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Iwasawa L-Functions of Varieties over Algebraic Number Fields

A First Approach

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A fascinating task in algebraic number theory is the study of the values of various complex L-functions (Dedekind zeta function, Artin L-function, Hasse-Weil L-function ...) at integer points. It has often turned out that these values are essentially rational numbers (see [7]). Therefore it is of course a fundamental problem to give arithmetic interpretations of these numbers. One possibility of attack on this problem seems to be the following: First interpolate these rational numbers by a p-adic L-function and then relate this function to the characteristic power series of an "Iwasawa module" which is naturally associated with the underlying arithmetic problem. But this program is extremely difficult and has been fully established only in special cases (recent work of Mazur/Wiles concerning the "main conjecture" in cyclotomic Iwasawa theory). For this reason, we pursue in this paper the much simpler problem of calculating, up to a p-adic unit, the values of the above mentioned characteristic power series at integer points. A considerable body of work has already been done in this direction. Of course, the results we obtain contain much of this earlier work, and also are compatible with known conjectures about the corresponding values of the complex L-functions.

Let X be a proper smooth scheme over an algebraic number field k; let $\bar{k}/k$ be an algebraic closure of k, $\bar{X} := X \times \bar{k}$, and $G_k := \mathrm{Gal}(\bar{k}/k)$ the absolute Galois group of k. Obviously $H^0_{et}(\bar{X}, \mathbb{Z})$ is a G_k-module finitely generated and free over $\mathbb{Z}$. It defines by duality an algebraic torus $T(X)$ over k (for example, $T(\mathrm{Spec}(k))$ is the multiplicative group $\mathbb{G}_m$ over k). In Part II of this paper we shall define and study the Iwasawa L-functions of an arbitrary algebraic torus T over k. In the case $T = T(X)$ they should be viewed as the 0-dimensional Iwasawa L-functions of X, because they depend only on the 0-cohomology of $\bar{X}$. Their complex analogue is the 0-dimensional L-function of X in the sense of Serre [32], which is nothing else but the Artin L-function (in the sense of [6] which differs slightly from the original one) associated with the representation of G_k on $H^0_{et}(\bar{X}, \mathbb{Q})$.

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It is well known that the Picard group scheme of X/k exists and that its connected component $A(X)$ is an abelian variety over k. The Iwasawa theory of an arbitrary abelian variety over k was initiated by Mazur [17]. We shall continue this work (in our context) in Part III of this paper. In particular, we shall discuss in detail a version of the conjecture of Birch and Swinnerton-Dyer for the Iwasawa L-functions of abelian varieties. For $A(X)$ we get in this way the 1-dimensional Iwasawa L-functions of X (see also the reasoning in [17], p. 190); their complex analogue is the Hasse-Weil L-function of $A(X)$, respectively, the 1-dimensional L-function of X in the sense of Serre.

In order to make the parallelism between $T(X)$ and $A(X)$ completely clear, we give the following equivalent description. Let $f: X \to \mathrm{Spec}(k)$ be the structure morphism, and let $R^q f_* \mathbb{G}_m$ denote the higher direct images of the sheaf represented by the multiplicative group $\mathbb{G}_m$ over X with respect to the global $fppf$-topologies. Then $T(X)$ represents $f_* \mathbb{G}_m = R^0 f_* \mathbb{G}_m$. On the other hand the Picard group scheme of X/k represents $R^1 f_* \mathbb{G}_m$. The status of the sheaves $R^q f_* \mathbb{G}_m$ for $q \geqq 2$ is not quite clear. They are not representable in general, but nevertheless they should be important for a theory of higher-dimensional Iwasawa L-functions of X.

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Contents

I. Preliminaries

§1. Notations and Conventions

For an abelian group M, let $\mathrm{Tor}\,M$ be the torsion subgroup and $M_{\mathrm{Tor}} := M/\mathrm{Tor}\,M$, let $\mathrm{Div}\,M$ be the maximal divisible subgroup and $M_{\mathrm{Div}} := M/\mathrm{Div}\,M$. We use the same notation for a homomorphism $f: M \to N$ between abelian groups, e.g. f_{Tor} denotes the induced map $M_{\mathrm{Tor}} \to N_{\mathrm{Tor}}$. Furthermore, f is called a quasi-isomorphism, if it has finite kernel and cokernel. Put $M_n := \{a \in M : na = 0\}$ for $n \in \mathbb{N}$ and $M(p) := \bigcup_j M_{p^j}$ for any prime number p. If M is finite, $\# M$ denotes the number of elements in M.

For a $\mathbb{Z}_p$-module M, let $M^* := \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ be the Pontrjagin dual of M. We say the $\mathbb{Z}_p$-torsion module M is of cofinite type if M^* is a finitely generated $\mathbb{Z}_p$-module; in this case put $\mathrm{corank}\, M := \mathrm{rank}_{\mathbb{Z}_p} M^*$.

For a commutative ring R with unit, let $R^\times$ be the group of invertible elements in R. We denote by $|\ |$, resp. $|\ |_p$, the usual real, resp. p-adic, absolute value.

Throughout the paper, k is a finite extension of $\mathbb{Q}$, $\bar{k}$ an algebraic closure of k, $G_k := \mathrm{Gal}(\bar{k}/k)$ the absolute Galois group of k, and $\mathfrak{o}$ the ring of integers in k. For any prime $\mathfrak{p}$ of k, let $\hat{k}_{\mathfrak{p}}$ be the completion of k at $\mathfrak{p}$. If $\mathfrak{p}$ is a finite prime then $\hat{\mathfrak{o}}_{\mathfrak{p}}$ denotes the ring of integers in $\hat{k}_{\mathfrak{p}}$, $\mathfrak{o}_{\mathfrak{p}}$ the Henselization of $\mathfrak{o}$ at $\mathfrak{p}$, and $k_{\mathfrak{p}}$ the quotient field of $\mathfrak{o}_{\mathfrak{p}}$.

By an S-group scheme $\mathcal{G}$ we always mean a commutative group scheme locally of finite presentation over the scheme S. The kernel $\mathcal{G}_n := \ker(\mathcal{G} \xrightarrow{n} \mathcal{G})$ of multiplication by $n \in \mathbb{N}$ is again an S-group scheme. For a prime number p, let $\mathcal{G}(p)$ be the ind-S-group scheme $\mathcal{G}(p) := \varinjlim \mathcal{G}_{p^j}$. Furthermore, $\mathbb{G}_{m/S}$ denotes the multiplicative group scheme over S; put $\mu_{n/S} := \ker(\mathbb{G}_{m/S} \xrightarrow{n} \mathbb{G}_{m/S})$ and $\mu(p)_{/S} := \varinjlim \mu_{p^j/S}$.

For the convenience of the reader we shall explain now in more detail our conventions concerning topology and cohomology. For any quasi-compact and quasi-separated scheme S (e.g. S affine) we consider the following three sites on S:

S_{fppf} site of all schemes locally of finite presentation over S with the $fppf$-topology;

S_{fpqf} site of all quasi-finite flat schemes of finite presentation over S with the $fppf$-topology;

(in both cases coverings are surjective families of flat morphisms)

S_{et} site of all etale schemes of finite presentation over S with the etale topology

(coverings are surjective families of morphisms). We have canonical morphisms of sites

$$S_{fppf} \xrightarrow{\pi} S_{fpqf} \xrightarrow{\sigma} S_{et}.$$

The direct image functor π_* between the corresponding categories of abelian sheaves is exact; similarly σ_* is left exact and commutes with pseudofiltered direct limits.

We denote by $H^i(S, .)$, resp. $H^i_{et}(S, .)$, the cohomology groups of abelian sheaves on S_{fpqf}, resp. S_{et}. They commute with pseudofiltered direct limits of sheaves; they also commute with certain projective limits of schemes (see SGA4VII.5.8 and [11], p. 172).

Let $\mathcal{G}$ be an S-group scheme. It represents an abelian sheaf on each of the three sites which we denote by

$$\mathcal{G}_{/S_{fppf}} \text{ resp. } \mathcal{G}_{/S_{fpqf}} \text{ resp. } \mathcal{G}_{/S_{et}}$$

if necessary. A sequence of S-group schemes is called exact if the corresponding sequence of abelian sheaves on S_{fppf} is exact. According to a theorem of Grothendieck we have $R^i \sigma_* \mathscr{G} = 0$ for $i > 0$ and therefore $H^i(S, \mathscr{G}) = H^i_{et}(S, \mathscr{G})$ for $i \geqq 0$ if $\mathscr{G}$ is smooth over S. As a further consequence we get that

$$0 \to \mathscr{G}_{/S_{et}} \to \mathscr{G}_{1/S_{et}} \to \mathscr{G}_{2/S_{et}} \to 0$$

is exact if

$$0 \to \mathscr{G} \to \mathscr{G}_1 \to \mathscr{G}_2 \to 0$$

is a short exact sequence of S-group schemes with $\mathscr{G}$ smooth over S.

We emphasize that we have $\pi_*(\varinjlim \mathscr{G}_{p^j/S_{fppf}}) = \varinjlim \mathscr{G}_{p^j/S_{fpqf}}$ although, in general, π_* does not commute with direct limits. If $f: T \to S$ is a morphism of quasi-compact and quasi-separated schemes then the inverse image functor f^* (for each of the three sites) is exact and commutes with direct limits. In the case that f or the structure morphism $\mathscr{G} \to S$ is a morphism in the site S. we have

$$f^*(\mathscr{G}_{/S.}) = (\mathscr{G} \underset{S}{\times} T)_{/T.}.$$

By $\mathrm{cd}_p S$ we always mean the cohomological p-dimension of S_{et}. Finally, if $S = \mathrm{Spec}(R)$ is affine we usually replace S by R in our notations. As a good reference for all questions concerning cohomology we recommend [22].

§ 2. *Arithmetic Homology*

We fix for the rest of this paper an odd prime number p. Put $k_n := k(\mu_{p^n})$ and $k_\infty := k(\mu(p)) = \bigcup_{n \in \mathbb{N}} k_n$; let $\mathfrak{o}_n$, resp. $\mathfrak{o}_\infty$, denote the ring of integers in k_n, resp. k_∞. For the Galois group $G := \mathrm{Gal}(k_\infty/k)$ we have the canonical decomposition

$$G = \Gamma \times \Delta$$

with $\Gamma := \mathrm{Gal}(k_\infty/k_1)$ and $\Delta := \mathrm{Gal}(k_1/k)$. The order $d := \# \Delta$ of Δ divides $p-1$, and Γ is isomorphic (as topological group) to the additive group of p-adic integers. The action of G on $\mu(p)$ is given by the "cyclotomic" character

$$\kappa: G \to \mathbb{Z}_p^\times$$

which induces a canonical isomorphism

$$\Gamma \xrightarrow{\cong} 1 + p^e \mathbb{Z}_p$$

for a certain $e \in \mathbb{N}$. We fix the topological generator $\gamma := \kappa^{-1}(1+p^e)$ of Γ.

Our arithmetic homology groups will be modules over the completed group ring $\mathbb{Z}_p[\![\Gamma]\!]$. Therefore, we first review some of the results in the theory of this modules (see [13]). Let M be a finitely generated $\mathbb{Z}_p[\![\Gamma]\!]$-torsion module. Because of the isomorphism

$$\mathbb{Z}_p[\![\Gamma]\!] \xrightarrow{\cong} \mathbb{Z}_p[\![t]\!], \quad \gamma \mapsto 1+t$$

we consider M as module over the power series ring $\mathbb{Z}_p[\![t]\!]$. According to the general structure theory we then have a quasi-isomorphism of $\mathbb{Z}_p[\![t]\!]$-modules

$$M \to \bigoplus_{i=1}^{r} \mathbb{Z}_p[\![t]\!]/\langle f_i(t) \rangle$$

where the $f_i(t) \in \mathbb{Z}_p[t]$ are powers of p or distinguished polynomials (i.e. monic and $\equiv t^{\lambda_i} \bmod p$). The polynomial

$$F_M(t) := \prod_{i=1}^{r} f_i(t)$$

depends only on M and is called the characteristic polynomial of M. This terminology is justified because of the following equivalent description: Namely, if

$$\mathrm{Tor}\, M \to \bigoplus_\alpha (\mathbb{Z}/p^{\mu_\alpha}\mathbb{Z})[\![t]\!]$$

is a quasi-isomorphism of $\mathbb{Z}_p[\![t]\!]$-modules, then we have

$$F_M(t) = p^{\Sigma \mu_\alpha} \cdot \det(t - (\gamma - 1);\ M \underset{\mathbb{Z}_p}{\otimes} \mathbb{Q}_p).$$

Let M^Γ, resp. M_Γ, be the Γ-invariants, resp. Γ-coinvariants, of M. Furthermore $M(n)$, $n \in \mathbb{Z}$, denotes the usual n-fold Tate twist of M (see [2] I, § 2). Now we have the following fundamental result about the values of $F_M(t)$.

Lemma 1. *For $n \in \mathbb{Z}$, the following three assertions are equivalent:*

i) *$M(n)^\Gamma$ is finite;*

ii) *$M(n)_\Gamma$ is finite;*

iii) $F_M((1+p^e)^{-n} - 1) \neq 0$.

If these assertions hold, then $|F_M((1+p^e)^{-n} - 1)|_p = \# M(n)^\Gamma / \# M(n)_\Gamma$.

Proof. We have

$$F_{M(n)}(t) = u_n \cdot F_M((1+p^e)^{-n}(t+1) - 1)$$

with $u_n \in \mathbb{Z}_p^\times$. Therefore it is enough to prove the lemma in the case $n=0$ for which we refer to [4] App. Lemma 9.

But we are also interested to analyze the case where $F_M(t)$ has a zero at $t=0$. Define $\rho(M) \geqq 0$ and $c(M) \in \mathbb{Z}_p$ by

$$[F_M(t) \cdot t^{-\rho(M)}]_{t=0} =: c(M) \neq 0.$$

The identity map on M induces the homomorphism

$$f_M:\ M^\Gamma \to M_\Gamma;$$

if f_M is a quasi-isomorphism, we put

$$q(M) := \# \mathrm{coker}\, f_M / \# \ker f_M.$$

Remark 2. i) Let $M \to N$ be a quasi-isomorphism of $\mathbb{Z}_p[\![\Gamma]\!]$-modules. Then, f_M is a quasi-isomorphism if and only if f_N is a quasi-isomorphism; in this case we have $q(M) = q(N)$.

ii) For $\mu \geqq 0$ and $M = \mathbb{Z}/p^\mu \mathbb{Z}[\![\Gamma]\!]$ we have: f_M is a quasi-isomorphism and $q(M) = p^\mu$.

Lemma 3. i) $\rho(M) \geqq \operatorname{rank}_{\mathbb{Z}_p} M^\Gamma = \operatorname{rank}_{\mathbb{Z}_p} M_\Gamma$;

ii) $\rho(M) = \operatorname{rank}_{\mathbb{Z}_p} M^\Gamma \Leftrightarrow f_M$ is a quasi-isomorphism; in this case we have $|c(M)|_p^{-1} = q(M)$.

Proof. This is an easy generalization of [37] lemma z.4 if one uses the above remark and the structure theory of $\mathbb{Z}_p[\![\Gamma]\!]$-modules.

An ind-p-group $\mathfrak{G}$ over $\mathfrak{o}$ is an inductive system

$$\mathfrak{G} = (\mathcal{G}_\nu, i_\nu)_{\nu \in \mathbb{N}},$$

where $\mathcal{G}_\nu$ is a quasi-finite flat $\mathfrak{o}$-group scheme, and, for each $\nu \in \mathbb{N}$,

$$0 \to \mathcal{G}_\nu \xrightarrow{i_\nu} \mathcal{G}_{\nu+1} \xrightarrow{p^\nu} \mathcal{G}_{\nu+1}$$

is an exact sequence of $\mathfrak{o}$-group schemes. $\mathfrak{G}$ obviously represents a sheaf on $\mathfrak{o}_{fpqf}$. We call

$$H_i(\mathfrak{G}) := H^i(\mathfrak{o}_\infty, \mathfrak{G})^*$$

the (p-adic) arithmetic homology groups of $\mathfrak{G}$. There is a natural action of G on $H_i(\mathfrak{G})$, and we have the canonical decomposition (as Γ-modules)

$$H_i(\mathfrak{G}) = \bigoplus_{j \bmod d} e_j H_i(\mathfrak{G}),$$

where $e_j H_i(\mathfrak{G})$ is the maximal $\mathbb{Z}_p$-submodule on which $\delta \in \Delta$ acts as multiplication by $\kappa(\delta)^j$. Of course, the $e_j H_i(\mathfrak{G})$ are compact $\mathbb{Z}_p[\![\Gamma]\!]$-modules. It is unknown in general whether they are finitely generated over $\mathbb{Z}_p[\![\Gamma]\!]$. To decide whether they are $\mathbb{Z}_p[\![\Gamma]\!]$-torsion modules seems to be a deep problem.

Lemma 4. *If $e_j H_i(\mathfrak{G})$ is a finitely generated $\mathbb{Z}_p[\![\Gamma]\!]$-torsion module and $n \in \mathbb{Z}$ an integer with $n \equiv j \bmod d$, then the following three assertions are equivalent:*

i) *$H^0(G, H^i(\mathfrak{o}_\infty, \mathfrak{G})(n))$ is finite;*

ii) *$H^1(G, H^i(\mathfrak{o}_\infty, \mathfrak{G})(n))$ is finite;*

iii) $F_{e_j H_i(\mathfrak{G})}((1+p^e)^n - 1) \neq 0$.

If, in addition, these assertions hold, then

$$|F_{e_j H_i(\mathfrak{G})}((1+p^e)^n - 1)|_p = \frac{\# H^1(G, H^i(\mathfrak{o}_\infty, \mathfrak{G})(n))}{\# H^0(G, H^i(\mathfrak{o}_\infty, \mathfrak{G})(n))}.$$

Proof. Because of

$$\begin{aligned} H^r(\Gamma, (e_j H_i(\mathfrak{G}))(-n)) &= H^r(G, H^i(\mathfrak{o}_\infty, \mathfrak{G})(n)^*) \\ &= H^{1-r}(G, H^i(\mathfrak{o}_\infty, \mathfrak{G})(n))^* \qquad (r = 0, 1) \end{aligned}$$

this is an easy consequence of Lemma 1.

Since we have a canonical topological generator γ of Γ, we can and shall identify the G-coinvariants of $H^i(\mathfrak{o}_\infty, \mathfrak{G})$ with $H^1(G, H^i(\mathfrak{o}_\infty, \mathfrak{G}))$. Again the identity map on $H^i(\mathfrak{o}_\infty, \mathfrak{G})$ induces a homomorphism

$$H^0(G, H^i(\mathfrak{o}_\infty, \mathfrak{G})) \to H^1(G, H^i(\mathfrak{o}_\infty, \mathfrak{G}))$$

which is nothing else than the Pontrjagin dual of $f_{e_0 H_i(\mathfrak{G})}$.

II. Algebraic Tori

§ 3. *The Basic Iwasawa Modules*

Let T be an algebraic torus over k of dimension $\dim T > 0$, and let $\mathcal{T}$ be its Néron model over $\mathfrak{o}$. In particular, $\mathcal{T}$ is a separated smooth $\mathfrak{o}$-group scheme. Furthermore we recall that the formation of the Néron model commutes with etale base change and with Henselization or completion in a closed point (see [27]). Let $\mathcal{T}^0$ denote the connected component of $\mathcal{T}$; it is an open subgroup scheme of $\mathcal{T}$ with connected fibres which is of finite type over $\mathfrak{o}$. The formation of the connected component commutes with arbitrary base change (see SGA3VI$_\text{B}$ §3).

Example. Let $\mathcal{G}_m$ denote the Néron model over $\mathfrak{o}$ of the multiplicative group $\mathbb{G}_{m/k}$. We have $\mathcal{G}_m^0 = \mathbb{G}_{m/\mathfrak{o}}$, and the closed fibres of $\mathcal{G}_m$ are extensions of the constant group scheme $\mathbb{Z}$ by the multiplicative group.

Our aim is to study the groups $H^i(\mathfrak{o}_\infty, \mathcal{T}(p))$; but we shall do this only under a certain restriction on p. First, let $\mathfrak{o}'$ resp. $\mathfrak{o}'_n$ resp. $\mathfrak{o}'_\infty$ denote the ring of p-integers in k resp. k_n resp. k_∞.

Remark 1. i) Above each finite prime of k there lie only finitely many primes of k_∞ ([3], Lemma 1).

ii) Among the finite primes of k exactly the primes above p are ramified in k_∞/k ([13], Lemma 4).

iii) $\mathfrak{o}'_\infty$ (but not $\mathfrak{o}_\infty$) is noetherian.

iv) $\mathcal{T} \underset{\mathfrak{o}}{\times} \mathfrak{o}'_n$ is the Néron model of $T \underset{k}{\times} k_n$ over $\mathfrak{o}'_n$.

v) $\mathcal{T}_{p^j/\mathfrak{o}'}$ is etale and quasi-finite over $\mathfrak{o}'$ (use SGA3VI$_\text{B}$1.3, VII$_\text{A}$8.4 and EGAIV 17.8.2).

The finite Galois extension k_T/k is defined to be the minimal splitting field of $T_{/k}$ (in $\bar{k}$). In this Part II we shall always assume the following condition on the odd prime number p to be fulfilled.

Hypothesis. All primes of k above p are unramified in k_T.

Of course, this excludes only finitely many prime numbers p.

Remark 2. With $\mathcal{T}^{(n)}$ as Néron model of $T \underset{k}{\times} k_n$ over $\mathfrak{o}_n$ we have:

i) The canonical map $\mathcal{T} \underset{\mathfrak{o}}{\times} \mathfrak{o}_n \to \mathcal{T}^{(n)}$ is an open immersion, which induces in particular an isomorphism $\mathcal{T}^0 \underset{\mathfrak{o}}{\times} \mathfrak{o}_n \cong \mathcal{T}^{(n)0}$. It also induces isomorphisms $(\mathcal{T}_m) \underset{\mathfrak{o}}{\times} \mathfrak{o}_n \cong (\mathcal{T}^{(n)})_m$, for each $m \in \mathbb{N}$, and $\mathcal{T}(p) \underset{\mathfrak{o}}{\times} \mathfrak{o}_n \cong \mathcal{T}^{(n)}(p)$.

ii) $\mathcal{T}_{p^j}$ is flat and quasi-finite over $\mathfrak{o}$; in particular, $\mathcal{T}(p)$ is an ind-p-group over $\mathfrak{o}$.

Proof. It is enough to prove the assertions over $\mathfrak{o}'$ and over $U := \mathrm{Spec}(\mathfrak{o}) \setminus \{\mathfrak{p}$ ramified in $k_T\}$ separately. Over $\mathfrak{o}'$, we simply use the first remark. Over U, by an etale base change it suffices to consider the multiplicative group for which the assertions obviously are fulfilled (compare SGA7IX3.1e).

The principal reason for our condition on p is that we have the following fundamental lemma. Namely, since $\mathrm{Spec}(\mathfrak{o}'_\infty) \to \mathrm{Spec}(\mathfrak{o}')$ is pro-etale with Galois group G, it enables us to use the Hochschild-Serre spectral sequence for the study of the groups $H^i(\mathfrak{o}_\infty, \mathcal{T}(p))$.

Lemma 3. $H^i_{et}(\mathfrak{o}'_\infty, \mathcal{T}(p)) = H^i(\mathfrak{o}_\infty, \mathcal{T}(p))$ *for* $i \geqq 0$.

Before we give the proof, we first establish two lemmas about the local cohomology of finite flat group schemes.

Lemma 4. *If $\mathcal{G}$ is a finite flat $\mathfrak{o}_\mathfrak{p}$-group scheme, then*

$$H^{i}_{\cdot}(\mathfrak{o}_\mathfrak{p}, \mathcal{G}) = H^{i}_{\cdot}(\hat{\mathfrak{o}}_\mathfrak{p}, \mathcal{G})$$

for $i \geqq 0$; both groups are zero for $i \neq 2, 3$.

Proof. First we recall that $H^{i}_{\cdot}(\ )$ denotes the relative *fpqf*-cohomology with respect to the closed point. The case $i=0$ is trivial because $\mathcal{G}$ is separated over $\mathfrak{o}_\mathfrak{p}$. For $i=1$ see [17], Lemma 5.1. There it is also shown that $H^{i}_{\cdot}(\mathfrak{o}_\mathfrak{p}, \mathcal{G}_0) = H^{i}_{\cdot}(\hat{\mathfrak{o}}_\mathfrak{p}, \mathcal{G}_0)$ holds true for $i>1$ and any smooth $\mathfrak{o}_\mathfrak{p}$-group scheme of finite type. According to [20], Proposition 5.1 we always have an exact sequence of $\mathfrak{o}_\mathfrak{p}$-group schemes

$$0 \to \mathcal{G} \to \mathcal{G}_0 \to \mathcal{G}_1 \to 0,$$

where $\mathcal{G}_0$ and $\mathcal{G}_1$ are smooth of finite type over $\mathfrak{o}_\mathfrak{p}$. Applying the five lemma to the corresponding long exact cohomology sequences gives $H^{i}_{\cdot}(\mathfrak{o}_\mathfrak{p}, \mathcal{G}) = H^{i}_{\cdot}(\hat{\mathfrak{o}}_\mathfrak{p}, \mathcal{G})$ for $i \geqq 2$. Finally, the vanishing of $H^{i}_{\cdot}(\mathfrak{o}_\mathfrak{p}, \mathcal{G})$ for $i \neq 2, 3$ is proved in [16] (1.10).

Lemma 5. *Let $\mathfrak{p}$ be a prime of k above p, let $\mathcal{G}$ be a finite flat $\mathfrak{o}_\mathfrak{p}$-group scheme of rank a power of p, and let $L/k_\mathfrak{p}$ be an infinitely ramified abelian extension with ring of integers R. We then have:*

i) $H^{i}_{\cdot}(R, \mathcal{G}) = 0$ *for* $i \neq 2$;

ii) $H^{2}_{\cdot}(R, \mathcal{G}) = 0$, *if the Cartier dual $\mathcal{G}^D$ of $\mathcal{G}$ is etale over $\mathfrak{o}_\mathfrak{p}$.*

Proof. Since cohomology commutes with filtered projective limits of affine schemes, Lemma 4 allows us to replace $\mathfrak{o}_\mathfrak{p}$, resp. $k_\mathfrak{p}$, by $\hat{\mathfrak{o}}_\mathfrak{p}$, resp. $\hat{k}_\mathfrak{p}$, in the statement, and, of course, it implies the first assertion for $i \neq 2, 3$. Furthermore, we can assume that $L/\hat{k}_\mathfrak{p}$ is a totally ramified infinite abelian p-extension. Namely, if the statement is proved in this case, it is obviously also true in case that L is a totally ramified p-extension of a finite extension of $\hat{k}_\mathfrak{p}$. The general case now follows again from a limit argument because L always is the union of fields of the latter type; this is seen with the help of the local class field theory.

According to [16] we have

$$H^{3}_{\cdot}(R, \mathcal{G}) = H^2_{et}(L, \mathcal{G}_{/L})$$

and

$$H^{2}_{\cdot}(R, \mathcal{G}) = \varinjlim H^{2}_{\cdot}(R_\alpha, \mathcal{G}) = (\varprojlim H^1(R_\alpha, \mathcal{G}^D))^*,$$

where R_α runs through the rings of integers in the subfields of L which are finite over $\hat{k}_\mathfrak{p}$. But $H^2_{et}(L, \mathcal{G}_{/L})$ vanishes, because L has a cohomological p-dimension $\leqq 1$ (see [31], proof of II-11, Proposition 9).

Now we assume that $\mathcal{G}^D$ is etale over $\hat{\mathfrak{o}}_\mathfrak{p}$. Let κ be the residue class field of $\hat{\mathfrak{o}}_\mathfrak{p}$. Since $L/\hat{k}_\mathfrak{p}$ is totally ramified, κ is the residue class field of each R_α. Therefore we have

$$H^1(R_\alpha, \mathcal{G}^D) = H^1_{et}(\kappa, \mathcal{G}^D_{/\kappa})$$

for all α (use [22] III 3.9 and VI 2.7), and one can easily check that the transition maps $H^1(R_\alpha, \mathcal{G}^D) \to H^1(R_\beta, \mathcal{G}^D)$, for $R_\beta \subseteq R_\alpha$, induce the multiplication by $[R_\alpha : R_\beta]$ on $H^1_{et}(\kappa, \mathcal{G}^D_{/\kappa})$. But $H^1_{et}(\kappa, \mathcal{G}^D_{/\kappa})$ is a finite p-group. Thus we get $H^{2}_{\cdot}(R, \mathcal{G}) = 0$.

Remark 6. If the finite prime $\mathfrak{p}$ of k is unramified in k_T, then, for all $n \in \mathbb{N}$, $\mathcal{T}_n \underset{\mathfrak{o}}{\times} \mathfrak{o}_\mathfrak{p}$ is finite flat over $\mathfrak{o}_\mathfrak{p}$ with etale Cartier dual.

Proof. By an etale base change and descent-theory (EGAIV2.7.1) it suffices to consider the multiplicative group. But μ_n is finite flat with etale Cartier dual over any base scheme.

We come back to the proof of Lemma 3: Let us consider the relative cohomology sequence

$$\to H^i(\mathfrak{o}_\infty, \mathcal{T}(p)) \to H^i(\mathfrak{o}'_\infty, \mathcal{T}(p)) \to \bigoplus_{\mathfrak{P}/p} H^{i+1}_{\cdot}(\mathfrak{o}_{\infty,\mathfrak{P}}, \mathcal{T}(p)) \to,$$

where $\mathfrak{o}_{\infty,\mathfrak{P}}$ denotes the Henselization of $\mathfrak{o}_\infty$ in $\mathfrak{P}/p$. Because of Lemma 5 and the above remark we have $H^{i}_{\cdot}(\mathfrak{o}_{\infty,\mathfrak{P}}, \mathcal{T}(p)) = 0$ for all $\mathfrak{P}/p$ and $i \geqq 0$. This implies

$$H^i(\mathfrak{o}_\infty, \mathcal{T}(p)) = H^i(\mathfrak{o}'_\infty, \mathcal{T}(p)) = H^i_{et}(\mathfrak{o}'_\infty, \mathcal{T}(p)),$$

since $\mathcal{T}(p)$ is ind-etale over $\mathfrak{o}'_\infty$.

Remarks. 1) Because of Lemma 3 it seems one should a priori consider only etale cohomology groups over $\mathfrak{o}'_\infty$ of ind-p-groups (over $\mathfrak{o}$). But in the situation which we shall study in Part III of this paper a similar lemma does not exist; and it will be clear that there the flat cohomology groups are the interesting ones.

2) Let $\mathcal{M}$ be the Galois group of the maximal abelian p-extension of k_∞, unramified outside p. Equivalently, $\mathcal{M}$ is the maximal abelian pro-p-factor group of the algebraic fundamental group of $\mathrm{Spec}(\mathfrak{o}'_\infty)$. Thus we have

$$H^1_{et}(\mathfrak{o}'_\infty, \mathbb{Q}_p/\mathbb{Z}_p)^* = \mathcal{M},$$

or because of Lemma 3

$$H_1(\mu(p)) = \mathscr{M}(-1).$$

The G-module $\mathscr{M}$ was studied in detail by Iwasawa in his work on cyclotomic fields ([13]). Therefore our arithmetic homology groups appear as a natural generalization of this "classical" Iwasawa module.

Lemma 7. i) $\mathrm{cd}_p \mathfrak{o}'_\infty \leqq 2$;
ii) *$H^2_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p))$ is divisible.*

Proof. i) All the residue class fields of $\mathfrak{o}'_\infty$ are p-closed. An easy modification of [35] II. 10.2.3 then shows

$$\mathrm{cd}_p \mathfrak{o}'_\infty \leqq \mathrm{cd}_p k_\infty + 1.$$

But according to [31] II-11, Proposition 9 we have $\mathrm{cd}_p k_\infty \leqq 1$.

ii) Define the sheaf $\mathscr{F}$ on $\mathfrak{o}'_{et}$ by the exact sequence

$$0 \to \mathscr{T}_{p/\mathfrak{o}'_{et}} \to \mathscr{T}(p)_{/\mathfrak{o}'_{et}} \xrightarrow{p} \mathscr{T}(p)_{/\mathfrak{o}'_{et}} \to \mathscr{F} \to 0.$$

$\mathscr{F}$ is a skyscraper sheaf because $T(p)_{/k_{et}} \xrightarrow{p} T(p)_{/k_{et}}$ is an epimorphism. Again a modification of [35] II. 10.1.2 implies

$$H^i_{et}(\mathfrak{o}'_\infty, \mathscr{F}) = 0$$

for $i > 0$. Using this and $H^3_{et}(\mathfrak{o}'_\infty, \mathscr{T}_p) = 0$ we derive the surjectivity of

$$H^2_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p)) \xrightarrow{p} H^2_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p))$$

from the above exact sequence.

Proposition 8. i) *$H_0(\mathscr{T}(p))$ is a finitely generated $\mathbb{Z}_p$-module;*
ii) *$H_1(\mathscr{T}(p))$ is a finitely generated $\mathbb{Z}_p[\![\Gamma]\!]$-module;*
iii) *$H_i(\mathscr{T}(p)) = 0$ for $i \geqq 2$.*

Proof. i) is clear.

ii) From the Hochschild-Serre spectral sequence we get the surjective edge homomorphism

$$H^1_{et}(\mathfrak{o}'_1, \mathscr{T}(p)) \to H^0(\Gamma, H^1_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p))),$$

because the cohomological dimension of Γ ist 1. According to Lemma 9i (applied to $\mathfrak{o}'_1$) the left term is of cofinite type. Therefore, $H_1(\mathscr{T}(p))_\Gamma$ is a finitely generated $\mathbb{Z}_p$-module, which implies the required result (see [13]).

iii) For $i > 2$ the assertion follows from Lemma 7i). The case $i = 2$ we shall treat in several steps.

Step 1: $T = \mathbb{G}_m$: From the exact sequences

$$0 \to \mu_{p^n} \to \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \to 0$$

on $(\mathfrak{o}'_\infty)_{et}$ we get the exact cohomology sequence

$$0 \to H^1_{et}(\mathfrak{o}'_\infty, \mathbb{G}_m) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^2_{et}(\mathfrak{o}'_\infty, \mu(p)) \to H^2_{et}(\mathfrak{o}'_\infty, \mathbb{G}_m)(p) \to 0.$$

But the Picard group $\mathrm{Pic}(\mathfrak{o}'_\infty) = H^1_{et}(\mathfrak{o}'_\infty, \mathbb{G}_m)$ is a torsion group, which means $H^1_{et}(\mathfrak{o}'_\infty, \mathbb{G}_m) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 0$. On the other hand we have $H^2_{et}(\mathfrak{o}'_\infty, \mathbb{G}_m)(p) \leqq H^2_{et}(k_\infty, \mathbb{G}_m)(p)$ (see [22] III 2.22) and $H^2_{et}(k_\infty, \mathbb{G}_m)(p) = 0$ because of $\mathrm{cd}_p k_\infty \leqq 1$. Therefore $H^2_{et}(\mathfrak{o}'_\infty, \mu(p))$ vanishes.

Step 2: $T = R_{K/k}\mathbb{G}_m =$ Weil restriction of the multiplicative group over a finite extension K/k: Let B, resp. B_∞, denote the ring of p-integers in K, resp. $K(\mu(p))$. The canonical morphism π: $\mathrm{Spec}(B) \to \mathrm{Spec}(\mathfrak{o}')$ is finite and faithfully flat. Therefore, if $\mathscr{G}_m$ is the Néron model of $\mathbb{G}_{m/K}$ over B, we have

$$\mathscr{T}_{/\mathfrak{o}'_{fppf}} = \pi_*(\mathscr{G}_{m/B_{fppf}})$$

(see [27] Prop. 2.5). In particular, we get

$$\mathscr{T}(p)_{/\mathfrak{o}'_{et}} = \pi_*(\mu(p)_{/B_{et}}).$$

But, for the etale topology, π_* is even exact ([22] II 3.6). This implies

$$H^2_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p)) = H^2_{et}(\mathfrak{o}'_\infty, \pi_*\mu(p)) = H^2_{et}(B_\infty, \mu(p))^a = 0$$

(with $a := [K \cap k_\infty : k]$).

Step 3: T is k-isogenous to a k-torus T_0 with $H^2_{et}(\mathfrak{o}'_\infty, \mathscr{T}_0(p)) = 0$, where $\mathscr{T}_0$ denotes the Néron model of T_0 over $\mathfrak{o}$: There exist k-homomorphisms

$$T \xrightarrow{\alpha} T_0 \xrightarrow{\beta} T$$

with $\beta \circ \alpha = n \cdot id$ for a certain $n \in \mathbb{N}$. Because of the universal property of Néron models this induces homomorphisms

$$H^2_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p)) \xrightarrow{H(\alpha)} H^2_{et}(\mathfrak{o}'_\infty, \mathscr{T}_0(p)) = 0 \xrightarrow{H(\beta)} H^2_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p))$$

with $H(\beta) \circ H(\alpha) = n \cdot id$, i.e. n annihilates $H^2_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p))$. Together with Lemma 7(ii) this means $H^2_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p)) = 0$.

The general case now results from the fact that we always have a k-isogeny

$$T^n \times \prod_{\nu=1}^{r} R_{k_\nu/k}\mathbb{G}_m \to \prod_{\nu=r+1}^{s} R_{k_\nu/k}\mathbb{G}_m$$

with appropriate $n, r, s \in \mathbb{N}$ and intermediate fields k_ν of k_T/k ([25] Theorem 1.5.1). q.e.d.

We have now to discuss which of the $e_j H_1(\mathscr{T}(p))$ are $\mathbb{Z}_p[\![\Gamma]\!]$-torsion modules. This requires some preparation.

Lemma 9. i) *$H^1_{et}(\mathfrak{o}', \mathscr{T}(p))$ is of cofinite type;*
ii) *the natural maps $H^i_{et}(\mathfrak{o}', \mathscr{T}^0(p)) \to H^i_{et}(\mathfrak{o}', \mathscr{T}(p))$ are quasi-isomorphisms;*
iii) *we have an exact sequence*

$$0 \to \mathscr{T}^0(\mathfrak{o}') \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^1_{et}(\mathfrak{o}', \mathscr{T}^0(p)) \to H^1_{et}(\mathfrak{o}', \mathscr{T}^0)(p) \to 0;$$

iv) *$H^1_{et}(\mathfrak{o}', \mathscr{T}^0)(p)$ is finite.*

Proof. i) We omit this proof because it is completely analogous to the proof of Lemma 6.8 in [17] (the finite set S of primes figuring there is in our case the set of finite primes of k which are ramified in k_T).

ii) Define the sheaf $\mathscr{F}$ on $\mathfrak{o}'_{et}$ by the exact sequence

$$0 \to \mathscr{T}^0(p)_{/\mathfrak{o}'_{et}} \to \mathscr{T}(p)_{/\mathfrak{o}'_{et}} \to \mathscr{F} \to 0.$$

We have to show that the groups $H^i_{et}(\mathfrak{o}', \mathscr{F})$ are finite. Obviously, $\mathscr{F}$ is a skyscraper sheaf, and, even more,

$$\mathscr{T}^0(p)_{/U} = \mathscr{T}(p)_{/U} \quad (\text{i.e. } \mathscr{F}_{/U_{et}} = 0)$$

holds true with $U := \mathrm{Spec}(\mathfrak{o}') \setminus \{\mathfrak{p} \text{ ramified in } k_T\}$. It is enough to verify this over the normalization V of U in k_T, because $V \to U$ is an etale covering; but, over V, both sides become isomorphic to $\mu(p)_{/V}$. Therefore, the cohomology groups of $\mathscr{F}$ are finite if the stalks of $\mathscr{F}$ are finite (see [35] II 10.1). This follows now by a further application of the method in the proof of Proposition 8 iii): Step 1: For $T = \mathbb{G}_m$ we even have $\mathscr{F} = 0$. Step 2: The formation of the connected component commutes with π_*. Step 3: The stalks of $\mathscr{F}$ are at least of cofinite type.

iii) The morphism $\mathscr{T}^0_{/\mathfrak{o}'} \xrightarrow{p} \mathscr{T}^0_{/\mathfrak{o}'}$ is etale and surjective (SGA 3 VI_B 3.11). Therefore we have the exact sequences

$$0 \to (\mathscr{T}^0_{p^j})_{/\mathfrak{o}'_{et}} \to \mathscr{T}^0_{/\mathfrak{o}'_{et}} \xrightarrow{p^j} \mathscr{T}^0_{/\mathfrak{o}'_{et}} \to 0,$$

and we get the required exact sequence by going to the direct limit in the corresponding cohomology sequences.

iv) This again follows by an application of the method in the proof of Proposition 8 iii): For $T = \mathbb{G}_m$ the assertion follows from the finiteness of the ideal class group of k. In general, i)–iii) imply that $H^1_{et}(\mathfrak{o}', \mathscr{T}^0)(p)$ is at least of cofinite type.

The character group $\tilde{T}$ of T is by definition the etale k-group scheme which represents the sheaf $\mathcal{H}om_{k_{fppf}}(T, \mathbb{G}_m)$ on k_{fppf}. We often identify $\tilde{T}$ with the discrete G_k-module $\tilde{T}(\bar{k})$; as $\mathbb{Z}$-module $\tilde{T}(\bar{k})$ is $\mathbb{Z}$-free of rank $\dim T$. For any finite prime $\mathfrak{p}$ of k let $T(\hat{k}_\mathfrak{p})^c$ denote the maximal compact subgroup of $T(\hat{k}_\mathfrak{p})$ (with respect to the $\mathfrak{p}$-adic topology).

Lemma 10. i) *$T(\hat{k}_\mathfrak{p})/T(\hat{k}_\mathfrak{p})^c$ is $\mathbb{Z}$-free of finite rank;*

ii) *$\mathscr{T}^0(\hat{\mathfrak{o}}_\mathfrak{p}) \leq T(\hat{k}_\mathfrak{p})^c$ of finite index;*

iii) *if $\mathfrak{p}$ is unramified in k_T, then $\mathscr{T}^0(\hat{\mathfrak{o}}_\mathfrak{p}) = T(\hat{k}_\mathfrak{p})^c$.*

Proof. We have

$$T(\hat{k}_\mathfrak{p})^c = \{x \in T(\hat{k}_\mathfrak{p}) : \chi(x) \in \hat{\mathfrak{o}}_\mathfrak{p}^\times \text{ for all } \chi \in \tilde{T}(\hat{k}_\mathfrak{p})\},$$

which immediately implies (i) and $\mathscr{T}^0(\hat{\mathfrak{o}}_\mathfrak{p}) \leq T(\hat{k}_\mathfrak{p})^c$ (we think of $\mathscr{T}^0(\hat{\mathfrak{o}}_\mathfrak{p})$ as being canonically embedded into $T(\hat{k}_\mathfrak{p})$). But $\mathscr{T}^0$ is Zariski open in $\mathscr{T}$; therefore, $\mathscr{T}^0(\hat{\mathfrak{o}}_\mathfrak{p})$ is open in $T(\hat{k}_\mathfrak{p})$, and the index $[T(\hat{k}_\mathfrak{p})^c : \mathscr{T}^0(\hat{\mathfrak{o}}_\mathfrak{p})]$ is finite. Let $\mathfrak{p}$ be unramified in k_T. If $L/\hat{k}_\mathfrak{p}$ is a finite unramified splitting field of $T_{/\hat{k}_\mathfrak{p}}$ with ring of integers B, we have

$$\mathscr{T}^0(B) = B^\times = \mathbb{G}_m(L)^c = T(L)^c.$$

The assertion iii) now follows by taking Galois invariants.

Definition. The torus $T_{/k}$ is called even, resp. odd, if k is totally real and all involutions in G_k act as multiplication by $+1$, resp. -1, on $\tilde{T}$.

Remark 11. i) $T_{/k}$ is even $\Leftrightarrow k_T$ is totally real $\Leftrightarrow k$ is totally real and $\tilde{T}(\hat{k}_\mathfrak{p}) = \tilde{T}$ for all $\mathfrak{p}/\infty$;

ii) $T_{/k}$ is odd $\Leftrightarrow k$ is totally real and $\tilde{T}(\hat{k}_\mathfrak{p}) = 0$ for all $\mathfrak{p}/\infty$.

Proposition 12. *$\mathscr{T}^0(\mathfrak{o}')$ is finitely generated of rank*

$$\sum_{\mathfrak{p}/p \text{ or } \infty} \mathrm{rank}_\mathbb{Z} \tilde{T}(\hat{k}_\mathfrak{p}) - \mathrm{rank}_\mathbb{Z} \tilde{T}(k).$$

Proof. In [33] the same assertion is proved for the p-unit group in $T(k)$ instead of $\mathscr{T}^0(\mathfrak{o}')$. The p-unit group is the subgroup of all elements in $T(k)$ which lie in $T(\hat{k}_\mathfrak{p})^c$ for all primes $\mathfrak{p}$ not dividing p. But Lemma 10 shows that $\mathscr{T}^0(\mathfrak{o}')$ is a subgroup of finite index in the p-unit group.

Theorem 13. i) *The $\mathbb{Z}_p[\![\Gamma]\!]$-rank of $H_1(\mathscr{T}(p))$ is $\frac{1}{2} \cdot \dim T \cdot [k_1 : \mathbb{Q}]$;*

ii) *if $T_{/k}$ is even, resp. odd, and $0 \leq j < d$ is odd, resp. even, then $e_j H_1(\mathscr{T}(p))$ is a $\mathbb{Z}_p[\![\Gamma]\!]$-torsion module.*

Proof. i) Put $\Gamma_n := \mathrm{Gal}(k_\infty/k_n)$. From the Hochschild-Serre spectral sequence and Lemma 3 we get the exact sequences

$$0 \to H^1(\Gamma_n, H^0(\mathfrak{o}_\infty, \mathscr{T}(p))) \to H^1_{et}(\mathfrak{o}', \mathscr{T}(p)) \to H^0(\Gamma_n, H^1(\mathfrak{o}_\infty, \mathscr{T}(p))) \to 0$$

(Γ has cohomological p-dimension 1). The groups $H^1(\Gamma_n, H^0(\mathfrak{o}_\infty, \mathscr{T}(p)))$ are finite, because the $T(k_n)(p)$ are. Together with Lemma 9 this implies

$$\begin{aligned}\mathrm{rank}_{\mathbb{Z}_p} H_1(\mathscr{T}(p))_{\Gamma_n} &= \mathrm{corank}\, H^1_{et}(\mathfrak{o}'_n, \mathscr{T}(p)) \\ &= \mathrm{corank}\, \mathscr{T}^0(\mathfrak{o}'_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p = \mathrm{rank}_\mathbb{Z} \mathscr{T}^0(\mathfrak{o}'_n).\end{aligned}$$

Since the k_n are totally imaginary, we conclude from Proposition 12

$$\mathrm{rank}_{\mathbb{Z}_p} H_1(\mathscr{T}(p))_{\Gamma_n} = \tfrac{1}{2} \cdot \dim T \cdot [k_1 : \mathbb{Q}] \cdot p^{n-1} + c_n$$

with $0 \leq c_n \leq \dim T \cdot s$, $s := \#\{\text{primes of } k_\infty \text{ above } p\}$. According to the general structure theory of finitely generated $\mathbb{Z}_p[\![\Gamma]\!]$-modules this means that the $\mathbb{Z}_p[\![\Gamma]\!]$-rank of $H_1(\mathscr{T}(p))$ is the asserted one.

ii) Let k_n^+ be the maximal totally real subfield of k_n, and let R_n be the ring of p-integers in k_n^+. An easy computation using Proposition 12 shows

$$\mathrm{rank}_\mathbb{Z} \mathscr{T}^0(\mathfrak{o}'_n)/\mathscr{T}^0(R_n) \leq \dim T \cdot s,$$

if $T_{/k}$ is even, resp.

$$\mathrm{rank}_\mathbb{Z} \mathscr{T}^0(R_n) \leq \dim T \cdot s,$$

if $T_{/k}$ is odd. For any Δ-module M define M^+, resp. M^-, to be the maximal submodule on which the complex conjugation in Δ acts as multiplication by $+1$, resp. -1. Obviously,

$$2M \leqq M^+ + M^- \quad \text{and} \quad M^+ \cap M^- \leqq M_2$$

holds true. Thus we get

$$\operatorname{corank}(\mathscr{T}^0(\mathfrak{o}'_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^+ = \operatorname{rank}_{\mathbb{Z}} \mathscr{T}^0(\mathfrak{o}'_n)^+ = \operatorname{rank}_{\mathbb{Z}} \mathscr{T}^0(R_n)$$

and

$$\operatorname{corank}(\mathscr{T}^0(\mathfrak{o}'_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^- = \operatorname{rank}_{\mathbb{Z}} \mathscr{T}^0(\mathfrak{o}'_n)^- = \operatorname{rank}_{\mathbb{Z}} \mathscr{T}^0(\mathfrak{o}'_n)/\mathscr{T}^0(R_n).$$

The rest of the proof can be dealt with exactly in the same way as in the first part.

For $T = \mathbb{G}_m$ this theorem (in the Galois theoretic version) was already obtained by Iwasawa ([13]).

§ 4. *Values of L-Functions*

We fix an even, resp. odd, torus $T_{/k}$ and an even, resp. odd, integer $0 \leqq j < d$. According to the results of the preceding paragraph the following definition makes sense. We call

$$L_p^{(j)}(T, s) := \prod_{i \geqq 0} F_{e_{j-1} H_i(\mathscr{T}(p))}((1+p^e)^{-s} - 1)^{(-1)^{i+1}}$$

$$= \frac{F_{e_{j-1} H_1(\mathscr{T}(p))}((1+p^e)^{-s} - 1)}{F_{e_{j-1} H_0(\mathscr{T}(p))}((1+p^e)^{-s} - 1)} \qquad (s \in \mathbb{Z}_p)$$

the j-th *Iwasawa L-function* of $T_{/k}$ (with respect to p). The aim of this and the next paragraph is to study the p-parts of the values of these L-functions at integer points. As (2.4) shows we have to consider the twisted groups $H^i(\mathfrak{o}_\infty, \mathscr{T}(p))(n)$ in order to do this. The idea now is to interprete these groups as cohomology groups of a twisted sheaf. Because of (3.3) it is enough to achieve this for the etale cohomology over $\mathfrak{o}'_\infty$. For any $n \in \mathbb{Z}$ define

$$\mathscr{T}_{p^j}(n) := \begin{cases} \mathscr{T}_{p^j/\mathfrak{o}'_{et}} & \text{if } n = 0, \\ \mathscr{T}_{p^j/\mathfrak{o}'_{et}} \otimes (\mu_{p^j/\mathfrak{o}'_{et}}^{\otimes n}) & \text{if } n > 0, \\ Hom_{\mathfrak{o}'_{et}}(\mu_{p^j}^{\otimes -n}, \mathscr{T}_{p^j}) & \text{if } n < 0, \end{cases}$$

$$\mathscr{T}(p)(n) := \varinjlim \mathscr{T}_{p^j}(n)$$

as sheaves on $\mathfrak{o}'_{et}$ (in fact, they are representable by etale quasi-finite separated $\mathfrak{o}'$-group schemes). We have canonical isomorphisms

$$H^i_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p)(n)) \cong H^i_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p))(n)$$

of G-modules (see [2], I § 2 or [22], p. 163).

Lemma 1. i) $H^i_{et}(\mathfrak{o}', \mathscr{T}(p)(n)) = 0$ for $i \geqq 3$;

ii) $H^0(G, H^0(\mathfrak{o}_\infty, \mathscr{T}(p))(n)) = H^0_{et}(\mathfrak{o}', \mathscr{T}(p)(n))$;

iii) $H^1(G, H^1(\mathfrak{o}_\infty, \mathscr{T}(p))(n)) = H^2_{et}(\mathfrak{o}', \mathscr{T}(p)(n))$;

iv) *we have an exact sequence*

$$0 \to H^1(G, H^0(\mathfrak{o}_\infty, \mathscr{T}(p))(n)) \to H^1_{et}(\mathfrak{o}', \mathscr{T}(p)(n))$$
$$\to H^0(G, H^1(\mathfrak{o}_\infty, \mathscr{T}(p))(n)) \to 0.$$

Proof. This follows easily from the Hochschild-Serre spectral sequence using (3.3) and (3.8 iii)).

Lemma 2. $H^0_{et}(\mathfrak{o}', \mathscr{T}(p)(n))$ *is finite for all* $n \neq -1$.

Proof. Let j: Spec$(k) \hookrightarrow$ Spec$(\mathfrak{o}')$ be the natural inclusion. Because of the universal property of the Néron model

$$j_*(T(p)_{/k_{et}}) = \mathscr{T}(p)_{/\mathfrak{o}'_{et}}$$

holds true. Since $\mu_{p^j/\mathfrak{o}'_{et}}$ is a locally free $\mathbb{Z}/p^j\mathbb{Z}$-sheaf of rank 1, we get

$$j_* T(p)(n) = \mathscr{T}(p)(n).$$

This implies $H^0_{et}(\mathfrak{o}', \mathscr{T}(p)(n)) = H^0_{et}(k, T(p)(n)) \leqq H^0_{et}(k_T, \mu(p)(n))^{\dim T}$, where the latter group obviously is finite for $n \neq -1$.

Proposition 3. $H^2_{et}(\mathfrak{o}', \mathscr{T}(p)(n))$ *is finite for* $n \geqq 1$.

Proof. Again we use the same method as in the proof of (3.8 iii)). Step 1: For $T = \mathbb{G}_m$ see [34] Theorem 5; in fact, the groups $H^2_{et}(\mathfrak{o}', \mu(p)(n))$ vanish for $n \geqq 1$. Step 2: Since $\mu_{p^j/\mathfrak{o}'_{et}}$ is a locally free $\mathbb{Z}/p^j\mathbb{Z}$-sheaf of rank 1, the twisting commutes with π_* (on the etale sites). Step 3: $H^2_{et}(\mathfrak{o}', \mathscr{T}(p)(n))$ is at least of cofinite type; this follows from Lemma 1 iii), (3.8 ii)), and the general fact that the Γ-invariants of a finitely generated $\mathbb{Z}_p[\![\Gamma]\!]$-module are of finite type over $\mathbb{Z}_p$.

We want to emphasize that we have not needed any hypothesis about $T_{/k}$ for these first three results.

Theorem 4. i) $L_p^{(j)}(T, -n) \neq \infty$ *for* $n \neq -1$;

ii) $L_p^{(j)}(T, -n) \neq 0$ *for* $n \in \mathbb{N}$;

iii) $|L_p^{(j)}(T, -n)|_p = \prod_{i \geqq 0} \# H^i_{et}(\mathfrak{o}', \mathscr{T}(p)(n))^{(-1)^i}$ *for* $n \in \mathbb{N}$ *with* $n \equiv j-1 \bmod d$.

Proof. i) Lemma 1 ii) and 2 imply the finiteness of $H_0(\mathscr{T}(p))(-n)_\Gamma$. The assertion now follows from (2.1). ii) This is similarly proved as i) using Proposition 3. iii) Combine (2.4), Lemma 1, Lemma 2, and Proposition 3.

Remarks. 1) A weaker version of the above theorem for $T = \mathbb{G}_m$ and k totally real is contained in [2] (6.1).

2) One possibility to state the Leopoldt conjecture is: $H^2_{et}(\mathfrak{o}', \mathbb{Q}_p/\mathbb{Z}_p) = 0$ ([29], p. 202). It is proved in case that k is abelian over $\mathbb{Q}$ or over an imaginary quadratic field. Therefore the proof of Proposition 3 gives the fol-

lowing result: $H^2_{et}(\mathfrak{o}', \mathcal{T}(p)(-1))$ is finite if k_T is abelian over $\mathbb{Q}$ or over an imaginary quadratic field, resp., is always finite if the Leopoldt conjecture holds true.

3) We conjecture that $H^2_{et}(\mathfrak{o}', \mathcal{T}(p)(n))$ is finite for all $n \neq 0$ (see [29] (5.7) for a very weak result in this direction). A consequence would be that the above theorem is correct for all $n \neq 0, 1$ (and $\equiv j-1 \bmod d$ in iii)).

4) Let $L(T, s)$ denote the Artin L-function associated with the representation of G_k on $\tilde{T}$ (for an arbitrary k-torus T), and let $n \in \mathbb{N}$ be a natural number. From the functional equation we derive

$$L(T, -n) \begin{cases} \neq 0 & \text{for } T_{/k} \text{ even and } n \text{ odd,} \\ & \text{or } T_{/k} \text{ odd and } n \text{ even,} \\ = 0 & \text{otherwise.} \end{cases}$$

According to Siegel the values $L(T, -n)$ are rational numbers, and Coates/Lichtenbaum ([6]) present a conjecture which expresses these numbers as etale Euler characteristics. We leave it to the reader as an exercise in applying the duality theorem of Artin/Verdier ([18]) to show that these Euler characteristics agree with the ones in Theorem 4iii).

Theorem 5. *Let ρ denote the multiplicity of the trivial representation in the representation of G_k on $\tilde{T} \otimes \mathbb{Q}$. Under the assumption $k_\infty \cap k_T = k$ we have:*

i) $F_{e_j H_0(\mathcal{T}(p))}(t) = \begin{cases} (t-(1+p^e)^{-1}+1)^\rho & \text{if } j \equiv -1 \bmod d, \\ 1 & \text{if } j \not\equiv -1 \bmod d; \end{cases}$

ii) *if $H^2_{et}(\mathfrak{o}'_1, \mathcal{T}(p)(-1))$ is finite and $j \neq 0$, then $L_p^{(j)}(T, 1) \neq 0, \infty$;*

iii) *if $T_{/k}$ is even and $H^2_{et}(\mathfrak{o}', \mathcal{T}(p)(-1))$ is finite (e.g. for k_T abelian over $\mathbb{Q}$), then*

$$\left| \left[\frac{L_p^{(0)}(T, s)}{((1+p^e)^{-s}-(1+p^e)^{-1})^\rho} \right]_{s=1} \right|_p = \prod_{i \geq 0} \# H^i_{et}(\mathfrak{o}', \mathcal{T}(p)(-1))_{\mathrm{Div}}^{(-1)^i}.$$

Proof. i) Because of $k_\infty \cap k_T = k$ we have

$$H^0(\mathfrak{o}_\infty, \mathcal{T}(p))(-1) = H^0_{et}(\mathfrak{o}', \mathcal{T}(p)(-1)). \qquad (*)$$

In particular, G acts trivially on the first term, and there is a quasi-isomorphism of G-modules $H^0(\mathfrak{o}_\infty, \mathcal{T}(p)) \to H^0(\mathfrak{o}_\infty, \mu(p))^\rho$.

ii) Using i) this follows exactly in the same way as Theorem 4ii).

iii) Combining $(*)$, Lemma 1, and (2.4) we get

$$\begin{aligned} &|F_{e_{-1} H_1(\mathcal{T}(p))}((1+p^e)^{-1}-1)|_p \\ &= \frac{\# H^1(G, H^1(\mathfrak{o}_\infty, \mathcal{T}(p))(-1))}{\# H^0(G, H^1(\mathfrak{o}_\infty, \mathcal{T}(p))(-1))} \\ &= \frac{\# H^2_{et}(\mathfrak{o}', \mathcal{T}(p)(-1))}{\# H^1_{et}(\mathfrak{o}', \mathcal{T}(p)(-1))_{\mathrm{Div}}} \cdot \# H^1(G, H^0(\mathfrak{o}_\infty, \mathcal{T}(p))(-1))_{\mathrm{Div}} \\ &= \prod_{i \geq 0} \# H^i_{et}(\mathfrak{o}', \mathcal{T}(p)(-1))_{\mathrm{Div}}^{(-1)^i}. \end{aligned}$$

Remarks. 1) The Galois theoretic version of the above theorem for $T = \mathbb{G}_m$ and k totally real was obtained by Coates in [4] (Theorem 1.13 and App. Lemma 8). More than that, he gives an interpretation of the corresponding Euler characteristic in "classical" terms (class number, Leopoldt regulator, ...). In this way he gets a complete analogue of the analytic class number formula for the L-function $L_p^{(0)}(\mathbb{G}_m, s)$.

2) Let ρ be as in Theorem 5. The action of G on $H_0(\mathcal{T}(p))(1) \otimes \mathbb{Q}_p$ factors through a finite quotient and is therefore semisimple. Consequently, $F_{e_{-1} H_0(\mathcal{T}(p))}(t)$ has a zero of exact multiplicity ρ at $t=0$. For $T_{/k}$ even this means that $L_p^{(0)}(T, s)$ has a pole of order $\leq \rho$ at $s=1$. Finally we recall that the Artin L-function has a pole of exact order ρ at $s=1$.

§ 5. *The Point $s=0$*

Let $T_{/k}$ be odd. Then the finitely generated groups $\mathcal{T}^0(\mathfrak{o}')$ and $\bigoplus_{\mathfrak{p}/p} \tilde{T}(\hat{k}_\mathfrak{p})$ have the same rank (see (3.12)). There exist two remarkable pairings between these groups. The first pairing:

$$\begin{aligned} (\ ,\ )_p \colon \mathcal{T}^0(\mathfrak{o}') \times \bigoplus_{\mathfrak{p}/p} \tilde{T}(\hat{k}_\mathfrak{p}) &\to \mathbb{Z} \\ (a, \oplus \chi_\mathfrak{p}) &\mapsto \sum_{\mathfrak{p}/p} v_\mathfrak{p} \circ \chi_\mathfrak{p}(a), \end{aligned}$$

where $v_\mathfrak{p} \colon \hat{k}_\mathfrak{p}^\times \to \mathbb{Z}$ denotes the normalized discrete valuation. The second and more important pairing:

$$\begin{aligned} \langle\ ,\ \rangle_p \colon \mathcal{T}^0(\mathfrak{o}') \times \bigoplus_{\mathfrak{p}/p} \tilde{T}(k_\mathfrak{p}) &\to \mathbb{Z}_p \\ (a, \oplus \chi_\mathfrak{p}) &\mapsto \sum_{\mathfrak{p}/p} \log_p \circ \mathrm{Norm}_{\hat{k}_\mathfrak{p}/\mathbb{Q}_p}(\chi_\mathfrak{p}(a)), \end{aligned}$$

where $\log_p \colon \mathbb{Q}_p^\times \to \mathbb{Z}_p$ is the extended ($\log_p p = 0$) p-adic logarithm.

For any prime $\mathfrak{p}$ of k above p define $K_\mathfrak{p}/\hat{k}_\mathfrak{p}$ to be a finite unramified splitting field of $T_{/\hat{k}_\mathfrak{p}}$ (our condition on p!). If $\sigma_\mathfrak{p} \in H_\mathfrak{p} := \mathrm{Gal}(K_\mathfrak{p}/\hat{k}_\mathfrak{p})$ denotes the Frobenius generator, we put

$$L_\mathfrak{p}(T, t) := \det(1 - \sigma_\mathfrak{p}^{-1} \cdot t; \tilde{T}(K_\mathfrak{p}) \otimes \mathbb{Q}) \in \mathbb{Z}[t].$$

Then $L_\mathfrak{p}(T, (N_\mathfrak{p})^{-s})^{-1}$, $N_\mathfrak{p} :=$ number of elements in the residue class field of $\mathfrak{p}$, is the Euler factor corresponding to $\mathfrak{p}$ in the Artin L-function $L(T, s)$. $L_\mathfrak{p}(T, t)$ has a zero of exact multiplicity $\mathrm{rank}_\mathbb{Z} \tilde{T}(\hat{k}_\mathfrak{p})$ at $t=1$, because $\sigma_\mathfrak{p}$ acts semisimply on $\tilde{T}(K_\mathfrak{p}) \otimes \mathbb{Q}$. Define

$$L_\mathfrak{p}(T) := [L_\mathfrak{p}(T, t) \cdot (1-t)^{-\mathrm{rank}\, \tilde{T}(\hat{k}_\mathfrak{p})}]_{t=1}$$

and

$$L_{p,\mathfrak{p}}(T, s) := (L_\mathfrak{p}(T) \cdot s^{\mathrm{rank}\, \tilde{T}(\hat{k}_\mathfrak{p})})^{-1} \qquad (s \in \mathbb{Z}_p).$$

Now we can state the main result of this paragraph.

Theorem 1. *For $T_{/k}$ odd we have:*

i) *$H^i(\mathfrak{o}, \mathcal{T}(p))$ is finite for all $i \geq 0$ and $=0$ for $i \geq 3$;*

ii) *$(\ ,\ )_p$ is nondegenerate;*

iii) $L_p^{(1)}(T,s)$ *has a zero of multiplicity* $\geqq \sum_{\mathfrak{p}|p} \operatorname{rank}_{\mathbb{Z}} \tilde{T}(\hat{k}_{\mathfrak{p}})$ *at* $s=0$; *equality holds if and only if* $\langle\,,\rangle_p$ *is nondegenerate;*

iv) *if* $\langle\,,\rangle_p$ *is nondegenerate, then*

$$|[L_p^{(1)}(T,s)\cdot\prod_{\mathfrak{p}|p} L_{p,\mathfrak{p}}(T,s)]_{s=0}|_p = \left|\frac{\det\langle\,,\rangle_p}{\det(\,,)_p}\right|_p \cdot \prod_{i\geqq 0} \# H^i(\mathfrak{o}, \mathscr{T}(p))^{(-1)^i}.$$

The proof will occupy the next pages and will be included in a series of lemmas. But first we want to make a remark on the value of $L(T,s)$ at $s=0$.

Remark. The Artin L-function $L(T,s)$ is holomorphic at $s=0$ with $L(T,0)\in\mathbb{Q}$. It is $L(T,0)\neq 0$ essentially in the both cases that $T_{/k}$ is odd or that $k=\mathbb{Q}$ and $T=\mathbb{G}_m$. In the first case a long and tedious computation starting from the results in [14] shows that

$$|L(T,0)|_p = \prod_{i\geqq 0} \# H^i(\mathfrak{o}, \mathscr{T}(p))^{(-1)^i}$$

(for all odd prime numbers p which are unramified in k_T/k). In the other case one has $L(\mathbb{G}_{m/\mathbb{Q}}, 0) = -\frac{1}{2}$.

If ρ denotes the multiplicity of the zero of $L_p^{(1)}(T,s)$ at $s=0$, then

$$\rho = \rho(e_0 H_1(\mathscr{T}(p))) - \rho(e_0 H_0(\mathscr{T}(p)))$$

and

$$[L_1^{(p)}(T,s)\cdot s^{-\rho}]_{s=0} = (-\log_p(1+p^e))^\rho \cdot \frac{c(e_0 H_1(\mathscr{T}(p)))}{c(e_0 H_0(\mathscr{T}(p)))}.$$

Because of (2.4) and (4.2) we have

$$\rho(e_0 H_0(\mathscr{T}(p)))=0, \qquad |c(e_0 H_0(\mathscr{T}(p)))|_p = \frac{\# H^1(G, H^0_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p)))}{\# H^0(\mathfrak{o}, \mathscr{T}(p))}.$$

From (2.3) and the final remark in § 2 we get

$$\rho(e_0 H_1(\mathscr{T}(p))) \geqq \operatorname{corank} H^0(G, H^1_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p))).$$

Furthermore, equality holds if and only if the homomorphism

$$f\colon H^0(G, H^1_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p))) \to H^1(G, H^1_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p)))$$

induced by the identity map on $H^1_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p))$ is a quasi-isomorphism; in that case

$$|c(e_0 H_1(\mathscr{T}(p)))|_p = \frac{\#\operatorname{coker} f}{\#\ker f}.$$

From (4.1) we get the exact sequence

$$0 \to H^1(G, H^0_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p))) \to H^1_{et}(\mathfrak{o}', \mathscr{T}(p)) \xrightarrow{\alpha} H^0(G, H^1_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p))) \to 0$$

and the isomorphism

$$H^1(G, H^1_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p))) \xrightarrow[\cong]{\beta} H^2_{et}(\mathfrak{o}', \mathscr{T}(p)).$$

According to (3.9) the canonical map

$$\mathscr{T}^0(\mathfrak{o}')\otimes\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\delta} H^1_{et}(\mathfrak{o}', \mathscr{T}(p))$$

is a quasi-isomorphism, and according to (3.12) we have

$$\operatorname{corank} \mathscr{T}^0(\mathfrak{o}')\otimes\mathbb{Q}_p/\mathbb{Z}_p = \sum_{\mathfrak{p}|p} \operatorname{rank}_{\mathbb{Z}} \tilde{T}(\hat{k}_{\mathfrak{p}}).$$

Combining all these formulas results in

$$\begin{aligned} \rho &\geqq \operatorname{corank} H^1(G, H^1_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p))) = \operatorname{corank} H^1_{et}(\mathfrak{o}', \mathscr{T}(p)) \\ &= \operatorname{corank} \mathscr{T}^0(\mathfrak{o}')\otimes\mathbb{Q}_p/\mathbb{Z}_p = \sum_{\mathfrak{p}|p} \operatorname{rank}_{\mathbb{Z}} \tilde{T}(\hat{k}_{\mathfrak{p}}), \end{aligned} \tag{1}$$

where equality holds if and only if f is a quasi-isomorphism; in that case

$$\begin{aligned} &|[L_p^{(1)}(T,s)\cdot\prod_{\mathfrak{p}|p} L_{p,\mathfrak{p}}(T,s)]_{s=0}|_p \\ &= |[L_p^{(1)}(T,s)\cdot s^{-\rho}]_{s=0}\cdot\prod_{\mathfrak{p}|p} L_{\mathfrak{p}}(T)^{-1}|_p \\ &= \prod_{\mathfrak{p}|p} |L_{\mathfrak{p}}(T)|_p^{-1}\cdot|\log_p(1+p^e)|_p^\rho\cdot \# H^0(\mathfrak{o}, \mathscr{T}(p))\cdot \frac{\#\operatorname{coker}\beta f\alpha}{\#\ker\beta f\alpha}. \end{aligned} \tag{2}$$

In order to analyze the map f we study the following commutative diagram.

$$\begin{tikzcd} \mathscr{T}^0(\mathfrak{o}')\otimes\mathbb{Q}_p/\mathbb{Z}_p \arrow[r] \arrow[d, "\delta"] & \bigoplus_{\mathfrak{p}|p} T(\hat{k}_{\mathfrak{p}})\otimes\mathbb{Q}_p/\mathbb{Z}_p \arrow[d] \\ H^1_{et}(\mathfrak{o}', \mathscr{T}(p)) \arrow[r] \arrow[d, "\alpha"] & \bigoplus_{\mathfrak{p}|p} H^1_{et}(\hat{k}_{\mathfrak{p}}, T(p)) \arrow[d] \\ H^0(G, H^1_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p))) \arrow[r] \arrow[d, "f"] & \bigoplus_{\mathfrak{p}|p} H^0(G, H^1_{et}(Y_{\mathfrak{p}}, T(p))) \arrow[d] \\ H^1(G, H^1_{et}(\mathfrak{o}'_\infty, \mathscr{T}(p))) \arrow[r] \arrow[d, "\beta", "\cong"'] & \bigoplus_{\mathfrak{p}|p} H^1(G, H^1_{et}(Y_{\mathfrak{p}}, T(p))) \arrow[d, "\cong"] \\ H^2_{et}(\mathfrak{o}', \mathscr{T}(p)) \arrow[r, "\delta_2"] & \bigoplus_{\mathfrak{p}|p} H^2_{et}(\hat{k}_{\mathfrak{p}}, T(p)) = \bigoplus_{\mathfrak{p}|p} (\tilde{T}(\hat{k}_{\mathfrak{p}})\otimes\mathbb{Z}_p)^* \end{tikzcd} \tag{3}$$

Here we have put $Y_{\mathfrak{p}} := \operatorname{Spec}(\mathfrak{o}'_\infty \otimes_{\mathfrak{o}'} \hat{k}_{\mathfrak{p}})$. The maps in the right column are defined completely in the same way as their analogues in the left column ($\operatorname{cd}_p Y_{\mathfrak{p}} \leqq 1$). Among the maps in the rows the first one is given by the inclusions $\mathscr{T}^0(\mathfrak{o}')\leqq T(\hat{k}_{\mathfrak{p}})$ and the other ones by a morphism of spectral sequences. The identification $H^2_{et}(\hat{k}_{\mathfrak{p}}, T(p)) = (\tilde{T}(\hat{k}_{\mathfrak{p}})\otimes\mathbb{Z}_p)^*$ comes from the local duality theorem ([31] II 5.8).

Lemma 2. *The composite map* $\neg\!\!\!\rightarrow: \mathcal{T}^0(\mathfrak{o}')\otimes\mathbb{Q}_p/\mathbb{Z}_p\to\bigoplus_{\mathfrak{p}/p}(\tilde{T}(\hat{k}_\mathfrak{p})\otimes\mathbb{Z}_p)^*$ *in* (3) *is a quasi-isomorphism if and only if* $\langle\ ,\ \rangle_p$ *is nondegenerate, in which case the map is surjective with kernel of order*

$$|\det\langle\ ,\ \rangle_p|_p^{-1}\cdot|\log_p(1+p^e)|_p^{\sum_{\mathfrak{p}/p}\operatorname{rank}\tilde{T}(\hat{k}_\mathfrak{p})}.$$

Proof. We fix a prime $\mathfrak{p}$ above p. We have to show that the composite map

$$\eta_T\colon T(\hat{k}_\mathfrak{p})\otimes\mathbb{Q}_p/\mathbb{Z}_p\to(\tilde{T}(\hat{k}_\mathfrak{p})\otimes\mathbb{Z}_p)^*$$

in the right column of (3) has the following description: There exists a unit $\varepsilon\in\mathbb{Z}_p^\times$ with

$$\log_p(1+p^e)\cdot(\eta_T(a\otimes x))(\chi)=\varepsilon\cdot(\log_p\circ\operatorname{Norm}_{\hat{k}_\mathfrak{p}/\mathbb{Q}_p}\circ\chi(a))\cdot x$$

for all $a\otimes x\in T(\hat{k}_\mathfrak{p})\otimes\mathbb{Q}_p/\mathbb{Z}_p$ and $\chi\in\tilde{T}(\hat{k}_\mathfrak{p})$. The definition of η_T does not depend on our assumption that $T_{/k}$ is odd. Thus we can consider the analogous map

$$\eta:=\eta_{\mathbb{G}_m}\colon \hat{k}_\mathfrak{p}^\times\otimes\mathbb{Q}_p/\mathbb{Z}_p=H^1_{et}(\hat{k}_\mathfrak{p},\mu(p))\to H^2_{et}(\hat{k}_\mathfrak{p},\mu(p))=\mathbb{Q}_p/\mathbb{Z}_p$$

(observe $H^1_{et}(\hat{k}_\mathfrak{p},\mathbb{G}_m)=0$ (Hilbert 90) and $\tilde{\mathbb{G}}_m(\hat{k}_\mathfrak{p})=\mathbb{Z}$). It is easy to check that

$$(\eta_T(a\otimes x))(\chi)=\eta(\chi(a)\otimes x)$$

holds true. Therefore it remains to prove

$$\ker(\log_p(1+p^e)\cdot\eta)=\ker\vartheta, \tag{4}$$

where $\vartheta\colon \hat{k}_\mathfrak{p}^\times\otimes\mathbb{Q}_p/\mathbb{Z}_p\to\mathbb{Q}_p/\mathbb{Z}_p$ is defined by

$$c\otimes x\mapsto(\log_p\circ\operatorname{Norm}_{\hat{k}_\mathfrak{p}/\mathbb{Q}_p}(c))\cdot x.$$

Let $\mathbb{Q}_p^\infty/\mathbb{Q}_p$ denote the unique $\mathbb{Z}_p$-extension which is contained in $\mathbb{Q}_p(\mu(p))$, and let $N\mathbb{Q}_p\leqq\mathbb{Q}_p^\times$, resp. $N\hat{k}_\mathfrak{p}\leqq\hat{k}_\mathfrak{p}^\times$, denote the subgroup of universal norms with respect to $\mathbb{Q}_p^\infty/\mathbb{Q}_p$, resp. $\mathbb{Q}_p^\infty\cdot\hat{k}_\mathfrak{p}/\hat{k}_\mathfrak{p}$. According to the local class field theory we have the commutative diagram

$$\begin{array}{ccc}\hat{k}_\mathfrak{p}^\times/N\hat{k}_\mathfrak{p} & \xrightarrow{\cong} & \operatorname{Gal}(\mathbb{Q}_p^\infty\cdot\hat{k}_\mathfrak{p}/\hat{k}_\mathfrak{p})\cong\mathbb{Z}_p\\ \downarrow{\scriptstyle\mathrm{Norm}} & & \downarrow\\ \mathbb{Q}_p^\times/N\mathbb{Q}_p & \xrightarrow{\cong} & \operatorname{Gal}(\mathbb{Q}_p^\infty/\mathbb{Q}_p)\cong\mathbb{Z}_p,\end{array}$$

where the isomorphisms in the rows are given by the reciprocity law. Thus the norm map $\hat{k}_\mathfrak{p}^\times/N\hat{k}_\mathfrak{p}\to\mathbb{Q}_p^\times/N\mathbb{Q}_p$ is injective with finite cokernel of order $[\mathbb{Q}_p^\infty\cap\hat{k}_\mathfrak{p}:\mathbb{Q}_p]$. From

$$N\mathbb{Q}_p=p^{\mathbb{Z}}\times\operatorname{Tor}\mathbb{Z}_p^\times=\ker(\log_p),\qquad \#\operatorname{coker}(\log_p)=p$$

(see [24] II 7.16) then follows

$$N\hat{k}_\mathfrak{p}=\ker(\log_p\circ\operatorname{Norm}_{\hat{k}_\mathfrak{p}/\mathbb{Q}_p})$$

and

$$\#\operatorname{coker}(\log_p\circ\operatorname{Norm}_{\hat{k}_\mathfrak{p}/\mathbb{Q}_p})=p\cdot[\mathbb{Q}_p^\infty\cap\hat{k}_\mathfrak{p}:\mathbb{Q}_p].$$

This implies

$$\ker\vartheta=\{z\in\hat{k}_\mathfrak{p}^\times\otimes\mathbb{Q}_p/\mathbb{Z}_p: p\cdot[\mathbb{Q}_p^\infty\cap\hat{k}_\mathfrak{p}:\mathbb{Q}_p]\cdot z\in N\hat{k}_\mathfrak{p}\otimes\mathbb{Q}_p/\mathbb{Z}_p\}. \tag{5}$$

Now we shall determine $\ker\eta$, which obviously is the kernel of the natural map

$$\hat{k}_\mathfrak{p}^\times\otimes\mathbb{Q}_p/\mathbb{Z}_p\to((\mathfrak{o}'_\infty\otimes_{\mathfrak{o}'}\hat{k}_\mathfrak{p})^\times\otimes\mathbb{Q}_p/\mathbb{Z}_p)_G.$$

Moreover, an easy argument shows

$$\ker\eta=\{z\in\hat{k}_\mathfrak{p}^\times\otimes\mathbb{Q}_p/\mathbb{Z}_p: s(\mathfrak{p})\cdot z\in\ker\bar{\eta}\} \tag{6}$$

with $s(\mathfrak{p}):=$ number of primes of k_∞ above $\mathfrak{p}$, $\hat{k}_\mathfrak{p}^\infty:=\mathbb{Q}_p^\infty\cdot\hat{k}_\mathfrak{p}$, and $\bar{\eta}$ the natural map

$$\bar{\eta}\colon \hat{k}_\mathfrak{p}^\times\otimes\mathbb{Q}_p/\mathbb{Z}_p\to((\hat{k}_\mathfrak{p}^\infty)^\times\otimes\mathbb{Q}_p/\mathbb{Z}_p)_{\operatorname{Gal}(\hat{k}_\mathfrak{p}^\infty/\hat{k}_\mathfrak{p})}.$$

But according to the next lemma we have

$$\ker\bar{\eta}=N\hat{k}_\mathfrak{p}\otimes\mathbb{Q}_p/\mathbb{Z}_p. \tag{7}$$

Now (4) follows from (5)–(7) and

$$p\cdot[\mathbb{Q}_p^\infty\cap\hat{k}_\mathfrak{p}:\mathbb{Q}_p]=|\log_p(1+p^e)\cdot s(\mathfrak{p})|_p^{-1}.$$

Lemma 3. *Let* $K/\mathbb{Q}_p$ *be a finite extension, and let* L/K *be a* $\mathbb{Z}_p$*-extension. If* $NK\leqq K^\times$ *denotes the subgroup of universal norms with respect to* L/K, *then* $NK\otimes\mathbb{Q}_p/\mathbb{Z}_p$ *is the kernel of the natural map*

$$K^\times\otimes\mathbb{Q}_p/\mathbb{Z}_p\to(L^\times\otimes\mathbb{Q}_p/\mathbb{Z}_p)_{\operatorname{Gal}(L/K)}.$$

Proof. Define L_n/K to be the unique subfield of L with $[L_n:K]=p^n$. For $c\otimes\frac{m}{p^i}\in NK\otimes\mathbb{Q}_p/\mathbb{Z}_p$ we choose $b\in L_i^\times$ such that $c=\operatorname{Norm}_{L_i/K}(b)$. Because of

$$\operatorname{Norm}_{L_i/K}(b^{p^i}\cdot c^{-1})=\operatorname{Norm}_{L_i/K}(b)^{p^i}\cdot c^{-p^i}=1$$

there exists an $a\in L_i^\times$ such that $b^{p^i}\cdot c^{-1}=\sigma a\cdot a^{-1}$ (Hilbert 90), where σ denotes a topological generator of $\operatorname{Gal}(L/K)$. We now have

$$c\otimes\frac{m}{p^i}=(b^{p^i}\cdot a\cdot(\sigma a)^{-1})\otimes\frac{m}{p^i}=a\otimes\frac{m}{p^i}-\sigma a\otimes\frac{m}{p^i}=(1-\sigma)\left(a\otimes\frac{m}{p^i}\right).$$

Thus it is shown that $NK\otimes\mathbb{Q}_p/\mathbb{Z}_p$ is contained in the kernel of that map. The inverse inclusion immediately follows from the following two assertions:

$$(\sigma-1)(L_n^\times\otimes\mathbb{Q}_p/\mathbb{Z}_p)\leqq NL_n\otimes\mathbb{Q}_p/\mathbb{Z}_p,$$

where $NL_n\leqq L_n^\times$ denotes the subgroup of universal norms with respect to $\operatorname{Gal}(L/L_n)$. Namely, since the action of $\operatorname{Gal}(L/K)$ on $\operatorname{Gal}(L/L_n)$ is trivial, the reciprocity law implies that $L_n^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p/N L_n \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$ as $\operatorname{Gal}(L/K)$-module.

$$(N L_n \otimes \mathbb{Q}_p/\mathbb{Z}_p) \cap (K^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p) = N K \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

Namely, the inclusion $K^\times \subseteq L_n^\times$ and the norm map $L_n^\times \to K^\times$ induce maps

$$\mathbb{Q}_p/\mathbb{Z}_p \cong K^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p/N K \otimes \mathbb{Q}_p/\mathbb{Z}_p \underset{N}{\overset{I}{\rightleftarrows}} L_n^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p/N L_n \otimes \mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Q}_p/\mathbb{Z}_p$$

with $N \circ I = p^n$. But again the reciprocity law implies $\#\ker N = p^n$. Therefore I is an isomorphism. q.e.d.

In order to study the maps δ and δ_2 in (3) we consider the relative cohomology sequence

$$\begin{aligned} 0 &\to H^1(\mathfrak{o}, \mathcal{T}(p)) \to H^1_{et}(\mathfrak{o}', \mathcal{T}(p)) \to \bigoplus_{\mathfrak{p}/p} H^2_\cdot(\hat{\mathfrak{o}}_\mathfrak{p}, \mathcal{T}(p)) \\ &\to H^2(\mathfrak{o}, \mathcal{T}(p)) \to H^2_{et}(\mathfrak{o}', \mathcal{T}(p)) \to \bigoplus_{\mathfrak{p}/p} H^3_\cdot(\hat{\mathfrak{o}}_\mathfrak{p}, \mathcal{T}(p)) \\ &\to H^3(\mathfrak{o}, \mathcal{T}(p)) \to 0. \end{aligned}$$

Use (3.4), (3.6) and (4.1 i)) for the zeros at the beginning and at the end. The same lemmas imply

$$H^i(\mathfrak{o}, \mathcal{T}(p)) = 0 \quad \text{for } i > 3. \tag{8}$$

Concerning the relative cohomology groups for $\mathfrak{p}/p$, we have

$$H^3_\cdot(\hat{\mathfrak{o}}_\mathfrak{p}, \mathcal{T}(p)) = H^2_{et}(\hat{k}_\mathfrak{p}, T(p))$$

and

$$H^2_\cdot(\hat{\mathfrak{o}}_\mathfrak{p}, \mathcal{T}(p)) = H^1_{et}(\hat{k}_\mathfrak{p}, T(p))/H^1(\hat{\mathfrak{o}}_\mathfrak{p}, \mathcal{T}(p))$$

([16]), and from the exact sequences of $\hat{\mathfrak{o}}_\mathfrak{p}$-group schemes

$$0 \to \mathcal{T}_{p^j} \to \mathcal{T}^0 \xrightarrow{p^j} \mathcal{T}^0 \to 0$$

(our condition on p!) we derive the exact sequence

$$0 \to \mathcal{T}^0(\hat{\mathfrak{o}}_\mathfrak{p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^1(\hat{\mathfrak{o}}_\mathfrak{p}, \mathcal{T}(p)) \to H^1(\hat{\mathfrak{o}}_\mathfrak{p}, \mathcal{T}^0)(p) \to 0.$$

Because of $H^1(\hat{\mathfrak{o}}_\mathfrak{p}, \mathcal{T}^0) = 0$ ([17] 5.1 iii)) and $\mathcal{T}^0(\hat{\mathfrak{o}}_\mathfrak{p}) = T(\hat{k}_\mathfrak{p})^c$ ((3.10 iii))) we thus get

$$H^2_\cdot(\hat{\mathfrak{o}}_\mathfrak{p}, \mathcal{T}(p)) = H^1_{et}(\hat{k}_\mathfrak{p}, T(p))/T(\hat{k}_\mathfrak{p})^c \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

Inserting this in the above cohomology sequence gives the exact sequence

$$\begin{aligned} 0 &\to H^1(\mathfrak{o}, \mathcal{T}(p)) \to H^1_{et}(\mathfrak{o}', \mathcal{T}(p)) \xrightarrow{\delta_1} \bigoplus_{\mathfrak{p}/p} H^1_{et}(\hat{k}_\mathfrak{p}, T(p))/T(\hat{k}_\mathfrak{p})^c \otimes \mathbb{Q}_p/\mathbb{Z}_p \\ &\to H^2(\mathfrak{o}, \mathcal{T}(p)) \to H^2_{et}(\mathfrak{o}', \mathcal{T}(p)) \xrightarrow{\delta_2} \bigoplus_{\mathfrak{p}/p} H^2_{et}(\hat{k}_\mathfrak{p}, T(p)) \\ &\to H^3(\mathfrak{o}, \mathcal{T}(p)) \to 0. \end{aligned} \tag{9}$$

Furthermore we need the commutative exact diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ \mathcal{T}^0(\mathfrak{o}') \otimes \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] \arrow[d, "\delta"] & \bigoplus_{\mathfrak{p}/p} (T(\hat{k}_\mathfrak{p})/T(\hat{k}_\mathfrak{p})^c) \otimes \mathbb{Q}_p/\mathbb{Z}_p \arrow[d] \\ H^1_{et}(\mathfrak{o}', \mathcal{T}(p)) \arrow[r, "\delta_1"] & \bigoplus_{\mathfrak{p}/p} H^1_{et}(\hat{k}_\mathfrak{p}, T(p))/T(\hat{k}_\mathfrak{p})^c \otimes \mathbb{Q}_p/\mathbb{Z}_p \arrow[d] \\ & \bigoplus_{\mathfrak{p}/p} H^1_{et}(\hat{k}_\mathfrak{p}, T)(p) \arrow[d] \\ & 0. \end{tikzcd} \tag{10}$$

The map in the first row is induced by the canonical homomorphism

$$\mathcal{T}^0(\mathfrak{o}') \subseteq T(k) \to \bigoplus_{\mathfrak{p}/p} T(\hat{k}_\mathfrak{p})/T(\hat{k}_\mathfrak{p})^c.$$

The short exact sequence in the right column is derived from the exact sequences of $\hat{k}_\mathfrak{p}$-group schemes $0 \to T_{p^j} \to T \xrightarrow{p^j} T \to 0$. We have already seen that δ is a quasi-isomorphism. According to the local duality theorem ([31] II 5.8) the groups $H^1_{et}(\hat{k}_\mathfrak{p}, T) = H^1_{et}(\hat{k}_\mathfrak{p}, \tilde{T})^*$ are finite.

Lemma 4. i) *δ_1 is a quasi-isomorphism;*

ii) *$(\ ,\ )_\mathfrak{p}$ is nondegenerate;*

iii) $|\prod_{\mathfrak{p}/p} L_\mathfrak{p}(T)^{-1} \cdot \det(\ ,\ )_\mathfrak{p}|_p = \dfrac{\#\operatorname{coker} \delta_1 \circ \delta}{\#\ker \delta_1 \circ \delta}.$

Proof. We consider the commutative diagram of pairings between free $\mathbb{Z}$-modules

$$\begin{array}{ccccc} T(K_\mathfrak{p})/T(K_\mathfrak{p})^c & \times & \tilde{T}(K_\mathfrak{p}) & \longrightarrow & \mathbb{Z} \\ \uparrow & & \uparrow & & \| \\ T(\hat{k}_\mathfrak{p})/T(\hat{k}_\mathfrak{p})^c & \times & \tilde{T}(\hat{k}_\mathfrak{p}) & \longrightarrow & \mathbb{Z} \end{array}$$

which are induced by $(a, \chi) \mapsto v_\mathfrak{p} \circ \chi(a)$ ($\mathfrak{p}/p$ and $v_\mathfrak{p}: K_\mathfrak{p}^\times \to \mathbb{Z}$ the normalized discrete valuation). Because $T_{/K_\mathfrak{p}}$ splits, one can easily check that the pairing in the first line has determinant ± 1. Taking Galois invariants resp. coinvariants we get the commutative diagram

$$\begin{array}{ccccc} (T(K_\mathfrak{p})/T(K_\mathfrak{p})^c)^{H_\mathfrak{p}} & \times & \tilde{T}(K_\mathfrak{p})_{H_\mathfrak{p}} & \longrightarrow & \mathbb{Z} \\ \uparrow \cong & & \uparrow & & \| \\ T(\hat{k}_\mathfrak{p})/T(\hat{k}_\mathfrak{p})^c & \times & \tilde{T}(\hat{k}_\mathfrak{p}) & \longrightarrow & \mathbb{Z}, \end{array}$$

where again the pairing in the top line has determinant ± 1. The map in the left column is an isomorphism because of $(T(K_\mathfrak{p})^c)^{H_\mathfrak{p}} = T(\hat{k}_\mathfrak{p})^c$ and $H^1(H_\mathfrak{p}, T(K_\mathfrak{p})^c) \leqq H^1(\hat{\mathfrak{o}}_\mathfrak{p}, \mathscr{T}^0) = 0$. According to a lemma of Tate ([37] z. 4) the map in the right column is injective with finite cokernel of order $|L_\mathfrak{p}(T)|$. Therefore, if the pairing in the bottom line is denoted by $(\,,\,)_\mathfrak{p}$, then $(\,,\,)_\mathfrak{p}$ is nondegenerate with

$$|\det(\,,\,)_\mathfrak{p}| = |L_\mathfrak{p}(T)| \cdot \#\operatorname{Tor}(\tilde{T}(K_\mathfrak{p})_{H_\mathfrak{p}})^{-1}.$$

$H_\mathfrak{p}$ being cyclic implies

$$\begin{aligned}\#\operatorname{Tor}(\tilde{T}(K_\mathfrak{p})_{H_\mathfrak{p}}) &= \# H^{-1}(H_\mathfrak{p}, \tilde{T}(K_\mathfrak{p})) = \# H^1(H_\mathfrak{p}, \tilde{T}(K_\mathfrak{p}))\\ &= \# H^1_{et}(\hat{k}_\mathfrak{p}, \tilde{T}) = \# H^1_{et}(\hat{k}_\mathfrak{p}, T)\end{aligned}$$

(use $\tilde{T}(K_\mathfrak{p})$ torsionfree and $H^1_{et}(K_\mathfrak{p}, \mathbb{Z}) = 0$. Thus we have established

$$|L_\mathfrak{p}(T)| = |\det(\,,\,)_\mathfrak{p}| \cdot \# H^1_{et}(\hat{k}_\mathfrak{p}, T).$$

Using (10) the lemma now follows from this formula and the fact that the kernel of the natural map

$$\mathscr{T}^0(\mathfrak{o}') \to \bigoplus_{\mathfrak{p}/p} T(\hat{k}_\mathfrak{p})/T(\hat{k}_\mathfrak{p})^c$$

is finite. In order to prove the last statement we observe that this kernel is contained in the "unit group"

$$\{a \in T(k) \colon a \in T(\hat{k}_\mathfrak{p})^c \text{ for all finite } \mathfrak{p}\}$$

of $T_{/k}$. According to [33] the unit group is finitely generated of rank

$$\sum_{\mathfrak{p}/\infty} \operatorname{rank}_\mathbb{Z} \tilde{T}(\hat{k}_\mathfrak{p}) - \operatorname{rank}_\mathbb{Z} T(k);$$

but this expression is zero, since $T_{/k}$ is odd.

Lemma 5. i) *$H^i(\mathfrak{o}, \mathscr{T}(p))$ is finite for all $i \geqq 0$ and $= 0$ for $i \geqq 3$;*

ii) *δ_2 is a quasi-isomorphism;*

iii) $$\Big|\prod_{\mathfrak{p}/p} L_\mathfrak{p}(T)^{-1} \cdot \det(\,,\,)_\mathfrak{p}\Big|_p \cdot \prod_{i>0} \# H^i(\mathfrak{o}, \mathscr{T}(p))^{(-1)^{i+1}} = \frac{\#\operatorname{coker}\delta}{\#\ker\delta} \cdot \frac{\#\operatorname{coker}\delta_2}{\#\ker\delta_2}.$$

Proof. In (8) we have already seen that $H^i(\mathfrak{o}, \mathscr{T}(p)) = 0$ holds true for $i \geqq 4$. We first show that the whole lemma follows from the vanishing of $H^3(\mathfrak{o}, \mathscr{T}(p))$. Namely, the groups $H^1_{et}(\mathfrak{o}', \mathscr{T}(p))$ and $H^2_{et}(\mathfrak{o}', \mathscr{T}(p))$ are of cofinite type and have the same corank; this is a consequence of (4.1 iii/iv)) using the general fact that the Γ-invariants and the Γ-coinvariants of a finitely generated $\mathbb{Z}_p[\![\Gamma]\!]$-torsion module are finitely generated $\mathbb{Z}_p$-modules of the same rank. This means

$$\begin{aligned}\operatorname{corank} H^2_{et}(\mathfrak{o}', \mathscr{T}(p)) &= \operatorname{corank} H^1_{et}(\mathfrak{o}', \mathscr{T}(p)) = \operatorname{rank} \mathscr{T}^0(\mathfrak{o}')\\ &= \sum_{\mathfrak{p}/p} \operatorname{rank}_\mathbb{Z} \tilde{T}(\hat{k}_\mathfrak{p}) = \sum_{\mathfrak{p}/p} \operatorname{corank} H^2_{et}(\hat{k}_\mathfrak{p}, T(p))\end{aligned}$$

and, therefore, using the exact sequence (9) we get that δ_2 is a quasi-isomorphism if $H^3(\mathfrak{o}, \mathscr{T}(p))$ vanishes. Under the same assumption the other assertions are now also easily derived from (9) and Lemma 4.

According to the flat arithmetic duality theorem of Artin and Mazur we have

$$H^3(\mathfrak{o}, \mathscr{T}(p))^* = \varinjlim \operatorname{Hom}_{\mathfrak{o}_{fppf}}(\mathscr{T}_{p^j}, \mathbb{G}_{m/\mathfrak{o}}) = \varinjlim \operatorname{Hom}_{\mathfrak{o}_{fppf}}(\mathscr{T}_{p^j}, \mathscr{G}_m)$$

(we remember that $\mathscr{G}_m$ denotes the Néron model of $\mathbb{G}_{m/k}$ over $\mathfrak{o}$). But the natural map $\operatorname{Hom}_{\mathfrak{o}_{fppf}}(\mathscr{T}_{p^j}, \mathscr{G}_m) \to \operatorname{Hom}_{k_{fppf}}(T_{p^j}, \mathbb{G}_m)$ is bijective. Over $\mathfrak{o}'$, this is a direct consequence of the universal property of Néron models, because $\mathscr{T}_{p^j/\mathfrak{o}'}$ is etale; over $\operatorname{Spec}(\mathfrak{o}) \setminus \{\mathfrak{p} \text{ ramified in } k_T\}$, by an etale base change it suffices to consider the case $\mathscr{T}_{p^j} = \mu_{p^j}$, where the required bijectivity is evident. Thus it remains to show $\varinjlim \operatorname{Hom}_{k_{fppf}}(T_{p^j}, \mathbb{G}_m) = 0$. Since $Ext^1_{k_{fppf}}(T, \mathbb{G}_m) = 0$ (SGA 7 VIII 3.3.1), the sequence $0 \to \tilde{T} \to \tilde{T} \to Hom_{k_{fppf}}(T_{p^j}, \mathbb{G}_m) \to 0$ of sheaves on k_{fppf} is exact. In the corresponding exact cohomology sequence

$$\tilde{T}(k) \to \operatorname{Hom}_{k_{fppf}}(T_{p^j}, \mathbb{G}_m) \to H^1(k, \tilde{T})_{p^j} \to 0$$

the first term $\tilde{T}(k)$ vanishes, because $T_{/k}$ is odd. On the other hand $H^1(k, \tilde{T})$ is annihilated by $[k_T : k]$ because of $H^1(k_T, \tilde{T}) = H^1(k_T, \mathbb{Z})^{\dim T} = 0$. We finally get

$$\varinjlim \operatorname{Hom}_{k_{fppf}}(T_{p^j}, \mathbb{G}_m) = \varinjlim H^1(k, \tilde{T})_{p^j} = 0. \quad \text{q.e.d.}$$

The proof of the theorem is now immediately accomplished by combining (1)–(3), Lemma 2, Lemma 4 ii), and Lemma 5.

Conjecture. (C) The pairing $\langle\,,\,\rangle_p$ is nondegenerate for any odd $T_{/k}$ (and any odd p which is unramified in k_T/k).

Under the assumption that (C) holds true (in appropriate situations; in which will be clear from the proofs) we can even determine the multiplicity at $s = 0$ of all Iwasawa L-functions $L_p^{(j)}(T, s)$.

Theorem 6. *Assume* (C) *to be true. If $T_{/k}$ is even, resp. odd, and $0 \leqq j < d$ is even, resp. odd, then $L_p^{(j)}(T, s)$ has a zero of exact multiplicity*

$$\operatorname{rank}_{\mathbb{Z}_p} e_{1-j}(\mathscr{T}^0(\mathfrak{o}'_1) \otimes \mathbb{Z}_p)$$

at $s = 0$.

Proof. Let $\rho^{(j)}(T_{/k})$ denote the multiplicity of $L_p^{(j)}(T, s)$ at $s = 0$. Because of $F_{e_{j-1}H_0(\mathscr{T}(p))}(0) \neq 0$ the number $\rho^{(j)}(T_{/k})$ is also the multiplicity of the zero of $F_{e_{j-1}H_1(\mathscr{T}(p))}(t)$ at $t = 0$. This means (compare (1))

$$\begin{aligned}\rho^{(j)}(T_{/k}) &\geqq \operatorname{rank}_{\mathbb{Z}_p} H^0(\Gamma, e_{j-1} H_1(\mathscr{T}(p)))\\ &= \operatorname{corank} H^0(\Gamma, e_{1-j} H^1(\mathfrak{o}_\infty, \mathscr{T}(p)))\\ &= \operatorname{corank} e_{1-j} H^1_{et}(\mathfrak{o}'_1, \mathscr{T}(p))\\ &= \operatorname{corank} e_{1-j}(\mathscr{T}^0(\mathfrak{o}'_1) \otimes \mathbb{Q}_p/\mathbb{Z}_p)\\ &= \operatorname{rank}_{\mathbb{Z}_p} e_{1-j}(\mathscr{T}^0(\mathfrak{o}'_1) \otimes \mathbb{Z}_p).\end{aligned}$$

But we have (for the notation see the proof of (3.13ii)))

$$\prod_{\substack{0\le j<d\\ j\,\text{even}}} L_p^{(j)}(T,s)=L_p^{(0)}(T\underset{k}{\times} k_1^+,s),$$

$$\sum \operatorname{rank} e_{1-j}(\mathcal{T}^0(\mathfrak{o}_1')\otimes\mathbf{Z}_p)=\operatorname{rank}(\mathcal{T}^0(\mathfrak{o}_1')\otimes\mathbf{Z}_p)^-$$

in case $T_{/k}$ is even, resp.

$$\prod_{\substack{0\le j<d\\ j\,\text{odd}}} L_p^{(j)}(T,s)=L_p^{(1)}(T\underset{k}{\times} k_1^+,s),$$

$$\sum \operatorname{rank} e_{1-j}(\mathcal{T}^0(\mathfrak{o}_1')\otimes\mathbf{Z}_p)=\operatorname{rank}(\mathcal{T}^0(\mathfrak{o}_1')\otimes\mathbf{Z}_p)^+=\operatorname{rank}_{\mathbf{Z}}\mathcal{T}^0(R_1)$$

in case $T_{/k}$ is odd. Therefore, it obviously suffices to prove the theorem under the additional assumption $k=k_1^+$. If $T_{/k}$ is odd, this is done in Theorem 1. For the rest of the proof let $T_{/k}$ be even and assume $k=k_1^+$. We have to show

$$\rho^{(0)}(T_{/k})=\operatorname{rank}(\mathcal{T}^0(\mathfrak{o}_1')\otimes\mathbf{Z}_p)^-. \tag{11}$$

First we consider the special case that all primes above p are unramified in k_1/k. If $\chi\colon G_k\to\Delta\to\{\pm1\}$ is the non-trivial character and if $T_{\chi/k}$ denotes the twist of T by χ, then we have:

(a) $T_{\chi/k}$ is odd;

(b) $k_{T_\chi}\le k_T\cdot k_1$; in particular, all primes above p are unramified in k_{T_χ}/k;

(c) $L_p^{(0)}(T,s)=L_p^{(1)}(T_\chi,s)$ and

$$\operatorname{rank}(\mathcal{T}^0(\mathfrak{o}_1')\otimes\mathbf{Z}_p)^-=\operatorname{rank}(\mathcal{T}_\chi^0(\mathfrak{o}_1')\otimes\mathbf{Z}_p)^+=\operatorname{rank}_{\mathbf{Z}}\mathcal{T}_\chi^0(\mathfrak{o}'),$$

where $\mathcal{T}_\chi$ denotes the Néron model of T_χ over $\mathfrak{o}$. Because of (a) and (b) Theorem 1 implies $\rho^{(1)}(T_{\chi/k})=\operatorname{rank}\mathcal{T}_\chi^0(\mathfrak{o}')$; because of (c) this is nothing else than (11).

Coming back to the general case it follows from the theorem of Grunwald-Hasse-Wang ([1]) that there exists a quadratic extension K/k with the following properties:

(d) K is totally real;

(e) each prime of k which is ramified in k_T is unramified in K;

(f) all primes above p are unramified in $K(\mu_p)/K$.

Let B resp. B_1 resp. B_∞ denote the ring of p-integers in K resp. $K(\mu_p)$ resp. $K(\mu(p))$. For any $\mathbf{Z}_p[\operatorname{Gal}(K(\mu_p)/k)]$-module M define $M^{\pm,\pm}$ to be the maximal submodule on which the non-trivial element in $\operatorname{Gal}(K(\mu_p)/K)$, resp. $\operatorname{Gal}(K(\mu_p)/k_1)$, acts as multiplication by the first sign, resp. the second sign.

First we notice that because of (e) the formation of the connected component $\mathcal{T}^0$ and the formation of the ind-$\mathfrak{o}$-group scheme $\mathcal{T}(p)$ (although not the formation of the Néron model $\mathcal{T}$) commute with the base change $\operatorname{Spec}(B)\to\operatorname{Spec}(\mathfrak{o}')$. Since $T_{/k}$ fulfils the condition of the above considered special case, we have

$$\begin{aligned}\rho^{(0)}(T_{/K})&=\operatorname{rank}(\mathcal{T}^0(B_1)\otimes\mathbf{Z}_p)^-\\&=\operatorname{rank}(\mathcal{T}^0(B_1)\otimes\mathbf{Z}_p)^{-,+}+\operatorname{rank}(\mathcal{T}^0(B_1)\otimes\mathbf{Z}_p)^{-,-}.\end{aligned}$$

We already know that

$$\rho^{(0)}(T_{/k})\ge\operatorname{rank}(\mathcal{T}^0(\mathfrak{o}_1')\otimes\mathbf{Z}_p)^-=\operatorname{rank}(\mathcal{T}^0(B_1)\otimes\mathbf{Z}_p)^{-,+}.$$

Since $\rho:=\rho^{(0)}(T_{/K})-\rho^{(0)}(T_{/k})$ is the multiplicity of the zero of $F_H(t)$, $H:=(H_{et}^1(B_\infty,\mathcal{T}(p))^*)^{-,-}$, at $t=0$, we get in a similar way that

$$\begin{aligned}\rho&\ge\operatorname{rank}H^0(\Gamma,H)=\operatorname{corank}H^0(\Gamma,H_{et}^1(B_\infty,\mathcal{T}(p))^{-,-})\\&=\operatorname{corank}H_{et}^1(B_1,\mathcal{T}(p))^{-,-}=\operatorname{rank}(\mathcal{T}^0(B_1)\otimes\mathbf{Z}_p)^{-,-}.\end{aligned}$$

Clearly (11) is a consequence of these formulas.

Corollary 7. *Assume (C) to be true. If k is totally real and $0\le j<d$ is even, then the multiplicity of the zero of $L_p^{(j)}(\mathbf{G}_{m/k},s)$ at $s=0$ is*

$$s(j):=\#\{\mathfrak{p}/p\colon [k_{\mathfrak{p}}(\mu_p):k_{\mathfrak{p}}]\mid(j-1)\}.$$

Proof. We have to show

$$\operatorname{rank}e_{1-j}((\mathfrak{o}_1')^\times\otimes\mathbf{Z}_p)=s(j).$$

On the one hand $p\ne2$ implies $\operatorname{rank}e_{1-j}(\mathfrak{o}_1^\times\otimes\mathbf{Z}_p)=0$ ([4], p. 285); on the other hand, according to the Dirichlet unit theorem, the normalized discrete valuations corresponding to the primes $\mathfrak{P}$ of k_1 above p induce a quasi-isomorphism of Δ-modules

$$(\mathfrak{o}_1')^\times/\mathfrak{o}_1^\times\otimes\mathbf{Z}_p\to\prod_{\mathfrak{P}/p}\mathbf{Z}_p\cong\prod_{\mathfrak{p}/p}\mathbf{Z}_p[\Delta/\Delta_{\mathfrak{p}}],$$

where we have set $\Delta_{\mathfrak{p}}:=\operatorname{Gal}(k_{\mathfrak{p}}(\mu_p)/k_{\mathfrak{p}})$. Since

$$e_{1-j}\mathbf{Z}_p[\Delta/\Delta_{\mathfrak{p}}]\cong\begin{cases}\mathbf{Z}_p & \text{if } \#\Delta_{\mathfrak{p}}\mid(j-1),\\ 0 & \text{otherwise,}\end{cases}$$

we consequently get

$$\operatorname{rank}e_{1-j}((\mathfrak{o}_1')^\times\otimes\mathbf{Z}_p)=\sum_{\mathfrak{p}/p}\operatorname{rank}e_{1-j}\mathbf{Z}_p[\Delta/\Delta_{\mathfrak{p}}]=s(j).\quad \text{q.e.d.}$$

It is a conjecture of Coates/Lichtenbaum ([6] Conjecture 2.2) that the above corollary is correct without the assumption that (C) holds true. Using transcendental methods Greenberg ([9]) has proved this in case that k is real abelian over $\mathbf{Q}$. In fact his proof shows that the pairing $\langle\,,\,\rangle_p$ is nondegenerate in certain situations (compare also [8]); we shall discuss this in another paper.

III. Abelian Varieties

§6. The Basic Iwasawa Module

Let A be an abelian variety over k of dimension $\dim A>0$, let $\mathcal{A}$ be its Néron model over $\mathfrak{o}$, and let $\mathcal{A}^0$ be the connected component of $\mathcal{A}$. We recall that $\mathcal{A}$

is of finite type over $\mathfrak{o}$. Again our task is to study the groups $H^i(\mathfrak{o}_\infty, \mathcal{A}(p))$, and again we have to impose restrictions on p. Firstly throughout Part III we always assume the following condition to be fulfilled.

Hypothesis. A has good reduction at all primes of k above p.

For brevity, we say A is ordinary at p, if A has ordinary good reduction at all primes of k above p. In § 8 we will introduce additional hypotheses on p.

Remark 1. i) $\mathcal{A}_{p^j}$ is flat and quasi-finite over $\mathfrak{o}$; in particular, $\mathcal{A}(p)$ is an ind-p-group over $\mathfrak{o}$.

ii) $\mathcal{A} \underset{\mathfrak{o}}{\times} \mathfrak{o}_n$ is the Néron model of $A \underset{k}{\times} k_n$ over $\mathfrak{o}_n$.

The duality theory for abelian varieties will be of fundamental importance to us. Let $\tilde{A}$ be the dual abelian variety over k ($\tilde{A}$ represents the sheaf $Ext^1_{k_{fppf}}(A, \mathbb{G}_m)$), and let $\tilde{\mathcal{A}}$ be the Néron model of $\tilde{A}$ over $\mathfrak{o}$. Since A and $\tilde{A}$ are k-isogenous, $\tilde{A}$ has good reduction at all primes above p too.

Remark 2. If A is ordinary at p, so, too, is $\tilde{A}$.

Proposition 3 (Artin/Mazur). i) *$H^i(\mathfrak{o}, \mathcal{A}_{p^j})$ is finite for all $i \geq 0$ and $=0$ for $i > 3$;*

ii) *the cup-product induces a complete duality*

$$H^i(\mathfrak{o}, \mathcal{A}_{p^j}) \times H^{3-i}(\mathfrak{o}, \tilde{\mathcal{A}}_{p^j}) \to \mathbb{Q}/\mathbb{Z}.$$

Proof. [17] (7.3).

Lemma 4. i) *$H^0(\mathfrak{o}, \mathcal{A}(p))$ is finite;*

ii) *$H^i(\mathfrak{o}, \mathcal{A}(p))$ is of cofinite type for $i = 1, 2$;*

iii) *$H^i(\mathfrak{o}, \mathcal{A}(p)) = 0$ for $i \geq 3$.*

Proof. i) is clear. ii) See [17] (6.8) for the case $i = 1$. But the same method of proof shows that $\varprojlim H^1(\mathfrak{o}, \tilde{\mathcal{A}}_{p^j})$ is a finitely generated $\mathbb{Z}_p$-module. Because of Proposition 3 ii) this implies that $H^2(\mathfrak{o}, \mathcal{A}(p)) = \varinjlim H^2(\mathfrak{o}, \mathcal{A}_{p^j})$ is of cofinite type. iii) This is a consequence of Proposition 3 using $\varprojlim H^0(\mathfrak{o}, \tilde{A}_{p^j}) = 0$.

Proposition 5. i) *$H_0(\mathcal{A}(p))$ is finite;*

ii) *$H_1(\mathcal{A}(p))$ is a finitely generated $\mathbb{Z}_p[[\Gamma]]$-module;*

iii) *$H_i(\mathcal{A}(p)) = 0$ for $i \geq 3$.*

Proof. i) [12]. ii) [17] (6.4) and (5.3); but it can also be deduced from the descent diagram in the next paragraph. iii) Using $H^i(\mathfrak{o}_\infty, \mathcal{A}(p)) = \varinjlim H^i(\mathfrak{o}_n, \mathcal{A}(p))$ this follows from Lemma 4 iii).

Mazur has conjectured that $H_1(\mathcal{A}(p))$ is even a $\mathbb{Z}_p[[\Gamma]]$-torsion module, if A is ordinary at p; he has some weak partial results in this direction ([17] (6.9) and [19] III.9). Without the assumption that A is ordinary at p this is certainly false ([28]). We now investigate the situation for $H_2(\mathcal{A}(p))$.

For j big enough, the sheaf theoretic image $\bar{\mathcal{A}}$ of $\mathcal{A}_{/\mathfrak{o}_{fpqf}} \xrightarrow{p^j} \mathcal{A}_{/\mathfrak{o}_{fpqf}}$ is independent of j. In the following we always assume that j is "big enough"! We then have the exact diagram

$$\begin{tikzcd} & & & 0 \arrow[d] & \\ 0 \arrow[r] & \mathcal{A}_{p^j/\mathfrak{o}_{fpqf}} \arrow[r] & \mathcal{A}_{/\mathfrak{o}_{fpqf}} \arrow[r, "\alpha_j"] \arrow[dr, "p^j"'] & \bar{\mathcal{A}} \arrow[r] \arrow[d, "\subseteq"] & 0 \\ & & & \mathcal{A}_{/\mathfrak{o}_{fpqf}}, & \end{tikzcd}$$

where α_j denotes the induced epimorphism (compare [17], p. 201).

Lemma 6. *We have an exact sequence*

$$0 \to A(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^1(\mathfrak{o}, \mathcal{A}(p)) \to H^1(\mathfrak{o}, \mathcal{A})(p) \to 0.$$

Proof. From the commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}_{p^{j+1}/\mathfrak{o}_{fpqf}} \arrow[r] & \mathcal{A}_{/\mathfrak{o}_{fpqf}} \arrow[r, "\alpha_{j+1}"] & \bar{\mathcal{A}} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{A}_{p^j/\mathfrak{o}_{fpqf}} \arrow[r] \arrow[u, "\subseteq"] & \mathcal{A}_{/\mathfrak{o}_{fpqf}} \arrow[r, "\alpha_j"] \arrow[u, equal] & \bar{\mathcal{A}} \arrow[r] \arrow[u, "p"'] & 0 \end{tikzcd}$$

we derive the commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \bar{\mathcal{A}}(\mathfrak{o})/p^{j+1}\mathcal{A}(\mathfrak{o}) \arrow[r] & H^1(\mathfrak{o}, \mathcal{A}_{p^{j+1}}) \arrow[r] & H^1(\mathfrak{o}, \mathcal{A})(p) \arrow[r, "\alpha_{j+1}"] & H^1(\mathfrak{o}, \bar{\mathcal{A}})(p) \\ 0 \arrow[r] & \bar{\mathcal{A}}(\mathfrak{o})/p^j\mathcal{A}(\mathfrak{o}) \arrow[r] \arrow[u, "p"'] & H^1(\mathfrak{o}, \mathcal{A}_{p^j}) \arrow[r] \arrow[u] & H^1(\mathfrak{o}, \mathcal{A})(p) \arrow[r, "\alpha_j"] \arrow[u, equal] & H^1(\mathfrak{o}, \bar{\mathcal{A}})(p). \arrow[u, "p"'] \end{tikzcd}$$

The assertion follows now by passing to the direct limit with respect to j. One only has to observe

$$\varinjlim \bar{\mathcal{A}}(\mathfrak{o})/p^j\mathcal{A}(\mathfrak{o}) = \varinjlim \mathcal{A}(\mathfrak{o})/p^j\mathcal{A}(\mathfrak{o}) = \mathcal{A}(\mathfrak{o}) \otimes \mathbb{Q}_p/\mathbb{Z}_p = A(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p,$$

because $\mathcal{A}(\mathfrak{o})/\bar{\mathcal{A}}(\mathfrak{o})$ is a finite p-group.

Lemma 7. *If $H^1(\mathfrak{o}, \mathcal{A})(p)$ is finite, then we have:*

i) $H^2(\mathfrak{o}, \mathcal{A}(p)) = (\tilde{\mathcal{A}}^0(\mathfrak{o}) \otimes \mathbb{Z}_p)^*$;

ii) $\operatorname{corank} H^1(\mathfrak{o}, \mathcal{A}(p)) = \operatorname{corank} H^2(\mathfrak{o}, \mathcal{A}(p)) = \operatorname{rank}_{\mathbb{Z}} A(k)$.

Proof. i) From the commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}_{p^{j+1}/\mathfrak{o}_{fpqf}} \arrow[r] \arrow[d] & \mathcal{A}_{/\mathfrak{o}_{fpqf}} \arrow[r, "\alpha_{j+1}"] \arrow[d, "p"] & \bar{\mathcal{A}} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{A}_{p^j/\mathfrak{o}_{fpqf}} \arrow[r] & \mathcal{A}_{/\mathfrak{o}_{fpqf}} \arrow[r, "\alpha_j"] & \bar{\mathcal{A}} \arrow[r] & 0 \end{tikzcd}$$

we derive the commutative exact diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \tilde{\mathscr{A}}(\mathfrak{o})/p^{j+1}\mathscr{A}(\mathfrak{o}) & \longrightarrow & H^1(\mathfrak{o}, \mathscr{A}_{p^{j+1}}) & \longrightarrow & H^1(\mathfrak{o}, \mathscr{A})(p) \\ & \downarrow & & \downarrow & & \downarrow p \\ 0 \longrightarrow & \tilde{\mathscr{A}}(\mathfrak{o})/p^{j}\mathscr{A}(\mathfrak{o}) & \longrightarrow & H^1(\mathfrak{o}, \mathscr{A}_{p^{j}}) & \longrightarrow & H^1(\mathfrak{o}, \mathscr{A})(p). \end{array}$$

Passing to the projective limit and applying Proposition 3ii) results in

$$H^2(\mathfrak{o}, \tilde{\mathscr{A}}(p))^* = \varprojlim H^1(\mathfrak{o}, \mathscr{A}_{p^j}) = \varprojlim \tilde{\mathscr{A}}(\mathfrak{o})/p^j\mathscr{A}(\mathfrak{o}).$$

Since $\mathscr{A}(\mathfrak{o})/\tilde{\mathscr{A}}(\mathfrak{o})$ is a finite p-group, we have $\varprojlim p^j\mathscr{A}(\mathfrak{o})/p^j\tilde{\mathscr{A}}(\mathfrak{o})=0$ and therefore

$$\varprojlim \tilde{\mathscr{A}}(\mathfrak{o})/p^j\mathscr{A}(\mathfrak{o}) = \varprojlim \tilde{\mathscr{A}}(\mathfrak{o})/p^j\tilde{\mathscr{A}}(\mathfrak{o}) = \tilde{\mathscr{A}}(\mathfrak{o})\otimes \mathbb{Z}_p$$

(according to Mordell and Weil, the group $A(k) \geqq \tilde{\mathscr{A}}(\mathfrak{o})$ is finitely generated). Finally, it is easy to see that $\tilde{\mathscr{A}}(\mathfrak{o})/\mathscr{A}^0(\mathfrak{o})$ is finite of order prime to p (compare [17], p. 200/201). Thus we get

$$H^2(\mathfrak{o}, \tilde{\mathscr{A}}(p))^* = \tilde{\mathscr{A}}(\mathfrak{o})\otimes\mathbb{Z}_p = \mathscr{A}^0(\mathfrak{o})\otimes\mathbb{Z}_p.$$

Because A and $\tilde{A}$ are k-isogenous, the group $H^1(\mathfrak{o}, \tilde{\mathscr{A}})(p)$ is finite if and only if $H^1(\mathfrak{o}, \mathscr{A})(p)$ is finite. The assertion follows now by reason of symmetry.

ii) This is a consequence of i) and Lemma 6 using

$$\operatorname{rank}_{\mathbb{Z}} \tilde{\mathscr{A}}^0(\mathfrak{o}) = \operatorname{rank}_{\mathbb{Z}} \tilde{A}(k) = \operatorname{rank}_{\mathbb{Z}} A(k).$$

Remark 8. $H^1(\mathfrak{o}, \mathscr{A})(p)$ is finite if and only if the p-primary component of the Tate-Šafarevič group of $A_{/k}$ is finite ([17], Appendix).

Proposition 9. *If $H_1(\mathscr{A}(p))$ is a $\mathbb{Z}_p[\![\Gamma]\!]$-torsion module and if the groups $H^1(\mathfrak{o}_n, \mathscr{A})(p)$ are finite for all $n\in\mathbb{N}$, then $H_2(\mathscr{A}(p))=0$.*

Proof. We deduce from Proposition 5i) and [17] (6.11) that $A(k_\infty)$ and therefore also $\tilde{A}(k_\infty)$ are finitely generated. Consequently we get $\varprojlim \tilde{\mathscr{A}}^0(\mathfrak{o}_n)\otimes\mathbb{Z}_p=0$, where the projective limit is taken with respect to the norm maps. Lemma 7i) now implies

$$H^2(\mathfrak{o}_\infty, \mathscr{A}(p)) = \varinjlim H^2(\mathfrak{o}_n, \mathscr{A}(p)) = (\varprojlim \tilde{\mathscr{A}}^0(\mathfrak{o}_n)\otimes\mathbb{Z}_p)^* = 0.$$

Remark. Using the descent diagram which we shall establish in the next paragraph one can show: If A is ordinary at p and if $H_2(\mathscr{A}(p))$ vanishes, then $H_1(\mathscr{A}(p))$ is a (finitely generated) $\mathbb{Z}_p[\![\Gamma]\!]$-torsion module.

Let $0\leqq j<d$ be an integer such that $e_{j-1}H_1(\mathscr{A}(p))$ is a $\mathbb{Z}_p[\![\Gamma]\!]$-torsion module; we call

$$L_p^{(j)}(A, s) := F_{e_{j-1}H_1(\mathscr{A}(p))}((1+p^e)^{1-s}-1) \quad (s\in\mathbb{Z}_p)$$

the j-th *Iwasawa L-function* of $A_{/k}$ (with respect to p). The rest of the paper is devoted to the study of $L_p^{(1)}(A, s)$ at $s=1$.

Remarks. 1) The substitution of variables we have made in the definition of the Iwasawa L-functions here and in § 4 follows the principle

$$t=(1+p^e)^{g-s}-1,$$

where $g=0$ or 1 is the "dimension" we spoke about in the introduction. This leads to the right analogies with the complex L-functions.

2) If A is ordinary at p and if $e_0H_1(\mathscr{A}(p))$ is a $\mathbb{Z}_p[\![\Gamma]\!]$-torsion module, then $L_p^{(1)}(A, s)$ has a functional equation with respect to $s\mapsto 2-s$ ([17] (7.8)).

§ 7. *The Descent Diagram*

The cohomological descent formalism we used in Part II was based on (3.3). But this lemma becomes false in the present context. Therefore we have to develop a more refined descent technique. As it is shown in the appendix there exist two spectral sequences

$$H^i(G, H^j(\mathfrak{o}_\infty, \mathscr{A}(p))) \Rightarrow E^{i+j} \qquad \text{(I)}$$

and

$$H^i(\mathfrak{o}, R^j\pi_G\mathscr{A}(p)) \Rightarrow E^{i+j} \qquad \text{(II)}$$

converging to the same abutment. The following fact enables us to use these spectral sequences for our purposes.

Lemma 1. $\mathscr{A}(p)_{/\mathfrak{o}_{fpqf}} = \pi_G(\mathscr{A}(p)_{/\mathfrak{o}_{fpqf}})$.

Proof. Put $G_n := \mathrm{Gal}(k_n/k)$, and let $\sigma_n\in G_n$ be a generator. Since

$$\pi_G(\mathscr{A}(p)_{/\mathfrak{o}_{fpqf}}) = \varinjlim_n \varinjlim_j \pi_{G_n}(\mathscr{A}_{p^j/\mathfrak{o}_{fpqf}})$$

holds true and since π_{G_n} is left exact, it suffices to prove

$$\mathscr{A}_{/\mathfrak{o}_{fpqf}} = \pi_{G_n}(\mathscr{A}_{/\mathfrak{o}_{fpqf}}).$$

The Galois group G_n acts in a natural way on the Weil restriction $R_{\mathfrak{o}_n/\mathfrak{o}}\mathscr{A}_{/\mathfrak{o}_n}$. The $\mathfrak{o}$-group scheme

$$\mathscr{B} := \ker(R_{\mathfrak{o}_n/\mathfrak{o}}\mathscr{A}_{/\mathfrak{o}_n} \xrightarrow{\sigma_n-1} R_{\mathfrak{o}_n/\mathfrak{o}}\mathscr{A}_{/\mathfrak{o}_n})$$

is separated and of finite type over $\mathfrak{o}$ and represents the sheaf $\pi_{G_n}(\mathscr{A}_{/\mathfrak{o}_{fpqf}})$. Furthermore, there exists a canonical exact sequence of $\mathfrak{o}$-group schemes

$$0\to\mathscr{A}\xrightarrow{\iota}\mathscr{B}.$$

Over $\mathrm{Spec}(\mathfrak{o})\setminus\{\mathfrak{p}/p\}$, ι is an isomorphism ([22] II 1.4). But there exists an open subscheme $U\subseteq\mathrm{Spec}(\mathfrak{o})$ such that U contains $\{\mathfrak{p}/p\}$ and $\mathscr{A}_{/U}$ is an abelian scheme. Over U, ι is a closed immersion (EGA IV 18.12.6). Thus ι is a closed immersion which induces an isomorphism in the generic fibres. Since $\mathscr{A}$ is flat over $\mathfrak{o}$, this implies that ι identifies $\mathscr{A}$ with the scheme-theoretic closure of the generic fibre of $\mathscr{B}$ in $\mathscr{B}$ (in the sense of EGA IV 2.8.5). In particular, we have $\mathscr{A}_{/\mathfrak{o}_f p_q f} = \mathscr{B}_{/\mathfrak{o}_f p_q f}$. q.e.d.

Under the assumption that $H^2(\mathfrak{o}_\infty, \mathscr{A}(p))$ vanishes we establish now the exact "descent diagram":

$$\begin{tikzcd}
& & 0 \arrow[d] & & \\
& & H^1(\mathfrak{o}, \mathscr{A}(p)) \arrow[d] \arrow[dr, dashed, "\alpha"] & & \\
0 \arrow[r] & H^1(G, A(k_\infty)(p)) \arrow[r] & E^1 \arrow[r] \arrow[d] & H^0(G, H^1(\mathfrak{o}_\infty, \mathscr{A}(p))) \arrow[r] & 0 \\
& & H^0(\mathfrak{o}, R^1\pi_G \mathscr{A}(p)) \arrow[d] & & \\
& & H^2(\mathfrak{o}, \mathscr{A}(p)) \arrow[d] \arrow[dr, dashed, "\beta"] & & \\
& & E^2 \arrow[r, equal] & H^1(G, H^1(\mathfrak{o}_\infty, \mathscr{A}(p))). &
\end{tikzcd}$$

The vertical sequence is the exact sequence of lower terms of (II), if one takes Lemma 1 into consideration. The horizontal sequence and the identification in the bottom row are derived from (I) using the above assumption and the fact that the cohomological p-dimension of G is 1. By α and β we simply denoted the induced maps.

The key problem is the computation of $H^0(\mathfrak{o}, R^1\pi_G \mathscr{A}(p))$.

Proposition 2. *If A is ordinary at p, then $H^0(\mathfrak{o}, R^1\pi_G \mathscr{A}(p))$ is finite of order*

$$\Big(\prod_{\mathfrak{p}/p} \# \mathscr{A}(\kappa_\mathfrak{p})(p)\Big)^2,$$

where $\kappa_\mathfrak{p}$ is the residue class field of $\mathfrak{p}$.

First we prove a more general lemma. Put $\hat{k}_{\mathfrak{p},\infty} := \hat{k}_\mathfrak{p}(\mu(p))$ and $G_\mathfrak{p} := \mathrm{Gal}(\hat{k}_{\mathfrak{p},\infty}/\hat{k}_\mathfrak{p})$.

Lemma 3.

$$\begin{aligned} H^0(\mathfrak{o}, R^1\pi_G \mathscr{A}(p)) &= \bigoplus_{\mathfrak{p}/p} \ker\big(H^1_{et}(\hat{k}_\mathfrak{p}, A) \to H^1_{et}(\hat{k}_{\mathfrak{p},\infty}, A)\big)(p) \\ &= \bigoplus_{\mathfrak{p}/p} H^1(G_\mathfrak{p}, A(\hat{k}_{\mathfrak{p},\infty}))(p). \end{aligned}$$

Proof. According to the appendix there exists a spectral sequence

$$H^i\Big(G, \bigoplus_{\mathfrak{P}/p} H^j_\cdot(\mathfrak{o}_{\infty,\mathfrak{P}}, \mathscr{A}(p))\Big) \Rightarrow E^{i+j}_{(p)} \tag{III}$$

and an exact sequence

$$E^1_{(p)} \to H^0(\mathfrak{o}, R^1\pi_G \mathscr{A}(p)) \to \bigoplus_{\mathfrak{p}/p} H^2_\cdot(\mathfrak{o}_\mathfrak{p}, \mathscr{A}(p)) \to E^2_{(p)} \tag{IV}$$

(which is derived from a second spectral sequence converging also to $E^\cdot_{(p)}$ – use Lemma 1); here $\mathfrak{o}_{\infty,\mathfrak{P}}$ denotes the Henselization of $\mathfrak{o}_\infty$ in $\mathfrak{P}/p$. If we apply (3.5) to (III), then we get $E^1_{(p)} = 0$ and $E^2_{(p)} = H^0(G, \bigoplus_{\mathfrak{P}/p} H^2_\cdot(\mathfrak{o}_{\infty,\mathfrak{P}}, \mathscr{A}(p)))$. Inserting this into (IV) leads to

$$\begin{aligned} H^0(\mathfrak{o}, R^1\pi_G \mathscr{A}(p)) &= \ker\Big(\bigoplus_{\mathfrak{p}/p} H^2_\cdot(\mathfrak{o}_\mathfrak{p}, \mathscr{A}(p)) \to \bigoplus_{\mathfrak{P}/p} H^2_\cdot(\mathfrak{o}_{\infty,\mathfrak{P}}, \mathscr{A}(p))\Big) \\ &= \bigoplus_{\mathfrak{p}/p} \ker\big(H^2_\cdot(\mathfrak{o}_\mathfrak{p}, \mathscr{A}(p)) \to H^2_\cdot(\mathfrak{o}_\mathfrak{p}(\mu(p)), \mathscr{A}(p))\big). \end{aligned}$$

Because of $H^2_\cdot(\mathfrak{o}_\mathfrak{p}, \mathscr{A}(p)) = H^1_{et}(\hat{k}_\mathfrak{p}, A)(p)$ for $\mathfrak{p}/p$ ([17] (5.2)) we thus have proved the assertion.

In order to prove Proposition 2 it therefore remains to show that $H^1(G_\mathfrak{p}, A(\hat{k}_{\mathfrak{p},\infty}))(p)$ is finite for $\mathfrak{p}/p$ of order $(\# \mathscr{A}(\kappa_\mathfrak{p})(p))^2$. We fix a prime $\mathfrak{p}$ above p in the following. First we introduce some notations. Let $L/\hat{k}_\mathfrak{p}$, resp. $\hat{L}$, be the maximal unramified extension, resp. its completion and set $L_\infty := L(\mu(p))$, $\hat{L}_\infty := \hat{L}(\mu(p))$, and $G^0_\mathfrak{p} := \mathrm{Gal}(L_\infty/L) = \mathrm{Gal}(\hat{L}_\infty/\hat{L})$:

$$\begin{tikzcd}
& L_\infty \arrow[r, dash] \arrow[d, dash, "G^0_\mathfrak{p}"] & \hat{L}_\infty \arrow[d, dash] \\
\hat{k}_{\mathfrak{p},\infty} \arrow[ur, dash] \arrow[d, dash, "G_\mathfrak{p}"'] & L \arrow[r, dash] & \hat{L} \\
\hat{k}_\mathfrak{p} \arrow[ur, dash] & &
\end{tikzcd}$$

Let $\hat{R}$, resp. $\hat{R}_\infty$, be the ring of integers in $\hat{L}$, resp. $\hat{L}_\infty$, and define $\bar{\kappa}_\mathfrak{p}$ to be the residue class field of $\hat{R}$, which is an algebraic closure of $\kappa_\mathfrak{p}$. Because of

$$H^i(\mathrm{Gal}(L/\hat{k}_\mathfrak{p}), A(L)) = H^i(\mathrm{Gal}(L_\infty/\hat{k}_{\mathfrak{p},\infty}), A(L_\infty)) = 0$$

for $i > 0$ ([17] (4.2) and (4.4) and the fact that the considered Galois groups are pro-cyclic) we easily deduce from the Hochschild-Serre spectral sequences that

$$\begin{aligned} H^1(G_\mathfrak{p}, A(\hat{k}_{\mathfrak{p},\infty})) &= H^1(\mathrm{Gal}(L_\infty/\hat{k}_\mathfrak{p}), A(L_\infty)) \\ &= H^0(\mathrm{Gal}(L/\hat{k}_\mathfrak{p}), H^1(G^0_\mathfrak{p}, A(L_\infty))). \end{aligned} \tag{1}$$

An appropriate generalization of [17] (5.1 ii)) (which is proved exactly in the same way; the finite extensions of L are namely still discretely valuated) shows

$$H^1(G^0_\mathfrak{p}, A(L_\infty)) = H^1(G^0_\mathfrak{p}, A(\hat{L}_\infty)).$$

This last cohomology group we now compute with the help of the exact sequence

$$0 \to \hat{\mathscr{A}}(\hat{R}_\infty) \to A(\hat{L}_\infty) \to \mathscr{A}(\bar{\kappa}_\mathfrak{p}) \to 0 \tag{2}$$

given by the reduction map; here $\hat{\mathscr{A}}$ is the formal completion of $\mathscr{A}$ over $\hat{\mathfrak{o}}_\mathfrak{p}$. Since A is ordinary at p, $\hat{\mathscr{A}}$ is isomorphic, over $\hat{R}$, to the (dim A)-fold product of the formal multiplicative group $\hat{\mathbb{G}}_m$ ([17](4.27)). In particular, we have an isomorphism of $G_{\mathfrak{p}}^0$-modules

$$\hat{\mathscr{A}}(\hat{R}_\infty) \cong \hat{\mathbb{G}}_m(\hat{R}_\infty)^{\dim A}.$$

Lemma.

$$H^i(G_{\mathfrak{p}}^0, \hat{\mathbb{G}}_m(\hat{R}_\infty))(p) = \begin{cases} \mathbb{Q}_p/\mathbb{Z}_p & \text{if } i=1, \\ 0 & \text{if } i>1. \end{cases}$$

Proof. We consider the exact sequences

$$0 \to \hat{\mathbb{G}}_m(\hat{R}_\infty) \to \hat{R}_\infty^\times \to \bar{\kappa}_{\mathfrak{p}}^\times \to 0$$

and

$$0 \to \hat{R}_\infty^\times \to \hat{L}_\infty^\times \xrightarrow{v} \mathbb{Q}' \to 0,$$

where v is that valuation of $\hat{L}_\infty$ whose restriction to $\hat{L}$ is normalized and $\mathbb{Q}'$ denotes the additive group of rational numbers whose denominators divide $p^m \cdot [L(\mu_p): L]$ for some $m \in \mathbb{N}$. The groups $\bar{\kappa}_{\mathfrak{p}}^\times$ and $\mathbb{Q}'$ are uniquely p-divisible and therefore p-cohomologically trivial. According to [15] Lemma 1 also the $G_{\mathfrak{p}}^0$-module $\hat{L}_\infty^\times$ is cohomologically trivial. Thus the corresponding cohomology sequences imply

$$H^1(G_{\mathfrak{p}}^0, \hat{\mathbb{G}}_m(\hat{R}_\infty))(p) = H^1(G_{\mathfrak{p}}^0, \hat{R}_\infty^\times)(p) = (\mathbb{Q}'/\mathbb{Z})(p) = \mathbb{Q}_p/\mathbb{Z}_p$$

and

$$H^i(G_{\mathfrak{p}}^0, \hat{\mathbb{G}}_m(\hat{R}_\infty))(p) = H^i(G_{\mathfrak{p}}^0, \hat{R}_\infty^\times)(p) = 0 \quad \text{for } i>1.$$

Using this lemma we see that the p-part of the cohomology sequence corresponding to (2) is the exact sequence of $\mathrm{Gal}(L/\hat{k}_{\mathfrak{p}})$-modules

$$0 \to N := (\mathbb{Q}_p/\mathbb{Z}_p)^{\dim A} \to H^1(G_{\mathfrak{p}}^0, A(L_\infty))\, A(L_\infty))(p) \to \mathrm{Hom}(G_{\mathfrak{p}}^0, \mathscr{A}(\bar{\kappa}_{\mathfrak{p}})(p)) \to 0, \quad (3)$$

where the Galois action on N is given by the twist matrix $U \in GL(\dim A, \mathbb{Z}_p)$ of A at $\mathfrak{p}$ (see [17], p. 216 or [15]). Because of $\det(U-1) \neq 0$ ([17](4.38)) the group $H^0(\mathrm{Gal}(L/\hat{k}_{\mathfrak{p}}), N)$ is finite of order $|\det(U-1)|_p^{-1}$ and $H^1(\mathrm{Gal}(L/\hat{k}_{\mathfrak{p}}), N)$ vanishes. Therefore, from (1) and the cohomology sequence corresponding to (3) we get that $H^1(G_{\mathfrak{p}}, A(\hat{k}_{\mathfrak{p},\infty}))(p)$ is finite of order

$$\begin{aligned} &|\det(U-1)|_p^{-1} \cdot \# H^0(\mathrm{Gal}(L/\hat{k}_{\mathfrak{p}}), \mathrm{Hom}(G_{\mathfrak{p}}^0, \mathscr{A}(\bar{\kappa}_{\mathfrak{p}})(p))) \\ &= |\det(U-1)|_p^{-1} \cdot \# \mathscr{A}(\kappa_{\mathfrak{p}})(p). \end{aligned}$$

But with the help of [23](§21, Theorem 4) and [17](4.34) (the constant c appearing there is a p-adic unit as is seen from the proof using $\det U \in \mathbb{Z}_p^\times$) one can easily show that

$$|\det(U-1)|_p^{-1} = \# \mathscr{A}(\kappa_{\mathfrak{p}})(p). \quad \text{q.e.d.}$$

§ 8. *An Analogue of the Conjecture of Birch and Swinnerton-Dyer*

In this paragraph we always assume that the following conditions are fulfilled.

Hypothesis (H): A is ordinary at $p (\neq 2)$,
$H^1(\mathfrak{o}_n, \mathscr{A})(p)$ is finite for all $n \geq 0$,
$H_1(\mathscr{A}(p))$ is a $\mathbb{Z}_p[\![\Gamma]\!]$-torsion module, and
$e_0 H_1(\mathscr{A}(p))$ has no finite Γ-submodules $\neq 0$.

Because of (6.9) we have the descent diagram at our disposal. First we collect the information which we can extract from that diagram. Let

$$f\colon H^0(G, H^1(\mathfrak{o}_\infty, \mathscr{A}(p))) \to H^1(G, H^1(\mathfrak{o}_\infty, \mathscr{A}(p)))$$

be the homomorphism induced by the identity map on $H^1(\mathfrak{o}_\infty, \mathscr{A}(p))$.

Lemma 1. i) *α and β are quasi-isomorphisms, and β is surjective;*
ii) $\operatorname{corank} H^0(G, H^1(\mathfrak{o}_\infty, \mathscr{A}(p))) = \operatorname{rank}_{\mathbb{Z}} A(k)$;
iii) $$\frac{\# \ker \mathrm{Div}\, \beta}{\# \ker \mathrm{Div}\, \alpha} \cdot \# \ker f_{\mathrm{Div}} = \prod_{i \geq 0} \# H^i(\mathfrak{o}, \mathscr{A}(p))_{\mathrm{Div}}^{(-1)^{i+1}} \cdot \Big(\prod_{\mathfrak{v}/p} \# \mathscr{A}(\kappa_{\mathfrak{v}})(p)\Big)^2.$$

Proof. Since the groups $H^1(G, A(k_\infty)(p))$ and $H^0(\mathfrak{o}, R^1 \pi_G \mathscr{A}(p))$ are finite (see (6.5 i)) and (7.2)), it follows from the descent diagram that α is a quasi-isomorphism and that β has finite kernel. Using that $H_1(\mathscr{A}(p))$ is a finitely generated $\mathbb{Z}_p[\![\Gamma]\!]$-torsion module we see that

$$\begin{aligned} \operatorname{corank} H^1(\mathfrak{o}, \mathscr{A}(p)) &= \operatorname{corank} H^0(G, H^1(\mathfrak{o}_\infty, \mathscr{A}(p))) \\ &= \operatorname{corank} H^1(G, H^1(\mathfrak{o}_\infty, \mathscr{A}(p))). \end{aligned}$$

But according to (6.7 ii)) we have

$$\operatorname{corank} H^1(\mathfrak{o}, \mathscr{A}(p)) = \operatorname{corank} H^2(\mathfrak{o}, \mathscr{A}(p)) = \operatorname{rank}_{\mathbb{Z}} A(k).$$

This implies the assertion ii) and that β is a quasi-isomorphism. Because $e_0 H_1(\mathscr{A}(p))$ has no finite Γ-submodules $\neq 0$, the group $H^1(G, H^1(\mathfrak{o}_\infty, \mathscr{A}(p)))$ is divisible. Therefore β must be surjective. An easy diagram chase finally results in assertion iii).

Now let ρ be the multiplicity of the zero of $L_p^{(1)}(A, s)$ at $s=1$. We have

$$\rho = \rho(e_0 H_1(\mathscr{A}(p)))$$

and

$$[L_p^{(1)}(A, s) \cdot (s-1)^{-\rho}]_{s=1} = (-\log_p(1+p^e))^\rho \cdot c(e_0 H_1(\mathscr{A}(p))).$$

From Lemma 1, (2.3), and the final remark in § 2 we get

$$\rho \geq \operatorname{rank}_{\mathbb{Z}} A(k), \quad (1)$$

where equality holds if and only if f is a quasi-isomorphism: in that case

$$|c(e_0 H_1(\mathscr{A}(p)))|_p^{-1} = \frac{\# \ker f}{\# \operatorname{coker} f} = \frac{\# \ker \operatorname{Div}(f \circ \alpha)}{\# \ker \operatorname{Div} \beta} \cdot \prod_{i \geq 0} \# H^i(\mathfrak{o}, \mathscr{A}(p))_{\mathrm{Div}}^{(-1)^{i+1}} \cdot \Big(\prod_{\mathfrak{p}/p} \# \mathscr{A}(\kappa_{\mathfrak{p}})(p)\Big)^2. \tag{2}$$

Next we define a pairing. For this purpose we consider the following sequence of maps:

$$\begin{tikzcd}
\tilde{A}(k) \otimes \mathbb{Z}_p & \operatorname{Hom}(A(k) \otimes \mathbb{Z}_p, \mathbb{Z}_p) \\
\cup\!\!\!\text{l} & \\
\tilde{\mathscr{A}}^0(\mathfrak{o}) \otimes \mathbb{Z}_p \arrow[d, equal] & \\
H^2(\mathfrak{o}, \mathscr{A}(p))^* & H^1(\mathfrak{o}, \mathscr{A}(p))^* \arrow[uuu] \\
H^1(G, H^1(\mathfrak{o}_\infty, \mathscr{A}(p)))^* \arrow[u, "\beta^*"] \arrow[r, "f^*"] & H^0(G, H^1(\mathfrak{o}_\infty, \mathscr{A}(p)))^* \arrow[u, "\alpha^*"']
\end{tikzcd}$$

The identification in the left column and the upper map in the right column are given by (6.7 i)) and (6.6). All maps besides eventually f^* are quasi-isomorphisms. Thus this sequence determines a unique pairing

$$\langle\ ,\ \rangle : A(k) \times \tilde{A}(k) \to \mathbb{Q}_p$$

which is nondegenerate if and only if f is a quasi-isomorphism; in that case

$$|\det \langle\ ,\ \rangle|_p^{-1} = \frac{\# \ker \operatorname{Div}(f \circ \alpha)}{\# \ker \operatorname{Div} \beta} \cdot |[\tilde{A}(k)_{\mathrm{Tor}} : \tilde{\mathscr{A}}^0(\mathfrak{o})_{\mathrm{Tor}}]|_p \tag{3}$$

holds true. Of course, our pairing depends on the chosen generator $\gamma \in \Gamma$. But we can easily avoid this by defining a canonical pairing

$$\langle\ ,\ \rangle_p : A(k) \times \tilde{A}(k) \to \mathbb{Q}_p$$

by $\langle a, b \rangle_p := \langle a, b \rangle \cdot \log_p(\kappa(\gamma))$. Combining (1)–(3) results in the following proposition.

Proposition 2. i) $\rho \geqq \operatorname{rank}_{\mathbb{Z}} A(k)$;

ii) $\rho = \operatorname{rank}_{\mathbb{Z}} A(k) \Leftrightarrow \langle\ ,\ \rangle_p$ *is nondegenerate; in that case*

$$\begin{aligned} &|[L_p^{(1)}(A, s) \cdot (s-1)^{-\rho}]_{s=1}|_p^{-1} \\ &= \prod_{i \geq 0} \# H^i(\mathfrak{o}, \mathscr{A}(p))_{\mathrm{Div}}^{(-1)^{i+1}} \cdot |[\tilde{A}(k)_{\mathrm{Tor}} : \tilde{\mathscr{A}}^0(\mathfrak{o})_{\mathrm{Tor}}]|_p^{-1} \\ &\quad \cdot |\det \langle\ ,\ \rangle_p|_p^{-1} \cdot \Big(\prod_{\mathfrak{p}/p} \# \mathscr{A}(\kappa_{\mathfrak{p}})(p)\Big)^2. \end{aligned}$$

We now want to interprete the Euler characteristic which appears in Proposition 2 in more "classical" terms. As usual we denote by $Ш_k(A)$ the Tate-Šafarevič group of $A_{/k}$. One of our assumptions means that $Ш_{k_n}(A)(p)$ is finite for all $n \geqq 0$ (compare (6.8)).

Lemma 3. $H^1(\mathfrak{o}, \mathscr{A}(p))_{\mathrm{Div}}$ *is canonically dual to* $H^1(\mathfrak{o}, \tilde{\mathscr{A}}^0)(p)_{\mathrm{Div}}$.

Proof (without assuming (H)). Since $\tilde{\mathscr{A}}^0_{/\mathfrak{o}_{fpqf}} \xrightarrow{p} \tilde{\mathscr{A}}^0_{/\mathfrak{o}_{fpqf}}$ is an epimorphism, we have the commutative diagram

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & H^1(\mathfrak{o}, \tilde{\mathscr{A}}^0)/p^{j+1} H^1(\mathfrak{o}, \tilde{\mathscr{A}}^0) \arrow[r] \arrow[d] & H^2(\mathfrak{o}, (\tilde{\mathscr{A}}^0)_{p^{j+1}}) \arrow[r] \arrow[d] & H^2(\mathfrak{o}, \tilde{\mathscr{A}}^0)_{p^{j+1}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(\mathfrak{o}, \tilde{\mathscr{A}}^0)/p^j H^1(\mathfrak{o}, \tilde{\mathscr{A}}^0) \arrow[r] & H^2(\mathfrak{o}, (\tilde{\mathscr{A}}^0)_{p^j}) \arrow[r] & H^2(\mathfrak{o}, \tilde{\mathscr{A}}^0)_{p^j} \arrow[r] & 0.
\end{tikzcd}$$

Passing to the projective limit (with respect to j) leads to

$$H^1(\mathfrak{o}, \tilde{\mathscr{A}}^0)(p)_{\mathrm{Div}} = \operatorname{Tor}(\varprojlim H^2(\mathfrak{o}, (\tilde{\mathscr{A}}^0)_{p^j})).$$

On the other hand from (6.3) we get

$$\begin{aligned} (H^1(\mathfrak{o}, \mathscr{A}(p))_{\mathrm{Div}})^* &= ((\varinjlim H^1(\mathfrak{o}, \tilde{\mathscr{A}}_{p^j}))_{\mathrm{Div}})^* \\ &= \operatorname{Tor}(\varprojlim H^2(\mathfrak{o}, \tilde{\mathscr{A}}_{p^j})). \end{aligned}$$

Therefore it remains to show that the canonical map

$$\varprojlim H^2(\mathfrak{o}, (\tilde{\mathscr{A}}^0)_{p^j}) \to \varprojlim H^2(\mathfrak{o}, \tilde{\mathscr{A}}_{p^j})$$

is an isomorphism. But this is a consequence of the fact that the quotient sheaf

$$\tilde{\mathscr{A}}_{p^j}/(\tilde{\mathscr{A}}^0)_{p^j} = (\tilde{\mathscr{A}}/\tilde{\mathscr{A}}^0)_{p^j}$$

on $\mathfrak{o}_{fpqf}$ is, for j big enough, independent of j (compare the discussion preceding (6.6)).

Remark. (6.6) and [17] (appendix) imply that $Ш_k(A)(p)_{\mathrm{Div}}$ is the image of the map

$$H^1(\mathfrak{o}, \mathscr{A}^0)(p)_{\mathrm{Div}} \to H^1(\mathfrak{o}, \mathscr{A})(p)_{\mathrm{Div}} = H^1(\mathfrak{o}, \mathscr{A}(p))_{\mathrm{Div}}.$$

Taking this into consideration one easily derives from Lemma 3 that $Ш_k(A)(p)_{\mathrm{Div}}$ is canonically dual to $Ш_k(\tilde{A})(p)_{\mathrm{Div}}$.

Lemma 4.

$$\begin{aligned} &\prod_{i \geq 0} \# H^i(\mathfrak{o}, \mathscr{A}(p))_{\mathrm{Div}}^{(-1)^{i+1}} \cdot |[\tilde{A}(k)_{\mathrm{Tor}} : \tilde{\mathscr{A}}^0(\mathfrak{o})_{\mathrm{Tor}}]|_p^{-1} \\ &= \frac{\# Ш_k(A)(p)}{\# A(k)(p) \cdot \# \tilde{A}(k)(p)} \cdot \prod_{\mathfrak{p} \not\mid \infty} \# \pi_{\mathfrak{p}}(A)(p), \end{aligned}$$

where $\pi_{\mathfrak{p}}(A)$ *is the group of connected components of* $A \underset{\mathfrak{o}}{\times} \kappa_{\mathfrak{p}}$.

Proof (assuming only the finiteness of $H^1(\mathfrak{o}, \mathscr{A})(p)$). Because of (6.7i)) we have

$$\# H^2(\mathfrak{o}, \mathscr{A}(p))_{\mathrm{Div}} = \# \tilde{A}^0(\mathfrak{o})(p).$$

Define the sheaf $\mathscr{F}$ on $\mathfrak{o}_{fpqf}$ by the exact sequence

$$0 \to \tilde{\mathscr{A}}^0_{/\mathfrak{o}_{fpqf}} \to \tilde{\mathscr{A}}_{/\mathfrak{o}_{fpqf}} \to \mathscr{F} \to 0.$$

The exact sequence of finite groups

$$0 \to (\tilde{A}(k)/\tilde{\mathscr{A}}^0(\mathfrak{o}))(p) \to \mathscr{F}(\mathfrak{o})(p) \to H^1(\mathfrak{o}, \tilde{\mathscr{A}}^0)(p) \to Ш_k(\tilde{A})(p) \to 0$$

is established in [17] (appendix). Using $\mathscr{F}(\mathfrak{o}) = \oplus \pi_{\mathfrak{p}}(\tilde{A})$ and Lemma 3 we get

$$\# H^1(\mathfrak{o}, \mathscr{A}(p))_{\mathrm{Div}} = \# Ш_k(\tilde{A})(p) \cdot \prod_{\mathfrak{p} \nmid \infty} \# \pi_{\mathfrak{p}}(\tilde{A})(p) \cdot |[\tilde{A}(k) : \tilde{\mathscr{A}}^0(\mathfrak{o})]|_p.$$

The assertion now follows if we take into consideration the equalities $\# \pi_{\mathfrak{p}}(\tilde{A})(p) = \# \pi_{\mathfrak{p}}(A)(p)$ (see [30], §1 Satz 10, or SGA 7 IX §11) and $\# Ш_k(\tilde{A})(p) = \# Ш_k(A)(p)$ (see the above remark or [36]).

The combination of Proposition 2 and Lemma 4 implies our main result.

Theorem 5. *Assuming (H), the multiplicity of the zero of $L_p^{(1)}(A, s)$ at $s=1$ is $\geqq \mathrm{rank}_{\mathbb{Z}} A(k)$; equality holds if and only if $\langle\ ,\ \rangle_p$ is nondegenerate, in which case we have*

$$|[L_p^{(1)}(A, s) \cdot (s-1)^{-\mathrm{rank}\, A(k)}]_{s=1}|_p^{-1} = \frac{\# Ш_k(A)(p) \cdot |\det \langle\ ,\ \rangle_p|_p^{-1}}{\# A(k)(p) \cdot \# \tilde{A}(k)(p)} \cdot \prod_{\mathfrak{p} \nmid \infty} \# \pi_{\mathfrak{p}}(A)(p) \cdot \Big(\prod_{\mathfrak{p} | p} \# \mathscr{A}(\kappa_{\mathfrak{p}})(p)\Big)^2.$$

Conjecture. *The pairing $\langle\ ,\ \rangle_p$ is nondegenerate.*

Remarks. 1) Conjecturally our assumptions (*H*) are always fulfilled if A is ordinary at p. Thus the above theorem can be viewed as a general result. In any case, for the definition of the pairing $\langle\ ,\ \rangle_p$ and for that part of the theorem which is not concerned with the explicit formula we do not need the assumption that $e_0 H_1(\mathscr{A}(p))$ has no finite Γ-submodules $\neq 0$.

2) From the proofs one can easily see that

$$\frac{\det \langle\ ,\ \rangle_p}{\# \tilde{A}(k)(p)} \cdot \prod_{\mathfrak{p} \nmid \infty} \# \pi_{\mathfrak{p}}(A)(p) \cdot \Big(\prod_{\mathfrak{p} | p} \# \mathscr{A}(\kappa_{\mathfrak{p}})(p)\Big)^2$$

is a p-adic integer.

3) First one should observe the astonishing analogy between the formula in Theorem 5 and the formula which Birch and Swinnerton-Dyer conjecture for the leading coefficient of the Hasse-Weil L-function of A at $s=1$. But undoubtedly the connection is much deeper. Let $A_{/\mathbb{Q}}$ be a Weil curve and let p be a prime number such that A is ordinary at p. Then Mazur and Swinnerton-Dyer ([21]) associate an analytically defined p-adic L-function $L_p(A, s)$ with A and p. Assuming the conjecture of Birch and Swinnerton-Dyer they prove:

$A(\mathbb{Q})$ is finite if and only if $L_p(A, 1) \neq 0$, in which case

$$|L_p(A, 1) \cdot c^{-2}|_p^{-1} = \# Ш_{\mathbb{Q}}(A)(p) \cdot \left(\frac{\# \mathscr{A}(\mathbb{F}_p)(p)}{\# A(\mathbb{Q})(p)}\right)^2 \cdot \prod_l \# \pi_l(A)(p);$$

here $c \in \mathbb{Q}^\times$ is a constant given by the Weil parametrization of A which conjecturally is equal to 1 (loc. cit. §2.3 and §9.6). We immediately realize that the above expression is completely in agreement with the formula in our theorem. This supports the so-called main conjecture of that theory (loc. cit. §9.5 Conj. 3) which asserts

$$L_p(A, s) = L_p^{(1)}(A, s) \cdot u((1+p)^{1-s} - 1)$$

with an appropriate invertible power series $u(t) \in \mathbb{Z}_p[[t]]^\times$.

4) Because of its abstract definition the pairing $\langle\ ,\ \rangle_p$ is not computable in practice. There exist several suggestions for an analytically defined p-adic height (Bernardi, Gross, Néron). Unfortunately they lead to a canonical pairing $A(k) \times \tilde{A}(k) \to \mathbb{Q}_p$ only in case that A has complex multiplication (but see [39]). What is the relationship between our pairing $\langle\ ,\ \rangle_p$ and p-adic heights?

Finally we want to point out another situation to which our methods can be applied. We assume that k contains an imaginary quadratic field F and that $A_{/k}$ is an elliptic curve with complex multiplication by the ring of integers in F; let $p \geqq 3$ be a prime number such that $(p) = \mathfrak{p} \cdot \bar{\mathfrak{p}}$ splits in F and A has good (and therefore ordinary) reduction at all primes above $\mathfrak{p}$. Changing our notations we put $k_\infty := k(A(\mathfrak{p}))$, $\mathfrak{o}_\infty$ the ring of integers in k_∞, and $G := \mathrm{Gal}(k_\infty/k)$. Then we are led to consider the G-module $H^1(\mathfrak{o}_\infty, \mathscr{A}(\mathfrak{p}))$. This leads to the definition of Iwasawa L-functions which are algebraic analogues of the Hecke L-function associated with the grossencharacter of $A_{/k}$. Now, as has already been done in a different language in [5] and [26], it is possible to discuss the values of these Iwasawa L-functions at integer points using the methods developed in this paper. We restrict ourselves to giving some indications only:

1) We have an analogue of (3.3). Furthermore, the basic Iwasawa module can be interpreted as a certain Galois group (compare the remark preceding (3.7)). Thus it is not astonishing that a modified form of the Leopoldt conjecture plays a central rôle in the study of that module. We refer the reader to [5].

2) At the moment one has no finiteness result like (4.3).

3) The vanishing of $H^2(\mathfrak{o}_\infty, \mathscr{A}(\mathfrak{p}))$ follows from the modified Leopoldt conjecture.

4) One can establish an analogue of Theorem 5. Indeed this is simpler because of the following facts:

a) The sequence $0 \to \mathscr{A}_{\mathfrak{p}} \to \mathscr{A} \xrightarrow{\mathfrak{p}} \mathscr{A} \to 0$ of $\mathfrak{o}$-group schemes is exact.

b) Instead of the descent diagram one simply uses the Hochschild-Serre spectral sequence associated with the Galois extension $\mathrm{Spec}(\mathfrak{o}'_\infty) \to \mathrm{Spec}(\mathfrak{o}')$ (here "$'$" means "$\mathfrak{p}$-integral").

c) One can define the analogue of the pairing $\langle\ ,\ \rangle_p$ without any hypotheses. Moreover, it is shown in [38] that $\langle\ ,\ \rangle_p$ is the same as the analytic height pairing, apart from a scalar multiple. The analogue of Theorem 5 now holds with the assumption that $Ш_k(A)(p)$ is finite and that the modified Leopoldt conjecture is valid.

Appendix. *Equivariant Cohomology*

Of course, the notion of equivariant cohomology can be developed in a very general context. Here we confine ourselves to that special case which is the most interesting one for our purposes. Let K/k be a Galois extension with Galois group G, let R be the ring of integers in K, and let π: Spec$(R) \to$ Spec$(\mathfrak{o})$ be the structure morphism. If $\mathscr{S}(\mathfrak{o})$ denotes the category of abelian sheaves on $\mathfrak{o}_{fpqf}$, we have the following three additive and left exact functors (which commute with pseudofiltered direct limits):

$$\begin{aligned} &\mathscr{S}(\mathfrak{o}) \to (\text{discrete } G\text{-modules}) \\ &\quad \mathscr{F} \mapsto H^0(R, \pi^*\mathscr{F}), \\ H^0(R/\mathfrak{o}, .)\colon\ &\mathscr{S}(\mathfrak{o}) \to (\text{abelian groups}) \\ &\quad \mathscr{F} \mapsto H^0(G, H^0(R, \pi^*\mathscr{F})), \\ \pi_G\colon\ &\mathscr{S}(\mathfrak{o}) \to \mathscr{S}(\mathfrak{o}) \\ &\quad \mathscr{F} \mapsto \pi_G\mathscr{F}(U_{/\mathfrak{o}}) := H^0(G, (\pi^*\mathscr{F})(U \times_{\mathfrak{o}} R)). \end{aligned}$$

Since π^* is exact and takes injective sheaves to $H^0(R, .)$-acyclic sheaves, the right derived functors of the first functor are the usual cohomology groups $H^i(R, \pi^*.)$ regarded as discrete G-modules. Define

$$H^i(R/\mathfrak{o}, .), \quad \text{resp. } R^i\pi_G.,$$

to be the right derived functors of $H^0(R/\mathfrak{o}, .)$, resp. π_G. Because of the commutative diagrams

$$\begin{tikzcd} \mathscr{S}(\mathfrak{o}) \arrow[rr, "H^0(R{,}\pi^*.)"] \arrow[dr, "H^0(R/\mathfrak{o}{,}.)"'] & & \begin{pmatrix}\text{discrete}\\ G\text{-modules}\end{pmatrix} \arrow[dl, "H^0(G{,}.)"] \\ & \begin{pmatrix}\text{abelian}\\ \text{groups}\end{pmatrix} & \end{tikzcd}, \qquad \begin{tikzcd} \mathscr{S}(\mathfrak{o}) \arrow[rr, "\pi_G"] \arrow[dr, "H^0(R/\mathfrak{o}{,}.)"'] & & \mathscr{S}(\mathfrak{o}) \arrow[dl, "H^0(\mathfrak{o}{,}.)"] \\ & \begin{pmatrix}\text{abelian}\\ \text{groups}\end{pmatrix} & \end{tikzcd}$$

there exist two spectral sequences

$$H^i(G, H^j(R, \pi^*\mathscr{F})) \Rightarrow H^{i+j}(R/\mathfrak{o}, \mathscr{F})$$

and

$$H^i(\mathfrak{o}, R^j\pi_G\mathscr{F}) \Rightarrow H^{i+j}(R/\mathfrak{o}, \mathscr{F}),$$

if we show that $H^0(R, \pi^*.)$, resp. π_G, takes injective sheaves to acyclic discrete G-modules, resp. flabby sheaves. By a direct limit argument (use [22] III 1.17(d) and 3.6(c)) it suffices to consider the case that G is finite.

Lemma 1. *If G is finite and $\mathscr{F} \in \mathscr{S}(\mathfrak{o})$ is an injective sheaf, then:*

i) *$H^0(R, \pi^*\mathscr{F})$ is an acyclic G-module;*

ii) *$\pi_G\mathscr{F}$ is a flabby sheaf.*

Proof. i) We regard G as a category with one object in the usual way. Let B_G be that site which has G as underlying category and arbitrary families of morphisms as coverings. We have the canonical morphism of topologies (in the sense of [35] I 1.2.2)

$$f\colon B_G \to \mathfrak{o}_{fpqf}$$

defined by $f(G) := $ Spec(R). The category of abelian sheaves on B_G is nothing else than the category of G-modules. Because of $H^0(R, \pi^*.) = f^s$ the assertion now is a consequence of [35] (I 3.7.2).

ii) The sheaf $\pi_*\pi^*\mathscr{F}$ is flabby according to [22] (III 1.11 and 2.13(b)). This means that the cochain complex

$$0 \to \mathscr{F}(U \times_{\mathfrak{o}} R) \to \prod_j \mathscr{F}(U_j \times_{\mathfrak{o}} R) \to \prod_{i,j} \mathscr{F}((U_i \times_U U_j) \times_{\mathfrak{o}} R) \to$$

is exact for any covering $U_j \to U$ in $\mathfrak{o}_{fpqf}$ ([22] III 2.12). An easy generalization of the assertion i) implies that each term in this sequence is an acyclic G-module. Therefore the sequence remains exact if we take the G-invariants, i.e. the cochain complex

$$0 \to \pi_G\mathscr{F}(U) \to \prod_j \pi_G\mathscr{F}(U_j) \to \prod_{i,j} \pi_G\mathscr{F}(U_i \times_U U_j) \to$$

is exact. Thus $\pi_G\mathscr{F}$ is flabby.

From now on we assume that $Z := \{\mathfrak{p}$ *ramified in* $K\} \subseteq$ Spec$(\mathfrak{o})$ is finite. Let $Y :=$ Spec$(\mathfrak{o}) \setminus Z$ denote the open complement and put $\bar{Z} := \pi^{-1}Z$ and $\bar{Y} := \pi^{-1}Y$.

Lemma 2. $R^i\pi_G\mathscr{F}_{/Y_{fpqf}} = 0$ for $\mathscr{F} \in \mathscr{S}(\mathfrak{o})$ and $i > 0$.

Proof. This results from the commutative diagram of functors

$$\begin{tikzcd} \mathscr{S}(\mathfrak{o}) \arrow[rr, "\pi_G"] \arrow[dr] & & \mathscr{S}(\mathfrak{o}) \arrow[dl] \\ & \mathscr{S}(Y) & \end{tikzcd}$$

([22] II 1.4) and the fact that the restriction functor $\mathscr{S}(\mathfrak{o}) \to \mathscr{S}(Y)$ is exact.

If we apply Lemma 2 to the relative cohomology sequence, then we get

$$H^i(\mathfrak{o}, R^j\pi_G\mathscr{F}) = H^i_Z(\mathfrak{o}, R^j\pi_G\mathscr{F}) \quad \text{for } j > 0. \tag{$*$}$$

Thus we are led to look for spectral sequences which compute the relative cohomology groups. We have the additive and left exact functors

$$\begin{aligned} &\mathscr{S}(\mathfrak{o}) \to (\text{discrete } G\text{-modules}) \\ &\quad \mathscr{F} \mapsto \ker(H^0(R, \pi^*\mathscr{F}) \to H^0(\bar{Y}, \pi^*\mathscr{F})), \\ H^0_Z(R/\mathfrak{o}, .)\colon\ &\mathscr{S}(\mathfrak{o}) \to (\text{abelian groups}) \\ &\quad \mathscr{F} \mapsto H^0(G, \ker(H^0(R, \pi^*\mathscr{F}) \to H^0(\bar{Y}, \pi^*\mathscr{F}))). \end{aligned}$$

Again the right derived functors of the first functor simply are the relative cohomology groups $H^i_Z(R, \pi^*.)$ regarded as discrete G-modules. Define

$$H^i_Z(R/\mathfrak{o}, .)$$

to be the right derived functors of $H^0_Z(R/\mathfrak{o}, .)$. Since flabby sheaves are $H^0_Z(\mathfrak{o}, .)$-acyclic, Lemma 1 ii) implies that π_G takes injective sheaves to $H^0_Z(\mathfrak{o}, .)$-acyclic sheaves.

Lemma 3. *If $\mathscr{F} \in \mathscr{S}(\mathfrak{o})$ is an injective sheaf, then $H^0_Z(R, \pi^*\mathscr{F})$ is an acyclic (discrete) G-module.*

Proof. Because of the injectivity of $\mathscr{F}$ we have $H^1_Z(R, \pi^*\mathscr{F})=0$ and thus the exact sequence of G-modules

$$0 \to H^0_Z(R, \pi^*\mathscr{F}) \to H^0(R, \pi^*\mathscr{F}) \to H^0(\bar{Y}, \pi^*\mathscr{F}) \to 0.$$

According to Lemma 1 the second and the third term are acyclic G-modules. Furthermore, $\pi_G\mathscr{F}$ is flabby which implies the surjectivity of the restriction map

$$H^0(G, H^0(R, \pi^*\mathscr{F})) = \pi_G\mathscr{F}(o) \to \pi_G\mathscr{F}(Y) = H^0(G, H^0(\bar{Y}, \pi^*\mathscr{F}))$$

(SGA 4 V 4.7). Therefore $H^0_Z(R, \pi^*\mathscr{F})$ must be acyclic too.

The commutative diagrams

$$\begin{tikzcd} \mathscr{S}(o) \arrow[rr, "H^0_Z(R,\pi^*.)"] \arrow[dr, "H^0_Z(R/o,.)"'] & & \begin{pmatrix}\text{discrete}\\ G\text{-modules}\end{pmatrix} \arrow[dl, "H^0(G,.)"] \\ & \begin{pmatrix}\text{abelian}\\ \text{groups}\end{pmatrix} & \end{tikzcd} \qquad \begin{tikzcd} \mathscr{S}(o) \arrow[rr, "\pi_G"] \arrow[dr, "H^0_Z(R/o,.)"'] & & \mathscr{S}(o) \arrow[dl, "H^0_Z(o,.)"] \\ & \begin{pmatrix}\text{abelian}\\ \text{groups}\end{pmatrix} & \end{tikzcd}$$

now imply the existence of the spectral sequences

$$H^i(G, H^j_Z(R, \pi^*\mathscr{F})) \Rightarrow H^{i+j}_Z(R/o, \mathscr{F})$$

and

$$H^i_Z(o, R^j\pi_G\mathscr{F}) \Rightarrow H^{i+j}_Z(R/o, \mathscr{F}).$$

Applying (*) to the sequence of lower terms of the latter spectral sequence results in the exact sequence

$$H^1_Z(R/o, \mathscr{F}) \to H^0(o, R^1\pi_G\mathscr{F}) \to H^2_Z(o, \pi_G\mathscr{F}) \to H^2_Z(R/o, \mathscr{F}).$$

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Added in Proof

In a forthcoming paper we shall weaken considerably the assumptions made in Theorem (8.5) and we shall identity the pairing $\langle\ ,\ \rangle_p$ with the p-adic height defined in [39].