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ON THE VALUES OF THE ZETA FUNCTION OF A VARIETY OVER A FINITE FIELD

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Let X be a projective smooth geometrically integral scheme of dimension d over the finite field $\mathbb{F}_q$. By $\zeta(X, s)$ we denote the zeta function of X (see [10]). For every integer n the numbers $\rho(n) \in \mathbb{Z}$ and $c(n) \in \mathbb{C}^*$ are defined through

$$\zeta(X, s) \sim c(n) \cdot (q^{-s} - q^{-n})^{\rho(n)} \quad \text{as} \quad s \to n.$$

In fact the $c(n)$ are rational numbers and the purpose of this paper is to compute them in cohomological terms associated with X. In the case $\rho(n) = 0$ Bayer/Neukirch in [1] have given an expression of $|\zeta(X, n)|_l$ (for every prime l not dividing q) as an l-adic Euler characteristic. On the other hand the case $d = 2$, $n = 1$ was studied by Tate in [12] (see also [5]). By combining the two methods we shall attack the general case. After some necessary preliminaries, we give in Section 1 a purely cohomological formula for $|c(n)|_l$ assuming that $\rho(n)$ has the 'right' value. This formula contains the determinants of certain Poincaré duality pairings. In Section 2 we discuss the relationship between these pairings and the intersection pairing on the algebraic cycles of X. Also, we study the special case of an abelian variety more fully in the last section. Finally I want to thank J. Coates for suggesting to me the study of this problem.

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Preliminaries

Throughout, $\bar{\mathbb{F}}_q$ denotes the algebraic closure of $\mathbb{F}_q$, Γ the Galois group of $\bar{\mathbb{F}}_q$ over $\mathbb{F}_q$, φ the Frobenius generator of Γ, and $\bar{X} := X_{\mathbb{F}_q} \times \bar{\mathbb{F}}_q$. We fix a prime l not dividing q. All cohomology groups are understood to be taken with respect to the etale topology.

According to Grothendieck ([6]) we have the following description of the zeta function of X. For every $i \geq 0$ define the polynomial

$$L_i(T) := \det(1 - \varphi^{-1}T; H^i(\bar{X}, \mathbb{Q}_l)).$$

For example, we have $L_0(T) = 1 - T$ and $L_{2d}(T) = 1 - q^dT$. Then

$$\zeta(X, s) = \prod_{i=0}^{2d} L_i(q^{-s})^{(-1)^{i+1}}.$$

Furthermore, by Deligne's proof of the Weil conjectures ([12]), the $L_i(T)$ have integer coefficients independent of l and their complex roots have absolute value $q^{-i/2}$. In particular, we see that $c(n) \in \mathbb{Q}^*$ and $\rho(n) \leq 0$ and strict inequality can only occur if $0 \leq n \leq d$. More precisely, one always has

$$\rho(n) \leq -\dim_{\mathbb{Q}_l} H^{2n}(\bar{X}, \mathbb{Q}_l(n))^{\Gamma};$$

moreover, equality would follow from the well-known conjecture that φ operates semisimply on the $\mathbb{Q}_l$-vectorspaces $H^i(\bar{X}, \mathbb{Q}_l(n))$. Here $\mathbb{Q}_l(n)$ denotes as usual the n-fold Tate twist of $\mathbb{Q}_l$ (see [1]), and M^{Γ} respectively M_{Γ}-are the invariants respectively coinvariants under Γ of any Γ-module M.

Finally we introduce more notation. For an abelian group A, let Tor A be the torsion subgroup and $A_{\mathrm{Tor}} := A/\mathrm{Tor}\, A$, let Div A be the maximal divisible subgroup and $A_{\mathrm{Div}} := A/\mathrm{Div}\, A$. For a homomorphism $f : A \to B$ between abelian groups let Tor f and f_{Tor} denote the induced homomorphisms Tor $A \to$ Tor B and $A_{\mathrm{Tor}} \to B_{\mathrm{Tor}}$; f is called a quasi-isomorphism, if it has finite kernel and cokernel, in which case we define $q(f) := \#\mathrm{coker}\, f / \#\ker f$.

1. The cohomological formula

We fix in the following the integer $n \in \mathbb{Z}$. As Γ has cohomological dimension 1, the Hochschild-Serre spectral sequence associated with the covering $\bar{X}/X$ degenerates and we get the short exact sequences

$$\vdots$$

$$0 \to H^i(\bar{X}, \mathbb{Z}_l(n))_{\Gamma} \to H^{i+1}(X, \mathbb{Z}_l(n)) \to H^{i+1}(\bar{X}, \mathbb{Z}_l(n))^{\Gamma} \to 0$$

(1)

$$0 \to H^{i-1}(\bar{X}, \mathbb{Z}_l(n))_{\Gamma} \to H^i(X, \mathbb{Z}_l(n)) \to H^i(\bar{X}, \mathbb{Z}_l(n))^{\Gamma} \to 0$$

$$\vdots$$

for every $i \geq 0$. In addition, we have $H^i(\bar{X}, \mathbb{Z}_l(n)) = 0$ for $i > 2d$ and $H^i(X, \mathbb{Z}_l(n)) = 0$ for $i > 2d + 1$ (see [6] VI.1). Now we use the well-known (see [1] (3.2) for a proof) result about the value of $L_i(q^{-n}) = \det(1 - \varphi^{-1}; H^i(\bar{X}, \mathbb{Q}_l(n)))$.

Lemma 1: *The following three assertions are equivalent: (i) $L_i(q^{-n}) \neq 0$; (ii) $H^i(\bar{X}, \mathbb{Z}_l(n))^{\Gamma}$ is finite; (iii) $H^i(\bar{X}, \mathbb{Z}_l(n))_{\Gamma}$ is finite. If these three conditions are valid, we have*

$$|L_i(q^{-n})|_l = \frac{\#H^i(\bar{X}, \mathbb{Z}_l(n))^{\Gamma}}{\#H^i(\bar{X}, \mathbb{Z}_l(n))_{\Gamma}}.$$

The equivalent conditions in this lemma are in fact fulfilled in the cases i odd, or $i \neq 2n$ even, or $i = 2n$ with n not a pole of $\zeta(X, s)$. Using (1) this gives the formulae

$$(2) \qquad |L_i(q^{-n})|_l = \begin{cases} \dfrac{\#H^i(\bar{X}, \mathbb{Z}_l(n))^{\Gamma}}{\#H^i(\bar{X}, \mathbb{Z}_l(n))_{\Gamma}} \text{ for } i \text{ odd}, \\ \dfrac{\#H^{i+1}(\bar{X}, \mathbb{Z}_l(n))^{\Gamma}}{\#H^{i-1}(\bar{X}, \mathbb{Z}_l(n))_{\Gamma}} \cdot \dfrac{\#H^i(X, \mathbb{Z}_l(n))}{\#H^{i+1}(X, \mathbb{Z}_l(n))} \\ \text{for } i \neq 2n \text{ even or } i = 2n, \rho(n) = 0. \end{cases}$$

In the case $\rho(n) = 0$ multiplying them all together we immediately obtain the result of Bayer/Neukirch in [1]:

Proposition 2: *For $\rho(n) = 0$, we have*

$$|\zeta(X, n)|_l = |c(n)|_l = \prod_i \#H^i(X, \mathbb{Z}_l(n))^{(-1)^{i+1}}.$$

From now on we assume $0 \leq n \leq d$. We have to investigate the two exact sequences from (1) in which infinite groups occur, namely

$$\begin{tikzcd}
0 \arrow[r] & H^{2n}(\bar{X}, \mathbb{Z}_l(n))_\Gamma \arrow[r, "\beta"] & H^{2n+1}(X, \mathbb{Z}_l(n)) \arrow[r] & H^{2n+1}(\bar{X}, \mathbb{Z}_l(n))^\Gamma \arrow[r] & 0 \\
0 \arrow[r] & H^{2n-1}(\bar{X}, \mathbb{Z}_l(n))_\Gamma \arrow[r] & H^{2n}(X, \mathbb{Z}_l(n)) \arrow[r, "\alpha"] & H^{2n}(\bar{X}, \mathbb{Z}_l(n))^\Gamma \arrow[r] \arrow[ull, dashed, "f"'] & 0.
\end{tikzcd} \tag{3}$$

The groups in the left lower and right upper corner are finite; therefore α and β are quasi-isomorphisms. The map f is induced by the identity on $H^{2n}(\bar{X}, \mathbb{Z}_l(n))$. Furthermore define $L(T) \in \mathbb{Q}[T]$ by

$$L_{2n}(T) = (T - q^{-n})^{-\rho(n)} \cdot L(T).$$

In particular, this means that $L(q^{-n}) \neq 0$ and

$$c(n) = \left(\prod_{i \neq 2n} L_i(q^{-n})^{(-1)^{i+1}}\right) \cdot L(q^{-n})^{-1}. \tag{4}$$

Lemma 3: $-\rho(n) = \dim H^{2n}(\bar{X}, \mathbb{Q}_l(n))^\Gamma =$ *rank* $H^{2n}(X, \mathbb{Z}_l(n))$ *if and only if f is a quasi-isomorphism, in which case*

$$|L(q^{-n})|_l^{-1} = q(f).$$

Proof: Clearly $-\rho(n) = \dim H^{2n}(\bar{X}, \mathbb{Q}_l(n))^\Gamma$ is equivalent to $|L(q^{-n})|_l = |\det(1-\varphi^{-1};\ H^{2n}(\bar{X}, \mathbb{Q}_l(n))/H^{2n}(\bar{X}, \mathbb{Q}_l(n))^\Gamma)|_l = |\det(1-\varphi^{-1};$ $(\varphi - 1)H^{2n}(\bar{X}, \mathbb{Q}_l(n)))|_l$. When this is true we have

$$\begin{aligned}
|L(q^{-n})|_l &= \frac{\#[(\varphi-1)H^{2n}(\bar{X}, \mathbb{Z}_l(n))]^\Gamma}{\#[(\varphi-1)H^{2n}(\bar{X}, \mathbb{Z}_l(n))]_\Gamma} \\
&= \frac{\#[(\varphi-1)H^{2n}(\bar{X}, \mathbb{Z}_l(n))]^\Gamma}{[(\varphi-1)H^{2n}(\bar{X}, \mathbb{Z}_l(n)) : (\varphi-1)^2 H^{2n}(\bar{X}, \mathbb{Z}_l(n))]} \\
&= \frac{\#\ker f}{\#\operatorname{coker} f} = q(f)^{-1}.
\end{aligned}$$

The equality $\dim H^{2n}(\bar{X}, \mathbb{Q}_l(n))^\Gamma = \operatorname{rank} H^{2n}(\bar{X}, \mathbb{Z}_l(n))^\Gamma = \operatorname{rank}$ $H^{2n}(X, \mathbb{Z}_l(n))$ is immediately seen from (3). Q.E.D.

Combining these results, we obtain the following lemma.

Lemma 4: *If* $-\rho(n) = \operatorname{rank} H^{2n}(X, \mathbb{Z}_l(n))$, *then*

$$|c(n)|_l = q((\beta f \alpha)_{\mathrm{Tor}}) \cdot \prod_i \#\mathrm{Tor}\, H^i(X, \mathbb{Z}_l(n))^{(-1)^{i+1}}.$$

Proof: From (3) and lemma 3, we conclude that

$$\begin{aligned}
|L(q^{-n})|_l^{-1} = q(f) &= \frac{q(\mathrm{Tor}(\beta f))}{q(\beta)} \cdot q((\beta f)_{\mathrm{Tor}}) \\
&= \frac{1}{\#H^{2n+1}(\bar{X}, \mathbb{Z}_l(n))^\Gamma} \cdot \frac{\#\mathrm{Tor}\, H^{2n+1}(X, \mathbb{Z}_l(n))}{\#\mathrm{Tor}\, H^{2n}(\bar{X}, \mathbb{Z}_l(n))^\Gamma} \cdot q((\beta f)_{\mathrm{Tor}}) \\
&= \frac{\#H^{2n-1}(\bar{X}, \mathbb{Z}_l(n))_\Gamma}{\#H^{2n+1}(\bar{X}, \mathbb{Z}_l(n))^\Gamma} \cdot \frac{\#\mathrm{Tor}\, H^{2n+1}(X, \mathbb{Z}_l(n))}{\#\mathrm{Tor}\, H^{2n}(X, \mathbb{Z}_l(n))} \cdot q((\beta f \alpha)_{\mathrm{Tor}}).
\end{aligned}$$

Inserting this and the formulae (2) into (4) gives the required statement.

Thus it remains to interpret the index $\#\operatorname{coker}(\beta f \alpha)_{\mathrm{Tor}} = q((\beta f \alpha)_{\mathrm{Tor}})$. For this we consider the following commutative diagram of pairings induced by cup-product

$$\begin{tikzcd}
H^{2n+1}(X, \mathbb{Z}_l(n))_{\mathrm{Tor}} \times & H^{2(d-n)}(X, \mathbb{Z}_l(d-n))_{\mathrm{Tor}} \arrow[r] \arrow[d, "\cong"] & \mathbb{Z}_l \arrow[d, equal] \\
(H^{2n}(\bar{X}, \mathbb{Z}_l(n))_\Gamma)_{\mathrm{Tor}} \times \arrow[u, "\beta"] & (H^{2(d-n)}(\bar{X}, \mathbb{Z}_l(d-n))^\Gamma)_{\mathrm{Tor}} \arrow[r] \arrow[d, equal] & \mathbb{Z}_l \arrow[d, equal] \\
(H^{2n}(\bar{X}, \mathbb{Z}_l(n))^\Gamma)_{\mathrm{Tor}} \times \arrow[u, "f"] & (H^{2(d-n)}(\bar{X}, \mathbb{Z}_l(d-n))^\Gamma)_{\mathrm{Tor}} \arrow[r] & \mathbb{Z}_l \arrow[d, equal] \\
H^{2n}(X, \mathbb{Z}_l(n))_{\mathrm{Tor}} \times \arrow[u, "\alpha", "\cong"'] & H^{2(d-n)}(X, \mathbb{Z}_l(d-n))_{\mathrm{Tor}} \arrow[r] \arrow[u, "\cong"'] & \mathbb{Z}_l
\end{tikzcd} \tag{5}$$

(recall $H^{2d+1}(X, \mathbb{Z}_l(d)) = H^{2d}(\bar{X}, \mathbb{Z}_l(d)) = \mathbb{Z}_l$, [6] VI.11). By Poincaré duality (loc. cit.) the pairing in the second line is nondegenerate, and therefore the pairing in the first line is too. If f is a quasi-isomorphism, then all pairings in the diagram must be nondegenerate. Assuming this to be the case we denote by $\Delta_n^{(0)}$, respectively $\Delta_n^{(1)}$, the determinant of the pairing in the top, respectively, the bottom line (both determinants are defined up to a unit in $\mathbb{Z}_l$), and we have

$$q((\beta f \alpha)_{\mathrm{Tor}}) = |\Delta_n^{(0)}/\Delta_n^{(1)}|_l.$$

Now we can state our first main result.

Theorem 5: *If* $0 \leq n \leq d$ *and* $\rho(n) = -$ *rank* $H^{2n}(X, \mathbb{Z}_l(n))$, *then*

$$\begin{aligned}
\left| c(n) \cdot \frac{\Delta_n^{(1)}}{\Delta_n^{(0)}} \right|_l &= \prod_i \#\mathrm{Tor}\, H^i(X, \mathbb{Z}_l(n))^{(-1)^{i+1}} \\
&= \prod_i \#H^i(X, \mathbb{Q}_l/\mathbb{Z}_l(n))_{\mathrm{Div}}^{(-1)^i}.
\end{aligned}$$

Proof: The first equality is just the combination of lemma 4 with the considerations above. The second equality follows easily from the exact cohomology sequence associated with the exact sequence of sheaves

$$0 \to \mathbb{Z}_l(n) \to \mathbb{Q}_l(n) \to \mathbb{Q}_l/\mathbb{Z}_l(n) \to 0.$$

Corollary 6: *For $n = 0$, d we have $\rho(n) = -1$ and.*

$$|c(n)|_l = \prod_i \#\mathrm{Tor}\, H^i(X, \mathbb{Z}_l(n))^{(-1)^{i+1}}.$$

Proof: That $\rho(n) = -1$ is clear because $L_0(T) = 1 - T$ and $L_{2d}(T) = 1 - q^d T$. Furthermore $(\beta f \alpha)_{\mathrm{Tor}}$ is an isomorphism in each case because we have $H^1(\bar{X}, \mathbb{Z}_l)^\Gamma = (\mathrm{Tor}\, H^1(\bar{X}, \mathbb{Z}_l))^\Gamma = (H^0(\bar{X}, \mathbb{Q}_l/\mathbb{Z}_l)_{\mathrm{Div}})^\Gamma = 0$ and $H^{2d+1}(\bar{X}, \mathbb{Z}_l(d)) = 0$.

Corollary 7: *If $d \geq 1$ and $\rho(1) = -\mathrm{rank}\, H^2(X, \mathbb{Z}_l(1))$, then*

$$\left| c(1) \cdot \frac{\Delta_1^{(1)}}{\Delta_1^{(0)}} \right|_l = \prod_i \# H^i(X, G_m)(l)_{\mathrm{Div}}^{(-1)^i},$$

where $H^i(X, G_m)(l)$ denotes the l-primary component of the cohomology of the multiplicative group G_m.

Proof: One simply passes to the direct limit (with respect to v) in the exact cohomology sequence associated with the Kummer sequence $0 \to \mathbb{Z}/l^v\mathbb{Z}(1) \to G_m \xrightarrow{l^v} G_m \to 0$, and uses theorem 5.

2. The intersection pairing

Again we assume $0 \leq n \leq d$. By $Z^n(X)$, respectively, $N^n(X)$, we denote the group of n-codimensional algebraic cycles on X, respectively its factor group modulo numerical equivalence. Then $N^n(X)$ is a finitely generated free abelian group (this follows from [6] VI.11.7 because of the fact, that $N^n(X)$ injects into $N^n(\bar{X})$), and the intersection product defines a nondegenerate pairing

$$N^n(X) \times N^{d-n}(X) \to \mathbb{Z};$$

let Δ_n be the determinant (defined up to sign) of this pairing. We want to relate Δ_n to the determinants introduced in the first section of this paper.

According to SGA $4\frac{1}{2}$ [Cycle] there exist canonical cycle maps $Z^n(X) \to H^{2n}(X, \mathbb{Z}/l^v\mathbb{Z}(n))$ for $v \geq 1$. For the further discussion we assume the following statement to be true (in fact this is a well-known conjecture of Tate, see [11]).

Hypothesis: The cycle maps induce an isomorphism

$$(*) \qquad N^n(X) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_l \xrightarrow{\simeq} H^{2n}(X, \mathbb{Q}_l(n)).$$

Obviously we have then also a canonical injection $N^n(X) \otimes \mathbb{Z}_l \to H^{2n}(X, \mathbb{Z}_l(n))_{\mathrm{Tor}}$ with finite cokernel, the order of which we denote by $t_l(n)$. Now we can prove

Lemma 8: *Assuming (*) for n and $d-n$, one has*

i) $\rho(n) = -\mathrm{rank}\, H^{2n}(X, \mathbb{Z}_l(n)) = -\mathrm{rank}\, N^n(X)$,

ii) $|\Delta_n^{(1)}|_l = |\Delta_n / t_l(n) \cdot t_l(d-n)|_l$,

iii) $|\Delta_n^{(0)}|_l = 1$.

Proof: The diagram of pairings

$$\begin{tikzcd}
N^n(X) \otimes \mathbb{Z}_l \arrow[d] & \times & N^{d-n}(X) \otimes \mathbb{Z}_l \arrow[d] \arrow[r] & \mathbb{Z}_l \arrow[d, equal] \\
H^{2n}(X, \mathbb{Z}_l(n))_{\mathrm{Tor}} & \times & H^{2(d-n)}(X, \mathbb{Z}_l(d-n))_{\mathrm{Tor}} \arrow[r] & \mathbb{Z}_l
\end{tikzcd}$$

is commutative (see SGA $4\frac{1}{2}$ [Cycle] and [9]). As the intersection pairing is nondegenerate, the pairing in the bottom line is nondegenerate too (by our hypothesis). This implies, of course, the statement (ii). But going back to the diagram (5) we see at once, that f must be a quasi-isomorphism, which implies the statement (i) by lemma 3. For (iii), we consider first the following commutative exact diagram induced by the cycle maps

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N^{d-n}(X) \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & H^{2(d-n)}(X, \mathbb{Z}_l(d-n))_{\mathrm{Tor}} \arrow[d] & \\
 & N^{d-n}(X) \otimes \mathbb{Q}_l \arrow[r, "\simeq"] \arrow[d] & H^{2(d-n)}(X, \mathbb{Q}_l(d-n)) \arrow[d] & \\
 & N^{d-n}(X) \otimes \mathbb{Q}_l/\mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Div}\, H^{2(d-n)}(X, \mathbb{Q}_l/\mathbb{Z}_l(d-n)) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

where the map in the middle is an isomorphism by our hypothesis. So the cokernel and kernel of the upper respectively lower map are finite of equal order. This means that in the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(H^{2(d-n)}(X, \mathbb{Z}_l(d-n)), \mathbb{Z}_l) \arrow[r] & \mathrm{Hom}(N^{d-n}(X)\otimes\mathbb{Z}_l, \mathbb{Z}_l) \arrow[dd, equal] \\
H^{2n+1}(X, \mathbb{Z}_l(n))_{\mathrm{Tor}} \arrow[u] \arrow[d, "\simeq"] & \\
\mathrm{Hom}(\mathrm{Div}\, H^{2(d-n)}(X, \mathbb{Q}_l/\mathbb{Z}_l(d-n)), \mathbb{Q}_l/\mathbb{Z}_l) \arrow[r] & \mathrm{Hom}(N^{d-n}(X)\otimes\mathbb{Q}_l/\mathbb{Z}_l, \mathbb{Q}_l/\mathbb{Z}_l)
\end{tikzcd}$$

the horizontal maps (again induced by the cycle maps) are injective with finite cokernels of equal order. Now the left vertical maps are given by cup-product, the Poincaré duality says that the lower one is an isomorphism and the upper one injective with finite cokernel of order $|\Delta_n^{(0)}|_l^{-1}$ (compare (5)). These facts together give immediately $|\Delta_n^{(0)}|_l = 1$, as required.

Theorem 5 and lemma 8 together imply the second main result.

Theorem 9: *If $0 \leq n \leq d$ and (*) is true for both n and $d-n$, then*

$$\left|c(n)\cdot\frac{\Delta_n}{t_l(n)\cdot t_l(d-n)}\right|_l = \prod_i \#\mathrm{Tor}\, H^i(X, \mathbb{Z}_l(n))^{(-1)^{i+1}}.$$

Remark: It is easy to see, that (*) is true for $n = 0, d$ and that $t_l(0) = 1$ and $|\Delta_0|_l = |\Delta_d|_l = |t_l(d)|_l = |\min\{m \in \mathbb{N}: X(\mathbb{F}_{q^m}) \neq \emptyset\}|_l$.

Lemma 10: *(*) for $n = 1 \leq d$ implies that $t_l(1) = 1$.*

Proof: According to the proof of lemma 8 it is enough to show the injectivity of $N^1(X)\otimes\mathbb{Q}_l/\mathbb{Z}_l \to H^2(X, \mathbb{Q}_l/\mathbb{Z}_l(1))$. But we have the commutative exact diagram (see SGA $4\frac{1}{2}$ [Cycle] 2.1)

$$\begin{tikzcd}
0 \arrow[r] & H^1(X, G_m)\otimes\mathbb{Q}_l/\mathbb{Z}_l \arrow[r] \arrow[d] & H^2(X, \mathbb{Q}_l/\mathbb{Z}_l(1)) \\
 & N^1(X)\otimes\mathbb{Q}_l/\mathbb{Z}_l \arrow[ur] \arrow[d] & \\
 & 0 &
\end{tikzcd}$$

where the injective horizontal map is induced by the connecting homomorphism in the exact cohomology sequence associated with the Kummer sequence Q.E.D.

3. The special case of an abelian variety

In this last section we assume that X is an abelian variety A over $\mathbb{F}_q$ of dimension $d > 0$ and $n \in \mathbb{Z}$ is arbitrary.

Lemma 11: *(i) $H^i(\bar{A}, \mathbb{Z}_l(n))$ is torsionfree for $i \geq 0$; (ii) the action of the Frobenius φ on $H^i(\bar{A}, \mathbb{Q}_l(n))$ is semisimple for $i \geq 0$.*

Proof: We give only short indications of proofs, because these assertions are well-known.
(i) We can $\bar{A}$ lift to characteristic 0 ([7]). Therefore, by the comparison theorem of étale cohomology, it is enough to show the torsion-freeness of the integral cohomology of abelian varieties over the complex numbers. But this is clear (see [8] §1).
(ii) For $i = 1$, see [8] p. 253. The general case then follows from the fact that

$$H^i(\bar{A}, \mathbb{Q}_l) = \bigwedge_i H^1(\bar{A}, \mathbb{Q}_l)$$

is the i-th exterior power of $H^1(\bar{A}, \mathbb{Q}_l)$ ([4] 2A8).

Lemma 12: $|\Delta_n^{(0)}|_l = 1$ *for* $0 \leq n \leq d$.

Proof: Lemma 11 (i) implies (by arguing modulo l) that the Poincaré duality

$$H^{2n}(\bar{A}, \mathbb{Z}_l(n)) \times H^{2(d-n)}(\bar{A}, \mathbb{Z}_l(d-n)) \to \mathbb{Z}_l$$

is dualizing, i.e. its determinant is a unit in $\mathbb{Z}_l$. Going back to the diagram (5) we see that the pairing in the second line is dualizing too. But again by lemma 11 (i) the map β_{Tor} is an isomorphism. This means that $|\Delta_n^{(0)}|_l = 1$.

Plainly lemma 11 (ii) gives $\rho(n) = -\dim H^{2n}(\bar{A}, \mathbb{Q}_l(n))^{\Gamma} = \mathrm{rank}$ $H^{2n}(A, \mathbb{Z}_l(n))$. Therefore simplifying the computations in Section 1 by making use of lemma 11 and 12 we get our last theorem.

Theorem 13: *(i)* $|L_i(q^{-n})|_l^{-1} = \#\mathrm{Tor}\, H^{i+1}(A, \mathbb{Z}_l(n))$ *unless* $i = 2n$, $0 \leq n \leq d$, *and* $\rho(n) \neq 0$; *(ii) for* $0 \leq n \leq d$ *we have*

$$\left|\left[\frac{L_{2n}(q^{-s})}{(q^{-s}-q^{-n})^{-\rho(n)}}\right]_{s=n}\right|_l^{-1} = \#\mathrm{Tor}\, H^{2n+1}(A, \mathbb{Z}_l(n)) \cdot |\Delta_n^{(1)}|_l^{-1}.$$

Corollary 14: The l-primary component $Br(A)(l)$ of the Brauer group of A is finite; furthermore we have $\rho(1) = -\mathrm{rank}\, N^1(A)$, and

$$\left|\left[\frac{L_2(q^{-s})}{(q^{-s}-q^{-1})^{-\rho(1)}}\right]_{s=1}\right|_l^{-1} = \#Br(A)(l) \cdot |\Delta_1^{(1)}|_l^{-1}.$$

Proof: The hypothesis (*) is fulfilled for A and $n = 1$ according to Tate [13]. From this it follows, of course, that

$$\mathrm{rank}\, H^2(A, \mathbb{Z}_l(1)) = \mathrm{rank}\, N^1(A),$$

and also the finiteness of the cokernel of the map $N^1(A) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^2(A, \mathbb{Q}_l/\mathbb{Z}_l(1))$. Now the proof of lemma 10 shows that this cokernel is equal to $H^2(A, G_m)(l)$. By [3] (2.6) and (3.3), we have $Br(A)(l) = H^2(A, G_m)(l)$ which means the finiteness of the first group. On the other hand $H^2(A, G_m)(l) = H^2(A, \mathbb{Q}_l/\mathbb{Z}_l(1))_{\mathrm{Div}} = \mathrm{Tor}\, H^3(A, \mathbb{Z}_l(1))$ is straightforward (see the proof of corollary 6). Thus we get

$$\mathrm{Tor}\, H^3(A, \mathbb{Z}_l(1)) = Br(A)(l).$$

Inserting this into theorem 13 (ii) gives the statement.

Finally we remark: The duality theory for abelian varieties allows to prove that hypothesis (*) for $n = d - 1$ implies $t_l(d-1) = 1$, that is $|\Delta_1^{(1)}|_l = |\Delta_1|_l$. Namely with the help of the correspondence (respectively its powers) defined by the Poincaré divisor of A one reduces to the consideration of the dual abelian variety at $n = 1$ and applies lemma 10.

Added in proof

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