn}(x))$$

hold (see [Se]).

Let now X be a smooth scheme, and $Y \hookrightarrow X$ be a closed subscheme. Then Soulê ([S], §4) has defined a λ-structure on the ring $K^Y(X)$ by globalizing the definition of the λ-ring structure in the affine case, discussed in [Se].
The crucial point is a new interpretation of the groups $K_p^Y(X)$ in terms of generalized sheaf cohomology in the sense of Brown [B]: For any commutative ring A with 1 one can define a pointed simplicial set $BGL(A)^+$ in a functorial way, and one then defines a pointed simplicial sheaf BGL^+ on X as the associated sheaf of the simplicial presheaf $U \mapsto BGL(\Gamma(U,\mathcal{O}_X))^+$. If $\mathbb{Z}\otimes BGL^+$ denotes the product of the constant sheaf $\mathbb{Z}$ and BGL^+, then one has a canonical isomorphism

$$K_p^Y(X) \xrightarrow{\cong} H_Y^{-p}(X,\mathbb{Z}\otimes BGL^+)$$

(see [G], Prop. 2.15., [BG], Th. 5). Using this cohomological description of $K_p^Y(X)$, Soulê constructs a canonical map

$$\varprojlim R_{\mathbb{Z}}(GL_N) \to \mathrm{Map}(K_p^Y(X),K_p^Y(X)) ,$$

where $R_{\mathbb{Z}}(GL_N)$ means the Grothendieck ring of representations of the group scheme GL_N over $\mathbb{Z}$, and where the projective limit is formed with respect to the standard inclusions $GL_N \to GL_{N+1}$, as transition maps. The exterior powers define a λ-ring structure on each $R_{\mathbb{Z}}(GL_N)$, compatible with the transition maps. Then for every $n \geq 1$ one defines the map

$$\lambda^n : K_p^Y(X) \to K_p^Y(X)$$

as the image of $\varprojlim(\lambda^n(\mathrm{id}_N - N))$ under the canonical map above.

<u>Theorem 3.1.</u>

i) If X is a smooth scheme and $Y \hookrightarrow X$ a closed subscheme, then the family $(\lambda^n)_{n\in\mathbb{N}}$ of maps $\lambda^n : K^Y(X) \to K^Y(X)$ defines on $K^Y(X)$ the structure of an augmented λ-algebra over the binomial λ-ring $H_Y^0(X,\mathbb{Z})$, whose associated Grothendieck filtration $(F^n K^Y(X))_{n\geqq 0}$ is locally nilpotent.

ii) If

$$\begin{array}{ccc} Y' & \hookrightarrow & X' \\ \downarrow & & \downarrow f \\ Y & \hookrightarrow & X \end{array}$$

is a Cartesian diagram, where the horizontal arrows are closed immersions into smooth schemes, then the induced map

$$f^* : K^Y(X) \to K^{Y'}(X')$$

is a morphism of augmented λ-algebras.

iii) If $Z \hookrightarrow Y \hookrightarrow X$ are closed subschemes of the smooth scheme X, then the arrows in the long exact sequence

$$\to K^Z(X) \to K^Y(X) \to K^{Y-Z}(X-Z) \xrightarrow{\partial} K^Z(X)$$

are morphisms of augmented λ-algebras.

Remarks.

1) The properties i), ii), and iii) for the map $K^Z(X) \to K^Y(X)$ suffice to prove the smooth Riemann-Roch theorem for K-theory with supports (see §§2,3).

2) Recall that the Grothendieck filtration on $K^Y(X)$ is defined as follows: $F^nK^Y(X)$ is the $H^0_Y(X,\mathbb{Z})$-submodule of $K^Y(X)$ generated by the elements

$$\gamma^{n_1}(x_1)\ \dots\ \gamma^{n_r}(x_r) \text{ with } \begin{cases} x_1,\dots,x_r \in \mathrm{Ker}(\varepsilon) \\ n_1+\dots+n_r \geqq n \end{cases}.$$

The locally nilpotence of the filtration means: For every $x \in F^1K^Y(X) = \mathrm{Ker}(\varepsilon)$, there exists an $N \in \mathbb{N}$, such that $\gamma^{n_1}(x)\ \dots\ \gamma^{n_r}(x) = 0$ whenever $n_1+\dots+n_r \geqq N$ (see [Se]).

4. The Chern character and the Todd homomorphism

Let more generally K be an augmented λ-algebra over a binomial λ-ring R (not necessarily with 1), let $\varepsilon : K \to R$ denote its augmentation. Let $(F^nK)_{n\geqq 0}$ be the associated Grothendieck filtration on K, and

$$Gr^{\cdot}K = \bigoplus_{n\geqq 0} F^nK/F^{n+1}K$$

be the associated graded object. The property $F^nK\cdot F^mK \subseteqq F^{n+m}K$ induces on $Gr^{\cdot}K$ the structure of a graded algebra over R.

For $n \geqq 1$ the n-th Chern class on K is defined to be the map

$$c_n : K \to Gr^nK$$
$$x \mapsto \gamma^n(x-\varepsilon(x)) \bmod F^{n+1}K .$$

Let $N_n(X_1,\dots,X_n)$ denote the n-th Newton polynomial, defined by $N_n(X_1,\dots,X_n) = Y_1^n + \dots + Y_n^n$, where X_i denotes the i-th elementary symmetric function in the indeterminates $Y_1,\dots,Y_n$. $N_n(X_1,\dots,X_n)$ is isobar of weight n. For every $x \in K$ it follows that

$$N_n(c_1(x),\dots,c_n(x)) \in Gr^n K$$

and hence

$$\frac{1}{n!} N_n(c_1(x),\dots,c_n(x)) \in Gr^n K \otimes \mathbb{Q} ,$$

where the tensor product is formed over $\mathbb{Z}$. We put

$$ch(x) = \varepsilon(x) + \sum_{n \geqq 1} \frac{1}{n!} N_n(c_1(x),\dots,c_n(x))$$

reading this as an element in the completion $\prod_{n \geqq 0} Gr^n K \otimes \mathbb{Q}$ of $Gr^{\cdot} K \otimes \mathbb{Q}$. The map

$$ch : K \to \prod_{n \geqq 0} Gr^n K \otimes \mathbb{Q}$$

is called the Chern character on K, and as a first property one has

<u>Lemma 4.1.</u>

The Chern character

$$ch : K \to \prod_{n \geqq 0} Gr^n K \otimes \mathbb{Q}$$

is a homomorphism of R-algebras.

We assume now that the Grothendieck filtration $(F^n K)_{n \geqq 0}$ on K is locally nilpotent. Then the Chern character takes its values in $Gr^{\cdot} K \otimes \mathbb{Q}$; so we have

$$ch : K \longrightarrow Gr^{\bullet} K \otimes Q \ ,$$

and we look at the induced homomorphism

$$ch : K \otimes Q \to Gr^{\bullet} K \otimes Q \ .$$

In doing so let us first recall the fundamental properties of the Adams operations ψ^k on $K \otimes Q$ (see [Se]):

Theorem 4.2.

In case of a locally nilpotent Grothendieck filtration $(F^n K)_{n \geqq 0}$ on K one has:

i) For every j the k^j-eigenspace $(K \otimes Q)^{(j)}$ of ψ^k on $K \otimes Q$ is independent of $k > 1$.

ii) For every $n \geqq 0$ one has

$$F^n K \otimes Q = \bigoplus_{j \geqq n} (K \otimes Q)^{(j)} \quad . -$$

Especially we have

$$K \otimes Q = \bigoplus_{j \geqq 0} (K \otimes Q)^{(j)} \ ,$$

and this decomposition into eigenspaces defines on $K \otimes Q$ the structure of a graded $R \otimes Q$-algebra, whose natural filtration coincides with the Grothendieck filtration $(F^n K \otimes Q)_{n \geqq 0}$. With respect to this graded structure on $K \otimes Q$ we have now the following

Theorem 4.3.

If the Grothendieck filtration $(F^n K)_{n \geqq 0}$ on K is locally nilpotent, then the Chern character

$$ch : K \otimes Q \to Gr^{\cdot} K \otimes Q$$

is an isomorphism of graded $R \otimes Q$-algebras; on the n-th homogeneous component $(K \otimes Q)^{(n)}$ of $K \otimes Q$, the Chern character is given by

$$ch(x) = x \bmod F^{n+1} K \otimes Q \ .$$

This theorem follows from the preceding theorem in connection with the well known formula

$$c_n(x) = (-1)^{n-1}(n-1)! \ x \bmod F^{n+1} K$$

for the n-th Chern class of elements $x \in F^n K$, and the formula $N_n(0,\ldots,0,X_n) = (-1)^{n-1} n \ X_n$.

We will now briefly recall the definition of the Todd homomorphism. We assume the considered augmented λ-algebra K over R to have an identity element.
Following Hirzebruch, to every power series $f(t) \in 1 + tQ[\![t]\!]$ one associates a map

$$td_f : K \to 1 + \prod_{n \geq 1} Gr^n K \otimes Q$$
$$x \mapsto \sum_{n \geq 0} H_{f,n}(c_1(x),\ldots,c_n(x)) \ ,$$

where the so called Hirzebruch polynomials

$$H_{f,n}(X_1,\ldots,X_n) \in Q[X_1,\ldots,X_n]$$

are defined in the following way: Let $q \geq n$, and let $Y_1,\ldots,Y_q$ be indeterminates over Q, and $X_1,\ldots,X_q$ be their elementary symmetric functions; then one puts

$$H_{f,n}(X_1,\ldots,X_n) = \text{coefficient of } t^n \text{ in } \prod_{i=1}^{g} f(Y_i t) \; .$$

Especially for line elements $x \in K$ one has $td_f(x) = f(c_1(x))$.

Lemma 4.4.

The map

$$td_f : K \to 1 + \prod_{n \geqq 1} Gr^n K \otimes \mathbb{Q}$$

is a homomorphism of abelian groups.

We now take the power series

(4.5) $\quad f(t) = B(t) \cdot e^t \; ,$

where $B(t)$ denotes the Bernoulli series, i.e.

$$B(t) = \frac{t}{e^t - 1} = \sum_{n \geqq 0} \frac{B_n}{n!} \; t^n \quad .$$

The associated homomorphism td_f is called the Todd homomorphism, and it will be denoted shortly by td. If the Grothendieck filtration on K is locally nilpotent, then the Todd homomorphism has its values in $Gr^{\cdot} K \otimes \mathbb{Q}$.

5. Let again X be a smooth scheme, and $Y \hookrightarrow X$ be a closed subscheme. By Theorem 3.1 the ring

$$K^Y(X) = \bigoplus_{p \geqq 0} K_p^Y(X)$$

carries the structure of an augmented λ-algebra over the binomial λ-ring $H_Y^0(X,\mathbb{Z})$, whose associated Grothendieck filtration $(F^n K^Y(X))_{n \geqq 0}$ is locally nilpotent. Hence by Theorem 4.3 the Chern character ch on $K^Y(X)$ induces an isomorphism

$$ch : K^Y(X) \otimes Q \xrightarrow{\cong} Gr^{\cdot} K^Y(X) \otimes Q$$

of graded $H_Y^o(X,Q)$-algebras, where the graduation on $K^Y(X) \otimes Q$ is given by the decomposition

$$K^Y(X) \otimes Q = \bigoplus_{n \geq 0} (K^Y(X) \otimes Q)^{(n)}$$

of $K^Y(X) \otimes Q$ into the k^n-eigenspaces of the Adams operations ψ^k.

Denoting the Grothendieck filtration of $K^Y(X)$ still by $(F^n K^Y(X))_{n \geqq 0}$, for all $n,p \geq 0$ we put

$$F^n K_p^Y(X) = F^n K^Y(X) \cap K_p^Y(X) .$$

It is not hard to see that we have the direct sum decomposition

$$(5.1) \qquad F^n K^Y(X) = \bigoplus_{p \geqq 0} F^n K_p^Y(X) .$$

Denoting the associated graded object of the filtration $(F^n K_p^Y(X))_{n \geqq 0}$ by $Gr^{\cdot} K_p^Y(X)$, from (5.1) we fet for every n:

$$Gr^n K^Y(X) = \bigoplus_{p \geqq 0} Gr^n K_p^Y(X) .$$

According to this decomposition the Chern class c_n on $K^Y(X)$ decomposes into a sum

$$c_n = \sum_{p \geqq 0} c_{p,n} .$$

The components $c_{p,n} : K^Y(X) \to Gr^n K_p^Y(X)$ are also called Chern classes; they live on $K_p^Y(X)$, and are additive for $p \geqq 1$.
In the same way the Chern character ch on $K^Y(X)$ decomposes into a sum

$$ch = \sum_{p \geqq 0} ch_p \quad .$$

The components $ch_p : K^Y(X) \to Gr^{\cdot} K_p^Y(x) \otimes \mathbb{Q}$ live on $K_p^Y(X)$, and for $p \geqq 1$ one has the simple formula

$$ch_p = \sum_{n \geqq 1} \frac{(-1)^{n-1}}{(n-1)!} c_{p,n} \quad .$$

Moreover it follows from the above that each of the Chern characters ch_p on $K_p^Y(X)$ induces an isomorphism

$$ch_p : K_p^Y(X) \otimes \mathbb{Q} \xrightarrow{\cong} Gr^{\cdot} K_p^Y(X) \otimes \mathbb{Q}$$

which maps the k^n-eigenspace $(K_p^Y(X) \otimes \mathbb{Q})^{(n)}$ of the ψ^k onto $Gr^n K_p^Y(X) \otimes \mathbb{Q}$.

§2. Riemann-Roch without denominators

1. Let be given a closed immersion

$$i : (Y,Z) \to (X,Z')$$

in the category $\mathcal{V}_*$ (§1, 2.), i.e. a commutative diagram

$$\begin{array}{ccc} Z & \hookrightarrow & Y \\ \downarrow & & \downarrow i \\ Z' & \hookrightarrow & X \end{array}$$

of schemes, where the horizontal arrows are closed immersions into smooth schemes, and where $i : Y \to X$ is a closed immersion too. By §1, 2.5, for $i : (Y,Z) \to (X,Z')$ there is defined the push-forward homomorphism

$$i_* : K^Z(Y) \to K^{Z'}(X) .$$

Let further be given a natural operation μ on the category of λ-ring with augmentation 0, i.e. $\mu(0) = 0$. The Riemann-Roch theorem without denominators describes the effect of i_* on the operation μ, acting on both of the λ-rings $K^Z(Y)$ and $K^{Z'}(X)$.

We consider the conormal sheaf N of $i : Y \to X$, defined by $N = I/I^2$, where I is the coherent ideal of $\mathcal{O}_X$ defining $i:Y \to X$. The sheaf N is a locally free $\mathcal{O}_Y$-module of finite rank, and hence defines an element $[N] \in K_o(Y)$ of finite λ-degree. We form the element

$$\lambda_{-1}([N]) = \sum_{p \geq 0} (-1)^p \lambda^p[N] \in K_o(Y) .$$

The given natural operation μ has a unique representation $\mu = f(\lambda^1, \lambda^2, \ldots)$ as a polynomial in the $\lambda^1, \lambda^2, \ldots$ with integer coefficients and vanishing constant coefficient. Using this well known fact, one obtains as an easy generalization of ([SGA6], V, 5.3) the following result about the action of μ on products $\lambda_{-1}([N]) \cdot y$ with $y \in K^Z(Y)$:

<u>Lemma 1.1.</u>
For every $y \in K^Z(Y)$, there exists an element $\mu([N],y) \in K^Z(Y)$ which is a universal polynomial in the $\lambda^p[N]$ and the λ^p_Y with integer coefficients, depending only on μ, such that

$$\mu(\lambda_{-1}([N]) \cdot y) = \lambda_{-1}([N]) \cdot \mu([N],y) .$$

In the following we write $\lambda_{-1}(N)$ and $\mu(N,y)$ instead of $\lambda_{-1}([N])$ and $\mu([N],y)$. We are now ready to state the theorem:

Theorem 1.2 (Riemann-Roch without denominators).

The diagram

$$\begin{array}{ccc} K^Z(Y) & \xrightarrow{\mu(N,-)} & K^Z(Y) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ K^{Z'}(X) & \xrightarrow{\mu} & K^{Z'}(X) \end{array}$$

commutes, i.e.: For every $y \in K^Z(Y)$, one has

$$\mu(i_*(y)) = i_*(\mu(N,y)).$$

Before proving the theorem we derive a corollary. We consider the Adams operations ψ^k for $k \geqq 1$. Then the elements $\psi^k(N,y)$ for $y \in K^Z(Y)$ can be computed as follows.

For $k \geqq 1$, the Botts' cannibalistic classes of N are defined to be the elements

$$\theta^k(N) := \psi^k(N,1)$$

in $K_o(Y)$. Then from the identity

$$\psi^k(\lambda_{-1}(N)y) = \lambda_{-1}(N)\psi^k(N,y)$$

one obtains the formula

$$\psi^k(N,y) = \theta^k(N)\psi^k(y) .$$

<u>Corollary 1.3</u> (Adams-Riemann-Roch without denominators).

The diagram

$$\begin{array}{ccc} K^Z(Y) & \xrightarrow{\theta^k(N)\psi^k} & K^Z(Y) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ K^{Z'}(X) & \xrightarrow{\psi^k} & K^{Z'}(X) \end{array}$$

commutes, i.e.: For every $y \in K^Z(Y)$, one has

$$\psi^k(i_*(y)) = i_*(\theta^k(N)\ \psi^k(y)).$$

2. <u>Proof of the theorem</u>

We begin with the following

<u>Remark 2.1.</u>

The given closed immersion $i : (Y,Z) \to (X,Z')$ in $\mathcal{V}_*$ is the composition of the closed immersions $(Y,Z) \to (X,Z)$ and $(X,Z) \to (X,Z')$. As the push-forward

$$K^Z(X) \to K^{Z'}(X)$$

for $(X,Z) \to (X,Z')$ commutes with the λ-operations (§1, 3.1), it suffices to prove the theorem for the closed immersion

$$i : (Y,Z) \to (X,Z).$$

Nevertheless, the more general formulation of the theorem will be essential in the last step of the proof.

As the first step we recall an appropriate intersection formula for excess dimension 0. Let

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ Y & \xrightarrow{i} & X \end{array}$$

be a Cartesian diagram with closed immersions of smooth schemes. Let N and N' denote the conormal sheaf of $Y \xrightarrow{i} X$ and $Y' \xrightarrow{i'} Y'$, respectively. Then we have a canonical surjection $j'^*N \to N'$. Its cernel is called the excess conormal sheaf, and the rank of the excess conormal sheaf is called the excess dimension of the diagram.

<u>Lemma 2.2</u> (Intersection formula for excess dimension 0). Assume that the above diagram has the excess dimension 0. Let Z be a closed subscheme of Y, and $Z' = j'^{-1}(Z)$. The following diagram commutes:

$$\begin{array}{ccc} K^{Z'}(Y') & \xrightarrow{i'_*} & K^{Z'}(X') \\ {\scriptstyle j'^*}\uparrow & & \uparrow{\scriptstyle j^*} \\ K^{Z}(Y) & \xrightarrow{i_*} & K^{Z}(X) \end{array} \quad .$$

<u>Proof.</u>

By a well known result ([SGA6], VII, 2.5), one has

$$\underline{\mathrm{Tor}}_q^{\mathcal{O}_X}(\mathcal{O}_Y,\mathcal{O}_{X'}) \cong \Lambda^q\ \underline{\mathrm{Tor}}_1^{\mathcal{O}_X}(\mathcal{O}_Y,\mathcal{O}_{X'})$$

for all q, and

$$\underline{\mathrm{Tor}}_1^{\mathcal{O}_X}(\mathcal{O}_Y,\mathcal{O}_{X'}) \cong \mathrm{Ker}(j'^*\, N \to N').$$

So our assumption implies

$$\underline{\mathrm{Tor}}_q^{\mathcal{O}_X}(\mathcal{O}_Y,\mathcal{O}_{X'}) = 0 \qquad \text{for } q > 0 ,$$

i.e. the schemes Y and X' are Tor independent over X. Now the intersection formula follows from [Q], §7, 2.11.

Remark 2.3.

Under the assumptions of Lemma 2.2 let us assume for a moment that

$$j'^* : K^Z(Y) \to K^{Z'}(Y')$$

is an isomorphism. As the push-forwards i_* and i'_* are isomorphisms, it follows from the intersection formula that $j^* : K^Z(X) \to K^{Z'}(X')$ is an isomorphism too. Remembering that j'^* and j^* are λ-morphisms (§1, 3.1), the intersection formula implies for elements $y \in K^Z(Y)$ and $y' = j'^*(y) \in K^{Z'}(Y')$ the equivalence of the formulas

$$\mu(i_*(y)) = i_*(\mu(N,y)) ,$$
$$\mu(i'_*(y')) = i'_*(\mu(N',y')) .$$

That means: If the Riemann-Roch theorem holds for one of the closed immersions

$$i : (Y,Z) \to (X,Z)$$
$$i' : (Y',Z') \to (X',Z') ,$$

then it holds for both of them.

Using the intersection formula and the homotopy property of K-theory, in our next steps we reduce the proof of the theorem for the given closed immersion

$$i : (Y,Z) \to (X,Z)$$

to that for the zero section

$$i' : (Y,Z) \to (\mathbb{P}(\mathcal{N} \oplus \mathcal{O}_Y),Z)$$

defined by the projection $\mathcal{N} \oplus \mathcal{O}_Y \to \mathcal{O}_Y$. This happens with the famous "deformation of $i : Y \to X$ into the zero section $i' : Y \to \mathbb{P}(\mathcal{N} \oplus \mathcal{O}_Y)$".

We start with the closed immersion $i : Y \to X$ with conormal sheaf $\mathcal{N}$. The extension $i : \mathbb{A}^1_Y \to \mathbb{A}^1_X$ is a closed immersion with conormal sheaf $p^*\mathcal{N}$, where $p : \mathbb{A}^1_Y \to Y$ is the structure morphism. Let $0 : Y \to \mathbb{A}^1_Y$ denote the zero section.
We consider the blowing up

$$W \to \mathbb{A}^1_X$$

of $\mathbb{A}^1_X$ along the closed immersion $Y \xrightarrow{0} \mathbb{A}^1_Y \xrightarrow{i} \mathbb{A}^1_X$. The conormal sheaf of this immersion is $\mathcal{N} \oplus \mathcal{O}_Y$. Hence the exceptional divisor of the blowing up, i.e. the inverse image of the center $Y \hookrightarrow \mathbb{A}^1_X$ under $W \to \mathbb{A}^1_X$, coincides with the Y-scheme $\mathbb{P}(\mathcal{N} \oplus \mathcal{O}_Y)$. Let

$$\mathbb{P}(\mathcal{N} \oplus \mathcal{O}_Y) \to W$$

denote the canonical immersion.
Now we consider the blowing up of $\mathbb{A}^1_Y$ along $Y \xrightarrow{0} \mathbb{A}^1_Y$. As $Y \xrightarrow{0} \mathbb{A}^1_Y$ is an effective Cartier divisor, this blowing up identifies canonically with $\mathbb{A}^1_Y$. Hence the immersion $\mathbb{A}^1_Y \xrightarrow{i} \mathbb{A}^1_X$ induces a closed $\mathbb{A}^1_X$-immersion

$$\mathbb{A}^1_Y \to W ,$$

and the diagram

$$(2.4) \qquad \begin{array}{ccc} Y & \xrightarrow{i'} & \mathbb{P}(N \oplus \mathcal{O}_Y) \\ {\scriptstyle 0}\downarrow & & \downarrow \\ \mathbb{A}^1_Y & \longrightarrow & W \end{array}$$

of closed immersions is Cartesian. The conormal sheaf of $\mathbb{A}^1_Y \to W$ is $p^*N \otimes \mathcal{O}_{\mathbb{A}^1_Y}(Y) = p^*N$, and the conormal sheaf of the zero section $Y \xrightarrow{i'} \mathbb{P}(N \oplus \mathcal{O}_Y)$ is N. Hence the Cartesian diagram (2.4) is of excess dimension 0.

Next we consider the one section $1 : X \to \mathbb{A}^1_X$. As the blowing up $W \to \mathbb{A}^1_X$ is an isomorphism over the complement of its center, the section $1 : X \to \mathbb{A}^1_X$ lifts uniquely to a closed immersion $X \to W$, and the diagram

$$(2.5) \qquad \begin{array}{ccc} Y & \xrightarrow{i} & X \\ {\scriptstyle i}\downarrow & & \downarrow \\ \mathbb{A}^1_Y & \longrightarrow & W \end{array}$$

is Cartesian of excess dimension 0.

Now we look at the closed subscheme $\mathbb{A}^1_Z$ of $\mathbb{A}^1_Y$. For every section $s : Y \to \mathbb{A}^1_Y$, we have $s^{-1}(\mathbb{A}^1_Z) = Z$. Using §1, 1.3 and the homotopy property ([Q], §7, 4.1), it follows that the map

$$s^* : K^{\mathbb{A}^1_Z}(\mathbb{A}^1_Y) \to K^Z(Y)$$

is an isomorphism. Thus both diagrams (2.4) and (2.5) satisfy the additional assumption of the remark (2.3) above. From this it follows that the Riemann-Roch theorem is true for both of the immersions

$$i : (Y,Z) \to (X,Z)$$

$$i' : (Y,Z) \to (\mathbb{P}(N \oplus \mathcal{O}_Y),Z) \quad ,$$

if it is true for one of them.

In the last step we prove the theorem for the zero section $i' : (Y,Z) \to (\mathbb{P}(N \oplus \mathcal{O}_Y), Z)$.
We put $X' = \mathbb{P}(N \oplus \mathcal{O}_Y)$. Let $p : X' \to Y$ be the structure morphism We consider the closed subscheme $Z' = p^{-1}(Z)$ of X'. Then the homomorphism

$$K^Z(X') \to K^{Z'}(X')$$

induced by $(X',Z) \to (X',Z')$ is injective. In fact, by §1, 2.2 the diagram

$$\begin{array}{ccc} K^Z(X') & \longrightarrow & K^{Z'}(X) \\ \downarrow \cong & & \downarrow \cong \\ K'(Z) & \longrightarrow & K'(Z') \end{array}$$

commutes. But, since the inclusion $Z \to Z'$ is a section of the morphism $Z' \to Z$ induced by $p : X' \to Y$, the map $K'(Z) \to K'(Z')$ is injective. As $K^Z(X') \to K^{Z'}(X')$ is a λ-homomorphism (§1, 3.1), it then suffices to prove the theorem for the zero section

$$i : (Y,Z) \to (X',Z') .$$

We consider the universal exact sequence

$$0 \to H \to p^*N \oplus \mathcal{O}_{X'} \to \mathcal{O}_{X'}(1) \to 0$$

on X'. The induced homomorphism $H \to \mathcal{O}_{X'}$ maps the universal hyperplane sheaf H onto the ideal I' of $\mathcal{O}_{X'}$, defining the zero section $i' : Y \to X'$, and the associated Koszul complex yields a finite locally free resolution

$$0 \to \Lambda^d H \to \dots \to \Lambda^1 H \to \mathcal{O}_X \to i_*^! \mathcal{O}_Y \to 0$$

of the sheaf $i_*^! \mathcal{O}_Y$ on X. After tensoring the exact sequence $\to \Lambda^2 H \to \Lambda^1 H \to I' \to 0$ by $\mathcal{O}_{X'}/I'$, one gets as conormal sheaf of $i' : Y \to X'$ the sheaf $i'^* H$, and from the universal exact sequence one obtains

(2.6) $\quad i'^* H = N.$

The Koszul resolution implies for the image of $1 \in K_0(Y)$ under $i_*^! : K_0(Y) \to K_0(X')$ the formula

(2.7) $\quad i_*^!(1) = \sum_{p \geq 0} (-1)^p [\Lambda^p H] = \lambda_{-1}(H)$.

Taking the closed subscheme $Z' = p^{-1}(Z)$ of X', the morphisms

$$i' : (Y,Z) \to (X',Z')$$
$$p : (X',Z') \to (Y,Z)$$

are both Cartesian, and we have the pull-back homomorphism (§1, 1.3)

$$i'^* : K^{Z'}(X') \to K^Z(Y)$$
$$p^* : K^Z(Y) \longrightarrow K^{Z'}(X') \quad .$$

By functoriality, we have $i'^* \cdot p^* = id$, and hence the map

$$i'^* : K^{Z'}(X') \to K^Z(Y)$$

is surjective.

Moreover, for $i' : (Y,Z) \to (X',Z')$ being Cartesian, we have the projection formula, i.e. the commutative diagram

$$\begin{array}{ccccc} K_o(Y) & \times & K^Z(Y) & \longrightarrow & K^Z(Y) \\ \Big\downarrow i'^* & & \Big\uparrow i'^* & & \Big\downarrow i'_* \\ K_o(X') & \times & K^{Z'}(X') & \longrightarrow & K^{Z'}(X') \end{array}$$

Using these fact, the Riemann-Roch theorem for $i' : (Y,Z) \to (X',Z')$, i.e. the formula

$$\mu(i'_*(y)) = i'_*(\mu(N,y)) \quad \text{for } y \in K^Z(Y)$$

now comes out by the following computation:

To a given $y \in K^Z(Y)$ choose an element $x \in K^{Z'}(X')$ with $i'^*(x) = y$.

Then:

$$\begin{aligned} \mu(i'_*(y)) &= \mu(i'_*(i'^*(x))) \\ &= \mu(i'_*(1)x) \\ &= \mu(\lambda_{-1}(H)x) \\ &= \lambda_{-1}(H)\mu(H,x) \\ &= i'_*(1)\ \mu(H,x) \\ &= i'_*(i'^*\mu(H,x)) \\ &= i'_*(\mu(i'^*H,i'^*(x))) \\ &= i'_*\ (\mu(N,y))\ . \end{aligned}$$

§3. The smooth Riemann-Roch theorem for K-theory with supports

1. We work with the category $\mathcal{V}_*$ introduced in §1, 2. It consists of pairs (X,Y) with X a smooth scheme and $Y \hookrightarrow X$ a closed subscheme, in which a morphism $f : (X',Y') \to (X,Y)$ is a commutative diagram

$$\begin{array}{ccc} Y' & \hookrightarrow & X' \\ \downarrow & & \downarrow f \\ Y & \hookrightarrow & X \end{array}$$

with a proper morphism $f : X' \to X$.
For every morphism $f : (X',Y') \to (X,Y)$ in $\mathcal{V}_*$, there is defined the push-forward homomorphism (§1, 2.5)

$$f_* : K^{Y'}(X') \to K^{Y}(X) \ .$$

For any object (X,Y) in $\mathcal{V}_*$, the augmented λ-ring structure on $K^Y(X)$ yields the Chern character (§1, 4)

$$ch : K^Y(X) \to Gr^{\cdot} K^Y(X) \otimes Q \ .$$

It induces an isomorphism (§1, 5)

$$ch : K^Y(X) \otimes Q \xrightarrow{\cong} Gr\ K^Y(X) \otimes Q \ .$$

Concerning the Riemann-Roch problem for the push-forward f_* above, this implies: If x is an element of $K^{Y'}(X')$, then to compute $ch(f_*(x))$ means to compute the image $f_*(x)$ modulo torsion.

For a smooth scheme X, the tangent sheaf $\mathcal{T}_{X/k} = (\Omega^1_{X/k})^{\vee}$ is a locally free $\mathcal{O}_X$-module of finite rank. One defines the Todd class of X to be

$$Td(X) := td([\mathcal{T}_{X/k}]) \in Gr^{\cdot} K_o(X) \otimes Q$$

where td denotes the Todd homomorphism on $K_o(X)$, see §1.4. Td(X) is a unit in $Gr^{\cdot} K_o(X) \otimes Q$ with component 1 in degree zero.

If $f : X' \to X$ is a morphism of schemes, then for a point $x' \in X'$ the integer

$$d_{x'}(f) = \dim_{x'}(X') - \dim_{f(x')}(X)$$

is called the virtual relative dimension of f in x'. The function $x' \mapsto d_{x'}(f)$ is locally constant on X'.

In what follows, it is convenient to have the Grothendieck filtration defined for all integers, so we let $F^n K = K$ for $n \leqq 0$.

Now we state the theorem:

Theorem 1.1 (Grothendieck-Riemann-Roch).

Let $f : (X',Y') \to (X,Y)$ be a morphism in $\mathcal{V}_*$ of constant virtual dimension d. Then one has:

i) The homomorphism $f_* : K^{Y'}(X') \to K^Y(X)$ has degree $-d$, i.e.

$$f_*(F^n K^{Y'}(X') \otimes Q) \subseteqq F^{n-d} K^Y(X) \otimes Q$$

for all integers n, and hence f_* induces a graded homomorphism

$$f_* : Gr\, K^{Y'}(X') \otimes Q \to Gr^{\cdot} K^Y(X) \otimes Q$$

of degree $-d$, the Gysin homomorphism.

ii) The diagram

$$\begin{array}{ccc} K^{Y'}(X') & \xrightarrow{\mathrm{Td}(X')\mathrm{ch}} & \mathrm{Gr}^{\cdot}K^{Y'}(X') \otimes \mathbb{Q} \\ \downarrow f_* & & \downarrow f_* \\ K^{Y}(X) & \xrightarrow{\mathrm{Td}(X)\mathrm{ch}} & \mathrm{Gr}^{\cdot}K^{Y}(X) \otimes \mathbb{Q} \end{array}$$

commutes, i.e.: For every $x \in K^{Y'}(X')$, one has

$$\mathrm{ch}(f_*(x)) = \mathrm{Td}(X)^{-1} f_*(\mathrm{Td}(X')\mathrm{ch}(x)).$$

Remark 1.2.

Because of the covariance of f_*, the theorem holds for a composition $g \circ f$ in $\mathcal{V}_*$, if it holds for g and f.

A given morphism $f : (X',Y') \to (X,Y)$ has a factorization

$$(X',Y') \xrightarrow{i} (\mathbb{P}^r_X, \mathbb{P}^r_Y) \xrightarrow{p} (X,Y)$$

into a closed immersion i and a projection p from a projective space. It suffices to prove the theorem for i and p.

2. Proof of Riemann-Roch for closed immersion

We consider a closed immersion

$$i : (Y,Z) \to (X,Z')$$

of constant codimension d, i.e. of constant virtual dimension -d. Let $\mathcal{N}$ be the conormal sheaf of $Y \xrightarrow{i} X$. $\mathcal{N}$ is a locally free $\mathcal{O}_Y$-module of rank d.

The crucial point of the proof is the Riemann-Roch theorem without denominators for the Adams operation ψ^k (see §2, 1.3):

For every $k \geq 1$, the diagram

$$\begin{array}{ccc} K^Z(Y) & \xrightarrow{\theta^k(N)\psi^k} & K^Z(Y) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ K^{Z'}(X) & \xrightarrow{\psi^k} & K^{Z'}(X) \end{array} ,$$

commutes.

In addition, we use the following fundamental formula, combining Adams, Bott, Chern, and Todd on $K_o(Y)$. Remember that the Chern character

$$ch : K_o(Y) \otimes Q \to Gr^{\cdot} K_o(Y) \otimes Q$$

is an isomorphism (§1, 4.3).

<u>Lemma 2.1.</u>

Let N be a locally free module of rank d on Y. Then in $K_o(Y) \otimes Q$ one has

$$\theta^k(N)\psi^k(ch^{-1}(td(\check{N}))) = k^d\ ch^{-1}(td(\check{N})) .$$

<u>Proof.</u>

Since the induced action of ψ^k on $Gr^n K_o(Y)$ is multiplication by k^n, the diagram

$$\begin{array}{ccc} K_o(Y) \otimes Q & \xrightarrow{ch} & Gr^{\cdot} K_o(Y) \otimes Q \\ \downarrow{\scriptstyle \psi^k} & & \downarrow{\scriptstyle \varphi^k} \\ K_o(Y) \otimes Q & \xrightarrow{ch} & Gr^{\cdot} K_o(Y) \otimes Q \end{array}$$

commutes, where φ^k is the graded algebra homomorphism which is multiplication by k^n on $Gr^n K_o(Y) \otimes Q$. Hence the stated formula is equivalent with the formula

$$ch(\theta^k(N))\ \varphi^k(td(\check{N})) = k^d\ td(\check{N}).$$

Let $p : \mathbb{D}(N) \to Y$ be the flag scheme of N over Y. In $K_o(\mathbb{D}(N))$ the class of p^*N decomposes into a sum of classes of invertible modules. Since the induced map $Gr^{\cdot}K_o(Y) \to Gr^{\cdot}K_o(\mathbb{D}(N))$ is injective ([SGA 6], VI, 5.5), it obviously suffices to prove the formula for an invertible sheaf N.
Let $a = c_1(N) \in Gr^{\cdot}K_o(Y)$ be the first Chern class of N. Then we have

$$\begin{aligned} ch(\theta^k(N)) &= ch(1 + [N] + \ldots + [N]^{k-1}) \\ &= 1 + e^a + \ldots + e^{(k-1)a} \quad , \end{aligned}$$

$$td(\check{N}) = \frac{-a}{e^{-a}-1}\, e^{-a} = \frac{-a}{1-e^a} \quad ,$$

$$\varphi^k(td(\check{N})) = \frac{-ka}{1-e^{ka}} \quad ,$$

and hence:

$$\begin{aligned} ch(\theta^k(N))\,\varphi^k(td(\check{N})) &= (1+e^a + \ldots + e^{(k-1)a}) \cdot \frac{-ka}{1-e^{ka}} \\ &= -ka\frac{1}{1-e^a} \\ &= k\ td(\check{N}) \ . \end{aligned}$$

We consider now the push-forward homomorphism

$$i_* : K^Z(Y) \otimes \mathbb{Q} \to K^{Z'}(X) \otimes \mathbb{Q}$$

induced by the given closed immersion $i : (Y,Z) \to (X,Z')$. As in §1, 4, we denote by $(K^Z(Y) \otimes \mathbb{Q})^{(j)}$ the k^j-eigenspace of the Adams operations ψ^k on $K^Z(Y) \otimes \mathbb{Q}$.

From the Riemann-Roch theorem without denominators and the formula above we deduce now as the essential fact:

Lemma 2.2.

By

$$y \mapsto i_*(ch^{-1}(td(\check{N}))\, y)$$

the eigenspace $(K^Z(Y) \otimes Q)^{(j)}$ is mapped into the eigenspace $(K^Z(X') \otimes Q)^{(j+d)}$.

Proof.

In fact, for an element $y \in (K^Z(Y) \otimes Q)^{(j)}$ we have

$$\begin{aligned} \psi^k(i_*(ch^{-1}(td(\check{N}))\, y) &= i_*(\theta^k(N)\psi^k(ch^{-1}(td(\check{N})))\psi^k(y)) \\ &= i_*(k^d ch^{-1}(td(\check{N}))\, k^j y) \\ &= k^{j+d} i_*(ch^{-1}(td(\check{N}))\, y), \end{aligned}$$

and hence $i_*(ch^{-1}(td(\check{N}))\, y) \in (K^{Z'}(X) \otimes Q)^{(j+d)}$.

From this lemma we shall now derive the Riemann-Roch theorem for $i : (Y,Z) \to (X,Z')$. We use the results §1, 4.2 and 4.3, especially the behavior of the Chern character on the eigenspaces of the Adams operations.

Lemma 2.3.

i) The homomorphism i_* has degree d, i.e.

$$i_*(F^n K^Z(Y) \otimes Q) \subseteq F^{n+d} K^{Z'}(X) \otimes Q$$

for all n.

ii) The diagram

$$\begin{array}{ccc} K^Z(Y) & \xrightarrow{Td(Y)ch} & Gr^{\cdot}K^Z(Y)\otimes Q \\ \downarrow i_* & & \downarrow i_* \\ K^{Z'}(X) & \xrightarrow{Td(X)ch} & Gr^{\cdot}K^{Z'}(X)\otimes Q \end{array}$$

where the vertical map on the right is the Gysin homomorphism, induced by i).

<u>Proof.</u>

We let $u = ch^{-1}(td(\check{N}))$; u is a unit in $K_o(Y)\otimes Q$ with augmentation 1.

i) Let $y \in F^nK^Z(Y)\otimes Q$ be given. We write $y = uy'$ with $y' = u^{-1}y \in F^nK^Z(Y)\otimes Q$. With respect to the decomposition (§1, 4.3)

$$F^nK^Y(Z)\otimes Q = \bigoplus_{j\geqq u} (K^Z(Y)\otimes Q)^{(j)}$$

we decompose

$$y' = \sum_{j\geqq n} y'_j$$

with $y'_j \in (K^Z(Y)\otimes Q)^{(j)}$. Then it follows

$$i_*(y) = i_*(uy') = \sum_{j\geqq u} i_*(uy'_j) \ .$$

By Lemma 2.2, we have $i_*(uy'_j) \in (K^{Z'}(X)\otimes Q)^{(j+d)}$ for all $j \geqq n$, and hence

$$i_*(y) \in \bigoplus_{j\geqq n} (K^{Z'}(X)\otimes Q)^{(j+d)} = F^{n+d}\,K^{Z'}(X)\otimes Q \ .$$

ii) We first prove the commutativity of the following diagram

$$\begin{array}{ccc} K^Z(Y) & \xrightarrow{td(\check{N})^{-1}ch} & Gr^{\cdot}K^Z(Y)\otimes Q \\ i_*\downarrow & & \downarrow i_* \\ K^{Z'}(X) & \xrightarrow{ch} & Gr^{\cdot}K^{Z'}(X)\otimes Q \end{array} \quad .$$

Let $y \in K^Z(Y)$ be given. We write again $y = uy'$ with $y' = u^{-1}y \in K^Z(Y) \otimes Q$. We have to prove

$$ch(i_*(uy')) = i_*(td(\check{N})^{-1}\ ch(uy')).$$

Now

$$\begin{aligned} i_*(td(\check{N})^{-1}ch(uy')) &= i_*(td(\check{N})^{-1}ch(u)ch(y')) \\ &= i_*(td(\check{N})^{-1}td(\check{N})ch(y')) \\ &= i_*(ch(y')) \ , \end{aligned}$$

and so we have to show: For every $y \in K^Z(Y) \otimes Q$ one has

$$ch(i_*(uy)) = i_*(ch(y)) \ .$$

In view of the decomposition of $K^Z(Y) \otimes Q$ into the eigenspaces $(K^Z(Y) \otimes Q)^{(n)}$, we may assume $y \in (K^Z(Y) \otimes Q)^{(n)}$. Then by §1, 4.3, we have

$$ch(y) = y \bmod F^{n+1}K^Z(Y) \otimes Q \ ,$$

and hence

(*) $$i_*(ch(y)) = i_*(y) \bmod F^{d+n+1}\ K^{Z'}(X) \otimes Q \ .$$

On the other hand, by Lemma 2.2 we have $i_*(uy) \in (K^{Z'}(X) \otimes Q)^{(d+n)}$ and hence again by §1, 4.3

(**) $$ch(i_*(uy)) = i_*(uy) \bmod F^{d+n+1}\ K^{Z'}(X) \otimes Q \ .$$

But now, since $u = ch^{-1}(td(\check{N})) \in K_o(Y) \otimes Q$ is a unit with augmentation 1, we have

$$uy \equiv y \bmod F^{n+1}\ K^Z(Y) \otimes Q$$

and therefore

$$i_*(uy) \equiv i_*(y) \bmod F^{d+n+1} K^{Z'}(X) \otimes Q .$$

So from (*) and (**) we get

$$ch(i_*(uy)) = i_*(ch(y)) ,$$

and the commutativity of the diagram above is proved.

Now the commutativity of

$$\begin{array}{ccc} K^Z(Y) & \xrightarrow{Td(Y)ch} & Gr^{\cdot}K^Z(Y) \otimes Q \\ \downarrow i_* & & \downarrow i_* \\ K^{Z'}(X) & \xrightarrow{Td(X)ch} & Gr^{\cdot}K^{Z'}(X) \otimes Q \end{array}$$

comes out as follows: For the closed immersion $i : \varphi \to X$, we have the exact sequence

$$0 \to N \to i^*\Omega^1_{X/k} \to \Omega^1_{Y/k} \to 0$$

and therefore

$$i^*(Td(X)) = Td(Y)td(\check{N}) .$$

Then for $y \in K^Z(Y)$ we have:

$$\begin{aligned} ch(i_*(y)) &= i_*(td(\check{N})^{-1} ch(y)) \\ &= i_*(i^*Td(X)^{-1}Td(Y)\ ch(y)) \\ &= Td(X)^{-1}\ i_*(Td(Y)\ ch(y)) , \end{aligned}$$

and hence

$$Td(X)\ ch(i_*(y)) = i_*(Td(Y)\ ch(y)) .$$

3. Proof of Riemann-Roch for $(\mathbb{P}^r_X, \mathbb{P}^3_Y) \to (X,Y)$.

The projection

$$p : (\mathbb{P}^r_X, \mathbb{P}^r_Y) \to (X,Y)$$

is of constant virtual dimension r. In the following we write $\mathbb{P}_X$ instead of $\mathbb{P}^r_X$, etc. We prove

Lemma 3.1.

i) The homomorphism $p_* : K^{\mathbb{P}_Y}(\mathbb{P}_X) \to K^Y(X)$ has degree $-r$, more precisely one has

$$p_*(F^n K^{\mathbb{P}_Y}(\mathbb{P}_X)) \subseteq F^{n-r}K^Y(X)$$

for all n.

ii) The diagram

$$\begin{array}{ccc} K^{\mathbb{P}_Y}(\mathbb{P}_X) & \xrightarrow{\ Td(\mathbb{P}_X)\,ch\ } & Gr^{\cdot} K^{\mathbb{P}_Y}(\mathbb{P}_X) \otimes \mathbb{Q} \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ K^Y(X) & \xrightarrow{\ Td(X)\,ch\ } & Gr^{\cdot} K^Y(X) \otimes \mathbb{Q} \end{array}$$

commutes.

Proof.

Let $\mathbb{P}$ denote the projective space of dimension r over k and let $q : \mathbb{P}_X \to \mathbb{P}$ be the projection. Then the canonical map

$$(3.2) \qquad \begin{array}{ccc} K^Y(X) \otimes K_o(\mathbb{P}) & \longrightarrow & K^{\mathbb{P}_Y}(\mathbb{P}_X) \\ x \otimes y & \longmapsto & p^*(x)q^*(y) \end{array}$$

is an isomorphism. In fact, since p is flat, we have the commutative diagram (§1, 2.3)

$$\begin{array}{ccc} K^{Y}(X) & \xrightarrow{p^*} & K^{\mathbb{P}Y}(\mathbb{P}_X) \\ \| & & \| \\ K'(Y) & \xrightarrow{p^*} & K'(\mathbb{P}_Y) \end{array}$$

in which the lower map p* between the K'-groups is induced by $p : \mathbb{P}_Y \longrightarrow Y$, and now it is a result of Quillen ([Q], §7, 4.3) that the canonical map

$$K'(Y) \otimes K_o(\mathbb{P}) \longrightarrow K'(\mathbb{P}_Y)$$
$$x \otimes y \longmapsto p^*(x)q^*(y)$$

is an isomorphism.

Using the isomorphism (3.2), the projection formula implies the commutativity of the diagram

$$\begin{array}{ccc} K^{Y}(X) \otimes K_o(\mathbb{P}) & \xrightarrow{\cong} & K^{\mathbb{P}Y}(\mathbb{P}_X) \\ \big\downarrow \mathrm{id} \otimes p_* & & \big\downarrow p_* \\ K^{Y}(X) \otimes \mathbb{Z} & \xrightarrow{\cong} & K^{Y}(X) \end{array}$$

in which on the left-hand side p denotes also the structure morphism $\mathbb{P} \rightarrow \mathrm{Spec}(k)$.

It is well known that $t \mapsto [\mathcal{O}_{\mathbb{P}}(-1)]$ induces a ring isomorphism

$$\mathbb{Z}[t]/(1-t)^{r+1} \xrightarrow{\cong} K_o(\mathbb{P}) \quad .$$

If we let $y = 1 - [\mathcal{O}_{\mathbb{P}}(-1)]$, then the elements $y^i \in F^iK_o(\mathbb{P})$, $i = 0,\ldots,r$, form a basis of $K_o(\mathbb{P})$, with the relation $y^{r+1} = 0$. As an easy generalization of SGA6, VI, 5.3, and with the same method of proof, one obtains for the Grothendieck filtration on $K^{\mathbb{P}Y}(\mathbb{P}_X)$ the result:

$$(3.2) \qquad \bigoplus_{i=0}^{r} F^{n-i}K^{Y}(X) \otimes \mathbb{Z}[y^i] \xrightarrow{\cong} F^nK^{\mathbb{P}Y}(\mathbb{P}_X)$$

From this and the projection formula we get

$$p_*(F^n K^{\mathbb{P}Y}(\mathbb{P}_X)) \subseteq \sum_{i=0}^{r} F^{n-i} K^Y(X)$$
$$= F^{n-r} K^Y(X) \ .$$

This proves the assertion i).

Furthermore, it follows from (3.3) that

$$\bigoplus_{i=0}^{r} Gr^{n-i} K^Y(X) \otimes Gr^i K_o(\mathbb{P}) \xrightarrow{\cong} Gr^n K^{\mathbb{P}Y}(\mathbb{P}_X)$$

and the projection formula shows that the Gysin homomorphism

$$p_* : Gr^n K^{\mathbb{P}Y}(P_X) \otimes \mathbb{Q} \to Gr^{n-r} K^Y(X) \otimes \mathbb{Q}$$

is induced by the corresponding

$$p_* : Gr^i K_o(\mathbb{P}) \otimes \mathbb{Q} \to Gr^{i-r} K_o(k) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{for } i = r \\ 0 & \text{otherwise} \end{cases} \ .$$

Now the assertion ii), i.e. the commutativity of the diagram

$$\begin{array}{ccc} K^{\mathbb{P}Y}(\mathbb{P}_X) & \xrightarrow{Td(\mathbb{P}_X)ch} & Gr^{\cdot} K^{\mathbb{P}Y}(\mathbb{P}_X) \otimes \mathbb{Q} \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ K^Y(X) & \xrightarrow{Td(X)ch} & Gr^{\cdot} K^Y(X) \otimes \mathbb{Q} \end{array}$$

is an immediate consequence of the classical Riemann-Roch for $\mathbb{P}/k$, i.e. the commutativity of

$$\begin{array}{ccc} K_o(\mathbb{P}) & \xrightarrow{Td(\mathbb{P})\, ch} & Gr^{\cdot} K_o(\mathbb{P}) \otimes \mathbb{Q} \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \mathbb{Z} & \xrightarrow{incl} & \mathbb{Q} \end{array} \ .$$

In fact, using (3.2), the projection formula, and $Td(\mathbb{P}_X) = p^*(Td(X))\, q^*(Td(\mathbb{P}))$, it follows for $x \in K^Y(X)$ and $y \in K_o(\mathbb{P})$:

$$\begin{aligned} p_*(Td(\mathbb{P}_X)\, ch(p^*(x)q^*(y))) &= \\ &= p_*(p^*(Td(X)\ ch(x))\ q^*(Td(\mathbb{P})\, ch(y))) \\ &= Td(X)\ ch(x)\ p_*(Td(\mathbb{P})\, ch(y)) \\ &= Td(X)\ ch(x)\ p_*(y) \\ &= Td(X)\ ch(p_*(p^*(x)q^*(y))) . \end{aligned}$$

§4. The singular Riemann-Roch

Througout this paragraph, we work with the category $\mathcal{V}$, so all schemes under consideration are quasi-projective schemes over the field k, and all morphisms of schemes are k-morphisms. For a scheme X, we put $K'(X) = \bigoplus_{p \geq 0} K'_p(X)$.
The singular Riemann-Roch is concerned with the push-forwards

$$f_* : K'(X) \to K'(Y)$$

for proper morphisms $f : X \to Y$. It is the collection of four theorems, which will be stated and proved seperately in the subsequent four sections.

1. The ascending filtration on $K'(X) \otimes \mathbb{Q}$ and the homomorphism τ

Let X be a scheme. We choose a closed immersion $X \hookrightarrow M$ of X into a smooth scheme M of pure dimension, say d. By the purity theorem (§1, 2.4), $X \hookrightarrow M$ induces an identification

$$K'(X) = K^X(M) .$$

We then define an ascending filtration $(F_n K'(X) \otimes \mathbb{Q})_{n \in \mathbb{Z}}$ on

$K'(X)\otimes\mathbb{Q}$ by

$$F_nK'(X)\otimes\mathbb{Q} = F^{d-n}K^X(M)\otimes\mathbb{Q}$$

and, if $Gr_{.}K'(X)\otimes\mathbb{Q}$ denotes the associated graded group, a homomorphism

$$\tau : K'(X) \to Gr_{.}K'(X)\otimes\mathbb{Q}$$

by the commutativity of

$$\begin{array}{ccc} K'(X) & \xrightarrow{\tau} & Gr_{.}K'(X)\otimes\mathbb{Q} \\ \| & & \| \\ K^X(M) & \xrightarrow{Td(M)\,ch} & Gr^{\cdot}K^X(M)\otimes\mathbb{Q} \end{array} \quad .$$

The first theorem asserts that the filtration $(F_nK'(X)\otimes\mathbb{Q})_{n\in\mathbb{Z}}$ and the homomorphism $\tau : K'(X) \to Gr_{.}K'(X)\otimes\mathbb{Q}$ are independent of the immersion $X \hookrightarrow M$. This is mainly a consequence of the Riemann-Roch theorem for K-theory with supports (§3, 1.1). For the proof we will need a finiteness property of the Grothendieck filtration on $K^X(M)$. Recall the decomposition (§1, 5.1)

$$F^nK^X(M) = \bigoplus_{p\geqq 0} F^nK^X_p(M) \quad .$$

<u>Proposition 1.1.</u>

For every $p \geqq 0$, there exists an $n(p)$, such that

$$F^nK^X_p(M)\otimes\mathbb{Q} = 0$$

for all $n \geqq n(p)$.

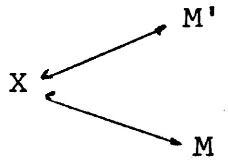
For an even more precise result, see [S], §4, Prop. 5.

Theorem 1.2 (Singular Riemann-Roch).

The filtration $(F_n K'(X) \otimes \mathbb{Q})_{n \in \mathbb{Z}}$ and the homomorphism $\tau : K'(X) \to Gr_{\bullet} K'(X) \otimes \mathbb{Q}$ are independent of the closed immersion $X \hookrightarrow M$.

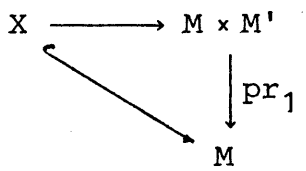
Proof.

Let

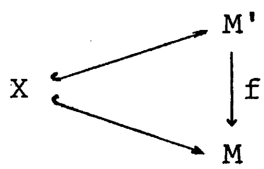


be two closed immersions of X into smooth schemes M and M' of pure dimension d and d'. We have to show that both immersions define the same filtration on $K'(X) \otimes \mathbb{Q}$ on the same map $\tau : K'(X) \to Gr_{\bullet} K'(X) \otimes \mathbb{Q}$.

The obvious commutative diagram



shows that we may assume the existence of a morphism $f : M' \to M$, such that



commutes.

$f : M' \to M$ being quasi-projective, it can be factored into an immersion $M' \to M''$ and a proper morphism $M'' \to M$, with M'' smooth of pure dimension. Hence it obviously suffices to prove the assertions in the two cases, where f is an open immersion, and where f is proper.

Let $f : M' \to M$ be an open immersion, then $d' = d$. By §1, 2.3, we have the commutative diagram

$$\begin{array}{ccc} K^X(M) & \xrightarrow{f^*} & K^X(M') \\ \wr\| & & \|\wr \\ K'(X) & \xrightarrow{id} & K'(X) \end{array}$$

in which f^* is a λ-morphism, respecting the augmentation (§1, 3.1). Hence we have

$$f^*(F^n K^X(M)) = F^n K^X(M')$$

for all n, and the diagram

$$\begin{array}{ccc} K^X(M) & \xrightarrow{Td(M)ch} & Gr^{\cdot} K^X(M) \otimes Q \\ f^* \downarrow & & \downarrow f^* \\ K^X(M') & \xrightarrow{Td(M')ch} & Gr^{\cdot} K^X(M') \otimes Q \end{array}$$

commutes, note $Td(M') = f_* \, Td(M)$. This proves the assertions in case of an open immersion $f : M' \to M$.

Let now $f : M' \to M$ be proper. Then by definition (§1, 2.5), the diagram

$$\begin{array}{ccc} K^X(M') & \xrightarrow{f_*} & K^X(M) \\ \wr\| & & \|\wr \\ K'(X) & \xrightarrow{id} & K'(X) \end{array}$$

commutes. The morphism f is of the constant virtual dimension $d' - d$. Then by the Riemann-Roch theorem we have:

i) $f_*(F^n K^X(M') \otimes Q) \subseteq F^{n+d-d'} K^X(M) \otimes Q$ for all n, and

ii) the diagram

$$\begin{array}{ccc} K^X(M')\otimes Q & \xrightarrow{Td(M')ch} & Gr^{\cdot}K^X(M')\otimes Q \\ f_*\downarrow & & \downarrow f_* \\ K^X(M)\otimes Q & \xrightarrow{Td(M)ch} & Gr^{\cdot}K^X(M)\otimes Q \end{array}$$

commutes.

In this diagram, the horizontal arrows are isomorphisms (see §1, 5, and note that the Todd classes are units), and the left vertical arrow too. Hence the Gysin homomorphism on the right is an isomorphism, that is for all n, the map

$$f_* : Gr^n K^X(M')\otimes Q \to Gr^{n+d-d'} K^X(M)\otimes Q$$

is an isomorphism.

This remains true for the p-components on both sides. Then using the finiteness property above (Lemma 1.1), it follows from i) that

$$f_*(F^n K^X(M')\otimes Q) = F^{n+d-d'} K^X(M)\otimes Q$$

for all n. This proves the assertions in case of a proper morphism $f : M' \to M$, and we are done.

Remark 1.3.

If for every $p \geqq 0$, we define the filtration $(F_n K'_p(X)\otimes Q)_{n\in\mathbb{Z}}$ on $K'_p(X)\otimes Q$ and the homomorphism $\tau_p : K'_p(X) \to Gr_{\cdot} K'_p(X)\otimes Q$ in the same way via $K'_p(X) = K^X_p(M)$, we have the decompositions

$$F_n K'(X)\otimes Q = \bigoplus_{p\geqq 0} F_n K'_p(X)\otimes Q ,$$

$$Gr_{\cdot} K'(X)\otimes Q = \bigoplus_{p\geqq 0} Gr_{\cdot} K'_p(X)\otimes Q ,$$

$$\tau = \sum_{p\geqq 0} \tau_p .$$

This follows from the corresponding decompositions for $K^X(M)$

and the Chern character (§1, 5.).

Definition 1.4.

For a scheme X, the twisted absolute homology is defined to be

$$H_p(X,j) := Gr_j K'_{p-2j}(X) \otimes \mathbb{Q} \quad .$$

Using this, the morphism τ_p may be written as

$$\tau_p : K'_p(X) \to \bigoplus_j H_{\underbrace{2j+p}}(X,j)$$
$$=: H_{2*+p}(X,*) \quad .$$

2. The Gysin-homomorphism and the covariance of τ.

We will now prove - again as a corollary of Riemann-Roch for K-theory with supports - the essential part of the singular Riemann-Roch, namely:

Theorem 2.1 (Singular Riemann-Roch).

Let $f : X \to Y$ be a proper morphism. Then one has:

i) The push-forward $f_* : K'(X) \to K'(Y)$ has degree 0 with respect to the filtration F_n, i.e.

$$f_*(F_n K'(X) \otimes \mathbb{Q}) \subseteq F_n K'(Y) \otimes \mathbb{Q}$$

for all n, and hence it induces a homomorphism

$$f_* : Gr_. K'(X) \otimes \mathbb{Q} \to Gr_. K'(Y) \otimes \mathbb{Q}$$

of graded groups, the Gysin homomorphism.

ii) The diagram

$$\begin{array}{ccc} K'(X) & \xrightarrow{\tau} & Gr_{\bullet}K'(X) \otimes Q \\ f_* \downarrow & & \downarrow f_* \\ K'(Y) & \xrightarrow{\tau} & Gr_{\bullet}K'(Y) \otimes Q \end{array}$$

is commutative.

Proof.

To the given proper morphism $f : X \to Y$, there exists a commutative diagram

$$\begin{array}{ccc} X & \hookrightarrow & M' \\ f \downarrow & & \downarrow f \\ Y & \hookrightarrow & M \end{array}$$

in which the horizontal maps are closed immersions into smooth schemes of pure dimension, and in which $f : M' \to M$ is proper. Such a diagram can be found by means of a factorization $f : X \hookrightarrow \mathbb{P}^r_Y \to Y$.

By definition (§1, 2.5), the diagram

$$\begin{array}{ccc} K^X(M') & \xrightarrow{f_*} & K^Y(M) \\ \wr\| & & \|\wr \\ K'(X) & \xrightarrow{f_*} & K'(Y) \end{array}$$

commutes. Let d' and d be the dimension of M' and M, respectively. Then, by the Riemann-Roch theorem for $f:(M',X)\to(M,Y)$, being of the virtual dimension $d' - d$, we have:

i) $f_* : K^X(M') \to K^Y(M)$ has degree $d - d'$, and hence

$$f_*(F^{d'-n}K^X(M') \otimes Q) \subseteq F^{d-n}K^Y(M) \otimes Q \ .$$

ii) The diagram

$$\begin{array}{ccc} K^X(M') & \xrightarrow{Td(M')ch} & Gr^{\cdot}K^X(M') \otimes Q \\ f_* \downarrow & & \downarrow f_* \\ K^Y(M) & \xrightarrow{Td(M)ch} & Gr^{\cdot}K^Y(M) \otimes Q \end{array}$$

commutes.

In view of the definitions of F_n and τ, this proves the theorem.

3. Behaviour of F_n and τ with respect to open immersions

If $f : X \to Y$ is a flat morphism, then the pull-back homomorphism

$$f^* : K'(Y) \to K'(X)$$

is defined. It is of interest to study the behaviour of the filtration F_n and of the transformation τ under f^*. Concerning the case of an open immersion, we prove the following trivial fact, which we nevertheless state as a theorem

Theorem 3.1 (Singular Riemann-Roch).
Let $f : X \to Y$ be an open immersion. Then one has

$$f^*(F_nK'(Y) \otimes Q) \subseteq F_nK'(X) \otimes Q$$

for all n, and the diagram

$$\begin{array}{ccc} K'(Y) & \xrightarrow{\tau} & Gr_{\cdot}K'(Y) \otimes Q \\ f^* \downarrow & & \downarrow f^* \\ K'(X) & \xrightarrow{\tau} & Gr_{\cdot}K'(X) \otimes Q \end{array}$$

with the induced map on the right, commutes.

Proof.

Obviously, there exists a cartesian diagram

$$\begin{array}{ccc} X & \hookrightarrow & M' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \hookrightarrow & M \end{array}$$

where the horizontal maps are closed immersions into smooth schemes of pure dimension and where $f : M' \to M$ is an open immersion. Then the diagram

$$\begin{array}{ccc} K^Y(M) & \xrightarrow{f^*} & K^X(M') \\ \wr\Vert & & \Vert\wr \\ K'(Y) & \xrightarrow{f^*} & K'(X) \end{array}$$

commutes (§1, 2.3). The upper f^* is a λ-morphism, respecting the augmentation (§1, 3.1). Hence we have

$$f^*(F^nK^Y(M)) \subseteq F^nK^X(M')$$

for all n, and the diagram

$$\begin{array}{ccc} K^Y(M) & \xrightarrow{Td(M)ch} & Gr^{\cdot}K^Y(M) \otimes \mathbb{Q} \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ K^X(M') & \xrightarrow{Td(M')ch} & Gr^{\cdot}K^X(M') \otimes \mathbb{Q} \end{array}$$

commutes. From this the theorem follows.

Remark.

As a much more interesting result the following generalization should be true (see [S], Th. 8, iii)):

Theorem 3.2.

For an etale morphism $f : X \to Y$, the assertions of the preceding Theorem 3.1 remain valid.

We only will say some few words concerning the filtration: The morphism $f : X \to Y$ admits a factorization of the form $f : X \xrightarrow{i} U \xrightarrow{j} \mathbb{P}^r_Y \xrightarrow{p} Y$, with i a closed immersion, j an open immersion and p the projection. Since f is etale, the immersion i is regular of the pure codimension r.

i is of finite Tor dimension, and so there is a pull-back homomorphism $i^* : K'(U) \to K'(X)$, see [Q], §7, 2.5. The morphisms j and p are flat, and $f^* : K'(Y) \to K'(X)$ has the factorization

$$f^* = i^* \cdot j^* \cdot p^* .$$

It is easy to see that

i) $p^*(F_n K'(Y) \otimes \mathbb{Q}) \subseteq F_{n+r} K'(\mathbb{P}^r_Y) \otimes \mathbb{Q}$.

In fact, this comes out with the cartesian diagram

$$\begin{array}{ccc} \mathbb{P}^r_Y & \hookrightarrow & \mathbb{P}^r_M \\ p \downarrow & & \downarrow p \\ Y & \hookrightarrow & M \end{array}$$

obtained by choosing a closed immersion of Y into a smooth scheme M of pure dimension.

It has been shown already that

ii) $j^*(F_n K'(\mathbb{P}^r_Y) \otimes \mathbb{Q}) \subseteq F_n K'(U) \otimes \mathbb{Q}$.

It is the crucial point to prove

iii) $i^*(F_n K'(U) \otimes \mathbb{Q}) \subseteq F_{n-r} K'(X) \otimes \mathbb{Q}$.

The proof is clear, if $i : X \to U$ can be imbedded into a cartesian diagram

$$\begin{array}{ccc} X & \hookrightarrow & M \\ {\scriptstyle i}\downarrow & & \downarrow \\ U & \hookrightarrow & M' \end{array}$$

where $X \hookrightarrow M$ and $U \hookrightarrow M'$ are closed immersions into smooth schemes of the pure dimension d and $d' = d+r$. Using a lemma of Fulton (see next section), such a diagram can easily be obtained in the case in which i is the zero section of $\mathbb{P}(\mathcal{N} \oplus \mathcal{O}_Y)$, $\mathcal{N}$ a locally free sheaf of rank r on Y. The general proof may be achieved by an intelligent deformation argument (see loc.cit.).

4. Chern, Riemann-Roch and the cap product

Before stating the theorem we first have to prove a result concerning the definition of a λ-structure on the ring (§1, 3.)

$$K(X) = \bigoplus_{p \geqq 0} K_p(X) ,$$

X being an arbitrary scheme of our category $\mathcal{V}$. For smooth schemes, this is done (§1, 3.1).

Lemma 4.1.

For every scheme X, the ring K(X) carries a λ-structure, uniquely determined by the following two properties:

i) The λ-structure is functorial in X with respect to pull-backs.

ii) For smooth X, the λ-structure coincides with the given one (§1, 3.1).

With respect to the canonical maps (§1.3)

$$H^{0}(X,\mathbb{Z}) \rightarrow K(X)$$

$$K(X) \longrightarrow H^{0}(X,\mathbb{Z}),$$

$K(X)$ is an augmented λ-algebra over $H^{0}(X,\mathbb{Z})$, whose associated Grothendieck filtration $(F^{n}K(X))_{n\in\mathbb{Z}}$ is locally nilpotent.

<u>Remark.</u>

For every scheme X, we now have the two homomorphisms

$$ch : K(X) \rightarrow Gr^{\cdot}K(X) \otimes \mathbb{Q}$$

$$\tau : K'(X) \rightarrow Gr_{\cdot}K'(X) \otimes \mathbb{Q} \quad .$$

They will be connected by the cap product

$$K(X) \times K'(X) \xrightarrow{\cap} K'(X) \quad .$$

For the proof of Lemma 4.1, let X be a scheme, and $I = I(X)$ be the category whose objects are morphisms $X \rightarrow M$ of X into smooth schemes M and whose arrows are commutative diagrams

$$X \begin{array}{l} \nearrow M \\ \quad \downarrow \\ \searrow M' \end{array} \quad .$$

The dual category I^{0} is filtering. We consider the functor $(X \rightarrow M) \mapsto K(M)$ from I^{0} into the category of commutative rings.

<u>Lemma 4.2.</u>

The canonical map

$$\varinjlim K(M) \rightarrow K(X)$$

is an isomorphism.

Since the category of λ-rings admits filtered inductive limits, this lemma implies the Lemma 4.1. The last assertion follows from §1, 3.1.

To prove Lemma 4.2, it suffices by [Q], §1, Prop. 3 to show that the functor

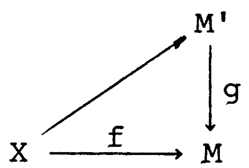
$$\varinjlim \; Q\mathcal{P}(M) \rightarrow Q\mathcal{P}(X)$$

is an isomorphism. I^{o} being filtered, it suffices to prove this for the objects and the arrows of the considered categories. In view of the definition of the Quillen category, the wanted isomorphism follows from the following lemma, due to Fulton ([F], 3.2.).

<u>Lemma 4.3.</u>

i) If $E^{\bullet}$ is a short exact sequence in $\mathcal{P}(X)$, then there exist an object $f : X \rightarrow M$ in I and a short exact sequence $F^{\bullet}$ in $\mathcal{P}(M)$, such that $f^{*}F^{\bullet} \cong E^{\bullet}$.

ii) If $f : X \rightarrow M$ is an object in I, and if $F_1^{\bullet}$ and $F_2^{\bullet}$ are short exact sequences in $\mathcal{P}(M)$ with $f^{*}F_1^{\bullet} = f^{*}F_2^{\bullet}$, then there exists a morphism



in I, such that $g^{*}F_1^{\bullet} \cong g^{*}F_2^{\bullet}$.

For later use, we will add the following facts concerning the canonical isomorphism $\varinjlim K(M) \cong K(X)$.

Remarks 4.4.

i) Let J be the full subcategory of $I = I(X)$ consisting of all closed immersions $X \to M$ into smooth schemes of pure dimension. Let $f : X \to M$ be any object of I. Then from a factorization $f : X \to \mathbb{P}^r_M \to M$ we get a commutative triangle

$$\begin{array}{ccc} & & M' \\ & \overset{i}{\nearrow} & \downarrow \\ X & \xrightarrow{\ f\ } & M \end{array}$$

where $i : X \to M'$ is a closed immersion into a smooth M'. Substituting the connected components M'_α of smaller dimension by affine spaces of suitable dimension over M_α, we arrive at a smooth scheme M'' of pure dimension. The zero sections give a closed immersion $j : X \to M''$, and the projections a commutative diagram

$$\begin{array}{ccc} & & M'' \\ & \overset{j}{\nearrow} & \downarrow \\ X & \xrightarrow{\ i\ } & M' \end{array} .$$

This shows that the dual J^o is a final subcategory of I^o, and hence from Lemma 4.2 we obtain

$$\varinjlim K(M) \cong K(X)$$

where now the inductive limit is formed on the category J^o.

ii) Since the index categories are filtering, the isomorphisms $\varinjlim K(M) \cong K(X)$ induce isomorphisms

$$\varinjlim F^n K(M) \cong F^n K(X)$$

for all n.

This last remark and §1, 5. imply

Remark 4.5.

Let

$$F^n K_p(X) = F^n K(X) \cap K_p(X) .$$

Then one has the decompositions

$$F^n K(X) = \bigoplus_{p \geqq 0} F^n K_p(X) ,$$

$$Gr^n K(X) = \bigoplus_{p \geqq 0} Gr^n K_p(X) .$$

Definition 4.6.

For a scheme X, the twisted absolute cohomology is defined by

$$H^p(X,j) := Gr^j K_{2j-p}(X) \otimes \mathbb{Q} .$$

The p-component $ch_p : K_p(X) \to Gr^{\cdot} K_p(X) \otimes \mathbb{Q}$ of the Chern character may then be read as a homomorphism

$$ch_p : K_p(X) \to H^{2*-p}(X,*)$$

where $H^{2*-p}(X,*) = \bigoplus_j H^{2j-p}(X,j)$.

For every scheme X, one has the cap product

$$K(X) \times K'(X) \xrightarrow{\cap} K'(X)$$

induced by the biexact functor $\mathcal{P}(X) \times \mathcal{M}(X) \to \mathcal{M}(X)$, see [G], §7.
We now state the theorem.

Theorem 4.7 (Singular Riemann-Roch).
Let X be a scheme, then:
i) The cap product $K(X) \times K'(X) \xrightarrow{\cap} K'(X)$ induces a composition

$$(F^n K(X) \otimes Q) \times (F_m K'(X) \otimes Q) \to F_{m-n} K'(X) \otimes Q$$

and hence a map

$$(Gr^{\cdot} K(X) \otimes Q) \times (Gr_{\cdot} K'(X) \otimes Q) \xrightarrow{\cap} Gr_{\cdot} K'(X) \otimes Q,$$

also called cap product.
ii) The diagram

$$\begin{array}{ccc} K(X) \times K'(X) & \xrightarrow{\quad\cap\quad} & K'(X) \\ \downarrow{\scriptstyle ch \times \tau} & & \downarrow{\scriptstyle \tau} \\ (Gr^{\cdot} K(X) \otimes Q) \times (Gr_{\cdot} K'(X) \otimes Q) & \xrightarrow{\quad\cap\quad} & Gr_{\cdot} K'(X) \otimes Q \end{array}$$

commutes.

Proof.
By 4.4.i) we have $K(X) = \varinjlim K(M)$ where the limit is over all closed immersions $i : X \to M$ into smooth schemes of pure dimension. For such an immersion $i : X \to M$, we have the commutative diagram

$$\begin{array}{ccc} K(M) \times K^X(M) & \xrightarrow{\quad\cup\quad} & K^X(M) \\ {\scriptstyle i^*}\downarrow \quad \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ K(X) \times K'(X) & \xrightarrow{\quad\cap\quad} & K'(X) \end{array}$$

where $\cup$ denotes the cup product for the K-groups ([S], 4.3.).

Using this and 4.4.ii), for the first assertion, it suffices to show that the cup product $K(M) \times K^X(M) \xrightarrow{\cup} K^X(M)$ induces maps

$$(F^n K(M) \otimes \mathbb{Q}) \times (F^k K^X(M) \otimes \mathbb{Q}) \xrightarrow{\cup} F^{n+k} K^X(M) \otimes \mathbb{Q}$$

for all n,k. But this is a result of Kratzer ([K], 6.4.) saying that the filtration, defined similar to the Grothendieck filtration but with cup products instead of products, coincides with the Grothendieck filtration modulo torsion.
For the second assertion, it suffices to prove the formula

$$ch(x \cup y) = ch(x) \cup ch(y)$$

for $x \in K(M)$ and $y \in K^X(M)$. In view of §1, 4.3 this follows from a second result of Kratzer ([K], 5.6.) saying that the Adams operations are compatible with the cup product, i.e.

$$\psi^k(x \cup y) = \psi^k(x) \cup \psi^k(y) \quad .$$

§5. Absolute cohomology and homology

We continue with the category $\mathcal{V}$. For a scheme X, the twisted absolute cohomology and the twisted absolute homology are defined by

$$H^p(X,j) = Gr^j K_{2j-p}(X) \otimes \mathbb{Q}$$
$$H_p(X,j) = Gr_j K'_{p-2j}(X) \otimes \mathbb{Q} \quad .$$

Recall that the Chern character yields an identification of $H^p(X,j)$ with the k^j-eigenspace of the Adams operations ψ^k on

$K_{2j-p}(X) \otimes Q$.

The absolute cohomology $H^p(X,j)$ is contravariant on $\mathcal{V}$. Concerning the absolute homology $H_p(X,j)$, we will now prove:

Theorem 5.1.

The absolute homology $H_p(X,j)$ satisfies the axioms of a twisted homology theory in the sense of Bloch-Ogus ([BO], 1.2), i.e.

i) $H_p(X,j)$ is covariant for proper morphisms.

ii) $H_p(X,j)$ is contravariant for open immersions (and more generally, for etale morphisms).

iii) If

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a Cartesian diagram in which the vertical arrows are proper and the horizontal ones are open immersions (or more generally, etale), then the diagram

$$\begin{array}{ccc} H_p(X,j) & \xrightarrow{g'^*} & H_p(X',j) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f'_*} \\ H_p(Y,j) & \xrightarrow{g^*} & H_p(Y',j) \end{array}$$

commutes.

iv) If $i : Y \to X$ is a closed immersion and $j : (X-Y) \to X$ the corresponding open immersion, then there is a long exact sequence

$$\to H_p(Y,j) \xrightarrow{i_*} H_p(X,j) \xrightarrow{j^*} H_p(X-Y,j) \xrightarrow{\partial} H_{p-1}(Y,j) \to$$

v) If

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ \downarrow & & \downarrow f \\ Y & \xrightarrow{i} & X \end{array}$$

is a commutative diagram in which the vertical arrows are proper and the horizontal arrows are closed immersions, and if $g : (X'-f^{-1}(Y)) \to (X'-Y')$ is the induced open immersion, then the diagram

$$\begin{array}{ccccccc} \to H_p(X',j) & \to & H_p(X'-Y',j) & \to & H_{p-1}(Y',j) & \to \\ \downarrow f_* & & \downarrow f_* g^* & & \downarrow f_* & \\ \to H_p(X,j) & \longrightarrow & H_p(X-Y,j) & \longrightarrow & H_{p-1}(Y,j) & \longrightarrow \end{array}$$

commutes.

<u>Proof.</u>

i) is Riemann-Roch (§4, 2.1).

ii) is Riemann-Roch (§4, 3.1). For etale morphisms, one needs §4, 3.2.

iii) Since $g : Y' \to Y$ is flat, the diagram

$$\begin{array}{ccc} K'_p(X) & \xrightarrow{g'^*} & K'_p(X') \\ f_* \downarrow & & \downarrow f'_* \\ K'_p(Y) & \xrightarrow{g^*} & K'_p(Y') \end{array}$$

commutes ([Q], §7, 2.11.). Using the fact that the homomorphisms

$$\tau_p : K'_p(X) \otimes Q \to Gr_{.} K'_p(X) \otimes Q$$

are isomorphims, it follows from Riemann-Roch (§4, 2.1 and 3.1) that the diagram

$$\begin{array}{ccc} Gr_{.}K'_p(X)\otimes Q & \xrightarrow{g'^*} & Gr_{.}K'_p(X')\otimes Q \\ f_* \downarrow & & \downarrow f_* \\ Gr_{.}K'_p(Y)\otimes Q & \xrightarrow{g^*} & Gr_{.}K'_p(Y')\otimes Q \end{array}$$

commutes, which proves i i). If g is etale, one needs §4, 3.2.

iv) follows in the same way from the exact localization sequence

$$\to K'_p(Y) \xrightarrow{i_*} K'_p(X) \xrightarrow{j^*} K'_p(X-Y) \xrightarrow{\partial} K'_{p-1}(Y) \longrightarrow$$

using the existence and commutativity of the diagram

$$(*) \qquad \begin{array}{ccc} K'_p(X-Y) & \xrightarrow{\partial} & K'_{p-1}(Y) \\ \downarrow \tau_p & & \downarrow \tau_{p-1} \\ Gr_{.}K'_p(X-Y)\otimes Q & \xrightarrow{\partial} & Gr_{.}K'_{p-1}(Y)\otimes Q \end{array}$$

which we will prove below.

v) follows from an appropriate diagram of K'-groups.

Proof of (*). We choose a closed immersion $X \hookrightarrow M$ of X into a smooth scheme M of pure dimension. Then the diagram

$$\begin{array}{ccc} K_p^{X-Y}(M-Y) & \xrightarrow{\partial} & K_{p-1}^{Y}(M) \\ \downarrow \cong & & \downarrow \cong \\ K'_p(X-Y) & \xrightarrow{\partial} & K'_{p-1}(Y) \end{array}$$

commutes (§1, 2.). Since the upper ∂ commutes with the λ-operations (§1, 3.1), the diagram

$$\begin{array}{ccc} K_p^{X-Y}(M-Y) & \xrightarrow{\partial} & K_{p-1}^{Y}(M) \\ \downarrow{\scriptstyle Td(M-Y)ch_p} & & \downarrow{\scriptstyle Td(M)ch_p} \\ Gr^{\cdot} K_p^{X-Y}(M-Y) \otimes Q & \xrightarrow{\partial} & Gr^{\cdot} K_{p-1}^{Y}(M) \otimes Q \end{array}$$

commutes. From this we obtain the required commutative diagram (*).

By Riemann-Roch (§4, 4.7), the cap product $K(X) \times K'(X) \xrightarrow{\cap} K'(X)$ induces a cap product

$$H^p(X,i) \times H_q(X,j) \xrightarrow{\cap} H_{q-p}(X,j-i) \ .$$

Theorem 5.2.

The cap product defines a Poincaré duality theory for the absolute cohomology and homology in the sense of Bloch-Ogus ([BO], 1.3), i.e.

i) If $f : X' \to X$ is an open immersion (or more generally, an etale morphism), then the diagram

$$\begin{array}{ccc} H^p(X,i) \times H_q(X,j) & \xrightarrow{\cap} & H_{q-p}(X,j-i) \\ \downarrow{\scriptstyle f^*} \quad \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ H^p(X',i) \times H_q(X',j) & \xrightarrow{\cap} & H_{q-p}(X',j-i) \end{array}$$

commutes.

ii) If $f : X \to Y$ is proper, then the diagram

$$\begin{array}{ccc} H^p(X,i) \times H_q(X,j) & \xrightarrow{\cap} & H_{q-p}(X,j-i) \\ {\scriptstyle f^*}\uparrow \quad \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ H^p(Y,i) \times H_q(Y,j) & \xrightarrow{\cap} & H_{q-p}(Y,j-i) \end{array}$$

commutes (Projection formula).

iii) If X is smooth of pure dimension d and if $\eta_X \in H_{2d}(X,d)$ denotes the fundamental class (i.e. the class of the structure sheaf), then the map

$$\cap\eta_X : H^p(X,i) \longrightarrow H_{2d-p}(X,d-i)$$

is an isomorphism.

Proof.

These properties follow from the corresponding properties of the cap product $K(X) \times K'(X) \xrightarrow{\cap} K'(X)$ by means of the isomorphism

$$ch_p : K_p(X) \otimes Q \xrightarrow{\cong} Gr^{\cdot} K_p(X) \otimes Q$$
$$\tau_p : K'_p(X) \otimes Q \xrightarrow{\cong} Gr_{\cdot} K'_p(X) \otimes Q$$

and Riemann-Roch (§4, 4.7).

From the assertions iii) of both theorems one immediately gets:

Corollary (Gysin sequence).

Let $Y \hookrightarrow X$ be a closed immersion of smooth schemes and assume that $Y \hookrightarrow X$ is of pure codimension d. Then there is a long exact sequence

$$\ldots \to H^{p-2d}(Y,j-d) \to H^p(X,j) \to H^p(X-Y,j) \to$$
$$\to H^{p+1-2d}(Y,j-d) \to \ldots \quad .$$

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