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Motivic Iwasawa Theory

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Dedicated to K. Iwasawa

The aim of this paper is to convince the reader that there is a general theory of p-adic L-functions for varieties over number fields which to a large extent parallels the theory of complex L-functions. We will develop a cohomological formalism which in some sense can be viewed as a theory of Iwasawa theoretic realizations of motives. From these realizations we then construct the p-adic L-functions via Iwasawa's abstract notion of a characteristic power series for an Iwasawa module (in order to make this work we unfortunately need, in the moment, some restrictive assumptions on the variety). We will establish various basic properties of these L-functions the most important of which is that they have a functional equation. We also will begin to discuss some of their finer properties like the location and multiplicities of their zeros and poles. Since such a task requires a careful motivation and justification in each step we decide to expand the introduction into a paragraph of its own right giving here only the list of the contents:

1. Motivation
2. The two approaches
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The paper is divided into two parts. This first part covers the paragraphs 1–5.

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We actually will present two a priori rather different approaches to our problem. The first one is a reformulation of ideas due to R. Greenberg. Its advantage is that the known etale cohomological formalism over number rings like Euler-Poincaré characteristics computations or duality theory can be applied directly. The second one is built upon Lichtenbaum's axiomatic theory of arithmetic cohomology. Therefore to a large extent it is hypothetical. But its advantage is that (at least conjecturally) it is directly related to the basic arithmetic invariants of a variety like its higher K-theory. Furthermore it is quite likely, as we will see, that both approaches are closely related so that they represent complementary aspects of the same theory.

What we call the p-adic L-function of a motive in this paper by its very construction will be only a class of functions defined up to multiplication by invertible functions. At least in the moment this ambiguity seems to be forced upon us by the nature. But it should be mentioned that there is another theory which associates true p-adic L-functions with certain automorphic representations by interpolating the values of the corresponding automorphic complex L-functions. Furthermore there is a p-adic version of Langlands' philosophy (usually called "main conjecture") which vaguely stated says that whenever an automorphic representation corresponds to a motive its automorphic p-adic L-function should be in the class of the p-adic L-function of the motive. It is this clear analogy to the theory of motivic and automorphic complex L-functions which we want to stress by calling motivic p-adic L-function what only is a class of functions.

There are two remarks in order about our general conventions. First, cohomology always is etale or ℓ-adic cohomology. Second, we use the word "motive" as a façon de parler. What we have in mind are motives defined via algebraic correspondences. But the reader always can think of the symbol $H^i(X)(n)$ which we are going to use throughout the paper (and which defines a motive only if one of Grothendieck's "standard conjectures" holds true) as being a notion for the family of the corresponding cohomological realizations.

I am very grateful to R. Greenberg for explaining his ideas to me. There certainly will be overlap with his forthcoming work and insofar I do not claim any originality. I have profited very much from conversations with S. Bloch, J.-M. Fontaine, S. Lichtenbaum, W. Messing, J. Stienstra, and K. Wingberg and I want to thank them heartily. Finally it should be mentioned that I learned a lot from exploring the paper [25] by Colliot-Thélène/Raskind for the purposes of Iwasawa theory.

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§ 1. Motivation

A main source of motivation is the analogy to the situation over a global function field. So we start by describing very briefly the theory there:

Let $X_{/k}$ be a projective smooth geometrically connected variety over the algebraic function field in one variable k with field of constants F_q.

The L-function of the motive

$$M := H^i(X)$$

is given by the Euler product

$$L(M, s) := \prod_{\substack{x \in S \\ \text{closed}}} \det(1 - \phi_x^{-1}(Nx)^{-s};\ H^i(\bar{X}, \mathbf{Q}_\ell)^{I_x})^{-1};$$

here $-\bar{X} := X \times_k$ (separable alg. closure of k),

$-S_{/F_q}$ is the projective smooth curve with function field k,

$-I_x$ is an inertia subgroup for x in the absolute Galois group of k,

$-\phi_x$ is the arithmetic Frobenius over the residue class field of x,

$-Nx :=$ number of elements in the residue class field of x, and

$-\ell$ is a prime number not dividing q.

In order to make sense out of this definition we silently assume that the characteristic polynomials which appear in the above Euler product have coefficients in $\mathbf{Q}$ which are independent of the choice of ℓ (this is conjectured always to be true and is known if X has good reduction at x). The basic fact about these L-functions is the following theorem of Grothendieck which provides a strong link between $L(M, s)$ and the *global* arithmetic of M: The groups $H^i(\bar{X}, \mathbf{Q}_\ell)^{I_x}$ are the stalks of the sheaf $g_* H^i(\bar{X}, \mathbf{Q}_\ell)$ on S_{et} and we have

$$(*) \qquad L(M, s) = \prod_{\nu=0}^{2} \det(1 - \phi^{-1} q^{-s};\ H^\nu(\bar{S}, g_* H^i(\bar{X}, \mathbf{Q}_\ell)))^{(-1)^{\nu+1}};$$

here $-g$: Spec$(k) \hookrightarrow S$ is the canonical inclusion,

$-\bar{S} := S \times_{F_q} \bar{F}_q$, and

$-\phi \in$ Gal$(\bar{F}_q/F_q)$ is the arithmetic Frobenius.

Furthermore we know:

a) $L(M, s)$ has a functional equation with respect to $s \mapsto i+1-s$.

b) (Deligne) If $m \in \boldsymbol{Z}$ then

$$L(M, m) = \begin{cases} 0 & \Longrightarrow \quad m = \dfrac{i+1}{2} \quad (i \text{ odd}) \\ \infty & \quad m = \dfrac{i}{2},\ \dfrac{i}{2}+1 \quad (i \text{ even}). \end{cases}$$

Now let $X_{/k}$ be a projective smooth geometrically connected variety over a finite extension $k/\boldsymbol{Q}$. The complex L-function of the motive

$$M := H^i(X)$$

is given by

$$L_\infty(M, s) := \prod_{\mathfrak{p}} \det(1 - \phi_{\mathfrak{p}}^{-1}(N\mathfrak{p})^{-s};\ H^i(\bar{X}, \boldsymbol{Q}_\ell)^{I_{\mathfrak{p}}})^{-1}$$

where $\mathfrak{p}$ now runs over the set of finite primes of k and where ℓ has to be chosen appropriately for each $\mathfrak{p}$. This last fact already destroys any naive hope for something like (*). But Iwasawa, in case X=point (see [4] for a survey), and Mazur and Coates/Wiles, in case X=curve (see [14]), realized that one can turn (*) into a very interesting definition. Since, for X a point, there appear some technical subtleties let us first consider the curve case:

Assume X to be a curve and the motive to be $M = H^1(X)$. We fix a prime number $p \neq 2$ and a $\boldsymbol{Z}_p$-extension k_∞/k. Let o_∞ be the ring of integers in k_∞ and $g\colon \operatorname{Spec}(k_\infty) \to \operatorname{Spec}(o_\infty)$ be the canonical inclusion and put $\Gamma := \operatorname{Gal}(k_\infty/k)$.

Key observation. If $A := \operatorname{Jac}(X)$ denotes the Jacobian of X then we have the exact sequence of Galois modules (or sheaves on k_{et})

$$0 \longrightarrow H^1(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(1)) \longrightarrow A(\bar{k}) \longrightarrow A(\bar{k}) \otimes \boldsymbol{Z}\left[\frac{1}{p}\right] \longrightarrow 0.$$

So, instead of considering the o_∞-cohomology of the sheaf $g_* H^1(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(1))$, which would be analogous to what one does in the function field case but which would be too naive here, we form the cohomology groups (g_* is not exact!)

$$H^\nu(o_\infty, \mathscr{H}_p(M(1)))$$

of the complex of sheaves

$$\mathscr{H}_p(M(1)) := \left[g_* A \longrightarrow g_* A \otimes \boldsymbol{Z}\left[\frac{1}{p}\right] \right].$$

Proposition. *The Pontrjagin duals $H^\nu(o_\infty, \mathscr{H}_p(M(1)))^*$ are finitely generated modules under the completed group ring $\boldsymbol{Z}_p[[\Gamma]]$ and are $=0$ for $\nu \geq 3$.*

In order to get further we have to restrict attention to the

cyclotomic $\boldsymbol{Z}_p$-extension k_∞/k.

Here we expect a particular well-behavior of the above $\boldsymbol{Z}_p[[\Gamma]]$-modules (one way to express this is that $H^2(o_\infty, \mathscr{H}_p(M(1)))$ should vanish); this would allow to associate with $H^1(o_\infty, \mathscr{H}_p(M(1)))^*$ some kind of

"characteristic" power series $\in \boldsymbol{C}_p[[T]]$
which is convergent on the open unit disk.

In order to be more precise let us assume that p is such that A has good ordinary reduction at the primes of k above p. Then the expected "well-behavior" amounts to the fact that $H^1(o_\infty, \mathscr{H}_p(M(1)))^*$ is a $\boldsymbol{Z}_p[[\Gamma]]$-torsion module. According to Iwasawa's structure theory of such modules this gives sense to the definition

$$L_p(M(1), s) := p^{\mu(\cdots)} \cdot \det(1 - \gamma^{-1}\kappa(\gamma)^{-s};\ H^1(o_\infty, \mathscr{H}_p(M(1)))^* \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p);$$

here γ is a fixed topological generator of Γ, $\kappa\colon \Gamma \to \boldsymbol{Z}_p^\times$ is the cyclotomic character, and $\mu(\cdots) \geq 0$ is a certain structural invariant of the module $H^1(o_\infty, \mathscr{H}_p(M(1)))$. We would call this the *p-adic L-function* of the motive $M(1)$.

Remarks. a) The definition of $L_p(M(1), s)$ depends on the $\boldsymbol{Q}_p/\boldsymbol{Z}_p$-cohomological realization of the variety $\bar{X}$. Examples show that the number $\mu(\cdots)$ is not invariant with respect to isogenies. It is therefore an abuse of language to speak about the p-adic L-function of the motive $M(1)$.

b) If we choose a different topological generator γ' of Γ then

$$L_p(M(1), s) \text{ w.r.t. } \gamma' = u(\kappa(\gamma')^{-s} - 1) \cdot L_p(M(1), s) \text{ w.r.t. } \gamma$$

with an appropriate unit power series $u(T) \in \boldsymbol{Z}_p[[T]]^\times$. In particular, the p-adic absolute value of $L_p(M(1), s)$ is independent of the choice of γ.

Let X again be arbitrary and $M := H^i(X)$. We see that the main problem in defining p-adic L-functions is to achieve the "extension" of the Galois modules $H^i(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(n))$, for $n \in \mathbf{Z}$, to complexes of sheaves (with Γ-action) on $\operatorname{Spec}(o_\infty)_{et}$. The analogy with the function field case suggests the following

Principle. The "extension" of $H^i(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(n))$ gives rise to the p-adic L-function $L_p(M(i+1-n), s)$ of the motive $M(i+1-n)$.

§ 2. The two approaches

I see two different approaches to this "extension" problem. The first one consists in a reformulation and generalization of ideas due to R. Greenberg. The second one is based on Lichtenbaum's axiomatic theory of arithmetic cohomology and therefore is hypothetical in the moment.

(A) Greenberg's approach

We begin with a simple but useful observation. Let F be any torsion sheaf on k_{et} and consider the complex of sheaves

$$Rg_*F \qquad \text{on } \operatorname{Spec}(o_\infty)_{et}.$$

Its cohomology is $H^*(o_\infty, Rg_*F) = H^*(k_\infty, F)$ and therefore gives nothing new. So we need to modify this complex in order to make it more interesting.

Lemma. *The complex Rg_*F is acyclic in degree ≥ 2.*

Proof. We have to show that $R^ig_*F=0$ for $i \geq 2$. But R^ig_*F, for $i \geq 1$, is a skyscraper sheaf and its stalk in a geometric point above a prime v of k_∞ is isomorphic to $H^i(k^{nr}_{\infty,v}, F)$ where $k^{nr}_{\infty,v}/k_{\infty,v}$ is the maximal unramified extension of the completion $k_{\infty,v}$ of k_∞ in v. It is well-known that the field $k^{nr}_{\infty,v}$ has cohomological dimension ≤ 1.

Therefore we have a canonical homomorphism (in the derived category)

$$Rg_*F \longrightarrow (R^1g_*F)[-1].$$

For any homomorphism of sheaves $\alpha: R^1g_*F \to G$ we then define the modified complex $\tau_\alpha Rg_*F$ to be the mapping cone

$$\begin{tikzcd}
 & \tau_\alpha Rg_*F \arrow[dl, "+1"'] & \\
Rg_*F[-1] \arrow[r] & (R^1g_*F)[-2] \arrow[r, "{\alpha[-2]}"'] & G[-2]. \arrow[ul]
\end{tikzcd}$$

For example, if $\alpha=0$ is the zero map, resp. $\alpha=\mathrm{id}$ is the identity map, we get the two "extreme" cases $\tau_0 Rg_*F \simeq Rg_*F$, resp. $\tau_{id} Rg_*F \simeq g_*F$. For arbitrary α we have

$$H^1(o_\infty, g_*F) \subseteq H^1(o_\infty, \tau_\alpha Rg_*F) \subseteq H^1(k_\infty, F),$$

so that we can hope for something new. Since we also want a Γ-action on our complex it is very natural to restrict attention to the following class of maps α. Suppose that, for each prime $\mathfrak{p} \mid p$ of k, there is given a subsheaf $F_\mathfrak{p} \subseteq F \mid k_\mathfrak{p}$ of F restricted to the Henselization (or completion) $k_\mathfrak{p}$ of k in $\mathfrak{p}$. If, for any finite prime v of k_∞, $i_v: v \hookrightarrow \operatorname{Spec}(o_\infty)$ denotes the natural inclusion we have the obvious Γ-equivariant homomorphism between skyscraper sheaves

$$\begin{array}{c}
\alpha = \alpha(F; F_\mathfrak{p}): R^1g_*F = \bigoplus_v i_{v*}H^1(k^{nr}_{\infty,v}, F) \\
\big\downarrow \\
(\bigoplus_{v \nmid p} i_{v*}H^1(k^{nr}_{\infty,v}, F)) \oplus (\bigoplus_{v \mid \mathfrak{p} \mid p} i_{v*}H^1(k^{nr}_{\infty,v}, F/F_\mathfrak{p})).
\end{array}$$

Remark. Let k_∞/k be the cyclotomic $\mathbf{Z}_p$-extension and let F be a p-primary torsion sheaf. For $\alpha = \alpha(F; F_\mathfrak{p})$ as above we have

$$\begin{aligned}
&H^1(o_\infty, \tau_\alpha Rg_*F) \\
&\quad = \ker\,(H^1(k_\infty, F) \longrightarrow (\bigoplus_{v \nmid p} H^1(k_{\infty,v}, F)) \oplus (\bigoplus_{v \mid \mathfrak{p} \mid p} H^1(k^{nr}_{\infty,v}, F/F_\mathfrak{p}))).
\end{aligned}$$

Proof. By the very definition of $\tau_\alpha Rg_*F$ the group $H^1(o_\infty, \tau_\alpha Rg_*F)$ is the kernel of the map

$$H^1(k_\infty, F) \longrightarrow (\bigoplus_{v \nmid p} H^0(Z_v, H^1(k^{nr}_{\infty,v}, F))) \oplus (\bigoplus_{v \mid \mathfrak{p} \mid p} H^1(k^{nr}_{\infty,v}, F/F_\mathfrak{p}))$$

where $Z_v := \operatorname{Gal}(k^{nr}_{\infty,v}/k_{\infty,v})$. Since the residue class field of k_∞ at any $v \nmid p$ is p-closed the pro-order of Z_v is prime to p which implies

$$H^0(Z_v, H^1(k^{nr}_{\infty,v}, F)) = H^1(k_{\infty,v}, F). \qquad \text{q.e.d.}$$

Greenberg originally defined his $\mathbf{Z}_p[[\Gamma]]$-modules in terms of the right hand side in the above remark.

Remark. Over the ring of p-integers in k_∞ the canonical map $g_*F \to \tau_\alpha Rg_*F$, for any $\alpha = \alpha(F; F_\mathfrak{p})$ as above, is a quasi-isomorphism.

Proof. By definition α is an isomorphism in the stalks above primes v not dividing p.

Now let us come back to our motive $M=H^i(X)$. Of course, we put

$$F:=H^i(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(n)).$$

The choice of the $F_\mathfrak{p}$ is more subtle. For any $\mathfrak{p}|p$, let $o_\mathfrak{p}$, resp. $\kappa_\mathfrak{p}$, denote the ring of integers, resp. its residue class field, in the completion $k_\mathfrak{p}$ of k in $\mathfrak{p}$. We will assume from now on in this approach that

all $\mathfrak{p}|p$ are ramified in k_∞ and that X has good reduction at all $\mathfrak{p}|p$,

and we fix a proper smooth model $\mathcal{X}_\mathfrak{p}\to\operatorname{Spec}(o_\mathfrak{p})$ of $X_{/k_\mathfrak{p}}$; let $Y_\mathfrak{p}:=\mathcal{X}_\mathfrak{p}\times_{o_\mathfrak{p}}\kappa_\mathfrak{p}$ be the reduction of $\mathcal{X}_\mathfrak{p}$ so that we have natural morphisms

$$Y_\mathfrak{p}\xrightarrow{h}\mathcal{X}_\mathfrak{p}\xleftarrow{f}X_{/k_\mathfrak{p}}.$$

If a bar indicates base extension to the algebraic or integral closure then the spectral sequence

$$E_2^{i,j}=H^i(\bar{Y}_\mathfrak{p}, \bar{h}^*R^j\bar{f}_*\mathbf{Z}_p(n))=H^i(\bar{\mathcal{X}}_\mathfrak{p}, R^j\bar{f}_*\mathbf{Z}_p(n))\Longrightarrow H^{i+j}(\bar{X}, \mathbf{Z}_p(n))$$

induces a $\operatorname{Gal}(\bar{k}_\mathfrak{p}/k_\mathfrak{p})$-equivariant decreasing filtration

$$F^{\cdot}_{(\mathfrak{p})}H^*(\bar{X}, \mathbf{Z}_p(n)).$$

If we set $d:=\dim X=\dim \bar{Y}_\mathfrak{p}$ then cohomological dimension arguments show that $E_2^{i,j}=0$ for $i>d$ or $j>d$ and consequently that

$$F^j_{(\mathfrak{p})}H^i(\bar{X}, \mathbf{Z}_p(n))=\begin{cases}H^i(\bar{X}, \mathbf{Z}_p(n)) & \text{if } j\leq\max(0, i-d),\\ 0 & \text{if } j>\min(i, d).\end{cases}$$

Using Poincaré duality

$$H^i(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(n))=\operatorname{Hom}(H^{2d-i}(\bar{X}, \mathbf{Z}_p(d-n)), \mathbf{Q}_p/\mathbf{Z}_p)$$

we also get a decreasing filtration

$$F^{\cdot}_{(\mathfrak{p})}H^*(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(n)):=F^{d+1-\cdot}_{(\mathfrak{p})}H^{2d-*}(\bar{X}, \mathbf{Z}_p(d-n))^{\perp}.$$

Again we have

$$F^j_{(\mathfrak{p})}H^i(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(n))=\begin{cases}H^i(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(n)) & \text{if } j\leq\max(0, i-d),\\ 0 & \text{if } j>\min(i, d).\end{cases}$$

We choose our $F_\mathfrak{p}$ now to be

$$\boxed{F_\mathfrak{p}:=F^{i+1-n}_{(\mathfrak{p})}H^i(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(n))}\,.$$

What distinguishes this particular step in the filtration? If $\operatorname{gr}^j_{(\mathfrak{p})}:=F^j_{(\mathfrak{p})}/F^{j+1}_{(\mathfrak{p})}$ is the associated graduation then S. Bloch conjectures that

$\operatorname{gr}^j_{(\mathfrak{p})}H^i(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(n))$ is isogenous to the $(j+n-i-1)$-th twist of the points of a certain connected p-divisible group over $o_\mathfrak{p}$ (provided the reduction $\bar{Y}_\mathfrak{p}$ is Hodge-Witt).

This would mean for us that we "truncate all nonnegative twists". In case the reductions $Y_\mathfrak{p}$, for $\mathfrak{p}|p$, are ordinary $H^1(o_\infty, \tau_a Rg_*F)$ with the above choice of F and $F_\mathfrak{p}$ is exactly the $\mathbf{Z}_p[[\Gamma]]$-module which Greenberg suggested to study; and he checked that, for $M=H^1$ (curve), it is the same as Mazur's $\mathbf{Z}_p[[\Gamma]]$-module which we described above.

We will come back to a more detailed discussion of the complexes $\tau_a Rg_*F$. But first we want to present the other approach since each of them sheds interesting light on the other one.

(B) The $\Gamma(n)$-approach

In [11] Lichtenbaum assumes the existence of complexes $\Gamma(n)$, for $n\geq 0$, of sheaves for the etale topology on any regular scheme which fulfill a number of fundamental axioms. The conclusions he then draws from these axioms show that the cohomology of the $\Gamma(n)$ may well be considered as some kind of universal arithmetic cohomology theory. In the following I list only those axioms which are needed in this section:

(L0) $\Gamma(0)=\mathbf{Z}$, $\Gamma(1)=\mathbf{G}_m[-1]$.

(L1) The formation of $\Gamma(n)$ commutes with proetale base change. (Convention: By a proetale morphism $S\to T$ we always mean a filtered projective limit of etale T-schemes of finite presentation $S_i\to T$ with affine transition morphisms.)

(L2) For any $m\geq 1$ prime to all residue field characteristics of the respective scheme there exists a distinguished triangle

$$\begin{tikzcd} & \mathbf{Z}/m\mathbf{Z}(n) \arrow[dl, "+1"'] & \\ \Gamma(n) \arrow[rr, "m"] & & \Gamma(n) \arrow[ul] \end{tikzcd}\ .$$

We will see that already these properties make the cohomology of the $\Gamma(n)$ well-suited to solve our "extension" problem. In [12] Lichtenbaum constructs a complex $\Gamma(2)$ which has the above properties (and also most of the not-listed ones) at least on regular schemes of finite type over a field.

Bloch in [2] gives candidates for all the $\Gamma(n)$ fulfilling some of the axioms; but, unfortunately, the property (L2) is not established for them.

If we pass in the above distinguished triangle, with $m=p^{\mu}$, to the cohomology and then to the direct, resp. projective, limit with respect to μ we get exact sequences

$$0\longrightarrow H^i(\bar{X}, \Gamma(n))\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p\longrightarrow H^i(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(n))$$
$$\longrightarrow H^{i+1}(\bar{X}, \Gamma(n))(p)\longrightarrow 0$$

and

$$0\longrightarrow \varprojlim H^i(\bar{X}, \Gamma(n))/p^{\mu}\longrightarrow H^i(\bar{X}, \boldsymbol{Z}_p(n))$$

where we now (in contrary to Greenberg's approach) have to restrict attention to nonnegative $n\geq 0$.

Proposition. *For $i\neq 2n$, the image of the natural map $H^i(\bar{X}, \Gamma(n))\rightarrow \varprojlim H^i(\bar{X}, \Gamma(n))/p^{\mu}$ is finite.*

Proof. It suffices to prove the assertion for the composed map

$$\theta: H^i(\bar{X}, \Gamma(n))\longrightarrow \varprojlim H^i(\bar{X}, \Gamma(n))/p^{\mu}\longrightarrow H^i(\bar{X}, \boldsymbol{Z}_p(n)).$$

Because of property (L1) we have

$$H^i(\bar{X}, \Gamma(n))=\varinjlim_{\substack{k\subseteq L\subseteq \bar{k}\\ L/k \text{ finite}}} H^i(X_{/L}, \Gamma(n)).$$

This implies that the image of θ is contained in the part of $H^i(\bar{X}, \boldsymbol{Z}_p(n))$ on which $\mathrm{Gal}(\bar{k}/k)$ acts discretely. But, by standard arguments, one derives from Deligne's proof of the Weil conjectures that this part is finite for $i\neq 2n$.

Remark. The kernel of $H^i(\bar{X}, \Gamma(n))\rightarrow \varprojlim H^i(\bar{X}, \Gamma(n))/p^{\mu}$ is p-divisible.

Proof. Since it maps injectively to $H^i(\bar{X}, \boldsymbol{Z}_p(n))$ the $\boldsymbol{Z}_p$-module $\varprojlim H^i(\bar{X}, \Gamma(n))/p^{\mu}$ is finitely generated; let p^s be the order of its torsion subgroup. If now x is in the kernel of the map under consideration then, for any $r\geq 0$, we find a $y_r\in H^i(\bar{X}, \Gamma(n))$ such that $x=p^r y_r$. In particular, we get $x=p(p^s y_{s+1})$ where $p^s y_{s+1}$ again lies in the kernel.

Corollary. *For $i\neq 2n$, we have $H^i(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(n))=H^{i+1}(\bar{X}, \Gamma(n))(p)$.*

Proof. As a consequence of the above facts we have, for $i\neq 2n$, that

$H^i(\bar{X}, \Gamma(n))\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p=0.$

We now make use of the following rather trivial construction which (under appropriate assumptions) makes a sheaf "almost" p-divisible without changing its p-primary torsion. For an arbitrary sheaf F we consider the natural map

$$\psi: F\longrightarrow \varprojlim F/p^{\mu}F$$

and we define

$$\Delta F:=\psi^{-1}\ (\text{torsion subsheaf of } \varprojlim F/p^{\mu}F).$$

Lemma. *If $p^s\cdot \Delta F\subseteq \ker\psi$ for some $s\geq 0$ then the sequence*

$$0\longrightarrow F(p)\longrightarrow \Delta F\longrightarrow (\Delta F)\otimes \boldsymbol{Z}\left[\frac{1}{p}\right]\longrightarrow 0$$

is exact.

Proof. We only have to check the surjectivity of the map $\Delta F\rightarrow \Delta F\otimes\boldsymbol{Z}[1/p]$. By exactly the same argument as in the above Remark one sees that $\ker\psi$ is p-divisible which implies the surjectivity of $\ker\psi\rightarrow\ker\psi\otimes\boldsymbol{Z}[1/p]$. But on the other hand we have $(\Delta F/\ker\psi)\otimes\boldsymbol{Z}[1/p]=0$.

Obviously we can apply this to the sheaf $F=H^{i+1}(\bar{X}, \Gamma(n))$ on k_{et}; furthermore, for $i\neq 2n$, the above Corollary identifies its p-torsion.

As a result we get the exact sequence

$$(*)\qquad 0\longrightarrow H^i(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(n))\longrightarrow \Delta H^{i+1}(\bar{X}, \Gamma(n))$$
$$\longrightarrow \Delta H^{i+1}(\bar{X}, \Gamma(n))\otimes\boldsymbol{Z}\left[\frac{1}{p}\right]\longrightarrow 0$$

of sheaves on k_{et} where from now on we assume once and for all (in this approach) that

$$i\neq 2n\ !$$

Remark. As a consequence of the above Proposition we even have

$$\Delta H^{i+1}(\bar{X}, \Gamma(n))=H^{i+1}(\bar{X}, \Gamma(n))\qquad \text{if } i\neq 2n-1.$$

If X is a curve and $i=n=1$ then $H^2(\bar{X}, \Gamma(1))=H^1(\bar{X}, \boldsymbol{G}_m)=\mathrm{Pic}\,(X)(\bar{k})$ and therefore

$$\Delta H^2(\bar{X}, \Gamma(1))=\mathrm{Jac}\,(X)(\bar{k})$$

so that (*) becomes precisely the exact sequence which we used in the construction of Mazur's $Z_p[[\Gamma]]$-module. So the analogy would suggest to take the complex

$$\left[g_*\Delta H^{i+1}(\bar{X}, \Gamma(n)) \longrightarrow g_*\Delta H^{i+1}(\bar{X}, \Gamma(n)) \otimes Z\left[\frac{1}{p}\right]\right]$$

on $\mathrm{Spec}(o_\infty)_{et}$ as solution to our "extension" problem. But as it is already clear from the first approach our "extension" should differ from the trivial extension $g_*H^i(\bar{X}, Q_p/Z_p(n))$ only in the primes above p. One soon realizes (take X=point and $i=0$, $n=1$) that the above complex is still too "big" for that. But all we have to do is to perform the "Δ-construction" a second time: Let o'_∞ denote the ring of p-integers in k_∞ and let g': $\mathrm{Spec}(k_\infty) \to \mathrm{Spec}(o'_\infty)$ be the canonical inclusion.

Proposition. *The sheaf $F := g'_*\Delta H^{i+1}(\bar{X}, \Gamma(n))$ on $\mathrm{Spec}(o'_\infty)_{et}$ fulfills, for $i \neq 2n$, the assumption in the above Lemma.*

Proof. We have exact sequences

$$H^i(\bar{X}, Z/p^\mu Z(n)) \longrightarrow \Delta H^{i+1}(\bar{X}, \Gamma(n)) \xrightarrow{p^\mu} \Delta H^{i+1}(\bar{X}, \Gamma(n))$$

where kernel, resp. cokernel, of the first, resp. the second, map are annihilated by some power of p which is independent of μ. If we work modulo sheaves which are annihilated by a power of p we deduce from that an injection

$$\varprojlim\, (g'_*\Delta H^{i+1}(\bar{X}, \Gamma(n)))/p^\mu \longrightarrow \varprojlim\, R^1g'_*H^i(\bar{X}, Z/p^\mu Z(n)).$$

We want to show that the torsion subsheaf of the right hand side is annihilated by some p^s. For that it suffices to look at the stalks of the right hand side in the sense of Z_p-sheaves. So, if v is any finite prime of k_∞ not dividing p we first pass to the stalk above v in the projective system $\{R^1g'_*H^i(\bar{X}, Z/p^\mu Z(n))\}_{\mu\geq 1}$ and then form the projective limit of those; the result is the group

$$H^1(I_v, H^i(\bar{X}, Z_p(n)))$$

where $I_v \subseteq \mathrm{Gal}(\bar{k}/k_\infty)$ denotes an inertia subgroup for v. Because of $v \nmid p$ the group I_v is an extension of $Z_p(1)$ by a group of pro-order prime to p. Therefore $H^1(I_v, H^i(\bar{X}, Z_p(n)))$ is isomorphic to a subquotient of the finitely generated Z_p-module $H^i(\bar{X}, Z_p)$. If X has good reduction at v, i.e., for the v above almost all primes of k, we even have, by the proper and smooth base change theorem,

$$H^1(I_v, H^i(\bar{X}, Z_p(n))) = H^i(\bar{X}, Z_p(n-1)) \cong H^i(\bar{X}, Z_p).$$

We see that the torsion subgroups of these stalks have orders which are bounded independently of v.

Corollary. *For $i \neq 2n$, we have the exact sequence*

$$0 \longrightarrow g'_*H^i(\bar{X}, Q_p/Z_p(n)) \longrightarrow \Delta g'_*\Delta H^{i+1}(\bar{X}, \Gamma(n)) \longrightarrow \Delta g'_*\Delta H^{i+1}(\bar{X}, \Gamma(n)) \otimes Z\left[\frac{1}{p}\right] \longrightarrow 0.$$

Remark. Assuming a certain standard conjecture about ℓ-adic representations (see [5], p. 319) one can show that

$$\Delta g'_*\Delta H^{i+1}(\bar{X}, \Gamma(n)) = g'_*H^{i+1}(\bar{X}, \Gamma(n)) \qquad \text{for } n > \frac{i}{2}+1.$$

Now let σ: $\mathrm{Spec}(o'_\infty) \longrightarrow \mathrm{Spec}(o_\infty)$ be the canonical inclusion. We then view the complex

$$\mathscr{H}_p(M) := \left[\sigma_*\Delta g'_*\Delta H^{i+1}(\bar{X}, \Gamma(n)) \longrightarrow \sigma_*\Delta g'_*\Delta H^{i+1}(\bar{X}, \Gamma(n)) \otimes Z\left[\frac{1}{p}\right]\right]$$

of sheaves on $\mathrm{Spec}(o_\infty)_{et}$ placed in degree 0 and 1 as solution to our "extension" problem for the motive $M = H^i(X)(n)$ (always in case $i \neq 2n$). For later use we explicitly state the obvious

Remark. The canonical map $g_*H^i(\bar{X}, Q_p/Z_p(n)) \to \mathscr{H}_p(M)$ is a quasi-isomorphism over $\mathrm{Spec}(o'_\infty)_{et}$.

§ 3. First results

Our ultimate aim, of course, is to understand the cohomology of these complexes we have constructed above. The Pontrjagin duals of their cohomology groups are, in a natural way, compact $Z_p[[\Gamma]]$-modules. In order to get some ground for the further discussion our first task therefore is to show that these $Z_p[[\Gamma]]$-modules are finitely generated and then to try to compute their $Z_p[[\Gamma]]$-ranks. Put $F := H^i(\bar{X}, Q_p/Z_p(n))$ and consider any α of the form $\alpha = \alpha(F; F_\mathfrak{p})$ where, for any $\mathfrak{p} | p$, $F_\mathfrak{p}$ is a given subsheaf of $F | k_\mathfrak{p}$. The complex $\tau_\alpha Rg_*F$ is defined by the distinguished triangle

$$\begin{tikzcd}[column sep=small] & \tau_\alpha Rg_*F \arrow[dl, "+1"'] & \\ R\sigma_*(g'_*F)[-1] \arrow[rr] & & \bigoplus_{v|\mathfrak{p}|p} i_{v*}H^1(k^{nr}_{\infty,v}, F/F_\mathfrak{p})[-2] \arrow[ul] \end{tikzcd}$$

(we keep all our above notations) so that we have to analyze the associated long exact cohomology sequence

$$\longrightarrow H^\nu(o_\infty, \tau_\alpha Rg_*F) \longrightarrow H^\nu(o'_\infty, g'_*F) \longrightarrow \bigoplus_{v|\mathfrak{p}|p} H^{\nu-1}(Z_v, H^1(k^{nr}_{\infty,v}, F/F_{\mathfrak{p}})) \longrightarrow$$

where $Z_v := \mathrm{Gal}(k^{nr}_{\infty,v}/k_{\infty,v})$. For simplicity and since we eventually would do so anyway we assume from now on that

$$k_\infty/k \text{ is the cyclotomic } \boldsymbol{Z}_p\text{-extension} \qquad (p \neq 2)!$$

Proposition. *$H^\nu(o'_\infty, g'_*F)^*$ is finitely generated over $\boldsymbol{Z}_p$ for $\nu=0$, is finitely generated over $\boldsymbol{Z}_p[[\Gamma]]$ for $\nu=1$ and 2, and $=0$ for $\nu \geq 3$. Furthermore, we have*

$$\mathrm{rank}_{\boldsymbol{Z}_p[[\Gamma]]} H^1(o'_\infty, g'_*F)^* - \mathrm{rank}_{\boldsymbol{Z}_p[[\Gamma]]} H^2(o'_\infty, g'_*F)^* = \sum_{\mathfrak{p}|\infty} \dim H^i(X_{\mathfrak{p}}(\boldsymbol{C}), \boldsymbol{Q})^{(-1)^{n-1}}.$$

Notation. For any infinite prime $\mathfrak{p}|\infty$ of k corresponding either to a real embedding ε or to a pair $(\varepsilon, \bar\varepsilon)$ of complex conjugate embeddings of k into $\boldsymbol{C}$ define

$$X_{\mathfrak{p}} := \begin{cases} X \underset{k,\varepsilon}{\times} \boldsymbol{C} & \text{if } \mathfrak{p} \text{ is real,} \\ (X \underset{k,\varepsilon}{\times} \boldsymbol{C}) \times (X \underset{k,\bar\varepsilon}{\times} \boldsymbol{C}) & \text{if } \mathfrak{p} \text{ is complex.} \end{cases}$$

On the singular cohomology $H^*(X_{\mathfrak{p}}(\boldsymbol{C}), \boldsymbol{Q})$ then acts the "Frobenius" $F_{\mathfrak{p}}$ induced by the complex conjugation on $X_{\mathfrak{p}}(\boldsymbol{C})$. We put

$$H^*(X_{\mathfrak{p}}(\boldsymbol{C}), \boldsymbol{Q})^{\pm} := F_{\mathfrak{p}}\text{-eigenspace of eigenvalue } \pm 1.$$

Proof. For $\nu=0$ the assertion is clear and for $\nu \geq 3$ it follows from the fact that the cohomological p-dimension of $\mathrm{Spec}(o'_\infty)_{et}$ is ≤ 2 (compare [16] § 3 Lemma 7). Now let o' be the ring of p-integers in k and let g'_0: $\mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(o')$ be the canonical inclusion. By SGA 4 VII 5.11 g'_*F is the inverse image of the sheaf $g'_{0*}F$ on $\mathrm{Spec}(o')_{et}$. The Hochschild-Serre spectral sequence for $\mathrm{Spec}(o'_\infty) \longrightarrow \mathrm{Spec}(o')$ then gives short exact sequences

$$0 \longrightarrow H^{\nu-1}(o'_\infty, g'_*F)_\Gamma \longrightarrow H^\nu(o', g'_{0*}F) \longrightarrow H^\nu(o'_\infty, g'_*F)^\Gamma \longrightarrow 0.$$

By the Nakayama lemma and the structure theory of $\boldsymbol{Z}_p[[\Gamma]]$-modules the other assertions of the proposition then are a consequence of the following lemma.

Lemma 1. i) *The groups $H^\nu(o', g'_{0*}F)$ are of cofinite type over $\boldsymbol{Z}_p$;*
ii) $\sum_{\nu \geq 0} (-1)^{\nu+1} \mathrm{corank}\, H^\nu(o', g'_{0*}F) = \sum_{\mathfrak{p}|\infty} \dim H^i(X_{\mathfrak{p}}(\boldsymbol{C}), \boldsymbol{Q})^{(-1)^{n-1}}$.

As usual denote by $o_{\mathfrak{p}}$, resp. $k_{\mathfrak{p}}$, resp. $\kappa_{\mathfrak{p}}$, the completion (or Henselization) of o in some finite prime $\mathfrak{p}$, resp. its quotient field, resp. its residue class field. Let $U \subseteq \mathrm{Spec}(o')$ be a nonempty open subset such that $(g'_{0*}F)|U$ is locally constant.

Lemma 2. *For $\mathfrak{p} \nmid p$ the groups $H^\nu(o_{\mathfrak{p}}, g'_{0*}F)$ are of cofinite type over $\boldsymbol{Z}_p$ with*

$$\sum_{\nu \geq 0} (-1)^\nu \mathrm{corank}\, H^\nu(o_{\mathfrak{p}}, g'_{0*}F) = 0.$$

Proof. By the local relative cohomology sequence it suffices to prove the analogous assertions separately for $H^\nu(o_{\mathfrak{p}}, g'_{0*}F)$ and for $H^\nu(k_{\mathfrak{p}}, F)$. Since $o_{\mathfrak{p}}$ is Henselian the first groups are of the type

$$H^\nu(o_{\mathfrak{p}}, g'_{0*}F) = H^\nu(\kappa_{\mathfrak{p}}, \text{Galois module of } \boldsymbol{Z}_p\text{-cofinite type}).$$

But for the groups on the right hand side our claim is obvious since $\kappa_{\mathfrak{p}}$ is a finite field. The assertion for the $H^\nu(k_{\mathfrak{p}}, F)$ is a consequence of the theorem about the local Euler-Poincaré characteristic (see [19] II–26 or [15] § 2 Satz 1).

From the relative cohomology sequence we see that we are reduced to prove the assertions of Lemma 1 for the groups $H^\nu(U, g'_{0*}F)$. Let G_S be the Galois group of the maximal extension of k which is unramified in all finite primes in U. Since $g'_{0*}F$ is locally constant over U_{et} we have

$$H^\nu(U, g'_{0*}F) = H^\nu(G_S, F).$$

For the right hand groups the assertions then follow from Tate's theorem about global Euler-Poincaré characteristics in the form

$$\sum_{\nu \geq 0} (-1)^{\nu+1} \mathrm{corank}\, H^\nu(G_S, F) = \sum_{\mathfrak{p}|\infty} \mathrm{corank}\, H^0(k_{\mathfrak{p}}, \tilde F)$$

with $\tilde F := \mathrm{Hom}(H^i(\bar X, \boldsymbol{Z}_p(n-1)), \boldsymbol{Q}_p/\boldsymbol{Z}_p)$ and where $k_{\mathfrak{p}}$ denotes the completion of k at $\mathfrak{p}|\infty$ (compare [15] § 3 Satz 1 or [23] Proposition 12). But the comparison theorem for etale cohomology implies

$$\mathrm{corank}\, H^0(k_{\mathfrak{p}}, \tilde F) = \dim H^0(k_{\mathfrak{p}}, H^i(\bar X, \boldsymbol{Q}_p(n-1))) = \dim H^i(X_{\mathfrak{p}}(\boldsymbol{C}), \boldsymbol{Q})^{(-1)^{n-1}}.$$

q.e.d.

Next we have to treat the local terms in our cohomology sequence.

Since here nothing depends on the particular Galois module structure of $F/F_\mathfrak{p}$ we do the computations in slightly greater generality: Let $K/\boldsymbol{Q}_p$ be a finite extension, K_∞/K be any $\boldsymbol{Z}_p$-extension with Galois group $\Gamma := \mathrm{Gal}(K_\infty/K)$, and M be a discrete $\mathrm{Gal}(\bar{K}/K)$-module which is of cofinite type over $\boldsymbol{Z}_p$; furthermore, let $Z := \mathrm{Gal}(K_\infty^{nr}/K_\infty)$ be the Galois group of the maximal unramified extension K_∞^{nr} of K_∞.

Proposition. *$H^\nu(Z, H^1(K_\infty^{nr}, M))^*$ is, for $\nu=0$, a finitely generated $\boldsymbol{Z}_p[[\Gamma]]$-module of rank equal to* corank $M\cdot[K: \boldsymbol{Q}_p]$, *and $=0$ for $\nu\geq 1$.*

Proof. Since the fields K_∞ and K_∞^{nr} and the group Z all have cohomological p-dimension ≤ 1 the Hochschild-Serre spectral sequence for the extension K_∞^{nr}/K_∞ has only three nonvanishing terms which fit into the exact sequence

$$0 \longrightarrow H^1(Z, H^0(K_\infty^{nr}, M)) \longrightarrow H^1(K_\infty, M) \longrightarrow H^0(Z, H^1(K_\infty^{nr}, M)) \longrightarrow 0.$$

Thus the vanishing statement in our proposition is clear and since $H^1(Z, H^0(K_\infty^{nr}, M))$ obviously is of cofinite type over $\boldsymbol{Z}_p$ it remains to show that $H^1(K_\infty, M)^*$ is a finitely generated $\boldsymbol{Z}_p[[\Gamma]]$-module of the asserted rank. Now, the Hochschild-Serre spectral sequence for the extension K_∞/K splits into the exact sequence

$$0 \longrightarrow H^0(K_\infty, M)_\Gamma \longrightarrow H^1(K, M) \longrightarrow H^1(K_\infty, M)^\Gamma \longrightarrow 0$$

and the identities

$$H^0(K_\infty, M)^\Gamma = H^0(K, M) \quad \text{and} \quad H^1(K_\infty, M)_\Gamma = H^2(K, M).$$

So it finally comes down to the assertion that the groups $H^\nu(K, M)$ are of cofinite type over $\boldsymbol{Z}_p$ with

$$\sum_{\nu\geq 0} (-1)^{\nu+1} \operatorname{corank} H^\nu(K, M) = \operatorname{corank} M\cdot[K: \boldsymbol{Q}_p].$$

But this is easily derived from the theorem about the local Euler-Poincaré characteristic (see [19] II–35 or [15] § 2 Satz 1 or [23] p. 127).

Later on we will also need the following technical fact the proof of which is left to the reader.

Lemma. *If $M \longrightarrow M'$ is a surjection of* $\mathrm{Gal}(\bar{K}/K)$*-modules then*

$$H^0(Z, H^1(K_\infty^{nr}, M)) \longrightarrow H^0(Z, H^1(K_\infty^{nr}, M'))$$

is a surjection, too.

Going back to our global situation we now have the following

Corollary. *$\oplus_{\mathfrak{w}|\mathfrak{p}|p} H^\nu(Z_\mathfrak{v}, H^1(k_{\infty,\mathfrak{v}}^{nr}, F/F_\mathfrak{p}))^*$ is, for $\nu=0$, a finitely generated $\boldsymbol{Z}_p[[\Gamma]]$-module of rank equal to*

$$\sum_{\mathfrak{p}|p} \operatorname{corank} F/F_\mathfrak{p}\cdot[k_\mathfrak{p}: \boldsymbol{Q}_p],$$

and $=0$ for $\nu\geq 1$.

If we combine these results we get a first basic information about the cohomology of our complexes.

Theorem. *$H^\nu(o_\infty, \tau_\alpha Rg_* F)^*$ is finitely generated over $\boldsymbol{Z}_p$ for $\nu=0$, is finitely generated over $\boldsymbol{Z}_p[[\Gamma]]$ for $\nu=1$ and 2, and $=0$ for $\nu\geq 3$. Furthermore, we have*

$$\operatorname{rank}_{\boldsymbol{Z}_p[[\Gamma]]} H^1(o_\infty, \tau_\alpha Rg_* F)^* - \operatorname{rank}_{\boldsymbol{Z}_p[[\Gamma]]} H^2(o_\infty, \tau_\alpha Rg_* F)^*$$
$$= \sum_{\mathfrak{p}|\infty} \dim H^i(X_\mathfrak{p}(\boldsymbol{C}), \boldsymbol{Q})^{(-1)^{n-1}} - \sum_{\mathfrak{p}|p} \operatorname{corank} H^i(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(n))/F_\mathfrak{p}\cdot[k_\mathfrak{p}: \boldsymbol{Q}_p].$$

Next we have to investigate the coranks of the particular subsheaves $F_\mathfrak{p}$ we have chosen above. We assume that

$$X \text{ has good reduction at all } \mathfrak{p}|p$$

and that we have fixed proper smooth models $\mathscr{X}_{\mathfrak{p}/o_\mathfrak{p}}$ of $X_{/k_\mathfrak{p}}$ so that the filtrations $F^{\cdot}_{(\mathfrak{p})}$ are defined. Then the $F_\mathfrak{p}$ are given by

$$F_\mathfrak{p} := F_{(\mathfrak{p})}^{i+1-n} H^i(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(n)).$$

In order to simplify notation we put, for this particular $\alpha=\alpha(F; F_\mathfrak{p})$,

$$\rho(H^i(X)(n)) := \operatorname{rank}_{\boldsymbol{Z}_p[[\Gamma]]} H^1(o_\infty, \tau_\alpha Rg_* F)^* - \operatorname{rank}_{\boldsymbol{Z}_p[[\Gamma]]} H^2(o_\infty, \tau_\alpha Rg_* F)^*.$$

We will see that these invariants are strongly tied up with the multiplicities of the complex L-function $L_\infty(H^i(X), s)$ at integer points. Of course, this can make sense only under the following hypothesis:

(A) $L_\infty(H^i(X), s)$ converges (and does not vanish) for $\operatorname{Re} s > (i/2)+1$, has analytic continuation and the conjectured functional equation with respect to $s \longmapsto i+1-s$.

(For the precise form of the functional equation see [21].) We put $\operatorname{ord}_{s=n} L_\infty(H^i(X), s) :=$ multiplicity of $L_\infty(H^i(X), s)$ at $s=n$ where poles are counted negatively. First there is an easy "stable" case to consider.

Proposition. *For $n>\min(i, \dim X)$ (i.e., $i+1-n\leq\max(0, i-\dim X)$) we have*

$$\rho(H^i(X)(n))=-\rho(H^i(X)(i+1-n))=\sum_{\mathfrak{p}|\infty}\dim H^i(X_\mathfrak{p}(C), Q)^{(-1)^{n-1}}$$

and, provided (A) *holds true, this number furthermore is equal to*

$$\begin{cases}\operatorname{ord}_{s=i+1-n} L_\infty(H^i(X), s) & \text{if } n>\frac{i}{2}+1,\\ \operatorname{ord}_{s=i/2} L_\infty(H^i(X), s)-\operatorname{ord}_{s=(i/2)+1} L_\infty(H^i(X), s) & \text{if } n=\frac{i}{2}+1.\end{cases}$$

Proof. For $\rho(H^i(X)(n))$, resp. $\rho(H^i(X)(i+1-n))$, we are in the situation that $F_\mathfrak{p}=F$, resp. $F_\mathfrak{p}=0$, for all $\mathfrak{p}|p$. The formula in the Theorem therefore specializes to

$$\rho(H^i(X)(n))=\sum_{\mathfrak{p}|\infty}\dim H^i(X_\mathfrak{p}(C), Q)^{(-1)^{n-1}},$$

resp.

$$\rho(H^i(X)(i+1-n))=\sum_{\mathfrak{p}|\infty}\dim H^i(X_\mathfrak{p}(C), Q)^{(-1)^{i+1-n}}.$$

But the right hand sides are equal since, for even i, the exponents are the same and, for odd i, the eigenspaces of $F_\mathfrak{p}$ have the same dimension. Finally, by an elementary computation with Hodge numbers one deduces from the functional equation for $L_\infty(H^i(X), s)$ that the asserted multiplicities are equal to the above right hand sides (compare [18]).

If we assume more about the reduction type of X at $\mathfrak{p}|p$ this result can be extended to all n.

Proposition. *If X has ordinary (good) reduction at all $\mathfrak{p}|p$ then we have*

$$\rho(H^i(X)(n))=-\rho(H^i(X)(i+1-n)) \qquad \text{for } n\in Z$$

and, provided (A) *holds true, furthermore*

$$\rho(H^i(X)(n))=\begin{cases}\operatorname{ord}_{s=i+1-n} L_\infty(H^i(X), s) & \text{if } n>\frac{i}{2}+1,\\ \operatorname{ord}_{s=i/2} L_\infty(H^i(X), s)-\operatorname{ord}_{s=(i/2)+1} L_\infty(H^i(X), s) & \text{if } n=\frac{i}{2}+1,\\ 0 & \text{if } n=\frac{i+1}{2}.\end{cases}$$

Proof. From the work of S. Bloch and K. Kato on the Hodge-Tate decomposition in [3] follows in particular the identity

$$\operatorname{rank}_{Z_p}\operatorname{gr}^j_{(\mathfrak{p})} H^i(\bar{X}, Z_p)=\dim H^j(X, \Omega^{i-j}),$$

or equivalently,

$$\operatorname{rank}_{Z_p} F^j_{(\mathfrak{p})} H^i(\bar{X}, Z_p)=\dim F^j H^i_{DR}(X)=\dim H^i_{DR}(X)/F^{i+1-j}.$$

Using this, the fact that the formation of our filtration $F^\cdot_{(\mathfrak{p})}$ commutes with Tate twist, and the strong Lefschetz theorem for de Rham cohomology we compute

$$\begin{aligned}\operatorname{corank} F/F^{i+1-n}_{(\mathfrak{p})}&=\operatorname{rank}_{Z_p} F^{d-i+n}_{(\mathfrak{p})}H^{2d-i}(\bar{X}, Z_p)\\ &=\dim F^{d-i+n}H^{2d-i}_{DR}(X)\\ &=\dim F^n H^i_{DR}(X)=\dim H^i_{DR}(X)/F^{i+1-n}.\end{aligned}$$

The above Theorem then specializes to

$$\rho(H^i(X)(n))=\sum_{\mathfrak{p}|\infty}(\dim H^i(X_\mathfrak{p}(C), Q)^{(-1)^{n-1}}-\dim F^n H^i_{DR}(X_\mathfrak{p})),$$

resp.

$$\rho(H^i(X)(i+1-n))=\sum_{\mathfrak{p}|\infty}(\dim H^i(X_\mathfrak{p}(C), Q)^{(-1)^{i-n}}-\dim H^i_{DR}(X_\mathfrak{p})/F^n).$$

But we have

$$\begin{aligned}&-\dim H^i(X_\mathfrak{p}(C), Q)^{(-1)^{i-n}}+\dim H^i_{DR}(X_\mathfrak{p})/F^n\\ &=\dim H^i(X_\mathfrak{p}(C), Q)^{(-1)^{i-(n-1)}}-\dim H^i(X_\mathfrak{p}(C), Q)+\dim H^i_{DR}(X_\mathfrak{p})/F^n\\ &=\dim H^i(X_\mathfrak{p}(C), Q)^{(-1)^{n-1}}-\dim F^n H^i_{DR}(X_\mathfrak{p}).\end{aligned}$$

So the first equality is established and consequently the vanishing of $\rho(H^i(X)((i+1)/2))$, too. Since above we have expressed $\rho(H^i(X)(n))$ in terms of Hodge numbers it again is an elementary computation to identify it with the respective multiplicity of $L_\infty(H^i(X), s)$ (compare [18]). q.e.d.

In general, the nature of the numbers corank $F_\mathfrak{p}$ is very complicated. We only are able to discuss, based on the already mentioned conjecture of Bloch, the case where

the reductions $Y_\mathfrak{p}$ of X, for all $\mathfrak{p}|p$, are Hodge-Witt in dimension i

which we therefore are going to assume for the rest of this paragraph.

(The reader may use [8] as an excellent survey of the main notions and results in the theory of the de Rham-Witt complex.) In its precise form the conjecture then says (see [1]):

(B) $\mathrm{gr}^j_{(\mathfrak{p})} H^i(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(n))$ is isogeneous to the $(j+n-i-1)$-th twist of the points of a connected p-divisible group over $o_\mathfrak{p}$ whose covariant Dieudonné module is $H^j_{\mathrm{Zar}}(Y_\mathfrak{p}, W\Omega^{i-j})$ / torsion.

We recall that the slope spectral sequence

$$E_1^{i,j}=H^j_{\mathrm{Zar}}(Y_\mathfrak{p}, W\Omega^i)\Longrightarrow H^{i+j}_{\mathrm{crys}}(Y_\mathfrak{p}/W)$$

degenerates up to torsion and that the filtration $F^{\cdot}H^*_{\mathrm{crys}}(Y_\mathfrak{p}/W)$ which it induces on the abutment, up to torsion, is the slope filtration

$$F^jH^i_{\mathrm{crys}}(Y_\mathfrak{p}/W)\otimes\boldsymbol{Q}=(H^i_{\mathrm{crys}}(Y_\mathfrak{p}/W)\otimes\boldsymbol{Q})_{[j,\infty)};$$

here M_I denotes, for any F-isocrystal M and any subset $I\subseteq\boldsymbol{Q}$, the maximal sub-F-isocrystal of M with slopes in I. The conjecture (B) consequently implies

$$\mathrm{corank}\, F/F^{i+1-n}_{(\mathfrak{p})}=\dim(H^i_{\mathrm{crys}}(Y_\mathfrak{p}/W)\otimes\boldsymbol{Q})_{[n,\infty)}.$$

Proposition. *If* (B) *holds true then we have, for* $n\in\boldsymbol{Z}$,

$$\rho(H^i(X)(n))+\rho(H^i(X)(i+1-n))=\sum_{\mathfrak{p}|p}[k_\mathfrak{p}:\boldsymbol{Q}_p]\cdot\dim(H^i_{\mathrm{crys}}(Y_\mathfrak{p}/W)\otimes\boldsymbol{Q})_{(n-1,n)};$$

in particular, if i is odd then

$$\rho\left(H^i(X)\left(\frac{i+1}{2}\right)\right)=\frac{1}{2}\sum_{\mathfrak{p}|p}[k_\mathfrak{p}:\boldsymbol{Q}_p]\cdot\dim(H^i_{\mathrm{crys}}(Y_\mathfrak{p}/W)\otimes\boldsymbol{Q})_{((i-1)/2,\,(i+1)/2)}.$$

Proof. By the same argument as in the previous proofs we have to show that

$$\mathrm{corank}\, F^n_{(\mathfrak{p})}-\mathrm{corank}\, F/F^{i+1-n}_{(\mathfrak{p})}=\dim(H^i_{\mathrm{crys}}(Y_\mathfrak{p}/W)\otimes\boldsymbol{Q})_{(n-1,n)}.$$

Conjecture (B) implies that the left hand side is equal to

$$\dim(H^i_{\mathrm{crys}}(Y_\mathfrak{p}/W)\otimes\boldsymbol{Q})_{(-\infty,i+1-n)}-\dim(H^i_{\mathrm{crys}}(Y_\mathfrak{p}/W)\otimes\boldsymbol{Q})_{[n,\infty)}.$$

If now

$$P((N\mathfrak{p})^{-s})=\det(1-\phi_\mathfrak{p}^{-1}(N\mathfrak{p})^{-s};\,H^i(\bar{X},\boldsymbol{Q}_\ell))$$

denotes the Euler factor at $\mathfrak{p}$ of the complex L-function $L_\infty(H^i(X),s)$ then we know that the slopes of $H^i_{\mathrm{crys}}(Y_\mathfrak{p}/W)\otimes\boldsymbol{Q}$ counted with multiplicities are precisely the p-adic valuations $v_p(a^{-1})$ where a runs through the roots of the integral polynomial $P(T)$ counted with multiplicities and where v_p is normalized in such a way that $v_p(N\mathfrak{p})=1$ (see [13] Theorem 2.2 and [7] 3.7.3). Since $P(T)$ fulfills a functional equation with respect to $T\mapsto(N\mathfrak{p})^{-i}T^{-1}$ we see that

$$\lambda\longmapsto i-\lambda$$

is a permutation of the set of slopes of $H^i_{\mathrm{crys}}(Y_\mathfrak{p}/W)\otimes\boldsymbol{Q}$. Consequently we get

$$\#\{\lambda<i+1-n\}-\#\{\lambda\geq n\}=\#\{\lambda>n-1\}-\#\{\lambda\geq n\}$$
$$=\#\{n-1<\lambda<n\}=\dim(H^i_{\mathrm{crys}}(Y_\mathfrak{p}/W)\otimes\boldsymbol{Q})_{(n-1,n)}.\qquad\text{q.e.d.}$$

In order to get a relation to the multiplicities of the complex L-function we need another hypothesis:

(H) The Hodge numbers $h^{j,i-j}:=\dim H^{i-j}(X,\Omega^j)$ of X are equal to the abstract Hodge numbers of the F-crystals $M_\mathfrak{p}:=H^i_{\mathrm{crys}}(Y_\mathfrak{p}/W)$/torsion, i.e.,

$$M_\mathfrak{p}/FM_\mathfrak{p}\cong\bigoplus_{j\geq0}[W(\kappa_\mathfrak{p})/p^jW(\kappa_\mathfrak{p})]^{h^{j,i-j}}\qquad\text{for }\mathfrak{p}\,|\,p.$$

Proposition. *If* (A), (B), *and* (H) *hold true then we have*

$$\rho(H^i(X)(n))-\sum_{\mathfrak{p}|p}[k_\mathfrak{p}:\boldsymbol{Q}_p]\cdot\dim H^{i+1-n}_{\mathrm{Zar}}(Y_\mathfrak{p},W\Omega^{n-1})/(\text{torsion}+F)$$

$$=\begin{cases}\mathrm{ord}_{s=i+1-n}L_\infty(H^i(X),s) & \text{if } n>\dfrac{i}{2}+1,\\ \mathrm{ord}_{s=i/2}L_\infty(H^i(X),s)-\mathrm{ord}_{s=(i/2)+1}L_\infty(H^i(X),s) & \text{if } n=\dfrac{i}{2}+1.\end{cases}$$

Proof. By our Theorem and (B) we have

$$\rho(H^i(X)(n))=\sum_{\mathfrak{p}|\infty}\dim H^i(X_\mathfrak{p}(\boldsymbol{C}),\boldsymbol{Q})^{(-1)^{n-1}}-\sum_{\mathfrak{p}|p}\dim(M_\mathfrak{p}\otimes\boldsymbol{Q})_{[n,\infty)}\cdot[k_\mathfrak{p}:\boldsymbol{Q}_p].$$

On the other hand we already know that the respective multiplicities are equal to

$$\sum_{\mathfrak{p}|\infty}\dim H^i(X_\mathfrak{p}(\boldsymbol{C}),\boldsymbol{Q})^{(-1)^{n-1}}-\sum_{\mathfrak{p}|p}\dim F^nH^i_{DR}(X)\cdot[k_\mathfrak{p}:\boldsymbol{Q}_p].$$

Therefore we have to compute

$$\dim F^n H^i_{DR}(X) - \dim (M_\mathfrak{p} \otimes \mathbf{Q})_{[n,\infty)}.$$

Since the slope spectral sequence degenerates up to torsion the second term equals

$$\sum_{j \geq n} \operatorname{rank} H^{i-j}_{\mathrm{Zar}}(Y_\mathfrak{p}, W\Omega^j)/\text{torsion}.$$

On the other hand, for $Y_\mathfrak{p}$ Hodge-Witt in dimension i, the abstract Hodge numbers $h^j_\mathfrak{p}$ of $M_\mathfrak{p}$ are given by

$$h^j_\mathfrak{p} = \dim H^{i-j}(W\Omega^j)/(\text{torsion}+V) + \dim H^{i-j+1}(W\Omega^{j-1})/(\text{torsion}+F)$$

(use [8] (3.4.1), (6.2.4), and (6.2.6)). Because of the relation

$$\dim H^*(W\Omega^\cdot)/(\text{torsion}+F) + \dim H^*(W\Omega^\cdot)/(\text{torsion}+V) = \operatorname{rank} H^*(W\Omega^\cdot)/\text{torsion}$$

we get

$$\sum_{j \geq n} \operatorname{rank} H^{i-j}_{\mathrm{Zar}}(Y_\mathfrak{p}, W\Omega^j)/\text{torsion} = \sum_{j \geq n} h^j_\mathfrak{p} - \dim H^{i+1-n}(W\Omega^{n-1})/(\text{torsion}+F).$$

But (H) implies

$$\sum_{j \geq n} h^j_\mathfrak{p} = \dim F^n H^i_{DR}(X).$$

So we have

$$\dim F^n H^i_{DR}(X) - \dim (M_\mathfrak{p} \otimes \mathbf{Q})_{[n,\infty)} = \dim H^{i+1-n}(W\Omega^{n-1})/(\text{torsion}+F).$$

Remarks. 1) In all the situations we considered above the invariants $\rho(H^i(X)(n))$ turned out to be independent of the particular choice of the reductions $Y_\mathfrak{p}$ for $\mathfrak{p}|p$. Later on, in connection with our $\Gamma(n)$-approach, we will discuss the question whether even the $\mathbf{Z}_p[[\Gamma]]$-modules $H^*(o_\infty, \tau_\alpha Rg_* F)$ (for our distinguished α) are independent of that choice.

2) The numbers ${}^j d^i_\mathfrak{p} := \dim H^j_{\mathrm{Zar}}(Y_\mathfrak{p}, W\Omega^{i-j})/(\text{torsion}+F)$ appearing in the above Proposition have two other interesting interpretations. Firstly, they can be expressed in terms of the slopes λ of the F-crystal $H^i_{\mathrm{crys}}(Y_\mathfrak{p}/W)$/torsion, namely

$${}^j d^i_\mathfrak{p} = \sum_{\lambda \in [i-j, i-j+1)} (\lambda - i + j)$$

(see [8] 6.2.3). Secondly, if, according to the conjecture (B), ${}^j\mathscr{G}^i_{/o_\mathfrak{p}}$ denotes a p-divisible group whose covariant Dieudonné module is $H^i_{\mathrm{Zar}}(Y_\mathfrak{p}, W\Omega^{i-j})$/torsion and whose points are isogeneous to $\mathrm{gr}^j_{(\mathfrak{p})} H^i(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(i+1-j))$ then

$${}^j d^i_\mathfrak{p} = \text{dimension of the } p\text{-divisible group dual to } {}^j\mathscr{G}^i_{/o_\mathfrak{p}}.$$

3) Kato ([26]) and Fontaine/Messing (unpublished) have a proof of the conjecture (B) in case $k_\mathfrak{p}/\mathbf{Q}_p$ is unramified and $i<p$. Fontaine/Messing also seem to have a formulation of a more general conjecture without the Hodge-Witt assumption. Hopefully this can be used for a computation of our invariants $\rho(\cdots)$ in general.

4) It seems not to be known whether the hypothesis (H) is fulfilled in general (always assuming the $Y_\mathfrak{p}$ are Hodge-Witt in dimension i) or not. In any case (H) holds true if X is an abelian variety or a surface or if the $Y_\mathfrak{p}$ are Hodge-Witt in dimension $i+1$ (and i) and $H^i_{\mathrm{crys}}(Y_\mathfrak{p}/W)$ and $H^{i+1}_{\mathrm{crys}}(Y_\mathfrak{p}/W)$ are torsionfree (see [8] 6.3).

§ 4. The rank conjecture in the ordinary case

In order to be able to define p-adic L-functions one first has to know the ranks of the individual $\mathbf{Z}_p[[\Gamma]]$-modules $H^\nu(o_\infty, \tau_\alpha Rg_* F)^*$. Absolutely fundamental for that is the following well-known conjecture of Iwasawa which we state in an appropriately reformulated way (compare [17] § 3 Lemma 8):

(I) $\operatorname{Spec}(o'_\infty)_{et}$ has cohomological p-dimension ≤ 1 (for any base field k).

It is all we need in the "stable" case.

Lemma. *If* (I) *holds true then* $H^1(o_\infty, \mathscr{F})$ *is finite for any constructible p-torsion sheaf* $\mathscr{F}$ *on* $\operatorname{Spec}(o_\infty)_{et}$.

Proof. We proceed in several steps.

Step 1. $\mathscr{F} = \mathbf{Z}/p\mathbf{Z}$: If μ_p denotes the sheaf of p-th roots of unity then we get from (I) that

$$H^2_{\mathrm{flat}}(o_\infty, \mu_p) = H^2(o'_\infty, \mu_p) = 0;$$

the first identity is a special case of Lemma 3.5 in [16]. By Kummer theory, this vanishing result implies the divisibility of the p-primary part $\operatorname{Pic}(o_\infty)(p)$ of the ideal class group of o_∞. According to Iwasawa ([9] Theorem 11) the finitely generated $\mathbf{Z}_p[[\Gamma]]$-torsion modules $\operatorname{Pic}(o_\infty)(p)^*$ and $H^1(o_\infty, \mathbf{Q}_p/\mathbf{Z}_p)^*$ have the same μ-invariant which consequently has to be zero. This shows that

$$H^1(o_\infty, \mathbf{Z}/p\mathbf{Z})^* = H^1(o_\infty, \mathbf{Q}_p/\mathbf{Z}_p)^*/p$$

is finite.

Step 2. $\mathscr{F}$ is constant: This is an immediate consequence of Step 1.

Step 3. $\mathscr{F}\,|\,U=0$ for a nonempty open subset U of Spec(o_∞): Let h: $Z\to\operatorname{Spec}(o_\infty)$ be the closed immersion of the complement $Z:=\operatorname{Spec}(o_\infty)\setminus U$ which is a finite set of closed points. We then have $\mathscr{F}=h_*h^*\mathscr{F}$ and therefore $H^1(o_\infty,\mathscr{F})=H^1(Z,h^*\mathscr{F})$. But from the fact that the residue class fields of o_∞ are either p-closed or finite we easily conclude that $H^1(Z,h^*\mathscr{F})$ is a finite direct sum of groups of the form $H^1(\kappa,M)$ where κ is a finite field of characteristic p and M is a finite p-torsion Galois module. Such groups obviously are finite.

Step 4. If the assertion holds true for $\mathscr{F}$ then also for any subsheaf of $\mathscr{F}$: This is a trivial consequence of the long exact cohomology sequence and the fact that the group of global sections of a constructible sheaf on Spec$(o_\infty)_{et}$ it finite.

Step 5. $\mathscr{F}$ arbitrary: First we find a nonempty open subset f: $U\to\operatorname{Spec}(o_\infty)$ such that $f^*\mathscr{F}$ is locally constant. Since the cokernel of the injective map of sheaves

$$0\longrightarrow f_!f^*\mathscr{F}\longrightarrow\mathscr{F}$$

is of the type considered in Step 3 it suffices to prove the assertion for sheaves of the form $f_!\mathscr{L}$ with a constructible locally constant p-torsion sheaf $\mathscr{L}$ on U_{et}. There is a finite extension of k_∞ with ring of integers R and structure morphism π: Spec$(R)\to$Spec(o_∞) such that

$$\tilde U:=\pi^{-1}(U)\xrightarrow{\ \pi\ }U \text{ is etale and } \pi^*\mathscr{L} \text{ is constant on } \tilde U_{et}.$$

If $\tilde f$: $\tilde U\to\operatorname{Spec}(R)$ denotes the corresponding open immersion we have an injective map of sheaves

$$f_!\mathscr{L}\longrightarrow f_!\pi_*\pi^*\mathscr{L}=\pi_*\tilde f_!(\pi^*\mathscr{L}).$$

By Step 4 we thus are reduced to show the assertion for $\pi_*\tilde f_!(\pi^*\mathscr{L})$. But the finiteness of π implies

$$H^1(o_\infty,\pi_*\tilde f_!(\pi^*\mathscr{L}))=H^1(R,\tilde f_!(\pi^*\mathscr{L})).$$

Simplifying notation again it therefore remains to consider a sheaf of the form $f_!\mathscr{L}$ with a constructible constant p-torsion sheaf $\mathscr{L}$ on U_{et}. This case finally is established by combining Step 2 and Step 4. q.e.d.

At this point it is convenient to adjust our notation in the following way: $M:=H^i(X)$ is our basic motive, $n\in\boldsymbol{Z}$ is an integer, and

$$\mathscr{G}_p(M(n)):=\tau_\alpha Rg_*H^i(\bar X,\boldsymbol{Q}_p/\boldsymbol{Z}_p(n))$$

with $\alpha=\alpha(H^i(\bar X,\boldsymbol{Q}_p/\boldsymbol{Z}_p(n))$; $F^{i+1-n}_{(\mathfrak{p})\dots}$) are the complexes Greenberg's approach provides us with.

Proposition. *If* (I) *holds true then we have*

i) $H^2(o_\infty,\mathscr{G}_p(M(n)))=0$ *for* $n>\min(i,\dim X)$;

ii) $H^1(o_\infty,\mathscr{G}_p(M(n)))^*$ *is finitely generated over* $\boldsymbol{Z}_p$ *for* $n\le\max(0,i-\dim X)$.

Proof. For $n>\min(i,\dim X)$ we have $F^{i+1-n}_{(\mathfrak{p})\dots}=H^i(\bar X,\boldsymbol{Q}_p/\boldsymbol{Z}_p(n))$ which implies $\mathscr{G}_p(M(n))=R\sigma_*(g'_*H^i(\bar X,\boldsymbol{Q}_p/\boldsymbol{Z}_p(n)))$ and therefore

$$H^2(o_\infty,\mathscr{G}_p(M(n)))=H^2(o'_\infty,g'_*H^i(\bar X,\boldsymbol{Q}_p/\boldsymbol{Z}_p(n)))=0.$$

Let us now assume that $n\le\max(0,i-\dim X)$. This implies $F^{i+1-n}_{(\mathfrak{p})\dots}=0$ and therefore that $\mathscr{G}_p(M(n))$ is quasi-isomorphic to $g_*H^i(\bar X,\boldsymbol{Q}_p/\boldsymbol{Z}_p(n))$.

Since kernel and cokernel of the multiplication by p on the sheaf $g_*H^i(\bar X,\boldsymbol{Q}_p/\boldsymbol{Z}_p(n))$ certainly are constructible p-torsion sheaves it is easily derived from the above Lemma that the kernel of the multiplication by p on $H^1(o_\infty,g_*H^i(\bar X,\boldsymbol{Q}_p/\boldsymbol{Z}_p(n)))$ is finite.

We now want to propose a conjecture which will enable us to compute the individual ranks for arbitrary n. But at the moment we only will consider the case where

$$X \text{ has ordinary good reduction at all } \mathfrak{p}\,|\,p$$

which we assume from now on in this paragraph. The reason for this is that, as we will see, in the "nonordinary" case our two approaches seem to lead to different $\boldsymbol{Z}_p[[\Gamma]]$-modules. So any more general conjecture has to be postponed till the relation between the complexes $\mathscr{G}_p$ and $\mathscr{H}_p$ has been clarified.

Conjecture. *We have (for any base field k):*

a) *If i is odd then $H^2(o_\infty,\mathscr{G}_p(M((i+1)/2)))^*$ is $\boldsymbol{Z}_p[[\Gamma]]$-torsion;*

b) *if i is even then $H^1(o_\infty,\mathscr{G}_p(M(i/2)))^*$ and $H^2(o_\infty,\mathscr{G}_p(M((i/2)+1)))^*$ are $\boldsymbol{Z}_p[[\Gamma]]$-torsion.*

For $i=1$ this Conjecture, as was explained at the beginning, comes down to Mazur's conjecture about the "well-behavior" of his $\boldsymbol{Z}_p[[\Gamma]]$-module. We also should remark that, presumably, the duality theory for $H^*(o_\infty,\cdot)$ which does not exist yet will show that the two assertions in part *b*. of the Conjecture are equivalent.

Proposition. *If the above Conjecture holds true then $H^\nu(o_\infty, \mathcal{G}_p(M(n)))^*$ is $\mathbf{Z}_p[[\Gamma]]$-torsion for $\nu=1$ and $n \leq i/2$ and for $\nu=2$ and $n \geq (i+1)/2$.*

Proof. Let ζ_p be a primitive p-th root of unity. Since the degree of $k(\zeta_p)/k$ is prime to p an easy descent argument shows that it suffices to prove the assertion for the base field $k(\zeta_p)$. Therefore let us assume that $\zeta_p \in k$. But then the formation of the cohomology $H^*(o_\infty, \cdot)$ commutes with Tate twist. By the results of the previous paragraph the long exact cohomology sequences consequently induce a commutative exact diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^1(o_\infty, \mathcal{G}_p(M(n)))(1) & \longrightarrow & H^1(o_\infty, \mathcal{G}_p(M(n+1))) \\
\downarrow & & \downarrow \\
H^1(o'_\infty, g'_* H^i(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(n)))(1) & = & H^1(o'_\infty, g'_* H^i(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(n+1))) \\
\downarrow & & \downarrow \\
\bigoplus_{v|\mathfrak{p}|p} L_v(H^i(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(n))/F^{i+1-n}_{(\mathfrak{p})})(1) & \longrightarrow & \bigoplus_{v|\mathfrak{p}|p} L_v(H^i(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(n+1))/F^{i-n}_{(\mathfrak{p})}) \rightarrow 0 \\
\downarrow & & \downarrow \\
H^2(o_\infty, \mathcal{G}_p(M(n)))(1) & \longrightarrow & H^2(o_\infty, \mathcal{G}_p(M(n+1))) \\
\downarrow & & \downarrow \\
H^2(o'_\infty, g'_* H^i(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(n)))(1) & = & H^2(o'_\infty, g'_* H^i(\bar{X}, \mathbf{Q}_p/\mathbf{Z}_p(n+1))) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where $L_v(\cdot) := H^0(Z_v, H^1(k^{nr}_{\infty,v}, \cdot))$. From that we obviously get an injection

$$0 \longrightarrow H^2(o_\infty, \mathcal{G}_p(M(n+1)))^* \longrightarrow H^2(o_\infty, \mathcal{G}_p(M(n)))(1)^*$$

and a surjection

$$H^1(o_\infty, \mathcal{G}_p(M(n+1)))^* \longrightarrow H^1(o_\infty, \mathcal{G}_p(M(n)))(1)^* \longrightarrow 0.$$

Since we have proved that $\rho(H^i(X)((i+1)/2))=0$ for odd i the assertion is immediate from these maps and our assumption.

Corollary. *If the above Conjecture holds true then we have*

$$\mathrm{rank}_{\mathbf{Z}_p[[\Gamma]]} H^1(o_\infty, \mathcal{G}_p(M(i+1-n)))^* = \mathrm{rank}_{\mathbf{Z}_p[[\Gamma]]} H^2(o_\infty, \mathcal{G}_p(M(n)))^* = 0$$

$$\text{if } n = \frac{i+1}{2} \quad \text{or} \quad \frac{i}{2}+1,$$

and, provided (A) *holds true,*

$$= \begin{cases} \mathrm{ord}_{s=n} L_\infty(M, s) & \text{if } n \neq \frac{i}{2}, \ \frac{i+1}{2}, \ \frac{i}{2}+1, \\ \mathrm{ord}_{s=n} L_\infty(M, s) - \mathrm{ord}_{s=n+1} L_\infty(M, s) & \text{if } n = \frac{i}{2}. \end{cases}$$

§ 5. p-adic L-functions in the ordinary case

Let H be a finitely generated $\mathbf{Z}_p[[\Gamma]]$-torsion module. Iwasawa's structure theory of such modules tells us that

$$H \underset{\mathbf{Z}_p}{\otimes} \mathbf{Q}_p \text{ is of finite dimension over } \mathbf{Q}_p$$

and that there is a $\mathbf{Z}_p[[\Gamma]]$-homomorphism

$$\mathrm{Tor}_{\mathbf{Z}_p} H \longrightarrow \bigoplus_i (\mathbf{Z}/p^{\mu_i}\mathbf{Z})[[\Gamma]]$$

with finite kernel and cokernel;

$$\mu(H) := \sum_i \mu_i$$

is called the μ-invariant of H. The characteristic polynomial of H then, by definition, is

$$P(T; H) := p^{\mu(H)} \cdot \det(1 - \gamma^{-1}T; H \underset{\mathbf{Z}_p}{\otimes} \mathbf{Q}_p)$$

where γ is our fixed topological generator of Γ. (Warning: In the literature usually $\det(-\gamma) \cdot P(T+1; H) = p^{\mu(H)} \cdot \det(T - (\gamma - 1); H \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)$ is called the characteristic polynomial of H.) We assume in this paragraph again that

$$X \text{ has ordinary good reduction at all } \mathfrak{p} | p.$$

We say that the integer $n \in \mathbf{Z}$ is *p-critical* for the motive $M = H^i(X)$ if the modules

$$H^\nu(o_\infty, \mathcal{G}_p(M(i+1-n)))^*, \quad \text{for } \nu = 1 \text{ and } 2, \text{ are } \mathbf{Z}_p[[\Gamma]]\text{-torsion}.$$

In this case we define the *p-adic L-function* of the motive $M(n)$ (by abuse of language) to be

$$L_p(M(n), s) := \prod_{\nu=0}^{2} P(\kappa(\gamma)^{-s}; H^\nu(o_\infty, \mathcal{G}_p(M(i+1-n)))^*)^{(-1)^{\nu+1}}$$

where $\kappa: \Gamma \to \boldsymbol{Z}_p^\times$ is the cyclotomic character. Since, in the moment, the nature of the μ-invariants

$$\mu(M(\cdot)) := \mu(H^1(o_\infty, \mathcal{G}_p(M(\cdot)))^*) - \mu(H^2(o_\infty, \mathcal{G}_p(M(\cdot)))^*)$$

is quite unclear it is convenient also to introduce the *reduced p-adic L-functions*

$$\tilde{L}_p(M(n), s) := L_p(M(n), s) \cdot p^{-\mu(M(i+1-n))}.$$

Remarks. 1) We will see later on that the reduced p-adic L-function most probably is independent of the particular choice of the reductions $Y_\mathfrak{p}$ for $\mathfrak{p} | p$.

2) It was Greenberg's marvellous idea that the notion of being p-critical should coincide with Deligne's notion (in [5]) of being critical. We have seen in the previous paragraph that this is exactly what would follow from our rank conjecture.

There are four major questions about these p-adic L-functions which we will at least begin to discuss in this paper:

—dependence on the twist
—location of poles and zeros
—functional equation
—p-adic regulators.

The dependence on the twist

Let ζ_p be a primitive p-th root of unity and put $\delta := [k(\zeta_p): k]$.

Theorem. *If m and n are p-critical for M with $m \equiv n \bmod \delta$ then we have*

$$\tilde{L}_p(M(m), s) = \tilde{L}_p(M(n), m-n+s).$$

Proof. The first assumption, in particular, says that $\rho(M(i+1-m)) = \rho(M(i+1-n))$. By the formula which we have established for these invariants in our first Theorem this amounts to

$$\begin{aligned}
&\sum_{\mathfrak{p}|\infty} \dim H^i(X_\mathfrak{p}(C), \boldsymbol{Q})^{(-1)^{i-m}} \\
&\quad - \sum_{\mathfrak{p}|p} \operatorname{corank} H^i(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(i+1-m))/F^m_{(\mathfrak{p})} \cdot [k_\mathfrak{p}: \boldsymbol{Q}_p] \\
&= \sum_{\mathfrak{p}|\infty} \dim H^i(X_\mathfrak{p}(C), \boldsymbol{Q})^{(-1)^{i-n}} \\
&\quad - \sum_{\mathfrak{p}|p} \operatorname{corank} H^i(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(i+1-n))/F^n_{(\mathfrak{p})} \cdot [k_\mathfrak{p}: \boldsymbol{Q}_p].
\end{aligned}$$

We claim that the left hand terms in this identity are equal. This is obvious by our second assumption if δ is even. But if δ is odd then the field k is totally imaginary so that we have

$$\dim H^i(X_\mathfrak{p}(C), \boldsymbol{Q})^{\pm} = \tfrac{1}{2} \dim H^i(X_\mathfrak{p}(C), \boldsymbol{Q}) \qquad \text{for any } \mathfrak{p} | \infty.$$

If $m \geq n$, say, the above identity therefore implies that

$$F^n_{(\mathfrak{p})} H^i(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(\cdot))/F^m_{(\mathfrak{p})} H^i(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(\cdot)) \text{ is finite for } \mathfrak{p} | p.$$

The kernel C of the natural map

$$\begin{aligned}
&\bigoplus_{v|\mathfrak{p}|p} L_v(H^i(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(i+1-m))/F^m_{(\mathfrak{p})})(m-n) \\
&\longrightarrow \bigoplus_{v|\mathfrak{p}|p} L_v(H^i(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(i+1-n))/F^n_{(\mathfrak{p})})
\end{aligned}$$

where $L_v(\cdot) = H^0(Z_v, H^1(k^{nr}_{\infty,v}, \cdot))$ consequently is annihilated by some power of p. Note that the formation of $L_v(\cdot)$, resp. of $H^*(o_\infty, \cdot)$, commutes with the $(m-n)$-th Tate twist since $m \equiv n \bmod \delta$. The long exact cohomology sequences induce an exact sequence (compare the big diagram in the last paragraph)

$$\begin{aligned}
0 &\longrightarrow H^1(o_\infty, \mathcal{G}_p(M(i+1-m)))(m-n) \longrightarrow H^1(o_\infty, \mathcal{G}_p(M(i+1-n))) \longrightarrow C \\
&\longrightarrow H^2(o_\infty, \mathcal{G}_p(M(i+1-m)))(m-n) \longrightarrow H^2(o_\infty, \mathcal{G}_p(M(i+1-n))) \longrightarrow 0
\end{aligned}$$

which translates into our assertion by the general fact that

$$P(\kappa(\gamma)^{-n}T; H) = P(T; H(n)).$$

Remark. This result suggests to extend the definition of the reduced p-adic L-function to any twist $M(m)$ such that there is a $n \equiv m \bmod \delta$ which is p-critical for M by

$$\tilde{L}_p(M(m), s) := \tilde{L}_p(M(n), m-n+s).$$

In case there is an n which is p-critical for $M_{/k(\zeta_p)}$ this procedure would even allow to define $\tilde{L}_p(M(m), s)$ for any $m \in \boldsymbol{Z}$. Note that, as a consequence of our rank conjecture, $n = (i+1)/2$, for odd i, always should be p-critical for $M_{/k(\zeta_p)}$. Since it seems artificial to make into a definition what should be a result we will not pursue this point of view.

At this point it is quite illustrative to discuss the case $X = \operatorname{Spec}(k)$

reformulating thereby Iwasawa's original definitions in our context: Let k be totally real and $M:=H^0(\mathrm{Spec}\,(k))$. Denote by K, resp. L, the maximal unramified outside p, resp. unramified, abelian p-extension of $k_\infty(\zeta_p)$ and put

$$\mathscr{M}:=\mathrm{Gal}\,(K/k_\infty(\zeta_p)) \quad \text{and} \quad \mathscr{X}:=\mathrm{Gal}\,(L/k_\infty(\zeta_p)).$$

Since δ is prime to p we have canonical decompositions of $\boldsymbol{Z}_p[[\Gamma]]$-modules

$$\mathscr{M}=\bigoplus_{j \bmod \delta} e_j\mathscr{M} \quad \text{and} \quad \mathscr{X}=\bigoplus_{j \bmod \delta} e_j\mathscr{X}$$

where

$e_j(\cdot):=$ maximal submodule of $\cdot$ on which $\mathrm{Gal}\,(k(\zeta_p)/k)$ acts via the j-th power of the cyclotomic character.

Our complexes are given by

$$\mathscr{G}_p(M(n))\sim\begin{cases} R\sigma_*(g'_*\boldsymbol{Q}_p/\boldsymbol{Z}_p(n)) & \text{if } n>0,\\ g_*\boldsymbol{Q}_p/\boldsymbol{Z}_p(n) & \text{if } n\le 0.\end{cases}$$

We consequently get

Fact 1. $$H^1(o_\infty, \mathscr{G}_p(M(n)))^*=\begin{cases}(e_n\mathscr{M})(-n) & \text{if } n>0,\\ (e_n\mathscr{X})(-n) & \text{if } n\le 0.\end{cases}$$

Fact 2. a) $H^2(o_\infty, \mathscr{G}_p(M(n)))=0$ for $n>0$;
b) for $n\le 0$, the $\boldsymbol{Z}_p[[\Gamma]]$-torsion submodule of $H^2(o_\infty, \mathscr{G}_p(M(n)))^*$ is $\boldsymbol{Z}_p(1-n)$ if $n\equiv 1 \bmod \delta$ and 0 otherwise.

Proof (sketch). We have

$$H^2(o_\infty, \mathscr{G}_p(M(n)))^*=\begin{cases}(e_n(H^2(o'_\infty(\zeta_p), \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*))(-n) & \text{if } n>0,\\ (e_n(H^2(o_\infty(\zeta_p), \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*))(-n) & \text{if } n\le 0.\end{cases}$$

It is well-known that $H^2(o'_\infty(\zeta_p), \boldsymbol{Q}_p/\boldsymbol{Z}_p)=0$ (compare [16] §3 Proof of Proposition 8 iii). On the other hand, by global duality and Kummer theory, we get

$$H^2(o_\infty(\zeta_p), \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*=\varprojlim\,(\text{units in } k_m(\zeta_p)\otimes \boldsymbol{Z}_p)$$

where k_m runs through the finite intermediate layers of k_∞/k. But the $\boldsymbol{Z}_p[[\Gamma]]$-torsion submodule of the right hand side is known to be $\boldsymbol{Z}_p(1)$ (compare [24] Theorem 8.17).

Fact 3. $$H^0(o_\infty, \mathscr{G}_p(M(n)))^*=\begin{cases}\boldsymbol{Z}_p(-n) & \text{if } n\equiv 0 \bmod \delta,\\ 0 & \text{otherwise.}\end{cases}$$

According to [9] Theorem 5 the module $\mathscr{X}$ is $\boldsymbol{Z}_p[[\Gamma]]$-torsion. Therefore our rank conjecture is fulfilled and we know that $n\in \boldsymbol{Z}$ is p-critical (for M) if and only if n is critical in the sense of Deligne if and only if n is positive and even or negative and odd. If we combine all these facts we get the following more explicit expression for the p-adic L-function:

$$L_p(M(n), 1-n+s)=\begin{cases} P(\kappa(\gamma)^{-s};\, e_{1-n}\mathscr{X}) & \text{if } n>0 \text{ even, } n\not\equiv 0 \bmod \delta,\\ \dfrac{P(\kappa(\gamma)^{-s};\, e_{1-n}\mathscr{X})}{1-\kappa(\gamma)^{-1-s}} & \text{if } n>0 \text{ even, } n\equiv 0 \bmod \delta,\\ P(\kappa(\gamma)^{-s};\, e_{1-n}\mathscr{M}) & \text{if } n<0 \text{ odd, } n\not\equiv 1 \bmod \delta,\\ \dfrac{P(\kappa(\gamma)^{-s};\, e_{1-n}\mathscr{M})}{1-\kappa(\gamma)^{-s}} & \text{if } n<0 \text{ odd, } n\equiv 1 \bmod \delta.\end{cases}$$

The characteristic polynomials on the right hand side are exactly those which Iwasawa suggested to view as p-adic L-functions of the field k. Furthermore, using Kummer theory, Iwasawa has proved (compare [4]) that, for even j,

$$(e_j\mathscr{M})(-1) \text{ is quasi-isomorphic to } (e_{1-j}\mathscr{X})^{\cdot}$$

where the dot indicates that the action of Γ has been inverted. Because of the general identity

$$P(T;\dot{H})=\det(-\gamma T)\cdot P\left(\frac{1}{T};\, H\right)$$

we easily deduce from that, for any p-critical n, the functional equation

$$L_p(M(n), s)=u_n\cdot v_n^s\cdot L_p(M(1-n), -s)$$

with appropriate units $u_n, v_n\in \boldsymbol{Z}_p^\times$.

Remark. This already indicates the general form of the functional equation. Let us call the motive $M:=H^i(X)(n)$ p-critical if n is p-critical for $H^i(X)$ and let $\check{M}=H^{2d-i}(X)(d-n)$ with $d=\dim X$ denote the dual motive. Later on we will show: M is p-critical if and only if $\check{M}(1)$ is p-critical; in this case we have a functional equation

$$L_p(M,s)=u\cdot v^s\cdot L_p(\check{M}(1), -s)$$

with appropriate units $u, v \in \boldsymbol{Z}_p^\times$. This is completely analogous to the conjectured functional equation for $L_\infty(M, s)$ (see [5] 1.2.3) besides the fact that there one always can write $L_\infty(\check{M}(1), -s) = L_\infty(\check{M}, 1-s)$ which we can do only if $\delta = 1$.

Remark. Accepting the artificial procedure in our previous Remark we can define the p-adic L-function of any Artin motive over k which can be written as a direct sum of twists of p-critical motives. (The Conjecture (I) says that in this context the relevant μ-invariants always vanish.) A typical example of such a motive is the following: Let K/k be a totally imaginary quadratic extension of the totally real field k and put $M_0 := H^0(\mathrm{Spec}(K))$; then the restriction of scalars M of M_0 to k is of this type. In addition, it seems reasonable to put

$$L_p(M_0, s) := L_p(M, s).$$

This would define the p-adic L-function $L_p(H^0(\mathrm{Spec}(K)), s)$ for any CM-field K. Almost by construction these p-adic L-functions have factorizations of the same form as those which in [6] are given for the automorphic p-adic L-functions of certain Hecke characters.

The denominator

We first discuss the polynomials $P(T; H^0(o_\infty, \mathscr{G}_p(M(n)))^*)$ where $M = H^i(X)$ and $n \in \boldsymbol{Z}$ is arbitrary. (Here our results are valid without any assumption about the reductions of X at $\mathfrak{p} \mid p$.) It is clear that the $\boldsymbol{Z}_p$-modules

$$H^0(o_\infty, \mathscr{G}_p(M(n)))^* = H^0(k_\infty, H^i(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(n)))^*$$

are finitely generated.

Theorem. i) *If i is odd then $H^0(k_\infty, H^i(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(n)))$ is finite;*
ii) *if i is even then there is, for an appropriate open subgroup $\Gamma' \subseteq \Gamma$, a quasi-isomorphism of Γ'-modules*

$$\boldsymbol{Q}_p/\boldsymbol{Z}_p\left(n - \frac{i}{2}\right) \oplus \cdots \oplus \boldsymbol{Q}_p/\boldsymbol{Z}_p\left(n - \frac{i}{2}\right) \longrightarrow H^0(k_\infty, H^i(\bar{X}, \boldsymbol{Q}_p/\boldsymbol{Z}_p(n))).$$

Proof. We have to show that

$$V := H^0(k_\infty, H^i(\bar{X}, \boldsymbol{Q}_p(n))) \begin{cases} = 0 \text{ for odd } i, \text{ resp.} \\ \cong \left(\oplus \boldsymbol{Q}_p\left(n - \frac{i}{2}\right)\right), \text{ as } \Gamma'\text{-modules, for even } i. \end{cases}$$

Faltings [(27)] has recently proved that $H^i(\bar{X}, \boldsymbol{Q}_p(n))$ and therefore also V have a Hodge-Tate decomposition at each $\mathfrak{p} \mid p$. By a theorem of Tate (see [20] III-7 complemented by [22] p. 171) V consequently is a locally algebraic Galois representation. This means (compare [10] p. 127) that there is an algebraic homomorphism $h: \boldsymbol{G}_{m/\boldsymbol{Q}_p} \to GL_V$ such that an appropriate open subgroup $\Gamma' \subseteq \Gamma$ acts semisimply on V via $h \circ \kappa$ (where κ, as always, denotes the cyclotomic character). By extending the base field we can assume without loss of generality that $\Gamma' = \Gamma$. Let now $\mathfrak{p} \nmid p$ be any finite prime at which X has good reduction and let $\phi_\mathfrak{p} \in \Gamma$ be the (arithmetic) Frobenius for $\mathfrak{p}$. Then the eigenvalues of $\phi_\mathfrak{p}$ on V are integral powers of $\kappa(\phi_\mathfrak{p})$. But Deligne's proof of the Weil conjecture implies that these eigenvalues have absolute value $|N\mathfrak{p}|^{n-i/2}$. Since the $\phi_\mathfrak{p}$ are dense in Γ we see that the semisimple action of Γ on V has to be via $\kappa^{n-(i/2)}$ if i is even whereas $V = 0$ if i is odd.

Corollary. i) *If i is odd then $P(T; H^0(o_\infty, \mathscr{G}_p(M(n)))^*) = 1$;*
ii) *if i is even then the only possible zero of $P(\kappa(\gamma)^{-s}; H^0(o_\infty, \mathscr{G}_p(M(n)))^*)$ at an integer point occurs at $s = n - (i/2)$.*

As part of our proof of the functional equation we will later on establish the following

Fact. If n is p-critical for $M = H^i(X)$ then

$$P(T; H^2(o_\infty, \mathscr{G}_p(M(i+1-n)))^*) = u \cdot T^a \cdot P\left(\frac{1}{T}; H^0(o_\infty, \mathscr{G}_p(M(n)))^*\right)$$

with an appropriate $u \in \boldsymbol{Z}_p$ and $a := \operatorname{corank} H^0(o_\infty, \mathscr{G}_p(M(n)))$.

The combination of these two results gives strong limitations to the possible poles of the p-adic L-functions.

Proposition. *If n is p-critical for M then we have:*
i) *If i is odd then there is a polynomial $P(T) \in \boldsymbol{Q}_p[T]$ such that*

$$L_p(M(n), s) = P(\kappa(\gamma)^{-s});$$

ii) *if i is even then the only possible poles of $L_p(M(n), s)$ at integer points occur at $s = (i/2) - n$ and $s = (i/2) + 1 - n$.*

Let $N^*(\bar{X})$ denote the graded group of algebraic cycles on $\bar{X}$ modulo numerical equivalence. Tate's conjecture says that the cycle map induces an isomorphism

$$(T) \qquad N^*(\bar{X}) \otimes \boldsymbol{Q}_p \xrightarrow{\cong} H^{2*}(\bar{X}, \boldsymbol{Q}_p(*))^{\text{discrete}}$$

where the right hand side is that part of $H^{2*}(\bar{X}, \boldsymbol{Q}_p(*))$ on which the absolute Galois group of k acts discretely. Let N^*, resp. N_∞^*, be the graded subgroup of cycle classes which contain a cycle defined over $k(\zeta_p)$, resp. $k_\infty(\zeta_p)$.

Proposition. *Let i be even and n be p-critical for M. If (T) holds true the multiplicity of the pole of $L_p(M(n), s)$ at $s=(i/2)-n$, resp. $s=(i/2)+1-n$, is $\leq \dim e_{(i/2)-n}(N^{i/2}\otimes\boldsymbol{Q}_p)$, resp. $\leq \dim e_{(i/2)+1-n}(N^{i/2}\otimes\boldsymbol{Q}_p)$.*

Proof. It is a straightforward consequence of (T) and the above Theorem that

$$H^0(o_\infty, \mathscr{G}_p(M(m)))^* \otimes \boldsymbol{Q}_p = \mathrm{Hom}_{\boldsymbol{Q}_p}(e_{(i/2)-m}(N_\infty^{i/2}\otimes \boldsymbol{Q}_p), \boldsymbol{Q}_p)\left(\frac{i}{2}-m\right)$$

holds true for any $m \in \boldsymbol{Z}$. By the nondegeneracy of the intersection pairing and by the hard Lefschetz theorem we have

$$\mathrm{Hom}_{\boldsymbol{Q}_p}(e_{(i/2)-m}(N_\infty^{i/2}\otimes \boldsymbol{Q}_p), \boldsymbol{Q}_p) \cong e_{m-(i/2)}(N_\infty^{i/2}\otimes \boldsymbol{Q}_p).$$

We therefore get

$$P(T; H^0(o_\infty, \mathscr{G}_p(M(i+1-n)))^*) = P(\kappa(\gamma)^{(i/2)+1-n}T; e_{(i/2)+1-n}(N_\infty^{i/2}\otimes \boldsymbol{Q}_p))$$

and, using the above Fact,

$$P(T; H^2(o_\infty, \mathscr{G}_p(M(i+1-n)))^*) = u\cdot P(\kappa(\gamma)^{(i/2)-n}T; e_{(i/2)-n}(N_\infty^{i/2}\otimes \boldsymbol{Q}_p))$$

with some $u \in \boldsymbol{Z}_p$. The assertion is immediate from these formulas since the involved Γ-actions are semisimple.

In the assertion of the above Proposition the equality sign occurs if and only if $P(\kappa(\gamma)^{-s}; H^1(o_\infty, \mathscr{G}_p(M(i+1-n)))^*)$ does not vanish at $s=(i/2)-n$ and $s=(i/2)+1-n$. Unfortunately this vanishing can happen due to the so-called "trivial zero" phenomenon which we will discuss later on. In any case the above result shows a behavior of the p-adic L-functions which is very similar to the behavior which, by another conjecture of Tate, one expects for

a) the complex L-function $L_\infty(M(n), s)$: the only possible pole at an integer point occurs for even i at $s=(i/2)+1-n$ and its multiplicity is rank $N^{i/2}(X)$ (there is a different kind of "trivial zero" phenomenon caused by the nature of the expected functional equation which allows to explain what happens at $s=(i/2)-n$);

b) the L-function $L(M(n), s)$ for a motive $M=H^i(X)$ over a global function field: the only possible poles at integer points occur for even i at $s=(i/2)-n$ and $s=(i/2)+1-n$ and both their multiplicities are equal to rank $N^{i/2}(X)$.

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