Reflection positivity in large-deformation limits of noncommutative field theories

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(joint work with Harald Grosse, Akifumi Sako)

1. Quantum field theory of matrix models

We investigate the possibility to construct quantum field theories as limits of models defined on some Euclidean noncommutative space. Such models are essentially matrix models with action $S(\Phi) = \mathrm{tr}(E\Phi^2 + \mathrm{pol}(\Phi))$, understood as limit of finite matrices. Here E is a positive selfadjoint operator which defines a dimension $D = \inf\{p > 0 \ : \ \mathrm{tr}((1+E)^{-p/2}) < \infty\}$, and $\mathrm{pol}(\Phi) = \sum_{k=1}^{r} \lambda_k \Phi^k$. The task could be to give a meaning to the limit measure $\frac{1}{\mathcal{Z}} e^{-V\,S(\Phi)} d\Phi$, where $V > 0$ represents the volume.

We do not suppose that the limit can be constructed. Instead we derive (for $\mathcal{N} \times \mathcal{N}$-matrices) equations between moments of the measure, simplify them by further Ward-Takahashi identities [1] resulting from the $U(\mathcal{N})$-group action, take the limit of the equations (which requires renormalisation $\Phi \mapsto \sqrt{Z}\Phi$ and suitable dependence $Z(\mathcal{N}), \lambda_k(\mathcal{N})$ on $\mathcal{N}, D$) and look for exact solutions of these Schwinger-Dyson equations.

This strategy was developed and investigated first for $\mathrm{pol}(\Phi) = \frac{\lambda}{4}\Phi^4$ in $D = 4$ and in a special limit $\mathcal{N}, V \to \infty$, with $\frac{\mathcal{N}}{V^{2/D}} = \Lambda^2$ fixed, followed by $\Lambda \to \infty$ [2]. In examples, this limit corresponds to a large-deformation limit of noncommutative geometries. We proved that this approach collapsed the tower of Schwinger-Dyson equations into a closed non-linear integral equation for the matricial 2-point function and a hierarchy of affine integral equations for all higher correlation functions. In fact higher functions were algebraically expressable in terms of fundamental building blocks, which in particular proved that the β-function in this matricial $\lambda\Phi^4$-model is identically zero (perturbatively proved in [1]). The equation for the 2-point function was reduced to a fixed point problem (for which we proved existence of a solution) for the boundary 2-point function $\mathbb{R}_+ \ni x \mapsto G(x, 0)$.

Recent highlight is the $\lambda\Phi^3$ model in $D = 2$ [3] and $D \in \{4, 6\}$ [4] where renormalisation requires (for $D = 6$) to consider

$$S(\Phi) = \mathrm{tr}\big(ZE\Phi^2 + (\kappa + \nu E + \zeta E^2)\Phi + \tfrac{1}{2}\mu_{bare}^2\Phi^2 + \tfrac{1}{3}\lambda_{bare}\Phi^3\big). \tag{1}$$

BPHZ normalisation conditions were directly implemented in the Schwinger-Dyson equation, leading to exact formulae for $Z, \kappa, \nu, \zeta, \mu_{bare}^2, \lambda_{bare}$ as function of $\mathcal{N}, V$ and the given spectrum of E. It turns out that λ_{bare} is $Z^{\frac{3}{2}}$ times a running coupling constant which corresponds to positive β-function for real λ, λ_{bare}. Nevertheless there is no Landau ghost; the model can be solved up to any scale Λ. After renormalisation we obtain a closed non-linear equation for the 1-point function. This equation is exactly solvable similar to the Makeenko-Semenoff solution [5] of $f^2(x) + \lambda^2 \int_a^b dt\ \rho(t)\frac{f(x)-f(t)}{x-t} = x$ by $f(x) = \sqrt{x+c} + \frac{\lambda^2}{2}\int_a^b \frac{dt\ \rho(t)}{(\sqrt{x+c}+\sqrt{t+c})\sqrt{t+c}}$ (together with a consistency condition on c). We prove: Let the spectrum of

E converge in the considered limit to a positive function $e(x)$, and let $X(x) := (2e(x)+1)^2$. Then for $D=6$ the 1-point function reads in the considered limit

$$
\begin{aligned}
(2)\quad G(x) &= \frac{\sqrt{(X+c)(1+c)}-c-\sqrt{X}}{2\lambda} \\
&\quad + \frac{\lambda}{4}\int_1^\infty \frac{dT\;(e^{-1}(\frac{\sqrt{T}-1}{2}))^2\;(\sqrt{X+c}-\sqrt{1+c})^2}{\sqrt{T}e'(e^{-1}(\frac{\sqrt{T}-1}{2}))(\sqrt{X+c}+\sqrt{T+c})(\sqrt{1+c}+\sqrt{T+c})^2\sqrt{T+c}},
\end{aligned}
$$

with $c(\lambda)$ the implicit solution of $-c = \lambda^2 \int_1^\infty \frac{dT\;(e^{-1}(\frac{\sqrt{T}-1}{2}))^2}{\sqrt{T}e'(e^{-1}(\frac{\sqrt{T}-1}{2}))(\sqrt{1+c}+\sqrt{T+c})^3\sqrt{T+c}}$. We checked that Taylor expansion to $\mathcal{O}(\lambda^3)$ agrees with renormalised Feynman graph computation. See [4].

Higher N-point functions can be viewed as representations of the permutation group. Every permutation is a product of cycles. Collecting permutations of the same cycle lengths $(N_1,\dots,N_B)$ leads to a decomposition of (total) N-point functions into $N_1+\dots+N_B$-point functions $G(x^1_1,\dots,x^1_{N_1}|\dots|x^B_1,\dots,x^B_{N_B})$. It was straightforward [3] to reduce them to $1+\dots+1$-point functions:

$$
\begin{aligned}
(3)\quad & G(x^1_1,\dots,x^1_{N_1}|\dots|x^B_1,\dots,x^B_{N_B}) \\
&= \frac{1}{\lambda^B}\sum_{k_1=1}^{N_1}\cdots\sum_{k_B=1}^{N_B}\underbrace{G(x^1_{k_1}|\dots|x^B_{k_B})}_{(*)}\prod_{\beta=1}^{B}\prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta}\frac{4\lambda}{(2e(x^\beta_{k_\beta})+1)^2-(2e(x^\beta_{l_\beta})+1)^2}\;,
\end{aligned}
$$

where for $B=1$ one should read $e(x^1_{k_1})+\frac{1}{2}+\lambda G(x^1_{k_1})$ instead of $(*)$. Solving the equations for the $1+\dots+1$-functions is a difficult combinatorial problem. Making essential use of Bell polynomials we proved (with $X(x^i)=(2e(x^i)+1)^2$):

$$
\begin{aligned}
G(x^1|x^2) &= \frac{4\lambda^2}{\sqrt{X(x^1)+c}\sqrt{X(x^2)+c}(\sqrt{X(x^1)+c}+\sqrt{X(x^2)+c})^2} \\
G(x^1|\dots|x^B) &= \frac{d^{B-3}}{dt^{B-3}}\Big(\frac{(-2\lambda)^{3B-4}}{(R(t))^{B-2}}\frac{1}{\sqrt{X(x^1)+c-2t}^3}\cdots\frac{1}{\sqrt{X(x^B)+c-2t}^3}\Big)\Big|_{t=0}
\end{aligned}
$$

for $B=2$ and $B\geq 3$, respectively, where $R(t)$ is an explicit integral [3, eq. (4.9)], which depends on $D,\lambda,e(.)$.

2. Schwinger functions and reflection positivity

It was speculated that space-time might be a noncommutative manifold. In its Euclidean formulation, a scalar field would be an element of a noncommutative algebra $\mathcal{A}$ which in many cases is approximated by matrices. A convenient example is the D-dimensional Moyal space, the Rieffel deformation of Schwartz functions by translation, $(f\star g)(\xi)=\int_{\mathbb{R}^{2D}}\frac{d(k,y)}{(2\pi)^D}f(\xi+\frac{1}{2}\Theta k)g(\xi+y)e^{\mathrm{i}\langle k,y}$, where Θ is a skew-symmetric real $D\times D$-matrix. We describe the transition to matrices in $D=2$ with $\Theta=\left(\begin{smallmatrix}0&\theta\\-\theta&0\end{smallmatrix}\right)$. The functions $f_{mn}(z)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\frac{2}{\theta}}z\Big)^{n-m}L^{n-m}_m\Big(\frac{2|z|^2}{\theta}\Big)e^{-\frac{|z|}{\theta}}$, with $z\in\mathbb{C}\equiv\mathbb{R}^2$, satisfy $(f_{mn}\star f_{kl})(z)=\delta_{nk}f_{ml}(z)$ and $\int_{\mathbb{C}}dz\,f_{mn}(z)=2\pi\theta\delta_{mn}$. Therefore, expanding an action functional for a scalar field $\phi=\sum_{\underline{m},\underline{n}}\Phi_{\underline{mn}}f_{\underline{mn}}$

on Moyal space, where $\underline{m} = (m_1, \dots, m_{D/2})$ and $f_{\underline{m}\underline{n}}(\underline{z}) = \prod_{i=1}^{D/2} f_{m_i n_i}(z_i)$, in this matrix basis leads back to the starting point (1) of a matricial QFT model. Connected Schwinger N-point functions on Moyal space can thus be obtained via (4)

$$S_c(\underline{z}_1, \dots, \underline{z}_N) := \lim_{\mathcal{N},V\to\infty} \sum_{\underline{m}_i,\underline{n}_i} f_{\underline{m}_1\underline{n}_1}(\underline{z}_1) \cdots f_{\underline{m}_N\underline{n}_N}(\underline{z}_N) \frac{(-\mathrm{i})^N \partial^N \log \hat{\mathcal{Z}}(J)}{\partial J_{\underline{m}_1\underline{n}_1} \dots \partial J_{\underline{m}_N\underline{n}_N}}\Big|_{J=0},$$

where formally $\hat{\mathcal{Z}}(J) = \int D\Phi \exp(-S(\Phi) + \mathrm{i}V \sum_{\underline{a}\underline{b}} \Phi_{\underline{a}\underline{b}} J_{\underline{a}\underline{b}})$. We proved that only the diagonals $G(x^1, \dots, x^1|\dots|x^B, \dots, x^B)$ of the rigorously constructed matricial correlation functions (3) contribute to the limit.

Inserting the explicit formulae we were able to check reflection positivity [6]. It should be known that reflection positivity implies the following for the momentum space Schwinger functions $\hat{S}$: *the temporal Fourier transform from independent energies p_j^0 to time differences $\tau_j > 0$ is, for all spatial momenta $\vec{p}_j$, a positive definite function on $\mathbb{R}_+^m$.* By the Hausdorff-Bernstein-Widder theorem, (i) positive definiteness is equivalent to (ii) being Laplace transform of a positive measure and to (iii) being a completely monotonic function. The latter property is what we check. We find that the 2-point function of $\lambda\Phi_D^3$ on Moyal space is reflection positive iff $D = 4, 6$ (not $D = 2$!) and $\lambda \in \mathbb{R}$ (where the partition function does not define a measure). The Källén-Lehmann measure was explicitly computed; it consists of a 'broadened mass shell' of width $2\mu^2\sqrt{-c}$ centred at $p^2 = \mu^2$ (with c given after (2), $|\lambda| \leq \lambda_c$ expressed in terms of the Lambert W-function) and a 'scattering part' supported on $p^2 \geq 2\mu^2$. See [4, Thm 6.1+6.2].

In unpublished work we prove that the projection to diagonal matricial correlation functions violates reflection positivity in higher Schwinger N-point functions. Hence, the above limit procedure needs modification. A natural suggestion would be to replace in (4) the pointwise product $f_{\underline{m}_1\underline{n}_1}(\underline{z}_1) \cdots f_{\underline{m}_N\underline{n}_N}(\underline{z}_N)$ by a state $\omega_{\underline{z}_1,\dots,\underline{z}_N}(f_{\underline{m}_1\underline{n}_1} \otimes \cdots \otimes f_{\underline{m}_N\underline{n}_N})$ on $\mathcal{A}^{\otimes N}$. It would be interesting to study whether the choice of state permits enough flexibility to rescue reflection positivity.

References

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