Smooth representations and Hecke modules in characteristic p

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Abstract

Let G be a p-adic Lie group and $I \subseteq G$ be a compact open subgroup which is a torsionfree pro-p-group. Working over a coefficient field k of characteristic p we introduce a differential graded Hecke algebra for the pair (G, I) and show that the derived category of smooth representations of G in k-vector spaces is naturally equivalent to the derived category of differential graded modules over this Hecke DGA.

Let G be a d-dimensional p-adic Lie group, and let k be any field. We denote by $\mathrm{Mod}_k(G)$ the category of smooth G-representations in k-vector spaces. It has enough injective objects (compare [Vig] I.5.9).

We fix a compact open subgroup $I \subseteq G$. In $\mathrm{Mod}_k(G)$ we then have the representation

$$\mathrm{ind}_I^G(1) := \text{all } k\text{-valued functions with finite support on } G/I$$

with G acting by left translations. The Hecke algebra of I by definition is the endomorphism ring

$$\mathcal{H}_I := \mathrm{End}_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1))^{opp} .$$

We let $\mathrm{Mod}(\mathcal{H}_I)$ denote the category of left unital $\mathcal{H}_I$-modules. There is the pair of adjoint functors

$$\begin{aligned} H^0 : \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Mod}(\mathcal{H}_I) \\ V &\longmapsto V^I \ , \end{aligned}$$

where $V^I = \mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), V)$ is the subspace of I-fixed vectors, and

$$\begin{aligned} T_0 : \mathrm{Mod}(\mathcal{H}_I) &\longrightarrow \mathrm{Mod}_k(G) \\ M &\longmapsto \mathrm{ind}_I^G(1) \otimes_{\mathcal{H}_I} M \ . \end{aligned}$$

If the characteristic of k does not divide the pro-order of I then the functor H^0 is exact. Moreover, it is well known that in this case H^0 induces a bijective correspondence between the irreducible smooth G-representations which have a nonzero I-invariant vector and the simple $\mathcal{H}_I$-modules. Since the structure of the algebra $\mathcal{H}_I$ some times is explicitly known this provides a very useful tool for the classification of irreducible smooth G-representations. On the other hand of course, since the prime number p always will divide the pro-order of I the functor H^0 certainly is not exact if k has characteristic p. It is unknown whether one still has this bijective correspondence between irreducible objects.

In these notes we assume from now on that the field k has characteristic p and that I is a torsionfree pro-p-group. It is obviously a pressing question to better understand the relation between the two categories $\mathrm{Mod}_k(G)$ and $\mathrm{Mod}(\mathcal{H}_I)$. Since the G-representation $\mathrm{ind}_I^G(1)$ is far from being cohomologically trivial both functors H^0 and T_0 now have a rather complicated behaviour. Since they are not exact it seems advantageous to work on the level of derived categories. We will show that by enlarging $\mathcal{H}_I$ to a certain differential graded algebra $\mathcal{H}_I^\bullet$ the derived category $D_{\mathcal{H}_I^\bullet}$ of differential graded $\mathcal{H}_I^\bullet$-modules is naturally equivalent to the (unbounded) derived category $D(\mathrm{Mod}_k(G))$. I thank W. Soergel for a very inspiring discussion about injective resolutions and DGA's.

We fix once and for all an injective resolution $\mathrm{ind}_I^G(1) \xrightarrow{\sim} \mathcal{I}^\bullet$ and introduce the differential graded algebra

$$\mathcal{H}_I^\bullet := \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{opp}$$

over k. We recall that

$$\mathcal{H}_I^n = \prod_{q\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathcal{I}^q, \mathcal{I}^{q+n})$$

with differential

$$(da)_q(x) = d(a_q(x)) - (-1)^n a_{q+1}(dx)$$

for $a = (a_q) \in \mathcal{H}_I^n$ and multiplication

$$(ba)_q := (-1)^{mn} a_{q+m} \circ b_q$$

for $a = (a_q) \in \mathcal{H}_I^n$ and $b = (b_q) \in \mathcal{H}_I^m$. The cohomology of $\mathcal{H}_I^\bullet$ is given by

$$h^*(\mathcal{H}_I^\bullet) = \mathrm{Ext}^*_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), \mathrm{ind}_I^G(1))$$

(compare [Har] I§6). In particular

$$h^0(\mathcal{H}_I^\bullet) = \mathcal{H}_I \ .$$

Correspondingly we have the derived category $D_{\mathcal{H}_I^\bullet}$ of differential graded left $\mathcal{H}_I^\bullet$-modules. We will construct a pair of adjoint functors

$$H : D(\mathrm{Mod}_k(G)) \longrightarrow D_{\mathcal{H}_I^\bullet}$$

and

$$T : D_{\mathcal{H}_I^\bullet} \longrightarrow D(\mathrm{Mod}_k(G)) \ .$$

According to [Laz] V.2.2.8 and [Ser] the group I has cohomological dimension d. This means that the higher derived functors of the left exact functor

$$\begin{aligned} \mathrm{Mod}_k(I) &\longrightarrow \mathrm{Vec}_k \\ E &\longmapsto E^I \end{aligned}$$

into the category Vec_k of k-vector spaces are zero in degrees $> d$. On the other hand the restriction functor

$$\begin{aligned} \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Mod}_k(I) \\ V &\longmapsto V|I \end{aligned}$$

is exact and respects injective objects. The latter is a consequence of the fact that compact induction

$$\begin{aligned} \mathrm{Mod}_k(I) &\longrightarrow \mathrm{Mod}_k(G) \\ E &\longmapsto \mathrm{ind}_I^G(E) \end{aligned}$$

is an exact left adjoint (compare [Vig] I.5.7). Hence the higher derived functors of the composed functor

$$\begin{aligned} H^0(I,.) : \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Vec}_k \\ V &\longmapsto V^I \end{aligned}$$

are given by $V \longmapsto H^i(I, V|I)$ and vanish in degrees $> d$. It follows that the total right derived functor

$$RH^0(I,.) : D(\mathrm{Mod}_k(G)) \longrightarrow D(\mathrm{Vec}_k)$$

between the corresponding (unbounded) derived categories exists ([Har] I.5.3). This discussion also shows the following.

Remark 1. *$h^*(\mathcal{H}_I^\bullet) = H^*(I, ind_I^G(1))$ and, in particular, $h^i(\mathcal{H}_I^\bullet) = 0$ for $i > d$.*

To compute $RH^0(I,.)$ we use the formalism of K-injective complexes as developed in [Spa]. We let $C(\mathrm{Mod}_k(G))$ and $K(\mathrm{Mod}_k(G))$ denote the category of unbounded complexes in $\mathrm{Mod}_k(G)$ with chain maps and homotopy classes of chain maps, respectively, as morphisms. The K-injective complexes form a full triangulated subcategory $K_{inj}(\mathrm{Mod}_k(G))$ of $K(\mathrm{Mod}_k(G))$. Exactly in the same way as [Spa] Prop. 3.11 one can show that any complex in $C(\mathrm{Mod}_k(G))$ has a right K-injective resolution (we emphasize that the category $\mathrm{Mod}_k(G)$ has inverse limits). Alternatively one may apply [Se] Thm. 3.13 based upon the following fact.

Lemma 2. *$Mod_k(G)$ is a Grothendieck category.*

Proof. The existence of arbitrary inductive limits and the exactness of filtered inductive limits in $\mathrm{Mod}_k(G)$ is clear. It remains to exhibit a generator. We define

$$X := \bigoplus_J \mathrm{ind}_J^G(1)$$

where J runs over all open subgroups in G. For any V in $\mathrm{Mod}_k(G)$ we have

$$\mathrm{Hom}_{\mathrm{Mod}_k(G)}(X, V) = \prod_J V^J \ .$$

Since $V = \bigcup_J V^J$ we easily deduce that X is a generator of $\mathrm{Mod}_k(G)$. □

The existence of K-injective resolutions means that the natural functor

$$K_{inj}(\mathrm{Mod}_k(G)) \xrightarrow{\simeq} D(\mathrm{Mod}_k(G))$$

is an equivalence of triangulated categories. We fix a quasi-inverse $\mathbf{i}$ of this functor. Then the derived functor $RH^0(I,.)$ is naturally isomorphic to the composed functor

$$D(\mathrm{Mod}_k(G)) \xrightarrow{\mathbf{i}} K_{inj}(\mathrm{Mod}_k(G)) \longrightarrow K(\mathrm{Vec}_k) \longrightarrow D(\mathrm{Vec}_k)$$

with the middle arrow given by

$$V^\bullet \mapsto \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), V^\bullet)\ .$$

For any K-injective complex $V^\bullet$ in $\mathrm{Mod}_k(G)$ the natural chain map

$$\mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet, V^\bullet) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), V^\bullet)$$

is a quasi-isomorphism. But the left hand term in a natural way is a differential graded left $\mathcal{H}_I^\bullet$-module. In fact we have the functor

$$\begin{aligned} K_{inj}(\mathrm{Mod}_k(G)) &\longrightarrow K_{\mathcal{H}_I^\bullet} \\ V^\bullet &\longmapsto \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet, V^\bullet) \end{aligned}$$

into the homotopy category $K_{\mathcal{H}_I^\bullet}$ of differential graded left $\mathcal{H}_I^\bullet$-modules which allows us to define the composed functor

$$H : D(\mathrm{Mod}_k(G)) \xrightarrow{\mathbf{i}} K_{inj}(\mathrm{Mod}_k(G)) \longrightarrow K_{\mathcal{H}_I^\bullet} \longrightarrow D_{\mathcal{H}_I^\bullet}\ .$$

The diagram

$$\begin{tikzcd}
D(\mathrm{Mod}_k(G)) \arrow[r, "H"] \arrow[dr, "RH^0(I,.)"'] & D_{\mathcal{H}_I^\bullet} \arrow[d, "\text{forget}"] \\
 & D(\mathrm{Vec}_k)
\end{tikzcd}$$

then is commutative up to natural isomorphism.

Explanation: Let $V^\bullet$ be a complex in $C(\mathrm{Mod}_k(G))$. To compute $RH^0(I,.)$ according to [Har] one chooses a quasi-isomorphism $V^\bullet \xrightarrow{\sim} C^\bullet$ into a complex consisting of objects which are acyclic for the functor $H^0(I,.)$. On the other hand let $V^\bullet \xrightarrow{\sim} A^\bullet$ be a quasi-isomorphism into a K-injective complex. By [Spa] Prop. 1.5.(c) we then have, in $K(\mathrm{Mod}_k(G))$, a unique commutative diagram:

$$\begin{tikzcd}
 & C^\bullet \arrow[dd] \\
V^\bullet \arrow[ur] \arrow[dr] & \\
 & A^\bullet
\end{tikzcd}$$

We claim that the induced map

$$(C^\bullet)^I \xrightarrow{\sim} (A^\bullet)^I$$

is a quasi-isomorphism. Choose quasi-isomorphisms

$$A^\bullet \xrightarrow{\sim} \tilde{C}^\bullet \xrightarrow{\sim} \tilde{A}^\bullet$$

where $\tilde{C}^\bullet$ consists of $H^0(I,.)$-acyclic objects and $\tilde{A}^\bullet$ is K-injective. By [Spa] Prop. 1.5.(b) the composite is an isomorphism in $K(\mathrm{Mod}_k(G))$ and hence induces a quasi-isomorphism $(A^\bullet)^I \xrightarrow{\sim} (\tilde{A}^\bullet)^I$. But by [Har] Thm. I 5.1 and Cor. I.5.3.(γ) the composite $C^\bullet \xrightarrow{\sim} A^\bullet \xrightarrow{\sim} \tilde{C}^\bullet$ also induces a quasi-isomorphism $(C^\bullet)^I \xrightarrow{\sim} (\tilde{C}^\bullet)^I$.

We recall that the kernel $\ker(F)$ of an exact functor $F : \mathfrak{A} \longrightarrow \mathfrak{B}$ between triangulated categories, i.e., the full subcategory of all objects which are mapped to a zero object by F, is a full triangulated subcategory of $\mathfrak{A}$ ([Ver] II.2.1.16).

Proposition 3. $\ker(H) = \ker(RH^0(I, .)) = 0$.

Proof. The completed group ring $\Omega := \Omega(I)$ of I over k is a pseudocompact local ring. If $\mathfrak{m} \subseteq \Omega$ denotes the maximal ideal then $\Omega/\mathfrak{m} = k$. Since Ω is noetherian ([Laz] V.2.2.4) its pseudocompact topology coincides with the $\mathfrak{m}$-adic topology. This implies that:

- $\Omega/\mathfrak{m}^j$ lies in $\mathrm{Mod}_k(I)$ for any $j \in \mathbb{N}$.
- For any E in $\mathrm{Mod}_k(I)$ we have

$$E = \bigcup_{j\in\mathbb{N}} E^{\mathfrak{m}^j=0} \qquad \text{where} \quad E^{\mathfrak{m}^j=0} := \{v \in E : \mathfrak{m}^j v = 0\}.$$

Because of

$$E^{\mathfrak{m}^j=0} = \mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, E)$$

we need to consider the left exact functors $\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, .)$ on $\mathrm{Mod}_k(I)$. Their right derived functors of course are $\mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, .)$. In particular

$$\mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}, .) = H^i(I, .)\ .$$

For any $j \in \mathbb{N}$ we have the short exact sequence

$$0 \longrightarrow \mathfrak{m}^j/\mathfrak{m}^{j+1} \longrightarrow \Omega/\mathfrak{m}^{j+1} \longrightarrow \Omega/\mathfrak{m}^j \longrightarrow 0$$

in $\mathrm{Mod}_k(I)$. Moreover, $\mathfrak{m}^j/\mathfrak{m}^{j+1} \cong k^{n(j)}$ for some $n(j) \geq 0$ since Ω is noetherian. The associated long exact Ext-sequence therefore reads

$$\ldots \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, .) \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^{j+1}, .) \longrightarrow H^i(I, .)^{n(j)} \longrightarrow \ldots$$

By induction with respect to j we deduce that:

- Each functor $\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, .)$ has cohomological dimension $\leq d$.
- Each $H^0(I, .)$-acyclic object in $\mathrm{Mod}_k(I)$ is $\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, .)$-acyclic for any $j \geq 1$.

It follows that the total right derived functors $R\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, .)$ on $D(\mathrm{Mod}_k(I))$ exist. More explicitly, let $E^\bullet$ be any complex in $D(\mathrm{Mod}_k(I))$ and choose a quasi-isomorphism $E^\bullet \xrightarrow{\sim} C^\bullet$ into a complex consisting of $H^0(I, .)$-acyclic objects. It then follows that we have the short exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, C^\bullet) \longrightarrow \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^{j+1}, C^\bullet) \longrightarrow \big((C^\bullet)^I\big)^{n(j)} \longrightarrow 0\ .$$

Suppose now that $RH^0(I, E^\bullet) = 0$. This means that the complex $(C^\bullet)^I$ is exact. By induction with respect to j we obtain the exactness of the complex

$$\mathrm{Hom}^\bullet_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, C^\bullet) = (C^\bullet)^{\mathfrak{m}^j=0}$$

for any $j \in \mathbb{N}$. Hence $C^\bullet$ and $E^\bullet$ are exact. □

For the functor T in the opposite direction we first note that $\mathcal{I}^\bullet$ naturally is a differential graded right $\mathcal{H}_I^\bullet$-module so that we can form the graded tensor product $\mathcal{I}^\bullet \otimes_{\mathcal{H}_I^\bullet} M^\bullet$ with any differential graded left $\mathcal{H}_I^\bullet$-module $M^\bullet$. This tensor product naturally is a complex in $C(\mathrm{Mod}_k(G))$. We now define T to be the composite

$$T : D_{\mathcal{H}_I^\bullet} \xrightarrow{\mathbf{p}} K_{pro,\mathcal{H}_I^\bullet} \xrightarrow{\mathcal{I}^\bullet \otimes_{\mathcal{H}_I^\bullet}} K(\mathrm{Mod}_k(G)) \longrightarrow D(\mathrm{Mod}_k(G)) \ .$$

Here $K_{pro,\mathcal{H}_I^\bullet}$ denotes the full triangulated subcategory of $K_{\mathcal{H}_I^\bullet}$ consisting of K-projective modules and $\mathbf{p}$ is a quasi-inverse of the equivalence of triangulated categories $K_{pro,\mathcal{H}_I^\bullet} \xrightarrow{\simeq} D_{\mathcal{H}_I^\bullet}$ (compare [BL] 10.12.2.9).

The usual standard computation shows that T is left adjoint to H.

Proposition 4. *The functor T is fully faithful.*

Proof. As explained in [Kel] p.5 and 2.5 this is, by infinite devissage, a formal consequence of the following two facts:

a. The adjunction map $\mathcal{H}_I^\bullet \xrightarrow{\cong} HT(\mathcal{H}_I^\bullet)$ is an isomorphism in $D_{\mathcal{H}_I^\bullet}$.

b. The functor H commutes with infinite direct sums.

Ad a.: As $\mathcal{I}^\bullet$ is K-injective and $\mathcal{H}_I^\bullet$ is K-projective $T(\mathcal{H}_I^\bullet)$ is given by the complex $\mathcal{I}^\bullet \otimes_{\mathcal{H}_I^\bullet} \mathcal{H}_I^\bullet = \mathcal{I}^\bullet$ and consequently $HT(\mathcal{H}_I^\bullet)$ is given by $\mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet) = \mathcal{H}_I^\bullet$.

Ad b.: It suffices to show commutation with infinite direct sums on cohomology, i.e., for the cohomology functors $H^i(I, E^\bullet) = \mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(k, E^\bullet)$ on $D(\mathrm{Mod}_k(I))$.

Let $\Omega = \Omega(I)$ again denote the completed group ring of I over k. Then $\mathrm{Mod}_k(I)$ is a Serre subcategory of the category $\mathrm{Mod}(\Omega)$ of (left) Ω-modules which is closed under infinite direct sums.

We recall that the functor $\mathrm{Hom}_{\mathrm{Mod}_k(I)}(k, .) = H^0(I, .)$ on $\mathrm{Mod}_k(I)$ is of finite cohomological dimension. The same is true for the functor $\mathrm{Hom}_{\mathrm{Mod}(\Omega)}(k, .)$ on $\mathrm{Mod}(\Omega)$ since the ring Ω is noetherian of finite global dimension ([Neu]). Hence in both cases the total derived functor can be computed by resolutions which are acyclic for the respective functor. But according to [Laz] V.1.2.6 and V.3.2.7 any $H^0(I, .)$-acyclic object in $\mathrm{Mod}_k(I)$ also is acyclic for $\mathrm{Hom}_{\mathrm{Mod}(\Omega)}(k, .)$ when viewed as an object in $\mathrm{Mod}(\Omega)$. It follows that

$$\mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(k, E^\bullet) \cong \mathrm{Ext}^i_{\mathrm{Mod}(\Omega)}(k, E^\bullet) \quad \text{for any } E^\bullet \text{ in } D(\mathrm{Mod}_k(I)).$$

This reduces us to showing that the functors $\mathrm{Ext}^i_{\mathrm{Mod}(\Omega)}(k, .)$ on $D(\mathrm{Mod}(\Omega))$ commute with infinite direct sums. Using again that Ω has finite global dimension we know from [Ver] III.3.1.4 that

$$\mathrm{Ext}^i_{\mathrm{Mod}(\Omega)}(k, E^\bullet) \cong \mathrm{Hom}_{D(\mathrm{Mod}(\Omega))}(k, E^\bullet[i]) \quad \text{for any } E^\bullet \text{ in } D(\mathrm{Mod}_k(I)).$$

What we need therefore is that the functor $\mathrm{Hom}_{D(\mathrm{Mod}(\Omega))}(k, .)$ commutes with infinite direct sums. This is equivalent (compare [Kel] Lemma on p.5) to k, viewed as an object of $D(\mathrm{Mod}(\Omega))$, being a perfect complex. Surely this follows from (once more) Ω having finite global dimension.

(In the references [Laz] and [Neu] the field k in fact is assumed to be the finite field $\mathbb{F}_p$. But the respective arguments generalize to arbitrary k. For everything except the fact that $\Omega(I)$ is noetherian one may alternatively use [Bru] Remark (1) on top of p. 452, Thm. 4.1, and Lemma 4.2(i).) $\square$

Theorem 5. *The functor H is an equivalence between triangulated categories*

$$D(Mod_k(G)) \xrightarrow{\sim} D_{\mathcal{H}_I^\bullet} .$$

Proof. We will show that T is a quasi-inverse for H. Let

$$\mathrm{ad}_{M^\bullet} : M^\bullet \longrightarrow H \circ T(M^\bullet) \qquad \text{for } M^\bullet \text{ in } D_{\mathcal{H}_I^\bullet}$$

and

$$\mathrm{ad}_{V^\bullet} : T \circ H(V^\bullet) \longrightarrow V^\bullet \qquad \text{for } V^\bullet \text{ in } D(\mathrm{Mod}_k(G))$$

denote the two adjunctions. Since T, by Prop. 4, is fully faithful the first adjunction

$$\mathrm{id}_{D_{\mathcal{H}_I^\bullet}} \xrightarrow{\sim} H \circ T$$

in fact is a natural isomorphism. Together with the identity

$$H(\mathrm{ad}_{V^\bullet}) \circ \mathrm{ad}_{H(V^\bullet)} = \mathrm{id}_{H(V^\bullet)}$$

this implies that $H(\mathrm{ad}_{V^\bullet})$ is an isomorphism for any object $V^\bullet$ in $D(\mathrm{Mod}_k(G))$. Using [Ver] II.2.1.16 it then follows from Prop. 3 that already $\mathrm{ad}_{V^\bullet}$ must be an isomorphism. □

A first step in the investigation of the DGA $\mathcal{H}_I^\bullet$ might be the computation of its cohomology algebra $h^*(\mathcal{H}_I^\bullet)$. By Remark 1 the latter is concentrated in degrees 0 to d. Of course the usual Hecke algebra $\mathcal{H}_I = h^0(\mathcal{H}_I^\bullet)$ is a subalgebra of $h^*(\mathcal{H}_I^\bullet)$. We finish this paper by determining the top cohomology $h^d(\mathcal{H}_I^\bullet)$ as a right $\mathcal{H}_I$-module.

Using the I-equivariant linear map

$$\begin{aligned} \pi_I : \mathrm{ind}_I^G(1) &\longrightarrow \mathrm{ind}_I^G(1)^I = \mathcal{H}_I \\ \phi &\longmapsto \Big[h \mapsto \sum_{g \in I/I\cap hIh^{-1}} \phi(gh)\Big] \end{aligned}$$

we obtain the map

$$\pi_I^* : h^*(\mathcal{H}_I^\bullet) = H^*(I, \mathrm{ind}_I^G(1)) \xrightarrow{H^*(I,\pi_I)} H^*(I, \mathcal{H}_I) = H^*(I, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{H}_I .$$

The last equality in this chain comes from the universal coefficient theorem which is applicable since I as a Poincaré group ([Laz] V.2.5.8) has finite cohomology $H^*(I, \mathbb{F}_p)$. Of course, as a ring $\mathcal{H}_I$ is a right module over itself. For our purposes we have to consider a modification of this module structure which is specific to characteristic p.

As a k-vector space $\mathrm{ind}_I^G(1)^I = \mathcal{H}_I$ has the basis $\{\chi_{IxI}\}_{x \in I\backslash G/I}$ consisting of the characteristic functions of the double cosets IxI. If we denote the multiplication in the algebra $\mathcal{H}_I$, as usual, by the symbol "$*$" for convolution then in this basis it is given by the formula

$$\chi_{IxI} * \chi_{IhI} = \sum_{y \in I\backslash G/I} c_{x,y;h} \chi_{IyI}$$

where the coefficients are

$$c_{x,y;h} = (\chi_{IxI} * \chi_{IhI})(y) = \sum_{y \in G/I} \chi_{IxI}(g)\chi_{IhI}(g^{-1}y) = |IxI \cap yIh^{-1}I/I| \cdot 1_k$$

with 1_k denoting the unit element in the field k. Of course, for fixed x and h we have $c_{x,y;h} = 0$ for all but finitely many $y \in I\backslash G/I$. But $IxI \cap yIh^{-1}I \neq \emptyset$ implies $IxI \subseteq IyIh^{-1}I$; by compactness the latter is a finite union of double cosets. Hence also for fixed y and h we have $c_{x,y;h} \neq 0$ for at most finitely many $x \in I\backslash G/I$. It follows that by combining the transpose of these coefficient matrices with the anti-automorphism

$$\begin{aligned} \mathcal{H}_I &\longrightarrow \mathcal{H}_I \\ \chi &\longmapsto \chi^*(g) := \chi(g^{-1}) \end{aligned}$$

we obtain through the formula

$$\chi_{IxI} *_\tau \chi_{IhI} := \sum_{y \in I\backslash G/I} c_{y,x;h^{-1}} \chi_{IyI}$$

a new right action of $\mathcal{H}_I$ on itself. We denote this new module by $\mathcal{H}_I^\tau$.

Comment: We compute

$$\begin{aligned} |IyI/I| \cdot c_{x,y;h} &= |IyI/I| \cdot (\chi_{IxI} * \chi_{IhI})(y) \\ &= \sum_{z \in G/I} \chi_{IyI}(z) \big(\chi_{IxI} * \chi^*_{Ih^{-1}I}\big)(z) \\ &= \big(\chi_{IyI} * (\chi_{IxI} * \chi^*_{Ih^{-1}I})^*\big)(1) \\ &= \big((\chi_{IyI} * \chi_{Ih^{-1}I}) * \chi^*_{IxI}\big)(1) \\ &= \sum_{z \in G/I} (\chi_{IyI} * \chi_{Ih^{-1}I})(z) \chi_{IxI}(z) \\ &= |IxI/I| \cdot (\chi_{IyI} * \chi_{Ih^{-1}I})(x) \\ &= |IxI/I| \cdot c_{y,x;h^{-1}} \ . \end{aligned}$$

This, of course, is valid with integral coefficients (instead of k). Moreover $|IxI/I|$ always is a power of p. It follows that over any field of characteristic different from p one has $\mathcal{H}_I^\tau \cong \mathcal{H}_I$. It also follows that $c_{x,y;h} = c_{y,x;h^{-1}}$ whenever both are nonzero.

It is straightforward to check that

$$\pi_I(\phi) *_\tau \chi_{IhI} = \pi_I(\phi * \chi_{IhI})$$

holds true for any $\phi \in \mathrm{ind}_I^G(1)$ and any $h \in G$. Hence

$$\pi_I : \mathrm{ind}_I^G(1) \longrightarrow \mathcal{H}_I^\tau \qquad \text{and} \qquad \pi_I^* : h^*(\mathcal{H}_I^\bullet) \longrightarrow H^*(I, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{H}_I^\tau$$

are maps of right $\mathcal{H}_I$-modules.

Proposition 6. *The map π_I^d is an isomorphism*

$$h^d(\mathcal{H}_I^\bullet) \xrightarrow{\cong} H^d(I, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{H}_I^\tau$$

of right $\mathcal{H}_I$-modules. By fixing a basis of the one dimensional $\mathbb{F}_p$-vector space $H^d(I, \mathbb{F}_p)$ we therefore obtain $h^d(\mathcal{H}_I^\bullet) \cong \mathcal{H}_I^\tau$ as right $\mathcal{H}_I$-modules.

Proof. It remains to show that π_I^d is bijective. We have the I-equivariant decomposition

$$\mathrm{ind}_I^G(1) = \bigoplus_{x \in I\backslash G/I} \mathrm{ind}_{I\cap xIx^{-1}}^I(1) \ .$$

The map π_I restricts to

$$\begin{aligned} \pi_I : \mathrm{ind}_{I\cap xIx^{-1}}^I(1) &\longrightarrow k \cdot \chi_{IxI} \subseteq \mathcal{H}_I \\ \phi &\longmapsto \Big(\sum_{y \in I/I\cap xIx^{-1}} \phi(y)\Big) \cdot \chi_{IxI} \ . \end{aligned}$$

Since $H^*(I,.)$ commutes with arbitrary direct sums it therefore suffices to show that the map

$$H^d(I, \phi \mapsto \sum_{y\in I/I\cap xIx^{-1}} \phi(y)) : H^d(I, \mathrm{ind}_{I\cap xIx^{-1}}^I(1_{\mathbb{F}_p})) \longrightarrow H^d(I,\mathbb{F}_p)$$

is bijective. Using Shapiro's lemma this latter map identifies (cf. [S-CG] Chap. I §2.5) with the corestriction map

$$\mathrm{Cor} : H^d(I \cap xIx^{-1}, \mathbb{F}_p) \longrightarrow H^d(I, \mathbb{F}_p)$$

which for Poincaré groups of dimension d is an isomorphism of one dimensional vector spaces ([S-CG] I-50(4)). ☐

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