AN INCOMPLETE OVERVIEW ABOUT THE KONTSEVICH MODEL

Abstract. In 1992, Kontsevich proved that intersection numbers of line bundles over the moduli space of complex curves can be computed by structures familiar from quantum field theory. Today we know that these structures describe the ϕ^3-model on a family of noncommutative geometries. Conversely, one can prove results about these noncommutative quantum field theories which by far exceed what can be done usually. I try to sketch how the ϕ^3-interaction arises in the Kontsevich model.

1. The moduli space of curves

We let $\mathcal{M}_{g,s}$ be the moduli space of equivalence classes of complex curves (1-dimensional oriented complex manifolds) of genus g with s distinct marked points, modulo biholomorphic reparametrisation. If the Euler characteristic $\chi = 2-2g-s$ is negative, $\mathcal{M}_{g,s}$ is locally parametrised by $d_{g,s} = (3g-3+s)$ complex parameters called moduli. $\mathcal{M}_{g,s}$ is an orbifold, not a manifold.

- The simplest case $\mathcal{M}_{0,3}$ consists of the Riemann sphere $\bar{\mathbb{C}}$ with three marked points z_1, z_2, z_3. The automorphisms of $\bar{\mathbb{C}}$ are the Möbius transforms $z \mapsto \frac{az+b}{cz+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$. There is a unique Möbius transform which maps z_1, z_2, z_3 into $0, 1, \infty$. Therefore, $\mathcal{M}_{0,3}$ is a point.
- A genus-1 Riemann surface is a torus which can be conformally mapped to the parallelogram with vertices $0, 1, \tau, 1+\tau$, where $\mathrm{Im}(\tau) > 0$ and opposite sides identified. The resulting common vertex is the distinguished marked point. Two such parallelograms τ, τ' define the same surface if $\tau' = \frac{a\tau+b}{c\tau+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. This includes the generators $\tau \mapsto \tau+1$ and $\tau \mapsto -\frac{1}{\tau}$ and restricts the parametrisation domain to $|\tau| \geq 1$ and $-\frac{1}{2} < \mathrm{Re}(\tau) \leq \frac{1}{2}$. Moreover, on the boundary $\tau = e^{\mathrm{i}\theta}$, with $\frac{\pi}{3} \leq \theta < \frac{2\pi}{3}$, we identify $e^{\mathrm{i}\theta}$ and $e^{\mathrm{i}(\pi-\theta)}$ which restricts the angle to $[\frac{\pi}{3}, \frac{\pi}{2}]$. In summary,

$$\mathcal{M}_{1,1} = \{-\tfrac{1}{2} < \mathrm{Re}(\tau) \leq \tfrac{1}{2}\} \cap \{\mathrm{Im}(\tau) > 0\} \cap \{|\tau| > 1\} \cup \{\tau = e^{\mathrm{i}\theta}\ ,\ \tfrac{\pi}{3} \leq \theta \leq \tfrac{\pi}{2}\}\ .$$

The spaces $\mathcal{M}_{g,s}$ are typically not compact. They can be compactified to $\overline{\mathcal{M}}_{g,s}$ by adding degenerate surfaces. In case of the torus class $\mathcal{M}_{1,1}$ one includes the pinched torus, which corresponds to a sphere with three marked points which is glued along two of them to a pinched torus: $\overline{\mathcal{M}}_{1,1} = \mathcal{M}_{1,1} \cup \mathcal{M}_{0,3}$. There is a stability condition: All connected components of degenerate surfaces have negative Euler characteristics. These Deligne-Mumford compactifications carry an analytic (hence smooth) structure, but they are no longer of constant dimension; they are called a stack.

2. Chern classes and intersection numbers

On the smooth space $\overline{\mathcal{M}}_{g,s}$ we have a natural family $\{\mathcal{L}_1, \dots, \mathcal{L}_s\}$ of complex line bundles (i.e. rank-1 vector bundles) over $\overline{\mathcal{M}}_{g,s}$. Every point $x \in \overline{\mathcal{M}}_{g,s}$ is a stable complex curve $x = \mathcal{C}$ with the same labelled regular points $z_1, \dots, z_s \in \mathcal{C}$. Take as the fibre of $\mathcal{L}_i$ at $x = \mathcal{C}$ the cotangent space $T^*_{z_i}\mathcal{C}$, which is well-defined because every z_i is a smooth point even for degenerate curves in $\overline{\mathcal{M}}_{g,s} \setminus \mathcal{M}_{g,s}$. It is known that smooth complex line bundles are classified by their first Chern class $c_1(\mathcal{L}_i)$, an element of the second cohomology group $H^2(\overline{\mathcal{M}}_{g,s}, \mathbb{Q})$. In case of manifolds M, the Chern classes take values in $H^2(M, \mathbb{Z})$; the Chern classes then count the number of linearly independent sections of a vector bundle. For orbispaces the rational cohomology group arises. A representative of $c_1(\mathcal{L}_i)$ is the curvature form Ω_i of any connection on $\mathcal{L}_i$. The class $[\Omega_i]$ does not depend on the choice of the connection.

The (commutative) wedge product of $\dim(\overline{\mathcal{M}}_{g,s}) = 3g - 3 + s$ of these 2-forms $c_1(\mathcal{L}_i)$ is of top degree $2(3g-3+s)$, equal to the real dimension of $\overline{\mathcal{M}}_{g,s}$. Therefore, the following integral is well-defined:

$$\langle \tau_{d_1} \cdots \tau_{d_s} \rangle := \int_{\overline{\mathcal{M}}_{g,s}} \prod_{j=1}^{s} \big(c_1(\mathcal{L}_j)\big)^{d_j} \tag{1}$$

which is non-zero only if $d_1 + \cdots + d_s = 3g - 3 + s$. The simplest cases turn out to be $\langle \tau_0 \tau_0 \tau_0 \rangle = 1$ for $\overline{\mathcal{M}}_{0,3}$ and $\langle \tau_1 \rangle = \frac{1}{24}$ for $\overline{\mathcal{M}}_{1,1}$. Sometimes one includes $\langle 1 \rangle = -\frac{1}{12}$ from the Euler characteristics of $\overline{\mathcal{M}}_{1,0}$. These rational numbers are called *intersection numbers*. They are topological invariants of $\overline{\mathcal{M}}_{g,s}$.

3. Witten's conjecture

Since the order of marked points does not matter, we write $\langle \tau_0 \tau_0 \tau_0 \rangle \equiv \langle \tau_0^3 \rangle$ and a general intersection number as $\langle \tau_0^{k_0} \tau_1^{k_1} \cdots \rangle$. Consider the generating function

$$F(t_0, t_1, \dots) = \Big\langle \exp\Big(\sum_{i=0}^{\infty} t_i \tau_i \Big) \Big\rangle = \sum_{k_0, k_1, \dots = 0}^{\infty} \langle \tau_0^{k_0} \tau_1^{k_1} \dots \rangle \prod_{i=0}^{\infty} \frac{t_i^{k_i}}{k_i!} , \tag{2}$$

which is a formal power series in the variables t_i. Conversely, the intersection numbers are easily extracted from F:

$$\langle \tau_{i_1}^{n_1} \cdots \tau_{i_r}^{n_r} \rangle = \frac{\partial^{n_1}}{\partial t_{i_1}^{n_1}} \cdots \frac{\partial^{n_r}}{\partial t_{i_r}^{n_r}} F(\{t\}) \Big|_{t_i = 0} \tag{3}$$

The simplest cases $F(t_0, t_1, \dots) = \frac{t_0^3}{3!} + \frac{t_1}{24} + \dots$ and an analogy to the *hermitean one-matrix model* led Witten to the following conjecture:

Conjecture 1 ([Witten, 1991]). (1) *The generating function F obeys the string equation*

$$\frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i} \ . \tag{4}$$

(2) *The second derivative $U(\{t\}) := \frac{\partial^2}{\partial t_0^2} F(\{t\})$ satisfies the Korteweg-de Vries equations*

$$\frac{\partial U}{\partial t_n} = \frac{\partial}{\partial t_0} R_{n+1}(U, \partial_{t_0} U, \partial_{t_0}^2 U, \dots) \ , \tag{5}$$

where the R_n are polynomials in U and their t_0-derivatives which are recursively defined by $R_1(U) = U$ and

$$\frac{\partial}{\partial t_0} R_{n+1} = \frac{1}{2n+1} \Big(R_n \frac{\partial U}{\partial t_0} + 2U \frac{\partial R_n}{\partial t_0} + \frac{1}{4} \frac{\partial^3 R_n}{\partial t_0^3} \Big) \ .$$

These equations (if true) allow to recursively evaluate all intersection numbers. For instance, the first KdV-equation reads

$$\frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3} \tag{6}$$

and determines the part in U that only depends on t_0, t_1 to $U(t_0, t_1, 0, 0, \dots) = \frac{t_0}{1-t_1}$. In particular, $\langle \tau_0^3 \tau_1^k \rangle = k!$.

4. The Kontsevich model

[Kontsevich, 1992] gave a proof of the Witten's conjecture 1 by relating F to the partition function of a new type of matrix model, the Kontsevich model. His construction uses closed *ribbon graphs* (or fatgraphs). These are graphs whose edges are thickened to bands to make an oriented 2-manifold with boundary embedded in $\mathbb{R}^3$. The boundary is a disjoint union of circles which we call faces. We let $\mathcal{RG}_s^3$ be the set of closed ribbon graphs made of 3-valent vertices with s labelled faces. Such a closed ribbon graph has an even number v of vertices, $\frac{3}{2}v$ edges and can be drawn without intersection on a surface of genus g given by $v - \frac{3}{2}v + s = 2 - 2g$.

Theorem 2 ([Kontsevich, 1992]). *The intersection numbers of line bundles on $\overline{\mathcal{M}}_{g,s}$ are generated by*

$$\sum_{d_1,\dots,d_s=0}^{\infty} \langle \tau_{d_1} \cdots \tau_{d_s} \rangle \prod_{i=1}^{s} \frac{(2d_i-1)!!}{\lambda_i^{2d_i+1}} = \sum_{\Gamma \in \mathcal{RG}_s^3} \frac{2^{-|V(\Gamma)|}}{\#\mathrm{Aut}(\Gamma)} \prod_{e \in E(\Gamma)} \frac{2}{\lambda'(e) + \lambda''(e)} \ , \tag{7}$$

where $V(\Gamma), E(\Gamma)$ are the sets of vertices and edges of Γ and $\#\mathrm{Aut}(\Gamma)$ is the order of the automorphism group of Γ. The faces are labelled by formal variables $\lambda_1, \dots, \lambda_s$, and $\lambda'(e), \lambda''(e)$ are the labels of the two faces separated by the edge e.

For example, the intersection number $\langle\tau_0\tau_0\tau_0\rangle$ arises from graphs in $\Gamma \in \mathcal{RG}_3^3$ with $v = 2$ vertices,

$$\frac{\langle\tau_0\tau_0\tau_0\rangle}{\lambda_1\lambda_2\lambda_3} = \frac{1}{4}\Big(\frac{8}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)} + \frac{8}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_1+\lambda_1)} + \frac{8}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_2+\lambda_2)} + \frac{8}{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)(\lambda_3+\lambda_3)}\Big)\,,$$

which yields $\langle\tau_0\tau_0\tau_0\rangle = 1$. For $v = 2$ vertices and $s = 1$ faces there is a single graph with symmetry factor $\mathrm{Aut}(\Gamma) = 6$:

$$\frac{\langle\tau_1\rangle}{\lambda_1^3} = \frac{1}{4\cdot 6}\Big(\frac{2}{\lambda_1+\lambda_1}\Big)^3\,,$$

which yields $\langle\tau_1\rangle = \frac{1}{24}$.

Several proofs are known today. Kontsevich started from a theorem by Strebel from the 1960s. We follow [Eynard, 2016]:

Theorem 3 ([Strebel, 1984]). *On any Riemann surface $\mathcal{C} \in \mathcal{M}_{g,s}$ with marked points $z_1, \dots, z_s$ there is, for any given $L_1, \dots, L_s \in \mathbb{R}_+$ (called perimeters), a unique quadratic differential $\Omega(z) = f(z)dz \otimes dz$ such that*

- *f is meromorphic on $\mathcal{C}$*
- *$\Omega(z) \propto -\frac{L_j^2}{(z-z_i)}(1 + O(z - z_j))dz \otimes dz$ near z_j.*

Moreover, the horizontal trajectories of Ω, defined by $\mathrm{Im}(\int^z \sqrt{\Omega}) = const$, are either circles about the marked points or critical trajectories which form a ribbon graph with s faces drawn on $\mathcal{C}$. The jth face has perimeter L_j when measured with the metric $\frac{1}{2\pi}|\sqrt{\Omega}|$.

Near a double pole z_j one has $\int^z \sqrt{\Omega} \sim \mathrm{i}L_j \log(z - z_j)$ so that the horizontal trajectories are (round) circles with centre z_j. By continuity they remain topological circles away from z_j, until a critical trajectory which goes through the zeros of Ω. Near a simple zero z_v of Ω one has $\int^z \sqrt{\Omega} \sim (z - z_v)^{\frac{3}{2}}$ so that three horizontal trajectories meet at a vertex.

For the single surface in $\mathcal{M}_{0,3}$ with marked points at $0, 1, \infty$, the Strebel differential is

$$\Omega(z) = -\frac{L_\infty^2 z^2 - (L_\infty^2 + L_0^2 - L_1^2)z + L_0^2}{z^2(z-1)^2} dz \otimes dz\,. \tag{8}$$

One has $\Omega(z_v) = 0$ at

$$z_v = \tfrac{1}{2L_\infty^2}\big(L_\infty^2 + L_0^2 - L_1^2 \pm \sqrt{L_0^2 + L_1^4 + L_\infty^2 - 2L_0^2L_1^2 - 2L_0^2L_\infty^2 - 2L_1^2L_\infty^2}\big)\,.$$

Generically 3-valent vertices corresponding to simple zeros arise. Only the three cases $L_0 = L_1 + L_\infty$, $L_1 = L_0 + L_\infty$, $L_\infty = L_0 + L_1$ produce a 4-valent ribbon graph.

The metric $\frac{1}{2\pi}|\sqrt{\Omega}|$ allows to measure the length ℓ_e of every edge of the ribbon graph Γ. 3-valent graphs with v vertices have $\frac{3}{2}v$ edges and Euler characteristics $\chi = 2-2g = v-\frac{3}{2}v+s$, or $\frac{3}{2}v = 3(2g+s-2) = 2(3g-3+s)+s$. Hence, Strebel's theorem assigns to a curve with perimeters $(\mathcal{C}, L_1, \dots, L_s) \in \mathcal{M}_{g,s} \times (\mathbb{R}_+^\times)^s$ a unique ribbon graph Γ with edge lengths $\{\ell_e\} \in (\mathbb{R}_+)^{s+2(3g-3+s)}$. This assignment turns out to be an isomorphism of orbifolds,

$$\mathcal{M}_{g,s} \times (\mathbb{R}_+^\times)^s \sim \bigcup_{\text{ribbon graphs}} (\mathbb{R}_+)^{s+2(3g-3+s)}$$

by which $\mathcal{M}_{g,s} \times (\mathbb{R}_+^\times)^s$ is stratified into cells labelled by genus-g 3-valent ribbon graphs of s faces. [*At some point still obscure to me the order of the automorphism group arises*]

In particular, the top degree differential forms must be proportional to each other. Kontsevich proved that

$$2^{2g-2+s} \bigwedge_{e=1}^{s+2(3g-3+s)} d\ell_e = \frac{2^{3-3g-s}}{(3g-3+s)!}\Big(\sum_{i=1}^{s} L_i^2 c_1(\mathcal{L}_i \times (\mathbb{R}_+^\times)^s)\Big)^{3g-3+s} \wedge dL_1 \wedge \cdots \wedge dL_s\,, \tag{9}$$

independently of the ribbon graph. This formula is the main achievement in [Kontsevich, 1992]. One has $c_1(\mathcal{L}_i \times (\mathbb{R}_+^\times)^s) = c_1(\mathcal{L}_i)$ because $(\mathbb{R}_+^\times)^s$ is trivial. It is written in this form because on $\mathcal{L}_i \times (\mathbb{R}_+^\times)^s$ Kontsevich can prove a formula for the connection form which gives rise to the Chern classes. A technically difficult aspect is that the stratification into ribbon graphs is only possible for true curves in $\mathcal{M}_{g,s}$ and not for the boundary of $\overline{\mathcal{M}}_{g,s}$. This difficulty was addressed, but we ignore it here.

The non-compactness of $(\mathbb{R}_+^\times)^s$ is addressed by a Laplace transform. One multiplies (9) by $\prod_{j=1}^s e^{-\lambda_j L_j}$ and integrates. On the rhs the multinomial formula gives

$$\begin{aligned}\text{rhs} &= \sum_{d_1+\cdots+d_s=3g-3+s} \Big(\prod_{j=1}^s \int_0^\infty dL_j \frac{L_j^{2d_j} e^{-L_j\lambda_j}}{2^{d_j} d_j!}\Big) \int_{\overline{M}_{g,s}} \prod_{j=1}^s (c_1(\mathcal{L}_j))^{d_j} \\ &= \sum_{d_1+\cdots+d_s=3g-3+s} \Big(\prod_{j=1}^s \frac{(2d_j-1)!!}{\lambda_j^{2d_j+1}}\Big) \langle \tau_1^{d_1}\cdots\tau_s^{d_s}\rangle\end{aligned}$$

On the lhs we have with $\prod_{i\in\text{faces}} e^{-\lambda_i L_i} = \prod_{e\in E(\Gamma)} e^{-\ell_e(\lambda'(e)+\lambda''(e))}$

$$\begin{aligned}\text{lhs} &= \sum_{\Gamma\in\mathcal{RG}_s^3} \frac{1}{\#\mathrm{Aut}(\Gamma)} \frac{1}{2^{2g-2+s}} \prod_{e\in E(\Gamma)} \int_0^\infty d\ell_e e^{-\ell_e(\lambda'(e)+\lambda''(e))} \\ &= \sum_{\Gamma\in\mathcal{RG}_s^3} \frac{2^{-V(\Gamma)}}{\#\mathrm{Aut}(\Gamma)} \prod_{e\in E(\Gamma)} \frac{2}{\lambda'(e)+\lambda''(e)}\end{aligned}$$

where $V(\Gamma)-|E(\Gamma)|+s=2-2g$ has been used. The factor $\#\mathrm{Aut}(\Gamma)$ is not yet clear to me. Altogether Theorem 2 is obtained.

5. Matrix model

A direct determination of the symmetry factor $\#\mathrm{Aut}(\Gamma)$ is not easy. It is therefore convenient to consider a generating function for ribbon graphs including symmetry factor. The solution is given by a partition function of a new type of matrix model.

Let $\Lambda=\mathrm{diag}(\lambda_1,\dots,\lambda_N)$ be a diagonal positive matrix. We normalise the translation-invariant Lebesgue measure on the space M_N^* of self-adjoint $N\times N$-matrices as follows:

$$W(\lambda):=\int_{M_N^*} dX\; e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)}=\prod_{i,j=1}^N \frac{1}{\sqrt{\lambda_i+\lambda_j}} \tag{10}$$

Note that $\frac{1}{2}\mathrm{Tr}(\Lambda X^2)=\sum_{i,j=1}^N \frac{\lambda_i+\lambda_j}{4}X_{ij}X_{ji}$. If $J=(J_{ij})$ is another self-adjoint matrix, then by translation invariance

$$\int_{M_N^*} dX\; e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)+\mathrm{Tr}(JX)}=\exp\Big(\sum_{i,j}\frac{J_{ij}J_{ji}}{\lambda_i+\lambda_j}\Big)\prod_{i,j=1}^N\frac{1}{\sqrt{\lambda_i+\lambda_j}}\;.$$

This gives

$$\begin{aligned}\int_{M_N^*} dX\; X_{ab}X_{cd}e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)}&=\frac{\partial^2}{\partial J_{ba}J_{dc}}\int_{M_N^*} dX\; e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)+\mathrm{Tr}(JX)}\Big|_{J=0}\\ &=\frac{2\delta_{ad}\delta_{bc}}{\lambda_a+\lambda_b}\cdot W(\lambda)\;.\end{aligned}$$

In other words, up to the prefactor $W(\lambda)$, the weight factor $\frac{2}{\lambda_a+\lambda_b}$ is precisely the covariance of the Gaußian matrix measure. Trivalent band graphs are generated as follows:

$$\begin{aligned}&\int_{M_N^*} dX\; e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)+\frac{\mathrm{i}}{6}\mathrm{Tr}(X^3)}\\ &=\sum_{v=0}^\infty\frac{(\mathrm{i}/2)^v}{v!}\Big(\frac{1}{3}\mathrm{Tr}\Big(\frac{\partial^3}{\partial J}\Big)\Big)^v\int_{M_N^*} dX\; e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)+\mathrm{Tr}(JX)}\Big|_{J=0}\\ &=\Big[\sum_{s=0}^\infty\frac{(-1)^s}{s!}\sum_{a_1,\dots,a_s=1}^N\sum_{\Gamma\in(\Pi\mathcal{RG}^3)_s}\frac{2^{-v}}{\#\mathrm{Aut}(\Gamma)}\prod_{e\in E(\Gamma)}\frac{2}{\lambda'(e)+\lambda''(e)}\Big]W(\lambda)\;.\end{aligned} \tag{11}$$

Here $\mathrm{i}^v=(-1)^{\frac{v}{2}}=(-1)^s$ was used. The sum runs over the set $(\Pi\mathcal{RG}^3)_s$ of not necessarily connected 3-valent ribbon graphs with together s faces labelled by

$\lambda_{a_1}, \dots, \lambda_{a_s}$. It is known from combinatorics that the logarithm of $[\dots]$ in (11) generates connected graphs:

$$
\begin{aligned}
&\log\left(\frac{\displaystyle\int_{M_N^*} dX\ e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)+\frac{\mathrm{i}}{6}\mathrm{Tr}(X^3)}}{\displaystyle\int_{M_N^*} dX\ e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)}}\right)\\
&= \sum_{s=1}^{\infty} \frac{(-1)^s}{s!} \sum_{a_1,\dots,a_s=1}^{N} \sum_{\Gamma\in\mathcal{RG}_s^3} \frac{2^{-v}}{\#\mathrm{Aut}(\Gamma)} \prod_{e\in E(\Gamma)} \frac{2}{\lambda'(e)+\lambda''(e)}\\
&= \sum_{s=1}^{\infty} \frac{1}{s!} \sum_{d_1,\dots,d_s=0}^{\infty} \langle\tau_{d_1}\cdots\tau_{d_s}\rangle \prod_{i=1}^{s}\sum_{a_i=1}^{N} \frac{-(2d_i-1)!!}{\lambda_{a_i}^{2d_i+1}}
\end{aligned}
\tag{12}
$$

where Kontsevich's theorem 2 is inserted. Collecting identical powers of Chern classes, we arrive at:

Theorem 4.

$$
\log\left(\frac{\displaystyle\int_{M_N^*} dX\ e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)+\frac{\mathrm{i}}{6}\mathrm{Tr}(X^3)}}{\displaystyle\int_{M_N^*} dX\ e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)}}\right) = \sum_{k_0,k_1,\dots=0}^{\infty} \langle\tau_0^{k_0}\tau_1^{k_1}\dots\rangle \prod_{i=0}^{\infty}\frac{t_i^{k_i}}{k_i!} = F(t_0,t_1,\dots)\ ,
$$

where $t_i := -(2i-1)!!\mathrm{Tr}(\Lambda^{-2i-1})$.

Independence of the t_i requires $N\to\infty$. The formula is understood asymptotically for $\lambda_a\to\infty$. It is not clear that the partition function converges.

6. Proof of the string equation

We follow [Witten, 1992]. To prove the string equation (4) one applies the operator $T := \sum_{a=1}^{N}\frac{1}{\lambda_a}\frac{\partial}{\partial\lambda_a}$ to

$$
We^F = \int_{M_N^*} dX\ e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)+\frac{\mathrm{i}}{6}\mathrm{Tr}(X^3)}\ ,\qquad \text{where } W=\prod_{i,j=1}^{N}\frac{1}{\sqrt{\lambda_i+\lambda_j}}\ .
$$

On the lhs one has

$$
TW = -\frac{1}{2}W\sum_{i,j=1}^{N}\frac{\frac{1}{\lambda_i}+\frac{1}{\lambda_j}}{\lambda_i+\lambda_j} = -\frac{1}{2}t_0^2W
$$

and $Te^F = \sum_{i=0}^{\infty} \frac{\partial F}{\partial t_i}(Tt_i)e^F$. With $Tt_i = -\sum_{a=1}^{N} \frac{1}{\lambda_a}\frac{\partial}{\partial \lambda_a}(2i-1)!!\sum_{b=1}^{N} \frac{1}{\lambda_b^{2i+1}} = +(2i+1)!!\sum_{b=1}^{N} \frac{1}{\lambda_b^{2i+3}} = -t_{i+1}$ we have

$$\begin{aligned} -We^F\Big(\sum_{i=0}^{\infty} t_{i+1}\frac{\partial F}{\partial t_i} + \frac{t_0^2}{2}\Big) &= \int_{M_N^*} dX\; T\big(e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)+\frac{\mathrm{i}}{6}\mathrm{Tr}(X^3)}\big) \\ &= -\frac{1}{2}\int_{M_N^*} dX \sum_{a,b=1}^{N} \frac{1}{\lambda_a} X_{ab}X_{ba} e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)+\frac{\mathrm{i}}{6}\mathrm{Tr}(X^3)} \\ &= \int_{M_N^*} dX \sum_{a=1}^{N} \frac{1}{\lambda_a}\Big(\mathrm{i}\frac{\partial}{\partial X_{aa}} + \mathrm{i}\lambda_a X_{aa}\Big) e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)+\frac{\mathrm{i}}{6}\mathrm{Tr}(X^3)} \\ &= \mathrm{i}\int_{M_N^*} dX\; \mathrm{Tr}(X) e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)+\frac{\mathrm{i}}{6}\mathrm{Tr}(X^3)}\,. \end{aligned} \tag{13}$$

Comparing with (4) it remains to prove

$$\int_{M_N^*} dX\; \mathrm{Tr}(\mathrm{i}X) e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)+\frac{\mathrm{i}}{6}\mathrm{Tr}(X^3)} = -\frac{\partial F}{\partial t_0}\cdot \int_{M_N^*} dX\; e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)+\frac{\mathrm{i}}{6}\mathrm{Tr}(X^3)}$$

This equation has a quantum field theoretical interpretation,

$$\frac{\partial F}{\partial t_0} = (-\mathrm{i})\sum_{a=1}^{n} \frac{\partial}{\partial J_{aa}}\Big(\log \int_{M_N^*} dX\; e^{-\frac{1}{2}\mathrm{Tr}(\Lambda X^2)+\frac{\mathrm{i}}{6}\mathrm{Tr}(X^3)+\mathrm{Tr}(JX)}\Big)\Big|_{J=0}\,.$$

It means that we have to prove that $\frac{\partial F}{\partial t_0}$ is the trace of the connected one-point function, which expands into open ribbon graphs with a distinguished external 1-valent vertex. We start from the generating function (2),

$$\frac{\partial F}{\partial t_0} = \sum_{k_0,k_1\dots} \langle \tau_0 \tau_0^{k_0}\tau_1^{k_1}\dots\rangle \prod_{i=0}^{\infty} \frac{t_i^{k_i}}{k_i!}$$

which has an additional insertion of τ_0. Starting from (7), we have

$$\sum_{d_1,\dots,d_s=0}^{\infty} \langle \tau_0\tau_{d_1}\cdots\tau_{d_s}\rangle \prod_{i=1}^{s} \frac{(2d_i-1)!!}{\lambda_i^{2d_i+1}} = \lim_{\lambda_0\to\infty} \lambda_0 \sum_{\Gamma\in\mathcal{RG}^3_{s+1}} \frac{2^{-|V(\Gamma)|}}{\#\mathrm{Aut}(\Gamma)} \prod_{e\in E(\Gamma)} \frac{2}{\lambda'(e)+\lambda''(e)}\,.$$

The sum is over ribbon graphs with face labels $\lambda_0, \lambda_1, \dots, \lambda_s$. The only contribution to the limit $\lambda_0 \to \infty$ is from graphs where $\frac{2}{\lambda_0+\lambda'}$ only arises once, i.e. from tadpole subgraphs. Including the vertex, but not the edge leading to the tadpole, the subgraph contributes weight $\frac{\mathrm{i}}{2}\cdot\frac{2}{\lambda_0+\lambda'}$, i.e. a factor i in the limit. The additional sign results from $(-1)^s$ versus $(-1)^{s+1}$. After summation over all λ_a one includes precisely the graphs with $\mathrm{Tr}(X)$-insertion. This finishes the proof of (4).

7. Relation to the Virasoro algebra

The non-linear KdV-equation for U can be solved by essentially the same strategy. We refer to [Witten, 1992]. We mention another result discussed in [Witten, 1992]. Define for $p \in \mathbb{Z} + \frac{1}{2}$ operators

$$a_p := \frac{(2p)!!}{\sqrt{2}} \frac{\partial}{\partial t_{p-\frac{1}{2}}} \;, \qquad p > 0$$
$$a_p := \frac{1}{(-2p-2)!!\sqrt{2}} \big(t_{-p-\frac{1}{2}} - \delta_{p,-\frac{3}{2}}\big) \;, \qquad p < 0$$

Then $[a_p, q_q] = p\delta_{p,-q}$. Define

$$L_n := \frac{1}{2} \sum_{p \in \mathbb{Z}+\frac{1}{2}} a_p a_{n-p} \qquad \text{for } n \neq 0 \;, \qquad L_0 := \frac{1}{16} + \sum_{p \in \mathbb{N}+\frac{1}{2}} a_{-p} a_p \;. \tag{14}$$

One can show that the L_n satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n} m(m^2-1) \;.$$

Then the KdV-equations for F are equivalent to $L_n e^F = 0$ for all $n \geq -1$, i.e. the partition function e^F is a highest-weight vector for the Virasoro algebra.

8. Outlook

In general it becomes necessary to study the expectation values

$$\langle X_{a_1 b_1} \cdots X_{a_n b_n} \rangle := \log \frac{\displaystyle\int_{M_N^*} dX \; X_{a_1 b_1} \cdots X_{a_n b_n} e^{-N\mathrm{Tr}(\Lambda X^2) + \frac{\mathrm{i}}{3} N \mathrm{Tr}(X^3)}}{\displaystyle\int_{M_N^*} dX \; e^{-N\mathrm{Tr}(\Lambda X^2) + \frac{\mathrm{i}}{3} N \mathrm{Tr}(X^3)}} \tag{15}$$

These are independently accessible via a *topological recursion* [Eynard, 2016] in the genus. For that purpose we have already rescaled Λ, X by a factor $(2N)^{\frac{1}{3}}$. This leads to an additional factor $(2N)^{v-e} = (2N)^{2-2g-s}$ in (12). The factor $\frac{1}{N^s}$ is absorbed in a new definition $t_i = -(2i-1)!! \frac{1}{N} \mathrm{Tr}(\Lambda^{-2i-1})$ which has a better large-N limit. The remaining N^{2-2g} provides the grading for the genus expansion $F = \sum_{g=0}^{\infty} N^{2-2g} F_g$. n-point functions (15) have exceeding unmarked faces which receive a factor $\frac{1}{N}$

$$\langle X_{a_1 a_1} \cdots X_{a_n a_n} \rangle = \sum_{g=0}^{\infty} N^{2-2g-n} G^{(g)}_{a_1|\ldots|a_n} \;.$$

For these functions $G^{(g)}_{a_1|\ldots|a_n}$ one can prove Schwinger-Dyson equations and solve them in terms of the eigenvalues λ_i or the functions t_i. Possibly up to numerical factors, the simplest case reads in terms of $W(x) = G_a^{(0)} - \lambda_a\big|_{\lambda_a^2 = Nx}$:

$$W^2(x) - \int_\alpha^\beta dt\; \rho(t) \frac{W(x) - W(t)}{x-t} = x \tag{16}$$

where $\rho(t) = \sum_{a=1}^\infty \delta(Nt - \lambda_a^2)$. Such an equation can be algebraically solved by techniques for boundary values of sectionally holomorphic functions [Makeenko-Semenoff, 1991]:

$$W(x) = \sqrt{x+c} - \frac{1}{2}\int_\alpha^\beta dt\; \frac{\rho(t)}{(\sqrt{x+c}+\sqrt{t+c})\sqrt{t+c}}\;, \qquad c = \int_\alpha^\beta dt\; \frac{\rho(t)}{\sqrt{t+c}} \tag{17}$$

All other g-homogeneous parts of n-point functions satisfy linear Schwinger-Dyson equations. They can be evaluated recursively. This is one of the rare cases where QFT-correlation functions can be computed exactly.

References

[Eynard, 2016] B. Eynard, "Counting Surfaces," Prog. Math. Phys. **70** (2016). doi:10.1007/978-3-7643-8797-6

[Kontsevich, 1992] M. Kontsevich, "Intersection theory on the moduli space of curves and the matrix Airy function," Commun. Math. Phys. **147** (1992) 1. doi:10.1007/BF02099526

[Strebel, 1984] K. Strebel, *Quadratic differentials*, Springer-Verlag Berlin, Heidelberg, New York (1984).

[Witten, 1992] E. Witten, "On the Kontsevich model and other models of two-dimensional gravity," IASSNS-HEP-91-24.

[Witten, 1991] E. Witten, "Two dimensional gravity and intersection theory on moduli space," Surveys in Diff. Geom. **1** (1991) 243–310.

[Makeenko-Semenoff, 1991] Y. Makeenko and G. W. Semenoff, "Properties of Hermitean matrix models in an external field," Mod. Phys. Lett. A **6** (1991) 3455. doi:10.1142/S0217732391003985