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Arithmetic of formal groups and applications I: Universal norm subgroups

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The theory of the norm map for the formal multiplicative group is well-known as local class field theory. In 1972 Mazur [14] in his generalization of Iwasawa theory to abelian varieties over number fields made quite clear that it is absolutely crucial for such a global theory first to understand the norm map for the formal groups attached to abelian varieties. He achieved this understanding in case of formal groups of multiplicative type by using the theory of proalgebraic groups. Later on, Lubin and Rosen [12] gave a completely elementary approach to Mazur's results. In the meantime Hazewinkel [8] had settled the case of one-dimensional commutative formal Lie groups; his method consists in an explicit and very complicated study of the properties of the power series coefficients of the logarithm of a formal group law by means of higher ramification theory. In the same spirit Vvedenskij [18] and Konovalov [10] reproved and slightly extended Hazewinkel's results.

Let $K/\mathbb{Q}_p$ be a finite extension with ring of integers R and residue class field κ and K_∞/K be a ramified $\mathbb{Z}_p$-extension with ring of integers R_∞ and $\Gamma := \mathrm{Gal}(K_\infty/K)$. Let $\mathscr{G}/R$ denote a smooth connected commutative formal R-group of finite dimension d (i.e., a commutative formal Lie group). If K_n/K is the intermediate layer of degree p^n in K_∞/K, R_n its ring of integers, and $G_n := \mathrm{Gal}(K_n/K)$ its Galois group then we know

– from the description of $\mathscr{G}$ as a formal Lie group that

$$\mathscr{G}(R_n)^{G_n} = \mathscr{G}(R), \quad \text{and}$$

– from the existence of a logarithm for $\mathscr{G}$ that $\mathscr{G}(R_n)$ is a finitely generated $\mathbb{Z}_p$-module with

$$\mathrm{rank}_{\mathbb{Z}_p} \mathscr{G}(R_n) = d \cdot [K_n : \mathbb{Q}_p].$$

Our main aim in this paper is to determine the subgroup

$$N\mathscr{G}(R) := \bigcap_{n \geq 0} \mathrm{Norm}_{K_n/K}\, \mathscr{G}(R_n) \subseteq \mathscr{G}(R)$$

of universal norms with respect to K_∞/K. There are two classes of groups $\mathscr{G}$ with extreme behaviour of the norm map, the groups of multiplicative type for which the subgroup $N\mathscr{G}(R)$ tends to be big, and the groups with connected Cartier dual for which $N\mathscr{G}(R)$ turns out to be zero. In the first paragraph we will show that any group $\mathscr{G}$ has a maximal closed subgroup $\mathscr{G}^{\mathrm{mult}}$ of multiplicative type such that $\mathscr{G}/\mathscr{G}^{\mathrm{mult}}$ exists and is coconnected, i.e., has connected Cartier dual. This will allow us later on to reduce the general case to a separate treatment of the two extreme cases. In the second paragraph we state our main result (Theorem 1) which computes the rank of $N\mathscr{G}(R)$ in terms of elementary invariants of $\mathscr{G}$, and derive consequences of it. For a coconnected $\mathscr{G}$ we will prove the slightly stronger fact that $N\mathscr{G}(R)$ vanishes, from which one easily deduces that the torsion subgroup in $\mathscr{G}(R_\infty)$ is finite. The first main application is to abelian varieties with good reduction over K where we get a formula for the rank of the corresponding universal norm subgroup (Theorem 2). Secondly we determine the $\mathbb{Z}_p[\![\Gamma]\!]$-corank of the relative flat cohomology over R_∞ of any p-divisible group over R and prove that the relative flat cohomology over R_∞ of any connected finite flat commutative R-group scheme vanishes (Theorem 3 and Corollary). This local result then is used to study the $\mathbb{Z}_p[\![\Gamma]\!]$-corank of certain global flat cohomology groups which in Iwasawa theory are naturally associated with an abelian variety over a number field (Theorems 4 and 5). In a subsequent paper we will develop the Iwasawa theory of abelian varieties at primes with nonordinary reduction using the computations in this paper.

The last two paragraphs are devoted to the proof of our main result. Making use of the structural fact established in the first paragraph, together with Mazur's results in the multiplicative type case (for which we include a very short independent treatment), of the properties of the logarithm map for $\mathscr{G}$, and of Tate's study of the local trace map in [17], we first reduce the proof to a problem about coconnected groups $\mathscr{G}$ over the residue class field κ. Over κ, we show that it is easy to obtain a partial answer in the case of the infinite dimensional formal Lie group of Witt covectors. The solution for $\mathscr{G}$ is then derived from this result, taking into account the well-known fact from the theory of Dieudonné modules that $\mathscr{G}$ has a "resolution" in terms of the Witt covectors.

§ 1. General facts about formal Lie groups

In this section we want to show that every commutative formal Lie group over R has a maximal closed subgroup which is a formal Lie group of multiplicative type and which retains its maximality after reduction to κ. Our basic reference is SGA 3 VII_B which we assume the reader is familiar with. In particular, by a formal R-group we always mean a group object in the category of formal R-varieties (loc. cit. § 2.1).

Definition. A commutative formal Lie group $\mathscr{G}$ over R or κ is of multiplicative type if, over the algebraic closure $\bar{\kappa}$ of κ, it is isomorphic to a product of formal multiplicative groups:

$$\mathscr{G}_{/\bar{\kappa}} \cong (\hat{\mathbb{G}}_{m/\bar{\kappa}})^d.$$

Our tool will be the theory of Cartier duals which we therefore have to recall in a form adapted to our purposes. Let $\mathfrak{m}$ denote the maximal ideal in R.

Definition. An affine commutative $\underline{\underline{R}}$-group scheme $G = (G_\nu)_{\nu\in\mathbb{N}}$ is a family of affine commutative $R/\mathfrak{m}^\nu$-group schemes G_ν, for $\nu\in\mathbb{N}$, together with morphisms

$$\begin{array}{ccc} G_\nu & \longrightarrow & G_{\nu+1} \\ \downarrow & & \downarrow \\ \operatorname{Spec}(R/\mathfrak{m}^\nu) & \longrightarrow & \operatorname{Spec}(R/\mathfrak{m}^{\nu+1}) \end{array}$$

such that the induced morphism

$$G_\nu \to G_{\nu+1} \underset{(R/\mathfrak{m}^{\nu+1})}{\times} (R/\mathfrak{m}^\nu)$$

is an isomorphism of $R/\mathfrak{m}^\nu$-group schemes. Furthermore, G is called flat if G_ν is flat over $R/\mathfrak{m}^\nu$ for every $\nu\in\mathbb{N}$.

Cartier duality $\mathscr{G}\mapsto\mathscr{G}^D$ gives an antiequivalence between the category of commutative topologically flat formal R-groups and the category of affine commutative flat $\underline{\underline{R}}$-group schemes (see SGA 3 VII_B 2.2.1 and 2.2.2). We will see that inside $\mathscr{G}^D$ we have its "connected component" which again is an affine commutative flat $\underline{\underline{R}}$-group scheme. Dualizing back then results in a canonical homomorphism $\mathscr{G}\to{}^0\mathscr{G}$ from $\mathscr{G}$ into a commutative topologically flat formal R-group ${}^0\mathscr{G}$ whose Cartier dual is "connected".

First, let $G_{/A}$ be an arbitrary affine commutative group scheme over an artinian local ring A. We write G as a filtered projective limit

$$G = \varprojlim X_i$$

of affine A-schemes X_i of finite type. The zero section of G induces a distinguished section of each X_i and we denote by X_i^0 the connected component of its image point in X_i which is an open and closed subscheme. Then

$$G^0 := \varprojlim X_i^0$$

is a closed subscheme of G.

Remark. (i) G^0 is connected;
(ii) if G is flat over A so, too, is G^0.

Proof. i. By [1] II § 4.3 we have to show that $\mathcal{O}(G^0) = \varinjlim \mathcal{O}(X_i^0)$ contains no idempotents other than 0 and 1. But this holds true for each $\mathcal{O}(X_i^0)$ since the X_i^0 are connected. ii. The closed immersion $G^0\to G$ is flat by [1] I § 2.7 Prop. 9.

The subscheme G^0 in G is defined intrinsically, i.e., independently of the particular choice of the system X_i, and it is a subgroup scheme, since it represents the subgroup functor $\underline{G}^0$ of G defined by

$$\underline{G}^0(S) := \{u \in G(S) \colon u(S) \subseteq \text{connected component of } 0 \text{ in } G\}$$

for any A-scheme S: The inclusion $\underline{G}^0 \subseteq G^0$ is clear from the construction of G^0 and the inclusion $G^0 \subseteq \underline{G}^0$ follows from the connectedness of G^0.

Remark. (SGA 3 VI_B 3.3.) The formation of G^0 commutes with base change.

For every affine commutative flat $\underline{\underline{R}}$-group scheme $G = (G_\nu)$, its "connected component" $G^0 := (G_\nu^0)$ therefore is defined and is again an affine commutative flat $\underline{\underline{R}}$-group scheme. We thus are led to consider, for a commutative topologically flat formal R-group $\mathscr{G}$, the canonical homomorphism

$$\mathscr{G} \to {}^0\mathscr{G} := ((\mathscr{G}^D)^0)^D.$$

But really this homomorphism only can be of use if it is topologically flat. In the following step we establish that fact for locally noetherian $\mathscr{G}$ (in the sense of EGA I (10.4.2)).

Remark. If S is a (commutative) pseudocompact ring which is local and noetherian then its topology is the $\mathfrak{n}$-adic one where $\mathfrak{n}$ denotes the maximal ideal in S.

Proof. Since S is pseudocompact, $\mathfrak{n}$ is a defining ideal for S (in the sense of EGA I (0.7.1.2)). But as finitely generated ideals the $\mathfrak{n}^j$ are closed in S ($\mathfrak{n}^j$ is the image of a continuous S-module homomorphism $S^m \to S$ and any such homomorphism has closed image according to [7] IV § 3 (after Prop. 11)). On the other hand the rings $S/\mathfrak{n}^j$ are artinian. Therefore $\mathfrak{n}^j$ is the intersection of finitely many open ideals in S and is consequently open itself.

Since the topological space which underlies $\mathscr{G}$ is discrete, we see that for $\mathscr{G}$ to be locally noetherian simply means that all the local rings of $\mathscr{G}$ are (complete local) noetherian rings. Put

$$\mathscr{G}^{\mathrm{mult}} := \ker(\mathscr{G} \to {}^0\mathscr{G}),$$

which at least is a commutative formal R-group. We begin by looking more closely at the situation over the residue class field κ of R.

Remark. ([5] III § 3.7.) The category $\mathfrak{A}\mathfrak{c}_\kappa$ of affine commutative group schemes over κ (or any field) is abelian. We furthermore have:

1) For any homomorphism $f \colon G \to G'$ in $\mathfrak{A}\mathfrak{c}_\kappa$, f is a monomorphism, resp. an epimorphism, if and only if f is a closed immersion, resp. faithfully flat, if and only if $\mathcal{O}(f) \colon \mathcal{O}(G') \to \mathcal{O}(G)$ is surjective, resp. injective.

2) A sequence in $\mathfrak{A}\mathfrak{c}_\kappa$ is exact if and only if the corresponding sequence of sheaves on the site of affine κ-schemes with the *fpqc*-topology (coverings are surjective families of flat morphisms; this gives a noetherian site) is exact.

3) For any G in $\mathfrak{A}\mathfrak{c}_\kappa$ the quotient G/G^0 is proetale, i.e., a filtered projective limit of etale finite groups in $\mathfrak{A}\mathfrak{c}_\kappa$.

Lemma 1. *Let $f \colon H \to G$ be a homomorphism in $\mathfrak{A}\mathfrak{c}_\kappa$; if f is a closed immersion then $f^D \colon G^D \to H^D$ is topologically flat and surjective.*

Proof. If $H = \mathrm{Spec}(C)$ and $G = \mathrm{Spec}(B)$ then the homomorphism $B \to C$ is surjective. By duality,

$$C^* := \mathrm{Hom}_\kappa(C, \kappa) \to B^* := \mathrm{Hom}_\kappa(B, \kappa)$$

then is injective and, by [6] I § 6.5, this implies that

$$G^D = Spf(B^*) \to H^D = Spf(C^*)$$

is topologically flat and that C^* is a direct factor of B^* as a C^*-module. For any closed maximal ideal $\mathfrak{n}$ of C^* therefore

$$C^*/\mathfrak{n} = C^* \hat{\otimes}_{C^*} (C^*/\mathfrak{n}) \to B^* \hat{\otimes}_{C^*} (C^*/\mathfrak{n}) = B^*/\overline{\mathfrak{n}B^*}$$

remains injective (for the identity on the right hand side see SGA 3 VII_B 0.3.2) which in particular means $\overline{\mathfrak{n}B^*} \neq B^*$. Since any proper closed ideal in B^* is contained in a closed maximal ideal we see that $Spf(B^*) \to Spf(C^*)$ is surjective.

Corollary 2. *$\mathscr{G}_{/\kappa} \to {}^0\mathscr{G}_{/\kappa}$ is topologically flat and surjective.*

Remark. For a (commutative) local pseudocompact ring S we have:

(i) If the maximal ideal $\mathfrak{n}$ in S is topologically finitely generated then S is noetherian;

(ii) if there is a non-zero topologically flat profinite $\underline{S}$-algebra S' which is noetherian then S is noetherian, too.

Proof. (i) Since in a pseudocompact ring any finitely generated ideal is closed the closed ideal $\mathfrak{n}$ is even finitely generated. Its powers $\mathfrak{n}^j$ consequently also are finitely generated which implies that they are closed and that $\mathfrak{n}/\mathfrak{n}^2$ is a finitely generated $S/\mathfrak{n}$-module. By EGA I (0.7.2.5) S then is noetherian.

(ii) Since S' is a non-zero topologically free S-module we have, for any closed ideal $\mathfrak{a}$ in S,

$$S'/\overline{\mathfrak{a}S'} = S' \hat{\otimes}_S (S/\mathfrak{a}) \cong \prod_{i \in I} (S/\mathfrak{a}) \quad \text{with } I \neq \emptyset$$

as $S/\mathfrak{a}$-modules. In particular, the map $S/\mathfrak{a} \to S'/\overline{\mathfrak{a}S'}$ is injective which means

$$\mathfrak{a} = \overline{\mathfrak{a}S'} \cap S.$$

We see, by our assumption that S' is noetherian, that every increasing sequence of closed ideals in S becomes stationary and that therefore every closed ideal in S is (topologically) finitely generated.

Lemma 3. *If $\mathscr{G}$ is locally noetherian so, too, is ${}^0\mathscr{G}$.*

Proof. Let S be a local ring of ${}^0\mathscr{G}$. By Corollary 2 and the above Remark ii. we know that $S \hat{\otimes}_R \kappa$ is noetherian. But

$$S \hat{\otimes}_R \kappa = S \hat{\otimes}_R (R/\mathfrak{m}) = S/\overline{\mathfrak{m}S} \neq 0.$$

Since $\mathfrak{m}$ is finitely generated any ideal in S which contains $\overline{\mathfrak{m}S}=\mathfrak{m}S$ (in particular the maximal ideal) therefore is finitely generated. Now apply the above Remark i.

Proposition 4. *If $\mathscr{G}$ is locally noetherian then $\mathscr{G}\to{}^0\mathscr{G}$ is topologically flat and surjective.*

Proof. We already know that $\mathscr{G}\to{}^0\mathscr{G}$ is surjective. If T, resp. S, is the local ring of $\mathscr{G}$, resp. ${}^0\mathscr{G}$, in a point, resp. its image point, then we have to show that T is a topologically free S-module. According to Lemma 3 the ring S is noetherian. By SGA VII$_B$ 0.3.8 we therefore equivalently have to show that T is a flat S-module. But since $\mathscr{G}$ and ${}^0\mathscr{G}$ are topologically flat over R the same reference says that T and S are flat R-modules. Furthermore, any continuous homomorphism between local pseudocompact rings is local. We thus can apply EGA I (0.6.6.19) and are reduced to show that $T\otimes_R\kappa$ is a flat $S\otimes_R\kappa$-module. Referring again to SGA 3 VII$_B$ 0.3.8 we finally have to establish that $T\hat{\otimes}_R\kappa$ is a topologically free $S\hat{\otimes}_R\kappa$-module which was done in Corollary 2.

Corollary 5. *If $\mathscr{G}$ is locally noetherian then $\mathscr{G}^{\text{mult}}_{/R}$ is topologically flat.*

Remark. If A is an artinian local ring then the above method gives the following:

1) Let $H\to G$ be a homomorphism of affine commutative flat A-group schemes; if $H\to G$ is a closed immersion and if G^D is locally noetherian then $G^D\to H^D$ is topologically flat and surjective.

2) Let G be an affine commutative flat A-group scheme; if G^D is locally noetherian then the quotient G/G^0 exists and is a proetale A-group scheme (if G itself is of finite type over A then G/G^0 exists by SGA 3 VI$_A$ and is etale).

Lemma 6. *If $\mathscr{G}$ is locally noetherian and connected then $\mathscr{G}_{/\kappa}=\mathscr{G}^{\text{mult}}_{/\kappa}\times{}^0\mathscr{G}_{/\kappa}$.*

Proof. According to [4] p. 36 the formal group $\mathscr{G}_{/\kappa}$ is ind-finite. Its Cartier dual $\mathscr{G}^D_{/\kappa}$ consequently is a profinite κ-group scheme. But for any finite commutative κ-group scheme $G_{/\kappa}$ we have a canonical splitting $G=G^0\times(G/G^0)$ (see [5] II § 5.2.4 or [4] p. 34).

Proposition 7. *If $\mathscr{G}$ is a commutative formal Lie group of finite dimension over R so, too, are $\mathscr{G}^{\text{mult}}$ and ${}^0\mathscr{G}$; furthermore, $\mathscr{G}^{\text{mult}}$ is of multiplicative type, and, for any profinite R-algebra S, the sequence*

$$0\to\mathscr{G}^{\text{mult}}(S)\to\mathscr{G}(S)\to{}^0\mathscr{G}(S)\to 0$$

is exact.

Proof. From the splitting in Lemma 6 one deduces by flatness arguments our first assertion (using Corollary 5) and (compare the proof of Prop. 4 in [17]) the existence of a section of the map $\mathscr{G}\to{}^0\mathscr{G}$. Since $\mathscr{G}^{\text{mult}}_{/\kappa}$ is smooth and its Cartier dual is proetale, the Frobenius map for $\mathscr{G}^{\text{mult}}_{/\kappa}$ is an isogeny and the Verschiebung is an isomorphism. Therefore multiplication by p on $\mathscr{G}^{\text{mult}}_{/\kappa}$ is an isogeny, i.e., $\mathscr{G}^{\text{mult}}_{/\kappa}$ comes from a p-divisible group whose dual p-divisible group is etale (see below). We conclude that $\mathscr{G}^{\text{mult}}_{/\kappa}$ is a product of formal multiplicative groups.

We finish this paragraph by shortly reviewing the connection between p-divisible groups and commutative formal Lie groups – a connection which is of use at several places in the paper. Our basic reference, of course, is [17]. If $\mathfrak{G}=(\mathfrak{G}_\nu)_{\nu\in\mathbb{N}}$ is a p-divisible group over R with $\mathfrak{G}_\nu=\operatorname{Spec}(S_\nu)$ then

$$\hat{\mathfrak{G}}:=Spf(\varprojlim S_\nu)$$

(as finitely generated free R-modules the S_ν have a natural topology and so has $\varprojlim S_\nu$) is a commutative formal R-group. On the other hand, the dual p-divisible group $\tilde{\mathfrak{G}}$ is defined by $\tilde{\mathfrak{G}}:=(\mathfrak{G}^D_\nu)_{\nu\in\mathbb{N}}$. The fundamental properties of these notions are the following:

1) The functor $\mathfrak{G}\mapsto\hat{\mathfrak{G}}$ induces an equivalence between the category of connected p-divisible groups over R and the category of divisible commutative formal Lie groups of finite dimension over R.

2) The relation between Cartier dual and dual p-divisible group is given by

$$\hat{\mathfrak{G}}^D=\varprojlim\tilde{\mathfrak{G}}_\nu.$$

3) As a finite flat ind-R-group scheme $\mathfrak{G}$ represents a sheaf on the $fpqf$-site on $\operatorname{Spec}(R)$. On the other hand, $\hat{\mathfrak{G}}$ represents a sheaf on the formal flat site on $Spf(R)$ (see SGA 3 VII$_B$ 1.5).

There is the following connection (in a special case)

$$\hat{\mathfrak{G}}(R)=\varprojlim\mathfrak{G}(R/\mathfrak{m}^i).$$

4) Any p-divisible group $\mathfrak{G}$ over R has a canonical filtration

$$0\subseteq\mathfrak{G}^{\text{mult}}\subseteq\mathfrak{G}^0\subseteq\mathfrak{G}$$

by (closed) p-divisible subgroups such that $(\mathfrak{G}^{\text{mult}})^\sim$ and $\mathfrak{G}/\mathfrak{G}^0$ are etale and $\mathfrak{G}^0/\mathfrak{G}^{\text{mult}}$ is local-local (i.e., connected with connected dual). Over the residue class field κ this filtration splits. Apparently we have $(\mathfrak{G}^{\text{mult}})^\wedge=(\hat{\mathfrak{G}})^{\text{mult}}$.

These notions naturally arise in the context of an abelian scheme $\mathscr{A}_{/R}$. We have its p-divisible group $\mathscr{A}(p)$ over R and the commutative formal Lie group $\hat{\mathscr{A}}$ over R which is the formal completion of $\mathscr{A}$ in the zero section of the closed fibre. The connection between the two is given by

$$\hat{\mathscr{A}}=(\mathscr{A}(p)^0)^\wedge.$$

§ 2. Statement of the result and applications

Our main result the proof of which will occupy the Paragraphs 3 and 4 is the following where from now on $\mathscr{G}_{/R}$ always denotes a commutative formal Lie group of finite dimension.

Theorem 1.

(i) $\operatorname{rank}_{\mathbb{Z}_p}\mathscr{G}(R)/N\mathscr{G}(R)=\dim{}^0\mathscr{G}\cdot[K:\mathbb{Q}_p]+\operatorname{rank}_{\mathbb{Z}_p}(\mathscr{G}^{\text{mult}})^D(\kappa)$;

(ii) *if $\mathscr{G}^{\text{mult}}=0$ then $N\mathscr{G}(R)=0$.*

We immediately note two easy consequences.

Corollary. *If $\mathcal{G}^{\text{mult}}=0$ then the torsion subgroup in $\mathcal{G}(R_\infty):=\bigcup_{n\geq 0}\mathcal{G}(R_n)$ is finite.*

Proof. In the language of [9] the discrete Γ-module $\operatorname{Tor}\mathcal{G}(R_\infty)$ is strictly Γ-finite. By the results of loc.cit. $\operatorname{Tor}\mathcal{G}(R_\infty)$ therefore has a Γ-submodule M of finite index for which the norm maps

$$H^0(\operatorname{Gal}(K_\infty/K_m), M)\to H^0(\operatorname{Gal}(K_\infty/K_n), M)$$

for $m\geq n\geq 0$ are surjective. From Theorem 1 ii. we see that M must vanish.

Corollary. *For any local-local p-divisible group $\mathfrak{G}_{/R}$ the group $\mathfrak{G}(R_\infty)$ is finite.*

Proof. $\mathfrak{G}(R_\infty)=\operatorname{Tor}\hat{\mathfrak{G}}(R_\infty)$.

The main application of the above theorem is the solution of the norm problem for abelian varieties with good reduction. If $A_{/K}$ is an abelian variety then, as the work in [14] and in [16] § 3 (see also below) shows, it is a basic arithmetic problem to determine the subgroup $NA(K)\subseteq A(K)$ of universal norms with respect to K_∞/K. If A has good reduction its Néron model $\mathcal{A}_{/R}$ is an abelian scheme such that we can apply our above result to the commutative formal Lie group $\hat{\mathcal{A}}_{/R}$ of dimension equal to $\dim A$.

Theorem 2. *If $A_{/K}$ is an abelian variety with good reduction then*

$$\operatorname{rank}_{\mathbb{Z}_p}A(K)/NA(K)=(\dim A-r)\cdot[K:\mathbb{Q}_p]$$

where r denotes the p-rank of the reduction of A.

Proof. (We recall that r is defined by

$$p^r=\#\{x\in\mathcal{A}(\bar{\kappa}): px=0\}.)$$

Because of our assumption of good reduction the formation of the Néron model commutes with base change such that the sequences

$$0\to\hat{\mathcal{A}}(R_n)\to A(K_n)\to\mathcal{A}(\kappa_n)\to 0$$

with κ_n the residue class field of R_n are exact for all $n\geq 0$. Since the $\mathbb{Z}_p$-extension K_∞/K is ramified we see that the sequence of finite groups $\mathcal{A}(\kappa_n)$ becomes stationary. Consequently, we have

$$\operatorname{rank}_{\mathbb{Z}_p}A(K)/NA(K)=\operatorname{rank}_{\mathbb{Z}_p}\hat{\mathcal{A}}(R)/N\hat{\mathcal{A}}(R).$$

The assertion therefore follows from Theorem 1 i. if we show that

$$\dim{}^0\hat{\mathcal{A}}=\dim A-r\quad\text{and}\quad\operatorname{rank}_{\mathbb{Z}_p}(\hat{\mathcal{A}}^{\text{mult}})^D(\kappa)=0$$

hold true. Making use of the relation $\hat{\mathcal{A}}=(\mathcal{A}(p)^0)^\wedge$ between $\hat{\mathcal{A}}$ and the p-divisible group $\mathcal{A}(p)$ of $\mathcal{A}$ we compute

$$\begin{aligned}\dim{}^0\hat{\mathcal{A}}&=\dim\hat{\mathcal{A}}-\dim\hat{\mathcal{A}}^{\text{mult}}=\dim A-\dim\mathcal{A}(p)^{\text{mult}}\\&=\dim A-\text{height }\mathcal{A}(p)^{\text{mult}}\end{aligned}$$

and

$$\operatorname{rank}_{\mathbb{Z}_p}(\hat{\mathcal{A}}^{\text{mult}})^D(\kappa)=\operatorname{corank}(\mathcal{A}(p)^{\text{mult}})^\sim(\kappa).$$

By the duality of abelian varieties we know that $\mathcal{A}(p)$ is isogeneous to its dual p-divisible group; thus

$$\begin{aligned}\text{height }\mathcal{A}(p)^{\text{mult}}&=\text{height }(\mathcal{A}(p)^{\text{mult}})^\sim\\&=\text{height }\mathcal{A}(p)/\mathcal{A}(p)^0\\&=\operatorname{corank}\mathcal{A}(p)(\bar{\kappa})=r\end{aligned}$$

and

$$\operatorname{corank}(\mathcal{A}(p)^{\text{mult}})^\sim(\kappa)=\operatorname{corank}\mathcal{A}(p)(\kappa)=0.$$

The second application concerns the rank of certain local $\mathbb{Z}_p[\![\Gamma]\!]$-modules and accomplishes work in [16]. If $\mathfrak{G}_{/R}$ is an arbitrary p-divisible group over R we are interested in the relative flat cohomology group $H^2_{\cdot}(R_\infty,\mathfrak{G})$ which, in a natural way, is a discrete $\mathbb{Z}_p[\![\Gamma]\!]$-module; let $\mathfrak{G}^{\text{et}}:=\mathfrak{G}/\mathfrak{G}^0$ be the etale part of $\mathfrak{G}$.

Theorem 3. *We have $H^2_{\cdot}(R_\infty,\mathfrak{G})=H^2_{\cdot}(R_\infty,\mathfrak{G}^{\text{et}})$, and the Pontrjagin dual $H^2_{\cdot}(R_\infty,\mathfrak{G})^*$ is a finitely generated $\mathbb{Z}_p[\![\Gamma]\!]$-module of rank equal to height $(\mathfrak{G}^{\text{et}})\cdot[K:\mathbb{Q}_p]$.*

Proof. Repeat the proof of [16] A § 3 Prop. 4 using now, of course, the above Theorem 1 ii.

Corollary. *For any connected finite flat commutative R-group scheme G we have $H^i_{\cdot}(R_\infty,G)=0$ for $i\geq 0$.*

Proof. For $i\neq 2$ see [15] Lemma (3.5) (which only is an appropriate combination of results in [13]). The case $i=2$ is a consequence of Theorem 3 since, by a theorem of Oort, any such G possesses a "resolution" by connected p-divisible groups (see [13] (2.5)).

The above theorem has consequences in the global arithmetic of abelian varieties. We fix

a finite extension $k/\mathbb{Q}$ with ring of integers $\mathcal{o}$,
an odd prime number p,
a $\mathbb{Z}_p$-extension k_∞/k with ring of integers $\mathcal{o}_\infty$ and
$\Gamma:=\operatorname{Gal}(k_\infty/k)$,
an abelian variety $A_{/k}$ with Néron model $\mathcal{A}_{/\mathcal{o}}$,

and we denote by Σ the finite set of primes of k which are ramified in k_∞/k. Furthermore we always assume that.

A has good reduction at all primes of k above p!

The central object in the arithmetic of A with respect to p is the $\mathbb{Z}_p[\![\Gamma]\!]$-module $H^1(\mathcal{o}_\infty,\mathcal{A}(p))$ where cohomology always is flat cohomology if not otherwise indicated and where $\mathcal{A}(p)$ denotes the p-primary torsion in $\mathcal{A}$. It follows from the weak Mordell-Weil theorem that the Pontrjagin dual $H^1(\mathcal{o}_\infty,\mathcal{A}(p))^*$ is finitely generated over $\mathbb{Z}_p[\![\Gamma]\!]$. Therefore, the first problem which arises is the computation of the $\mathbb{Z}_p[\![\Gamma]\!]$-rank ρ of $H^1(\mathcal{o}_\infty,\mathcal{A}(p))^*$. In [16] § 3 we began the discussion of ρ and in the following we simply restate the results of that paper proved there under additional assumptions on A which now, in the light of Theorem 3, turn out to be superfluous. There is an additional global invariant coming in which is

$$\rho':=\operatorname{rank}_{\mathbb{Z}_p[\![\Gamma]\!]}H^2(\mathcal{o}_\infty,\mathcal{A}(p))^*.$$

Remark. The structure of $H^2(\mathfrak{o}_\infty, \mathscr{A}(p))^*$ is almost completely determined by ρ'. We namely have an injective quasi-isomorphism

$$H^2(\mathfrak{o}_\infty, \mathscr{A}(p))^* \to \mathbb{Z}_p[\![\Gamma]\!]^{\rho'} \oplus \varprojlim A(k_\infty)_{p^\nu}.$$

Proof. See [16] §1 Prop. 8; the assumption made there is fulfilled (for n big enough which suffices as the proof of that proposition shows) by Theorem 3 and the proof of [16] §3 Lemma 6.

For $\mathfrak{p} \in \Sigma$ let $k_\mathfrak{p}$ be the completion of k at $\mathfrak{p}$ and put $r_\mathfrak{p} := p$-rank of the reduction of A at $\mathfrak{p}$.

Proposition. $\rho = \rho' + \sum_{\mathfrak{p} \in \Sigma} (\dim A - r_\mathfrak{p}) \cdot [k_\mathfrak{p} : \mathbb{Q}_p]$.

Proof. Theorem 3 and [16] §3 Lemma 2.

As a consequence we at least have a lower bound for ρ in terms of local invariants of A.

Theorem 4. *If $A(k)$ and the p-component of the Tate-Šafarevič group of $A_{/k}$ are finite then $\rho' = 0$ and*

$$\rho = \sum_{\mathfrak{p} \in \Sigma} (\dim A - r_\mathfrak{p}) \cdot [k_\mathfrak{p} : \mathbb{Q}_p].$$

Proof. Theorem 3 and [16] §3 Lemma 6.

Examples show that the behaviour of ρ' may be rather complicated in general. The situation changes if we restrict ourselves to a consideration of the cyclotomic $\mathbb{Z}_p$-extension only - which we do from now on - since it almost certainly has the following additional property.

Conjecture. *If $\mathfrak{o}'_\infty$ is the ring of p-integers in the cyclotomic $\mathbb{Z}_p$-extension of k then*

$$cd_p(\mathfrak{o}'_\infty)_{\text{et}} \leqq 1.$$

We have seen in [16] §3 Lemma 8 that the above conjecture for all k is equivalent to Iwasawa's conjecture about "$\mu = 0$" (this μ is the μ-invariant of the $\mathbb{Z}_p[\![\Gamma]\!]$-module $H^1_{\text{et}}(\mathfrak{o}'_\infty, \mathbb{Q}_p/\mathbb{Z}_p)^*$) for all k. The latter conjecture was proved by Ferrero/Washington for abelian $k/\mathbb{Q}$.

Theorem 5. *If $r_\mathfrak{p} = 0$ for $\mathfrak{p} | p$ and if $cd_p(\mathfrak{o}'_\infty)_{\text{et}} \leqq 1$ then $H^1(\mathfrak{o}_\infty, \mathscr{A}(p))^*$ is $\mathbb{Z}_p$-torsion free and has $\mathbb{Z}_p[\![\Gamma]\!]$-rank $\rho = \dim A \cdot [k : \mathbb{Q}]$.*

Proof. From Theorem 3 and our first assumption we deduce

$$H^i(\mathfrak{o}_\infty, \mathscr{A}(p)) = H^i_{\text{et}}(\mathfrak{o}'_\infty, \mathscr{A}(p)) \quad \text{for } i \geqq 0;$$

compare the proof of §3 Lemma 7 in [16]. Now, on $(\mathfrak{o}'_\infty)_{\text{et}}$ we have an exact sequence of sheaves

$$0 \to \mathscr{A}_p \to \mathscr{A}(p) \xrightarrow{p} \mathscr{A}(p) \to \mathscr{F} \to 0$$

where $\mathscr{F}$ is a skyscraper sheaf. Since the residue class fields of $\mathfrak{o}'_\infty$ are p-closed we have

$$H^i_{\text{et}}(\mathfrak{o}'_\infty, \mathscr{F}) = 0 \quad \text{for } i > 0$$

and therefore the exact commutative diagram

$$\begin{tikzcd}
H^1_{\text{et}}(\mathfrak{o}'_\infty, \mathscr{A}(p)) \arrow[r] \arrow[dr, "p"'] & H^1_{\text{et}}(\mathfrak{o}'_\infty, p\mathscr{A}(p)) \arrow[r] \arrow[d] & H^2_{\text{et}}(\mathfrak{o}'_\infty, \mathscr{A}_p) \arrow[r] & H^2_{\text{et}}(\mathfrak{o}'_\infty, \mathscr{A}(p))_p \arrow[r] & 0 \\
& H^1_{\text{et}}(\mathfrak{o}'_\infty, \mathscr{A}(p)) \arrow[d] & & & \\
& 0 & & &
\end{tikzcd}$$

Our second assumption implies $H^2_{\text{et}}(\mathfrak{o}'_\infty, \mathscr{A}_p) = 0$ and consequently, on the one hand side, the vanishing of $H^2_{\text{et}}(\mathfrak{o}'_\infty, \mathscr{A}(p))_p$ which means $\rho' = 0$ and on the other side the divisibility of $H^1_{\text{et}}(\mathfrak{o}'_\infty, \mathscr{A}(p))$ which means that its Pontrjagin dual is $\mathbb{Z}_p$-torsion free.

Remarks. 1) Let k_0 be the fixed field of a p-Sylow subgroup in $\mathrm{Gal}(k(A_p)/k)$ where $k(A_p)$ denotes the field k adjoined all p-torsion points on A. By a closer look at the "method of the trace" in SGA 4 IX §5 one can weaken the assumption "$cd_p(\mathfrak{o}'_\infty)_{\text{et}} \leqq 1$" in the above theorem to the assumption that Iwasawa's conjecture about "$\mu = 0$" holds true for the field k_0.

2) If we drop the first assumption in the theorem then we only get an upper bound for ρ: If Iwasawa's conjecture about "$\mu = 0$" holds true for the field k_0 then $\rho \leqq \dim A \cdot [k : \mathbb{Q}]$. (Compare [11].)

3) Let $A_{/k}$ be an elliptic curve with supersingular reduction at all primes above p; assume that there is an abelian extension $K/\mathbb{Q}$ with $K \supseteq k$ such that $A(K)$ contains a nontrivial point of order p. Then the conclusion of Theorem 5 holds true. In fact, by the first remark and the Ferrero-Washington theorem, it suffices to show that $k_0/\mathbb{Q}$ is abelian in this situation. But, since we can assume that K also contains a primitive p-th root of unity, we have an exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to A_p \to \mathbb{Z}/p\mathbb{Z} \to 0$$

of Galois modules over K; consequently, $K(A_p)/K$ is a p-extension which implies $k_0 \subseteq K$.

We strongly suspect that (in case of the cyclotomic $\mathbb{Z}_p$-extension) ρ' always vanishes which by the above proposition would lead to an explicit and simple formula for ρ. Additional evidence in case of ordinary reduction at p is provided by the theory of p-adic height pairings.

§3. Reduction to a problem over the residue class field

We begin by proving our Theorem 1 in the multiplicative type case. Instead of deducing this case from [14] §4 or [12] we prefer to give a short independent proof. If we view $\mathscr{G} = \hat{\mathfrak{G}}$ as associated with a p-divisible group $\mathfrak{G}_{/R}$ whose dual p-divisible group $\tilde{\mathfrak{G}}$ is etale then we have to show

$$\mathrm{rank}_{\mathbb{Z}_p} \hat{\mathfrak{G}}(R)/N\hat{\mathfrak{G}}(R) = \mathrm{corank}\, \tilde{\mathfrak{G}}(\kappa).$$

According to [13] (1.10(ii) and p. 357) the relative cohomology sequence

$$0 \to H^1(R, \tilde{\mathfrak{G}}) \to H^1(K, \tilde{\mathfrak{G}}) \to H^2_!(R, \tilde{\mathfrak{G}}) \to 0$$

is exact and there is the canonical isomorphism

$$H^2_!(R, \tilde{\mathfrak{G}})^* = \hat{\mathfrak{G}}(R).$$

Since $\tilde{\mathfrak{G}}_{/R}$ is etale we have $H^1(R, \tilde{\mathfrak{G}}) = H^1(\kappa, \tilde{\mathfrak{G}})$. If we fix $n \geqq 0$ such that K_n/K is unramified and K_∞/K_n is totally ramified we therefore get the commutative exact diagram

$$\begin{tikzcd}
 & H^2(G_n, \tilde{\mathfrak{G}}(\kappa_n)) & & & \\
0 \arrow[r] & H^1(\kappa_n, \tilde{\mathfrak{G}})^\Gamma \arrow[r] \arrow[u] & H^1(K_\infty, \tilde{\mathfrak{G}})^\Gamma \arrow[r] & ((\varprojlim \hat{\mathfrak{G}}(R_n))_\Gamma)^* & \\
0 \arrow[r] & H^1(\kappa, \mathfrak{G}) \arrow[r] \arrow[u] & H^1(K, \mathfrak{G}) \arrow[r] \arrow[u] & \hat{\mathfrak{G}}(R)^* \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & H^1(G_n, \tilde{\mathfrak{G}}(\kappa_n)) \arrow[r] \arrow[u] & H^1(\Gamma, \tilde{\mathfrak{G}}(R_\infty)) \arrow[r] \arrow[u] & (\hat{\mathfrak{G}}(R)/N\hat{\mathfrak{G}}(R))^* \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where κ_n is the residue class field of R_n. Since the groups $H^i(G_n, \tilde{\mathfrak{G}}(\kappa_n))$ for $i \geqq 1$ are finite we conclude

$$\begin{aligned}
\operatorname{rank}_{\mathbb{Z}_p} \hat{\mathfrak{G}}(R)/N\hat{\mathfrak{G}}(R) &= \operatorname{corank} H^1(\Gamma, \tilde{\mathfrak{G}}(R_\infty)) \\
&= \operatorname{corank} H^0(\Gamma, \tilde{\mathfrak{G}}(R_\infty)) = \operatorname{corank} \tilde{\mathfrak{G}}(R) = \operatorname{corank} \tilde{\mathfrak{G}}(\kappa).
\end{aligned}$$

In the next paragraph we will establish the following assertion about formal Lie groups over the finite field κ.

Key Proposition. *For any commutative formal Lie group $\mathscr{G}_{/\kappa}$ of finite dimension over κ with $\mathscr{G}^{\mathrm{mult}} = 0$ we have $\varprojlim \mathscr{G}(R_n \underset{R}{\otimes} \kappa) = 0$.*

Remark. The easiest way to see that the groups $\mathscr{G}(R_n \underset{R}{\otimes} \kappa)$ form a projective system with respect to the norm maps seems to be to observe that $\mathscr{G}_{/\kappa}$ is liftable to R (see [6] p. 184).

Let $\mathscr{G}_{/R}$ be again arbitrary. In order to deduce Theorem 1 from the Key Proposition we make use of the existence of a logarithm map for $\mathscr{G}$. Fix in the following an R-isomorphism

$$\mathscr{G} \cong Spf(R[\![X_1, \ldots, X_d]\!]) \quad \text{with } d = \dim \mathscr{G}. \tag{$*$}$$

There are three basic facts to recall. First, we have the exact sequences of reduction

$$0 \to \mathscr{G}(\mathfrak{m}R_n) \to \mathscr{G}(R_n) \to \mathscr{G}(R_n \underset{R}{\otimes} \kappa) \to 0 \tag{1}$$

where $\mathfrak{m}$ denotes the maximal ideal in R and $\mathscr{G}(\mathfrak{m}^i R_n)$ are the obvious congruence subgroups of $\mathscr{G}(R_n)$. Secondly, if e is the absolute ramification index of K and if we put $\varepsilon := \left[\frac{e}{p-1}\right] + 1$ then the logarithm map for $\mathscr{G}$ induces isomorphisms

$$\mathscr{G}(\mathfrak{m}^\varepsilon R_n) \xrightarrow{\cong} (\mathfrak{m}^\varepsilon R_n)^d \tag{2}$$

(see [17] p. 169 or [2] III § 7.6). Finally, according to [2] III § 7.4 Prop. 5(iv) there are isomorphisms

$$\mathscr{G}(\mathfrak{m}^i R_n)/\mathscr{G}(\mathfrak{m}^{i+1} R_n) \xrightarrow{\cong} (\mathfrak{m}^i R_n/\mathfrak{m}^{i+1} R_n)^d \quad \text{for } i \geqq 1. \tag{3}$$

The maps in (2) and (3) are natural once we have fixed ($*$). In addition, we rely on the known behaviour of the usual trace map. Let $|\,|$ be the absolute value of K canonically extended to K_∞.

Lemma (Tate). *There is a constant c independent of n such that*

$$|\mathrm{Trace}_{K_n/K}(x)| \leqq |p|^{n-c} \cdot |x| \quad \textit{for } x \in K_n.$$

Proof. See Corollary 3 on p. 171 in [17].

Lemma. $\varprojlim \mathscr{G}(R_n) = \varprojlim \mathscr{G}(R_n \underset{R}{\otimes} \kappa)$.

Proof. Since the reduction sequences (1) are exact sequences of compact groups passing to the projective limit is exact and gives the exact sequence

$$0 \to \varprojlim \mathscr{G}(\mathfrak{m}R_n) \to \varprojlim \mathscr{G}(R_n) \to \varprojlim \mathscr{G}(R_n \underset{R}{\otimes} \kappa) \to 0.$$

By combining (2), (3), and the above Lemma of Tate we easily see that $\varprojlim \mathscr{G}(\mathfrak{m}R_n) = 0$.

In case $\mathscr{G}^{\mathrm{mult}} = 0$ we therefore get as a consequence of the Key Proposition that

$$\varprojlim \mathscr{G}(R_n) = 0;$$

from the exact sequence

$$0 \to \varprojlim H^{-1}(G_n, \mathscr{G}(R_n)) \to (\varprojlim \mathscr{G}(R_n))_\Gamma \to \mathscr{G}(R) \to \mathscr{G}(R)/N\mathscr{G}(R) \to 0$$

then follows

$$N\mathscr{G}(R) = 0 \quad \text{and} \quad \varprojlim H^{-1}(G_n, \mathscr{G}(R_n)) = 0.$$

In order to settle the general case we use Proposition 7 in § 1 which gives the exact cohomology sequence

$$\begin{aligned}
0 = \varprojlim H^{-1}(G_n, {}^0\mathscr{G}(R_n)) &\to \mathscr{G}^{\mathrm{mult}}(R)/N\mathscr{G}^{\mathrm{mult}}(R) \to \\
&\to \mathscr{G}(R)/N\mathscr{G}(R) \to {}^0\mathscr{G}(R)/N\,{}^0\mathscr{G}(R) \to 0.
\end{aligned}$$

Since Theorem 1 already is proved for $\mathscr{G}^{\mathrm{mult}}$ and for ${}^0\mathscr{G}$ the vanishing of the left hand term shows that it also holds true for $\mathscr{G}$.

§ 4. Solution of the problem

Here we give the proof of the Key Proposition which, as we have seen, implies Theorem 1. A main tool will be the infinite dimensional commutative formal Lie group $\hat{CW}_{/\mathbb{Z}_p}$ of Witt covectors over $\mathbb{Z}_p$ (see [6]). Let $W_{m/\mathbb{Z}}$ be the commutative $\mathbb{Z}$-group scheme of Witt vectors of length m (relative to p) and let $\hat{W}_{m/\mathbb{Z}_p}$ be its formal completion in the zero section of the fibre above p; $\hat{W}_m$ is a commutative formal Lie group of dimension m over $\mathbb{Z}_p$. We put

$$\hat{CW} := \varinjlim \hat{W}_m.$$

A convenient way to write down the underlying $\mathbb{Z}_p$-formal functor is

$$\hat{CW}(A) = \{(\ldots, a_{-i}, \ldots, a_{-1}, a_0) : a_{-i} \in \mathrm{radical}(A) \text{ and } a_{-i} = 0 \text{ for almost all } i \geq 0\}$$

for any (commutative) finite $\mathbb{Z}_p$-algebra A. (Warning: In [6], $\hat{CW}$ denotes a bigger formal group of which our $\hat{CW}$ is the unipotent and connected part.) The Verschiebung on $\hat{CW}_{/\mathbb{F}_p}$ has the following explicit description

$$V((\ldots, a_{-i}, \ldots, a_0)) = (\ldots, a_{-i-1}, \ldots, a_{-1}).$$

Lemma 1. *The Verschiebung induces an automorphism of* $\varprojlim \hat{CW}(R_n \otimes_R \kappa)$.

Proof. Since the left hand terms in the exact sequences

$$0 \to \hat{\mathbb{G}}_a(R_n \otimes_R \kappa) \to \hat{CW}(R_n \otimes_R \kappa) \xrightarrow{V} \hat{CW}(R_n \otimes_R \kappa) \to 0$$

are finite passing to the projective limit is exact and shows that we have to prove

$$\varprojlim \hat{\mathbb{G}}_a(R_n \otimes_R \kappa) = 0$$

(which, of course, is our Key Proposition in case $\mathscr{G} = \hat{\mathbb{G}}_a$). But this is an easy consequence of the two lemmata in § 3.

Let now $\mathscr{G}_{/\kappa}$ be a commutative formal Lie group of finite dimension over κ with $\mathscr{G}^{\mathrm{mult}} = 0$. The use of the Witt covectors consists in the fact that under this assumption there is an exact sequence of commutative formal κ-groups

$$0 \to \mathscr{G} \to (\hat{CW}_{/\kappa})^s \to (\hat{CW}_{/\kappa})^t$$

with appropriate $s, t \in \mathbb{N}$. This is a consequence of the theory of Dieudonné modules (see [3] Remark 4.2.a) or [6] Chap. III and Remark 3 on p. 92). Because of the functoriality of the Verschiebung Lemma 1 applied to the above "resolution" of $\mathscr{G}$ then leads to an analogous assertion for $\mathscr{G}$.

Lemma 2.

$$V^m : \varprojlim \mathscr{G}^{(p^m)}(R_n \otimes_R \kappa) \to \varprojlim \mathscr{G}(R_n \otimes_R \kappa)$$

is an isomorphism for any $m \geq 1$.

A second consequence of our assumption $\mathscr{G}^{\mathrm{mult}} = 0$ is that for every $n \geq 0$ there exists a $m \geq 1$ such that

$$V^m : \mathscr{G}^{(p^m)}(R_n \otimes_R \kappa) \to \mathscr{G}(R_n \otimes_R \kappa)$$

is the zero map. In the projective limit this gives the following property.

Lemma 3. *For any open subgroup U of the profinite group $\varprojlim \mathscr{G}(R_n \otimes_R \kappa)$ there is a $m \geq 1$ such that*

$$V^m(\varprojlim \mathscr{G}^{(p^m)}(R_n \otimes_R \kappa)) \subseteq U.$$

The combination of Lemma 2 and 3 obviously shows the vanishing of $\varprojlim \mathscr{G}(R_n \otimes_R \kappa)$, i.e., proves our Key Proposition.

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