Quantum field theory on noncommutative geometries

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Constructive renormalisation of quantum field theories was very successful in low dimensions but a complete failure in in four dimensions. The reason is that the only candidate, Yang-Mills theory, is too complicated. Simplifications such as QED or ϕ^4_4-theory would be treatable, but they do not exist due to the Landau ghost problem (resp. triviality). In previous work with Harald Grosse we noticed that if the ϕ^4_4-model is put on a (particular) noncommutative Euclidean space, the β-function is modified so that the model should exist non-perturbatively. There is a realistic chance to prove this.

We consider the quantum field theory defined by the action

$$S = \int d^4x \Big(\frac{1}{2}\phi(-\Delta + \Omega^2\tilde{x}^2 + \mu^2)\phi + \frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(x) \;. \tag{1}$$

Here, $\star$ refers to the Moyal product parametrised by the antisymmetric 4×4-matrix Θ, and $\tilde{x} = 2\Theta^{-1}x$. We have whown in [1] that (1) gives rise to a renormalisable quantum field theory. The action is covariant under the Langmann-Szabo duality transformation and becomes self-dual at $\Omega = 1$. Evaluation of the β-functions for the coupling constants Ω, λ in first order of perturbation theory leads to a coupled dynamical system which indicates a fixed-point at $\Omega = 1$, while λ remains bounded [2, 3]. The vanishing of the β-function at $\Omega = 1$ was next proven in [4] at three-loop order and finally by Disertori, Gurau, Magnen and Rivasseau [5] to all orders of perturbation theory. It implies that there is no infinite renormalisation of λ, and a non-perturbative construction seems possible. The Landau ghost problem is solved. The action (1) also arises by sign-reversal of μ^2 in the spectral action for the harmonic oscillator spectral triple [6, 7].

The action functional (1) is most conveniently expressed in the matrix base of the Moyal algebra [1]. For $\Omega = 1$ it simplifies to

$$S = \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V(\phi) \;, \tag{2}$$

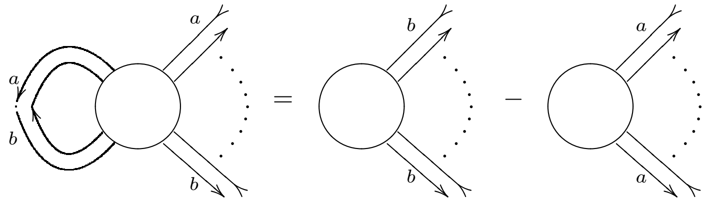
$$H_{mn} = Z\big(\mu^2_{bare} + |m| + |n|\big) \;, \qquad V(\phi) = \frac{Z^2\lambda}{4}\sum_{m,n,k,l\in\mathbb{N}^2_\Lambda} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm} \;, \tag{3}$$

The model only needs wavefunction renormalisation $\phi \mapsto \sqrt{Z}\phi$ and mass renormalisation $\mu_{bare} \to \mu$, but no renormalisation of the coupling constant [5] or of $\Omega = 1$. All summation indices $m, n, \dots$ belong to $\mathbb{N}^2$, with $|m| := m_1 + m_2$, and $\mathbb{N}^2_\Lambda$ refers to a cut-off in the matrix size.

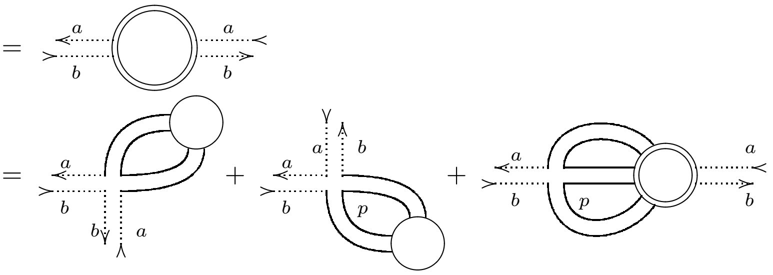
The key step in the proof [5] that the β-function vanishes is the discovery of a Ward identity induced by inner automorphisms $\phi \mapsto U\phi U^\dagger$. Inserting into the connected graphs one special insertion vertex

$$V^{ins}_{ab} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na} \tag{4}$$

is the same as the difference of graphs with external indices b and a, respectively, $Z(|a|-|b|)G^{ins}_{[ab]\dots} = G_{b\dots} - G_{a\dots}$:

(5) $\quad Z(|a|-|b|)$ 

The Schwinger-Dyson equation for the one-particle irreducible two-point function Γ^{ab} reads

(6) $\quad \Gamma_{ab} =$ 

The sum of the last two graphs can be reexpressed in terms of the two-point function with insertion vertex,

$$\begin{aligned}(7)\qquad \Gamma_{ab} &= Z^2\lambda\sum_p \Big(G_{ap} + G^{-1}_{ab}G^{ins}_{[ap]b}\Big) = Z^2\lambda\sum_p \Big(G_{ap} - G^{-1}_{ab}\frac{G_{bp}-G_{ba}}{Z(|p|-|a|)}\Big)\\ &= Z^2\lambda\sum_p\Big(\frac{1}{H_{ap}-\Gamma_{ap}} + \frac{1}{H_{bp}-\Gamma_{bp}} - \frac{1}{H_{bp}-\Gamma_{bp}}\frac{(\Gamma_{bp}-\Gamma_{ab})}{Z(|p|-|a|)}\Big)\;.\end{aligned}$$

This is a closed equation for the two-point function alone. It involves the divergent quantities Γ_{bp} and Z, μ_{bare} in H (3). Introducing the renormalised planar two-point function Γ^{ren}_{ab} by Taylor expansion $\Gamma_{ab} = Z\mu^2_{bare} - \mu^2 + (Z-1)(|a|+|b|) + \Gamma^{ren}_{ab}$, with $\Gamma^{ren}_{00} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$, we obtain a coupled system of equations for Γ^{ren}_{ab}, Z and μ_{bare}. It leads to a closed equation for the renormalised function Γ^{ren}_{ab} alone, which is further analysed in the integral representation.

We replace the indices in $a, b, \dots \mathbb{N}$ by continuous variables in $\mathbb{R}_+$. Equation (7) depends only on the length $|a| = a_1 + a_2$ of indices. The Taylor expansion respects this feature, so that we replace $\sum_{p\in\mathbb{N}^2_\Lambda}$ by $\int_0^\Lambda |p|\,dp$. After a convenient change of variables $|a| =: \mu^2\frac{\alpha}{1-\alpha}$, $|p| =: \mu^2\frac{\rho}{1-\rho}$ and

$$(8)\qquad \Gamma^{ren}_{ab} =: \mu^2\frac{1-\alpha\beta}{(1-\alpha)(1-\beta)}\Big(1 - \frac{1}{G_{\alpha\beta}}\Big)\;,$$

and using an identity resulting from the symmetry $G_{0\alpha} = G_{\alpha 0}$, we arrive at [8]:

Theorem 1. *The renormalised planar connected two-point function $G_{\alpha\beta}$ of self-dual noncommutative ϕ^4_4-theory satisfies the integral equation*

$$G_{\alpha\beta} = 1 + \lambda\Big(\frac{1-\alpha}{1-\alpha\beta}\big(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}\big) + \frac{1-\beta}{1-\alpha\beta}\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}\big) + \frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1\Big)\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0}\big) - \frac{\alpha(1-\beta)}{1-\alpha\beta}\big(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}\big) + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta} - 1)\mathcal{Y}\Big) , \tag{9}$$

where $\alpha, \beta \in [0, 1)$,

$$\mathcal{L}_\alpha := \int_0^1 d\rho\, \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho} , \quad \mathcal{M}_\alpha := \int_0^1 d\rho\, \frac{\alpha\, G_{\alpha\rho}}{1-\alpha\rho} , \quad \mathcal{N}_{\alpha\beta} := \int_0^1 d\rho\, \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho-\alpha} ,$$

and $\mathcal{Y} = \lim_{\alpha\to 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}$.

The non-trivial renormalised four-point function fulfils a linear integral equation with the inhomogeneity determined by the two-point function [8].

These integral equations are the starting point for a perturbative solution. In this way, the renormalised correlation functions are directly obtained, without Feynman graph computation and further renormalisation steps. On the other hand, the implicit function theorem in Banach spaces or the Nash-Moser theorem in Fréchet spaces might be used to prove existence and uniqueness of the solution in a neighbourhood of the free theory.

References

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