Renormalisation scalar quantum field theory on 4D-Moyal plane

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(joint work with Harald Grosse; Vincent Rivasseau, Fabien Vignes-Tourneret)

Quantum field theories on the Moyal plane characterised by the $\star$-product (in D dimensions)

$$(1) \qquad (a \star b)(x) := \int d^D y \frac{d^D k}{(2\pi)^D} \, a(x + \tfrac{1}{2}\theta \cdot k) \, b(x+y) \, \mathrm{e}^{\mathrm{i}ky} \, , \qquad \theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R}$$

became fashionable after their appearance in string theory [1] and the discovery of the UV/IR-mixing problem [2]. The UV/IR-mixing contains a clear message: If we make the world noncommutative at very short distances, we must at the same time modify the physics at large distances. The required modification is, to the best of our knowledge, unique: It is given by an harmonic oscillator potential for the free field action. In fact, we can prove the following

Theorem 1. *The quantum field theory associated with the action*

$$(2) \quad S = \int d^4 x \Big(\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{\Omega^2}{2} (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{\mu^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \Big)(x) \, ,$$

for $\tilde{x}_\mu := 2(\theta^{-1})_{\mu\nu} x^\nu$, ϕ-real, Euclidean metric, is perturbatively renormalisable to all orders in λ.

There are by now three different proofs of the theorem.

(1) by exact renormalisation group equations in a matrix base of the Moyal algebra [3],
(2) by multi-scale analysis again in a matrix base of the Moyal algebra [4]
(3) by multi-scale analysis directly in position space [5].

First proof [H. Grosse and R. Wulkenhaar] The $\star$-product (1) leads in momentum space to oscillating phase factors which result for some non-planar Feynman graphs in convergent but not absolutely convergent integrals. Our starting point was the conjecture that the selection of a certain order of integration by the renormalisation scheme is at the origin of the UV/IR-mixing problem. Thus, our idea was to invent a regularisation where the model with cut-off is well-defined and no ambiguity in the order of integration appears.

We selected the renormalisation group approach as the right strategy. In order to make use of the simplicity of renormalisation proofs based on exact renormalisation group equations, it was necessary to have amplitudes for vertices and the propagator which are manifestly positive, not oscillating. This lead us to the use of the matrix base of the Moyal algebra where, with respect to that base (given by Laguerre polynomials), the $\star$-product becomes a product of infinite matrices. The price for this achievement was a rather complicated kinetic matrix of the ϕ^4-action. We eventually succeeded in computing the propagator in the matrix base by identifying the eigenvectors of the kinetic matrix as Meixner polynomials. At

the end, the matrix propagator was expressed as a finite sum over hypergeometric functions. See [3].

The adaptation of the renormalisation group equation framework to the action (2) requires two steps:

- Prove bounds for the cut-off propagator.
- Compute the amplitude of a graph as a function of these bounds.

We started with the second step. In [6] we proved a power counting theorem for general dynamical matrix models, characterised by a scaling behaviour of the propagator, in the exact renormalisation group approach. The proof is by induction in the number of vertices and loops. As Feynman graphs for matrix models are ribbon graphs characterised by their topology, i.e. genus and number of holes of the Riemann surface, the difficulty was:

- to guess the power counting theorem in terms of the topology,
- to prove that the scaling behaviour (in terms of the topology) is independent of the history in which a graph of given topology arises from smaller graphs with their topology.

The second step alone goes over 20 pages!

It turned out numerically [3] that the propagator obtained for $\Omega = 0$ in (2) has scaling properties which make the perturbative renormalisation impossible. Of course, this is a manifestation of UV/IR-mixing. With inclusion of the harmonic oscillator piece, also motivated by a duality argument [7], the scaling behaviour together with the power-counting theorem [6] implied that all non-planar graphs and all graphs with more than 4 external legs are irrelevant. See [3].

As a result, only planar graphs with two or four external legs can be relevant or marginal. However, these graphs are labelled by an infinite number of matrix indices. Here, we invented a discrete Taylor expansion in the matrix indices of the external legs which decomposes the (infinite number of) planar two- and four-leg graphs into a linear combination of four relevant or marginal base functions and an irrelevant remainder. The explicit realisation of the propagator in terms of hypergeometric functions was essential in this estimation. These four universal base functions have the same index dependence as the original action in matrix formulation, which implies the renormalisability of the model [3]. A summary of the main ideas and techniques can be found in [8].

We have also computed in [9] the one-loop β-functions of the model which describe the dependence of the bare coupling constant and the bare oscillator frequency on the cut-off matrix size. It turned out that $\frac{\lambda}{\Omega^2}$ remains constant under the renormalisation flow. As $|\Omega|$ is bounded by 1, the running coupling constant can be kept arbitrarily small over all scales for a sufficiently small renormalised coupling constant. This is a sign that a constructive renormalisation of the noncommutative ϕ_4^4-model is possible.

Second Proof [V. Rivasseau, F. Vignes-Tourneret, R. Wulkenhaar] In [3] the asymptotic properties of the propagator are only numerically determined. This shortcoming was cured in [4] where we proved these bounds rigorously, for Ω large

enough. The idea was to use the Schwinger representation of the matrix propagator and to cut it into slices $M^{-i} \leq \alpha \leq M^{-i+1}$. We proved bounds in the matrix indices as a function of the scale index i.

These bounds confirmed the previous numerically estimation, but also gave rise to a different renormalisation proof. For given attribution of scale indices to each propagator, we were able to sum all independent matrix indices of the graph, thus giving the amplitude in terms of the scale attribution. In order to determine the independent matrix summation indices, the $SO(2) \times SO(2)$ symmetry of the model (2) was used, which is most conveniently realised in the dual of the graph. Then, the lines of the dual graph are distinguished into tree lines (chosen according to the scale attribution) and loop lines. The bounds implied that summation over the loop angular momenta do not cost anything so that the power-counting degree of divergence boils down to twice the number of completely inner vertices of the dual graph minus the number of propagators. This is precisely the topological degree of divergence found in [3]!

Third Proof [R. Gurau, J. Magnen, V. Rivasseau and F. Vignes-Tourneret] Renormalisation should be basis independent. In particular, the model (2) should also be renormalisable in position space (or, which by duality [7] is the same, momentum space). This was indeed confirmed in [5]. The advantage of position space is that the propagator is simple: it is given by the Mehler kernel. The price to pay are the oscillating phase factors in the vertex. The problem is elegantly circumvented by first proving that non-orientable graphs (which are always non-planar) are irrelevant even if one bounds the oscillating phases by 1. And for orientable graphs (which can be planar or non-planar) the phases can be globally handled.

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