Introduction to Hopf algebras in renormalization and noncommutative geometry *

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Abstract

We review the appearance of Hopf algebras in the renormalization of quantum field theories and in the study of diffeomorphisms of the frame bundle important for index computations in noncommutative geometry.

1 Introductory remarks

This contribution focuses on two applications discovered during the last two years of the Hopf algebra of rooted trees. They suggest an amazing link between mathematics and physics. There exists an excellent review [1] of these topics, written by the authors of these ideas. In what follows I am going to explain parts of this development which I was able to understand. I hope it can be useful to somebody else.

In mathematics, foliations provide a large class of examples of noncommutative spaces and lead to an index problem for the transverse hypoelliptic operator [2]. The computation of the cocycles in the local index formula turned out to be extremely lengthy even in dimension one. Alain Connes and Henri Moscovici [3] were looking for an organizing principle for that calculation, which they found in the cyclic cohomology of a Hopf algebra $\mathcal{H}_T$ obtained by the action of vector fields on a crossed product of functions by diffeomorphisms.

Concerning physics, Dirk Kreimer [4] discovered that a perturbative quantum field theory carries in a natural way a Hopf algebra structure $\mathcal{H}_R$ given by operations on Feynman graphs. The antipode reproduces precisely the combinatorics of renormalization, i.e. it produces the local counterterms to make the divergent integral corresponding to the Feynman graph finite.

Noticing that both Hopf algebras have formally a very similar structure, Connes and Kreimer gave the precise relation [5] between $\mathcal{H}_T$ and $\mathcal{H}_R$. This is very transparent in the language of rooted trees they used. The commutative Hopf subalgebra $\mathcal{H}^1$ of Connes–Moscovici is (in dimension 1) a Hopf subalgebra of Kreimer's Hopf algebra for a quantum field theory with a single primitively divergent graph.

Recently it was pointed out [6] that the same algebra of rooted trees plays a role in Runge–Kutta methods of numerical analysis.

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2 The Hopf algebra of Connes–Moscovici

In principle, the Hopf algebra of Connes and Moscovici can be understood from classical differential geometry [7]. We give here a somewhat shortened version of the derivation and refer to [8] for more details. We recommend [9] for a useful introduction to Hopf algebras and related topics.

We regard the frame bundle F^+ of a manifold M and in particular the vector fields on F^+. There is a natural notion of vertical vector fields, these are the tangent vectors to curves in F^+ obtained by the right action of the group $Gl^+(n)$ of $n \times n$ matrices with positive determinant. The horizontal vector fields are not canonically given, they are determined once a connection is specified. For our purpose we can work in local coordinates.

Let $\{x^\mu\}_{\mu=1\dots,n}$ be the coordinates of $x \in M$ within a local chart of M and $\{y_i^\mu\}_{\mu,i=1,\dots n}$ be the coordinates of n linearly independent vectors of the tangent space T_xM with respect to the basis ∂_μ. On F^+ there exist the following geometrical objects, written in terms of the local coordinates (x^μ, y_i^μ) of $p \in F^+$:

1) an $\mathbb{R}^n$-valued (soldering) 1-form α with $\alpha^i = (y^{-1})^i_\mu dx^\mu$,

2) a $gl(n)$-valued (connection) 1-form ω with $\omega^i_j = (y^{-1})^i_\mu (dy^\mu_j + \Gamma^\mu_{\alpha\beta}\, y^\alpha_j dx^\beta)$, where $\Gamma^\mu_{\alpha\beta}$ depends only on x^ν ,

3) n^2 vertical vector fields $Y^i_j = y^\mu_j \partial^i_\mu$,

4) n horizontal (with respect to ω) vector fields $X_i = y^\mu_i(\partial_\mu - \Gamma^\nu_{\alpha\mu} y^\alpha_j \partial^j_\nu)$.

A local diffeomorphism ψ of M has a lift $\tilde{\psi} : (x^\mu, y^\mu_i) \mapsto (\psi(x)^\mu, \partial_\nu \psi(x)^\mu y^\nu_i)$ to the frame bundle and induces the following transformations of the previous geometrical objects:

1′) $(\tilde{\psi}^*\alpha)\big|_p = \alpha\big|_p$.

2′) $(\tilde{\psi}^*\omega)\big|_p = (y^{-1})^i_\mu (dy^\mu_j + \tilde{\Gamma}^\mu_{\alpha\beta}\, y^\alpha_j dx^\beta)$ is again a connection form, with $\tilde{\Gamma}^\mu_{\alpha\beta}\big|_x = ((\partial\psi(x))^{-1})^\mu_\gamma\, \Gamma^\gamma_{\delta\epsilon}\big|_{\psi(x)}\, \partial_\alpha \psi(x)^\delta \partial_\beta \psi(x)^\epsilon + ((\partial\psi(x))^{-1})^\mu_\gamma\, \partial_\beta \partial_\alpha \psi(x)^\gamma$,

3′) $(\tilde{\psi}_* Y^j_i)\big|_p = Y^j_i\big|_p$,

4′) $(\tilde{\psi}^{-1}_* X_i)\big|_p = y^\mu_i(\partial_\mu - \tilde{\Gamma}^\nu_{\alpha\mu} y^\alpha_j \partial^j_\nu)$ is horizontal to $\tilde{\psi}^*\omega$.

We refer to [8] for the proof.

Given these tools of classical differential geometry, the new idea is to apply the vector fields X, Y to a crossed product $\mathcal{A} = C^\infty_c(F^+) \rtimes \Gamma$ of the algebra of smooth functions on F^+ with compact support by the action of the pseudogroup Γ of local diffeomorphisms of M. As a set, $\mathcal{A}$ can be regarded as the tensor product of $C^\infty_c(F^+)$ with Γ. It is generated by the monomials

$$fU^*_\psi \,, \qquad f \in C^\infty_c(\mathrm{Dom}(\tilde{\psi})) \,, \quad \psi \in \Gamma \,, \tag{1}$$

where $\tilde{\psi}$ is the diffeomorphism of F^+ obtained as the lift of $\psi \in \Gamma$. As an algebra, the multiplication rule in $\mathcal{A}$ is defined by

$$f_1 U^*_{\psi_1} \; f_2 U^*_{\psi_2} := f_1 (f_2 \circ \tilde{\psi}_1) U^*_{\psi_2 \psi_1} \; . \tag{2}$$

Here, the function $f_1(f_2 \circ \tilde{\psi}_1)$ evaluated at p (in the domain of definition) gives $f_1(p)\, f_2(\tilde{\psi}_1(p))$, i.e. we have a non-local product on the function algebra.

The action of vector fields on $\mathcal{A}$ is defined as the action on the function part. Interesting is the application to the product (2), because the non-locality in the function part leads to a deviation from the Leibniz rule. For V being a vector field on F^+ one computes

$$V(f_1 U^*_{\psi_1} \; f_2 U^*_{\psi_2}) = V(f_1 U^*_{\psi_1}) \; f_2 U^*_{\psi_2} + f_1 U^*_{\psi_1} \; \big(\tilde{\psi}_{1*}(V)\big)\big(f_2 U^*_{\psi_2}\big) \; . \tag{3}$$

Since diffeomorphisms and right group action commute, we get the unchanged Leibniz rule for the vertical vector fields,

$$Y_i^j(ab) = Y_i^j(a)\, b + a\, Y_i^j(b) \;, \qquad a, b \in \mathcal{A} \; . \tag{4}$$

For the horizontal vector fields, however, there will be an additional term $a(\psi_{1*} X_i - X_i)(b)$. Comparing 4), 4′) and 3) above we have $\psi_{1*} X_i - X_i = \tilde{\delta}^k_{ji} Y^j_k$, for some function $\tilde{\delta}^k_{ji}$. Using (2) we commute this function in front of a and obtain

$$X_i(ab) = X_i(a)\, b + a\, X_i(b) + \delta^k_{ji}(a)\, Y^j_k(b) \;, \qquad a, b \in \mathcal{A} \; . \tag{5}$$

The operator δ^k_{ji} on $\mathcal{A}$ is computed to

$$\delta^k_{ji}(f U^*_\psi) = (\tilde{\Gamma}^\nu_{\alpha\mu} - \Gamma^\nu_{\alpha\mu}) y^\alpha_j y^\mu_i (y^{-1})^k_\nu f U^*_\psi \;, \tag{6}$$

where $\tilde{\Gamma}^\nu_{\alpha\mu}$ are the connection coefficients belonging to $\tilde{\psi}^*\omega$. It turns out that δ^k_{ji} is a derivation:

$$\delta^k_{ji}(ab) = \delta^k_{ji}(a)\, b + a\, \delta^k_{ji}(b) \; . \tag{7}$$

These formulae can now be interpreted in the dual sense, for instance $X_i(ab) = \Delta(X_i)\,(a \otimes b)$, which leads to a structure of a coalgebra on the linear space $\mathbb{R}(1, X_i, Y^j_k, \delta^k_{ji})$,

$$\begin{aligned} \Delta(Y^j_k) &= Y^k_j \otimes 1 + 1 \otimes Y^j_k \;, \\ \Delta(X_i) &= X_i \otimes 1 + 1 \otimes X_i + \delta^k_{ji} \otimes Y^j_k \;, \\ \Delta(\delta^k_{ji}) &= \delta^k_{ji} \otimes 1 + 1 \otimes \delta^k_{ji} \;, \\ \Delta(1) &= 1 \otimes 1 \;, \end{aligned} \tag{8}$$

with 1 being the identity on $\mathcal{A}$. Coassociativity $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ is easy to check.

Vector fields form a Lie algebra, so the next step is to ask whether $\mathbb{R}(1, X_i, Y^j_k, \delta^k_{ji})$ close under the Lie bracket. The first commutators are OK,

$$\begin{aligned} [Y^i_j, Y^k_l](f U^*_\psi) &= (\delta^i_l Y^k_j - \delta^k_j Y^i_l)(f U^*_\psi) \;, \\ [Y^k_j, X_i](f U^*_\psi) &= \delta^k_i X_j (f U^*_\psi) \;, \\ [Y^i_j, \delta^k_{lm}](f U^*_\psi) &= (\delta^i_l \delta^k_{jm} + \delta^i_m \delta^k_{lj} - \delta^k_j \delta^i_{lm})(f U^*_\psi) \; . \end{aligned} \tag{9}$$

The next one between horizontal fields

$$[X_i, X_j] = R^k_{lij} Y^l_k + \Theta^k_{ij} X_k \tag{10}$$

leads to new generators, because curvature R and torsion Θ are no structure 'constants'. Therefore, one uses a different strategy and considers instead of $\mathcal{A}$ a *Morita equivalent* algebra $\mathcal{A}'$ based on a *flat* manifold $N = \coprod U_\alpha$ – the disjoint union of the charts U_α of M. Now, there is neither curvature nor torsion, and horizontal vector fields commute. There remain the commutators of X with δ, which lead indeed to new generators of the Lie algebra:

$$\begin{aligned}\delta^k_{ji,\ell_1\dots\ell_n}(fU^*_\psi) &:= [X_{\ell_n}, \dots, [X_{\ell_1}, \delta^k_{ji}]\dots](fU^*_\psi) \\ &= \partial_{\lambda_n}\dots\partial_{\lambda_1}\Big(((\partial\psi(x))^{-1})^\nu_\beta\,\partial_\mu\partial_\alpha\psi(x)^\beta\Big) y^\mu_j y^\alpha_i (y^{-1})^k_\nu\, y^{\lambda_1}_{\ell_1}\cdots y^{\lambda_n}_{\ell_n}\; fU^*_\psi\,.\end{aligned} \tag{11}$$

All these generators $\delta^k_{ji,\ell_1\dots\ell_n}$ commute with each other.

Now having established a Lie algebra, we call $\mathcal{H}$ its enveloping algebra, i.e. the algebra of polynomials in $\{1, X_i, Y^k_j, \delta^k_{ji}, \delta^k_{ji,\ell_1\dots\ell_n\dots}\}$, with the commutation relations inherited from the Lie algebra. With the coproduct Δ on the Lie algebra, $\mathcal{H}$ becomes automatically a bialgebra, where the coproduct is defined via the algebra homomorphism axiom:

$$\Delta(h^1h^2) = \Delta(h^1)\,\Delta(h^2) := \sum h^1_1 h^1_2 \otimes h^2_1 h^2_2\;, \qquad \Delta(h_i) = \sum h^1_i \otimes h^2_i\;, \tag{12}$$

for $h_1, h_2 \in \mathcal{H}$. The counit $\epsilon : \mathcal{H} \to \mathbb{C}$ is defined by

$$\varepsilon(1) = 1_{\mathbb{C}}\;, \qquad \varepsilon(h) = 0 \quad \forall h \neq 1\;. \tag{13}$$

The counit axiom $(\varepsilon \otimes \mathrm{id}) \circ \Delta(h) = (\mathrm{id} \otimes \varepsilon) \circ \Delta(h) = h$ is straightforward to check.

There also exists an antipode on $\mathcal{H}$ which makes it to a Hopf algebra. The antipode is the unique antiautomorphism of $\mathcal{H}$ satisfying

$$\begin{aligned} S(h_1h_2) &= S(h_2)S(h_1)\;, \\ m \circ (S \otimes \mathrm{id}) \circ \Delta(h) &= 1\varepsilon(h) = m \circ (\mathrm{id} \otimes S) \circ \Delta(h)\;, \end{aligned} \tag{14}$$

for $h, h_1, h_2 \in \mathcal{H}$, and where m denotes the multiplication. From the second line and (8) one easily obtains

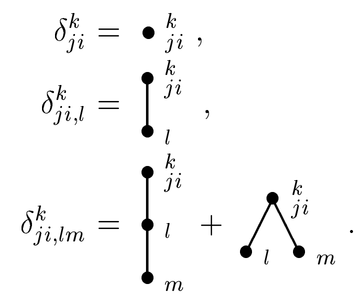
$$\begin{aligned} S(1) &= 1\;, \\ S(Y^j_k) &= -Y^j_k\;, \\ S(\delta^k_{ji}) &= -\delta^k_{ji}\;, \\ S(X_i) &= -X_i + \delta^k_{ji} Y^j_k\;. \end{aligned} \tag{15}$$

The action of S on the other generators of $\mathcal{H}$ can be derived from (14).

The purpose of this Hopf algebra $\mathcal{H}$ is to ease the computation [3] of cocycles in the local index formula [2] of Connes and Moscovici. So far I did not study this calculation for myself, but I think a good way to learn it would be to consult [10].

3 Rooted trees

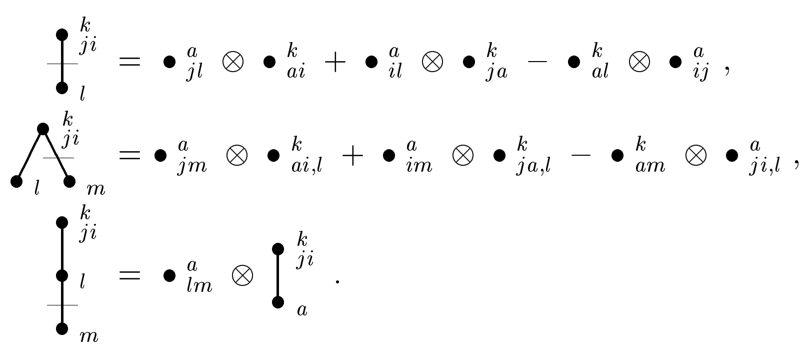
Coproduct and antipode for the generators $\delta^k_{ji,\ell_1\ldots\ell_n\ldots}$ are only recursively defined via the axioms of coproduct and antipode. Now we are going to present an explicit solution – via the concept of rooted trees. This was introduced by Connes and Kreimer [5] to clarify the relation between the two Hopf algebras in the theory of foliations and in perturbative quantum field theory. We generalize [8] their construction from dimension 1 to arbitrary dimension of the manifold M. To the first three classes of δ's we associate the following trees:



(16)

The rule is obvious. A symbol $\delta^k_{ji,A\ell}$, for A a string of $|A|$ indices, is obtained from $\delta^k_{ji,A} = \sum_{a=1}^{|A|!} t_a^{|A|}$ by attaching to each of its trees $t_a^{|A|}$ a new vertex with label ℓ successively to the right of each vertex. The root (with three indices) remains the same and order is important.

Coproduct and antipode require the definition of cuts of a tree. An elementary cut along a chosen edge splits a tree into two – the trees above (trunk) and below (cut branch) the cut. It is clear that we have to add 2 indices to complete the root of the cut branch. This will be a pair of summation indices. We define the action of a cut as the shift of one index of the vertex above the cut to the first position of the new root of the cut branch. The remaining position to complete the root of the cut branch is filled with a summation index and the same summation index is put into the vacant position of the trunk. In the case of cutting immediately below the root, we have to sum over the three possibilities of picking up indices of the root, adding a minus sign if we pick up the unique upper index. The following examples illustrate the definition of a cut, where we write the trunk as the rhs of the tensor product and the cut branch as the lhs:



(17)

A multiple cut consists of several elementary cuts, where *the order of cuts is from top to bottom and from left to right*. An *admissible* cut is a multiple cut such that on the path from any vertex to the root there is at most one elementary cut.

The product of all cut branches forms the lhs of the tensor product, whereas the trunk alone containing the old root serves as the rhs.

The purpose of these definitions is to give an explicit formula for coproduct and antipode. Indeed, by induction one can prove the following:

Proposition 1 *The coproduct of* $\delta^k_{ji,A} = \sum_{a=1}^{|A|!} t_a^{|A|}$ *is given by*

$$\Delta(\delta^k_{ji,A}) = \delta^k_{ji,A} \otimes 1 + 1 \otimes \delta^k_{ji,A} + \sum_{a=1}^{|A|!} \sum_{\mathcal{C}} P^{\mathcal{C}}(t_a^{|A|}) \otimes R^{\mathcal{C}}(t_a^{|A|}) \;, \tag{18}$$

where for each $t_a^{|A|}$ *the sum is over all admissible cuts* $\mathcal{C}$ *of* $t_a^{|A|}$. *In eq. (18),* $R^{\mathcal{C}}(t_a^{|A|})$ *is the trunk and* $P^{\mathcal{C}}(t_a^{|A|})$ *the product of cut branches obtained by cutting* $t_a^{|A|}$ *via the multiple cut* $\mathcal{C}$.

Proof. We start from

$$\begin{aligned}\Delta(\delta^k_{ji,A\ell}) &= [\Delta(\delta^k_{ji,A}), \Delta(X_\ell)] = \delta^k_{ji,A\ell} \otimes 1 + 1 \otimes \delta^k_{ji,A\ell} + R^k_{ji,A\ell} \;,\\ R^k_{ji,A\ell} &= [X_\ell \otimes 1 + 1 \otimes X_\ell, R^k_{ji,A}] + [\delta^m_{n\ell} \otimes Y^n_m, R^k_{ji,A} + (1 \otimes \delta^k_{ji,A})] \in \mathcal{H} \otimes \mathcal{H} \;.\end{aligned}$$

By definition of the tree, the commutator with X_ℓ attaches a vertex ℓ successively to all previous vertices, where $X_\ell \otimes 1$ attaches to the cut branches and $1 \otimes X_\ell$ attaches to the trunk. Next, the commutator with $\delta^m_{n\ell} \otimes Y^n_m$ puts for each vertex of the trunk (due to the commutator with Y) a cut branch consisting of a single vertex to the lhs of the tensor product. Both contributions together yield precisely all admissible cuts of the trees corresponding to $\delta^k_{ji,A\ell}$. □

The antipode is obtained by applying the antipode axiom $m \circ (S \otimes \mathrm{id}) \circ \Delta = 0$ to (18). By recursion one proves

Proposition 2 *The antipode* S *of* $\delta^k_{ji,A} = \sum_{a=1}^{|A|!} t_a^{|A|}$ *is given by*

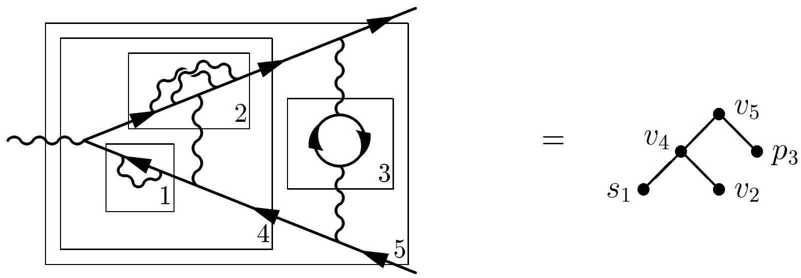
$$S(\delta^k_{ji,A}) = -\delta^k_{ji,A} - \sum_{a=1}^{|A|!} \sum_{\mathcal{C}_a} (-1)^{|\mathcal{C}_a|}\, P^{\mathcal{C}_a}(t_a^{|A|})\, R^{\mathcal{C}_a}(t_a^{|A|}) \;, \tag{19}$$

where the sum is over the set of all non-empty multiple cuts $\mathcal{C}_a$ *of* $t_a^{|A|}$ *(multiple cuts on paths from bottom to the root are allowed) consisting of* $|\mathcal{C}_a|$ *individual cuts.* □

4 Feynman graphs and rooted trees

In a perturbative quantum field theory it is convenient to symbolize contributions to Green's functions by Feynman graphs. These Feynman graphs stand for analytic expressions of momentum variables. Internal momentum variables have to be integrated out. Very often some of these integrations formally yield infinity. The art of obtaining meaningful results out of these integrals is called renormalization. A central problem is the existence of subdivergences which cannot be regularized by a simple subtraction of the divergent part. Bogoliubov [11] found a recursion formula for the regularization of Feynman graphs with subdivergences and Zimmermann gave an explicit solution – the forest formula [12].

In 1997 Dirk Kreimer discovered [4] that there is the structure of a Hopf algebra behind this art of renormalization, with the combinatorics of the forest formula produced by the antipode. Kreimer's idea was to visualize the divergence structure of Feynman graphs in terms of parenthesized words, which are in 1:1 correspondence to rooted trees [5]. Let us exemplify this idea by a Feynman graph from QED:

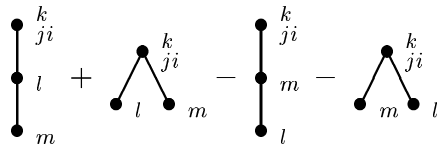


(20)

Straight lines stand for fermions and wavy lines for bosons, and the boxes contain divergent sectors. A criterion for superficial divergence of a region confined in a box is power counting. If a box has n_B bosonic and n_F fermionic outgoing legs, the power counting degree of divergence d is (in four dimensions) defined by $d := 4 - n_B - \frac{3}{2}n_F \geq 0$. Owing to symmetries the actual degree of divergence of one graph or a sum of graphs can be lower than d, see [13]. The construction of the rooted tree from the Feynman graphs with identified divergent sectors is clear: The outermost (superficial) divergence (5) is the root v_5. The box (5) contains the boxes (3) and (4) as immediate subdivergences, hence we connect two vertices p_3 and v_4 directly to the root v_5. The box (4) contains the subdivergences (1) and (2), so we attach the vertices s_1 and v_2 to v_4. This works as long as there are no overlapping divergences, which must be resolved before in terms of disjoint and nested ones and lead to a sum of rooted trees [14, 15].

Having identified the trees to Feynman graphs, it are the same cutting operations on trees as before which give us coproduct and antipode. Here, a cut splits a Feynman graph into several subgraphs – a standard operation in renormalization. It is very remarkable that the antipode obtained in this way reproduces the combinatorics of renormalization [4]. These surprising facts have been extended to a complete renormalization of a toy model [16], which we review in the next section.

Before, let us ask an interesting question: What is the role of the operators $\delta^k_{ji,\ell_1\dots\ell_n}$ in quantum field theory, and what is the meaning of the individual trees for diffeomorphisms? I am not aware of an answer, but there is an interesting observation [8] concerning the relation of the *decorated* rooted trees (16) to Feynman graphs. The trees emerging from the Connes–Moscovici Hopf algebra are decorated by spacetime indices (three for the root) whereas in QFT the decoration is a label for divergent Feynman graphs without subdivergences. Although the operators δ are invariant under permutation of the indices after the comma, for instance $\delta^k_{ji,lm} = \delta^k_{ji,ml}$, see (11), this symmetry is lost on the level of individual trees. That leads us to speculate that *the sum of Feynman graphs according to* *the collection of rooted trees to δ's has more symmetry than the individual Feynman graphs.* This should be checked in QFT calculations. Another interpretation would be the observation from (16)



$$= 0 \;, \qquad (21)$$

which could possibly be regarded as a relation between Feynman graphs similar to those derived in [17]. According to a private communication by Kreimer, (21) is satisfied in QFT for the leading divergences, as it can be derived from sec. V.C in [18]. For non-leading singularities there will be (probably systematic) modifications.

In mathematics, Connes and Kreimer extended the investigation of the commutative Hopf subalgebra $\mathcal{H}^1$ in [3] to the level of individual trees [5]. They showed that the Hopf algebra of rooted trees $\mathcal{H}_R$ is the solution of a universal problem in Hochschild cohomology. We recall [3] that $\mathcal{H}^1$ is the dual of the enveloping algebra of the Lie algebra $\mathcal{L}^1$ of formal vector fields on $\mathbb{R}$ vanishing to order 2 at the origin, and that $\mathcal{H}^1$ itself is isomorphic to the Hopf algebra of coordinates on the group of diffeomorphisms of $\mathbb{R}$ of the form $\psi(x) = x + o(x)$. By analogy, Connes and Kreimer regard $\mathcal{H}_R$ as the Hopf algebra of coordinates on a nilpotent formal group $\mathcal{G}$ whose Lie algebra $\mathcal{L}^1$ they succeed to compute. This group was recently found to be related to the Butcher group in numerical analysis [6]. It will certainly contain precious information for quantum field theory because the antipode in $\mathcal{H}_R$ governing renormalization is the dual of the inversion operation in $\mathcal{G}$. Renormalization seems to provide a new mathematical calculus which generalizes differential calculi.

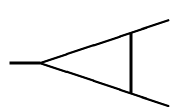
5 A toy model: iterated integrals

In the spirit of Kreimer [16] we are going to give the reader a feeling for renormalization by considering a toy model. The toy model is given by iterated divergent integrals, in close analogy to QFT. The only difference is that the integrals are very simple to compute.

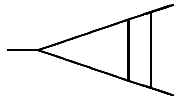
Let us take the integral

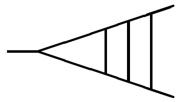
$$\Gamma^1(t) = \int_t^\infty \frac{dp_1}{p_1^{1+\epsilon}} \;, \qquad (22)$$

which diverges logarithmically for $\epsilon \to 0$. We can regard it as the analytic expression to the Feynman graph

$\bullet$ = 

To a Feynman graph with subdivergence there corresponds an iterated integral:

 $=$  $\longleftrightarrow \quad \Gamma^2(t) = \int_t^\infty \frac{dp_1}{p_1^{1+\epsilon}} \int_{p_1}^\infty \frac{dp_2}{p_2^{1+\epsilon}}$,

 $=$  $\longleftrightarrow \quad \Gamma^3(t) = \int_t^\infty \frac{dp_1}{p_1^{1+\epsilon}} \int_{p_1}^\infty \frac{dp_2}{p_2^{1+\epsilon}} \int_{p_2}^\infty \frac{dp_3}{p_3^{1+\epsilon}}$. (23)

Clearly, these iterated integrals form a Hopf algebra of rooted trees without side branches, and the coproduct is given by the admissible cuts of the trees. The renormalization of these integrals requires an algebra homomorphisms ϕ_a on iterated integrals, which represents a certain way of evaluation under "a set of conditions a". For our purpose we take

$$\phi_a\Big(\prod_{i\in I} \Gamma^i(t)\Big) := \prod_{i\in I} \Gamma^i(a) \;, \tag{24}$$

the evaluation of the integrals at $t = a$. In QFT, a should be regarded as an energy scale, and ϕ_a evaluates the Feynman graphs at this scale.

The essential idea [16] is now to consider the *convolution product* of these homomorphisms, defined via the Hopf algebra structure:

$$(\phi\star\psi)(h) := m\circ(\phi\otimes\psi)\circ\Delta(h) \;, \qquad h\in\mathcal{H} \;. \tag{25}$$

The antipode axiom can be written in the compact form $S\star\mathrm{id} = 1\,\varepsilon$. It is however more interesting to consider the following modification:

$$\varepsilon_{a,b} = S_a\star\mathrm{id}_b := (\phi_a\circ S)\star\phi_b \;. \tag{26}$$

Due to the Hopf algebra properties, the $\varepsilon_{a,b}$ satisfy a groupoid law. We give the derivation in full detail, using 1) associativity of m and coassociativity of Δ, 2) the antipode axiom, 3) homomorphism property of ϕ, 4) $\phi\circ 1\epsilon = 1\epsilon$, 5) the counit axiom:

$$\begin{aligned}
\varepsilon_{a,b}\star\varepsilon_{b,c} &= m\circ\Big(\big(m\circ(S_a\otimes\phi_b)\circ\Delta)\big)\otimes\big(m\circ(S_b\otimes\phi_c)\circ\Delta)\big)\Big)\circ\Delta \\
&= m\circ(m\otimes m)\circ\Big(S_a\otimes\phi_b\otimes S_b\otimes\phi_c\Big)\circ(\Delta\otimes\Delta)\circ\Delta \\
&= m\circ(\mathrm{id}\otimes m)\circ(m\otimes\mathrm{id}\otimes\mathrm{id})\circ\Big(S_a\otimes\phi_b\otimes S_b\otimes\phi_c\Big)\circ \\
&\qquad\qquad \circ(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta)\circ\Delta \\
&=^1 m\circ(m\otimes\mathrm{id})\circ(m\otimes\mathrm{id}\otimes\mathrm{id})\circ\Big(S_a\otimes\phi_b\otimes S_b\otimes\phi_c\Big)\circ \\
&\qquad\qquad \circ(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\circ(\Delta\otimes\mathrm{id})\circ\Delta \\
&= m\circ(\Big(m\circ(m\otimes\mathrm{id})\circ(S_a\otimes\phi_b\otimes S_b)\circ(\Delta\otimes\mathrm{id})\circ\Delta\Big)\otimes\phi_c)\circ\Delta \\
&=^1 m\circ(\Big(m\circ(\mathrm{id}\otimes m)\circ(S_a\otimes\phi_b\otimes S_b)\circ(\mathrm{id}\otimes\Delta)\circ\Delta\Big)\otimes\phi_c)\circ\Delta
\end{aligned}$$

$$
\begin{aligned}
&= m \circ \Big(\Big(m \circ \{S_a \otimes \big(m \circ (\phi_b \otimes \phi_b) \circ (\mathrm{id} \otimes S) \circ \Delta\big)\} \circ \Delta\Big) \otimes \phi_c\Big) \circ \Delta \\
&=^{2,3} m \circ \Big(\Big(m \circ \{S_a \otimes \big(\phi_b \circ 1\epsilon\big)\} \circ \Delta\Big) \otimes \phi_c\Big) \circ \Delta \\
&=^{4} m \circ \Big(\Big(m \circ (S_a \otimes \mathrm{id}) \circ (\mathrm{id} \otimes 1\epsilon) \circ \Delta\Big) \otimes \phi_c\Big) \circ \Delta \\
&= m \circ (m \otimes \mathrm{id}) \circ (S_a \otimes \mathrm{id} \otimes \phi_c) \circ (\mathrm{id} \otimes 1\epsilon \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ \Delta \\
&=^{1,4} m \circ (\mathrm{id} \otimes m) \circ (S_a \otimes \phi_c \otimes \phi_c) \circ (\mathrm{id} \otimes 1\epsilon \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \circ \Delta \\
&=^{3} m \circ (S_a \otimes \phi_c) \circ \big(\mathrm{id} \otimes \big(m \circ (1\epsilon \otimes \mathrm{id}) \circ \Delta\big)\big) \circ \Delta \\
&=^{5} m \circ (S_a \otimes \phi_c) \circ \Delta \\
&= \varepsilon_{a,c} \ .
\end{aligned}
$$

We apply now the $\varepsilon_{a,b}$ operation to the divergent integrals to compute $\varepsilon_{a,b}(\Gamma^i(t)) = \Gamma^i_{a,b}$:

$$
\begin{aligned}
\Gamma^1_{a,b} &= m \circ (\phi_a \otimes \phi_b) \circ (S \otimes \mathrm{id}) \circ \Delta(\bullet) \\
&= m \circ (\phi_a \otimes \phi_b) \circ \big(-\bullet \otimes 1 + 1 \otimes \bullet\big) \\
&= -\Gamma^1(a) + \Gamma^1(b) = \int_b^a \frac{dp}{p^{1+\epsilon}} \ .
\end{aligned}
$$

The result $\Gamma^1_{a,b}$ is finite for $\epsilon \to 0$ and vanishes for $a = b$. We proceed with the next integral, using the definition of Δ as given by the admissible cuts and S as given by all cuts (with sign from the number of elementary cuts) of the graphs:

$$
\begin{aligned}
\Gamma^2_{a,b} &= m \circ (\phi_a \otimes \phi_b) \circ (S \otimes \mathrm{id}) \circ \Delta\Big(\overset{\bullet}{\underset{\bullet}{|}}\Big) \\
&= m \circ (\phi_a \otimes \phi_b) \circ \Big(S\Big(\overset{\bullet}{\underset{\bullet}{|}}\Big) \otimes 1 + S(\bullet) \otimes \bullet + 1 \otimes \overset{\bullet}{\underset{\bullet}{|}}\Big) \\
&= m \circ (\phi_a \otimes \phi_b) \circ \Big(-\overset{\bullet}{\underset{\bullet}{|}} \otimes 1 + \bullet\,\bullet \otimes 1 - \bullet \otimes \bullet + 1 \otimes \overset{\bullet}{\underset{\bullet}{|}}\Big) \\
&= -\Gamma^2(a) + \Gamma^1(a)\Gamma^1(a) - \Gamma^1(b)\Gamma^1(a) + \Gamma^2(b) \\
&= \Big(-\int_a^\infty \int_{p_1}^\infty + \int_a^\infty \int_a^\infty - \int_b^\infty \int_a^\infty + \int_b^\infty \int_{p_1}^\infty\Big) \frac{dp_1}{p_1^{1+\epsilon}} \frac{dp_2}{p_2^{1+\epsilon}} \\
&= \int_b^a \frac{dp_1}{p_1^{1+\epsilon}} \int_{p_1}^a \frac{dp_2}{p_2^{1+\epsilon}} \ .
\end{aligned}
$$

Again, the result is finite. Note that in $\bullet \otimes \bullet$ the root which stands for the p_1 integration is the right vertex and hence is evaluated at $t = b$. The computation for $\Gamma^3_{a,b}$ is left as an exercise.

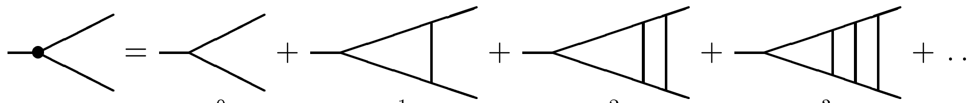
From the identity $\varepsilon_{a,b} \star \varepsilon_{b,c} = \varepsilon_{a,c}$ and the coproduct rule given by admissible cuts of a tree without side branches we get Chen's Lemma [19]:

$$
\Gamma^i_{a,c} = \Gamma^i_{a,b} + \Gamma^i_{b,c} + \sum_{j=1}^{i-1} \Gamma^j_{a,b} \Gamma^{i-j}_{b,c} \ . \tag{27}
$$

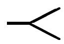
For $i = 2$ it reads

$$\int_c^a \frac{dp_1}{p_1} \int_{p_1}^a \frac{dp_2}{p_2} = \int_b^a \frac{dp_1}{p_1} \int_{p_1}^a \frac{dp_2}{p_2} + \int_c^b \frac{dp_1}{p_1} \int_{p_1}^b \frac{dp_2}{p_2} + \int_c^b \frac{dp_1}{p_1} \int_b^a \frac{dp_2}{p_2} \;.$$

The purpose of these considerations was the renormalization of a QFT. Let us assume a theory where all contributions to the coupling constant come from the following ladder diagrams:



$$\Gamma \quad = \quad \Gamma^0 \quad + \quad \Gamma^1 \quad + \quad \Gamma^2 \quad + \quad \Gamma^3 \quad + \ldots$$

Formally, this series evaluates to infinity, but this infinity can be renormalized to a finite but *undetermined* value. That value has to be adapted to experiment and yields a *normalization condition*. At some energy scale a we are allowed to fix the coupling constant $\Gamma_a = \Gamma^0(a)$. But suppose we measure now the value of the coupling constant at another energy scale b. The normalization condition is fixed so that in the diagrams we have to use in all vertices the normalized coupling constant  $= \Gamma_a$. Since the renormalization removing the infinities was scale dependent, the loop diagrams Γ^i now give a contribution, and this contribution is precisely $\Gamma^i_{a,b}$. Hence,

$$\Gamma_b = \Gamma_a + \Gamma^1_{a,b} + \Gamma^2_{a,b} + \Gamma^3_{a,b} + \ldots \tag{28}$$

Assuming the series converges, we get a finite shift of the coupling constant. In realistic quantum field theories, the agreement of this value with experiment is overwhelming. In particular, in first order we recover the familiar logarithmic energy dependence of the coupling constant. We also learn from (28) that one can completely avoid talking about infinities.

As it is clear from our model, the running coupling constants resulting from renormalization are governed by the Hopf algebra structure together with the convolution product. The Hopf algebra structure not only produces the combinatorics of the forest formula, it also allows to compare different renormalization schemes, which arise from each other by a finite re-normalization. The theory is consistent without a preferred scale or preferred renormalization scheme. They are always related by the convolution identity $\varepsilon_{a,c} = \varepsilon_{ab} \star \varepsilon_{bc}$, where a, b, c stand for parameterizations of different renormalization schemes. Applications of these ideas to QFT calculations are starting [18].

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References

[1] A. Connes & D. Kreimer, *Lessons from quantum field theory*, Lett. Math. Phys. **48** (1999) 85–96, hep-th/9904044.

[2] A. Connes & H. Moscovici, *The local index formula in noncommutative geometry*, GAFA **5** (1995) 174–243.

[3] A. Connes & H. Moscovici, *Hopf algebras, cyclic cohomology and the transverse index theory*, Commun. Math. Phys. **198** (1998) 198–246, math.DG/9806109.

[4] D. Kreimer, *On the Hopf algebra structure of perturbative quantum field theories*, Adv. Theor. Math. Phys. **2.2** (1998) 303–334, q-alg/9707029.

[5] A. Connes & D. Kreimer, *Hopf algebras, renormalization and noncommutative geometry*, Commun. Math. Phys. **199** (1998) 203–242, hep-th/9808042.

[6] C. Brouder, *Runge-Kutta methods and renormalization*, hep-th/9904014

[7] Y. Choquet-Bruhat, C. De Witt-Morette & M. Dillard-Bleick, *Analysis, Manifolds and Physics*, North Holland, Amsterdam (1982).

[8] R. Wulkenhaar, *On the Connes–Moscovici Hopf algebra associated to the diffeomorphisms of a manifold*, math-ph/9904009.

[9] S. Majid, *Quasitriangular Hopf algebras and Yang–Baxter equations*, Int. J. Mod. Phys. A **5** (1990) 1–91.

[10] M. Crainic, *Cyclic cohomology of Hopf algebras, and non-commutative Chern–Weil theory*, math.QA/9812113.

[11] N. N. Bogoliubov & D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, J. Wiley (1980).

[12] W. Zimmermann, *Convergence of Bogoliubov's method of renormalization in momentum space*, Commun. Math. Phys. **15** (1969) 208–234.

[13] C. Itzykson and J. B. Zuber, *Quantum Field Theory*, McGraw-Hill (1980).

[14] T. Krajewski & R. Wulkenhaar, *On Kreimer's Hopf algebra structure of Feynman graphs*, Eur. Phys. J. C **7** (1999) 697–708, hep-th/9805098.

[15] D. Kreimer, *On overlapping divergences*, Commun. Math. Phys. **204** (1999) 669–689, hep-th/9810022.

[16] D. Kreimer, *Chen's iterated integral represents the operator product expansion*, Adv. Theor. Math. Phys. **3.3**, hep-th/9901099.

[17] D. Kreimer, *Weight systems from Feynman diagrams*, J. Knot Theor. Ramifications **7** (1998) 61–85, hep-th/9612011.

[18] D. Kreimer & R. Delbourgo, *Using the Hopf algebra structure of QFT in calculations*, Phys. Rev. D **60** (1999) 105025, hep-th/9903249.

[19] K.-T. Chen, *Iterated path integrals*, Bull. Amer. Math. Soc. **83** (1977) 831–879.