VON NEUMANN ALGEBRAS IN PHYSICS: A ROUGH SKETCH OF ALGEBRAIC QUANTUM FIELD THEORY

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Abstract. These are the notes from a talk I gave during a seminar about von Neumann algebras. It deals with the role of type III-factors in algebraic quantum field theory, explains the KMS condition and the characterization of type III-factors via their modular conjugation group.

1. Introduction

We give a short introduction into the main aspects of quantum field theory, in particular algebraic quantum field theory via nets of von Neumann algebras. We assume no familiarity with the concepts of mechanics and quantum mechanics, which we will briefly sketch in the beginning. We pay particular attention to the case, where the local algebras are type III_1-factors, which is supported by most examples from physics. In this case states of the net can be changed locally to eigenstates without disturbing them on the causal complement. A thought experiment by Fermi will be discussed, which shows that quantum mechanics alone does not have these features and is therefore a strictly non-relativistic theory [7, 1].

In a second part we will explain the KMS condition and how it can be used to characterize the modular group. This also reveals a nice characterization of type III-factors as those with modular conjugation, which is outer for at least one state and at some value $t \in \mathbb{R}$. The material for this part is taken from [4].

2. From classical mechanics to quantum field theory

2.1. **Classical mechanics.** In classical Hamiltonian mechanics a physical system with energy conservation is described by functions on the phase space X. The latter captures the information about the space coordinates and the momenta of all particles. For n point particles moving in $\mathbb{R}^3$ this would be $\mathbb{R}^{6n}$. But it may be more complex, e.g. the particles may be confined to a manifold M, in which case the phase space can be identified with the cotangent bundle T^*M. The collection of all possible space coordinates is often called the configuration space of the system. We will assume $X = \mathbb{R}^{6n}$ and denote the points by (x_i^j, p_i^j) with $x_i^j, p_i^j \in \mathbb{R}$ the ith space coordinate and ith momentum coordinate of the jth particle. The dynamics of the system are described by

(1) the energy or Hamilton function on the phase space, often denoted by $H\colon X \to \mathbb{R}$.
(2) the Hamiltonian equations

$$\dot{x}_i^j = \frac{\partial H}{\partial p_i^j} \quad , \quad \dot{p}_i^j = -\frac{\partial H}{\partial x_i^j}$$

where the dot denotes the time derivative as is common in physics.

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(3) initial conditions for these differential equations.

Example 2.1. Let us assume we have one particle with mass m moving along the real axis with dynamics described by $H(x,p) = \frac{p^2}{2m}$. Then we get

$$\dot{x} = \frac{p}{m} =: v \quad , \quad \dot{p} = 0 \quad \Rightarrow \quad x(t) = v \cdot t + x_0 \ ,$$

which describes the movement of a *free particle* with initial velocity v and initial coordinate x_0.

Example 2.2. Assume the particles moves along the real axis, but this time with dynamics described by $H(x,p) = \frac{p^2}{2m} + \frac{x^2}{2}$ and initial momentum $p(0) = 0$. Then

$$\dot{x} = \frac{p}{m} \quad , \quad \dot{p} = -x \quad \Rightarrow \quad x(t) = x_0 \cos\left(\frac{t}{\sqrt{m}}\right) \ .$$

This is the movement of the so called *harmonic oscillator*. The particle oscillates around a fixed coordinate with starting point x_0.

2.2. **Quantum mechanics.** Quantum mechanics arose from observations that showed that classical mechanics breaks down at an atomic level. For example the movement of an electron around the nucleus seemed to be confined to certain discrete energy levels. A phenomenon that could not be explained using Hamiltonian mechanics. Bohr, Schrödinger, Heisenberg and many others worked out the basics of quantum mechanics. The idea is to replace the phase space X by the Hilbert space $\mathbb{H} = L^2(M, \mathbb{C})$ of complex valued square integrable functions on the configuration space M and to replace the energy function H by an operator in the following way

(1) exchange x_i^j by the multiplication operator with the coordinate x_i^j.
(2) exchange the momentum coordinate by the differential operator $\frac{\hbar}{i}\frac{\partial}{\partial x_i^j}$, where the latter is understood as a densely defined unbounded operator on $\mathbb{H}$.

Note that this procedure is far from being unique or even unambiguous due to the fact that the variables x_i^j and p_i^j commute whereas the operators do not. Nevertheless, for most physical systems there is a sensible way to write down a Hamiltonian operator H. The dynamics are now described by the Schrödinger equation for a state $\psi \in \mathbb{H}$:

$$\left(i\hbar\frac{\partial}{\partial t}\psi\right)(x,t) = (H\psi)(x,t)$$

and an initial state $\psi_0 \in \mathbb{H}$. The quantity $\hbar$ is Planck's constant. If H is time independent, the solution is given by

$$\psi(x,t) = \left(e^{-\frac{i}{\hbar}tH}\psi_0\right)(x) \ .$$

So, we need the eigenspace decomposition of the operator H to compute the right hand side. The interpretation of ψ is now of a stochastical nature. If ψ is a unit vector, then the norm-square $|\psi(x,t)|^2$ is the probability density for the presence of the particle at the point x and time t. Thus, if $U \subset M$ is a (measurable) subset of the configuration space, I is a time interval and χ_U, χ_I are the characteristic functions of these sets, then

$$\langle \psi, \chi_U \cdot \chi_I \cdot \psi \rangle = \int_{U \times I} \chi_U(x)\chi_I(t)|\psi(x,t)|^2 dxdt$$

is the probability to find the particle inside of U during the interval I. Likewise, any other observable quantity is represented by a self-adjoint operator and its expectation value is given by $\langle \psi, A\,\psi \rangle$. For an observable A the only values that will appear in a measurement are its eigenvalues. Since these may well be discrete, it solves the problems sketched at the beginning of this section.

Example 2.3. Let us review the example of the free particle, but now consider the interval $[0, 1]$ as the configurations space, i.e. the particle is contained in a box. The Hilbert space is $\mathbb{H} = L^2([0, 1], \mathbb{C})$. At the boundary of the box we impose the conditions $\psi_0(0) = \psi_0(1) = 0$. The Hamiltonian for the free particle is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \ .$$

Its eigenvalues E_n are indeed discrete and labeled by the natural numbers, precisely we have

$$E_n = n^2 \cdot \frac{\hbar^2 \pi^2}{2m} \ .$$

So the energy in our toy model is quantized. If we think about ψ as a (probability) wave, then our boundary conditions force this wave to reflect at the points 0 and 1, therefore it interferes with itself and either annihilates or strengthens itself depending on its energy, which is basically its frequency.

Let us take a closer look at the expectation value of an observable A. It can be rewritten as

$$\langle \psi, A\,\psi \rangle = \langle \psi_0, e^{\frac{i}{\hbar}tH}\, A\, e^{-\frac{i}{\hbar}tH}\, \psi_0 \rangle \ .$$

Thus, instead of seeing the wave functions as dynamical quantities, we may as well think about the observables as our fundamental objects! The time evolution yields

$$\alpha_t(A) = e^{\frac{i}{\hbar}tH}\, A\, e^{-\frac{i}{\hbar}tH}$$

mapping the observables into themselves and the expectation value becomes a linear functional

$$\omega(A) = \langle \psi_0, A\,\psi_0 \rangle \ .$$

So, why not study the algebra generated by the observables with the expectation values being linear functionals on the algebra and the time evolution inducing an automorphism of it? This is the Heisenberg picture of quantum mechanics, whereas the one that focusses on wave functions is called the Schrödinger picture. The way to get from an abstract to a concrete algebra is via its representation as operators on Hilbert spaces. We can represent the space coordinate X by a multiplication operator and the momentum P by a derivative. But the question remains, if there are other interesting representations of the algebra generated by X and P. A theorem of Stone and von Neumann tells us that this is not the case.

Theorem 2.4 (Stone-von Neumann)**.** *The representation of the algebra generated by P and X is unique up to unitary equivalence.*

For example, we may simply exchange the roles of P and X and represent P by multiplication and X by a differential operator, then the two pictures are related by the unitary equivalence given by Fourier transform.

2.3. **Quantum field theory.** As powerful as quantum mechanics is to describe physical systems at an atomic scale, there still are problems: It is limited to systems with finitely many degrees of freedom. So it is not possible to quantize continuously distributed observables like the energy density of an electromagnetic field and there is no description of thermodynamic limits. Moreover, it is inherently non-relativistic, even worse we have separated the time evolution from the spacial part. In a picture compatible with special relativity they should be treated on an equal footing.

To overcome these difficulties is the goal of quantum field theory. As mentioned above, in a field theory the observables themselves are densities on the configuration space. By what we learned, we have to replace them by operator valued densities or operator valued distributions. It is not enough to simply work with operator valued functions, since an energy measurement at a single point would need infinitely many energy, so H would not be defined everywhere. Let $\mathcal{S}(\mathbb{R}^4)$ be the space of Schwartz functions on the Minkowski space $\mathbb{R}^4$. A field ϕ would then be a linear functional

$$\phi\colon \mathcal{S}(\mathbb{R}^4) \to UB(\mathbb{H})$$

into the unbounded operators on a separable Hilbert space $\mathbb{H}$. Of course we omit many details about domains of definition here. The test function from $\mathcal{S}(\mathbb{R}^4)$ can be seen as characterizing a region over which the measurement is smeared out or averaged.

Wightman developed a set of axioms for those fields. The most important of these are

- **transformation behavior:** There is a unitary representation U of the Poincaré group G on $\mathbb{H}$ and the fields are *covariant* in the sense that $U(g)\psi(f)U(g)^* = \psi(g \cdot f)$ for $g \in G$.
- **locality:** If the supports of $f, h \in \mathcal{S}(\mathbb{R}^4)$ are space-like to each other, the corresponding field operators commute, i.e. $[\psi(f), \psi(h)] = 0$. The reasoning here is of course that measurements, which happen in space-like separated regions are not allowed to influence each other by special relativity.

Indeed, this leads to a setup for quantum field theory nowadays known as axiomatic qft. Due to the analytical difficulties, it is quite hard to deal with. But there is another approach we will sketch in the next paragraph.

2.3.1. *Algebraic quantum field theory.* As we have seen in the last chapters, the physical content of a theory and its dynamics are contained in the algebra of observables and its automorphisms. But working with unbounded densely defined operators forces us to take care about domains all the time. One way to circumvent these difficulties is to study the algebra of *bounded functions* of observables, which yields an algebra of *bounded operators*. This lead to the definition of a net of operator algebras as developed by Haag and Kastler. An introduction into this subject can be found in [3]. Before getting into the details of their definition, we take a closer look at sensible regions in space-time, in which our measurements will take place.

Suppose we have a detector, which we switch on at a space-time point (t_1, x_1). Any event that can be influenced by this lies in the forward light cone that has its tip at this point. We switch off our detector at (t_2, x_2). Any event that could have influenced our detector up to then lies in the backward light cone at (t_2, x_2). The intersecting region is called the causal closure of the two points or a double cone. The term causal closure refers to the fact that these are all points lying space-like to all points, which are space-like to both of the two

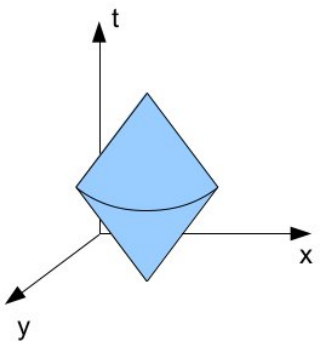

FIGURE 1. A double cone in three-dimensional Minkowski space

points. The set $\mathcal{C}$ of all open double cones in Minkowski space forms a basis for its topology and will be of fundamental importance for the next definition.

Definition 2.5. Let $\mathbb{H}$ be a fixed separable Hilbert space and let vNa($\mathbb{H}$) be the set of all von Neumann algebras inside $B(\mathbb{H})$. A map

$$\mathcal{A}\colon \mathcal{C} \to \text{vNa}(\mathbb{H}) \quad , \quad \mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$$

will be called a *net of von Neumann algebras* if it has the following properties

- **isotony:** Let $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{C}$. If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ as a unital subalgebra.
- **locality:** If $\mathcal{O}_1$ is space-like to $\mathcal{O}_2$, then $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$.
- **covariance:** There exists a strongly continuous unitary representation of the Poincaré group G on $\mathbb{H}$, such that $U(g)\mathcal{A}(\mathcal{O})U(g)^* = \mathcal{A}(g\,\mathcal{O})$.
- **positivity:** The generator of translations has spectrum lying in the forward light cone.

Clearly, the locality and covariance postulate were motivated by the Wightman axioms. Isotony seems plausible: It should not matter, if I measure an observable in a "bigger laboratory". The positivity assumption ensures the positivity of the energy operator. This is only the lowest common denominator among all possible axioms, there may be more, but this is the basic setup.

An astounding observation is that in all (irreducible) examples from physics, *the local algebras $\mathcal{A}(\mathcal{O})$ turn out to be isomorphic to the unique hyperfinite type III_1-factor.* (This has not been proven directly from the axioms, but a slightly weaker property called the Borchers property holds.)

2.4. **Type III-factors.** During the seminar we have met von Neumann algebras of type I_n for $n \in \mathbb{N} \cup \{\infty\}$, type II_1 and type II_∞. Let us review their definitions: Let M be a factor.

- M is of **type I** if there exists a minimal projection in M.
- M is of **type $\mathbf{II}_1$** if there is a normal normalized trace $\text{tr}\colon M \to \mathbb{C}$ on M. This is equivalent to saying that all projections are finite, which is equivalent to $1 \in M$ being finite.
- M is of **type $\mathbf{II}_\infty$** if $1 \in M$ is infinite, but M contains at least one finite projection.

As we see, if a factor M is of **type III**, i.e. does not belong to the list above, all projections in M are infinite. An important corollary of this is the following:

Corollary 2.6. *If M is a type III-factor on a separable Hilbert space $\mathbb{H}$, then all projections in M are (Murray-von Neumann) equivalent to $1 \in M$.*

Proof. Let $e \in M$ be a projection. By the above considerations it is infinite. Therefore $\exists u \in M$ such that $u^*u = e$ and $e_1 := uu^* < e$. Let $e_n = ue_{n-1}u^*$. Observe that $e_2 = ue_1u^* < uu^* = e_1$, thus by induction

$$e_{n+1} = ue_nu^* < ue_{n-1}u^* = e_n \quad \text{if } e_n < e_{n-1} \ .$$

The e_i form a decreasing sequence of projections. Let $p_n = e_n - e_{n+1}$. These are orthogonal projections, which are all equivalent to p_1 and with $p_\infty = e - e_1$ we have

$$e = p_\infty + \sum_{n=1}^{\infty} p_n \ .$$

Let f_i be a maximal system of orthogonal projections with $f_i \precsim p_1$. Suppose $f_0 := 1 - \sum_i f_i > 0$. Then there is a projection $0 \neq \widetilde{p} \in M$ with $\widetilde{p} \leq 1 - \sum_i f_i$ and $\widetilde{p} \precsim p_1$, which is a contradiction. Thus, we can represent $1 \in M$ as a sum of mutually orthogonal projections $f_i \in M$ with $f_i \precsim p_1$. Therefore

$$1 = \sum_i f_i \precsim \sum_n p_n \leq \sum_n p_n + p_\infty = e \ .$$

Since we also have $e \leq 1$, we get $e \sim 1$ and we are done. □

Let M be a type III-factor with separable predual. As we have seen in the chapter about Tomita-Takesaki theory, a choice of a cyclic and separating vector $\Omega \in \mathbb{H}$ for M or equivalently of a normal faithful state ϕ on M yields an unbounded positive operator Δ_ϕ and an anti-unitary operator J_ϕ with $J_\phi M J_\phi = M'$ and $\Delta_\phi^{it} M \Delta_\phi^{-it} = M$. The spectrum $\sigma(\Delta_\phi)$ depends on the choice of state ϕ. Thus, consider

$$S(M) = \bigcap_\phi \sigma(\Delta_\phi) \subset \mathbb{R}_+ \ .$$

Connes proved that $S(M)$ is in fact a *multiplicaltive subgroup* of $\mathbb{R}_+$. Since it is also closed, the only possibilities are $S(M) = \{0, 1\}$, $S(M) = \{0\} \cup \{\lambda^n \mid n \in \mathbb{Z}\}$ for some $0 < \lambda < 1$ or $S(M) = \mathbb{R}_+$. This yields a finer classification of type III-factors. A type III-factor M is called

- of **type $\mathbf{III}_0$** if $S(M) = \{0, 1\}$,
- of **type $\mathbf{III}_\lambda$** if $S(M) = \{0\} \cup \{\lambda^n \mid n \in \mathbb{Z}\}$,
- of **type $\mathbf{III}_1$** if $S(M) = \mathbb{R}_+$.

In the last case Connes and Størmer proved the following result:

Theorem 2.7 (Connes-Størmer [2])**.** *If M is a type III_1-factor with separable predual, then we have: For every $\epsilon > 0$ and every two normal states ϕ and ψ there exists a unitary $u \in M$ with*

$$\|\phi(u \ \cdot \ u^*) - \psi\| < \epsilon \ .$$

2.4.1. *Type III-factors and algebraic quantum field theory.* Let $\mathcal{A}$ be a net of von Neumann algebras with all $\mathcal{A}(\mathcal{O})$ isomorphic to a type III_1-factor. To any such net there is an associated C^*-algebra $\mathfrak{A}$ defined via

$$\mathfrak{A} = \overline{\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})}^{\|\cdot\|} .$$

We can consider states of the net, by which we mean a continuous linear functional on $\mathfrak{A}$ which is normal on the local algebras $\mathcal{A}(\mathcal{O}) \subset \mathfrak{A}$. Let ω be such a state and let $p \in \mathcal{A}(\mathcal{O})$ be a projection. By our hypothesis, there is a partial isometry $v \in \mathcal{A}(\mathcal{O})$ with $v^*v = 1$ and $vv^* = p$. Now consider

$$\omega_v(a) = \omega(v^*av) ,$$

which is again a state of the net. For $a' \in \mathcal{A}(\mathcal{O})'$ we have

$$\omega_v(a') = \omega(v^*a'v) = \omega(v^*va') = \omega(a') .$$

In particular, this holds for $a' \in \mathcal{A}(\mathcal{O}') \subset \mathcal{A}(\mathcal{O})'$, if $\mathcal{O}'$ is space-like with respect to $\mathcal{O}$. Moreover, $\omega_v(p) = 1$. The lesson to learn from this construction is that – due to type III – we can alter the state ω *locally* inside the region $\mathcal{O}$ without affecting anything that lies space-like to it. In our case ω_v has become an eigenstate for p. By theorem 2.7 we can approximate any state in any region $\mathcal{O}$ by any other state if the local algebras are of type III_1.

2.4.2. *A thought experiment by Fermi.* To see that these approximation properties and the locality of states fail in the case of quantum mechanics, let us look at a thought experiment by Fermi that shows its acausality. Suppose a and b are two atoms separated by a distance R. Initially b is in an excited state and a is in its ground state. We would like to describe the energy transfer from b to a. The Hilbert space for this problem would be

$$\mathbb{H} = \mathbb{H}_a \otimes \mathbb{H}_b \otimes \mathbb{H}_c ,$$

where $\mathbb{H}_a$ is the state space of a, $\mathbb{H}_b$ that of b and $\mathbb{H}_c$ captures all information about the radiation field. The algebra is $B(\mathbb{H})$. The initial state of the system can be described by

$$\omega_0 = \omega_a \otimes \omega_b \otimes \omega_c ,$$

where ω_a is the ground state of a, ω_b the excited state of b and ω_c is the ground state of the radiation field. As we have seen above, we get the state ω_t at time t via

$$\omega_t = \omega_0 \left(e^{itH} \ \cdot \ e^{-itH}\right)$$

for some Hamiltonian operator H. Suppose ω_a is given by a vector $\psi_a \in \mathbb{H}_a$. Let $F_a = \psi_a \langle \psi_a, \cdot \rangle$ be the rank 1 projection onto the subspace generated by ψ_a and set

$$E_a = (1 - F_a) \otimes 1_b \otimes 1_c \in B(\mathbb{H}) .$$

The probability to find a in an excited state at time t is then given by $P(t) := \omega_t(E_a)$. If quantum mechanics were a relativistic theory, $P(t)$ should vanish for $t < \frac{R}{c}$. But

$$t \mapsto E_a e^{itH} \phi_a$$

may be continued to an analytic function in the upper half plane. Thus, if it vanishes on an interval, it vanishes on the full real line! We get $\omega_t(E_a) = 0$ for all $t \in \mathbb{R}$. The only option left is that the change of state occurs instantaneously.

We have seen in the last section that states on nets of von Neumann algebras are much more flexible: In a net one would consider regions $\mathcal{O}_a$,$\mathcal{O}_b$ centered around a and b that are

space-like to each other. The time evolved state ω_t can be the ground state on $\mathcal{A}(\mathcal{O}_a)$ and the excited state of the radiation field in $\mathcal{A}(\mathcal{O}_b)$ without any problems. There still is a projection E_a, but this is not contained in the local algebra $\mathcal{A}(\mathcal{O}_a)$, but in a larger one. For a detailed treatment of this problem see [7, 1].

3. The KMS-condition

Now that we have explained the importance of type III-factors in quantum field theory, the question may arise, how to detect whether a particular algebra is of type III. In this section we will sketch a way to attack this problem via modular theory. We start with a formal calculation: Let M be a von Neumann algebra in $B(\mathbb{H})$. Let $\Omega \in \mathbb{H}$ be a cyclic and separating vector and define $\phi(x) = \langle x\,\Omega, \Omega\rangle$. Denote the modular automorphism group by $\sigma_t^\phi(x) = \Delta^{it}\, x\, \Delta^{-it}$. Suppose we could extend the domain of definition for the modular group from $\mathbb{R}$ to the whole complex plane. Then

$$\begin{aligned}\phi(xy) &= \langle xy\,\Omega, \Omega\rangle = \langle y\,\Omega, x^*\,\Omega\rangle = \langle y\,\Omega, J\Delta^{\frac{1}{2}}\, x\,\Omega\rangle \\ &= \langle y\,\Omega, J\Delta^{-\frac{1}{2}}\,\Delta\, x\,\Delta^{-1}\,\Omega\rangle = \langle \Delta\, x\,\Delta^{-1}\,\Omega, S\, y\,\Omega\rangle \\ &= \langle y\,\Delta\, x\,\Delta^{-1}\,\Omega, \Omega\rangle = \phi(y\,\sigma_{-i}^\phi(x))\end{aligned}$$

If we set $F(t) = \phi(\sigma_t^\phi(x)\, y)$ we have $F(t+i) = \phi(y\,\sigma_t^\phi(x))$ by the above calculation. We turn this property into a definition.

Definition 3.1. Let α_t be a strongly continuous 1-parameter automorphism subgroup of $\mathrm{Aut}(M)$ and let ϕ be a faithful normal state on M. α_t satisfies the KMS condition (KMS stands for Kubo, Martin and Schwinger [5, 6]) for ϕ if

- $\phi(\alpha_t(x)) = \phi(x)$
- for every $x, y \in M$ there exists a continuous function $F\colon \mathbb{C} \to \mathbb{C}$ which is bounded on the strip $\{z \in \mathbb{C} \mid 0 \le \mathrm{Im}(z) \le 1\}$ and analytic in its interior such that for all $t \in \mathbb{R}$:
$$F(t) = \phi(\sigma_t^\phi(x)\, y) \quad , \quad F(t+i) = \phi(y\,\alpha_t(x))\ .$$

The next theorem shows that our naive thinking above is not too far away from the truth and that the KMS condition yields a nice characterization of the modular automorphism group.

Theorem 3.2. *If ϕ is a faithful normal state on a von Neumann algebra M, then σ_t^ϕ is the unique one parameter subgroup satisfying the KMS condition for ϕ.*

Instead of proving this theorem, we give an important corollary.

Corollary 3.3. *Let ϕ and M be as in theorem 3.2. For $a \in M$ the following statements are equivalent:*
a) $\phi(ax) = \phi(xa)$ for all $x \in M$.
b) $\sigma_t^\phi(a) = a$ for all $t \in \mathbb{R}$.

Proof. Let $\Omega \in \mathbb{H}$ be the cyclic and separating vector obtained from the GNS construction of ϕ, i.e. $\phi(x) = \langle x\,\Omega, \Omega\rangle$. To prove that a) implies b) we show first, that $a\,\Omega \in \mathrm{Dom}(S^*)$. This follows from

$$\langle S\, x\,\Omega, a\,\Omega\rangle = \langle x^*\,\Omega, a\,\Omega\rangle = \langle \Omega, x\, a\,\Omega\rangle \overset{a)}{=} \langle \Omega, a\, x\,\Omega\rangle = \langle a^*\,\Omega, x\,\Omega\rangle$$

and the Cauchy-Schwarz inequality. We also get $S^*(a\Omega) = a^*\Omega = S(a\Omega)$ and therefore

$$J\Delta^{-\frac{1}{2}}\, a\,\Omega = J\Delta^{\frac{1}{2}}\, a\,\Omega \quad \Rightarrow \quad \Delta\, a\,\Omega = a\,\Omega\ .$$

But this implies $\sigma_t^\phi(a)\Omega = \Delta^{it}a\Delta^{-it}\Omega = a\Omega$. Since Ω is separating, the claim follows.

To see that b) implies a) observe that

$$\phi(\sigma_t^\phi(x)\cdot a) \overset{b)}{=} \phi(\sigma_t^\phi(xa)) = \phi(xa)\ .$$

Therefore $F(t)$ is constant. By the Schwarz reflection principle $F(t)$ has an analytic continuation into the strip with $-i \leq \mathrm{Im}(z) \leq i$. Since it is constant along the real axis, it has to be constant on the whole strip. But this implies a). □

We are now in the situation to compare the two different modular automorphisms obtained from two different states.

Theorem 3.4. *Let M be a von Neumann algebra and ϕ, ψ two normal faithful states on M. Then there is a strongly continuous map $u\colon \mathbb{R} \to U(M)$ mapping t to u_t such that*

$$\sigma_t^\phi = \mathrm{Ad}_{u_t} \circ \sigma_t^\psi\ .$$

Proof. We will use a matrix trick to see this: Let Φ be the following normal faithful state on $M_2(\mathbb{C}) \otimes M = M_2(M)$

$$\Phi\colon M_2(\mathbb{C}) \otimes M \to \mathbb{C} \quad , \quad (x_{ij})_{ij} \mapsto \frac{1}{2}\left(\phi(x_{11}) + \psi(x_{22})\right)\ .$$

Moreover let

$$p = e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad q = e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\ .$$

Since $\Phi(p\,X) = \Phi(X\,p)$, the modular automorphism σ_t^Φ fixes p by corollary 3.3. Thus, σ_t^Φ acts on $p\,M_2(M)\,p$ and satisfies the KMS condition. By theorem 3.2 we get

$$\sigma_t^\Phi(x \otimes e_{11}) = \sigma_t^\phi(x) \otimes e_{11}\ .$$

and similarly $\sigma_t^\Phi(x \otimes e_{22}) = \sigma_t^\psi(x) \otimes e_{22}$. Now set

$$v_t = \sigma_t^\Phi(1 \otimes e_{21})\ ,$$

which satisfies $v_t v_t^* = \sigma_t^\Phi(q) = q$ and $v_t^* v_t = p$. A small calculation shows that it can only be of the form

$$v_t = \begin{pmatrix} 0 & 0 \\ u_t & 0 \end{pmatrix}$$

for a unitary u_t. The calculation

$$\begin{pmatrix} 0 & 0 \\ 0 & u_t\,\sigma_t^\phi(x)u_t^* \end{pmatrix} = v_t\,\sigma_t^\Phi \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} v_t^* = \sigma_t^\Phi \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_t^\psi(x) \end{pmatrix}$$

implies the claim. □

The characterization of type III-factors alluded to in the beginning is now a corollary.

Corollary 3.5. *Let M be a factor, such that σ_t^ϕ is an outer automorphism for any ϕ and $t \in \mathbb{R}$, then M is of type III.*

Proof. We have to exclude the other cases. In case of type I_n for $n \in \mathbb{N} \cup \{\infty\}$, all automorphisms are inner. In case of type II_1, the trace provides a state with inner modular conjugation. By theorem 3.4, every modular conjugation then acts via inner automorphisms. In case of type II_∞ we can choose a faithful normal state ϕ on $B(\mathbb{H})$ and use $\mathrm{tr} \otimes \phi$ and the fact that the modular conjugation on the whole algebra is the tensor product of the conjugations of the two factors and therefore also inner. □

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