The dynamical system for the integer quantum Hall effect

References

[1] J. Bellissard, "Gap Labelling Theorems for Schrödinger Operators", in: From Number Theory to Physics, Springer-Verlag (1993), pp 538–630
`http://people.math.gatech.edu/~jeanbel/publijb.html`

[2] J. Bellissard, H. Schulz-Baldes and A. van Elst, "The Non Commutative Geometry of the Quantum Hall Effect," J. Math. Phys. **35** (1994) 5373–5471.

[3] J. Bellissard, "On the Noncommutative Geometry of Aperiodic Solids: 1) Gap Labeling Theorems", lecture at INDAM Meeting Noncommutative Geometry, Index Theory and Applications , Cortona, Italy (2012).
`http://people.math.gatech.edu/~jeanbel/talksjbE.html`

[4] M. Reed and B. Simon, "Functional Analysis," Academic Press, 1980.

1 Quantum Hall effect

Consider a flat electrically conducting probe contained in the x-y-plane and with electric current I in x-direction. If the probe is additionally submersed in a constant magnetic field B in z-direction, the Lorentz force on the charge carriers results in a Hall voltage U in y-direction. Elementary physical considerations give for the Hall resistance

$$R_H := \frac{U}{I} = \frac{B}{ne}$$

where e is the electron charge and n the charge carrier density which varies with the material. The linear behaviour in B was observed by Hall in 1879. 100 years later, in 1980, von Klitzing discovered that in very strong magnetic fields the behaviour is different: The Hall resistence is a piecewise constant (globally non-decreasing) function of the magnetic field. Over a certain range of the magnetic field, the Hall resistance takes discrete values

$$\frac{1}{R_H} = \nu \frac{e^2}{2\pi\hbar}$$

where $\hbar$ is Planck's constant, and ν is to extraordinary precision an integer, later also a certain fraction such as $\nu \in \{\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{2}{3}, \frac{3}{5}, \frac{1}{5}, \frac{2}{9}, \frac{3}{13}, \dots\}$. These values are independent of the material, in particular of the charge carrier density n. Therefore they are used today do *define* the ratio $\frac{e^2}{2\pi\hbar}$ of the constants of Nature.

It was soon realised that the exact quantisation must be of topological origin. For the integer case a complete description which explains all features at once was given by Jean Bellissard within noncommutative geometry. This seminar is devoted to this approach. As Jean explained two weeks ago, the magnetic translation operators are important. They generate a rotation algebra which is sensitive to whether the magnetic flux is rational or irrational in natural units. So the theory has to explain, among others, that rational and irrtional fluxes give the same value for the Hall resistance. Roughly speaking, it is the stability of K-theory which does the job.

If I understand it correctly, one can use for the integer case an effective theory for a single electron. Typically a cm^3 of material contains $> 10^{23}$ particles, namely ions which may or may not sit at lattice sites and up to 10^{22} free electrons in metals, 10^{10} free electrons in pure semiconductors. These form an effective potential experienced by a single free electron which otherwise is considered as non-interacting with the other 10^{23} particles! The fractional quantum Hall effects relies on the interaction between many electrons and is not yet understood. Our SFB project C4 tries to improve this situation.

2 Hamiltonian

Time evolution in quantum physics is described by a strongly-continuous 1-parameter family $\{U(t)\}_{t\in\mathbb{R}}$ of unitary operators on a Hilbert space $\mathcal{H}$, $U(t)U(t') = U(t+t')$. By Stones' theorem, this family defines a distinguished self-adjoint unbounded operator H defined (in the strong operator topology) by $H\psi = -\mathrm{i}\lim_{t\to 0} \frac{1}{t}(U(t)\psi - \psi)$ for a dense set of ψ. Unbounded self-adjoint operators are a dangerous subject which can be very differnt from bounded self-adjoint operators! It is *not* sufficient that $\langle H\phi, \psi\rangle = \langle \phi, H\psi\rangle$ for all ϕ, ψ in the domain of H; in addition $\mathrm{dom}(H) = \mathrm{dom}(H^*)$ is required which is not automatic and not always possible! To circumvent the domain issues it is often more convenient to work with the resolvent operators. The resolvent set of a closed (e.g. self-adjoint) operator H is the set of $z \in \mathbb{C}$ for which $(z - H) : \mathrm{dom}(H) \to \mathcal{H}$ is bijective; this set is open. Its complement in $\mathbb{C}$ is the spectrum denoted $\mathrm{sp}(H)$. By the closed graph theorem, the resolvent $R_z(H) := (z - H)^{-1}$ is a bounded operator for every $z \notin \mathrm{sp}(H)$. In many situations, for instance in perfect crystals, the resolvents are even compact operators.

We have to introduce some notation. Let $\mathcal{K}$ be the space of the compact operators on $L^2(\mathbb{R}^D)$. Let $(T_y\psi)(x) := \psi(x+y)$ be the translation by $y \in \mathbb{R}^D$. In a perfect crystal the potential is periodic, $(T_aV)(x) = V(x+a) = V(x)$ for some $a \in \Gamma$, where $\Gamma \subset \mathbb{R}^D$ is a discrete subgroup. Its dual is the group $\Gamma^* = \{b \in \mathbb{R}^D \ : \ \langle a, b\rangle \in 2\pi\mathbb{Z}\ \forall a \in \Gamma\}$; it defines the Brillouin zone $\boldsymbol{B} = \mathbb{R}^D/\Gamma^*$. One has the following result:

Theorem 1 *Let $H = -\frac{\hbar^2}{2m}\Delta + V$ be the Hamiltonian for a Γ-periodic potential V (Δ the Laplace operator and m the (possibly renormalised) mass of the electron). Then the C^*-algebra $C(H)$ generated by the family $\{T_yR_z(H)T_y^{-1} \ : \ y \in \mathbb{R}^3\}$ is, for any $z \notin \mathrm{sp}(H)$, equal to $C(\boldsymbol{B}) \otimes \mathcal{K}$, i.e. the stabilised continuous functions on the Brillouin zome.*

This fact has a far-reaching generalisation: The construction of the C^*-algebra $C(H)$ as generated by $T_yR_z(H)T_y^{-1}$ extends to the non-periodic case and in presence of magnetic fields. First remark that this is the right C^*-algebra of observables in macroscopic physics. On large scales the material looks homogeneous so that averaging $\{T_yHT_y^{-1} \ : \ y \in \mathbb{R}^D\}$ of the microscopic Hamiltonian H is anyway necessary. We first extend this averaging to the case relevant in presence of magnetic fields:

Definition 2 *Let $\mathcal{H}$ be a separable Hilbert space and G be a locally compact group with a unitary, projective, strongly continuous representation on $\mathcal{H}$, i.e. $U(a)U(b) = U(a+b)e^{i\phi(a,b)}$ for some $\phi(a,b) \in \mathbb{R}$ and $G \ni a \mapsto U(a)\psi$ continuous for every $\psi \in \mathcal{H}$. A self-adjoint operator H on $\mathcal{H}$ is* homogeneous *with respect to G if for any $z \notin \mathrm{sp}(H)$ the family $S(z) = \{U(a)R_z(H)U(a)^{-1} \ : \ a \in G\}$ admits a compact strong closure.*

Compact strong closure means the following approximation property: For every $\epsilon > 0$ and $\psi_1, \ldots, \psi_N \in \mathcal{H}$ there exist finitely many $a_1, \ldots, a_n \in G$ such that for every $a \in G$ there is an i with $\|U(a)R_z(H)U(a)^{-1}\psi_j - U(a_i)R_z(H)U(a_i)^{-1}\psi_j\| < \epsilon$ for all $j = 1, \ldots, N$. Let $\Omega(z)$ be the strong closure of $S(z)$, which is a compact metrisable space equipped with an action (called T) of G. The resolvent identities guarantee that any two $\Omega(z), \Omega(z')$ are homeomorphic so that identifying them gives rise to an abstract compact metrisable space Ω. The dynamical system (Ω, G, T) associated with a G-homogeneous operator H is called the *hull*.

The hull Ω to a homogeneous self-adjoint operator H involves not only the translated resolvents but also limit points. Every $\omega \in \Omega$ defines a self-adjoint operator H_ω on $L^2(\mathbb{R}^D)$ via the Trotter-Kato theorem for strong resolvent limits. Then the map $\Omega \ni \omega \mapsto (z - H_\omega)^{-1}$ is strongly continuous and $U(a)H_\omega(U(a)^{-1}) = H_{T_a\omega}$.

3 Magnetic field

The effect of an magnetic field B is captured by a Hamiltonian

$$H_B = H_{B,0} + V \;, \qquad H_{B,0} = \frac{1}{2m}\sum_{\mu=1}^{D}\Big(\frac{\hbar}{\mathrm{i}}\partial_\mu - eA_\mu\Big)^2 \;. \tag{1}$$

We have $D = 2$ for the quantum Hall effect, but also $D = 3$ might be interesting. The A_μ are the components of a 1-form A which defines the magnetic field $B = dA$. In $D = 2$ it is constant (times $dx_1 \wedge dx_2$) for $A = \frac{B}{2}(x_1 dx_2 - x_2 dx_1)$. The free Hamiltonian $H_{B,0}$ is the $2D$-harmonic oscillator with an additional angular momentum term. One can introduce[1] an independent pair of creation-annihilation operators $a^\dagger, a, b^\dagger, b$ in which $H_{B,0} = \frac{eB\hbar}{m}(a^\dagger a + \frac{1}{2})$. Hence, $H_{B,0}$ is self-adjoint with discrete but infinitely degenerate spectrum so that its resolvent is not compact on $L^2(\mathbb{R}^2)$. By the Kato-Rellich theorem, H_B is self-adjoint for every essentially bounded function $V \in L^\infty(\mathbb{R}^D)$. The interaction V may resolve the degeneracy and lead to compact resolvents.

The free Hamiltonian $H_{B,0}$ is clearly not translation invariant. We pass to magentic translation operators

$$\begin{aligned}(U(a)\psi)(x) &= \exp\Big(-\frac{\mathrm{i}e}{\hbar}\int_{x-a}^{x} A\Big)\psi(x-a)\\ &\overset{B=\text{const}}{=} \exp\Big(-\frac{\mathrm{i}eB}{2\hbar}(x_1 a_2 - x_2 a_1)\Big)\psi(x-a)\;. \end{aligned}\tag{2}$$

At least for constant B it is easy to check that $U(a)$ commutes with $\frac{\hbar}{\mathrm{i}}\partial_\mu - eA_\mu$ so that $U(a)H_{B,0}U(a)^{-1} = H_{B,0}$. Since V is a multiplication operator, we get $(U(a)VU(a)^{-1})(x) = V_a(x) := V(x-a)$ almost everywhere. Moreover, $U(a)U(b) = e^{\frac{\mathrm{i}eB}{2\hbar}(a_1 b_2 - a_2 b_1)}U(a+b)$ so that a projective representation of $\mathbb{R}^2$ is obtained. The magnetic translations by base vectors $u = U(e_1)$ and $v = U(e_2)$ thus satisfy the commutation relation $uv = e^{2\pi\mathrm{i}\theta}vu$ with $\theta = \frac{eB}{2\pi\hbar}$ of the rotation algebra.

Then:

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[1]

$$\begin{aligned}\hat{a} &= \frac{1}{2}\sqrt{\frac{eB}{2\hbar}}(\hat{x} - \mathrm{i}\hat{y}) + \frac{\mathrm{i}}{2}\sqrt{\frac{2}{eB\hbar}}(\hat{p}_x - \mathrm{i}\hat{p}_y)\;, \quad & \hat{a}^\dagger &= \frac{1}{2}\sqrt{\frac{eB}{2\hbar}}(\hat{x} + \mathrm{i}\hat{y}) - \frac{\mathrm{i}}{2}\sqrt{\frac{2}{eB\hbar}}(\hat{p}_x + \mathrm{i}\hat{p}_y)\\ \hat{b} &= \frac{1}{2}\sqrt{\frac{eB}{2\hbar}}(\hat{x} + \mathrm{i}\hat{y}) + \frac{\mathrm{i}}{2}\sqrt{\frac{2}{eB\hbar}}(\hat{p}_x + \mathrm{i}\hat{p}_y)\;, \quad & \hat{b}^\dagger &= \frac{1}{2}\sqrt{\frac{eB}{2\hbar}}(\hat{x} - \mathrm{i}\hat{y}) - \frac{\mathrm{i}}{2}\sqrt{\frac{2}{eB\hbar}}(\hat{p}_x - \mathrm{i}\hat{p}_y)\end{aligned}$$

Theorem 3 *Let the potential be essentially bounded, $V \in L^\infty(\mathbb{R}^D)$. Then the operator $H_B = H_{B,0} + V$ is homogeneous with respect to magnetic translations.*

I think this requires a restriction to finite volume or further assumptions on V to resolve the degeneracy!

Idea of the proof. $H_{B,0}$ is already homogeneous (under assumptions!), and all V_a are essentially bounded. The ball in $L^\infty(\mathbb{R}^D)$ with radius $\|V\|$ is weak-* compact. It remains to show that the map $L^\infty(\mathbb{R}^D) \ni V \mapsto (z - H_{B,0} - V)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^D))$ is continuous with respect to the weak-* topology on $L^\infty(\mathbb{R}^D)$ and the strong operator topology on $\mathcal{B}(L^2(\mathbb{R}^D))$. $\square$

A C^*-algebra $\mathcal{A}_B = C(\Omega) \rtimes \mathbb{R}^2$, called the *noncommutative Brillouin zone*, is associated with the hull Ω of $H_B = H_{B,0} + V$ as follows. Define on the space $C_K(\Omega \times \mathbb{R}^D)$ of continuous, compactly supported functions product and involution by

$$\begin{aligned}(fg)(\omega, x) &:= \int_{\mathbb{R}^2} dy\, f(\omega, y) g(T_{-y}\omega, x - y) \exp\left(\frac{\mathrm{i}e}{2\hbar}\langle x, By\rangle\right) , \\ f^*(\omega, x) &:= \overline{f(T_{-x}\omega, -x)} .\end{aligned} \tag{3}$$

Here B is viewed as skew-symmetric matrix. This *-algebra is represented on $L^2(\mathbb{R}^2)$ via the family of left-regular representations

$$(\pi_\omega(f)\psi)(x) := \int_{\mathbb{R}^2} dy f(T_{-x}\omega, y - x) \exp\left(\frac{\mathrm{i}e}{2\hbar}\langle y, Bx\rangle\right)\psi(y) . \tag{4}$$

It satisfies $\pi_\omega(fg) = \pi_\omega(f)\pi_\omega(g)$, $\pi_\omega(f^*) = (\pi_\omega(f))^*$ and $U(a)\pi_\omega(f)(U(a))^{-1} = \pi_{T_a\omega}(f)$.

The noncommutative Brillouin zone $\mathcal{A}_B = C(\Omega) \rtimes \mathbb{R}^2$ is then the completion of $C_K(\Omega \times \mathbb{R}^2)$ with respect to the norm $\|f\| = \sup_{\omega\in\Omega} \|\pi_\omega(f)\|$.

Theorem 4 (Bellissard 1986) *If H_B is homogeneous, it is affiliated to its C^*-algebra $\mathcal{A}_B$, i.e there is a $*$-homomorphism $C_0(\mathbb{R}) \ni f \mapsto f(H_B) \in \mathcal{A}_B$ such that $\pi_\omega(f(H)) = f(H_\omega)$ for all $\omega \in \Omega$.*

According to the spectral theorem there is a 1:1-correspondence between a self-adjoint operator H and a family $\{P_\Sigma\}_{\Sigma\subset\mathbb{R}}$ of projection-valued measures[2] indexed by Borel sets of $\mathbb{R}$. These pv-measures define an integral which allows to write $H = \int_{-\infty}^{\infty} \lambda\, dP_\lambda$ and $f(H) = \int_{-\infty}^{\infty} f(\lambda)\, dP_\lambda$ for any bounded Borel

[2]Every P_Σ is a projection, $P_\Sigma P_{\Sigma'} = P_{Sigma\cap\Sigma'}$ is a projection, $P_\varnothing = 0$ and $P_\mathbb{R} = 1$. If $\Sigma = \bigcup_{i=1}^\infty \Sigma_i$ for pairwise disjoint Σ_i, then $P_\Sigma = \text{s-lim}_{n\to\infty} \sum_{i=1}^n P_{\Sigma_i}$.

function f on $\mathbb{R}$. Stones' theorem expresses certain projections in terms of resolvents:

$$\tfrac{1}{2}(P_{[a,b]} + P_{]a,b[}) = \operatorname*{s-lim}_{\epsilon\searrow 0} \frac{1}{2\pi\mathrm{i}} \int_a^b d\lambda \Big(R_{\lambda-\mathrm{i}\epsilon}(H) - R_{\lambda+\mathrm{i}\epsilon}(H) \Big) \tag{5}$$

The characteristic function $P_{]-\infty,E]} = \chi(H \le E) = \int_{-\infty}^{E} dP_\lambda$ is a projector that belongs to the von Neumann algebra, but in general not to the C^*-algebra associated with H.

We let the C^*-algebra spectrum $\mathrm{Sp}(H)$ of H to be the union of the spectra of all H_ω, i.e. $\mathrm{Sp}(H) := \bigcup_{\omega\in\Omega} \mathrm{sp}(H_\omega)$.

Theorem 5 (Bellissard 1986) *If the orbit of $\omega \in \Omega$ is dense then $\mathrm{Sp}(H) = \mathrm{sp}(H_\omega)$. If there is a periodic orbit in Ω then $\mathrm{Sp}(H)$ cannot be nowhere dense.*

A *gap* is a connected component of $\mathbb{R}\setminus\mathrm{Sp}(H)$. For E in a gap one can deform the characteristic function that defines $\chi(H \le E)$ to a continuous function, thereby showing that $\chi(H_B \le E) \in \mathcal{A}_B$ if (and only if) E is in a gap. This will allow to characterise the gaps via K-theory $K_0(\mathcal{A}_B)$

4 Integrated density of states (IDS)

Let $\mathbb{P}$ be a probability measure on the hull Ω, invariant and ergodic under $\mathbb{R}^D$-action. Such measures exist by abstract properties of dynamical systems. Then

$$\tau(f) := \int_\Omega \mathbb{P}(d\omega) f(\omega, 0) \ , \qquad f \in C_K(\Omega \times \mathbb{R}^2)$$

defines a trace on compactly supported functions. It can be used to define various completions, a Hilbert space via GNS construction and a weak closure to a von Neumann algebra. The trace can also be expressed as:

Theorem 6

$$\tau(f) = \lim_{\Lambda\nearrow\mathbb{R}^2} \frac{1}{|\Lambda|} \mathrm{Tr}_\Lambda(\pi_\omega(f))$$

for $\mathbb{P}$-almost every $\omega \in \Omega$. Here, Tr_Λ denotes the restriction of the L^1-trace on $\mathcal{K}(L^2(\mathbb{R}^2))$ to Λ.

Proof. The integral kernel of $f \in C_K(\Omega \times \mathbb{R}^2)$ is $F(x,y) = f(T_{-x}\omega, y - x)\exp(\mathrm{i}e/(2\hbar)\langle y, Bx\rangle)$, see (4), hence its trace $\mathrm{Tr}_\Lambda(\pi_\omega(f)) = \int_\Lambda dx\ F(x,x) = \int_\Lambda dx\ f(T_{-x}\omega, 0)$. The result follows from Birkhoff's ergodic theorem. □

Let H_Λ be the restriction of the Hamiltonian H_B given in (1) to a rectangular box $\Lambda \subset \mathbb{R}^D$, with certain boundary conditions. Then H_Λ has discrete spectrum bounded from below. Let $N_\Lambda(E)$ be the number of eigenvalues of H_Λ less or equal to E. It is physically plausible that $N_\Lambda(E)$ is, up to $o(|\Lambda|)$-corrections, invariant under translation of Λ and additive when gluing disjoint Λ's. Therefore, the limit

$$\mathcal{N}(E) := \lim_{\Lambda \nearrow \mathbb{R}^D} \frac{1}{|\Lambda|} N_\Lambda(E) \ ,$$

called the *integrated density of states (IDS)*, should exist.

For H_Λ of discrete spectrum one has equivalently $N_\Lambda(E) = \mathrm{Tr}(\chi(H_\Lambda \leq E))$ for the number of eigenvalues of H_Λ less or equal to E.

Definition 7 *The Hamiltonian H obeys Schubin's formula if $\mathcal{N}(E) = \tau(\chi(H \leq E))$, where τ is the trace per volume in the C^*-algebra of H.*

This is not automatic. It was proved by Shubin (1979) for uniformly elliptic differential operators with almost-periodic coefficients and generalised by Bellissard (1986).

Theorem 8 *Let H be a homogeneous self-adjoint Hamiltonian (e.g. the one in (1)) and $\mathcal{A}$ the C^*-algebra associated with its hull. Let τ be a translation-invariant trace on $\mathcal{A}$ for which H obeys Shubin's formula. Then its IDS $\mathcal{N}(E)$ is a non-negative non-decreasing function which is constant on each gap of* $\mathrm{Sp}(H)$*. If τ is faithful, then conversely the spectrum of H coincides with the set of points $E \in \mathbb{R}$ in the vicinity of which the IDS is not constant.*

Proof. The first claim is clear, the second follows from

$$\mathcal{N}(E+\delta) - \mathcal{N}(E-\delta) = \tau(\chi(E-\delta < H \leq E+\delta)) \ ,$$

which is strictly positive if τ is faithful. □

Any discontinuity point corresponds to an eigenvalue of H with infinite multiplicity (there are such examples!). It is believed that, under certain assumptions, the IDS $\mathcal{N}(E)$ is continuous.

5 The current-current correlation

Let X_μ be the position operator, $(X_\mu\psi)(x) = x_\mu\psi(x)$. The current operator is then defined as

$$J_\mu(B) := \frac{e}{\mathrm{i}\hbar}[X_\mu, H_\omega(B)] = \frac{e}{m}\Big(\frac{\hbar}{\mathrm{i}}\partial_\mu - eA_\mu\Big) \ .$$

It depends on the magnetic field but not on the interaction potential. The current operators are used to define the following correlation functions between functions Φ, Φ' of the Hamiltonian:

$$\langle \Phi \Phi' \rangle_{\mu\nu} := \tau\big(J_\mu(B) \Phi(H_\omega) J_\nu(B) \Phi'(H_\omega) \big) \ . \tag{6}$$

Theorem 9 (Khorunzhy-Pastur, 1993) *There exists a Radon measure $dm_{\mu\nu}$ on $\mathbb{R}^2$ such that $\langle \Phi\Phi' \rangle_{\mu\nu} = \frac{e^2}{\hbar^2} \int_{\mathbb{R}\times\mathbb{R}} dm_{\mu\nu}(E, E')\ \Phi(E)\Phi(E')$. The support of the measure is $\Sigma \times \Sigma$, where Σ is the $\mathbb{P}$-almost sure non-random spectrum of $H_\omega(B)$.*

Important physical quantities such as the conductivity and the Anderson localisation length can be expressed as integrals which involve the current-current correlation measure.