Lambert-W solves the noncommutative Φ^4-model

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This abstract is based on [1] where we give strong evidence for

Conjecture 1. *The non-linear integral equation for a function* $G_\lambda : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$,

$$
\begin{aligned}
(1+a+b)G_\lambda(a,b) = 1 + \lambda \int_0^\infty dp\, \Big(\frac{G_\lambda(p,b)-G_\lambda(a,b)}{p-a} + \frac{G_\lambda(a,b)}{1+p}\Big) \\
+ \lambda \int_0^\infty dq\, \Big(\frac{G_\lambda(a,q)-G_\lambda(a,b)}{q-b} + \frac{G_\lambda(a,b)}{1+q}\Big) \\
- \lambda^2 \int_0^\infty dp \int_0^\infty dq\, \frac{G_\lambda(a,b)G_\lambda(p,q)-G_\lambda(a,q)G_\lambda(p,b)}{(p-a)(q-b)}\;,
\end{aligned}
\tag{1}
$$

is for any real coupling constant $\lambda > -1/(2\log 2) \approx -0.721348$ *solved by*

$$
G_\lambda(a,b) = G_\lambda(b,a) = \frac{(1+a+b)\exp(N_\lambda(a,b))}{\big(b+\lambda W\big(\frac{1}{\lambda}e^{(1+a)/\lambda}\big)\big)\big(a+\lambda W\big(\frac{1}{\lambda}e^{(1+b)/\lambda}\big)\big)}\;, \quad \textit{where} \tag{2}
$$

$$
N_\lambda(a,b) := \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^\infty dt\; \log\Big(1 - \frac{\lambda\log(\frac{1}{2}-\mathrm{i}t)}{b+\frac{1}{2}+\mathrm{i}t}\Big) \frac{d}{dt} \log\Big(1 - \frac{\lambda\log(\frac{1}{2}+\mathrm{i}t)}{a+\frac{1}{2}-\mathrm{i}t}\Big)\;. \tag{3}
$$

Here, W *denotes the Lambert function, more precisely its principal branch* W_0 *for* $\lambda > 0$ *and the other real branch* W_{-1} *for* $-1 < \lambda < 0$ *of the solution of* $W(z)e^{W(z)} = z$*. The function* $N_\lambda(a,b)$ *defined for* $\lambda > -1/(2\log 2)$ *has a perturbative expansion into Nielsen polylogarithms.*

Equation (1) arises from the Dyson-Schwinger equation for the 2-point function of the $\lambda\phi^{\star 4}$-model with harmonic propagation on 2-dimensional noncommutative Moyal space in a special limit where the matrix size and the deformation parameter are simultaneously sent to infinity. We refer to [1, 2] for details and treat here only the solution of (1).

Starting point is the observation that (1) is equivalent to a boundary value problem. Define by

$$
\Psi_\lambda(z,w) := 1 + z + w - \lambda\log(-z) - \lambda\log(-w) + \lambda^2 \int_0^\infty dp \int_0^\infty dq\, \frac{G_\lambda(p,q)}{(p-z)(q-w)}
$$

a function holomorphic on $(\mathbb{C} \setminus [0,\Lambda^2])^2$. Then (1) is equivalent to

$$
\Psi_\lambda(a+\mathrm{i}\epsilon, b+\mathrm{i}\epsilon)\Psi_\lambda(a-\mathrm{i}\epsilon, b-\mathrm{i}\epsilon) = \Psi_\lambda(a+\mathrm{i}\epsilon, b-\mathrm{i}\epsilon)\Psi_\lambda(a-\mathrm{i}\epsilon, b+\mathrm{i}\epsilon)\;. \tag{4}
$$

Therefore, there is a real function $\tau_a(b)$ with

$$
\Psi_\lambda(a+\mathrm{i}\epsilon, b+\mathrm{i}\epsilon)e^{-\mathrm{i}\tau_a(b)} = \Psi_\lambda(a+\mathrm{i}\epsilon, b-\mathrm{i}\epsilon)e^{\mathrm{i}\tau_a(b)}\;. \tag{5}
$$

The Plemelj formulae give (after introducing a common cut-off Λ in the integrals (1)) two equations for the real and imaginary part of (5). Both are Carleman-type singular integral equations for $G_\lambda(a,b)$ and for $\mathcal{G}_\lambda(a,b) := \frac{1}{\lambda\pi} + \mathcal{H}_a[G_\lambda(\bullet,b)]$, where $\mathcal{H}_a[f(\bullet)] := \frac{1}{\pi} \fint_0^{\Lambda^2} dp \, \frac{f(p)}{p-a}$ is the one-sided Hilbert transform (we denote by $\fint$ the Cauchy principal value). The equation for $G_\lambda(a,b)$ is easily solved by

$$G_\lambda(a,b) = \frac{\sin \tau_a(b)}{\lambda\pi} e^{\mathcal{H}_b[\tau_a(\bullet)]} \,. \tag{6}$$

The solution for $\mathcal{G}_\lambda(a,b)$ is the symmetric partner $a \leftrightarrow b$ of the $G_\lambda(a,b)$-equation provided that

$$\begin{aligned} &\lambda\pi \cot \tau_b(a) = 1 + a + b - \lambda \log a + I_\lambda(a) \,, \quad \text{where} \\ &I_\lambda(a) := \frac{1}{\pi} \int_0^\infty dp \left(e^{-\mathcal{H}_p[\tau_a(\bullet)]} \sin \tau_a(p) - \frac{\lambda\pi}{1+p} \right) . \end{aligned} \tag{7}$$

A solution of (7) as formal power series in λ leads surprisingly far. Using the HyperInt package [3] we convinced ourselves that whereas $\mathcal{H}_p[\tau_a(\bullet)]$ recursively evaluates to polylogarithms and more complicated hyperlogarithms, $I_\lambda(a)$ itself remains extremely simple and only contains powers of $\log(1+a)$. The results for $I_\lambda(a)$ are of such striking simplicity and structure that we could obtain an explicit formula. Concretely,

$$\begin{aligned} I_\lambda(a) &= -\lambda \log(1+a) + \sum_{n=1}^\infty \lambda^{n+1} \Big(\frac{(\log(1+a))^n}{n a^n} + \frac{(\log(1+a))^n}{n(1+a)^n} \Big) \\ &+ \sum_{n=1}^\infty \frac{(n-1)!\lambda^{n+1}}{(1+a)^n} \sum_{j=1}^{n-1} \sum_{k=0}^{n} (-1)^j \frac{s_{j,n-k}}{k!j!} \Big(\Big(\frac{1+a}{a}\Big)^{n-j} + 1 \Big) \big(\log(1+a)\big)^k \end{aligned} \tag{8}$$

correctly reproduces the first 10 terms of the expansion in λ. We conjecture that it holds true to all orders. By $s_{n,k}$ we denote the Stirling numbers of the first kind, with sign $(-1)^{n-k}$. Using generating functions of Stirling numbers, (8) is simplified to

$$I_\lambda(a) = \sum_{n=1}^\infty \frac{\lambda^n}{n!} \frac{d^{n-1}}{da^{n-1}} (-\log(1+a))^n - \lambda \sum_{n=1}^\infty \frac{\lambda^n}{n!} \frac{d^{n-1}}{da^{n-1}} \frac{(-\log(1+a))^n}{a} \,. \tag{9}$$

This structure is covered by the *Lagrange inversion theorem* which shows that the first sum in (9) is the inverse $w(\lambda) = \sum_{n=1}^\infty \frac{\lambda^n}{n!} \frac{d^{n-1}}{dw^{n-1}} (\phi(w))^n \big|_{w=0}$ of the function $\lambda(w) = \frac{w}{\phi(w)}$ if we set $\phi(w) = -\log(1+a+w)$. On the other hand, $\lambda(w) = -\frac{w}{\log(1+a+w)}$ is easily inverted to the Lambert-W function which solves $W(z)e^{W(z)} = z$. The second sum in (9) (without the preceeding $-\lambda$), written as $\sum_{n=1}^\infty \frac{\lambda^n}{n!} \frac{d^{n-1}}{dw^{n-1}} \big(H'(w)\phi(w)^n\big)\big|_{w=0}$ for $H(w) = \log(1+w/a)$, is by the *Lagrange-Bürmann formula* equal to $H(w(\lambda))$ for the same $w(\lambda)$ as above. Putting everything together, we we have resummed (9) to

$$I_\lambda(a) = \lambda W\Big(\frac{1}{\lambda} e^{\frac{1+a}{\lambda}}\Big) - \lambda \log\Big(\lambda W\Big(\frac{1}{\lambda} e^{\frac{1+a}{\lambda}}\Big) - 1\Big) - 1 - a + \lambda \log a \,. \tag{10}$$

A closer discussion shows that for $\lambda \geq 0$ the principal branch W_0 of the Lambert function is selected and for $-1 < \lambda < 0$ the other real branch W_{-1}. It can be shown that (10) is analytic at any $\lambda > -1$.

This result gives $\tau_b(a)$ via (7). For $G_\lambda(a,b)$ we need according to (6) the Hilbert transform of that function. A lengthy calculation leads to

$$\mathcal{H}_a[\tau_b(\bullet)] = \log\sqrt{(b+\lambda W(\tfrac{1}{\lambda}e^{(1+a)/\lambda}) - \lambda\log(\lambda W(\tfrac{1}{\lambda}e^{(1+a)/\lambda})-1))^2 + (\lambda\pi)^2}$$
$$+ \log\Big(\frac{(1+a+b)\exp(N_\lambda(a,b))}{(b+\lambda W(\frac{1}{\lambda}e^{(1+a)/\lambda}))(a+\lambda W(\frac{1}{\lambda}e^{(1+b)/\lambda}))}\Big)\,,$$

(11)

$$N_\lambda(a,b) := \frac{1}{2\pi\mathrm{i}}\int_{\gamma_\epsilon} dz \log\Big(1-\frac{\lambda\log(-z)}{1+b+z}\Big)\frac{d}{dw}\log\Big(1-\frac{\lambda\log(1+z+w)}{1+a-(1+z+w)}\Big)\Big|_{w=0}\,,$$

where γ_ϵ is the curve in the complex plane which encircles the positive real axis clockwise at distance ϵ. Equation (11) holds for $\lambda > -1$ and can be rearranged for $\lambda > -\frac{1}{2\log 2}$ into (3). In particular, formula (2) follows.

Further information is obtained from the generating function $R_{\alpha,\beta}(a,b;w)$ defined by $N_\lambda(a,b) = \sum_{m,n=1}^\infty \frac{(-\lambda)^{m+n}}{m!n!}\partial_a^{m-1}\partial_b^{n-1}\partial_\alpha^m\partial_\beta^n\partial_w R_{\alpha,\beta}(a,b;w)\big|_{\alpha=\beta=w=0}$,

$$R_{\alpha,\beta}(a,b;w) = \frac{(1+w)^{\alpha+\beta}}{(1+a+b-w)}\frac{\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)}\Big\{-1$$

(12)

$$+\frac{(1+b)^\beta}{(1+w)^\beta}{}_2F_1\Big(\begin{matrix}-\alpha,\beta\\1-\alpha\end{matrix}\Big|\frac{w-b}{1+w}\Big) + \frac{(1+a)^\alpha}{(1+w)^\alpha}{}_2F_1\Big(\begin{matrix}-\beta,\alpha\\1-\beta\end{matrix}\Big|\frac{w-a}{1+w}\Big)\Big\}\,.$$

The hypergeometric function generates precisely the Nielsen polylogarithms

$${}_2F_1\Big(\begin{matrix}-x,y\\1-x\end{matrix}\Big|z\Big) = 1 - \sum_{n,p\geq 1} S_{n,p}(z)x^n y^p\,,$$

(13)

$$S_{n,p}(z) := \frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 dt\,\frac{(\log(t))^{n-1}(\log(1-zt))^p}{t}\,,$$

and $\frac{\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)} = \exp\big(\sum_{k=2}^\infty((\alpha+\beta)^k-\alpha^k-\beta^k)\frac{\zeta(k)}{k}\big)$ gives rise to Riemann zeta values.

Our result now permits to complete the exact solution of the whole $\lambda\phi^{\star 4}$-model on Moyal space [2]. Moreover, all experience shows that solving a non-linear problem such as (1) by generalised radicals (here $W(z), N_\lambda(a,b)$) can only be expected in presence of a hidden symmetry. We consider it worthwhile to explore the corresponding integrable structure.

References

[1] E. Panzer and R. Wulkenhaar, "Lambert-W solves the noncommutative Φ^4-model," arXiv:1807.02945 [math-ph].

[2] H. Grosse and R. Wulkenhaar, "Self-dual noncommutative ϕ^4-theory in four dimensions is a non-perturbatively solvable and non-trivial quantum field theory," Commun. Math. Phys. **3**29 (2014) 1069–1130 [arXiv:1205.0465 [math-ph]].

[3] E. Panzer, "Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals," Comput. Phys. Commun. **188** (2015) 148–166 [arXiv:1403.3385 [hep-th]].