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The cyclic homology of p-adic reductive groups

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The Baum-Connes conjecture is about a canonical isomorphism between the equivariant K-homology of a locally compact group G and the K-theory of the associated reduced C^*-algebra ([BCH]). The corresponding homology version should be an isomorphism between the equivariant homology of an appropriate G-space X and the periodic cyclic homology of an appropriately defined group algebra. Let G be a p-adic reductive group. Then the G-space in question is the Bruhat-Tits building X of G. For the group algebra one eventually would like to take the Schwartz algebra of G. But in this paper we take the smaller and purely algebraically defined Hecke algebra of G and establish an isomorphism of the kind as described above. In doing so we restrict to the case of a semisimple and simply connected group G. In the general case everything would go through if one would work with a fine enough barycentric subdivision of the building. But this is somewhat unnatural. In fact, in a forthcoming paper we will show that for a general reductive group the most natural version of that isomorphism can be obtained by working with cohomology instead of homology. The reason is that the G-equivariant cohomology of the building X can be defined in the more functorial and flexible context of G-equivariant sheaf theory on X.

After having completed this paper I learned that Higson and Nistor also have obtained the same results. The methods are basically the same and are inspired by the older work of Nistor ([Nis]) on the corresponding problem for real Lie groups. From the point of view of reductive groups the way the arguments are arranged in this paper appears to be slightly more natural since we work with the canonical polysimplicial structure of the building.

I want to express my sincere thanks to M.-F. Vigneras and to R. Plymen. The former suggested that the theory of coefficient systems as developped in [SS] might be useful in the present context. The latter patiently explained the Baum-Connes conjecture to me. I am equally thankful to the MSRI (Berkeley) where this paper was written for providing the stimulating atmosphere which I enjoyed so much.

0. Background

Let G be the group of K-rational points of a connected reductive group over a non-archimedean local field K. To simplify the presentation we restrict to the case that the centre of G is compact. Let $\mathscr{H}$ denote the space of all locally constant complex valued functions with compact support on G. We fix a Haar measure dg on G. Then $\mathscr{H}$ is an associative $\mathbb{C}$-algebra without unit – the Hecke algebra of G – via the convolution product

$$(\psi * \phi)(h) := \int_G \psi(g)\,\phi(g^{-1}h)\,dg\,.$$

A representation of G in a $\mathbb{C}$-vector space V is called smooth if the stabilizer of each vector is open in G. Let $\mathrm{Alg}(G)$ denote the category of smooth G-representations. The algebra $\mathscr{H}$ acts from the left on any smooth representation V through

$$\psi * v := \int_G \psi(g)\cdot gv\,dg\,.$$

This actually sets up an equivalence of categories between $\mathrm{Alg}(G)$ and the category of nondegenerate left $\mathscr{H}$-modules. In this paper we consider $\mathscr{H}$ as an object of $\mathrm{Alg}(G)$ in two different ways: (1) Through the left translation action of G in which case we simply write $\mathscr{H}$; (2) through the adjoint action of G in which case we write $\mathscr{H}^{\mathrm{ad}}$.

The category $\mathrm{Alg}(G)$ has enough projective objects. Hence the homology groups $\mathrm{H}_*(G, V)$, for V in $\mathrm{Alg}(G)$, exist as the left derived functors of taking G-coinvariants ([Cas], App.). Let us collect various facts about the relation between group homology and other homological invariants of $\mathscr{H}$.

– Let d be the K-rank of G and let St denote the Steinberg representation of G. Then we have natural duality isomorphisms $\mathrm{Ext}_G^{d-*}(St, V) \cong \mathrm{H}_*(G, V)$ for any V in $\mathrm{Alg}(G)$ where Ext_G^* stands for the Ext-functor in the category $\mathrm{Alg}(G)$ ([SS], III. 3).

– $\mathrm{H}_*(G, \mathscr{H}^{\mathrm{ad}})$ coincides with the Hochschild homology $\mathrm{HH}_*(\mathscr{H})$ of the algebra $\mathscr{H}$ ([BB], 2.3); in particular both vanish for $* > d$.

– The cyclic homology $\mathrm{HC}_*(\mathscr{H})$ of $\mathscr{H}$ is related to the Hochschild homology through the Connes spectral sequence

$$E^1_{r,s} = \mathrm{HH}_{s-r}(\mathscr{H}) \Rightarrow \mathrm{HC}_{r+s}(\mathscr{H})\,.$$

– The periodic cyclic homology $\mathrm{HC}_*^{\mathrm{per}}(\mathscr{H})$ of $\mathscr{H}$ is related to the cyclic homology through the exact sequence

$$0 \to \varprojlim_n{}^1 \mathrm{HC}_{*+2n+1}(\mathscr{H}) \to \mathrm{HC}_*^{\mathrm{per}}(\mathscr{H}) \to \varprojlim_n \mathrm{HC}_{*+2n}(\mathscr{H}) \to 0$$

where the transition maps in the respective projective systems are given by the periodicity map S.

One of our goals in this paper is to get a better understanding of these two sequences. The following observation turns out to be very useful for that purpose. Consider the algebra (with pointwise multiplication)

$$R^\infty(G) := \text{locally constant class functions on } G\,.$$

It acts by pointwise multiplication on the cyclic double complex ([BB], 3.5) and hence on the Hochschild and (periodic) cyclic homology. Let c be a specific function in $R^\infty(G)$. Then

$$G = \dot{\bigcup_{\delta \in \mathbb{C}}} G_{c=\delta} \quad \text{with } G_{c=\delta} := \{g \in G : c(g) = \delta\}$$

is an open covering by conjugation-invariant subsets and hence

$$\mathscr{H}^{\mathrm{ad}} = \bigoplus_{\delta \in \mathbb{C}} \mathscr{H}_{c=\delta} \quad \text{with } \mathscr{H}_{c=\delta} := \{\psi \in \mathscr{H} : \operatorname{supp} \psi \subseteq G_{c=\delta}\}$$

is a direct sum decomposition in $\mathrm{Alg}(G)$. It follows easily that the whole cyclic double complex decomposes into eigenspaces with respect to the action of c. Correspondingly the Connes spectral sequence decomposes into the direct sum of the spectral sequences

$$E^1_{r,s} = \mathrm{HH}_{s-r}(\mathscr{H})^{c=\delta} \Rightarrow \mathrm{HC}_{r+s}(\mathscr{H})^{c=\delta}.$$

Moreover, as discussed in [BB], § 3, we have natural isomorphisms

$$\mathrm{HH}_*(\mathscr{H})^{c=\delta} \cong \mathrm{H}_*(G, \mathscr{H}_{c=\delta}).$$

There is no formal reason for the periodic cyclic homology to decompose into eigenspaces. But at least we have

$$\mathrm{HC}^{\mathrm{per}}_*(\mathscr{H}) = \mathrm{HC}^{\mathrm{per}}_*(\mathscr{H})^{c=0} \oplus \mathrm{HC}^{\mathrm{per}}_*(\mathscr{H})_c$$

(the subscript "c" in the second summand denotes the localization in c) and the exact sequences

$$0 \to \varprojlim_n{}^1 \mathrm{HC}_{*+2n+1}(\mathscr{H})^{c=0} \to \mathrm{HC}^{\mathrm{per}}_*(\mathscr{H})^{c=0} \to \varprojlim_n \mathrm{HC}_{*+2n}(\mathscr{H})^{c=0} \to 0$$

and

$$0 \to \varprojlim_n{}^1 \mathrm{HC}_{*+2n+1}(\mathscr{H})_c \to \mathrm{HC}^{\mathrm{per}}_*(\mathscr{H})_c \to \varprojlim_n \mathrm{HC}_{*+2n}(\mathscr{H})_c \to 0.$$

In the next section we will further explore these facts for a particular function in $R^\infty(G)$.

1. The function D

Let X denote the Bruhat-Tits building of G (see [SS], I. 1 for a quick overview). For any facet F in X let $P_F^\dagger$, resp. P_F, denote the stabilizer, resp. the pointwise stabilizer, of F. These are compact open subgroups of G. Put

$$G_0^\dagger := \bigcup_F P_F^\dagger \quad \text{and} \quad G_0 := \bigcup_F P_F.$$

By the fixed point theorem of Bruhat and Tits $G_0^\dagger$ is the set of all compact elements in G. If G is semisimple and simply connected then $G_0^\dagger = G_0$.

Lemma 1. *$G_0^\dagger$ is an open and closed subset of G.*

Proof. Obviously $G_0^\dagger$ is open. By realizing G as a Zariski closed subgroup of some general linear group $\mathrm{GL}_N(K)$ the question of being closed is reduced to showing that the subset of all compact elements in $\mathrm{GL}_N(K)$ is closed. But an element in the latter group is compact if and only if its eigenvalues have absolute value 1. This clearly is a closed condition.

As a consequence

$$D := \text{characteristic function of } G \backslash G_0^\dagger$$

lies in $R^\infty(G)$ and we may apply the general discussion in the previous section. In order to simplify the notation we write

$$\mathscr{H}_0 := \mathscr{H}_{D=0} = \{\psi \in \mathscr{H} : \operatorname{supp} \psi \subseteqq G_0^\dagger\}.$$

We then have the decomposition

$$\mathrm{HC}_*^{\mathrm{per}}(\mathscr{H}) - \mathrm{HC}_*^{\mathrm{per}}(\mathscr{H})^{D=0} \oplus \mathrm{HC}_*^{\mathrm{per}}(\mathscr{H})_D,$$

the spectral sequence

$$E_{r,s}^1 = \mathrm{H}_{s-r}(G, \mathscr{H}_0) = \mathrm{HH}_{s-r}(\mathscr{H})^{D=0} \Rightarrow \mathrm{HC}_{r+s}(\mathscr{H})^{D=0},$$

and the exact sequence

$$0 \to \varprojlim_n{}^1 \mathrm{HC}_{*+2n+1}(\mathscr{H})^{D=0} \to \mathrm{HC}_*^{\mathrm{per}}(\mathscr{H})^{D=0} \to \varprojlim_n \mathrm{HC}_{*+2n}(\mathscr{H})^{D=0} \to 0.$$

Our first main result is the following.

Theorem I. *Under the assumption that G is semisimple and simply connected we have:*

(A) *The Connes spectral sequence induces an isomorphism*

$$\mathrm{HC}_*(\mathscr{H})^{D=0} \cong \bigoplus_{i \geqq 0} \mathrm{H}_{*-2i}(G, \mathscr{H}_0).$$

(B) $\mathrm{HC}_*^{\mathrm{per}}(\mathscr{H})^{D=0} = \bigoplus_{i \in \mathbb{Z}} \mathrm{H}_{*+2i}(G, \mathscr{H}_0)$.

Since (A) implies that the transition maps in the projective systems

$$\{\mathrm{HC}_{*+2n}(\mathscr{H})^{D=0}\}_n$$

are surjective it is clear that (B) is a consequence of (A). We will prove (A) in section 3.

Theorem II. *Under the assumption that K has characteristic* 0 *we have:*

(A) *The periodicity map* $\mathrm{HC}_{*+2}(\mathscr{H})_D \xrightarrow{S} \mathrm{HC}_*(\mathscr{H})_D$ *is zero.*

(B) $\mathrm{HC}_*(\mathscr{H})_D = 0$ *for* $* \geqq d$.

(C) $\mathrm{HC}_*^{\mathrm{per}}(\mathscr{H})_D = 0$.

The assertion (C) should be considered as the strongest possible form of the abstract Selberg principle.

Obviously (B) implies (C). Also it follows from (A) and Connes' long exact sequence that $HC_*(\mathscr{H})_D$ is contained in $HH_{*+1}(\mathscr{H})_D$. But the latter groups vanish for $*+1 > d$. Hence (B) is a consequence of (A). The first step in the proof of (A) consists in showing that it suffices to establish a localized version of the statement. For this let us point out some elementary facts about the ring $R := R^\infty(G)$. For any open and closed conjugation-invariant subset $U \subseteqq G$ let $1_U \in R$ denote its characteristic function. It is completely trivial that any function $c \in R$ can be written as $c = 1_U \cdot c'$ where $U := G \backslash c^{-1}(0)$ and where $c' \in R^\times$ is a unit.

Lemma 2. (i) *Any prime ideal in $R^\infty(G)$ is maximal.*

(ii) $\operatorname{Spec}(R^\infty(G))$ *is a compact and totally disconnected Hausdorff space.*

(iii) *The localization of $R^\infty(G)$ in any maximal ideal $\mathfrak{m}$ is equal to the residue class field $R^\infty(G)/\mathfrak{m}$.*

Proof. (i) Let $\mathfrak{p} \subseteqq R$ be a prime ideal and let $\mathfrak{m} \subseteqq R$ be a maximal ideal containing $\mathfrak{p}$. We claim that $\mathfrak{p} = \mathfrak{m}$. It suffices to show that any function $c \in \mathfrak{m}$ of the form $c = 1_U$ already lies in $\mathfrak{p}$. Because of $1_U + 1_{G\backslash U} = 1$ we have $1_{G\backslash U} \notin \mathfrak{m}$. Hence $1_U \cdot 1_{G\backslash U} = 0$ implies that $1_U \in \mathfrak{p}$.

(ii) This follows from the observation that the zero set $V(Rc) \subseteqq \operatorname{Spec}(R)$ of any principal ideal $Rc \subseteqq R$ is open. Namely, writing $c = 1_U \cdot c'$ with $c' \in R^\times$ we have the direct decomposition $R = R1_U \times R1_{G\backslash U}$.

(iii) Consider any function $1_U \in \mathfrak{m}$. Then $1_{G\backslash U} \notin \mathfrak{m}$ which means that $1_{G\backslash U}$ becomes invertible in $R_\mathfrak{m}$. Because of $1_U \cdot 1_{G\backslash U} = 0$ this forces 1_U to become zero in $R_\mathfrak{m}$.

We have the continuous map

$$j : G \to \operatorname{Spec}(R^\infty(G)),$$
$$g \mapsto \mathfrak{m}_g := \{c \in R : c(g) = 0\}.$$

Proposition 3. *The support of $\mathscr{H}^{\mathrm{ad}}$ as an $R^\infty(G)$-module is contained in $j(G)$.*

Proof. Let $\psi \in \mathscr{H}$ be any function and let $C \subseteqq G$ denote the support of ψ. Also let $\mathfrak{m} \subseteqq R$ be a maximal ideal not in the image of j. For any $g \in G$ we then find a function $c_g \in \mathfrak{m}$ such that $c_g(g) \neq 0$. In other words there is an open and closed conjugation-invariant neighbourhood $U(g) \subseteqq G$ of g such that $1_{U(g)} \in \mathfrak{m}$. Since C is compact we have

$$C \subseteqq U := U(g_1) \cup \cdots \cup U(g_r)$$

for appropriate elements $g_1, \ldots, g_r \in C$. It follows that 1_U which differs from

$$1_{U(g_1)} + \cdots + 1_{U(g_r)}$$

by a unit lies in $\mathfrak{m}$. Hence $\psi = 1_U \cdot \psi \in \mathfrak{m}\mathcal{H}$.

Corollary 4. *The support of the $R^\infty(G)$-module $\mathrm{HC}_*(\mathcal{H})_D$ is contained in $j(G \setminus G_0^\dagger)$.*

Proof. First of all the argument in the previous proof obviously generalizes to any term in the cyclic double complex for $\mathcal{H}$ showing that $\mathrm{HC}_*(\mathcal{H})$ has support in $j(G)$. The direct summand $\mathrm{HC}_*(\mathcal{H})_D$ is a module for the quotient $R/R1_{G_0^\dagger}$. Hence its support is contained in $j(G) \cap V(R1_{G_0^\dagger}) = j(G \setminus G_0^\dagger)$.

One knows (from the Jacobson-Morosov theorem) that the semisimple part s of an element $g \in G$ lies in the closure of the orbit of g. It follows that $\mathfrak{m}_g = \mathfrak{m}_s$. Altogether we see that Theorem II. A is implied by the following localized statement:

(A′) *If K has characteristic* 0 *then the periodicity map*

$$\mathrm{HC}_{*+2}(\mathcal{H})/\mathfrak{m}_g \xrightarrow{S} \mathrm{HC}_*(\mathcal{H})/\mathfrak{m}_g$$

is zero for any semisimple element $g \in G \setminus G_0^\dagger$.

This will be proved in section 4.

2. The equivariant homology of the building

The natural tool to compute the group homology of $\mathcal{H}_0$ is a certain coefficient system on the building X (for the general notion of a coefficient system or cosheaf on X compare [SS], II. 2 or [BCH], § 5). Recall that P_F, for any facet F in X, is a compact open subgroup of G. Hence

$$\mathcal{H}_F := \{\psi \in \mathcal{H} : \psi \text{ has support in } P_F\}$$

is a P_F-invariant subspace of $\mathcal{H}^{\mathrm{ad}}$ as well as a subalgebra of $\mathcal{H}$. We have

$$\sum_F \mathcal{H}_F \subseteq \mathcal{H}_0$$

with equality if G is semisimple and simply connected. With "extension by zero" as transition maps these spaces form a coefficient system of associative algebras (without unit)

$$\mathcal{C} : F \to \mathcal{H}_F$$

on X. In the following we compute its (equivariant) homology. The homology $\mathrm{H}_*(X, C)$ of an arbitrary coefficient system $C = (C_F)_F$ on X is defined to be the homology of the chain complex

$$\mathrm{C}_c^{\mathrm{or}}(X_{(d)}, C) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathrm{C}_c^{\mathrm{or}}(X_{(0)}, C).$$

Here $(X_{(q)})X_q$, for any $0 \leqq q \leqq d$, denotes the set of all (oriented) q-dimensional facets of X. The space of oriented q-chains of C by definition is

$$\mathrm{C}_c^{\mathrm{or}}(X_{(q)}, C) := \mathbb{C}\text{-vector space of all maps } \omega : X_{(q)} \to \dot{\bigcup_{F \in X_q}} C_F$$

such that

- ω has finite support,
- $\omega((F, c)) \in C_F$, and, if $q \geqq 1$,
- $\omega((F, -c)) = -\omega((F, c))$ for any $(F, c) \in X_{(q)}$.

The boundary map ∂ is the obvious one (compare [SS], V or [BCH], §5; we prefer not to fix an orientation of X). In this paper C is called G-equivariant if it carries a G-action ([SS], V) with the property that the induced action of P_F on C_F, for any facet F, is smooth. If C is G-equivariant then its chain complex is a complex in $\mathrm{Alg}(G)$ ([SS], V) so that it makes sense to speak of its group hyperhomology

$$\mathrm{H}_*(X, G; C) := \mathrm{H}_*(G, \mathrm{C}_c^{\mathrm{or}}(X_{(\cdot)}, C))$$

which we call the equivariant homology of C.

Remark 1. If C is G-equivariant then $\mathrm{C}_c^{\mathrm{or}}(X_{(q)}, C)$, for any $0 \leqq q \leqq d$, is a projective object in $\mathrm{Alg}(G)$.

Proof. The representation in question is a finite direct sum of subrepresentations of the following form. Let F be a fixed q-facet and put

$\mathrm{C}_c^{\mathrm{or}}(F, C) :=$ subspace of all those chains with support in the union of the G-orbits of the oriented facets with underlying facet F.

In order to see that this latter representation is projective we let $P_F^\dagger$ denote, as before, the stabilizer of F, we fix an orientation (F, c) of F, and we let $\varepsilon_F : P_F^\dagger \to \{\pm 1\}$ be the unique character such that $g((F, c)) = (F, \varepsilon_F(g) \cdot c)$. Moreover we define, for any $b \in C_F$, a chain $\omega_b \in \mathrm{C}_c^{\mathrm{or}}(F, C)$ by

$$\omega_b((F', c')) := \begin{cases} b & \text{if } (F', c') = (F, c), \\ -b & \text{if } q \geqq 1 \text{ and } (F', c') = (F, -c), \\ 0 & \text{otherwise}. \end{cases}$$

One checks that, for any V in $\mathrm{Alg}(G)$, the map

$$\mathrm{Hom}_G(\mathrm{C}_c^{\mathrm{or}}(F, C), V) \xrightarrow{\cong} \mathrm{Hom}_{P_F^\dagger}(C_F \otimes \varepsilon_F, V),$$
$$A \mapsto (b \mapsto A(\omega_b))$$

is an isomorphism. Note that P_F is a normal subgroup of finite index in $P_F^\dagger$ so that C_F is a smooth $P_F^\dagger$-representation. It therefore suffices to see that $C_F \otimes \varepsilon_F$ is projective in $\mathrm{Alg}(P_F^\dagger)$. But $P_F^\dagger$ being compact the latter category actually is semisimple.

As a consequence we obtain that the equivariant homology $\mathrm{H}_*(X, G; C)$ of an equivariant coefficient system C is the homology of the complex of G-coinvariants

$$\mathrm{C}_c^{\mathrm{or}}(X_{(d)}, C)_G \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathrm{C}_c^{\mathrm{or}}(X_{(0)}, C)_G .$$

Obviously $\mathscr{C}$ is G-equivariant. It is an easy exercise to see that the groups $\mathrm{H}_*(X, G; \mathscr{C})$ are naturally isomorphic to the groups defined in [BCH], (6.9) and called there the equivariant homology of the building X.

Proposition 2. *If G is semisimple and simply connected we have $\mathrm{H}_*(X, \mathscr{C}) = 0$ for $* \neq 0$ and $= \mathscr{H}_0$ for $* = 0$.*

Proof. There is the obvious augmentation map

$$\mathrm{C}_c^{\mathrm{or}}(X_{(0)}, \mathscr{C}) \to \mathscr{H}_0 ,$$
$$\omega \mapsto \sum_{F \in X_{(0)}} \omega(F) .$$

We have to show that the augmented chain complex of $\mathscr{C}$ is an exact resolution of $\mathscr{H}_0$. Since G-equivariance is unimportant here we fix some orientation on X and work with the corresponding chain complex. It is the complex of global sections with compact support of the complex of sheaves

$$0 \to \bigoplus_{F \in X_d} i_{F*}\mathbb{C} \to \cdots \to \bigoplus_{F \in X_0} i_{F*}\mathbb{C} \to \mathbb{C} \to 0$$

on G_0. Here $i_F : P_F \hookrightarrow G_0$ denotes the inclusion map. All sheaves in this complex are c-soft. Hence it suffices to check that this complex of sheaves is exact which can be done stalkwise. But it is clear that the complex of stalks in an element $g \in G_0$ computes the singular homology of the nonempty polysimplicial subcomplex

$$X^{(g)} := \text{all facets which are fixed pointwise by } g$$

of X. We therefore are reduced to show that $X^{(g)}$ is contractible. But under our assumption on G the subcomplex $X^{(g)}$ coincides with the fixed point set X^g. The latter clearly is contractible since the G-action on X respects geodesics.

Corollary 3. *If G is semisimple and simply connected then $\mathrm{H}_*(X, G; \mathscr{C}) = \mathrm{H}_*(G, \mathscr{H}_0)$.*

By combining Theorems I and II and Corollary 3 we obtain our second main result.

Theorem III. *Let G be semisimple and simply connected.*

(i) $\mathrm{HC}_*^{\mathrm{per}}(\mathscr{H})^{D=0} = \bigoplus_{i \in \mathbb{Z}} \mathrm{H}_{*+2i}(X, G; \mathscr{C})$.

(ii) *If K has characteristic* 0 *we have*

$$\mathrm{HC}_*^{\mathrm{per}}(\mathscr{H}) = \bigoplus_{i \in \mathbb{Z}} \mathrm{H}_{*+2i}(X, G; \mathscr{C}) .$$

This is a cyclic homology analog of the Baum-Connes conjecture for G ([BCH], (6.1)).

3. Proof of Theorem I.A

In this section G is assumed to be semisimple and simply connected. We are going to make use of the following concept introduced in [Nis]. A quasi-cyclic vector space is a pre-simplicial $\mathbb{C}$-vector space

$$\cdots \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} E_2 \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} E_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} E_0$$

together with endomorphisms

$$t_n : E_n \to E_n \quad \text{for } n \geqq 0$$

such that

$$d_i t_n = \begin{cases} t_{n-1} d_{i-1} & \text{for } 1 \leqq i \leqq n, \\ d_n & \text{for } i = 0. \end{cases}$$

It is called precyclic if in addition

$$t_n^{n+1} = \mathrm{id} \quad \text{for } n \geqq 0$$

holds true. All the basic constructions in cyclic homology can be done for precyclic vector spaces ([Lod], 2.5.6).

The quasi-cyclic vector spaces of interest in this paper arise in the following way. Let $U \subseteq G$ be any open subset. Then $\mathscr{C}.(U)$ defined by

$$\mathscr{C}_n(U) := \text{space of all } \mathbb{C}\text{-valued locally constant functions with compact support on } U \times G^{n+1},$$

$$(d_i \varphi)(h, g_0, \ldots, g_n) := \int_G \varphi(h, g_0, \ldots, g_{i-1}, g, g_i, \ldots, g_n)\, dg,$$

and

$$(t_n \varphi)(h, g_0, \ldots, g_n) := \varphi(h, g_1, \ldots, g_n, hg_0)$$

is a quasi-cyclic vector space. If $U' \subseteq U \subseteq G$ are open then extending functions by zero induces a homomorphism of quasi-cyclic vector spaces

$$\mathscr{C}.(U') \to \mathscr{C}.(U).$$

For any $g \in G$ the map

$$g : \mathscr{C}.(U) \to \mathscr{C}.(gUg^{-1}),$$
$$\varphi \mapsto (g\varphi)(h, g_0, \ldots, g_n) := \varphi(g^{-1}hg, g^{-1}g_0, \ldots, g^{-1}g_n)$$

is an isomorphism of quasi-cyclic vector spaces. It follows that

$$\mathscr{C}. : F \mapsto \mathscr{C}.(P_F)$$

constitutes a G-equivariant coefficient system of quasi-cyclic vector spaces on X. By passing to chains we obtain the augmented complex of quasi-cyclic smooth G-representations

$$\mathrm{C}_c^{\mathrm{or}}(X_{(d)}, \mathscr{C}.) \to \cdots \to \mathrm{C}_c^{\mathrm{or}}(X_{(0)}, \mathscr{C}.) \to \mathscr{C}.(G_0)\,.$$

Taking G-coinvariants further leads to the augmented complex of quasi-cyclic vector spaces

$$\mathrm{C}_c^{\mathrm{or}}(X_{(d)}, \mathscr{C}.)_G \to \cdots \to \mathrm{C}_c^{\mathrm{or}}(X_{(0)}, \mathscr{C}.)_G \to \mathscr{C}.(G_0)_G\,. \tag{1}$$

Lemma 1. *$\mathscr{C}.(P_F)_{P_F}$, for any facet F, as well as $\mathscr{C}.(G_0)_G$ are precyclic vector spaces.*

Proof. Since $\mathscr{C}.(G_0)_G$ is a quotient of $\bigoplus_{F \in X_{(0)}} \mathscr{C}.(P_F)_{P_F}$ we only have to consider a $\mathscr{C}.(P_F)_{P_F}$. Any function in $\mathscr{C}_n(P_F)$ is a linear combination of characteristic functions of compact open subsets in $P_F \times G^{n+1}$ of the form $h_0 U \times U\gamma_0 U \times \cdots \times U\gamma_n U$ where $U \subseteqq P_F$ is a compact open normal subgroup and $h_0 \in P_F$, $\gamma_i \in G$. If φ is the characteristic function of such a subset then one easily checks that both $t_n^{n+1}\varphi$ and $h_0^{-1}\varphi$ are equal to the characteristic function of the subset $h_0 U \times U h_0^{-1}\gamma_0 U \times \cdots \times U h_0^{-1}\gamma_n U$.

As a consequence we have that (1) actually is a complex of precyclic vector spaces.

For any precyclic vector space $E.$ let $B..(E.)$ denote the double complex

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
b\downarrow & & -b'\downarrow & & b\downarrow & & -b'\downarrow & & \\
E_2 & \xleftarrow{1-t_2} & E_2 & \xleftarrow{N} & E_2 & \xleftarrow{1-t_2} & E_2 & \xleftarrow{N} & \cdots \\
b\downarrow & & -b'\downarrow & & b\downarrow & & -b'\downarrow & & \\
E_1 & \xleftarrow{1+t_1} & E_1 & \xleftarrow{N} & E_1 & \xleftarrow{1+t_1} & E_1 & \xleftarrow{N} & \cdots \\
b\downarrow & & -b'\downarrow & & b\downarrow & & -b'\downarrow & & \\
E_0 & \xleftarrow{1-t_0} & E_0 & \xleftarrow{N} & E_0 & \xleftarrow{1-t_0} & E_0 & \xleftarrow{N} & \cdots
\end{array}$$

with

$$b := \sum_{i=0}^{n} (-1)^i d_i, \quad b' := b - (-1)^n d_n,$$

and

$$N := \sum_{i=0}^{n} (-1)^{ni} t_n^i$$

which defines the cyclic homology of $E.$ by

$$\mathrm{HC}_*(E.) := \mathrm{h}_*(\mathrm{Tot}\, B..(E.)).$$

Applying this to (1) we obtain the triple complex

$$(2') \qquad B..(\mathrm{C}_c^{\mathrm{or}}(X_{(d)}, \mathscr{C}.)_G) \to \cdots \to B..(\mathrm{C}_c^{\mathrm{or}}(X_{(0)}, \mathscr{C}.)_G) \to \to B..(\mathscr{C}.(G_0)_G),$$

resp. the double complex

$$(2) \qquad \operatorname{Tot} B..(\mathrm{C}_c^{\mathrm{or}}(X_{(d)}, \mathscr{C}.)_G) \to \cdots \to \operatorname{Tot} B..(\mathrm{C}_c^{\mathrm{or}}(X_{(0)}, \mathscr{C}.)_G) \to \to \operatorname{Tot} B..(\mathscr{C}.(G_0)_G).$$

Proposition 2. $\mathrm{HC}_*(\mathscr{C}.(G_0)_G) = \mathrm{HC}_*(\mathscr{H})^{D=0}$.

Proof. (Compare [Nis], § 3.1.) The right hand side is, by definition, the cyclic homology of the precyclic vector space $\mathscr{H}.(G_0)$ given by

$$\mathscr{H}_n(G_0) := \text{space of all } \mathbb{C}\text{-valued locally constant functions on } G^{n+1} \text{ with compact support contained in } (G^{n+1})_0 := \{(g_0, \ldots, g_n) \in G^{n+1} : g_0 \cdot \ldots \cdot g_n \in G_0\},$$

$$(d_i\varphi)(g_0, \ldots, g_n) := \int_G \varphi(g_0, \ldots, g_{i-1}, g, g^{-1}g_i, g_{i+1}, \ldots, g_n)\, dg \quad \text{for } 0 \leqq i \leqq n,$$

$$(d_{n+1}\varphi)(g_0, \ldots, g_n) := \int_G \varphi(g^{-1}g_0, g_1, \ldots, g_n, g)\, dg$$

and

$$(t_n\varphi)(g_0, \ldots, g_n) := \varphi(g_1, \ldots, g_n, g_0).$$

A straightforward computation shows that

$$\pi : \mathscr{C}.(G_0)_G \to \mathscr{H}.(G_0),$$
$$\varphi \mapsto (\pi\varphi)(g_0, \ldots, g_n) := \int_G (g\varphi)(g_0 \cdots g_n, g_0, g_0g_1, \ldots, g_0 \cdots g_n)\, dg$$

is a homomorphism of precyclic vector spaces. We claim that π is an isomorphism. Fix a degree $n \geqq 0$ and put

$$\mathscr{C}'_n(G_0) := \text{space of all } \mathbb{C}\text{-valued locally constant functions on } G^{n+2} \text{ with compact support contained in } \{(g, g_0, \ldots, g_n) \in G^{n+2} : g_0 \cdot \ldots \cdot g_n \in G_0\}.$$

We let G act on $\mathscr{C}'_n(G_0)$ by left translations in the first variable. Then

$$\mathscr{C}_n(G_0) \xrightarrow{\cong} \mathscr{C}'_n(G_0),$$
$$\varphi \mapsto [(g, g_0, \ldots, g_n) \mapsto \varphi(gg_0 \cdots g_n g^{-1}, gg_0, \ldots, gg_0 \cdots g_n)]$$

is a G-equivariant isomorphism. The map π transforms into

$$\mathscr{C}'_n(G_0)_G \to \mathscr{H}_n(G_0)\,,$$
$$\psi \mapsto [(g_0,\dots,g_n) \mapsto \int_G \psi(g^{-1},g_0,\dots,g_n)\,dg]\,.$$

It is clear that $\mathscr{C}'_n(G_0) \cong \mathscr{H} \otimes \mathscr{H}_n(G_0)$ where G acts on the right hand side through the first factor. Therefore our assertion finally comes down to the fact that

$$\mathscr{H}_G \xrightarrow{\cong} \mathbb{C}\,,$$
$$\psi \mapsto \int_G \psi(g^{-1})\,dg$$

is an isomorphism which is nothing else than the unicity of the Haar measure on G.

The next step is to show that all the rows in (2) are exact.

Proposition 3. *For any $n \geqq 0$ the complex*

$$0 \to C_c^{\mathrm{or}}(X_{(d)},\mathscr{C}_n)_G \to \cdots \to C_c^{\mathrm{or}}(X_{(0)},\mathscr{C}_n)_G \to \mathscr{C}_n(G_0)_G \to 0$$

is exact.

Proof. It is sufficient to show that

$$0 \to C_c^{\mathrm{or}}(X_{(d)},\mathscr{C}_n) \to \cdots \to C_c^{\mathrm{or}}(X_{(0)},\mathscr{C}_n) \to \mathscr{C}_n(G_0) \to 0$$

is an exact sequence of projective objects in $\mathrm{Alg}(G)$. The projectivity of all but the last term follows from Remark 2.1. Concerning the projectivity of the last term we have seen in the previous proof that we have

$$\mathscr{C}_n(G_0) \cong \mathscr{H} \otimes \mathscr{H}_n(G_0)$$

where G acts on the right hand side through the first factor. According to a result of Blanc (see [Cas], A. 4) $\mathscr{H}$ is projective in $\mathrm{Alg}(G)$.

On the other hand the above complex is isomorphic to the complex

$$0 \to C_c^{\mathrm{or}}(X_{(d)},\mathscr{C}) \to \cdots \to C_c^{\mathrm{or}}(X_{(0)},\mathscr{C}) \to \mathscr{H}_0 \to 0$$

which was shown to be exact in Prop. 2.2 tensorized by $\mathscr{H}^{\otimes n+1}$.

By feeding the information from the Propositions 2 and 3 into the double complex (2) we obtain the following consequence.

Corollary 4. $\mathrm{HC}_*(\mathscr{H})^{D=0}$ *is the homology of the complex*

$$(3) \qquad \mathrm{Tot}\,[\mathrm{Tot}\,B_{\cdot\cdot}(C_c^{\mathrm{or}}(X_{(d)},\mathscr{C}_\cdot)_G) \to \cdots \to \mathrm{Tot}\,B_{\cdot\cdot}(C_c^{\mathrm{or}}(X_{(0)},\mathscr{C}_\cdot)_G)]\,.$$

It turns out that the complex (3) is quasi-isomorphic to a much simpler complex.

First of all note that any vector space E may be viewed as a constant precyclic vector space by defining $E_n := E$ and $d_i := t_n := \mathrm{id}$ for all i and n. In the following we will always view the coefficient system $\mathscr{C}$ from section 2 as a coefficient system of constant precyclic vector spaces on X. Let us compute right away the homology of the complex

$$(4) \qquad \mathrm{Tot}\,[\mathrm{Tot}\,B..(\mathrm{C}_c^{\mathrm{or}}(X_{(d)},\mathscr{C})_G) \to \cdots \to \mathrm{Tot}\,B..(\mathrm{C}_c^{\mathrm{or}}(X_{(0)},\mathscr{C})_G)]\,.$$

If E is any vector space viewed as a constant precyclic vector space then it is a trivial exercise to see that

$$\mathrm{HC}_*(E) = \begin{cases} E & \text{if } * \geqq 0 \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

Nevertheless let us look into this. The double complex $B..(E)$ turns out to be the direct sum over $n \geqq 0$ of the complexes

$$\begin{array}{cccccccc}
E & & & & & & & \\
{\scriptstyle 1}\downarrow & & & & & & & \\
E & \xleftarrow{\;2\;} & E & & & & & \\
 & & {\scriptstyle -1}\downarrow & & & & & \\
 & & E & \xleftarrow{\;n-1\;} & E & & & \\
 & & & & {\scriptstyle 1}\downarrow & & & \\
 & & & & E & \xleftarrow{\;2\;} & E & \\
 & & & & & & {\scriptstyle -1}\downarrow & \\
 & & & & & & E & \xleftarrow{\;n-3\;} \; E \\
 & & & & & & & \ddots \\
 & & & & & & & E \xleftarrow{\;1\;} E
\end{array}$$

with the upper left, resp. lower right, corner in degree $(0, 2n)$, resp. $(2n, 0)$. It follows that

$$\mathrm{Tot}\,B..(E) \cong \bigoplus_{n \geqq 0} b_n(E)[1-2n]$$

where $b_n(E)$ denotes the complex

$$\begin{array}{ccc} \prod\limits_{m=1}^{2n} & \leftarrow & \prod\limits_{m=1}^{2n+1} E, \\ (e_m - e_{m+1})_m & \leftarrow & (e_m)_m \end{array}$$

in degrees 0 and 1. All this is functorial in E. Hence if

$$E_d \to \cdots \to E_0$$

is a complex of vector spaces with homology groups $\mathrm{h}_*(E)$ then we obtain that the homology of the complex

$$\mathrm{Tot}[\mathrm{Tot}\,B..(E_d) \to \cdots \to \mathrm{Tot}\,B..(E_0)]$$
$$\cong \bigoplus_{n \geqq 0} \mathrm{Tot}[b_n(E_d) \to \cdots \to b_n(E_0)][1-2n]$$
$$\sim \bigoplus_{n \geqq 0} [E_d \to \cdots \to E_0][-2n]$$

(here $\sim$ stands for quasi-isomorphic) is isomorphic to $\bigoplus_{n \geqq 0} \mathrm{h}_{*-2n}(E.)$.

Applying this to the complex

$$\mathrm{C}_c^{\mathrm{or}}(X_{(d)}, \mathscr{C})_G \to \cdots \to \mathrm{C}_c^{\mathrm{or}}(X_{(0)}, \mathscr{C})_G$$

whose homology is $\mathrm{H}_*(G, \mathscr{H}_0)$ we obtain the following result.

Lemma 5. *The homology of* (4) *is naturally isomorphic to* $\bigoplus_{i \geqq 0} \mathrm{H}_{*-2i}(G, \mathscr{H}_0)$.

For the proof of Theorem I.A it therefore remains to exhibit a natural quasi-isomorphism between the complexes (3) and (4). (We will leave it to the reader as an exercise in diagram chasing to check that the splitting so obtained is compatible with the Connes spectral sequence.) Consider the map

$$\mathfrak{J} : \mathscr{C}_n(P_F) \to \mathscr{H}_F,$$
$$\varphi \mapsto (\mathfrak{J}\varphi)(h) := \int_{G^{n+1}} \varphi(h, g_0, \ldots, g_n)\, dg_0 \cdots dg_n .$$

It is straightforward to see that $\mathfrak{J}$ actually is a G-equivariant homomorphism

$$\mathfrak{J} : \mathscr{C}. \to \mathscr{C}$$

of coefficient systems of quasi-cyclic vector spaces on X. Hence it induces a homomorphism of complexes

$$\mathfrak{J} : (3) \to (4) .$$

The final step in our proof now is the following.

Proposition 6. *The homomorphism* $\mathfrak{J} : (3) \to (4)$ *is a quasi-isomorphism.*

Proof. By the general formalism of double complexes it suffices to show that $\mathfrak{J}$ induces a quasi-isomorphism between the columns of the cyclic double complexes which appear in (3) and (4). This amounts to checking that the sequences

$$\cdots \to \mathrm{C}_c^{\mathrm{or}}(X_{(q)}, \mathscr{C}_n)_G \xrightarrow{b} \cdots \xrightarrow{b} \mathrm{C}_c^{\mathrm{or}}(X_{(q)}, \mathscr{C}_0)_G \xrightarrow{\mathfrak{J}} \mathrm{C}_c^{\mathrm{or}}(X_{(q)}, \mathscr{C})_G \to 0$$

and

$$\cdots \to \mathrm{C}_c^{\mathrm{or}}(X_{(q)}, \mathscr{C}_n)_G \xrightarrow{b'} \cdots \xrightarrow{b'} \mathrm{C}_c^{\mathrm{or}}(X_{(q)}, \mathscr{C}_0)_G \to 0$$

are exact. By the same projectivity argument which is based on Remark 2.1 and which we have used before we can do so before passing to the G-coinvariants. The sequences then in question arise by tensorizing with $C_c^{or}(X_{(q)}, \mathscr{C})$ the sequences

$$\text{(a)} \qquad \cdots \to \mathscr{H}^{\otimes n+1} \xrightarrow{b} \cdots \xrightarrow{b} \mathscr{H} \xrightarrow{\mathrm{i}} \mathbb{C} \to 0$$

and

$$\text{(b)} \qquad \cdots \to \mathscr{H}^{\otimes n+1} \xrightarrow{b'} \cdots \xrightarrow{b'} \mathscr{H} \to 0\,.$$

Recall that

$$b = \sum_{i=0}^{n} (-1)^i d_i \quad \text{and} \quad b' = \sum_{i=0}^{n-1} (-1)^i d_i$$

with

$$d_i = 1 \otimes \cdots \otimes \mathrm{j} \otimes \cdots \otimes 1 \quad \text{and} \quad \mathrm{j} : \mathscr{H} \to \mathbb{C}\,, \quad \varphi \mapsto \int_G \varphi(g)\, dg\,.$$

We see that (b) is the tensor product of (a) by $\mathscr{H}$. Therefore all we need to know is the exactness of (a). But this is a standard fact.

4. Proof of Theorem II.A′

In this section we assume K to be of characteristic 0. We will only sketch the arguments since they are exactly parallel to the corresponding proofs in [Nis]. Recall that g is a semisimple element in $G \backslash G_0^\dagger$. In particular, by the fixed point theorem of Bruhat and Tits, g is noncompact. Note that the centralizer $Z(g)$ of g in G again is a connected reductive group ([SpSt], II. 3.9) and that there exists a continuous section $Z(g)\backslash G \to G$ of the projection map ([BB], 4.1). Exactly the same proof as for [Nis], Prop. 4.4 (iii) then shows that by replacing G by $Z(g)$ it suffices to treat the case of a noncompact element g in the centre of G. In this situation we can write G as $G = G^1 \times \mathbb{Z}$ in such a way that the projection $a \in \mathbb{Z}$ of g is nonzero. We then have the decomposition

$$\mathscr{H} = \mathscr{H}^1 \otimes \mathbb{C}[\mathbb{Z}]$$

where $\mathscr{H}^1$ is the Hecke algebra of G^1 and $\mathbb{C}[\mathbb{Z}]$ is the group ring of $\mathbb{Z}$. Moreover via the projection $G \twoheadrightarrow \mathbb{Z}$ we may view $R^\infty(\mathbb{Z})$ as a subring of $R^\infty(G)$. The above decomposition is $R^\infty(\mathbb{Z})$-linear if $R^\infty(\mathbb{Z})$ acts on the left side through its inclusion into $R^\infty(G)$ and on the right side through the second factor. Clearly it suffices to show that

$$\mathrm{HC}_{*+2}(\mathscr{H})/\mathrm{m}_a \xrightarrow{S} \mathrm{HC}_*(\mathscr{H})/\mathrm{m}_a$$

is zero. Since $\mathscr{H}$ is a crossed product of $\mathscr{H}^1$ by $\mathbb{Z}$ this can be found in [Ni], 2.9.

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