V. F. R. JONES: THE TYPE OF CROSSED PRODUCT VON NEUMANN ALGEBRAS

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Abstract. This is an exposition of a section in the lecture notes by Jones. We discuss the type of von Neumann algebras obtained from the group-measure space construction.

Throughout the notes we consider only σ-finite measure spaces. Moreover Γ will always be a discrete group. For most purposes it is convenient to assume Γ to be countable, but we will indicate the steps where this is needed.

1. A type III-action

Let us recall the following definitions.

Definition 1.1. *Let Γ be a discrete group acting on the measure space (X, μ). The action is called*

a) (essentially) transitive if there exists $x \in X$ such that $\mu(\Gamma \cdot x) = \mu(X)$.

b) (essentially) free if for every $e \neq \gamma \in \Gamma$ we have

$$\mu(\{x \in X | \gamma \cdot x = x\}) = 0$$

c) ergodic if for every measurable subset $A \subset X$ satisfying

$$\mu(A\Delta(\gamma \cdot A)) = 0$$

for all $\gamma \in \Gamma$ we have either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

If Γ acts ergodically on X and $Y \subset X$ is a Γ-invariant measurable subset then we have either $\mu(Y) = 0$ or $\mu(X \setminus Y) = 0$. Although not needed in the sequel, let us verify that ergodicity is in fact equivalent to this apparently weaker condition if the group is countable.

Lemma 1.2. *If Γ is a countable group acting on the measure space (X, μ) then the action is ergodic iff for every Γ-invariant measurable subset $Y \subset X$ we have either $\mu(Y) = 0$ or $\mu(X \setminus Y) = 0$.*

Proof. Assume first that the action is ergodic and let $A \subset$ be measurable with $\mu(A\Delta(\gamma \cdot A)) = 0$ for all $\gamma \in \Gamma$. Let

$$B = \{x \in A | \gamma \cdot x \in A \text{ for all } \gamma \in \Gamma\} = \{x \in X | \forall \gamma \in \Gamma\ \exists x_\gamma \in A : \gamma \cdot x_\gamma = x\}.$$

Then

$$\mu(B) = \mu\Big(\bigcap_{\gamma \in \Gamma} \gamma \cdot A\Big) = \mu\Big(A \setminus \bigcup_{\gamma \in \Gamma} A\Delta(\gamma \cdot A)\Big) = \mu(A)$$

since $\bigcup_{\gamma \in \Gamma} A\Delta(\gamma \cdot A)$ is a μ-null set. Here we use that Γ is countable. Since B is Γ-invariant we obtain $\mu(A) = 0$ or $\mu(X \setminus A) = 0$ as claimed.

Conversely let $Y \subset X$ be Γ-invariant. Then $Y\Delta\gamma \cdot Y = \emptyset$ for all $\gamma \in \Gamma$. Hence the condition implies $\mu(Y) = 0$ or $\mu(X \setminus Y) = 0$. □

Lemma 1.3. *If the discrete group Γ acts ergodically on the measure space (X, μ) preserving the σ-finite measure μ then any other Γ-invariant measure ν on X which is absolutely continuous to μ is of the form $\nu = \lambda\mu$ for some $\lambda > 0$.*

Proof. Since ν is assumed to be absolutely continuous to μ we can consider the Radon-Nikodym derivative $f = d\nu/d\mu$. Recall that $f : X \to [0,\infty)$ is a measurable function such that

$$\nu(A) = \int_A f d\mu$$

for all measurable sets $A \subset X$. Since both μ and ν are Γ-invariant we have

$$\int_A (\gamma \cdot f) d\mu = \int_{\gamma^{-1}\cdot A} f d\mu = \nu(\gamma^{-1} \cdot A) = \nu(A) = \int_A f d\mu$$

for all A and every $\gamma \in \Gamma$. By the uniqueness assertion of the Radon-Nikodym theorem we conclude that $\gamma \cdot f = f$ almost everywhere for all $\gamma \in \Gamma$.
Since f is takes values in $[0,\infty)$ we find $c > 0$ such that

$$A = \{x \in X | f(x) \leq c\}$$

has measure $\mu(A) > 0$. By our above considerations $\mu((\gamma \cdot A)\Delta A) = 0$ for all $\gamma \in \Gamma$. Since the action is ergodic and $\mu(A) > 0$ we conclude that $\mu(X \setminus A) = 0$. This means that f is essentially bounded by c. In particular, $f \in L^\infty(X,\mu)^\Gamma = \mathbb{C}$. Hence $f = \lambda$ is a constant function, and this yields the claim. □

Let now $\Gamma = \mathbb{Q} \rtimes \mathbb{Q}^*$ be the $ax + b$-group. That is, $\Gamma = \mathbb{Q} \times \mathbb{Q}^*$ as a set with multiplication

$$(b_1, a_1) \cdot (b_2, a_2) = (b_1 + a_1 b_2, a_1 a_2).$$

Let λ denote the Lebesgue measure on $\mathbb{R}$ and consider the action of Γ on $(\mathbb{R}, \lambda)$ given by

$$(b, a) \cdot x = ax + b.$$

We collect some properties of this action in the following lemma.

Lemma 1.4. *The natural action of the $ax+b$-group $\Gamma = \mathbb{Q} \rtimes \mathbb{Q}^*$ on $(\mathbb{R}, \lambda)$ defined above is free and ergodic, and there is no Γ-invariant measure on $\mathbb{R}$ equivalent to the Lebesgue measure λ.*

Proof. We show that the additive subgroup $\mathbb{Q} \subset \Gamma$ acts ergodically on $(\mathbb{R}, \lambda)$. Assume that $f \in L^\infty(\mathbb{R})$ is invariant under translations by $\mathbb{Q}$. Then f satisfies in particular $f(x) = f(x + 1)$ almost everywhere, and it suffices to show that the corresponding function on $\mathbb{T}$, again denoted by f, is constant. Applying Fourier decomposition to $f \in L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$ we can write

$$f = \sum_{n\in\mathbb{Z}} f_n z^n$$

for some l^2-sequence f_n. Now $r \in \mathbb{Q}/\mathbb{Z}$ acts by

$$r \cdot f = \sum_{n\in\mathbb{Z}} f_n e^{2\pi i n r} z^n$$

We conclude $f_n e^{2\pi i n r} = f_n$ for all $r \in \mathbb{Q}$ and hence $f_n = 0$ for $n \neq 0$. This means $f = f_e$ is a constant function.
For freeness of the action observe that $(a, b) \cdot x = ax + b = x$ means $(a - 1)x = b$. If $a = 1$ we obtain $b = 0$ and hence $(a, b) = e$. For $a \neq 1$ we see that x is uniquely determined. In particular, for $(a, b) \neq e$ the set $\{x \in X | (a, b) \cdot x = x\}$ contains only one element and has therefore measure zero.
Assume that ν is a Γ-invariant measure on $\mathbb{R}$. Then ν is in particular $\mathbb{Q}$-invariant. We have seen above that $\mathbb{Q}$ acts ergodically on $(\mathbb{R}, \lambda)$. According to lemma 1.3 this means that ν is a scalar multiple of Lebesgue measure λ. However, the Lebesgue measure is not Γ-invariant since the multiplicative subgroup $\mathbb{Q}^* \subset \Gamma$ does not preserve λ. □

2. Conditional expectations

Let Γ be a discrete group acting on the von Neumann algebra M. Then the projection $p_e : \mathcal{H} \otimes l^2(\Gamma) \to \mathcal{H} \otimes l^2(\Gamma)$ onto the closed subspace $\mathcal{H} \otimes \mathbb{C}e$ induces an ultraweakly continuous linear map $E : M \rtimes \Gamma \to M$ given by $E(x) = p_e x p_e$. Explicitly we find

$$E\left(\sum_{\gamma\in\Gamma} x_\gamma \gamma\right) = x_e,$$

and hence E takes indeed values in M. The map E is called the conditional expectation from $M \rtimes \Gamma$ onto M.

Lemma 2.1. *Let Γ be a discrete group acting on a von Neumann algebra M. The conditional expectation $E : M \rtimes \Gamma \to M$ has the following properties.*

a) E is unital and faithful, that is $E(1) = 1$ and $E(x^*x) = 0$ implies $x = 0$.
b) E is a projection of norm one in the Banach space sense, that is, $E^2 = E$ and E has norm one as Banach space operator.
c) E is an M-bimodule map, that is $E(axb) = aE(x)b$ for all $x \in M \rtimes \Gamma$ and $a, b \in M \subset M \rtimes \Gamma$.

Proof. *a*) Clearly we have $E(1) = 1$. Assume that $x \in M \rtimes \Gamma$ satisfies $E(x^*x) = 0$. We may write $x = \sum_{\gamma\in\Gamma} x_\gamma\gamma$ for some $x_\gamma \in M$ and find

$$E(x^*x) = \sum_{\gamma\in\Gamma} x_\gamma^* x_\gamma.$$

Hence $E(x^*x) = 0$ implies $x_\gamma = 0$ for all γ and hence $x = 0$.
b) The formula $E^2 = E$ is obvious. From $E(x) = p_e x p_e$ we see that E has norm $||E|| \leq 1$, and since $E(1) = 1$ it follows that $||E|| = 1$.
c) Since $p_e \in M' \subset \mathbb{L}(\mathcal{H} \otimes l^2(\Gamma))$ we find

$$E(axb) = p_e axb p_e = ap_e x p_e b = aE(x)b$$

for $x \in M \rtimes \Gamma$ and $a, b \in M$ as claimed. □

3. Semifinite crossed products

Theorem 3.1. *Let Γ be an infinite countable discrete group acting freely and ergodically on the σ-finite measure space (X, μ) preserving the measure μ.*

a) If μ is a finite measure then $L^\infty(X, \mu) \rtimes \Gamma$ is a type II_1-factor.
b) If μ is an infinite measure and Γ acts non-transitively then $L^\infty(X, \mu) \rtimes \Gamma$ is a type II_∞-factor.
c) If μ is an infinite measure and Γ acts transitively then $L^\infty(X, \mu) \rtimes \Gamma$ is a type I_∞-factor.

Proof. *a*) We prove a slightly more general statement. Assume that M is a finite factor with normalized trace tr and assume that Γ preserves tr. Let $E : M \rtimes \Gamma \to M$ be the conditional expectation and consider $Tr = \mathrm{tr} \circ E$. Then Tr is an ultraweakly continuous positive linear map. The computation

$$\begin{aligned} Tr(xu_\gamma y u_\eta) &= \delta_{\gamma,\eta^{-1}} Tr(x(\gamma \cdot y)) = \delta_{\gamma,\eta^{-1}} \mathrm{tr}(x(\gamma \cdot y)) = \delta_{\gamma,\eta^{-1}} \mathrm{tr}((\gamma \cdot y)x) \\ &= \delta_{\gamma,\eta^{-1}} \mathrm{tr}(y(\gamma^{-1} \cdot x)) = Tr(yu_\eta x u_\gamma) \end{aligned}$$

together with ultraweak continuity shows that Tr is in fact a normalized trace on $M \rtimes \Gamma$. Hence the factor $M \rtimes \Gamma$ is finite. We cannot obtain a finite type I-factor since Γ was assumed to be infinite. Hence $L^\infty(X, \mu) \rtimes \Gamma$ is of type II_1.
b) We have to assume here that (X, μ) is a standard measure space. If Γ acts non-transitively, there cannot be atoms in (X, μ). Otherwise $\Gamma \cdot x$ for $x \in X$ of positive measure would be a Γ-invariant set so $\mu(\Gamma \cdot x) = \mu(X)$ by ergodicity, contradicting

the assumption that Γ acts non-transitively. Let $A \subset X$ be a measurable subset with $0 < \mu(A) < \infty$ and $\xi = \chi_A \otimes \delta_e \in L^2(X,\mu) \otimes l^2(\Gamma)$. Then

$$\omega_\xi(fu_\gamma) = \langle \xi, fu_\gamma \xi \rangle = \delta_{\gamma,e} \int_A f(x) d\mu(x)$$

and for $p = \chi_A u_e \in L^\infty(X,\mu) \rtimes \Gamma$ we obtain

$$\begin{aligned} \omega_\xi((pfu_\gamma p)(pgu_\eta p)) &= \omega_\xi(\chi_A f(\gamma \cdot \chi_A)(\gamma \cdot g) u_{\gamma\eta}) \\ &= \delta_{\gamma,\eta^{-1}} \int_{A \cap \gamma \cdot A} f(\gamma \cdot g) d\mu \\ &= \delta_{\gamma,\eta^{-1}} \int_{\gamma^{-1} A \cap A} (\gamma^{-1} \cdot f) g d\mu \\ &= \delta_{\gamma,\eta^{-1}} \int_{A \cap \eta \cdot A} g(\eta \cdot f) d\mu \\ &= \omega_\xi((pgu_\eta p)(pfu_\gamma p)) \end{aligned}$$

using the Γ-invariance of μ. It follows that ω_ξ is a trace on $p(L^\infty(X,\mu) \rtimes \Gamma)p$, and hence $p(L^\infty(X,\mu) \rtimes \Gamma)p$ is a finite factor. Since (X,μ) is a standard measure space then A, having no atoms, contains subsets of arbitrary measure smaller than $\mu(A)$. Hence the factor $p(L^\infty(X,\mu) \rtimes \Gamma)p$ cannot be of type I since it contains $L^\infty(A,\mu)$. If $L^\infty(X,\mu) \rtimes \Gamma$ itself were finite with finite trace tr then $\nu(Y) = \mathrm{tr}(\chi_Y)$ would give a finite Γ-invariant measure on X absolutely continuous to μ. According to lemma 1.3 this means $\nu = \lambda\mu$ for some $\lambda > 0$ and hence $\nu(X) = \infty$, a contradiction. Hence $L^\infty(X,\mu) \rtimes \Gamma$ is of type II_∞.
c) We may assume that $X = \Gamma$. Since Γ is countable it follows that μ is a multiple of the counting measure. The crossed product $L^\infty(\Gamma,\mu) \rtimes \Gamma$ is unitarily equivalent to $(L^\infty(\Gamma,\mu)\mathcal{L}(\Gamma))'' \subset \mathbb{L}(l^2(\Gamma)$. Direct computation shows that the latter contains all matrix units $e_{\gamma\eta}$ for $\gamma, \eta \in \Gamma$ and hence $L^\infty(\Gamma,\mu) \rtimes \Gamma \cong \mathbb{L}(l^2(\Gamma))$. ☐

4. Type III-crossed products

We need some prelimiaries on lower semicontinuous functions.

Definition 4.1. *Let X be a topological space. A function $f : X \to [-\infty,\infty] = \mathbb{R} \cup \{-\infty,\infty\}$ is called lower semicontinuous if*

$$f^{-1}((K,\infty]) = \{x \in X | f(x) > K\}$$

is an open set for every $K \in \mathbb{R}$.

Let X be a topological space and let $f : X \to [-\infty,\infty]$ be a function. Clearly f is lower semicontinous iff the set $f^{-1}([-\infty,K])$ is closed for every $K \in \mathbb{R}$. If $x \in X$ we say that f is lower semicontinuous at x if either $f(x) = -\infty$ or $f(x) = \infty$ and for every $K > 0$ we find an open neighborhood U of x such that $f(u) > K$ for all $u \in U$, or $f(x) \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists an open neighborhood U of x such that

$$f(u) > f(x) - \epsilon$$

for all $u \in U$.

Lemma 4.2. *Let X be a topological space. For a function $f : X \to [-\infty,\infty] = \mathbb{R} \cup \{-\infty,\infty\}$ the following conditions are equivalent.*

a) f is lower semicontinuous.
b) f is lower semicontinuous at every $x \in X$.

Proof. *a)* $\Rightarrow$ *b)* Assume that $-\infty < f(x) < \infty$ and let $\epsilon > 0$. By lower semicontinuity, the set $U = f^{-1}((f(x) - \epsilon, \infty])$ is an open neighborhood of x.
b) $\Rightarrow$ *a)* Let $K \in \mathbb{R}$ and consider $x \in f^{-1}((K, \infty])$. If $f(x) = \infty$ we find an open neighborhood U of x such that $f(u) > K$ for all $u \in U$, hence $U \subset f^{-1}((K, \infty])$. If $f(x) < \infty$ we choose $\epsilon > 0$ such that $f(x) - \epsilon > K$. Then there is an open neighborhhood U of x such that $f(u) > f(x) - \epsilon > K$ so that $U \subset f^{-1}((K, \infty])$ as well. Hence $f^{-1}((K, \infty])$ is open which means that f is lower semicontinuous. □
We collect some basic facts on lower semicontinuous functions.

Lemma 4.3. *Let X be a compact space and let $f : X \to [-\infty, \infty]$ be a lower semicontinuous function. Then f attains its minimum on X.*

Proof. *b)* If $-\infty$ is in the image of f of $f(x) = \infty$ for all $x \in X$ there is nothing to prove. Hence we may assume that $f(X) \subset (-\infty, \infty]$ and $f(x_0) < \infty$ for some $x_0 \in X$. The set $K = \{x \in X | f(x) \leq f(x_0)\}$ is closed by lower semicontinuity, and it clearly suffices to show that the restriction of f to K attains its minimum. In other words, we may restrict to the case that $f : X \to \mathbb{R}$ takes values in $\mathbb{R}$. Fix $\epsilon > 0$ and let U_x for $x \in X$ be an open set such that $f(u) > f(x) - \epsilon$ for all $u \in U_x$. Then $(U_x)_{x \in X}$ is an open cover of X, and since X is compact there exist $x_1, \dots, x_n$ such that $U_{x_1} \cup \cdots \cup U_{x_n} = X$. It follows that f is bounded below, and we denote by r the infimum of the set $f(X)$. The nonempty sets $A_n = f^{-1}([r, r + 1/n])$ are closed for all $n \in \mathbb{N}$. Using again that X is compact we find a point y in the intersection of all A_n. We conclude $f(y) = r$ and this yields the claim. □

Lemma 4.4. *If $(f_j)_{j \in J}$ is a family of lower semicontinuous functions from the topological space X to $[-\infty, \infty]$ and $\bigvee_{j \in J} f_j : X \to [-\infty, \infty]$ is defined by*

$$\bigvee_{j \in J} f_j(x) = \sup_{j \in J} f_j(x),$$

then $\bigvee_{j \in J} f_j$ is again lower semicontinuous.

Proof. Let $K \in \mathbb{R}$. Then

$$\left(\bigvee_{j \in J} f_j\right)^{-1}((K, \infty]) = \bigcup_{j \in J} f_j^{-1}((K, \infty])$$

is an open set by lower semicontinuity of the f_j. □

Lemma 4.5. *Let $\mathcal{H}$ be a Hilbert space and let $\xi \in \mathcal{H}$. Then $t_\xi(x) = ||x\xi||$ defines a lower semicontinuous function from $\mathbb{L}(\mathcal{H})$ with the weak topology to $\mathbb{R}$.*

Proof. For arbitrary $K \in \mathbb{R}$ we have to show that the set

$$U_K = \{x \in \mathbb{L}(\mathcal{H}) | ||x\xi|| > K\}$$

is weakly open in $\mathbb{L}(\mathcal{H})$. Clearly U_K is strongly open. Hence

$$C_K = \mathbb{L}(\mathcal{H}) \setminus U_K = \{x \in \mathbb{L}(\mathcal{H}) | ||x\xi|| \leq K\}$$

is strongly closed. Since C_K is convex this means that C_K is weakly closed by the Hahn-Banach theorem. Hence U_K is weakly open as desired. □

Lemma 4.6. *Let M be a semifinite factor with unit ball M_1 and let* $\operatorname{tr} : M_+ \to [0, \infty]$ *be a semifinite trace on M. Then for each $K > 0$ the set*

$$M(K) = \{x \in M_1 : \operatorname{tr}(x^* x) \leq K\}$$

is weakly compact.

Proof. Let us first consider the case that M is finite. We may assume without loss of generality that tr is the normalized trace - recall that the normalized trace on a finite factor is unique, see 7.1.19.
Using the GNS-construction for tr we can write

$$\operatorname{tr}(x) = \langle \Lambda(1), x\Lambda(1)\rangle$$

and hence $\operatorname{tr}(x^*x) = ||x\Lambda(1)||^2$ for all $x \in M$. According to lemma 4.5 we see that the map $t : M \to [0, \infty)$ given by $t(x) = \operatorname{tr}(x^*x)$ is weakly lower semicontinuous. Hence $t^{-1}([0, K]) = \{x \in M | \operatorname{tr}(x^*x) \leq K\} \subset M$ is weakly closed. By the Kaplansky density theorem the unit ball M_1 of M is weakly compact. We conclude that

$$M(K) = M_1 \cap t^{-1}([0, K])$$

is weakly compact. This yields the claim for finite M.
Now assume that M is a type I_∞-factor or a type II_∞-factor. Then we may write

$$M \cong N \otimes \mathbb{L}(l^2(\mathbb{N})) \subset L^2(M, \tau) \otimes \mathbb{L}(l^2(\mathbb{N}))$$

with $N = \mathbb{C}$ in the first case or N a type II_1-factor in the second case. In both cases $\tau : N \to \mathbb{C}$ denotes the normalized trace. Then, up to a scalar,

$$\operatorname{tr}(x) = \sum_{j=1}^{\infty} \langle (\Lambda(1) \otimes e_j), x(\Lambda(1) \otimes e_j)\rangle$$

for $x \in M_+$.
We want to show that the function $t : M \to [0, \infty]$ given by

$$t(x) = \operatorname{tr}(x^*x) = \sum_{j=1}^{\infty} \langle x(\Lambda(1) \otimes e_j), x(\Lambda(1) \otimes e_j)\rangle = \sum_{j=1}^{n} ||x(\Lambda(1) \otimes e_j)||$$

is weakly lower semicontinuous. For this consider the function $t_n : M \to [0, \infty]$ given by

$$t_n(x) = \sum_{j=1}^{n} \langle x(\Lambda(1) \otimes e_j), x(\Lambda(1) \otimes e_j)\rangle = \sum_{j=1}^{n} ||x(\Lambda(1) \otimes e_j)||.$$

Obviously we have

$$\bigvee_{n \in \mathbb{N}} t_n = t,$$

and according to lemma 4.5 the maps t_n are weakly lower semicontinuous for all n. Hence due to lemma 4.4 the function t is indeed weakly lower semicontinuous. Now the same argument as in the finite case finishes the proof. □

Proposition 4.7. *Let M be a semifinite factor with unit ball M_1 and let* $\operatorname{tr} : M_+ \to [0, \infty]$ *be a semifinite trace on M. As above we write*

$$M(K) = \{x \in M_1 : \operatorname{tr}(x^*x) \leq K\}$$

for $K > 0$. Let $N \subset M$ be a von Neumann subalgebra. If $x \in M(K)$ let us denote by $W(x)$ the weak closure of all convex combinations of elements of the form uxu^* for $u \in N$ unitary. Then $W(x) \subset M(K)$ and if $t : W(x) \to [0, \infty]$ is the function

$$t(y) = \operatorname{tr}(y^*y)$$

then t attains its minimum at a unique point $e(x)$ of $W(x)$.

Proof. Note that a convex combination of elements $u_j x u_j^*$ with $u_j \in N$ unitary is a finite sum of the form

$$c = \sum_{j=1}^{n} \lambda_j u_j x u_j^*$$

where $\sum_{j=1}^{n} \lambda_j = 1$, $\lambda_i > 0$ for all i. It is clear that the norm of such a convex combination is bounded by 1 since $x \in M_1$. Moreover the trace of c is clearly bounded by K. Since $M(K)$ is weakly compact we see that $W(x)$ is a weakly compact convex subset of $M(K)$.
From the proof of lemma 4.6 we know that t is a weakly lower semicontinuous function. Hence according to lemma 4.3 there is a point $e(x) \in W(x)$ where t attain its minimum.
Next recall that the GNS-construction for tr is the Hilbert space completion $\mathcal{H}$ of the linear space

$$\mathcal{N} = \{z \in M : \mathrm{tr}(z^*z) < \infty\} \subset M$$

with respect to the inner product $\langle y, z\rangle = \mathrm{tr}(y^*z)$. The function t extends to the function $t : \mathcal{H} \to [0, \infty)$ given by $t(\xi) = ||\xi||^2$. Since $t(y) \geq t(e(x))$ for all $y \in W(x)$ and t is continuous for the norm topology of $\mathcal{H}$, we also have $t(\xi) \geq t(e(x))$ for all ξ in the norm closure $\overline{W(x)}$ of $W(x)$. Since $\overline{W(x)} \subset \mathcal{H}$ is a convex closed subset, the function $t : \overline{W(x)} \to [0, \infty)$ has a unique minimum by basic Hilbert space geometry. □

Proposition 4.8. *Suppose that* Γ *acts freely and ergodically on* $L^\infty(X, \mu)$ *such that* $M = L^\infty(X, \mu) \rtimes \Gamma$ *is a semifinite factor. Let tr be a semifinite trace on* M *and let* $p \in M$ *be a nonzero projection with* $\mathrm{tr}(p) < \infty$*. If* $E : L^\infty(X, \mu) \rtimes \Gamma \to L^\infty(X, \mu)$ *denotes the canonical conditional expectation then*

$$e(p) = E(p)$$

and

$$0 < \mathrm{tr}(e(p)^2) \leq \mathrm{tr}(p)$$

where $e(p) \in M$ *is defined as above.*

Proof. By the uniqueness of $e(p) \in M$ it follows that $e(p)$ commutes with every unitary in $L^\infty(X, \mu)$. Since $N = L^\infty(X, \mu)$ is maximal abelian in the crossed product, by 11.2.11 it follows that $e(p) \in L^\infty(X, \mu)$.
If $x = \sum_{j=1}^{n} \lambda_j u_j p u_j^* \in W(p)$ for $u_j \in N$ we clearly have

$$E(x) = \sum_{j=1}^{n} \lambda_j E(u_j p u_j^*) = \sum_{j=1}^{n} \lambda_j u_j E(p) u_j^* = E(p)$$

by the bimodule property of E and the fact that $L^\infty(X, \mu)$ is abelian. Since E is ultraweakly continuous we have in fact $E(x) = E(p)$ for all $x \in W(p)$. Moreover we have $e(p) \in W(p)$ and together with our observation $e(p) \in L^\infty(M, \mu)$ above we therefore obtain

$$e(p) = E(e(p)) = E(p).$$

Since $E(p) \leq p$ we conclude

$$\mathrm{tr}(e(p)^2) = \mathrm{tr}(e(p)^* e(p)) = t(e(p)) \leq t(p) = \mathrm{tr}(p).$$

Finally $E(p) = E(p^2)$ is a positive non-zero element of M and hence $e(p)^2 = E(p)^2$ must have non-zero trace. □

Theorem 4.9. *Let the countable discrete group* Γ *act freely and ergodically on the countably separated* σ*-finite measure space* (X, μ)*. If the factor* $L^\infty(X, \mu) \rtimes \Gamma$ *is semifinite there exists a* σ*-finite* Γ*-invariant measure on* X *which is absolutely continuous with respect to* μ*.*

Proof. Define a measure ν on X by $\nu(A) = \mathrm{tr}(\chi_A)$ for measurable subsets $A \subset X$. Then ν has to be finite and nonzero on some A. Indeed, choose a nonzero projection $p \in L^\infty(X, \mu) \rtimes \Gamma$ with $\mathrm{tr}(p) < \infty$. Then according to proposition 4.8 the function $E(p)^2 \in L^\infty(X, \mu)$ has finite positive measure with respect to ν. By ergodicity of the action we see that the complement of the Γ-invariant set $\bigcup_{\gamma\in\Gamma}\gamma\cdot A$ has measure zero so that ν is σ-finite. From the relation

$$\nu(\gamma Y) = \mathrm{tr}(\gamma\cdot\chi_Y) = \mathrm{tr}(u_\gamma\chi_Y u_\gamma^{-1}) = \mathrm{tr}(\chi_Y) = \nu(Y)$$

for measurable $Y\subset X$ we see that ν is Γ-invariant. □

As a consequence we obtain examples of factors which are not semifinite. Such factors are sometimes called purely infinite. Since being semifinite is the same things as being type I or II the following terminology is equivalently used.

Definition 4.10. *A factor is of type III if it is not of type I or II.*

According to theorem 4.9 we obtain a type III-factor from any example of a free ergodic group action on a countably separated, σ-finite measure space (X,μ) such that there is no invariant σ-finite invariant measure absolutely continuous with respect to μ. Hence lemma 1.4 gives the following result.

Corollary 4.11. *The crossed product $L^\infty(\mathbb{R},\lambda)\rtimes\Gamma$ for the natural action of the $ax+b$-group $\Gamma=\mathbb{Q}\rtimes\mathbb{Q}^*$ on $(\mathbb{R},\lambda)$ is a type III-factor.*

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