

## ***p*-Adic Height Pairings I**

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Let  $A$  and  $\tilde{A}$  be dual abelian varieties over the finite algebraic number field  $k$ . We have the well known Néron-Tate height pairing ([9])

$$A(k) \times \tilde{A}(k) \rightarrow \mathbb{R}, \tag{*}$$

the determinant of which plays an important rôle in the fine form of the conjecture of Birch and Swinnerton-Dyer. Our aim in this paper is to define a pairing

$$A(k) \times \tilde{A}(k) \rightarrow \mathbb{Q}_p \tag{*_p}$$

which can be regarded as a  $p$ -adic analogue of (\*). The basic idea is to imitate Bloch's description of (\*) in [2]. After the preparatory Sects. 1 and 2 we shall give the definition in Sect. 3 under a certain condition on the prime  $p$ . In Sect. 4 it is shown that (\*<sub>p</sub>) possesses a decomposition into a product of local factors. The final section contains a computation of (\*<sub>p</sub>) in terms of  $p$ -adic theta functions in the case that  $A$  is a Tate curve at  $p$ . In a second part of this paper we plan to discuss the relationship between our pairing (\*<sub>p</sub>) and the cohomologically defined pairing in [14]. It should be mentioned that, for  $A$  an elliptic curve with complex multiplication, there exist already several definitions of  $p$ -adic height pairings in the literature (see [1, 4, 11, 12]).

### **1. *p*-Adic Valuations**

Let  $K/\mathbb{Q}_l$  be a finite extension and denote by  $|\cdot|_l: \mathbb{Q}_l^\times \rightarrow \mathbb{Q}^\times$  the normalized ( $|\cdot|_l = l^{-v}$ ) absolute value of  $\mathbb{Q}_l$ . Once for all we fix the prime number  $p$ . The homomorphism  $v_K: K^\times \rightarrow \mathbb{Z}_p$  (the " $p$ -adic valuation" of  $K$ ) is defined by

$$v_K := \begin{cases} \log_p \circ \text{Norm}_{K/\mathbb{Q}_p} & \text{if } l = p, \\ \log_p \circ |\cdot|_l \circ \text{Norm}_{K/\mathbb{Q}_p} & \text{if } l \neq p. \end{cases}$$

Here  $\log_p: \mathbb{Q}_p^\times \rightarrow \mathbb{Z}_p$  denotes the  $p$ -adic logarithm extended by the usual convention  $\log_p p = 0$ .

For a finite extension  $k/\mathbb{Q}$  and any finite prime  $p$  of  $k$ , let  $k_p$  denote the completion of  $k$  at  $p$  and put  $v_p := v_{k_p}$ . Moreover, let  $A := A_k$  be the group of finite adèles of  $k$ . We have the homomorphism

$$v = v_k : \mathbb{G}_m(A) \rightarrow \mathbb{Z}_p$$

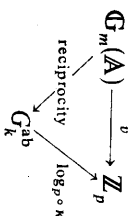
$$(a_p)_p \mapsto \sum_p v_p(a_p).$$

**Lemma.** (Product formula). For  $a \in k^\times$  we have  $v(a) = 0$ .

*Proof.* Because of the properties of the norm maps it suffices to prove the lemma in case  $k = \mathbb{Q}$  where it is an easy exercise.

Define the cyclotomic character  $\kappa : G_k^{\text{ab}} \rightarrow \mathbb{Z}^p$  on the maximal abelian factor group  $G_k^{\text{ab}}$  of the absolute Galois group  $G_k$  of  $k$  by  $\sigma \zeta = \zeta^{\kappa(\sigma)}$  for  $\sigma \in G_k^{\text{ab}}$  and any  $p$ -power root of unity  $\zeta$ .

**Proposition.** The diagram



is commutative.

*Proof.* The proof again reduces to the case  $k = \mathbb{Q}$  by applying the norm map. But for  $k = \mathbb{Q}$  we can check the assertion by explicit computation (compare the last formula on p. 240 in [10]).

**Corollary.**  $[\mathbb{Z}_p : \text{im } v] = p \cdot [(\mathbb{Q}(\mu(p)) \cap k : \mathbb{Q}]_p^{-1}$  if  $p \neq 2$ .

**2. Universal Norms**

Let  $K/\mathbb{Q}_p$  be a finite extension, define  $K_\infty/K$  to be the  $\mathbb{Z}_p$ -extension which is contained in the field  $K(\mu(p))$  of all  $p$ -power roots of unity over  $K$ , and put  $\Gamma := \text{Gal}(K_\infty/K)$ .

*Problem.* Determine the subgroup  $NG(K) \leq G(K)$  of universal norms with respect to  $K_\infty/K$  for any commutative  $K$ -group scheme  $G/K$ .

**Proposition 1.** i)  $NG_m(K) = \ker v_K$ ;  
 ii)  $[\mathbb{Z}_p : \text{im } v_K] = p \cdot [(\mathbb{Q}_p(\mu(p)) \cap K : \mathbb{Q}_p]_p^{-1}$  if  $p \neq 2$ .

*Proof.* This is a local version of the proposition in the preceding section. See also the proof of Lemma (5.2) in [14].

We now want to study more closely the case when  $G = A$  is an abelian variety over  $K$ . Let  $\mathcal{A}/R$  be the Néron model of  $A$  over the ring of integers  $R$  in  $K$ , and let  $F$  denote the residue class field of  $R$ .

**Proposition 2.** If  $A$  has ordinary good reduction, then  $N\tilde{A}(K)$  is a subgroup of finite index in  $A(K)$ ; for  $p \neq 2$  this index is equal to  $(\# \mathcal{A}(F)(p))^2$ .

*Proof.* The first part of the assertion is proved in [6]. In [14] §7 we have shown that  $H^1(\Gamma, \tilde{A}(K_\infty))$  is a finite group of order  $(\# \mathcal{A}(F)(p))^2$ . Here  $\mathcal{A}$  denotes the Néron model of the dual abelian variety  $\tilde{A}$ . According to Tate's local duality theorem the group  $H^1(\Gamma, \tilde{A}(K_\infty))$  is canonically dual to the factor group  $A(K)/N\tilde{A}(K)$ . On the other hand we have  $\# \mathcal{A}(F) = \# \mathcal{A}(F)$  since  $\mathcal{A}/F$  and  $\mathcal{A}/F$  are isogenous.

*Remark.* Let  $A$  have purely multiplicative reduction. Using the  $p$ -adic parametrization of  $A$  (due to McCabe/Mumford/Raynaud/Tate) one can establish precise conditions under which  $N\tilde{A}(K)$  is of finite index in  $A(K)$ . In case  $\dim A = 1$  this is worked out in [8]. For example, if  $A$  is an elliptic curve with split-multiplicative reduction and period  $q \in K^\times$ , then  $N\tilde{A}(K)$  is of finite index in  $A(K)$  if and only if  $v_K(q) \neq 0$ . Mainin [5] p. 41) poses the question whether this condition is always true when  $A$  is already defined over a finite extension of  $\mathbb{Q}$ . In the next sections we need some information how the formation of  $NG(K)$  is behaved with respect to short exact sequences. Let

$$0 \rightarrow \mathbb{G}_m \rightarrow X \rightarrow A \rightarrow 0$$

be an exact sequence of commutative  $K$ -group schemes.

**Lemma 3.** If  $N\tilde{A}(K)$  is of finite index in  $A(K)$ , then the sequence

$$0 \rightarrow NG_m(K) \rightarrow NX(K) \rightarrow N\tilde{A}(K) \rightarrow 0$$

is exact.

*Proof.* Obviously it is enough to show the exactness of

$$0 \rightarrow K^\times / N\mathbb{G}_m(K) \rightarrow X(K) / NX(K) \rightarrow A(K) / N\tilde{A}(K) \rightarrow 0.$$

Let  $K_n$  denote the  $n$ -th intermediate layer in  $K_\infty/K$  and  $G_n := \text{Gal}(K_n/K)$ . Because of Hilbert 90 we have the exact sequence of  $G_n$ -modules

$$0 \rightarrow K_n^\times \rightarrow X(K_n) \rightarrow A(K_n) \rightarrow 0,$$

and therefore the exact sequence of Tate cohomology groups

$$\begin{aligned} \hat{H}^{-1}(G_n, A(K_n)) &\rightarrow \hat{H}^0(G_n, K_n^\times) \rightarrow \hat{H}^0(G_n, X(K_n)) \\ &\rightarrow \hat{H}^0(G_n, A(K_n)) \rightarrow H^1(G_n, K_n^\times) = 0. \end{aligned}$$

Since these groups are finite, passing to the projective limit is exact. In this way we get the required exact sequence up to the zero at the beginning. But the injectivity of  $K^\times / N\mathbb{G}_m(K) \rightarrow X(K) / NX(K)$  follows from the fact that on the one hand  $K^\times / N\mathbb{G}_m(K) \cong \mathbb{Z}_p$  is torsion free and that on the other hand, according to Tate's local duality theorem, we have

$$\begin{aligned} \# \hat{H}^{-1}(G_n, A(K_n)) &= \# H^1(G_n, A(K_n)) = \# \hat{H}^0(G_n, \tilde{A}(K_n)) \\ &\leq [\tilde{A}(K) : N\tilde{A}(K)]. \end{aligned}$$

Our assumption namely implies that  $N\tilde{A}(K)$  is of finite index in  $\tilde{A}(K)$  since the dual abelian variety  $\tilde{A}$  is  $K$ -isogenous to  $A$ .

**3. The Pairing**

Let  $A_k$  be an abelian variety over the finite number field  $k$  such that  $N(A(k_p))$  is of finite index in  $A(k_p)$  for all primes  $p$  of  $k$  dividing  $p$ . We are going to define a pairing

$$A(k) \times \tilde{A}(k) \rightarrow \mathbb{Q}_p,$$

where  $\tilde{A}$  denotes the dual abelian variety over  $k$ . Let  $\mathfrak{o}$  be the ring of integers in  $k$ , let  $\mathcal{A}$ , resp.  $\mathcal{A}^\vee$ , be the Néron model of  $A$ , resp.  $\tilde{A}$ , over  $\mathfrak{o}$ , and let  $\mathcal{A}^\circ$  be the connected component of  $\mathcal{A}$  (in the sense of SGA3 VI<sub>§</sub> §3). First we derive from [7] (5.1) the canonical identification

$$\tilde{A}(k) = \text{Ext}_{\mathfrak{o}_p}^1(\mathcal{A}^\circ, \mathbb{G}_m),$$

where the Ext-group is taken with respect to the category of abelian sheaves on the *fppf*-site of  $\mathfrak{o}$  (compare Lemma 9 in [15]).

For the moment let us fix an element  $\tilde{a} \in \tilde{A}(k)$ ,  $\tilde{a} = (0 \rightarrow \mathbb{G}_m \rightarrow \mathcal{X} \rightarrow \mathcal{A}^\circ \rightarrow 0)$ . The descent theory implies that  $\mathcal{X}$  is represented by a smooth separated commutative  $\mathfrak{o}$ -group scheme. We now consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker v & \longrightarrow & \ker v \cdot \mathcal{X}(\Delta)_0 & \longrightarrow & \mathcal{A}^\circ(\Delta)_0 \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & \mathbb{G}_m(\Delta) & \longrightarrow & \mathcal{X}(\Delta) & \longrightarrow & \mathcal{A}^\circ(\Delta) \longrightarrow 0 \\ & & \downarrow v & & & & \downarrow \\ & & \mathbb{Z}_p & & & & \end{array}$$

where, for any separated commutative  $\mathfrak{o}$ -group scheme  $\mathcal{G}$ , the subgroup  $\mathcal{G}(\Delta)_0 \leq \mathcal{G}(\Delta)$  is defined by

$$\mathcal{G}(\Delta)_0 := \prod_{p \nmid p} \mathcal{G}(\mathfrak{o}_p) \times \prod_{p \mid p} N\mathcal{G}(k_p)$$

( $\mathfrak{o}_p :=$  ring of integers in  $k_p$ ). Because of (2.1i) we have  $\mathbb{G}_m(\Delta)_0 \leq \ker v$ . This together with (2.3) and the fact that  $H^1(\text{Spec } R, \mathbb{G}_m)$  vanishes for any local  $R$  implies the exactness of the rows. Furthermore, by our assumption, the subgroup  $\mathcal{A}^\circ(\Delta)_0$  is of finite index in  $\mathcal{A}^\circ(\Delta)$ . Thus the above diagram implies the existence of a uniquely determined homomorphism

$$v_a: \mathcal{X}(\Delta) \rightarrow \mathbb{Q}_p$$

with the following properties:

- a)  $v_a|_{\mathbb{G}_m(\Delta)} = v$ ,
- b)  $v_a|_{\mathcal{X}(\Delta)_0} = 0$ .

Because of Hilbert 90 also the sequence

$$0 \rightarrow k^\times \rightarrow \mathcal{X}(k) \rightarrow A(k) \rightarrow 0$$

is exact. But we have  $k^\times \leq \ker v$  (product formula). Therefore restricting  $v_a$  to global rational points induces a homomorphism

$$v_a: A(k) \rightarrow \mathbb{Q}_p.$$

We define

$$\langle \cdot, \cdot \rangle_p := \langle \cdot, \cdot \rangle_p: A(k) \times \tilde{A}(k) \rightarrow \mathbb{Q}_p$$

$$(a, \tilde{a}) \mapsto v_a(a).$$

Left-additivity holds by definition, and right-additivity follows from an easy computation using the Baer sum. Obviously, the pairing is also functorial in  $A$ . We call  $\langle \cdot, \cdot \rangle_p$  the *p-adic height pairing* for  $A_k$ . The fundamental question which we cannot answer at present is the following.

*Problem.* Is the bilinear form  $(A(k) \otimes \mathbb{Q}_p) \times (\tilde{A}(k) \otimes \mathbb{Q}_p) \xrightarrow{\langle \cdot, \cdot \rangle_p} \mathbb{Q}_p$  nondegenerate?

*Remark.* Let us assume that  $A$  has good ordinary reduction at all primes of  $k$  dividing  $p$ . Then the pairing  $\langle \cdot, \cdot \rangle_p$  is defined and we have a rather explicit knowledge about its integrality properties (see (2.2)). As Mazur and Tate remarked the above methods can be used to construct the following more general (but less explicit) version of a height pairing: Let  $G(p)$  denote the Galois group of the maximal abelian  $p$ -extension of  $k$ -unramified outside  $p$ . Then there is a canonical pairing

$$A(k) \times \tilde{A}(k) \rightarrow G(p) \otimes \mathbb{Q}_p.$$

**4. The Local Decomposition**

We keep the notations and assumptions of the preceding section. Let  $p$  be a finite prime of  $k$ . Any divisor  $\Delta$  on  $A_{k_p}$  algebraically equivalent to 0 represents a point  $a_\Delta \in \tilde{A}(k_p) = \text{Ext}_{\mathfrak{o}_p}^1(\mathcal{A}^\circ, \mathbb{G}_m)$  and thus an extension  $0 \rightarrow \mathbb{G}_m \rightarrow \mathcal{X}_\Delta \rightarrow \mathcal{A}^\circ \rightarrow 0$  over  $\mathfrak{o}_p$ . As in the global situation we have the exact commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{G}_m(k_p)_0 & \longrightarrow & \mathcal{X}_\Delta(k_p)_0 & \longrightarrow & \mathcal{A}^\circ(k_p)_0 \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & k_p^\times & \longrightarrow & \mathcal{X}_\Delta(k_p) & \longrightarrow & A(k_p) \longrightarrow 0 \\ & & \downarrow v_p & & & & \downarrow \\ & & \mathbb{Z}_p & & & & \end{array}$$

where, for any separated commutative  $\mathfrak{o}_p$ -group scheme  $\mathcal{G}$ , the subgroup  $\mathcal{G}(k_p)_0 \leq \mathcal{G}(k_p)$  is defined by

$$\mathcal{G}(k_p)_0 := \begin{cases} \mathcal{G}(\mathfrak{o}_p) & \text{if } p \nmid p, \\ N\mathcal{G}(k_p) & \text{if } p \mid p. \end{cases}$$

And as in the global situation this leads to a uniquely determined homomorphism

$$v_{p,\Delta}: \mathcal{G}_\Delta(k_p) \rightarrow \mathcal{Q}_p$$

with the properties  $v_{p,\Delta}|_{k_p^\times} = v_p$  and  $v_{p,\Delta}|_{\mathcal{G}_\Delta(k_p)_0} = 0$ .

On the other hand there is the commutative diagram of morphisms over  $k_p$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathcal{G}_\Delta & \longrightarrow & A \longrightarrow 0 \\ & & & & \nearrow \sigma_{p,\Delta} & & \downarrow \cong \\ & & & & & & A \setminus \Delta \end{array}$$

( $|\Delta| := \text{supp } \Delta$ ) where  $\sigma_{p,\Delta}$  is canonical up to translation by  $\mathbb{G}_m(k_p) = k_p^\times$  (see [2]). If  $Z_p^0(A)$  denotes the group of zero cycles on  $A_{/k_p}$  of degree 0 with  $k_p$ -rational components, we obviously get a canonical homomorphism

$$\sigma_{p,\Delta}: \{z \in Z_p^0(A) : |z| \cap |\Delta| = \emptyset\} \rightarrow \mathcal{G}_\Delta(k_p).$$

Thus, if  $D_p^g(A)$  denotes the group of divisors on  $A_{/k_p}$  algebraically equivalent to 0, we have the map

$$\langle \cdot, \cdot \rangle_p: \{(z, \Delta) \in Z_p^0(A) \times D_p^g(A) : |z| \cap |\Delta| = \emptyset\} \rightarrow \mathcal{Q}_p$$

defined by  $\langle z, \Delta \rangle_p := v_{p,\Delta} \circ \sigma_{p,\Delta}(z)$ .

**Lemma.** i)  $\langle \cdot, \cdot \rangle_p$  is bilinear;

ii) if  $\Delta = (f)$  is a principal divisor, then  $\langle z, \Delta \rangle_p = v_p(f(z))$ .

**Proposition.** Let  $\Delta$  be a divisor on  $A_{/k}$  algebraically equivalent to 0 and representing  $\tilde{a} \in \tilde{A}(k)$ , and let  $z$  be a zero cycle on  $A_{/k}$  of degree 0 with  $k$ -rational components which maps to  $a \in A(k)$ ; assume further that  $|z| \cap |\Delta| = \emptyset$ . We then have

$$\langle a, \tilde{a} \rangle_p = \sum_p \langle z, \Delta \rangle_p.$$

We omit the proofs since they are completely analogous to the proofs of (2.7), (2.12), and (2.13) in [2].

**5. An Explicit Formula**

Let  $E_{/K}$  be the Tate curve of period  $q \in K^\times$  over the finite extension  $K/\mathcal{Q}_l$ . We then have the analytic parametrization

$$0 \rightarrow q^{\mathbb{Z}} \rightarrow K^\times \xrightarrow{\cong} E(K) \rightarrow 0.$$

Let  $\Delta$  be a divisor on  $E_{/K}$  of degree 0. The corresponding canonical exact sequence

$$0 \rightarrow K^\times \rightarrow \mathcal{G}_\Delta(K) \rightarrow E(K) \rightarrow 0$$

can be completed to an exact commutative diagram of  $K$ -holomorphic groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & K^\times & \longrightarrow & K^\times \times K^\times & \longrightarrow & K^\times \longrightarrow 0 \\ & & \parallel & & \downarrow \varphi & & \downarrow \pi \\ 0 & \longrightarrow & K^\times & \longrightarrow & \mathcal{G}_\Delta(K) & \longrightarrow & E(K) \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

where  $\alpha \in K^\times$  is determined by  $\Delta$  up to power of  $q$ . This follows from Theorem 1 in [3]. We now recall the definition and some properties of non-archimedean theta functions; all this is well explained in [3] and in Chap. II.5 of [13].

**Definition.** A theta function  $\theta(x)$  of type  $(\Delta, \alpha)$  is a  $K$ -meromorphic function on  $K^\times$  with  $\theta(qx) = \alpha\theta(x)$  and  $\text{div } \theta = \Delta$ .

**Lemma.** i) There exist theta functions of type  $(\Delta, \alpha)$  and any two of them only differ by a constant in  $K^\times$ ;

ii) if  $\theta(x)$  is a theta function of type  $(\Delta, \alpha)$  then  $x^m\theta(x)$  is a theta function of type  $(\Delta, \alpha q^m)$ ;

iii) for  $x_1, \dots, x_r \in K^\times$  and  $m_1, \dots, m_r \in \mathbb{Z}$  with  $\sum m_i = 0$  put  $\Delta := \sum m_i(\pi(x_i))$  and  $\alpha := q^m \cdot \prod x_i^{m_i}$ ; then

$$\theta(x) := x^m \cdot \prod_{i=1}^r \theta_0\left(\frac{x}{x_i}\right)^{m_i}$$

with  $\theta_0(x) := \prod_{n \geq 0} (1 - q^n x) \cdot \prod_{n \geq 1} (1 - q^n x^{-1})$  is a theta function of type  $(\Delta, \alpha)$ .

Our section  $\sigma_{K,\Delta}: E_{/K} \setminus \Delta \rightarrow \mathcal{G}_{\Delta/K}$  lifts to the map  $x \mapsto (\theta(x), x)$  with a theta function  $\theta(x)$  of type  $(\Delta, \alpha)$  on  $K^\times$ . We define

$$\theta_\Delta(x) := v_K(\theta(x)) - \frac{v_K(x)}{v_K(q)}$$

for  $x \in K^\times \setminus \pi^{-1}(|\Delta|)$ . In order to make sense of this we assume from now on that  $v_K(q) \neq 0$  (as we know this is only a condition in case  $l = p$  where it is furthermore equivalent to  $[E(K) : NE(K)] < \infty$ ). One immediately checks that  $\theta_\Delta(x)$  is  $q$ -periodic and therefore induces a map

$$\theta_\Delta: \{z \in Z_K^0(E) : |z| \cap |\Delta| = \emptyset\} \rightarrow \mathcal{Q}_p$$

which depends only on  $\Delta$  (and not on the special choice of  $\alpha$  and  $\theta$ ). (The subscript "p" which in the notations of the preceding section indicated the base field  $K = k_p$  is now always replaced by " $K$ ".)

**Proposition.**  $\langle z, \Delta \rangle_K = \theta_\Delta(z)$ .

*Proof.* We have to verify that

$$v_{K,\Delta} \circ \varphi((\theta(x), x)) = v_K(\theta(x)) \frac{v_K(\alpha)}{v_K(q)} \cdot v_K(x)$$

holds true for any  $x \in K^\times \setminus \pi^{-1}(\Delta)$ . Because of  $v_{K,\Delta} \circ \varphi((y, 1)) = v_K(y)$  for  $y \in K^\times$  it suffices to show

$$v_{K,\Delta} \circ \varphi((1, x)) = -\frac{v_K(\alpha)}{v_K(q)} \cdot v_K(x).$$

The kernel of  $v_{K,\Delta} \circ \varphi$  obviously is equal to the closure of the subgroup  $(\mathbb{G}_m \times \mathbb{G}_m)(K)_0 \cdot (\alpha, q)^{\mathbb{Z}}$  in  $K^\times \times K^\times$ . We now choose  $n \geq 1$  and  $\xi \in \mathbb{Z}_p$  (resp.  $\in \mathbb{Z}$  if  $l \neq p$ ) such that

$$x^n \equiv q^\xi \pmod{\mathbb{G}_m(K)_0}, \quad \text{i.e. } v_K(x)/v_K(q) = \xi/n.$$

Then the element  $(\alpha^{-\xi}, q^{-\xi}) \in K^\times \times K^\times$  is well-defined modulo  $(\mathbb{G}_m \times \mathbb{G}_m)(K)_0$  and lies in the kernel of  $v_{K,\Delta} \circ \varphi$ . We thus get

$$\begin{aligned} v_{K,\Delta} \circ \varphi((1, x)) &= \frac{1}{n} \cdot v_{K,\Delta} \circ \varphi((1, x^n)) \\ &= \frac{1}{n} \cdot v_{K,\Delta} \circ \varphi((\alpha^{-\xi}, 1)) = \frac{1}{n} v_K(\alpha^{-\xi}) \\ &= -\frac{\xi}{n} v_K(\alpha) = -\frac{v_K(\alpha)}{v_K(q)} \cdot v_K(x). \end{aligned}$$

*Remark.* Assume  $l \neq p$ . For  $|q| < |x| \leq 1$  (which is a "fundamental domain" for  $E(K)$ ) we have  $v_K(\theta_0(x)) = v_K(1-x)$ . Therefore in this case one can establish in an evident way a further simplified expression for  $\langle \cdot, \cdot \rangle_K$ . Of course, this gives nothing else than a transcription of Tate's explicit formula for the corresponding local factor of the Néron-Tate height.

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Oblatum VI-1982