Integrability in a 4D QFT model

RAIMAR WULKENHAAR
(joint work with Harald Grosse)

We start from a regularisation of the $\lambda\phi_4^4$-model on noncommutative Moyal space in finite volume [1],

(1)
$$S[\phi] = \frac{1}{64\pi^2}\int d^4x \Big(\frac{Z}{2}\phi\star\big(-\Delta+\Omega^2\|2\Theta^{-1}x\|^2 + \mu_{bare}^2\big)\phi + \frac{\lambda_{bare}Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)\;,$$

where $Z, \lambda_{bare}, \mu_{bare}$ are functions of renormalised values λ, μ and of the regulators $\Omega, \Theta, \mathcal{N}$ encoded in the oscillator potential and the $\star$-product. We expand $\phi(x) = \sum_{\underline{m},\underline{n}\in\mathbb{N}^2} \Phi_{\underline{mn}} f_{m_1n_1}(x^0,x^1) f_{m_2n_2}(x^3,x^4)$ in the matrix basis of the Moyal product

(2)
$$f_{mn}(y^0,y^1) = 2(-1)^m\sqrt{\tfrac{m!}{n!}}\Big(\sqrt{\tfrac{2}{\theta}}y\Big)^{n-m} L_m^{n-m}\Big(\tfrac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}},$$

which satisies $f_{\underline{mn}} \star f_{\underline{kl}} = \delta_{\underline{nk}} f_{\underline{ml}}$ and $\int \frac{dx}{64\pi^2} f_{\underline{mn}}(x) = V\delta_{\underline{mn}}$ with $V := (\frac{\theta}{4})^2$. At the special point $\Omega = 1$ one then obtains a matrix model $S[\Phi] = V\mathrm{Tr}(ZE\Phi^2 + \frac{Z^2\lambda}{4}\Phi^4)$ with $E = (E_{\underline{m}}\delta_{\underline{mn}}) = \frac{\mu_{\rm bare}^2}{2} + \frac{1}{\sqrt{V}}\mathrm{diag}(0,1,1,2,2,2,\dots)$ which admits a natural cut-off $\mathcal{N}$. The resulting partition function $\mathcal{Z}[J] = \int \mathcal{D}\Phi \exp(-S[\Phi] + \mathrm{tr}(J\Phi))$ is merely considered as a device to extract the equations of motions, i.e. Schwinger-Dyson equations. The matrix model structure induces a refinement of N-point functions into partitions $N = N_1 + \cdots + N_B$ and a corresponding expansion

(3)
$$V^{-2}\log\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \sum_{B=1}^{\infty}\sum_{1\le N_1\le\dots\le N_B}^{\infty}\sum_{\underline{p}_1^1,\dots,\underline{p}_{N_B}^B}\frac{G_{|\underline{p}_1^1\dots\underline{p}_{N_1}^1|\dots|\underline{p}_1^B\dots\underline{p}_{N_B}^B|}}{S_{N_1\dots N_B}}\prod_{\beta=1}^{B}\frac{J_{\underline{p}_1^\beta\underline{p}_2^\beta}\cdots J_{\underline{p}_{N_\beta}^\beta\underline{p}_1^\beta}}{VN_\beta}.$$

The Ward identity for the $U(\mathcal{N})$ group action [2] is used to collapse — in a coupled limit $\sqrt{V},\mathcal{N}\to\infty$ with their ratio fixed — the tower of Schwinger-Dyson equations into a self-consistent formula for the 2-point function alone,

(4)
$$G_{|\underline{ab}|} = \frac{1}{Z(E_{\underline{a}}+E_{\underline{b}})} - \frac{Z\lambda}{(E_{\underline{a}}+E_{\underline{b}})}\frac{1}{V}\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}}\Big(G_{|\underline{ab}|}G_{|\underline{ap}|} - \frac{G_{|\underline{pb}|}-G_{|\underline{ab}|}}{Z(E_{\underline{p}}-E_{\underline{a}})}\Big),$$

and a hierarchy of linear equations for all higher correlation functions [3]. These equations are algebraic if one $N_i > 2$, e.g. $G_{|\underline{abcd}|} = (-\lambda)\frac{G_{|\underline{ab}|}G_{|\underline{cd}|}-G_{|\underline{ad}|}G_{|\underline{cb}|}}{(E_{\underline{a}}-E_{\underline{c}})(E_{\underline{b}}-E_{\underline{d}})}$ which proves that the β-function is zero, otherwise (e.g. for $G_{|\underline{ab}|\underline{cd}|}$) complicated but linear.

In a scaling limit $\mathcal{N}, V\to\infty$ with $\frac{\mathcal{N}}{\sqrt{V\mu^4}} = \Lambda$ fixed, sums over $\underline{p}\in\mathbb{N}^2_{\mathcal{N}}$ converge to Riemann integrals of continuous variables $a,b\in[0,\Lambda^2]$, and the finite Hilbert transform $\mathcal{H}_a^\Lambda(f) = \frac{1}{\pi}\mathcal{P}\int_0^{\Lambda^2}\frac{f(p)\,dp}{p-a}$ arises. The limit $\Lambda\to\infty$ requires renormalisation which, because of the vanishing β-function, can be directly implemented in (4). Noticing that the difference $G(a,b)-G(a,0)$ satisfies a linear equation, the solution theory of Carleman-Tricomi gives the renormalised limiting function $G(a,b)$ in terms of the boundary $G(a,0)$:

Theorem 1 ([3, 4]). *Define* $\tau_b(a) := \arctan_{[0,\pi]}\Big(\frac{|\lambda|\pi a}{b+\frac{1+\lambda\pi a\mathcal{H}_a^\Lambda[G(\bullet,0)]}{G(a,0)}}\Big)$. *Then*

$$(5)\qquad G(a,b)=\frac{\sin(\tau_b(a))}{|\lambda|\pi a}\mathrm{e}^{\mathrm{sign}(\lambda)(\mathcal{H}_0[\tau_0(\bullet)]-\mathcal{H}_a[\tau_b(\bullet)])}\begin{cases}1 & \text{for } \lambda<0,\\ \big(1+\frac{Ca+bF(b)}{\Lambda^2-a}\big) & \text{for } \lambda>0.\end{cases}$$

Surprisingly, instantons corresponding to solutions of the homogeneous equation, parametrised by a constant C and a function $F(b)$, live at $\lambda>0$. This reversal is a consequence of renormalisation, to be discussed below. The remaining equation for $G(a,0)$ reduces to symmetry $G(b,0)=G(0,b)$. For $\lambda<0$ one has

$$(6)\qquad G(b,0)=\frac{1}{1+b}\exp\Big(-\lambda\int_0^b dt\int_0^\infty\frac{dp}{(\lambda\pi p)^2+(t+\frac{1+\lambda\pi\mathcal{H}_p[G(\bullet,0)]}{G(p,0)})^2}\Big).$$

A numerical iteration of (6) converges and shows a phase transition at $\lambda_c\approx-0.39$ [4]. For $\lambda>0$ the symmetry $G(a,b)=G(b,a)$ is violated if the instantons are ignored. In [5] we have proved by the Schauder fixed point theorem that a C_0^1-solution $\frac{1}{(1+b)^{1-|\lambda|}}\leq G(0,b)\leq\frac{1}{(1+b)^{1-\frac{|\lambda|}{1-2|\lambda|}}}$ exists for $-\frac{1}{6}\leq\lambda<0$.

Returning to the original formulation (1) in position space, we define connected Schwinger functions on $\mathbb{R}^4$ as

$$(7)\qquad \mu^N S_c(\mu x_1,\dots,\mu x_N) := \lim_{\mathcal{N},V\to\infty}\sum_{\underline{m}_i,\underline{n}_i\in\mathbb{N}^2_{\mathcal{N}}} f_{\underline{m}_1\underline{n}_1}(x_1)\cdots f_{\underline{m}_N\underline{n}_N}(x_N)\frac{(V\mu^4)^{-2}\mu^{4N}\partial^N\log\mathcal{Z}[J]}{\partial J_{\underline{m}_1\underline{n}_1}\dots\partial J_{\underline{m}_N\underline{n}_N}}\Big|_{J=0}.$$

Inserting (3) one gets a partition into $f_{\underline{mn}}$-cycles. Expressing the correlation functions as Laplace-Fourier transform produces $\sum_{m_1,\dots,m_N=0}^\infty\prod_{i=1}^N z_i^{m_i}L_{m_i}^{m_{i+1}-m_i}(r_i)$ which we evaluated in [6]. For the choice of z_i, the $V\to\infty$ limit is $\sim V^0$ for N odd but $\sim V^1$ for N even. Together with the V^{-1}-prefactor in (3) for every B one arrives at:

Theorem 2 ([6]). *Defining* $\mathcal{Y}:=\lim_{b\to0}\frac{(1-G(0,b))}{b}$ *and* $s_\beta:=N_1+\dots+N_{\beta-1}$, *the connected Schwinger functions are given by*

$$(8)\qquad S_c(\mu x_1,\dots,\mu x_N) =\frac{1}{64\pi^2}\sum_{\substack{N_1+\dots+N_B=N\\ N_\beta\text{ even}}}\sum_{\sigma\in\mathcal{S}_N}\Big(\prod_{\beta=1}^B\frac{4^{N_\beta}}{N_\beta}\int_{\mathbb{R}^4}\frac{d^4p_\beta}{4\pi^2\mu^4}\,e^{\mathrm{i}\langle p_\beta,\sum_{i=1}^{N_\beta}(-1)^{i-1}x_{\sigma(s_\beta+i)}\rangle}\Big) \times\frac{1}{S_{N_1\dots N_B}}G\big(\underbrace{\tfrac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})},\cdots,\tfrac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1}\big|\dots\big|\underbrace{\tfrac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})},\cdots,\tfrac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}\big).$$

Only the face-diagonal matrix correlation functions contribute to the Schwinger functions in position space. This can be viewed as confinement of noncommutativity: Whereas interactions involve the complete matrix structure, Schwinger functions depend only on the projection to the diagonal. Euclidean symmetry is manifest. The Schwinger functions show a restricted kinematics where scattering is such that particle momenta are individually conserved, as it is the case in any integrable model.

We have also pointed out in [6] that reflection positivity of the 2-point function amounts to a Stieltjes representation $G(a,a) = \int_0^\infty dt \frac{\rho(t)}{t+a}$ for a positive measure ρ. This is excluded for $\lambda > 0$, whereas we accumulated a lot of evidence that this is the case for $\lambda_c < \lambda \leq 0$. The preferrence of $\lambda < 0$ is a renormalisation effect. The Stieltjes property is related to the anomalous dimension η in $\hat{S}_2(p) \sim \frac{1}{(\|p\|^2+\mu^2)^{1-\eta/2}}$. Naïvely we have $\eta > 0$, in fact $\eta = +\infty$, for $\lambda > 0$. It turns out that the renormalised anomalous dimension is positive for $\lambda < 0$. Consequently, there is no hope to construct a rigorous measure for the partition function, which is why we based our approach on Schwinger-Dyson equations made rigorous.

Our best results so far (not yet published) start from an ansatz $G(0,x) = {}_4F_3\big(\begin{smallmatrix} a,b_1,b_2,b_3 \\ c_1,c_2,c_3 \end{smallmatrix}\big| -x\big)$ with $0 < a < 1$ and $1 < b_i < c_i$, which is a Stieltjes function. Optimising for a, b_i, c_i we came to the conjecture $a = 1 - \frac{1}{\pi}\arcsin(|\lambda|\pi)$ which we were able to prove. Consequently, we expect the critical coupling constant to be exactly $\lambda_c = -\frac{1}{\pi}$. Such a hypergeometric function ansatz solves the fixed point equation (6) up to an error of 10^{-8}. We can plug it into Theorem 1 and notice that $G(\frac{x}{2}, \frac{x}{2})$ is very close, but not exactly equal, to $G(0,x)$. We thus expect that also $G(\frac{x}{2}, \frac{x}{2})$ is Stieltjes, with an intriguing behaviour of the Källén-Lehmann spectral measure ρ: There is a mass gap $[0, \mu^2[$ but *no further gap* $]\mu^2, 4\mu^4[$! Absence of this second gap — remnant of the cured UV/IR-mixing problem — circumvents several triviality theorems.

The exact solution of the model, its restricted kinematics, the vanishing of the β-function and the striking value $\lambda_c = -\frac{1}{\pi}$ of the critical coupling constant all support the conjecture that these results are due to a hidden integrable structure.

References

[1] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base," Commun. Math. Phys. **256** (2005) 305–374 [hep-th/0401128].

[2] M. Disertori, R. Gurau, J. Magnen and V. Rivasseau, "Vanishing of beta function of non commutative ϕ_4^4 theory to all orders," Phys. Lett. B **649** (2007) 95–102 [hep-th/0612251].

[3] H. Grosse and R. Wulkenhaar, "Self-dual noncommutative ϕ^4-theory in four dimensions is a non-perturbatively solvable and non-trivial quantum field theory," Commun. Math. Phys. **329** (2014) 1069–1130 [arXiv:1205.0465 [math-ph]].

[4] H. Grosse and R. Wulkenhaar, "Solvable 4D noncommutative QFT: phase transitions and quest for reflection positivity," arXiv:1406.7755 [hep-th].

[5] H. Grosse and R. Wulkenhaar, "On the fixed point equation of a solvable 4D QFT model," Vietnam J. Math. **44** (2016) 153–180 [arXiv:1505.05161 [math-ph]].

[6] H. Grosse and R. Wulkenhaar, "Solvable limits of a 4D noncommutative QFT," arXiv:1306.2816 [math-ph].