Quantum field theory on noncommutative spaces

Raimar Wulkenhaar

Abstract This survey tries to give a rigorous definition of Euclidean quantum field theory on a fairly large class of noncommutative geometries, namely nuclear AF Fréchet algebras. After a review of historical developments and current trends we describe in detail the construction of the Φ^3-model and explain its relation to the Kontsevich model. We review the current status of the construction of the Φ^4-model and present in an outlook a possible definition of Schwinger functions for which the Osterwalder-Schrader axioms can be formulated.

1 Introduction

1.1 Quantum field theory and gravity

History of sciences culminates in the discovery that the enormous complexity of structures observed on earth and in its nearby part[1] of the universe derives, through a hierarchy of models, from a tiny set of rules that we call the standard model of particle physics coupled to Einstein gravity. This standard model is described elsewhere in this collection of surveys. Here we stress that it has to be built in two stages: The first stage is classical field theory, which has an elegant mathematical formulation in terms of (traditional or noncommutative) differential geometry. The dynamics is governed by field equations which can be derived from an action functional. The second stage is quantisation, i.e. the implementation of field equations between operators on Hilbert space which satisfy natural axioms. *This is not yet achieved.*

Raimar Wulkenhaar
Mathematisches Institut der Westfälischen Wilhelms-Universität
Einsteinstraße 62, 48149 Münster, Germany
e-mail: raimar@math.uni-muenster.de

[1] The large-scale dynamics of the universe seems to require ‘dark matter’ which is not at all understood.

It is true that remarkable approximations have been established, for instance the agreement to 11 significant digits between quantum field theoretical prediction and measurement of the magnetic moment of the electron (classical field theory agrees to one digit with measurement).

The problem is twofold. There is first the challenge to construct an interacting quantum field theory on four-dimensional Euclidean or Minkowskian space, a difficult mathematical problem. Several approaches seem possible; we give more details in Sec. 2.

However, even if one of these programmes succeeds, there remains a profound physical problem: gravity has to be taken into account. We should distinguish at least four levels:

1. Gravity is ignored, the universe is flat Minkowski space. See above.
2. Quantum field theory in an external classical gravitational background field, i.e. on a curved Lorentz manifold, but without any back-reaction of the quantum field theoretical objects to the manifold. The local quantum field theory approach which goes back to Haag-Kastler can cope with this generality. The free quantum field on a general Lorentz manifold is under control.
3. Gravity is still described by classical Einstein theory in which quantum field theoretical objects influence the metric via the stress-energy tensor. Discussed below.
4. True quantum gravity. A driving force in contemporary mathematical physics, with many ideas on the market. Work on these programmes has produced spectacular mathematical results, and will continue to do so. But a solution is not in sight. We will not discuss it in this survey.

This survey is about a conceptional problem which arises already in level 3. Quantum fields are operator-valued distributions, smeared over the support of a test function (see Sec. 2). How "small" can we make the support? According to Heisenberg's principle, the extension Δx of support in position space and the extension Δp of support in momentum space are correlated by $\Delta x \Delta p \geq \hbar/2$ (where $\hbar$ is Planck's constant). We can sharply localise Δx at expense of undetermined momentum. In a certain sense, it is this momentum uncertainty which manifests itself in the experimentally confirmed quantum corrections. However, all this breaks down in dynamical gravity. A momentum uncertainty Δp comes with an uncertainty in the stress-energy tensor which, by Einstein's field equation, creates an uncertainty in the metric tensor. For a rough estimate we translate $\Delta p = c\Delta m$ in a mass uncertainty (c is the speed of light) which induces an uncertainty $\Delta x_s = \frac{2G\Delta m}{c^2}$ of the Schwarzschild radius (where G is Newton's constant). Its influence on the original geometry in which we localised the support of our quantum field to Δx can only be ignored as long as

$$\Delta x \gg \Delta x_s = \frac{2G}{c^3}\Delta p > \frac{G\hbar}{c^3}\frac{1}{\Delta x} .$$

In other words, we cannot localise the support of quantum fields better than the Planck length $\ell_P = \sqrt{\frac{G\hbar}{c^3}}$ if (classical!) gravity is taken into account.

These restrictions on the localisability of quantum fields must be incorporated into quantum field theory itself. This is what quantum field theories on noncommutative geometries try to do.

1.2 Noncommutativity

We know from quantum mechanics that any measurement uncertainty (enforced by principles of Nature and not due to lack of experimental skills) goes hand in hand with noncommutativity. To the best of my knowledge, the possibility that geometry looses its meaning in quantum physics was first[2] considered by Schrödinger [Sch34]. On the other hand, Heisenberg suggested to use coordinate uncertainty relations to ameliorate the short-distance singularities in the quantum theory of fields. His idea (which appeared later [Hei38]) inspired Peierls in the treatment of electrons in a strong external magnetic field [Pei33]. Via Pauli and Oppenheimer the idea came to Snyder who was the first to write down uncertainty relations between coordinates [Sny47]. The mutual interaction of quantum-mechanical and gravitational disturbances was first discussed by Wheeler [Whe55] in his model of 'geons'.

The uncertainty relations for coordinates were revived by Doplicher, Fredenhagen and Roberts [DFR95] as a means to avoid gravitational collapse when localising events with extreme precision. According to [DFR95], the coordinate uncertainties Δx^μ have to satisfy $\Delta x^0(\Delta x^1+\Delta x^2+\Delta x^3) \geq \ell_P^2$ and $\Delta x^1\Delta x^2+\Delta x^2\Delta x^3+\Delta x^3\Delta x^1 \geq \ell_P^2$. These uncertainty relations are induced by noncommutative coordinate operators $\hat{x}^\mu = (\hat{x}^\mu)^*$ satisfying $[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\hat{\Theta}^{\mu\nu}$, where $\hat{\Theta}^{\mu\nu}$ are the components of a 2-form $\hat{\Theta}$ which is central and normalised to $\langle\hat{\Theta},\hat{\Theta}\rangle = 0$ and $\langle\hat{\Theta},*\hat{\Theta}\rangle = 8\ell_P^4$. Moreover, in [DFR95] first steps are taken towards a perturbative quantum field theory on the resulting (Minkowskian) quantum space-time.

The previous discussion suggests that space itself, and not only the phase space of quantum mechanics, should be noncommutative. The corresponding mathematical framework of *noncommutative geometry* [Con94] has been developed. It is today an integral part of mathematics. Noncommutative geometry is the reformulation of geometry and topology in an algebraic and functional-analytic language, thereby permitting an enormous generalisation. Many of its facets are presented in this collection of surveys.

1.3 Structure of the survey

The main structures and techniques in noncommutative geometry are already presented in other surveys so that we will not repeat them. We therefore start in Sec. 2

[2] Actually, Riemann himself speculated in his famous Habilitationsvortrag [Rie92] about the possibility that the hypotheses of geometry lose their validity in the infinitesimal small.

with an informal introduction into basic concepts of quantum field theory (QFT) in its traditional sense, i.e. on ordinary Minkowskian or Euclidean space. After a sketch of the Wightman axioms we give a few details of the Euclidean formulation of QFT. In this framework we describe the crucial concept of renormalisation. For completeness we also sketch Feynman graphs, but our philosophy is to avoid them.

Section 3 generalises the Euclidean formulation to a class of noncommutative geometries. We argue that — to implement renormalisation — this should be the class of nuclear AF Fréchet algebras. They are the analogue of AF C^*-algebras, but with closure in a locally-convex topology rather than in norm topology. Our construction heavily uses two classical theorems: The Bochner-Minlos theorem 2.3 provides us with a measure on the space of Euclidean quantum fields. The Kōmura-Kōmura theorem 3.3 achieves the coexistence of QFT on a discrete noncommutative algebra with an apparently continuous universe.

We describe in Sec. 4 a couple of examples of such noncommutative geometries and also mention a few other popular geometries which are not of this class. One should retain from these discussions that we work with sequences of matrix algebras. As such, QFT on noncommutative geometry is closely related to matrix models. After Secs. 3+4 the reader may directly jump to Sec. 7 where we introduce the main techniques and a prominent example in a class of matrix models we have in mind. In between we try to give in Sec. 5 a review of more physically oriented work on noncommutative quantum field theories. In Sec. 6 we give more details on a particular direction which branched on one hand into the axiomatic formulation of Sec. 3 and on the other hand into a novel approach to quantum gravity. Both Secs. 5+6 are nearly independent from the others and could be skipped or exclusively read.

After a preparation on matrix models in Sec. 7, we present in Secs. 8+9 two Euclidean QFT models on noncommutative geometries for which our programme succeeded completely (Sec. 8) or should be successfully finished within a few years (Sec. 9). In a more speculative Sec. 10 we outline how a QFT on noncommutative geometry could possibly be projected onto a true QFT on Minkowski space. That section should be read with caution; the sketched ideas might go into a void direction.

1.4 Disclaimer

This survey has severe limitations. They are partly unavoidable because several lines of research were developed in parallel, whereas only a sequential presentation can be given. More severely, since an enormous amount of publications has been produced, we can only review a tiny fraction. Our apologies go to everyone who is not adequately acknowledged.

2 Quantum field theory

2.1 Axiomatic and algebraic quantum field theory

It is fair to say that noncommutative geometry, operator algebras and quantum field theory have a common root in von Neumann's axiomatic characterisation of quantum mechanics [vN32]. Quantum field theory (QFT) is an extension of quantum mechanics to infinitely many degrees of freedom which at the same time takes care of interactions between particles mediated by quantised fields. In fact the distinction between particles and fields is abandoned in favour of a unifying framework. The first spectacular confirmation of the new concept was Bethe's explanation [Bet47] of the Lamb shift.

In the early 1950's, Gårding and Wightman gave a rigorous mathematical foundation to quantum field theory by casting the unquestionable physical principles (locality, covariance, stability, unitarity) into a set of axioms. These ideas were published years later [Wig56, WG64, SW64]. There are several possibilities to group the data, but essentially one wants for a single scalar field:

Definition 2.1 A *scalar quantum field* φ on D-dimensional Minkowski space $\mathbb{R}^{1,D-1}$ is an unbounded operator-valued distribution, i.e. $\varphi(f), \varphi^*(f) : \mathcal{D} \to \mathcal{D}$ are linear on a dense subspace $\mathcal{D} \subset \mathcal{H}$ of a Hilbert space, for any test function $f \in \mathcal{S}(\mathbb{R}^{1,D-1})$. Moreover:

1. *Covariance.* There is a representation of the Poincaré group $\mathcal{P}_+^\uparrow \ni (t, R)$ by unitaries $U(t, R)$ in $\mathcal{H}$, which preserves $\mathcal{D}$ and satisfies $U(t, R)\varphi(f)U(t, R)^{-1} = \varphi(f_{t,R})$ with $f_{t,R}(x) := f(R^{-1}(x - t))$.
2. *Spectrum condition.* The joint spectrum of the generators of the translation subgroup of $\mathcal{P}_+^\uparrow$ lies in the forward lightcone $V_+ = \{(p_0, \mathbf{p}) \in \mathbb{R}^{1,D-1} \;:\; p_0 \geq 0,\; p_0^2 \geq \|\mathbf{p}\|^2\}$.
3. *Locality.* For f, g causally independent, $[\varphi(f), \varphi(g)] = 0$.

It is convenient to require that the subspace of $\mathcal{P}_+^\uparrow$-invariant vectors in $\mathcal{D}$ is one-dimensional, and that for a $\mathcal{P}_+^\uparrow$-invariant unit vector Ω (the vacuum), the generated subspace span(polynomials($\varphi(f_i), \varphi(f_j)^*)\Omega$) is dense in $\mathcal{H}$.

From these data one builds the *Wightman functions*, i.e. vacuum expectation values of field operators

$$(\mathcal{S}(\mathbb{R}^{1,D-1}))^N \ni (f_1, \dots, f_N) \mapsto W(f_1, \dots, f_N) := \langle \Omega, \Phi(f_1) \cdots \Phi(f_N)\Omega \rangle \,. \tag{2.1}$$

They are also called N-point functions or correlation functions. The Wightman axioms induce covariance, locality, positivity, spectrum and cluster properties for the Wightman functions. Conversely, Wightman's reconstruction theorem allows to reconstruct the data of Definition 2.1 from Wightman functions with these properties. Some fundamental theorems such as PCT theorem and spin-statistics theorem can be proved in this framework. See [SW64]. The Wightman theory is the basis for a rigorous theory of scattering processes (Haag-Ruelle theory [Haa58, Rue62]) — and this is exactly what the large accelerator facilities detect.

Unfortunately, it turned out very difficult to provide examples richer than the free field which satisfy these (very natural) Wightman axioms. This difficulty motivated the development of equivalent or more general frameworks. In particular, the Wightman axioms are not made for gauge fields so that generalisation is indeed necessary. One powerful generalisation is Algebraic QFT (or better *Local QFT*) which shifts the focus from the field operators to the Haag-Kastler net of operator algebras assigned to open regions in space-time [HK64]. Fields merely provide coordinates on the algebra. This has the advantage to work with (C^*, von Neumann) algebras of bounded operators where powerful mathematical tools are available. Over the years this point of view turned out to be very fruitful [Haa96]. It is, in particular, possible to describe quantum field theory on curved space-time [BFV03].

2.2 Euclidean QFT

As consequence of the spectrum condition 2. in Definition 2.1, Wightman functions (2.1) admit an analytic continuation in time. At purely imaginary time they become the Schwinger functions [Sch59] of a Euclidean quantum field theory. Symanzik emphasised the powerful Euclidean-covariant functional integral representation [Sym64]. In this way the Schwinger functions become the moments of a statistical physics probability distribution. This tight connection between Euclidean quantum field theory and statistical physics led to a fruitful exchange of concepts and methods, most importantly that of the renormalisation group [WK74].

It is sometimes possible to rigorously prove the existence of a Euclidean quantum field theory or of a statistical physics model without knowing or using that this model derives from a true relativistic quantum field theory. Sufficient conditions on the Euclidean model which guarantee the Wightman axioms were first given by Nelson [Nel73b, Nel73a]. These conditions based on Markoff fields turned out to be too strong or inconvenient. Shortly later, Osterwalder and Schrader established a set of axioms [OS73, OS75] by which the Euclidean quantum field theory is (up to a regularity subtlety) equivalent to a Wightman theory. In simplified terms, the following data are necessary:

Definition 2.2 Let $\mathcal{S}_{N0} \subset \mathcal{S}(\mathbb{R}^{ND})$ be the subspace of test functions which vanish, with all derivatives, on any diagonal $x_i = x_j$, for $1 \leq i < j \leq N$. For $x_i =: (x_i^0, \mathbf{x}_i) \in \mathbb{R}^D$, let $\mathcal{S}_{N0+} \subset \mathcal{S}_{N0}$ be the subspace of test functions with support in the cone $\{x_i^0 \geq 0\}$ for all $i = 1, \dots, N$. Moreover, we let $f^\sigma(x_1, \dots, x_N) := f(x_{\sigma(1)}, \dots, x_{\sigma(N)})$ be a permutation and $f^r(x_1, \dots, x_N) := f((-x_1^0, \mathbf{x}_1), \dots, (-x_N^0, \mathbf{x}_N))$ be the reflection of all first components.

A *Euclidean quantum field theory* consists of a family $\{S_N\}$ of *Schwinger N-point distributions*, where S_N is a linear functional on $\mathcal{S}_{N0}$ which satisfies

1. *Euclidean invariance.* $S_N(f) = S_N(f_{t,R})$ for any $f \in \mathcal{S}_{N0}$ and $(t, R) \in \mathbb{R}^D \rtimes \mathrm{SO}(D)$, where $f_{t,R}(x_1, \dots, x_N) = f(R^{-1}(x_1 - t), \dots, R^{-1}(x_N - t))$.
2. *Reflection positivity.* For any tuple $(f_0, f_1, \dots, f_K)$ of $f_k \in \mathcal{S}_{k0+}$, one has $\sum_{k,l=1}^{K} S_{k+l}(\overline{f_k^r} \times f_l) \geq 0$.
   [here, $(\overline{f_k^r} \times f_l)(x_1, \dots, x_{k+l}) = \overline{f_k^r}(x_1, \dots, x_k) f_l(x_{k+1}, \dots, x_{k+l})$]
3. *Symmetry.* $S_N(f) = S_N(f^\sigma)$ for any $f \in \mathcal{S}_{N0}$ and any permutation σ.

The Osterwalder-Schrader theorem asserts that Schwinger functions according to Definition 2.2 of factorial growth (i.e. $|S_N(f)| \leq c^N (N!)^L \|f\|_{\mathcal{S}_{N0}}$ for some semi-norm defining $\mathcal{S}_{N0}$) are Laplace-Fourier transforms of Wightman functions in a relativistic quantum field theory. The properties of the vacuum Ω follow if the Schwinger functions cluster, $\lim_{t\to\infty} S_{k+l}((f_k)_{t,1} \times f_l) = S_k(f_k) S_l(f_l)$.

2.3 The free Euclidean scalar field

We describe in some detail the free Euclidean scalar field because it serves as example for the construction in the noncommutative setting. We start from the Schwinger 2-point function and its corresponding Schwinger distribution

$$S_2(x, y) := \int_{\mathbb{R}^D} \frac{dq}{(2\pi)^D} \frac{e^{i\langle x-y, q\rangle}}{\|q\|^2 + \mu^2}, \quad S_2(f \times g) = \int_{\mathbb{R}^{2D}} d(x, y)\, S_2(x, y) f(x) g(y) . \tag{2.2}$$

It satisfies reflection positivity on tuples $(0, f, 0, \dots, 0)$, where $f \in \mathcal{S}_{10+}$:

$$\begin{aligned} & S_2(\overline{f^r} \times f) \qquad (2.3) \\ &= \int_{\mathbb{R}^{2D}} d(x, y) \int_{\mathbb{R}^D} \frac{dq}{(2\pi)^D} \frac{e^{i(x^0-y^0)q^0 + i\langle \mathbf{x}-\mathbf{y}, \mathbf{q}\rangle}}{\|q\|^2 + \mu^2} \overline{f(-x^0, \mathbf{x})} f(y_0, \mathbf{y}) \\ &= \int_{(\mathbb{R}_+)^2} d(x^0, y^0) \int_{-\infty}^{\infty} dq^0 \int_{\mathbb{R}^{3(D-1)}} \frac{d(\mathbf{x}, \mathbf{y}, \mathbf{q})}{(2\pi)^D} \frac{e^{-i(x^0+y^0)q^0 + i\langle \mathbf{x}-\mathbf{y}, \mathbf{q}\rangle}}{q_0^2 + \|\mathbf{q}\|^2 + \mu^2} \overline{f(x^0, \mathbf{x})} f(y_0, \mathbf{y}) \\ &= \int_{\mathbb{R}^{D-1}} \frac{d\mathbf{q}}{(2\pi)^{D-1} \cdot 2\omega_\mu(\mathbf{q})} \left| \int_0^\infty dx_0 \int_{\mathbb{R}^{D-1}} d\mathbf{x}\, e^{-x^0 \omega_\mu(\mathbf{q}) - i\langle \mathbf{x}, \mathbf{q}\rangle} f(x_0, \mathbf{x}) \right|^2 \geq 0 , \end{aligned}$$

where $\omega_\mu(\mathbf{q}) := \sqrt{\|\mathbf{q}\|^2 + \mu^2}$. Here, from the 2nd to 3rd line, after $x^0 \mapsto -x^0$, the support property of $f \in \mathcal{S}_{10+}$ has been used. This allows to evaluate the q_0-integral via the residue theorem, resulting in the last line.

Next we introduce one of our most important tools:

Theorem 2.3 (Bochner-Minlos) *Let X be a real nuclear vector space. Let a continuous map $\mathcal{F} : X \to \mathbb{R}$ with $\mathcal{F}(0) = 1$ be of positive type, i.e. for any $x_1, \dots, x_K \in X$ and $c_1, \dots, c_K \in \mathbb{C}$ one has $\sum_{i,j=1}^{K} c_i \bar{c}_j \mathcal{F}(x_i - x_j) \geq 0$. Then there exists a unique Radon probability measure $d\mathcal{M}$ on the dual space X' with*

$$\mathcal{F}(x) = \int_{X'} e^{i\phi(x)} d\mathcal{M}(\phi) \; . \tag{2.4}$$

The theorem was proved in this generality by Minlos [Min59]. A proof can be found e.g. in [GJ87, §A6], starting from Bochner's theorem [Boc32] which covers the case $X = X' = \mathbb{R}$.

Recall that the Schwartz spaces $\mathcal{S}(\mathbb{R}^D)$ are nuclear. Consider for $f \in \mathcal{S}_{10+}$ the continuous functional $\mathcal{F}(f) = \exp(-\frac{1}{2}S_2(f \times f))$ defined by the Schwinger 2-point function (2.2). Then

$$\sum_{i,j=1}^{K} c_i \overline{c_j} \mathcal{F}(f_i - f_j) = \sum_{i,j=1}^{K} \sum_{n=0}^{\infty} C_i \overline{C_j} \frac{S_2(f_i \times f_j)^n}{n!} \tag{2.5}$$

with $C_i := c_i e^{-\frac{1}{2}S_2(f_i \times f_i)}$. Since $\langle f_i, f_j \rangle = S_2(f_i \times f_i)$ has all properties of a scalar product, $(S_2(f_i \times f_j))_{ij}$ is a positive definite Gram matrix. By the Schur product theorem, $(S_2(f_i, f_j))^n$ is, as Hadamard product of positive matrices, again positive.

In this way we have constructed out of a Schwinger 2-point function (2.2) a measure $d\mathcal{M}(\phi)$ on the space $(\mathcal{S}(\mathbb{R}^D))'$ of *Euclidean quantum fields*. It gives rise to Schwinger N-point distributions via

$$\begin{aligned} S_N(f_1 \times \cdots \times f_N) &:= \int_{(\mathcal{S}(\mathbb{R}^D))'} \phi(f_1) \cdots \phi(f_N) \, d\mathcal{M}(\phi) \\ &= (-i)^N \frac{\partial^N}{\partial t_1 \cdots \partial t_N} \mathcal{F}(t_1 f_1 + \cdots + t_N f_N)\Big|_{t_i=0} \; . \end{aligned} \tag{2.6}$$

The family $\{S_N\}$ satisfies all Osterwalder-Schrader axioms.

2.4 The interacting scalar field

For a polynomial P bounded from below, we would like to define an interacting scalar QFT by a "deformed measure" on $(S(\mathbb{R}^D))'$:

$$d\mathcal{M}_{\text{int}}(\phi) := \frac{d\mathcal{M}(\phi) \; \exp\big(-\int_{\mathbb{R}^D} dx \; \lambda(x) P(\phi(x))\big)}{\int_{X'} d\mathcal{M}(\phi) \; \exp\big(-\int_{\mathbb{R}^D} dx \; \lambda(x) P(\phi(x))\big)} \, , \tag{2.7}$$

where $d\mathcal{M}(\phi)$ is the previous Bochner-Minlos measure and λ a test function which, to achieve Euclidean invariance, eventually is sent to a coupling constant. The "definition" (2.7) has *very many* problems. It is motivated by the successful quantum-mechanical Feynman-Kac formula [Kac49] which constructs deformations of the Wiener measure on the space of Hölder-continuous paths.

The problems with (2.7) are related to the fact that the pointwise product of distributions is not necessarily a distribution. Therefore, the integral in (2.7), and integrals involving $d\mathcal{M}_{\text{int}}(\phi)$, are meaningless; we have to modify the rules. Many such modification procedures are known; they are (or should be) all equivalent. The following steps are typical:

Programme 2.4 1. *Infrared regularisation*. Restrict $\mathbb{R}^D$ to a compact subset $\mathcal{K} \subset \mathbb{R}^D$. If $\mathcal{K}$ is a cube, the Schwinger 2-point function (2.2) will contain a sum over discrete $\{q_n\}$ instead of an integral. After integrating over $\mathcal{K}$, now safe, the summation over $\{q_n\}$ might still diverge. Therefore it is necessary to introduce an

2. *Ultraviolet regularisation*. Restrict to finitely many discrete momenta $\|q_n\| \leq \Lambda$. Both regularisations together give rise to a *finite-dimensional problem*, which is important to retain for the treatment of quantum field theories on noncommutative geometries in the next section.
3. *Identify parameters*. After regularisation one hopes to see how to modify parameters in order to achieve a well-defined limit $\Lambda \to \infty$ and $\mathcal{K} \to \mathbb{R}^D$. These parameters can be the scalar coefficients in the polynomial $P(\phi)$, the mass μ^2 in (2.2) and a global field redefinition $\phi \mapsto \sqrt{Z}\phi$. Let us collectively call them $\lambda_1(\mathcal{K},\Lambda),\dots,\lambda_r(\mathcal{K},\Lambda)$.
4. *Renormalisation.* Identify r moments (here a lot of experience is necessary to decide which) of the regularised measure to be kept constant. We say they are normalised to values $M_1,\dots,M_r$ which all depend on $\lambda_1(\mathcal{K},\Lambda),\dots,\lambda_r(\mathcal{K},\Lambda)$. By the implicit function theorem, this dependence can generically be inverted to

$$\lambda_1(\mathcal{K},\Lambda,M_1,\dots,M_r),\dots,\lambda_r(\mathcal{K},\Lambda,M_1,\dots,M_r)\ .$$

   In this way *all* moments of the regularised measure, i.e. our regularised Schwinger functions, depend on $\mathcal{K},\Lambda,M_1,\dots,M_r$; we say they are *renormalised*.

Here comes the

Challenge 2.5 (of QFT)

1. Prove that after these preparations the limit $\Lambda \to \infty$ and $\mathcal{K} \to \mathbb{R}^D$ (understood as convergence of nets) of all moments exists.
2. Prove that the resulting Schwinger functions satisfy the Osterwalder-Schrader axioms.

This programme succeeded in a few cases, all in dimension $D < 4$. In $D = 2$ dimensions and for *arbitrary* polynomial $P(\phi)$ bounded from below, this was achieved in groundbreaking works by Simon [Sim74] and Glimm-Jaffe-Spencer [GJS74]. See also [GJ87]. They proved that it is essentially enough to replace $P(\phi)$ by its *normal ordering* $:P(\phi):$, a well-defined procedure which for monomials reads $:\phi^k: = \sum_{k_2+\dots+k_s=k} c_{k_1,\dots,k_s}\phi^{k_1}(\int \phi^{k_2}d\mathcal{M}(\phi))\cdots(\int \phi^{k_s}d\mathcal{M}(\phi))$ for certain integers $c_{k_1,\dots,k_s}$. The resulting polynomial is no longer bounded from below so that the very assumption of the Feynman-Kac formula is lost. That the Challenge 2.5 is solvable for this so-called $P(\phi)_2$-model (the subscript 2 refers to $D = 2$ dimensions) is a highly non-trivial result.

For the ϕ^4_3-model, which means $P(\phi) = \phi^4$ in $D = 3$ dimensions, a similar existence proof can be established, but the work is already harder. The strategy fails for the ϕ^4_4-model (i.e. in 4 dimensions). Here the only possibility is to expand $e^{-\lambda\int_{\mathbb{R}^4} dx\,(\phi(x))^4} = \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}(\int_{\mathbb{R}^4} dx\;(\phi(x))^4)^n$ into a power series and to *formally* exchange sum $\sum_n$ and integration $\int_{X'} d\mathcal{M}(\phi)$. By a procedure known as renormalised perturbation theory, briefly sketched below, one can give a meaning to the $\int_{X'} d\mathcal{M}(\phi)$ order by order in λ^n. However, the resulting series necessarily has zero radius of convergence[3]. In principle there exist summation techniques for series where $\lambda = 0$ is a boundary point of the holomorphicity domain. But in case of ϕ^4_4 this is also expected to fail because of the so-called triviality conjecture [Aiz81, Frö82]. The problem was first discovered by Landau et al for QED [LAK54]; it almost killed renormalised quantum field theory (rescued by the discovery of asymptotic freedom in QCD [GW73, Pol73]). Later we come back to that point.

Here we only mention that Yang-Mills theory in 4 dimensions is conjecturally free of the triviality problem and should exist as a quantum field theory. The proof is one of the millennium prize problems [JW00], left for the far future.

2.5 Feynman graphs and Feynman integrals

Here we briefly discuss interesting structures which arise when exchanging sum and integral of the expanded interaction term $e^{-\frac{\lambda}{4!}\int_{\mathbb{R}^D} dx\,(\phi(x))^4}$. The integral in footnote 3 serves as a warning that this is not what one ultimately wants. Step 1. of Challenge 2.5 is often in reach order by order in λ^n; for that it is helpful that the implicit function theorem has an easy Taylor approximation. Step 2. is meaningless as long as the convergence of the series remains obscure.

The various contributions are conveniently organised into *Feynman graphs* [Fey49]. These arise because the (divergent) integral

$$\lambda^n \int_{\mathbb{R}^{nD}} d(y_1,\dots,y_N) f_1(y_1)\cdots f_N(y_N) \int_{\mathbb{R}^{nD}} d(x_1,\dots,x_n) \\ \times \int_{X'} d\mathcal{M}(\phi)\; \phi(y_1)\cdots\phi(y_N)(\phi(x_1))^4\cdots(\phi(x_n))^4$$

contributing to the perturbative order-n Schwinger N-point function factors into all possible pairings of $\phi(x_i)$ with $\phi(x_j)$ into $S_2(x_1,x_2)$. Up to combinatorial factors which we do not discuss, this is given by the sum over *Feynman graphs* Γ with

[3] It is instructive to look at the following integral, which is a sort of ϕ^4_0-model:

$$I(\lambda) = \int_{-\infty}^{\infty} d\phi\; e^{-\phi^2-\lambda\phi^4} \text{“="} \sum_{k=0}^{\infty}\int_{-\infty}^{\infty} d\phi\;\frac{(-\lambda)^k\phi^{4k}}{k!}e^{-\phi^2} = \sum_{k=0}^{\infty}(-\lambda)^k\frac{\Gamma(2k+\frac{1}{2})}{\Gamma(k+1)}\,,$$

where “=” results when exchanging sum and integral. The series diverges for any $\lambda \neq 0$, but the lhs is perfectly defined for $\mathrm{Re}(\lambda) \geq 0$ and evaluates into $I(\lambda) = \frac{1}{2\sqrt{\lambda}}\exp(\frac{1}{8\lambda})K_{\frac{1}{4}}(\frac{1}{8\lambda})$, where K_ν is a modified Bessel function.

- n four-valent vertices located at $x_1, \dots, x_n \in \mathbb{R}^D$, each with a factor λ assigned;
- N one-valent vertices located at $y_1, \dots, y_N \in \mathbb{R}^D$;
- with factor $S_2(x_a, x_b)$ assigned to every edge between x_a, x_b (which can be x's and y's, also $x_a = x_b$ is allowed);
- integrated over $x_1, \dots, x_n$ (will diverge unless restricted to $\mathcal{K}$ and Λ);
- integrated against test functions $f_1(y_1), \dots, f_N(y_N)$ over $y_1, \dots, y_N$.

The resulting *Feynman integral* can be rearranged in several ways. One can keep the momentum variables q from (2.2) in every edge and move the Fourier kernels to the vertices, where the x-integrals give Dirac-δ distributions. The momentum rules are thus:

- assign oriented momentum q_{ab} to every (arbitrarily oriented) edge from vertex a to vertex b, assign $\frac{1}{\|q_{ab}\|^2+\mu^2}$ to that edge;
- assign $(2\pi)^D \lambda \delta(q_1 + \dots + q_4)$ to every 4-valent vertex a, where $q_i := q_{bi}$ if the edge arrives from vertex b and $q_i := -q_{ib}$ if the edge goes to b.

Writing the weight factors as $\frac{1}{\|q\|^2+\mu^2} = \int_0^\infty d\alpha \;\; e^{-\alpha(\|q\|^2+\mu^2}$ and returning to $(2\pi)^D \delta(q_1, \dots, q_4) = \int_{\mathbb{R}^D} dx \;\; e^{i\langle x, (q_1+\dots+q_r)\rangle}$, all x-integrations and q-integrations for edges between 4-valent vertices are Gaußian and give rise to the *parametric representation* which can immediately be deduced from:

Theorem 2.6 *A connected graph* Γ *with* L *edges between* 4*-valent vertices contributes with weight*

$$\int_{\mathbb{R}^D} \frac{dp_1 \; \hat{f}_1(p)}{|p_1\|^2 + \mu^2} \cdots \int_{\mathbb{R}^D} \frac{dp_1 \; \hat{f}_N(p_N)}{|p_N\|^2 + \mu^2} \delta(p_1 + \ldots + p_N) \mathcal{A}_\Gamma(p_1, \dots, p_N) , \qquad \textit{where}$$

$$\mathcal{A}_\Gamma(p_1, \dots, p_N) = \int_{(\mathbb{R}_+)^L} d(\alpha_1, \dots, \alpha_L) \; \frac{e^{-\mu^2(\alpha_1+\dots+\alpha_L) - \frac{V_\Gamma(\alpha_\ell, p_v)}{U_\Gamma(\alpha_\ell)}}}{(U_\Gamma(\alpha_\ell))^{\frac{D}{2}}} , \tag{2.8}$$

$$U_\Gamma(\alpha_\ell) = \sum_{T_1 \in \Gamma} \prod_{\ell \notin T_1} \alpha_\ell , \qquad V_\Gamma(\alpha_\ell) = \sum_{T_2 \in \Gamma} \Big(\prod_{\ell \notin T_2} \alpha_\ell \Big) \Big\| \sum_{v \in T_{21}} p_v \Big\|^2 .$$

Here the sum in U_Γ *runs over all spanning trees* T_1 *of* Γ *(trees which meet every* 4*-valent vertex); the sums in* V_Γ *runs over all forests* T_2 *of exactly two trees* T_{21} *and* T_{22} *which together contain all* 4*-valent vertices, and each of them at least one vertex with some incoming momentum* p_v*.*

The U_Γ, V_Γ are referred to as the Kirchhoff-Symanzik polynomials of Γ. At vanishing external momenta p_v, the amplitude diverges for $\alpha \to 0$. This divergence is best controlled by a decomposition into Hepp sectors $\alpha_{\pi(1)} < \dots < \alpha_{\pi(L)}$, which give rise to iterated integrals. Such iterated integrals are fascinating objects [Bro13b]. They form a Hopf algebra [Kre00], which relates them to noncommutative geometry [CK98], and they evaluate (for $\mu = 0$ and $p_v = 0$) to special values of analytic number theory, typically multiple zeta values [Bro13a]. The Symanzik polynomials provide connections between algebraic varieties and Feynman integrals [Blo15]. According to the Goncharov-Manin conjecture [GM04], there is a relation between periods in mixed Tate motives and residues of Feynman integrals.

The perturbative regularisation and renormalisation methods for the above Feynman integrals produce two problems which make a resummation of the perturbation series impossible. The same will apply to the perturbative treatment of QFTs on noncommutative geometries. One can do better, both in traditional and noncommutative QFT. First, the usual renormalisation at a single scale $p_v = 0$ produces amplitudes which grow as $\log \|p_v\|$. Inserting n such renormalised graphs γ as subgraphs into a bigger, convergent graph Γ with n vertices leads to amplitudes $\mathcal{A}_\Gamma = \mathcal{O}(n!)$. This is the *renormalon problem*; it arises because one repairs too much. Constructive renormalisation theory [Riv91] avoids the renormalon problem by slicing the α-integrals and external momenta p_v into multiple scales and only repairs if the α-scale is higher than the p-scale. The second problem is that the number of Feynman graphs with n vertices grows too fast with n. This problem is addressed by a reorganisation of the perturbation series into trees instead of graphs [GRM09]. The idea goes as follows: A Schwinger function is a sum of amplitudes indexed by graphs, $S = \sum_\Gamma S_\Gamma$. Let $T \subset \Gamma$ be the spanning trees, and assume there is a weight function with $\sum_{T\subset\Gamma} w(\Gamma, T) = 1$. Then formally $S = \sum_\Gamma \sum_{T\subset\Gamma} w(\Gamma, T)S_\Gamma = \sum_T S_T$ with $S_T = \sum_{\Gamma\supset T} w(\Gamma, T)S_\Gamma$.

3 Euclidean quantum fields on noncommutative geometries

3.1 Nuclear AF Fréchet algebras

A noncommutative geometry is for us a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ [Con95] consisting of an associative $*$-algebra represented on Hilbert space $\mathcal{H}$, together with a selfadjoint unbounded operator $\mathcal{D}$ such that $[\mathcal{D}, a]$ extends to a bounded operator for all $a \in \mathcal{A}$. Often compactness of $a(D + \mathrm{i})^{-1}$ is required, and various topologies on and closures of $\mathcal{A}$ are considered.

In this survey we restrict ourselves to *Euclidean quantum field theory* which we intend to construct by analogy with Secs. 2.3 and 2.4. The construction of a (Euclidean) QFT on a noncommutative geometry $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a compromise between two contradictory requirements. Steps 1.+2. of Programme 2.4 require a reduction to a finite-dimensional problem where everything is well-defined. This allows to adjust parameters so that a limit can be studied. Finite-dimensional algebras are matrix algebras, and the limiting procedure seems at first sight to be what is known as AF C^*-algebras [Bra72]. An important (non-unital) example is the algebra of compact operators which is the norm closure of finite-rank operators. On the other hand, for the initial definition of a free Euclidean scalar field via the Bochner-Minlos theorem 2.3 we need a scalar product on a *nuclear* vector space. But the vector space of compact operators is not nuclear, and consequently the norm closure is the wrong concept for our purpose.

What we rather need is a class of algebras $\mathcal{A}$ which we would like to call *nuclear AF Fréchet-algebras*:

Definition 3.1 A *Fréchet space* is a locally convex vector space X topologised by a countable increasing family (p_n) of seminorms, which make X metrisable and complete. The Fréchet space is *nuclear* if

1. the topology is defined by a countable family (p_n) of Hilbert seminorms, i.e. for every n there is an inner product $\langle\ ,\ \rangle_n$ on X with $(p_n(x))^2 = \langle x, x\rangle_n$.
2. If X_n denotes the closure of X with respect to $\langle\ ,\ \rangle_n$, then for any p_n there is a larger p_m such that the natural map from X_m to X_n is trace-class.

Definition 3.2 A *(nuclear) Fréchet algebra* is an algebra $\mathcal{A}$ that, as a vector space, is a (nuclear) Fréchet space and in which the multiplication $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is continuous.

Nuclear Fréchet algebras can always be understood as a certain space of *smooth functions with deformed product*. Namely,

Theorem 3.3 *For a Fréchet space X are equivalent:*

1. *X is nuclear.*
2. *X is isomorphic to a closed subspace of $C^\infty(U)$, for any open $U \subset \mathbb{R}^D$.*

The equivalence is essentially due to T. and Y. Kōmura [KK66] (for $D = 1$; the general case can be found in the literature, see e.g. [Vog00]). Given now a Fréchet algebra $\mathcal{A}$, then an isomorphism ι_U of vector spaces between $\mathcal{A}$ and a closed subspace of $C^\infty(U)$ induces a deformed product $\star_U$ on $\iota_U(\mathcal{A})$ by

$$\iota_U(a) \star_{\iota_U} \iota_U(b) := \iota_U(ab)\ . \tag{3.1}$$

We will mainly be interested in $U = \mathbb{R}^D$. In case that $\iota_{\mathbb{R}^D}(\mathcal{A})$ is invariant under translations by $\mathbb{R}^D$, or even under the Euclidean group $\mathbb{R}^D \rtimes SO(D)$, or under a subgroup of them, we can define a corresponding group action on $\mathcal{A}$ by

$$\alpha_{t,R}(a) := \iota^{-1}_{\mathbb{R}^D}((\iota_{\mathbb{R}^D} a)_{t,R})\ , \qquad \text{where } f_{t,R}(x) = f(R^{-1}(x-t))\ . \tag{3.2}$$

Such a group action is very important for us because it is needed to formulate an analogue of the Osterwalder-Schrader axioms, see Sec. 10. There are clearly examples for Fréchet algebras carrying an action of the Euclidean group, but we do not know how generic they are. So let us formulate:

? Question 3.4

Which assumptions on a Fréchet algebra guarantee the existence of an isomorphism to a closed subspace of $C^\infty(\mathbb{R}^D)$ that is invariant under (a subgroup of) the Euclidean group?

It remains to define our approximation property needed for renormalisation:

Definition 3.5 A nuclear Fréchet algebra $\mathcal{A}$ is called *AF* (for approximately finite-dimensional) if there is an increasing sequence $\mathcal{A}^0 \subset \mathcal{A}^1 \subset \mathcal{A}^2 \subset \dots$ of finite-dimensional subalgebras, embedded into each other by $*$-homomorphisms $\iota_N : \mathcal{A}^N \to \mathcal{A}^{N+1}$, such that $\bigcup_{N\in\mathbb{N}} \mathcal{A}^N$ is dense in $\mathcal{A}$ in the locally-convex topology induced by the Hilbert seminorms $\{p_n\}$ of $\mathcal{A}$.

This definition is inspired by the corresponding definition AF-C^*-algebras [Bra72], but we do not require $\mathcal{A}$ to be unital and we close in the locally-convex topology. The class of unital AF-C^*-algebras is very rich and classified by K-theoretic data [Ell76]. We have no idea about the corresponding landscape in the locally-convex setup:

? Question 3.6

1. How rich is the class of nuclear AF Fréchet algebras?
2. How much do they depend on the choice of Hilbert seminorms?
3. Is there any chance to classify them, possibly under extra conditions?
4. Is there any relation to limits of compact quantum metric spaces [Rie04]?

Remark 3.7 The Kōmura-Kōmura theorem 3.3 is a typical example for the coexistence of the discrete and the continuum in noncommutative geometry [Con95]. It is undeniable that our universe is very close to a continuous space. But we cannot conclude that our universe is a manifold; it just means that smooth functions on a manifold are the universal model for a nuclear Fréchet space which could very well be inherently discrete.

3.2 The free Euclidean scalar field on a noncommutative geometry

Every nuclear Fréchet algebra $\mathcal{A}$ admits a free Euclidean scalar field. The vector space $\mathcal{A}_*$ of self-adjoint elements of $\mathcal{A}$ is a real nuclear vector space. Consider a continuous symmetric positive-semidefinite bilinear form C on $\mathcal{A}_*$, called the *covariance*. Such a C always exists; take for instance the inner product defining any of the Hilbert seminorms p_n on $\mathcal{A}$. In the same way as in Sec. 2.3, the continuous linear map $\mathcal{F}_C : \mathcal{A}_* \to \mathbb{R}$ defined by

$$\mathcal{F}_C(a) := \exp\Big(-\frac{1}{2}C(a,a)\Big)\,, \qquad a = a^* \in \mathcal{A}_*\,, \tag{3.3}$$

satisfies the assumptions of the Bochner-Minlos theorem 2.3. Consequently, there exists a unique Radon probability measure $d\mathcal{M}_C$ on the dual space $\mathcal{A}'_*$ of *Euclidean scalar fields* with

$$\mathcal{F}_C(a) = \int_{\mathcal{A}'_*} \mathrm{d}\mathcal{M}_C(\Phi)\; e^{\mathrm{i}\Phi(a)}\,. \tag{3.4}$$

It is straightforward to generalise this construction to finitely generated projective modules over $\mathcal{A}$, but for the sake of clarity we will not spell it out. The moments of $\mathrm{d}\mathcal{M}_C$ are, as before, given by

$$\begin{aligned}\int_{\mathcal{A}'_*} \mathrm{d}\mathcal{M}_C(\Phi)\ \Phi(a_1)\cdots\Phi(a_N) \\ &= (-\mathrm{i})^N \frac{\partial^N}{\partial t_1\cdots\partial t_N}\mathcal{F}(t_1a_1+\cdots+t_Na_N)\Big|_{t_i=0} \\ &= \begin{cases} \displaystyle\sum_{\text{pairings of } [N]} C(a_{i_1},a_{j_1})\cdots C(a_{i_{N/2}},a_{j_{N/2}}) & \text{for } N \text{ even.} \\ 0 & \text{for } N \text{ odd,}\end{cases}\end{aligned} \tag{3.5}$$

A pairing is a partition of $\{1,2,\dots,N\}$ into $\frac{N}{2}$ subsets $(i_1,j_1),\dots,(i_{N/2},j_{N/2})$ with $i_k < j_k$. We interpret $S_N(a_1\otimes\cdots\otimes a_N) := \int_{\mathcal{A}'_*}\mathrm{d}\mathcal{M}_C(\Phi)\ \Phi(a_1)\cdots\Phi(a_N)$ as the *Schwinger N-point function of the free scalar field* of covariance C on the noncommutative algebra $\mathcal{A}$.

We postpone a discussion of Osterwalder-Schrader axioms for the free field to Sec. 10.

3.3 Towards an interacting scalar field on noncommutative geometry

Switching on interaction is, as in ordinary quantum field theory, a very hard problem. We proceed as in Sec. 2.4 and formally define correlation functions as moments of a Feynman-Kac perturbation of $\mathrm{d}\mathcal{M}_C$. Given a functional S_{int} on $\mathcal{A}'_*$, bounded from below, we define

$$\begin{aligned}\langle a_1\otimes\cdots\otimes a_N\rangle &:= \frac{\displaystyle\int_{\mathcal{A}'_*}\mathrm{d}\mathcal{M}_C(\Phi)\ \Phi(a_1)\cdots\Phi(a_N)\exp(-S_{\text{int}}(\Phi))}{\displaystyle\int_{\mathcal{A}'_*}\mathrm{d}\mathcal{M}_C(\Phi)\ \exp(-S_{\text{int}}(\Phi))} \\ &= \frac{(-\mathrm{i})^N}{\mathcal{Z}_C(0)}\frac{\partial^N\mathcal{Z}_C(t_1a_1+\cdots+t_Na_N)}{\partial t_1\cdots\partial t_N}\Big|_{t_i=0},\end{aligned} \tag{3.6}$$

where

$$\mathcal{Z}_C(J) = \int_{\mathcal{A}'_*}\mathrm{d}\mathcal{M}_C(\Phi)\ \mathrm{e}^{\mathrm{i}\Phi(J)-S_{\text{int}}(\Phi)} \tag{3.7}$$

is the *partition function*. For the free theory $S_{\text{int}}(\Phi)\equiv 0$, it coincides with the characteristic function $\mathcal{F}_C(J)$ defined in (3.3), with $J\in\mathcal{A}_*$. For $S_{\text{int}}(\Phi)\neq 0$, however, these naïve correlation or partition functions do not make any sense. One meets the usual divergences whose treatment requires regularisation and renormalisation. As stressed in Sec. 2.4, we have to restrict in a first step to finite-dimensional subspaces, at least for a rigorous (non-perturbative) treatment. In perturbation theory one may hope to do less, but this depends on the situation.

Inspired by renormalisation in usual QFT, sketched in Sec. 2.4, we proceed as follows:

Programme 3.8 For a given nuclear AF Fréchet algebra $\mathcal{A}$, consider covariances $C^{(\mathcal{N})}$ and interaction functionals $S^{(\mathcal{N})}_{\mathrm{int}}$ on the finite-dimensional subspace $\mathcal{A}^{\mathcal{N}}_*$ of $\mathcal{A}_*$. Parametrise them by real numbers $\lambda_1(\mathcal{N}),\dots,\lambda_r(\mathcal{N})$, define correlation functions (3.6) by integrals with measure $d\mathcal{M}_{C^{(\mathcal{N})}}$ over $(\mathcal{A}^{\mathcal{N}}_*)'$. Identify r of these moments $M_1,\dots,M_r$, all functions of $\lambda_1(\mathcal{N}),\dots,\lambda_r(\mathcal{N})$, but considered as fixed. The implicit function theorem generically allows to invert to $\lambda_1(\mathcal{N},M_1,\dots,M_r),\dots,\lambda_r(\mathcal{N},M_1,\dots,M_r)$. Consequently, *all* level-$\mathcal{N}$ correlation functions depend on $\mathcal{N}$ and $M_1,\dots,M_r$.

Now we face part 1. of the fundamental Challenge 2.5: Prove that with these preparations, under the embeddings $\iota_{\mathcal{N}}$ and corresponding embeddings of $(C^{(\mathcal{N})})$ and $(S^{(\mathcal{N})}_{\mathrm{int}})$, the limit $\mathcal{N}\to\infty$ of all moments (3.6) exists, thereby defining Schwinger functions of an interacting scalar QFT on $\mathcal{A}$.

Discussion of step 2., the Osterwalder-Schrader axioms, has to be postponed.

We describe the parametrisation by $\lambda_1(\mathcal{N}),\dots,\lambda_r(\mathcal{N})$ for the most relevant case of tracial interaction functionals. As with C^*-algebras, every $\mathcal{A}^{\mathcal{N}}$ is a direct sum of finitely many matrix algebras, $\mathcal{A}^{\mathcal{N}}=\bigoplus_i M_{n_{\mathcal{N},i}}(\mathbb{C})$. Let $(e^{(i)}_{kl})$ be the standard matrix basis of $M_{n_{\mathcal{N},i}}(\mathbb{C})$. We extend the real linear functionals Φ to $\mathcal{A}$ via $\Phi(a+\mathrm{i}b):=\Phi(a)+\mathrm{i}\Phi(b)$. Then the restriction of $\Phi\in\mathcal{A}'_*$ to $\mathcal{A}^{\mathcal{N}}$ is uniquely specified by the complex numbers $\Phi^{(i)}_{kl}:=\Phi(e^{(i)}_{kl})\equiv\overline{\Phi^{(i)}_{lk}}$, and the following defines a functional on $(\mathcal{A}^{\mathcal{N}}_*)'$:

$$S^{(\mathcal{N})}_{\mathrm{int}}(\Phi):=\sum_i\sum_{p_i}\frac{\lambda_{i,p_i}(\mathcal{N})}{p_i}\sum_{k^i_1,\dots,k^i_{p_i}=1}^{n_{\mathcal{N},i}}\Phi^{(i)}_{k^i_1k^i_2}\Phi^{(i)}_{k^i_2k^i_3}\cdots\Phi^{(i)}_{k^i_{p_i-1}k^i_{p_i}}\Phi^{(i)}_{k^i_{p_i}k^i_1}\,. \tag{3.8}$$

Additional $\lambda_c(\mathcal{N})$ will parametrise the covariances and possible field redefinitions $\Phi^{(i)}_{kl}\mapsto\sqrt{Z_i(\mathcal{N})}\Phi^{(i)}_{kl}$.

An investigation in this generality has not yet been performed. We describe in Sec. 4.1 below the simplest case given by a single summand i (hence omitted) and an embedding $\iota_{\mathcal{N}}$ analogous to compact operators.

4 Some noncommutative geometries for QFT

4.1 Simplest example: Moyal algebra

Let $\mathcal{A}_\theta=\{a=(a_{kl})\;:\;k,l=1,2,3,\dots\}$ be the vector space of double-indexed sequences, with involution $(a^*)_{kl}=\overline{a_{lk}}$, completed in the Fréchet topology induced by the family of inner products

$$\langle a,b\rangle_m := \sum_{k,l=1}^{\infty} \theta^{2m}(k+\tfrac{1}{2})^m(l+\tfrac{1}{2})^m\overline{a_{kl}}b_{kl}\ . \tag{4.1}$$

Let e_{rs} be the terminating sequence in $\mathcal{A}_\theta$ defined by $(e_{rs})_{kl} = \delta_{rk}\delta_{sl}$, then $(b^{(m)}_{rs})$ with $b^{(m)}_{rs} := \theta^{-m}(r+\frac{1}{2})^{-m/2}(s+\frac{1}{2})^{-m/2}e_{rs}$ form an orthonormal basis with respect to $\langle\ ,\ \rangle_m$. Let $\mathcal{A}_{\theta,m}$ be the closure of $\mathcal{A}_\theta$ with respect to $\langle\ ,\ \rangle_m$. Every $a \in \mathcal{A}_{\theta,m+3}$ has a representation

$$a = \sum_{r,s=1}^{\infty}\langle b^{(m+3)}_{rs},a\rangle_{m+3}b^{(m+3)}_{rs} = \sum_{r,s=1}^{\infty}\frac{\langle b^{(m+3)}_{rs},a\rangle_{m+3}}{\theta^3(r+\frac{1}{2})^{\frac{3}{2}}(s+\frac{1}{2})^{\frac{3}{2}}}b^{(m)}_{rs}\ ,$$

which shows that, for any m, the natural map $\mathcal{A}_{\theta,m+3} \ni a \mapsto a \in \mathcal{A}_{\theta,m}$ is trace-class. Hence, $\mathcal{A}_\theta$ is a nuclear Fréchet space.

For $a = (a_{kl}), b = (b_{kl}) \in \mathcal{A}_\theta$ we consider $[ab]_{kl} := \sum_{n=1}^\infty a_{kn}b_{nk}$. Using Cauchy-Schwarz inequality several times one has

$$\begin{aligned}(p_m(ab))^2 &\leq \Big(\sum_{k,n=1}^{\infty}\theta^m(k+\tfrac{1}{2})^m|a_{kn}|^2\Big)\Big(\sum_{l,n'=1}^{\infty}\theta^m(l+\tfrac{1}{2})^m|b_{n'l}|^2\Big)\\ &\leq \frac{1}{\theta^{2m}}(p_m(a))^2(p_m(b))^2\ .\end{aligned} \tag{4.2}$$

This shows that the sequence $ab := ([ab]_{kl})$ belongs to $\mathcal{A}_\theta$ and that the corresponding multiplication $\mathcal{A}_\theta \times \mathcal{A}_\theta \ni (a,b) \mapsto ab \in \mathcal{A}_\theta$ is continuous. Hence, $\mathcal{A}_\theta$ is a nuclear Fréchet algebra. The terminating sequences (e_{rs}) introduced above satisfy $e_{rs}e_{tu} = \delta_{st}e_{ru}$. Thus they play the rôle of matrix bases, and we will often expand $a = \sum_{k,l=1}^\infty a_{kl}e_{kl}$ where it is understood that $e_{kl} \in \mathcal{A}_\theta$ and $a_{kl} \in \mathbb{C}$.

By Theorem 3.3 there exists an isomorphism $\iota_\theta := \iota_{\mathbb{R}^2}$ of vector spaces between $\mathcal{A}_\theta$ and a closed subspace of $C^\infty(\mathbb{R}^2)$. A particular realisation is given by $\iota_\theta(e_{kl}) := f_{k-1,l-1}$, where

$$f^{(\theta)}_{kl}(x_1,x_2) := 2(-1)^k\sqrt{\frac{k!}{l!}}\Big(\sqrt{\frac{2}{\theta}}(x_1+\mathrm{i}x_2)\Big)^{l-k}L^{l-k}_k\Big(\frac{2\|x\|^2}{\theta}\Big)\mathrm{e}^{-\frac{\|x\|^2}{\theta}}\ , \tag{4.3}$$

for $x = (x_1,x_2)$. The $L^\alpha_m(t)$ are associated Laguerre polynomials of degree m in t. One has $\int_{\mathbb{R}^2}\mathrm{d}x\ f^{(\theta)}_{mn}(x) = 2\pi\theta\delta_{mn}$. By linear extension we obtain an isomorphism between $\mathcal{A}_\theta$ and the nuclear vector space $\mathcal{S}(\mathbb{R}^2)$ of Schwartz functions. According to (3.1), this isomorphism induces an associative product $\star_\theta$ on $\mathcal{S}(\mathbb{R}^2)$, the *Moyal product*. One can verify [GBV88]

$$(\phi\star_\theta\psi)(x) = \int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{\mathrm{d}k\,\mathrm{d}y}{(2\pi)^2}\ \phi(x+\tfrac{1}{2}\Theta y)\psi(x+y)\mathrm{e}^{\mathrm{i}\langle y,k\rangle}\ , \tag{4.4}$$

where $\Theta = \begin{pmatrix}0 & \theta\\ -\theta & 0\end{pmatrix}$. This makes the Moyal product an example of a strict deformation quantisation by action of $\mathbb{R}^2$ [Rie93]. In particular, the action of the Euclidean group

$\mathbb{R}^2 \rtimes SO(2)$ on $\mathcal{S}(\mathbb{R}^2)$ induces via (3.2) a corresponding group action $\alpha_{t,R}$ on $\mathcal{A}_\theta$ which has the important property to commute with the multiplication:

$$\alpha_{t,R}(ab) = (\alpha_{t,R}a)(\alpha_{t,R}b)\ . \tag{4.5}$$

Remains to describe the AF structure of $\mathcal{A}_\theta$. We let $\mathcal{A}_\theta^{\mathcal{N}} := \mathrm{span}(e_{kl}\ :\ 1 \leq k,l \leq \mathcal{N})$, then every $\mathcal{A}_\theta^{\mathcal{N}}$ is a subalgebra for the multiplication in $\mathcal{A}^{\mathcal{N}}$. The natural identification $\mathcal{A}_\theta^{\mathcal{N}} \equiv M_{\mathcal{N}}(\mathbb{C})$ defines via

$$\iota_{\mathcal{N}} : \mathcal{A}_\theta^{\mathcal{N}} = M_{\mathcal{N}}(\mathbb{C}) \ni a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_{\mathcal{N}+1}(\mathbb{C}) = \mathcal{A}_\theta^{\mathcal{N}+1}$$

the connecting $*$-homomorphism $\iota_{\mathcal{N}}$. Given a finite family $(p_{n_1}, \dots, p_{n_K})$ of Hilbert seminorms and $\epsilon > 0$, for every $a = (a_{kl}) \in \mathcal{A}_\theta$ the absolute convergence $p_{n_i}(a) < \infty$ guarantees the existence of an $N_{\epsilon;n_1,\dots,n_K} \in \mathbb{N}$ with

$$p_{n_i}\Big(a - \sum_{k,l=1}^{N_{\epsilon;n_1,\dots,n_K}} a_{kl}e_{kl}\Big) < \epsilon \qquad \text{for all } i = 1, \dots, K\ .$$

Hence, $\bigcup_{\mathcal{N}=1}^{\infty} \mathcal{A}_\theta^{\mathcal{N}}$ is dense in $\mathcal{A}_\theta$ for the Fréchet topology, and $\mathcal{A}_\theta$ is indeed a nuclear AF Fréchet algebra.

4.2 Quantum fields on the Moyal algebra

We describe possible choices of covariances and interaction functionals on $\mathcal{A}_\theta$. The interaction functionals (3.8) specify to

$$S_{\mathrm{int}}^{(\mathcal{N})}(\Phi) = \sum_p \frac{\lambda_p(\mathcal{N})}{p} \sum_{k_1,\dots,k_p=1}^{\mathcal{N}} \Phi_{k_1k_2}\Phi_{k_2k_3}\cdots\Phi_{k_{p-1}k_{k_p}}\Phi_{k_pk_1}\ , \tag{4.6}$$

where $\Phi_{kl} := \Phi(e_{kl})$ and the sum over p is finite. The particularly relevant cases $p = 3$ and $p = 4$ are discussed in secs. 8 and 9. Passing via ι_θ to Schwartz functions and taking the Fourier transform, the functional (4.6) becomes in the limit $\mathcal{N} \to \infty$

$$\begin{aligned} &S_{\mathrm{int}}(\Phi) \\ &= \sum_p \frac{\lambda_p}{p} \int_{\mathbb{R}^{2p}} \mathrm{d}(q_1, \dots, q_p)\ \delta(q_1 + \dots + q_p) \mathrm{e}^{\frac{\mathrm{i}}{2}\sum_{1\leq k<l\leq p}\langle q_k, \Theta q_l\rangle} \hat{\Phi}(q_1)\cdots\hat{\Phi}(q_p)\ , \end{aligned} \tag{4.7}$$

assuming that $\lambda_p(\mathcal{N})$ has a limit (which is rarely the case in practice). This form has often been used in a perturbative treatment of QFTs on Moyal space (see Sec. 5). It is not suitable for a rigorous construction.

The covariances (3.3) could be chosen arbitrarily, but for a convenient interpretation we assume that they arise from a family $\{d_\nu\}_{\nu=1,\dots,s}$ of continuous linear maps $d_\nu : \mathcal{A}_\theta \to \mathcal{A}_\theta$ of geometrical significance, for instance induced by the Dirac operator $\mathcal{D}$ of a spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$. For $a = \sum_{k,l=1}^{\mathcal{N}} a_{kl}e_{kl} \in \mathcal{A}_\theta^{\mathcal{N}} \equiv M_{\mathcal{N}}(\mathbb{C})$, consider

$$\sum_{\nu=1}^{s} \mathrm{Tr}((d_\nu a)^* d_\nu a) =: \sum_{k,l,m,n=1}^{\mathcal{N}} D_{kl;mn} a_{kl} a_{mn} \;. \tag{4.8}$$

The covariance is the inverse of that matrix, $\sum_{m',n'=1}^{\mathcal{N}} C^{(\mathcal{N})}(e_{kl}, e_{m'n'}) D_{n'm';mn} = \delta_{kn}\delta_{lm} = \sum_{m',n'=1}^{\mathcal{N}} D_{kl;m'n'} C^{(\mathcal{N})}(e_{n'm'}, e_{mn})$.

The following choices are of particular relevance:

- For an arbitrary sequence (E_k) of positive real numbers, define $d_E(e_{kl}) = \sqrt{E_k + E_l}\, e_{kl}$. It follows $C_E(e_{kl}, e_{mn}) = \frac{\delta_{kn}\delta_{lm}}{E_k+E_l}$. This covariance together with $p = 3$ in (4.6) defines the Kontsevich model [Kon92] which is of paramount importance in algebraic geometry. We discuss it in Sec. 7.2 and 8. The same covariance but with quartic interaction $p = 4$ has also received considerable attention and will be discussed in Sec. 9.
- The action $\alpha_{t,1}$ of translations by $t \in \mathbb{R}^2$ defined in (3.2) can be shown to be generated by

$$\begin{aligned}(\partial_1 - \mathrm{i}\partial_2)(e_{kl}) &= \sqrt{\frac{2}{\theta}}(\sqrt{l-1}\, e_{k,l-1} - \sqrt{k}\, e_{k+1,l}) \;, \\ (\partial_1 + \mathrm{i}\partial_2)(e_{kl}) &= \sqrt{\frac{2}{\theta}}(\sqrt{k-1}\, e_{k-1,l} - \sqrt{l}\, e_{k,l+1}) \;. \end{aligned}\tag{4.9}$$

  In this way the covariance of the Laplacian $\langle a, -\Delta a\rangle = \sum_{\nu=1}^{2} \mathrm{Tr}((\partial_\nu a)^* \partial_\nu a)$ can be defined. The calculation is lengthy; one has to diagonalise the resulting matrix $\Delta_{kl;mn}$ via Meixner polynomials. We briefly describe these steps in Sec. 6.1.
- Pointwise multiplication $(M_1\phi)(x) = x_1\phi(x)$ and $(M_2\phi)(x) = x_2\phi(x)$ of Schwartz functions defines continuous linear maps which translate via ι_θ^{-1} into the following action on the matrix bases:

$$\begin{aligned}(M_1 + \mathrm{i}M_2)(e_{kl}) &= \sqrt{\frac{\theta}{2}}(\sqrt{l-1}\, e_{k,l-1} + \sqrt{k}\, e_{k+1,l}) \;, \\ (M_1 - \mathrm{i}M_2)(e_{kl}) &= \sqrt{\frac{\theta}{2}}(\sqrt{k-1}\, e_{k-1,l} + \sqrt{l}\, e_{k,l+1}) \;. \end{aligned}\tag{4.10}$$

  Instead of the Laplacian one can consider the slightly more general covariance of the operator $\langle a, H^\Omega a\rangle = \sum_{\nu=1}^{2} \mathrm{Tr}((\partial_\nu a)^* \partial_\nu a + \frac{4\Omega^2}{\theta^2}(M_\nu a)^* M_\nu a)$. See Sec. 6.1.

Remark 4.1 The Moyal product (4.4) has its origin in quantum mechanics, in particular in Weyl's operator calculus [Wey28]. Wigner introduced the useful concept of the phase space distribution function [Wig32]. Then, Groenewold [Gro46] and Moyal

[Moy49] showed that quantum mechanics can be formulated on classical phase space using the *twisted product* concept. In particular, Moyal proposed the "sine-Poisson bracket" (nowadays called Moyal bracket), which is the analogue of the quantum mechanical commutation relation. The twisted product was extended from Schwartz class functions to (appropriate) tempered distributions by Gracia-Bondía and Várilly [GBV88, VGB88]. The programme of Groenewold and Moyal culminated in the axiomatic approach of *deformation quantisation* [BFF$^+$78a, BFF$^+$78b]. The problem to lift a given Poisson structure to an associative $\star$-product was solved by Kontsevich [Kon03]. Cattaneo and Felder [CF00] found a physical derivation of Kontsevich's formula in terms of a path integral quantisation of a Poisson sigma model [SS94]. The Moyal product is a strict deformation by $\mathbb{R}^D$-action [Rie93], not only a formal deformation. The Moyal plane is a spectral triple [GGBI$^+$04], and the spectral action has been computed [Vas04, GI05].

4.3 4-dimensional Moyal space

The generalisation of the Moyal algebra introduced in Sec. 4.1 to 4 dimensions is achieved by double-double-indexed sequences $\mathcal{A}_\Theta = \{a = (a_{\underline{k}\underline{l}}) \;:\; \underline{k}, \underline{l} \in \mathbb{N}^2_{\geq 1}\}$ with $(a^*)_{\underline{k}\underline{l}} = \overline{a_{\underline{l}\underline{k}}}$ and completed in the Fréchet topology induced by

$$\langle a, b\rangle_m := \sum_{\underline{k},\underline{l}\in\mathbb{N}^2_{\geq 1}} \Big(\theta_1^{2m}(k_1+\tfrac{1}{2})^m(l_1+\tfrac{1}{2})^m + \theta_2^{2m}(k_2+\tfrac{1}{2})^m(l_2+\tfrac{1}{2})^m\Big)\overline{a_{\underline{k}\underline{l}}}b_{\underline{k}\underline{l}}\;. \tag{4.11}$$

We introduce already here the block-diagonal matrix $\Theta = \mathrm{diag}(\Theta_1, \Theta_2)$ with $\Theta_i = \begin{pmatrix} 0 & \theta_i \\ -\theta_i & 0 \end{pmatrix}$. A multiplication on the nuclear Fréchet space $\mathcal{A}_\Theta$ is again introduced via multi-indexed matrix bases $(e_{\substack{r_1\; s_1\\ r_2\; s_2}})_{\substack{k_1\; l_1\\ k_2\; l_2}} = \delta_{r_1k_1}\delta_{r_2k_2}\delta_{s_1l_1}\delta_{s_2l_2}$ and $e_{\substack{k_1\; l_1\\ k_2\; l_2}} e_{\substack{m_1\; n_1\\ m_2\; n_2}} := \delta_{l_1m_1}\delta_{l_2m_2}e_{\substack{k_1\; n_1\\ k_2\; n_2}}$. For the isomorphism $\iota_\Theta := \iota_{\mathbb{R}^4}$ of Theorem 3.3 we arrange $\iota_\Theta : \mathcal{A}_\Theta \to \mathcal{S}(\mathbb{R}^4)$ by defining $\iota_\Theta(e_{\substack{k_1\; l_1\\ k_2\; l_2}}) := f^{(\theta_1)}_{k_1-1,l_1-1} \times f^{(\theta_2)}_{k_2-1,l_2-1}$ and linear extension, where $(f^{(\theta_1)}_{kl} \times f^{(\theta_2)}_{mn})(x_1,x_2,x_3,x_4) := f^{(\theta_1)}_{kl}(x_1,x_2)f^{(\theta_1)}_{mn}(x_3,x_4)$. Then the resulting $\star$-product (3.1) on $\mathcal{S}(\mathbb{R}^4)$ takes the form

$$(\phi \star_\Theta \psi)(x) = \int_{\mathbb{R}^4\times\mathbb{R}^4} \frac{dk\, dy}{(2\pi)^4}\; \phi(x+\tfrac{1}{2}\Theta y)\psi(x+y)e^{i\langle y,k\rangle}\;,$$

generalising (4.4). The AF-structure is obtained via the Cantor polynomial which implements the bijection between $\mathbb{N}^2$ and $\mathbb{N}$. After a shift, the Cantor bijection reads $P(\substack{k_1\\ k_2}) := \frac{1}{2}((k_1+k_2)^2 - 3k_1 - k_2 + 2)$. Accordingly, we identify $e_{\substack{k_1\; l_1\\ k_2\; l_2}}$ with the standard matrix basis $e_{\frac{1}{2}((k_1+k_2)^2-3k_1-k_2+2),\frac{1}{2}((l_1+l_2)^2-3l_1-l_2+2)}$. Symmetry between both components selects $\mathcal{A}^{\mathcal{N}}_\Theta \equiv M_{\mathcal{N}(\mathcal{N}+1)/2}(\mathbb{C})$; the embedding $\iota_{\mathcal{N}} : \mathcal{A}^{\mathcal{N}}_\Theta \to \mathcal{A}^{\mathcal{N}+1}_\Theta$ is given by filling up with zeros.

? Question 4.2

Is it true that $\lim_{\mathcal{N}\to\infty}\mathcal{A}_\Theta^{\mathcal{N}}$ described here is, as nuclear AF Fréchet algebra, different from $\lim_{\mathcal{N}\to\infty}\mathcal{A}_\theta^{\mathcal{N}}$ introduced in Sec. 4.1?

Under the isomorphism ι_Θ we obtain finite-dimensional sub-algebras of the 4-dimensional Moyal algebras $\iota_\Theta(\mathcal{A}_\Theta^{\mathcal{N}}) = \mathrm{span}(f^{(\theta_1)}_{k_1 l_1} \times f^{(\theta_2)}_{k_2 l_2} : k_1 + l_1 \leq \mathcal{N}-1\,,\ k_2 + l_2 \leq \mathcal{N}-1)$. Interestingly, this dependence on the length $|{}^{k_1}_{k_1}| := k_1 + k_2$ of double indices will be respected by the covariances chosen in secs. 6, 8 and 9.

4.4 Gauge models

Gauge models arise very naturally in noncommutative geometry [CR87]. With spectral triples either the older formulation [Con94, Con95] via the Dixmier trace or the spectral action [Con96, CC97] is available. Everything works for the Moyal space [Gay03, Vas04, GI05]. But this defines only the classical action which does *not* give rise to a covariance for the free gauge field. Gauge fixing [FP67] is required and can be implemented in noncommutative geometry [Wul00, Per07].

In D dimensions one needs D gauge fields $A_1, \dots, A_D \in \mathcal{A}$ which have to be extended by a Faddeev-Popov ghost c (a Maurer-Cartan form for the BRST-differential s) and two auxiliary objects $\bar{c}, B$. Then a covariance for $(A_1, \dots, A_D, B)$ and for $(c, \bar{c})$ exists, and quantum gauge theory can *formally* be defined along the same lines as before. Partial results for perturbative renormalisation have been achieved, also numerical results have been obtained, but nothing rigorous. Some of these investigations will be reviewed in Sec. 5.

4.5 Fuzzy spaces

The fuzzy sphere [Mad92] is one of the simplest noncommutative spaces. It is obtained by truncating representations of $su(2)$. The algebra $S^2_{\mathcal{N}}$ is identified with mappings from the representation space $\frac{\mathcal{N}}{2}$ of $su(2)$ to itself, thus with the algebra $M_{\mathcal{N}+1}(\mathbb{C})$. The fuzzy sphere $S^2_{\mathcal{N}}$ is generated by $\hat{X}_\nu$, $\nu = 1, 2, 3$, which form an $su(2)$-Lie algebra with suitable rescaling, identified by the requirement that the Casimir operator still fulfils the defining relation of the two-sphere as an operator:

$$[\hat{X}_\mu, \hat{X}_\nu] = \sum_{\kappa=1}^{3} \mathrm{i}\lambda\epsilon_{\mu\nu\kappa}\hat{X}_\kappa\,, \qquad \sum_{\nu=1}^{3} \hat{X}_\nu\hat{X}_\nu = R^2\,, \qquad \frac{R}{\lambda} = \sqrt{\frac{\mathcal{N}}{2}\Big(\frac{\mathcal{N}}{2}+1\Big)}\,. \qquad (4.12)$$

The philosophy about the limit $\mathcal{N} \to \infty$ is quite different than before, namely, the ordinary commutative sphere should arise in the limit. The necessary framework was worked out by Rieffel [Rie04]. Its main steps are the realisation of the $S^2_{\mathcal{N}}$ as compact quantum metric spaces, where Lipschitz seminorms are relevant. The embeddings of $S^2_{\mathcal{N}}$ into bigger structures are achieved via "bridges", and in the end it is shown that the sequence of $S^2_{\mathcal{N}}$ converges to S^2 in the Gromov-Hausdorff topology.

The Lie algebra $su(2)$ generated by J_ν, $\nu = 1,2,3$, acts on $a \in S^2_{\mathcal{N}}$ by the adjoint action $J_\nu a = \frac{1}{\lambda}[\hat{X}_\nu, a]$. Thus, an element $a \in S^2_{\mathcal{N}}$ can be represented by $a = \sum_{l=0}^{\mathcal{N}} \sum_{m=-l}^{l} a_{lm}\Psi_{lm}$, where

$$\sum_{\nu=1}^{3} J_\nu^2 \Psi_{lm} = l(l+1)\Psi_{lm}\,, \quad J_3\Psi_{lm} = m\Psi_{lm}\,, \quad \frac{4\pi}{\mathcal{N}+1}\mathrm{Tr}(\overline{\Psi_{lm}}\Psi_{l'm'}) = \delta_{ll'}\delta_{mm'}\,. \tag{4.13}$$

Other fuzzy spaces include the fuzzy $\mathbb{C}P^2$ [GS99, ABIY02] and the q-deformed fuzzy sphere [GMS01, GMS02].

Remark 4.3 Any quantum field theory shows divergences in some way. The first step to treat them is regularisation. Typically a regularisation destroys the symmetries of the theory so that the limit $\mathcal{N} \to \infty$ is considered. Fuzzy noncommutative spaces [Mad92, Mad95] achieve a regularisation of quantum field theory models without loosing symmetry [GM92, GKP96b, GKP96a, GS99]. Of course, the usual divergences of a quantum field theory on S^2 will reappear in the limit $\mathcal{N} \to \infty$. This limit was investigated in [CMS01]. For the one-loop self-energy in the ϕ^4-model, a finite but non-local difference between the $\mathcal{N} \to \infty$ limit of the fuzzy sphere and the ordinary sphere was found. See [Haw99] for similar calculations. Gauge models on the fuzzy sphere have been studied e.g. in [IKTW01, Ste04]. Another approach to finite quantum field theories on noncommutative spaces is point-splitting via tensor products [CHMS00, BDFP03].

4.6 A non-example: the noncommutative torus

The noncommutative torus A_θ is the universal C^*-algebra generated by two unitaries U, V satisfying $UV = \mathrm{e}^{2\mathrm{i}\pi\theta}VU$, for some $\theta \in \mathbb{R}\setminus\mathbb{Q}$. Several equivalent presentations are known, for instance as irrational rotation algebra. It is probably the best-studied noncommutative space [Rie90]. We are more interested in a Fréchet subalgebra of A_θ which consists of elements of the form

$$a = \sum_{q_1,q_2\in\mathbb{Z}} a_{q_1q_2}U^{q_1}V^{q_2}\,, \qquad \langle a,b\rangle_n := \sum_{q_1,q_2\in\mathbb{Z}} (1+|q_1|+|q_2)^n\overline{a_{q_1q_2}}b_{q_1q_2} < \infty\,. \tag{4.14}$$

The noncommutative torus is not an AF algebra (all AF-algebras have trivial K_1-group, whereas $K_1(A_\theta) = \mathbb{Z}$). However, there is an AF algebra into which A_θ embeds.

The construction relies on the approximative continued fraction expansion of θ and is explained in [LLS01]. It is also shown there that objects of a quantum field theory on the noncommutative torus can conveniently be constructed as limits of finite-dimensional problems. This formulation is employed in [LLS04] to construct matrix models which approximate field theories on the noncommutative torus.

The computation of the spectral action [EILS08] and renormalisation of scalar fields [DPV16] on the noncommutative torus are quite involved. QFTs on projective modules over the noncommutative torus were treated in [GJKW07].

4.7 Other (non-) examples

Many other noncommutative spaces have been studied. A formal definition of free Euclidean scalar fields is mostly possible. An overview goes beyond the scope of this survey, but a few examples can be flashed: κ-deformation [IMSS11], quantum groups [ILS09].

5 QFT on NCG: the first years[4]

5.1 Very short overview about QFT on deformed Minkowski space

The initial work by Doplicher-Fredenhagen-Roberts [DFR95] mentioned in Sec. 1.2 also introduced free relativistic quantum fields on quantum space time and prepared for a perturbative treatment of interactions. Later the Euclidean approach (see Sec. 5.2), formally obtained by a Wick rotation, became much more popular.

It was pointed out in [BDFP02] that a simple Wick rotation does *not* give a meaningful theory on Minkowskian space-time, first of all because formal (i.e. wrong) Wick rotation destroys unitarity [GM00, AGBZ01]. To obtain a consistent Minkowskian quantum field theory, it was proposed in [BDFP02] to iteratively solve the field equations à la Yang-Feldman. See also [Bah03]. Another possibility is time-ordered perturbation theory [LS02c, LS02b]. See also [BFG$^+$03, DS03]. Unfortunately, the resulting Feynman rules become so complicated that apart from tadpole-like diagrams [BFG$^+$03] it seems impossible to perform perturbative calculations in time-ordered perturbation theory. Moreover, it seems impossible to preserve Ward identities [ORZ04], and dispersion relations are severely distorted [Zah06].

A fascinating re-import of quantum fields on deformed Minkowski space back into usual Minkowski space was initiated by Grosse and Lechner [GL07]. They considered a family of free quantum fields indexed by the noncommutativity parameter and related by a Lorentz transform. They showed that the family can be

[4] This is a slight adaptation from a previous review [Wul06].

considered as wedge-local on ordinary Minkowski space, satisfying the axioms, and possessing a non-trivial two-particle S-matrix. Their construction was generalised to the Haag-Kaster setting in [BS08] and termed "warped convolution". After further investigation in the Wightman setting [GL08], it was shown in [BLS11] that warped convolution is an isometric representation of Rieffel's strict deformation quantisation [Rie93] of C^*-dynamical systems.

5.2 Perturbative QFT on deformed Euclidean space

The Euclidean approach started with Filk [Fil96] who showed that the planar graphs of a field theory on the Moyal plane are identical to the commutative theory (and thus have the same divergences). Another achievement in [Fil96] was the definition of the *intersection matrix* of a graph which is read-off from its reduction to a rosette. Later in [VGB99] the persistence of divergences was rephrased in the framework of noncommutative geometry, based on the general definition of a dimension and the noncommutative formulation of external field quantisation. At about the same time, Connes, Douglas and Schwarz [CDS98] investigated the possibility that M-theory is compactified on the noncommutative torus instead of on a ordinary torus. M-theory lives in higher dimensions so that some of them must be compactified to give a realistic model. Compactifying on a noncommutative instead of a commutative torus amounts to turn on a constant background 3-form C. An alternative interpretation based on D-branes on tori in presence of a Neveu-Schwarz B-field was given by Douglas and Hull [DH98].

Knowing that divergences persist in quantum field theories on the Moyal plane, the question arises whether these models are renormalisable. Martín and Sánchez-Ruiz [MSR99] investigated U(1) Yang-Mills theory on the noncommutative $\mathbb{R}^4$ (the same as the Moyal space) at the one-loop level. They found that all one-loop pole terms of this model in dimensional regularisation[5] can be removed by multiplicative renormalisation (minimal subtraction) in a way preserving the BRST symmetry. This is completely analogous to the situation on the noncommutative 4-torus [KW00] where ζ-function techniques and cocycle identities are used to extract pole parts of Feynman graphs, thereby proving multiplicative renormalisation of the initial action and verifying the Ward identities. Around the same time there appeared also an investigation of $(2+1)$-dimensional super-Yang-Mills theory with the two-dimensional space being the noncommutative torus [SJ99].

Inspired by [CDS98] and its companion [DH98], Schomerus [Sch99] observed that in string theory with D-branes, and a magnetic field on the branes, the field theory limit of string theory produces Kontsevich's formal $\star$-product [Kon03] of deformation quantisation. There are also other noncommutative spaces which arise as limiting cases of string theory [ARS99].

[5] There is of course a problem extending Θ to complex dimensions, this is however discussed in [MSR99].

Shortly later, the appearance of noncommutative field theory in the zero-slope limit of type-II string theory was thoroughly investigated by Seiberg and Witten [SW99]. They noticed that passing to the zero-slope limit in two different regularisation schemes (point-splitting and Pauli-Villars) gives rise to a Yang-Mills theory either on noncommutative or on commutative $\mathbb{R}^D$. Since the regularisation scheme cannot matter, Seiberg and Witten argued that both theories must be gauge-equivalent. More general, under an infinitesimal transformation of θ one has to require that gauge-invariant quantities remain gauge-invariant. This requirement leads to the Seiberg-Witten differential equation

$$\frac{dA_\mu}{d\theta_{\rho\sigma}} = -\frac{1}{8}\{A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\}_\star + \frac{1}{8}\{A_\sigma, \partial_\rho A_\mu + F_{\rho\mu}\}_\star \,, \tag{5.1}$$

where $\{a, b\}_\star = a \star b + b \star a$.

The Seiberg-Witten paper [SW99] made the connection between string theory and noncommutative geometry extremely popular. Several lines of research appeared. One important question concerns the extension of the one-loop renormalisation proof of quantum field theories on noncommutative $\mathbb{R}^D$ to any loop order. The main contributions to this programme are due to Chepelev and Roiban [CR00]. Their work uses ribbon graphs in an essential manner. Ribbon graphs were invented by t'Hooft [tH74] for strong interactions and were first employed for noncommutative field theories in [Haw99]. Such ribbon graphs can be drawn on an (oriented) Riemann surface with boundary to which the external legs of the graph are attached. See Sec. 7.1. Chepelev and Roiban derived the parametric representation of a ribbon graph Γ and found the analogues of the Symanzik polynomials U_Γ, V_Γ of Theorem 2.6, which now contain θ in a manner that depends on the topology of Γ. This makes the identification of divergent Hepp sectors more involved. The first conclusion in [CR00] was that a noncommutative field theory is renormalisable iff its commutative counterpart is renormalisable. However, by computing the non-planar one-loop graphs explicitly, Minwalla, Van Raamsdonk and Seiberg pointed out a serious problem in the renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ and ϕ^3-theory on noncommutative $\mathbb{R}^6$ [MVRS00]. It turned out that this problem was simply overlooked in the first version of [CR00], with the power-counting analysis being correct. A refined proof of the power-counting theorem was given in [CR01]. Roughly, the problem discovered in [MVRS00] is the following: Non-planar graphs are regulated by the phase factors in the $\star$-product (4.7), but only if the external momenta of the graph are non-exceptional. Inserting non-planar graphs (declared as regular) as subgraphs into bigger graphs, external momenta of the subgraph are internal momenta for the total graph. As such, exceptional external momenta for the subgraph are realised in the loop integration, resulting in an divergent integral for the total graph. This is the so-called *UV/IR-mixing* problem [MVRS00].

The UV/IR mixing problem received considerable attention. In the following months an enormous number of articles doing (mostly) one-loop computations of all kind of models appeared of which only a few key results should be mentioned in this survey: the two-loop calculation of ϕ^4-theory [ABK00b]; the renormalisation of complex $\phi \star \phi^* \star \phi \star \phi^*$ theory [ABK00a], later explained by a topological analysis [CR01]; computations in noncommutative QED [Hay00]; the calculation of noncommutative $U(1)$ Yang-Mills theory [MST00], with an outlook to super-Yang-Mills theory; the one-loop analysis of noncommutative $U(N)$ Yang-Mills theory [BS01]. Several reviews of these activities appeared, for instance by Konechny and Schwarz with focus on compactifications of M-theory on noncommutative tori as well as on instantons and solitons on noncommutative $\mathbb{R}^D$ [KS02], by Douglas and Nekrasov [DN01] as well as by Szabo [Sza03], both with focus on field theory on noncommutative spaces in relation to string theory. For reviews which include results discussed in Sec. 6, see [Wul06, Riv07b].

It was also investigated whether for gauge theories on noncommutative $\mathbb{R}^D$ the Seiberg-Witten θ-expansion defined in (5.1) can be helpful. In that approach one solves (5.1) as a formal power series in θ, in fact often truncated to finite order. The result is a local quantum field theory which has no relation to the original problem. Anyway, the Seiberg-Witten approach was made popular in [JSSW00] where it was argued that this is the only way to obtain a finite number of degrees of freedom in non-Abelian noncommutative Yang-Mills theory. The solution of (5.1) to all orders in θ and lowest order in $A^{(0)}$ was given in [Gar00]. A generating functional for the complete solution was derived in [JSW01]. The quantum field theoretical treatment of θ-expanded field theories was initiated in [BGP+02]. In [BGG+01] it was shown that the superficial divergences in the photon self-energy are field redefinitions to all orders in θ and any loop order [BGG+01]. However, this fails for more complicated sectors [Wul02]. In fact, in the class of formal power series in θ, quantum field theoretical quantities are (up to field redefinition) the same with or without the Seiberg-Witten map [GW02]. Thus, the Seiberg-Witten map is merely an unphysical (but convenient) change of variables.

In [Ste07] an alternative interpretation of the UV/IR-mixing of $U(1)$-gauge fields was proposed: It does not describe a noncommutative photon but a sort of graviton. See [GSW08] and, for a review, [Ste10]. Phenomenological investigations of θ-expanded field theories have also been performed [CJS+02, BDD+03].

5.3 Numerical simulations

There is another approach which goes back to older work on the large-N limit of two-dimensional $SU(N)$ lattice gauge theory. Here the number of degrees of freedom is reduced and corresponds to a zero-dimensional model [EK82], under the condition that no spontaneous breakdown of the $[U(1)]^4$-symmetry appears. As shown in [GAO83], a spontaneous symmetry breakdown does not appear when twisted boundary conditions are used. This construction was adapted in [AII+00] to type-IIB matrix models. It was shown in [AMNS99] that, imposing a natural constraint for the (finite) matrices, the twisted Eguchi-Kawai construction [GAO83] can be generalised to noncommutative Yang-Mills theory on a toroidal lattice. The appearing gauge-invariant operators are the analogues of Wilson loops [Wil74]. This formulation enabled numerical simulations [BHN02, BHN03] of the various limiting procedures which confirmed conjectures [GS01] about striped and disordered patterns in the phase diagram of spontaneously broken noncommutative ϕ^4-theory.

6 Renormalisation of noncommutative ϕ^4-theory to all orders

With Harald Grosse we started in summer 2002 an investigation of the UV/IR-mixing problem in the matrix basis (4.3) [GBV88] of the Moyal space. In combination with Polchinski's implementation [Pol84] of Wilson's renormalisation group equations [WK74], we hoped to disentangle various limit procedures which occur in the renormalisation of Feynman graphs. Our programme succeeded, but not for the anticipated reason.

6.1 QFT with harmonic oscillator covariance on Moyal space

The Laplace kernel defined in (4.8) with (4.9) takes for the $\mathcal{A}_\Theta^{\mathcal{N}}$-approximation of 4-dimensional Moyal space (see Sec. 4.3) the form [GW05b]

$$\sum_{\nu=4}^{2}\mathrm{Tr}((\partial_\nu a)^*\partial_\nu a) + \mu^2\mathrm{Tr}(X^*X) =: \sum_{k_i,l_i,m_i,n_i=1}^{\mathcal{N}} \Delta_{\begin{smallmatrix}k_1 & l_1\\ k_2 & l_2\end{smallmatrix};\begin{smallmatrix}m_1 & n_1\\ m_2 & n_2\end{smallmatrix}}\, a_{\begin{smallmatrix}k_1 & l_1\\ k_2 & l_2\end{smallmatrix}}\, a_{\begin{smallmatrix}m_1 & n_1\\ m_2 & n_2\end{smallmatrix}}\;,$$

$$\begin{aligned}\Delta_{\begin{smallmatrix}k^1 & l^1\\ k^2 & l^2\end{smallmatrix};\begin{smallmatrix}m^1 & n^1\\ m^2 & n^2\end{smallmatrix}} &= \big(\mu^2+\tfrac{2}{\theta}(m^1+n^1+m^2+n^2+2)\big)\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2} \\ &\quad - \tfrac{2}{\theta}\big(\sqrt{k^1l^1}\,\delta_{n^1+1,k^1}\delta_{m^1+1,l^1}+\sqrt{m^1n^1}\,\delta_{n^1-1,k^1}\delta_{m^1-1,l^1}\big)\delta_{n^2k^2}\delta_{m^2l^2} \\ &\quad - \tfrac{2}{\theta}\big(\sqrt{k^2l^2}\,\delta_{n^2+1,k^2}\delta_{m^2+1,l^2}+\sqrt{m^2n^2}\,\delta_{n^2-1,k^2}\delta_{m^2-1,l^2}\big)\delta_{n^1k^1}\delta_{m^1l^1}\;.\end{aligned} \tag{6.1}$$

We call the line (6.1) the local interaction, the last two lines the nearest-neighbour interaction. When deriving Feynman rules for ribbon graphs on assigns to the edges the covariance, which is the inverse of $\Delta_{\begin{smallmatrix}k^1 & l^1\\ k^2 & l^2\end{smallmatrix};\begin{smallmatrix}m^1 & n^1\\ m^2 & n^2\end{smallmatrix}}$. As shown in [GW05a], renormalisability requires a sufficiently fast decay of the covariance $C(e_{\begin{smallmatrix}k^1 & l^1\\ k^2 & l^2\end{smallmatrix}}, e_{\begin{smallmatrix}m^1 & n^1\\ m^2 & n^2\end{smallmatrix}})$ with $\max(k_i, l_i, m_i, n_i)$ and a bound on partial sums $\sum_{k^1,k^2}\max_{l_i,m_i} C(e_{\begin{smallmatrix}k^1 & l^1\\ k^2 & l^2\end{smallmatrix}}, e_{\begin{smallmatrix}m^1 & n^1\\ m^2 & n^2\end{smallmatrix}})$. It turned out that this would be the case if only the local interaction was present, but the nearest-neighbour interaction spoils it. We therefore decided to scale down, completely ad hoc, the nearest-neighbour terms. The resulting kernel

$$\begin{aligned}H^{\Omega}_{\begin{smallmatrix}m^1 & n^1\\ m^2 & n^2\end{smallmatrix};\begin{smallmatrix}k^1 & l^1\\ k^2 & l^2\end{smallmatrix}} &= \big(\mu^2+\tfrac{2+2\Omega^2}{\theta}(m^1+n^1+m^2+n^2+2)\big)\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2} \\ &\quad - \tfrac{2-2\Omega^2}{\theta}\big(\sqrt{k^1l^1}\,\delta_{n^1+1,k^1}\delta_{m^1+1,l^1}+\sqrt{m^1n^1}\,\delta_{n^1-1,k^1}\delta_{m^1-1,l^1}\big)\delta_{n^2k^2}\delta_{m^2l^2} \\ &\quad - \tfrac{2-2\Omega^2}{\theta}\big(\sqrt{k^2l^2}\,\delta_{n^2+1,k^2}\delta_{m^2+1,l^2}+\sqrt{m^2n^2}\,\delta_{n^2-1,k^2}\delta_{m^2-1,l^2}\big)\delta_{n^1k^1}\delta_{m^1l^1}\end{aligned} \tag{6.2}$$

turned out to describe the harmonic oscillator Schrödinger operator

$$\sum_{\nu=1}^{4}\Big(\mathrm{Tr}((\partial_\nu a)^*\partial_\nu a)+\frac{4\Omega}{\theta^2}\mathrm{Tr}((M_\nu a)^*M_\nu a)\Big)+\mu^2\mathrm{Tr}(a^*a) =: \sum_{k_i,l_i,m_i,n_i=1}^{\mathcal{N}} H^{\Omega}_{\substack{k_1\\k_2}\substack{l_1\\l_2};\substack{m_1\\m_2}\substack{n_1\\n_2}}\, a_{\substack{k_1\\k_2}\substack{l_1\\l_2}}\, a_{\substack{m_1\\m_2}\substack{n_1\\n_2}}\;, \tag{6.3}$$

where M_ν is the pointwise multiplication introduced in (4.10). The introduction of Ω was completely ad hoc. It has, however, one appealing property. The interaction $\mathrm{Tr}(\Phi^n)$, for n even, is invariant under a duality transformation of the Moyal product discovered by Langmann and Szabo [LS02a]. This transformation transforms $\sum_{\nu=1}^{4}\mathrm{Tr}((\partial_\nu\Phi)^*\partial_\nu\Phi)$ into $\sum_{\nu=1}^{4}\mathrm{Tr}((M_\nu\Phi)^*M_\nu\Phi)$ and vice-versa, thus achieving duality-covariance of the model with Ω-term.

For renormalisation a fine control of the covariance is necessary. To invert (6.3) one first diagonalises (6.2) by noticing that the corresponding 3-term relation defines the Meixner polynomials [KS96]. Then the inverse is computed to

$$\begin{aligned}
&C(e_{\substack{m^1\\m^2}\substack{n^1\\n^2}}, e_{\substack{k^1\\k^2}\substack{l^1\\l^2}}) = \frac{\theta}{2(1+\Omega)^2}\delta_{m^1+k^1,n^1+l^1}\delta_{m^2+k^2,n^2+l^2}\\
&\times \sum_{v^1=\frac{|m^1-l^1|}{2}}^{\frac{m^1+l^1}{2}}\;\sum_{v^2=\frac{|m^2-l^2|}{2}}^{\frac{m^2+l^2}{2}} B\big(1+\tfrac{\mu^2\theta}{8\Omega}+\tfrac{1}{2}(m^1+k^1+m^2+k^2)-v^1-v^2,\,1+2v^1+2v^2\big)\\
&\times {}_2F_1\Bigg(\begin{matrix}1+2v^1+2v^2,\;\frac{\mu^2\theta}{8\Omega}-\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2\\ 2+\frac{\mu^2\theta}{8\Omega}+\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2\end{matrix}\Bigg|\frac{(1-\Omega)^2}{(1+\Omega)^2}\Bigg)\\
&\times \prod_{i=1}^{2}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{2v^i}\sqrt{\binom{n^i}{v^i+\frac{n^i-k^i}{2}}\binom{k^i}{v^i+\frac{k^i-n^i}{2}}\binom{m^i}{v^i+\frac{m^i-l^i}{2}}\binom{l^i}{v^i+\frac{l^i-m^i}{2}}}\;.
\end{aligned}\tag{6.4}$$

The free theory is now under control. In the limit $\Omega\to 0$ to the covariance of the Laplacian a confluent hypergeometric function arises.

Formally we define an interacting QFT via the perturbation (3.6) of the Bochner-Minlos measure $d\mathcal{M}_C$ associated with the covariance (6.4):

$$G_{\substack{k^1\\m^1}\substack{l^1\\n^1};\dots;\substack{k^N\\m^N}\substack{l^N\\n^N}} := \frac{\displaystyle\int d\mathcal{M}_C(\Phi)\;\Phi_{\substack{k^1\\m^1}\substack{l^1\\n^1}}\cdots\Phi_{\substack{k^N\\m^N}\substack{l^N\\n^N}}\;\exp\Big(-\frac{\lambda}{4}\mathrm{Tr}(\Phi^4)\Big)}{\displaystyle\int d\mathcal{M}_C(\Phi)\;\exp\Big(-\frac{\lambda}{4}\mathrm{Tr}(\Phi^4)\Big)}\;, \tag{6.5}$$

where $\Phi_{\substack{k^1\\m^1}\substack{l^1\\n^1}} := \Phi(e_{\substack{k^1\\m^1}\substack{l^1\\n^1}})$.

At this stage we are interested in a perturbative expansion as formal power series $G_{\substack{k^1\\m^1}\substack{l^1\\n^1};\dots;\substack{k^N\\m^N}\substack{l^N\\n^N}} = \sum_{v=0}^{\infty}\lambda^v\sum_{g=0}^{\infty}\sum_{B=1}^{N}G^{(v,B,g)}_{\substack{k^1\\m^1}\substack{l^1\\n^1};\dots;\substack{k^N\\m^N}\substack{l^N\\n^N}}$ in which we collect contributions of ribbon graphs with v vertices, B boundary components and genus g. We postpone a discussion of such ribbon graphs to Sec. 7.1. As pointed out in Programme 3.8, for renormalisation we have to restrict to finite matrix size $\mathcal{N}$ and to take a conditional limit $\mathcal{N} \to \infty$ where certain correlation functions (6.5) are held fixed. In the original work [GW05b], instead of a sharp restriction $\mathcal{N}$ a smooth cut-off of matrix indices near $\theta\Lambda^2$ was chosen. This allows to derive first-order Polchinski differential equations [Pol84] which describe the flow of correlation functions when varying the scale Λ. The key strategy is to integrate these differential equation for mixed boundary conditions. Finitely many correlation functions which are termed relevant or marginal are integrated from $\Lambda = 0$ to Λ. These are $G^{(v,1,0)}_{\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0}}$, $G^{(v,1,0)}_{\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0}}$ (which has a relevant and a marginal contribution) as well as $G^{(v,1,0)}_{\genfrac{}{}{0pt}{}{1\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,1}{0\,0}} = G^{(v,1,0)}_{\genfrac{}{}{0pt}{}{0\,0}{1\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,1}} = G^{(v,1,0)}_{\genfrac{}{}{0pt}{}{0\,1}{0\,0};\genfrac{}{}{0pt}{}{1\,0}{0\,0}} = G^{(v,1,0)}_{\genfrac{}{}{0pt}{}{0\,0}{0\,1};\genfrac{}{}{0pt}{}{0\,0}{1\,0}}$. The remaining infinitely many irrelevant correlation functions are integrated from $\Lambda = \infty$ down to Λ. Here a subtlety has be taken into account: A local planar four-point function (which is not the generic case; adjacent indices are the same) must be split as

$$
\begin{aligned}
&G^{(v,1,0)}_{\genfrac{}{}{0pt}{}{k^1\,k^2}{m^1\,m^2};\genfrac{}{}{0pt}{}{k^2\,k^3}{m^2\,m^3};\genfrac{}{}{0pt}{}{k^3\,k^4}{m^3\,m^4};\genfrac{}{}{0pt}{}{k^4\,k^1}{m^4\,m^1}} \\
&\equiv \Big(G^{(v,1,0)}_{\genfrac{}{}{0pt}{}{k^1\,k^2}{m^1\,m^2};\genfrac{}{}{0pt}{}{k^2\,k^3}{m^2\,m^3};\genfrac{}{}{0pt}{}{k^3\,k^4}{m^3\,m^4};\genfrac{}{}{0pt}{}{k^4\,k^1}{m^4\,m^1}} - G^{(v,1,0)}_{\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0}}\Big) + G^{(v,1,0)}_{\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0}} \,.
\end{aligned}
$$

The final term is marginal and integrated from 0 to Λ, whereas the difference of the first two terms must be proved to be irrelevant, integrated from ∞ down to Λ. Similar mixed integrations are necessary for the local and nearest-neighbour planar 2-point function. After all one achieves, order by order in the coupling constant, bounds which allow to take the limit $\Lambda \to \infty$ of any Λ-dependent correlation function. Here bounds on the covariance enter, which in [GW05b] were only numerically achieved. In [RVTW06] rigorous analytic bounds for Ω close to 1 were proved.

Renormalisability of the two-dimensional case has been proved in [GW03]. In this case the oscillator frequency required in intermediate steps can be switched of at the end.

6.2 The β-function

In total we have four marginal and relevant correlation functions integrated from initial values at $\Lambda_R = 0$ to Λ. They can be interpreted as produced by Λ-dependent parameters in the $S_{\text{int}}(\Phi)$-perturbed measure. These are

1. a scale-dependent mass $\mu(\Lambda)$,
2. a scale-dependent oscillator frequency $\Omega(\Lambda)$ in (6.2) or (6.4),
3. a wave-function renormalisation $\Phi \mapsto \sqrt{Z(\Lambda)}\Phi$ which induces a global prefactor $\frac{1}{Z(\Lambda)}$ in front of (6.4), and
4. a combined factor $\lambda \mapsto \lambda(\lambda)Z(\Lambda)^2$ in $S_{\text{int}}(\Phi)$.

The logarithmic derivatives $\beta_\Omega = \Lambda\frac{\partial}{\partial\Lambda}\Omega(\mu_R, \Omega_R, \lambda_R, \Lambda)$ and $\beta_\lambda = \Lambda\frac{\partial}{\partial\Lambda}\lambda(\mu_R, \Omega_R, \lambda_R, \Lambda)$ are referred to as β-functions (of oscillator frequency and coupling constant). Here $\mu_R, \Omega_R, \lambda_R$ are the initial values of mass, oscillator frequency and coupling constant corresponding to the moments held fixed at $\Lambda_R = 0$. At one-loop order one finds [GW04]

$$\beta_\Omega = \frac{\lambda_R \Omega_R}{96\pi^2} \frac{(1-\Omega_R^2)}{(1+\Omega_R^2)^3} \qquad\qquad \beta_\lambda = \frac{\lambda_R^2}{48\pi^2} \frac{(1-\Omega_R^2)}{(1+\Omega_R^2)^3} \ . \tag{6.6}$$

These relations have far-reaching consequences. Namely, $\frac{\lambda(\Lambda)}{\Omega^2(\Lambda)}$ is *constant* under the renormalisation group flow (first noticed by David Broadhurst). Solving the coupled system of differential equations one finds $\lim_{\Lambda\to\infty} \Omega(\Lambda) = 1$ and consequently $\lim_{\Lambda\to\infty} \lambda(\Lambda) = \frac{\lambda_R}{\Omega_R^2}$. The finiteness of $\lambda(\infty)$ is in sharp contrast with the usual (commutative) ϕ^4_4-model which is believed to suffer from the triviality problem. Strictly speaking, triviality is only proved in $4+\epsilon$ dimensions [Aiz81, Frö82], but the perturbative renormalisation group flow indicates triviality also in 4 dimensions. Triviality means that the running coupling constant $\lambda(\Lambda)$ diverges already at finite Λ_0, referred to as the Landau pole [LAK54]. The only possibility to extend the model to $\Lambda \to \infty$ is to let the initial coupling $\lambda_R \to 0$, resulting in a free (or trivial) field theory.

The one-loop absence of the triviality problem had considerable impact on the further development of the subject. It seemed that implementing the constructive (as opposed to perturbative) approach [Riv91] to quantum field theory, the Φ^4-model on 4-dimensional Moyal space could possibly become the first constructed interacting quantum field theory model in 4 dimensions. The first step, the multiscale-slicing of the covariance, was introduced in [RVTW06]. We describe in the next subsection the further development into this direction. Here we focus on the progress which the enlarged community achieved for the β-function.

At the fixed point $\Omega = 1$ of the renormalisation group flow, the covariance simplifies enormously: The matrix Schrödinger operator (6.2) becomes $H^{\Omega=1}_{\substack{m^1\,n^1\\m^2\,n^2};\substack{k^1\,l^1\\k^2\,l^2}} = (\mu^2+\frac{4\Omega^2}{\theta}(m^1+n^1+m^2+n^2+2))\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2}$ with inverse

$$C^{\Omega=1}(e_{\substack{m^1\,n^1\\m^2\,n^2}}, e_{\substack{k^1\,l^1\\k^2\,l^2}}) = \frac{\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2}}{\mu^2+\frac{4\Omega^2}{\theta}(m^1+n^1+m^2+n^2+2)} \ . \tag{6.7}$$

This simplification has early been exploited by Langmann, Szabo and Zarembo [LSZ03, LSZ04] to make contact with the theory of matrix models [DFGZJ95]. We return to this point in sec 7. Also the perturbative calculation of the β-function simplifies enormously. Disertori and Rivasseau proved in [DR07] that at $\Omega = 1$ the β-function remains zero up to three-loop order.

This result clearly suggested the existence of a symmetry transformation which implies $\beta_\lambda = 0$ to all orders in perturbation theory. The transformation was soon identified by Disertori, Gurau, Magnen and Rivasseau in [DGMR07], inspired by one-dimensional Fermi liquid [BM04]. We will derive these Ward-Takahashi identities in slightly generalised form in Sec. 7.3. In [DGMR07] a special case was considered and thereby proved that the divergent part of the 4-point function is, graph by graph, completely determined by the divergent part of the 2-point function. Therefore, it is enough to renormalise the 2-point function; no infinite renormalisation of the coupling constant λ is necessary. This means that the β-function at $\Omega = 1$ vanishes to all orders in perturbation theory.

Research bifurcated at this point. With H. Grosse we developed a solution strategy for models in the Kontsevich class (to be reviewed in secs. 8 and 9). The authors of [DGMR07] tailored constructive renormalisation theory to the noncommutative situation. We briefly review some achievements in the next subsection.

6.3 Constructive renormalisation

Several aspects of constructive renormalisation are best understood in position space. The bosonic covariance is the Mehler kernel [GRVT06],

$$C(x,y) = \frac{\Omega^2}{\pi^2\theta^2}\int_0^\infty \frac{\mathrm{d}t}{\sin^2\frac{4\Omega t}{\theta}}\,\mathrm{e}^{\left(-\frac{\Omega}{2\theta}\|x-y\|^2\coth(\frac{2\Omega}{\theta})-\frac{\Omega}{2\theta}\|x+y\|^2\tanh(\frac{2\Omega}{\theta})-\mu^2 t\right)}\;. \tag{6.8}$$

In [GRVT06] also the fermionic covariance was evaluated, which was used in [VT07] to prove renormalisability to all orders of the orientable noncommutative Gross-Neveu model. In [GR07] the parametric representation was derived, which in particular identified the analogues of the Symanzik polynomials (see Theorem 2.6). Besides linking QFT to algebraic geometry, these Symanzik polynomials are also particular multivariate versions of the Tutte polynomial in graph theory [KRTW10]. The graph-theoretical interpretation of the noncommutative analogue of the Symanzik polynomial (Bollobás-Riordan polynomials) was clarified in [KRVT11].

Traditional bosonic constructive renormalisation employs two technical tools: the cluster expansion and the Mayer expansion [GJS74, BK87]. They are designed for usual Euclidean space which is divided into cubes to test the localisation of vertices. Since vertices of a QFT on Moyal space are not localised, these traditional tools cannot be applied. In [Riv07a], Rivasseau developed the *loop vertex expansion* which serves as substitute for cluster and Mayer expansion. It was made for constructive matrix theory, but it is also a conceptional simplification for traditional constructive renormalisation [MR08]. The loop vertex expansion combines the Hubbard-Stratonovich transform, the BKAR forest formula [BK87, AR95] and the replica trick. For $\mathcal{N}\times\mathcal{N}$-matrices Φ, the Hubbard-Stratonovich transform is based on the following identity:

$$\mathrm{e}^{-\lambda/4\,\mathrm{Tr}(\Phi^4)}\int \mathrm{d}\sigma\;\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}(\sigma^2)} = \int \mathrm{d}\sigma\;\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}(\sigma^2)-\mathrm{i}\sqrt{\lambda/2}\,\mathrm{Tr}(\sigma\Phi^2)}\;,$$

where $d\sigma$ is the translation-invariant Lebesgue measure on $\mathbb{R}^{\mathcal{N}^2}$. Then the Euclidean scalar field Φ is integrated with measure $\mathrm{d}\mathcal{M}_C(\Phi)$, resulting in an effective potential $\mathrm{e}^{-V(\sigma)}$, called the loop vertex, for the intermediate field σ. In the limit $\mathcal{N}\to\infty$ divergences reappear and must be treated by multiscale slicing [GR15]. The factorisation over slices is automated by Grassmann integrals over fermionic variables. Then, in an expansion of the exponential $e^{-W} = \sum_{n=0}^{\infty} \frac{1}{n!}(-W)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{a=1}^{n}(-W_a)\Big|_{W_a=W}$, one artificially distinguishes the factors (replica trick). The replica measure is degenerate, consisting of an $n \times n$-matrix with all entries 1. The BKAR forest formula [BK87, AR95] allows to write such matrices as sum of positive matrices indexed by forests. Here a two-level forest formula, bosonic and fermionic, is necessary. Taking the logarithm amounts to restricting the forest to a sum of trees. In this way the organisation of the perturbative series into trees, briefly outlined at the end of Sec. 2.5, is achieved. For an overview about this and other new methods in constructive QFT, see [GRS14]. In [RW12, RW15] this new constructive renormalisation method was successfully applied to the ϕ^4_2-model. The considerably harder problem, the constructive renormalisation of the Φ^4-model on two-dimensional Moyal space with harmonic propagation of critical frequency $\Omega = 1$, was achieved by Wang [Wan18]. He proved that the logarithm of the partition function is the Borel sum of its perturbation series, analytic in a cardioid domain $|\lambda| < \rho \cos^2(\frac{1}{2}\arg(\lambda))$, excluding the negative reals.

6.4 Other developments

In [dGWW07] (using Mehler kernels in position space) and in [GW07] (using the matrix basis) an induced gauge theory with Ω-term was derived by coupling quantum scalar fields to classical gauge fields A_μ. The induced class of actions can be formulated using covariant coordinates [MSSW00] $X_\mu(x) = (\Theta^{-1})_{\mu\nu} x^\nu + A_\mu$ as (with Einstein's sum convention)

$$S = \int_{\mathbb{R}^4} dx \,\Big(c_1 F_{\mu\nu} \star F^{\mu\nu} + c_2 \{X_\mu, X_\nu\}_\star \star \{X^\mu, X^\nu\}_\star + c_3 X_\mu \star X^\mu\Big)(x)\,, \qquad (6.9)$$

where $F_{\mu\nu} = (\Theta^{-1})_{\mu\nu} - i[X_\mu, X_\nu]_\star = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star$ is the curvature and $[a, b]_\star := a \star b - b \star a$, $\{a, b\}_\star := a \star b + b \star a$. These gauge models were abandoned because of their complicated vacuum structure.

In [GMRT09] another possibility to cure the UV/IR-mixing problem on Moyal space was suggested. It relies on a covariance which in position coordinates reads $C(x, y) = \int dq \, \frac{e^{i\langle q, x-y\rangle}}{\|q\|^2 + \mu^2 + \frac{\alpha}{\theta^2\|q\|^2}}$. Renormalisability to all orders was also proved in [GMRT09]. It was generalised to gauge theories in [BGK+08], where however the renormalisation is much more involved [BKR+10].

The extension of the harmonic oscillator potential to Minkowski space was discussed in [FS09]. Corresponding field theory models are problematic [Zah11].

A spectral noncommutative geometry which leads to (6.3) was analysed in [GW12, GW13a]. It lives in a Clifford algebra of doubled dimension which unites the standard Dirac operator with the “Feynman slash”, central in a new proposal for quanta of geometry [CCM15, CCM14].

6.5 Tensor models

Work on quantum field theories on noncommutative geometries inspired a new research topic: coloured random tensor models. Tensor models were introduced in [ADJ91] to extend the success of matrix models in describing 2-dimensional quantum gravity [DFGZJ95] (see also Sec. 7.2) to higher dimension. However, they were essentially useless because no analogue of the $1/N$-expansion [tH74] was found. In 2009, Gurau [Gur11a] introduced the *colouring* of tensor models. The colouring allowed Gurau [Gur11b, Gur12] and with Rivasseau in [GR11] shortly after to show that the tensor models have an analytically controlled $1/N$-expansion indexed by a positive integer called the *degree*. Then, Ben Geloun and Rivasseau proved in [BGR13] that a certain rank-4 tensor model is renormalisable to all orders in perturbation theory. The proof uses multiscale analysis and relies on experience with topological aspects in QFTs on noncommutative geometries. Soon many more renormalisable tensor models have been found; and they generically show asymptotic freedom [BG14]. First analytical results were established in [BGRR11]: tensor models undergo a phase transition to a theory of continuous random spaces when tuning to criticality. Also the loop vertex expansion [Riv07a] was generalised to tensor models [Gur14].

With these initial achievements, the subject of coloured tensor field theory took a spectacular development. For more information we refer to several reviews already available. There is an early review [GR12] by Gurau and Ryan, a sequence of "Tensor track" lectures by Rivasseau (e.g. [Riv14]), Gurau's book [Gur17]. Most recently connections to the Sachdev-Ye-Kitaev model [SY93, Kit15a, Kit15b, MS16] have been established [Wit16]. The enormous activity goes beyond this survey; we only refer to recent lectures [KPT18].

7 Structures and techniques in matrix models

7.1 Riemann surfaces and ribbon graphs

A *Riemann surface* is a complex-analytic manifold of complex dimension 1 (hence real dimension 2). We are only interested in compact (and connected) Riemann surfaces on which we distinguish a (possibly empty) set $x_1, \dots, x_s$ of marked points. Two such Riemann surfaces are isomorphic if there exists a biholomorphic map between them which sends marked points into marked points. Two (marked) Riemann surfaces are homeomorphic if and only if they have the same number s of marked points and the same genus $g \in \mathbb{N}$. Their common Euler characteristics is $\chi_{g,s} = 2 - 2g - s$. The isomorphism classes of Riemannian manifolds of genus g with s marked points form for $\chi_{g,s} < 0$ a complex orbifold $\mathcal{M}_{g,s}$ of complex dimension $d_{g,s} = 3g - 3 + s$, called the *moduli space of complex curves*.

As conjectured by Witten [Wit91] and proved by Kontsevich [Kon92] the topology of the moduli spaces $\mathcal{M}_{g,s}$ is deeply related to matrix models, and therefore, as argued in the previous sections, to QFT on noncommutative geometries. For the Witten-Kontsevich relation we have to introduce two further structures: a compactification $\overline{\mathcal{M}}_{g,s}$ of the moduli spaces (which we sketch in Sec. 7.2) and ribbon graphs drawn on Riemann surfaces.

A *ribbon graph* is a simplicial 2-complex Γ made of $|\mathcal{V}_\Gamma|$ vertices, $|\mathcal{E}_\Gamma|$ edges and $|\mathcal{F}_\Gamma|$ faces. An edge connects two vertices (possible the same) and separates two faces (possibly the same). Ribbon graphs arise in several variants, depending on presence of marked faces or boundaries. First consider absent boundaries and a ribbon graph Γ with a total number $|\mathcal{F}_\Gamma|$ of faces, s of them marked. This ribbon graph can be drawn on a compact genus-g Riemann surface $\Sigma_{g,s}$ with s marked points. The genus is determined by $\chi = 2-2g-s = |\mathcal{V}_\Gamma| - |\mathcal{E}_\Gamma| + (|\mathcal{F}_\Gamma| - s)$. The drawing partitions $\Sigma_{g,s}$ into $|\mathcal{F}_\Gamma|$ closed subsurfaces, each topologically a disk, and with s of these disks containing precisely one marked point. Conversely, a Riemann surface is the gluing of topological disks into the faces of a ribbon graph. See the left picture in fig. 7.1 for an example.

Closely related are Riemann surfaces and ribbon graphs with boundaries. They arise by removing from the Riemann surface a small open disk (inside the disks glued into the ribbon graph) around a marked point. The previously marked face thus becomes an annulus (see central picture in fig. 7.1).

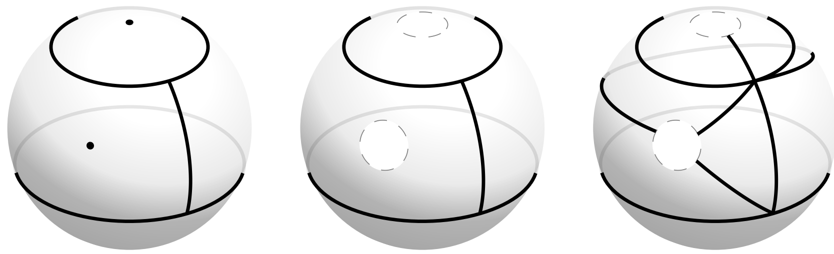

Fig. 7.1 LEFT: a ribbon graph with $|\mathcal{V}| = 2$ vertices (both tri-valent), $|\mathcal{E}| = 3$ edges and $|\mathcal{F}| = 3$ faces of which $s = 2$ are marked, drawn on a sphere (a compact Riemann surface of genus $g = 0$). One marked point is the north pole, the other one near the equator. The Euler characteristics is $\chi = |\mathcal{V}| - |\mathcal{E}| + (|\mathcal{F}| - s) = 2 - 3 + (3-2) = 0 = 2-2g-s$.
CENTRE: Removing small open disks around the marked points produces a surface with boundary, here of $B = 2$ components (topologically a cylinder). The marked faces become an annulus.
RIGHT: 4 half-edges, each connecting a vertex to one of the boundary components, are added to the central picture. In total there are $|\mathcal{V}| = 2$ vertices (one 4-valent and one 6-valent), $|\mathcal{E}| = 7$ (half-)edges and $|\mathcal{F}| = 5$ faces (4 of them external; the remaining internal face coincides with the unmarked face on the left). The Euler characteristics is $\chi = |\mathcal{V}| - |\mathcal{E}| + |\mathcal{F}| = 2 - 7 + 5 = 0$. We later say that this ribbon graph describes a contribution to the planar (1+3)-point function in a QFT-model with both Φ^4- and Φ^6-interactions.

We extend the previous ribbon graphs by admitting half-edges in the annulus. Half-edges connect with its true end to a vertex on the previous marked face and with the other virtual end to the boundary. Crossings of half-edges with other (half-) edges are forbidden. See the right picture in fig. 7.1 for an example. We have two equivalent interpretations of the Euler characteristics. Either we ignore the half-edges (consider them as amputated), or we count them as ordinary edges but also include the additional external faces between half-edges and parts of the boundary.

Ribbon graphs with half-edges ending at boundary components can be contracted by subsequently gluing a pair of half-edges to form a true edge. Two cases must be distinguished:

I. half-edges ending at different boundary components (of the same surface or of disconnected surfaces) are glued; see fig. 7.2;

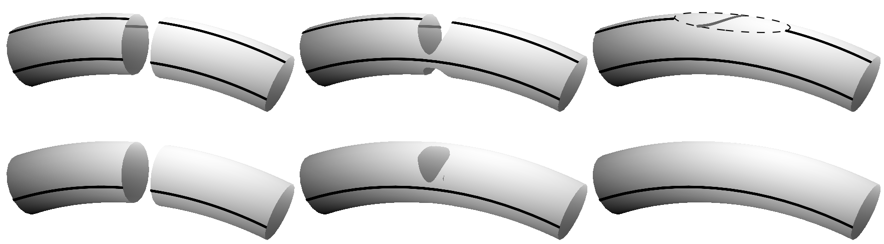

Fig. 7.2 Gluing of half-edges from different boundary components
(Top) [at least one of the previous boundary components carries ≥ 2 half-edges]: The neighbourhoods of the boundary components where we want to glue half-edges can be deformed to a half-cylinder (left). We glue two half-edges to an edge and also the faces bordering the previous half-edges (centre). The result is deformed to a single common boundary component (right). The total number of (half-)edges is reduced by 1 ($\Delta|\mathcal{E}| = -1$), the number of faces is reduced by 2, ($\Delta|\mathcal{F}| = -2$), the number of boundary components is reduced by 1 ($\Delta B = -1$). Vertices are unchanged. If boundary components of a connected surface are glued this way, its Euler characteristics changes by $\Delta|\mathcal{V}| - \Delta|\mathcal{E}| + \Delta|\mathcal{F}| = 0 - (-1) + (-2) = -2(\Delta g) - \Delta B$, i.e. the genus is increased by $\Delta g = +1$. If two disconnected surfaces of topology (g_1, B_1) and (g_2, B_2) are glued, the resulting connected surface has Euler characteristics $(|\mathcal{V}_1| + |\mathcal{V}_2|) - (|\mathcal{E}_1| + |\mathcal{E}_2| - 1) + (|\mathcal{F}_1| + |\mathcal{F}_2| - 2) = (2 - 2g_1 - B_1) + (2 - 2g_2 - B_2) - 1 = 2 - 2(g_1 + g_2) - (B_1 + B_2 - 1)$. Hence, the genus is additive.
(Bottom) [both previous boundary components carry a single half-edge]: After deforming the neighbourhood to a half-cylinder (left), we glue both half-edges to an edge and also the faces bordering the previous half-edges (centre). The resulting boundary component no longer carries half-edges and by convention is shrunk to the empty set (right). We have $\Delta|\mathcal{E}| = -1$, $\Delta|\mathcal{F}| = -1$, $\Delta B = -2$. In case of the same surface, the Euler characteristics changes by $\Delta|\mathcal{V}| - \Delta|\mathcal{E}| + \Delta|\mathcal{F}| = 0 - (-1) + (-1) = -2(\Delta g) - \Delta B$, i.e. the genus is increased by $\Delta g = +1$. In case of different surfaces, $(|\mathcal{V}_1| + |\mathcal{V}_2|) - (|\mathcal{E}_1| + |\mathcal{E}_2| - 1) + (|\mathcal{F}_1| + |\mathcal{F}_2| - 1) = (2 - 2g_1 - B_1) + (2 - 2g_2 - B_2) = 2 - 2(g_1 + g_2) - (B_1 + B_2 - 2)$. Hence, the genus is additive.

II. half-edges ending at the same boundary component are glued; see fig. 7.3.

We see that the subcase where all boundary components carry exactly one half-edge corresponds to the usual framework of bordisms. This framework is relevant for the Atiyah-Segal formulation [Ati88, Seg01] of *topological quantum field theory (TQFT)* [Wit88].

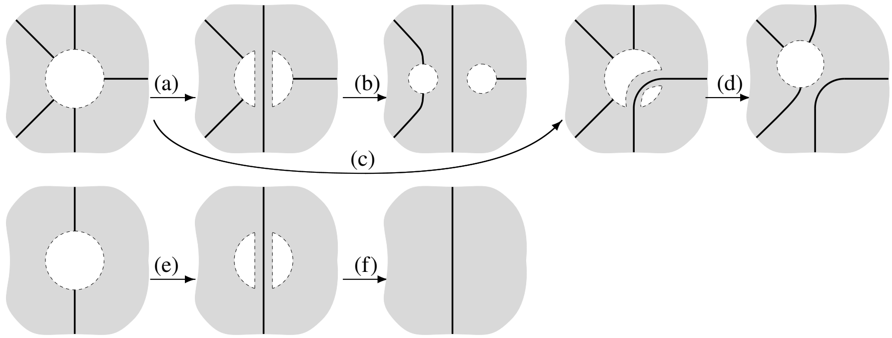

Fig. 7.3 Gluing of a pair of half-edges at the same boundary component which carries N half-edges (Top) (a) We glue for $N \geq 4$ two non-neighboured half-edges to an edge and also join the faces bordering the previous half-edges. The boundary component splits into two. (b) The result is deformed into two disjoint boundary components with at least one and in total $N-2$ half edges. (c) We glue for $N \geq 3$ two neighboured half-edges to an edge and also join the faces bordering the previous half-edges. The boundary components splits into two, but one of them no longer carries any half-edge. (d) The boundary component without half-edge is shrunk to the empty set, the other one deformed into a boundary component with $N-2$ half-edges.

(Bottom) (e) We glue for $N = 2$ both half-edges to an edge and also join the faces bordering the previous half-edges. The boundary component splits into two, but none of them contains any half-edge. (f) Both boundary component without half-edges are shrunk to the empty set.

We consider it worthwhile to extend this axiomatisation to the richer case where several half-edges end at the boundary. Namely, individual ribbon graphs correspond to a single contribution to the perturbative expansion of correlation functions (3.6) in a QFT on noncommutative geometries (see Sec. 3). For a non-perturbative formulation we are interested in the sum over all contributions encoded in ribbon graphs with the same boundary structure, or better we do not want to perturbatively expand at all. This means we encode a non-perturbative amplitude of a QFT on noncommutative geometries in a Riemann surface with boundary and *defects* on the boundary components. Such surfaces can be glued *along the defects*, not along the boundary as a whole. The corresponding rules can be read off from figs. 7.2 and 7.3 by reduction to the end points of half edges. To such a surface $\Sigma^g_{N_1,\dots,N_B}$ of genus g with B boundaries of $N_1,\dots,N_B$ defects, all $N_\beta \geq 1$, we associate an amplitude

$$G^{(g)}_{N_1,\dots,N_B} : \bigotimes_{\beta=1}^{B} \underbrace{\mathcal{A}_* \otimes_c \cdots \otimes_c \mathcal{A}_*}_{N_\beta} \to \mathbb{C} \tag{7.1}$$

where $\otimes_c$ is a cyclic tensor product and the leading $\bigotimes$ a symmetric tensor product. As such we have the first ingredient of a hypothetical functor from the category of Riemann surface with boundary and defects to the category of vector spaces. The gluing of such surfaces along defects is mapped to tensor products with contraction of vector spaces. Such an axiomatic setting analogous to TQFT could be called *noncommutative quantum field theory (NCQFT)* because it exactly captures the non-perturbative formulation of Sec. 3.

? Question 7.1

Can these ideas be turned into a consistent axiomatisation? Is it useful in other areas?

In practice we have more structures on the vector space side:

- The vector spaces we are interested in have trace functionals $T_n(a_1, \ldots, a_n) = \lambda_n \mathrm{Tr}(a_1 \cdots a_N)$. An n-valent vertex in a ribbon graph is mapped to T_n. These vertices alone do not describe any surface, but they encode another building block: an elementary n-disk, i.e. a sphere ($g = 0$) with one boundary component ($B = 1$) and n defects on it. As part of the rules one has to implement the removal of an elementary n-disk with at least one of its defects located on a boundary component. This removal translates to the Dyson-Schwinger equations in quantum field theory. See secs. 7.3, 8 and 9.
- The Ward-Takahashi identities present on the vector space side (see Corollary 7.6 later) should also be transferred to the category of surfaces.

We will see later that Dyson-Schwinger equations and Ward-Takahashi identities completely determine the NCQFT models, at least for Φ^4- and Φ^3-interaction.

7.2 The Kontsevich model

This section gives a short introduction into the Kontsevich model [Kon92]. It became a classical topic which is reviewed and discussed in nearly every book and review on matrix models and 2-dimensional quantum gravity. More details than given here can be found e.g. in the books by Lando and Zvonkin [LZ04], by Eynard [Eyn16] as well as in the review [DFGZJ95] by Di Francesco, Ginsparg and Zinn-Justin. The Kontsevich model comes close to a quantum field theory; it ignores however renormalisation and is understood as a formal power series only. In Sec. 8 we show how to non-perturbatively construct renormalised correlation functions out of a quantum field theory closely related to the Kontsevich model. This construction heavily uses prior work on the original Kontsevich model, most importantly an exact solution [MS91] of a non-linear integral equation and the topological recursion [EO07, Eyn14, Eyn16].

Euclidean quantum gravity is an attempt to give a meaning to the partition function

$$\sum_{\text{topologies}} \int_{\text{metrics}} \mathrm{d}g \ \exp(-S_{EH}(g)) ,$$

where $S_{EH}(g)$ is the Einstein-Hilbert action with cosmological constant. In $D = 2$ dimensions, where the Einstein-Hilbert action reduces (by the Gauß-Bonnet theorem) to the Euler characteristics and (from the cosmological constant) the area of the surface, it was argued at the end of [LPW88] and further elaborated in [Wit90, MP90, Wit91] that topological gravity in 2 dimensions reduces to topological data of the moduli spaces $\{\mathcal{M}_{g,s}\}$ (more precisely their compactifications). Particularly significant are the intersection numbers which we briefly introduce below. More details can be found in [Wit91, LZ04, Eyn16].

For the *Deligne-Mumford compactification* on adds to $\mathcal{M}_{g,s}$ degenerate surfaces, so-called *nodal curves*. They arise from gluing (smaller) Riemann surfaces $\Sigma_1 \cup \cdots \cup \Sigma_\ell$ of Euler characteristics $\chi_i = 2 - 2g_i - s_i < 0$ along each two of their marked points. The resulting nodal curve contributes to $\overline{\mathcal{M}}_{g,s}$ if $2 - 2g - s = \sum_{i=1}^{\ell} \chi_i$ and s is the number of marked points of $\Sigma_1, \dots, \Sigma_\ell$ which are not glued. For example, a sphere with three marked points glued along two of them gives rise to a pinched torus of genus $g = 1$ and one remaining marked point: $\overline{\mathcal{M}}_{1,1} = \mathcal{M}_{1,1} \cup \mathcal{M}_{0,3}$. In general, $\overline{\mathcal{M}}_{g,s}$ has subsets of smaller dimension than $d_{g,s}$; it is called a stack.

On $\overline{\mathcal{M}}_{g,s}$ there is a natural family $\{\mathcal{L}_i\}_{i=1,\dots,s}$ of complex line bundles obtained by taking as fibre of $\mathcal{L}_i$ at $x \in \overline{\mathcal{M}}_{g,s}$ the cotangent space $T^*_{z_i} C$ at the marked point z_i of the curve $x \equiv C$. Complex line bundles are classified by their first Chern class $c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,s}, \mathbb{Q})$. The (commutative) wedge product of $\dim(\overline{\mathcal{M}}_{g,s}) = 3g-3+s$ of these 2-forms $c_1(\mathcal{L}_i)$ is of top degree $2(3g - 3 + s)$, equal to the real dimension of $\overline{\mathcal{M}}_{g,s}$. So the following integral is meaningful:

$$\langle \tau_{d_1} \cdots \tau_{d_s} \rangle := \int_{\overline{\mathcal{M}}_{g,s}} \prod_{j=1}^{s} (c_1(\mathcal{L}_j))^{d_j} , \tag{7.2}$$

which is non-zero only if $d_1 + \cdots + d_s = 3g - 3 + s$. These rational numbers are called *intersection numbers* and provide topological invariants of $\overline{\mathcal{M}}_{g,s}$.

Since the order of marked points does not matter, the intersection numbers can be collected to $\langle \tau_0^{k_0} \tau_1^{k_1} \cdots \rangle$. Their generating function is defined by

$$F(t_0, t_1, \dots) = \sum_{k_0, k_1, \dots = 0}^{\infty} \langle \tau_0^{k_0} \tau_1^{k_1} \dots \rangle \prod_{i=0}^{\infty} \frac{t_i^{k_i}}{k_i!} . \tag{7.3}$$

The simplest cases and an analogy to the *Hermitean one-matrix model* [BK90, DS90, GM90] led Witten to the following conjecture:

Conjecture 7.2 ([Wit91])

1. F obeys the string equation

$$\frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i} . \tag{7.4}$$

2. $U(\{t\}) := \frac{\partial^2}{\partial t_0^2} F(\{t\})$ *satisfies the Korteweg-de Vries equations*

$$\frac{\partial U}{\partial t_n} = \frac{\partial}{\partial t_0} R_{n+1}(U, \partial_{t_0} U, \partial_{t_0}^2 U, \dots) , \tag{7.5}$$

where the R_n are polynomials in U and their t_0-derivatives which are recursively defined by $R_1(U) = U$ and

$$\frac{\partial}{\partial t_0} R_{n+1} = \frac{1}{2n+1} \Big(R_n \frac{\partial U}{\partial t_0} + 2U \frac{\partial R_n}{\partial t_0} + \frac{1}{4} \frac{\partial^3 R_n}{\partial t_0^3} \Big) .$$

Kontsevich [Kon92] achieved a proof of the Witten's conjecture 7.2 by relating F to the partition function of a new type of matrix model, nowadays called the Kontsevich model. Starting point is a theorem by Strebel [Str67] which provides a stratification of decorated moduli spaces by ribbon graphs:

Theorem 7.3 ([Str84]) *On any Riemann surface $C \in \mathcal{M}_{g,s}$ (where $s > 0$ and $\chi_{g,s} < 0$) with marked points $z_1, \dots, z_s$ there is, for any given perimeters $L_1, \dots, L_s \in \mathbb{R}_+$, a unique quadratic differential $\Omega(z) = f(z)(\mathrm{d}z)^2$ such that*

- *f is meromorphic on C, with poles of order 2 at z_i, and no other poles;*
- *horizontal trajectories of Ω, defined by $\mathrm{Im}(\int^z \sqrt{\Omega}) = const$, are either circles about the marked points or critical trajectories which form a ribbon graph with s faces drawn on C.*

The jth face has perimeter L_j when measured with the metric $\frac{1}{2\pi}|\sqrt{\Omega}|$.

A k-fold zero of the quadratic differential gives rise to a $(k+2)$-valent vertex of the ribbon graph of critical trajectories. The ribbon graph of a generic surface $C \in \mathcal{M}_{g,s}$ has only 3-valent vertices (corresponding to simple zeros) and $2(3g-3+s)+s$ edges (combine $2-2g = v-e+s$ with $3v = 2e$) whose lengths $\ell_1, \dots, \ell_{2(3g-3+s)+s} > 0$ are measured by $\frac{1}{2\pi}|\sqrt{\Omega}|$. An edge between different vertices of valencies k_1, k_2 may have degenerate length 0; this collapses the vertices to a single (k_1+k_2-2)-valent vertex and corresponds to a (k_1+k_2-4)-fold zero of the quadratic differential. The topology (g, s) is unchanged by such contractions. It turns out that this assignment of ribbon graphs with 3-valent vertices and (possibly degenerate) edge lengths ℓ_e to a Riemann surface with face perimeters L_i defines (for $s \geq 1$) an isomorphism of orbifolds (see [LZ04, Eyn16])

$$\mathcal{M}_{g,s} \times (\mathbb{R}_+^\times)^s \sim \bigcup_{\mathcal{RG}_{g,s}^3} (\mathbb{R}_+)^{s+2(3g-3+s)} , \tag{7.6}$$

where we denote by $\mathcal{RG}_{g,s}^3$ the set of (connected) genus-g ribbon graphs with s faces and only 3-valent vertices. In particular, the top degree differential forms must be proportional to each other. Kontsevich proved in [Kon92] that

$$\frac{2^{3-3g-s}}{(3g-3+s)!} \Big(\sum_{i=1}^{s} L_i^2 c_1(\mathcal{L}_i) \Big)^{3g-3+s} \wedge \mathrm{d}L_1 \wedge \dots \wedge \mathrm{d}L_s = 2^{2g-2+s} \bigwedge_{e=1}^{s+2(3g-3+s)} \mathrm{d}\ell_e , \tag{7.7}$$

independently of the ribbon graph. Since $L_i = \sum_{e \in \text{edges around face } i} \ell_e$ and every edge e separates two faces $i(e), i'(e)$ (possibly the same), one has $\prod_{i=1}^s \mathrm{e}^{-E_i L_i} = \prod_{e\in\mathcal{E}_\Gamma} \mathrm{e}^{-\ell_e(E_{i(e)}+E_{i'(e)})}$. Inserted into the cell decomposition (7.6) gives after integration with volume forms (7.7) the following

Theorem 7.4 ([Kon92]) *The intersection numbers of line bundles on $\overline{\mathcal{M}}_{g,s}$ are generated by*

$$\sum_{d_1+\cdots+d_s=3g-3+s} \langle \tau_{d_1}\cdots\tau_{d_s}\rangle \prod_{i=1}^{s} \frac{(2d_i-1)!!}{E_i^{2d_i+1}} = \sum_{\Gamma\in\mathcal{RG}^3_{g,s}} \frac{2^{2g+s-2}}{\#\mathrm{Aut}(\Gamma)} \prod_{e\in\mathcal{E}_\Gamma} \frac{1}{E_{i(e)}+E_{i'(e)}}\,, \tag{7.8}$$

where $\mathcal{E}_\Gamma$ denotes the set of edges of Γ and $\#\mathrm{Aut}(\Gamma)$ is the order of the automorphism group of Γ. The faces are labelled by positive real numbers $E_1,\dots,E_s$, and $E_{i(e)}, E_{i'(e)}$ are the labels of the two faces $i(e), i'(e)$ separated by the edge e.

The sum over ribbon graphs with weight $\frac{1}{E_i+E_{i'}}$ for an edge separating faces i,i' is easily interpreted as the perturbative expansion of a partition function. For a diagonal $\mathcal{N}\times\mathcal{N}$-matrix $E=(E_i\delta_{ij})$ and $\mathrm{d}\Phi$ the usual Lebesgue measure on the vector space $M_\mathcal{N}(\mathbb{C})_* \simeq \mathbb{R}^{\mathcal{N}^2}$ of self-adjoint matrices, the Gauß measure

$$\mathrm{d}\mathcal{M}_{C_E}(\Phi) := \frac{\mathrm{d}\Phi\ \exp\Big(-\mathcal{N}\mathrm{Tr}(E\Phi^2+\frac{1}{3}\lambda\Phi^3)\Big)}{\int_{(M_\mathcal{N}(\mathbb{C})_*)'}\mathrm{d}\Phi\ \exp\Big(-\mathcal{N}\mathrm{Tr}(E\Phi^2)\Big)} \tag{7.9}$$

is precisely the Borel measure measure on $(M_\mathcal{N}(\mathbb{C})_*)'$ for a covariance $C_E(e_{kl},e_{mn}) = \frac{\delta_{lm}\delta_{kn}}{\mathcal{N}(E_k+E_l)}$ introduced in Sec. 3.2. In particular, its moments are given by (3.6). It is then a combinatorial exercise to establish

$$\begin{aligned}&\log\Big(\int_{(M_\mathcal{N}(\mathbb{C})_*)'} \mathrm{d}\mathcal{M}_{C_E}(\Phi)\ \mathrm{e}^{-\frac{\lambda}{3}\mathcal{N}\,\mathrm{Tr}(\Phi^3)}\Big)\\ &= \sum_{g=0}^\infty\sum_{s=1}^\infty \frac{1}{s!}\Big(\frac{\lambda^2}{\mathcal{N}}\Big)^{2g-2+s} \sum_{i_1,\dots,i_s=1}^{\mathcal{N}} \Big[\sum_{\Gamma\in\mathcal{RG}_{g,s}} \frac{1}{\#\mathrm{Aut}(\Gamma)}\prod_{e\in\mathcal{E}_\Gamma}\frac{1}{E_{i(e)}+E_{i'(e)}}\Big]\,,\end{aligned} \tag{7.10}$$

where the innermost sum is over labelled ribbon graphs Γ of genus g with s faces labelled $E_{i_1},\dots,E_{i_s}$. These face labels are subsequently summed over its indices from 1 to $\mathcal{N}$.

Inserting (7.8) for [ ] on the rhs of (7.10) shows that the intersection numbers $\langle\tau_{d_1}\cdots\tau_{d_s}\rangle$ are generated by the cubic matrix model (7.9). Strictly speaking, independence of the formal variables t_i in (7.3) is only achieved in the limit $\mathcal{N}\to\infty$. On the other hand, convergence of the sums over $i_1,\dots,i_s$ on the rhs and a meaningful integral on the lhs of (7.10) are not guaranteed for $\mathcal{N}\to\infty$. For these reasons the Kontsevich model is not yet a quantum field theory, but as shown in Sec. 8, it can be turned into one. We remark that (7.10) describes only the vacuum contributions. True correlation functions do arise in the proof [Wit92, DFIZ93] of the string equation (7.4) and the KdV equation (7.5). Some of these correlation functions have a topological interpretation as κ-classes [AC96].

7.3 The Ward-Takahashi identity in matrix models

The Ward-Takahashi identities to be derived in this section play a key rôle in the exact solutions of QFT models in Sec. 8 and 9.

Lemma 7.5 *Let $\mathcal{A}$ be a nuclear AF Fréchet algebra (see Sec. 3) with matrix basis (e_{kl}). Let $\mathcal{F}_{C_E}(J)$ be the Bochner-Minlos characteristic function (3.3) for a covariance $C_E(e_{kl}, e_{mn}) = \frac{\delta_{kn}\delta_{lm}}{E_k+E_l}$, where (E_k) is a sequence of positive real numbers and $J \in \mathcal{A}_*$. Then*

$$(E_k - E_l)\frac{\partial^2 \mathcal{F}_{C_E}(J)}{\partial J_{kn}\partial J_{nl}} = J_{ln}\frac{\partial \mathcal{F}_{C_E}(J)}{\partial J_{kn}} - J_{nk}\frac{\partial \mathcal{F}_{C_E}(J)}{\partial J_{nl}} . \tag{7.11}$$

Proof Expanding $J = \sum_{p,q} J_{pq} e_{pq}$, the characteristic function reads

$$\mathcal{F}_{C_E}(J) = \int_{\mathcal{A}'_*} d\mathcal{M}_{C_E}(\Phi)\; e^{i\sum_{p,q} J_{pq}\Phi(e_{pq})} ,$$

where $\Phi(e_{pq}) := \frac{1}{2}\Phi(e_{pq} + e_{qp}) - \frac{i}{2}\Phi(ie_{pq} - ie_{qp})$. Then

$$\begin{aligned}(E_k - E_l)\frac{\partial^2 \mathcal{F}_{C_E}(J)}{\partial J_{kn}\partial J_{nl}} &= \frac{\partial}{\partial J_{nl}}(E_k + E_n)\frac{\partial \mathcal{F}_{C_E}(J)}{\partial J_{kn}} - \frac{\partial}{\partial J_{kn}}(E_l + E_n)\frac{\partial \mathcal{F}_{C_E}(J)}{\partial J_{nl}} \\ &= \frac{\partial}{\partial J_{nl}}\int_{\mathcal{A}'_*} d\mathcal{M}_{C_E}(\Phi)\; i(E_k + E_n)\Phi(e_{kn})e^{i\sum_{p,q} J_{pq}\Phi(e_{pq})} \\ &\quad - \frac{\partial}{\partial J_{kn}}\int_{\mathcal{A}'_*} d\mathcal{M}_{C_E}(\Phi)\; i(E_l + E_n)\Phi(e_{nl})e^{i\sum_{p,q} J_{pq}\Phi(e_{pq})} .\end{aligned}$$

Now observe that, expanding the exponential and evaluating the pairings (3.5), we have

$$\int_{\mathcal{A}'_*} d\mathcal{M}_{C_E}(\Phi)\; \Phi(e_{kn})e^{i\sum_{p,q} J_{pq}\Phi(e_{pq})} = \frac{iJ_{nk}}{E_k + E_n}\int_{\mathcal{A}'_*} d\mathcal{M}_{C_E}(\Phi)\; e^{i\sum_{p,q} J_{pq}\Phi(e_{pq})}$$

and similarly for the other term. The $(E_k + E_n)$ and $(E_l + E_n)$ terms cancel, and derivative and multiplication with J commute up to a term which also cancels. We end up in (7.11). □

It is now remarkable that, at least formally, Lemma 7.5 extends to interacting QFT models. Namely, the partition function (3.7) can be realised as a derivative operator applied to the characteristic function:

$$\mathcal{Z}_E(J) = \int_{\mathcal{A}'_*} d\mathcal{M}_C(\Phi)\, e^{i\Phi(J) - S_{\text{int}}(\{\Phi(e_{pq})\})} = \exp\Big(-S_{\text{int}}\Big(\Big\{\frac{\partial}{i\partial J_{pq}}\Big\}\Big)\Big)\mathcal{F}_{C_E}(J)\;. \tag{7.12}$$

Since the derivative operator commutes with the lhs of (7.11), we conclude:

Corollary 7.6 *Under the conditions of Lemma 7.5, the partition function (7.12) satisfies*

$$\begin{aligned}(E_k - E_l)\frac{\partial^2 \mathcal{Z}_E(J)}{\partial J_{kn}\partial J_{nl}} &= J_{ln}\frac{\partial \mathcal{Z}_E(J)}{\partial J_{kn}} - J_{nk}\frac{\partial \mathcal{Z}_E(J)}{\partial J_{nl}} \\ &+ \Big[J_{ln}, S_{\text{int}}\Big(\Big\{\frac{\partial}{i\partial J_{pq}}\Big\}\Big)\Big]\frac{\partial \mathcal{Z}_E(J)}{\partial J_{kn}} - \Big[J_{nk}, S_{\text{int}}\Big(\Big\{\frac{\partial}{i\partial J_{pq}}\Big\}\Big)\Big]\frac{\partial \mathcal{Z}_E(J)}{\partial J_{nl}}\;.\end{aligned} \tag{7.13}$$

If $S_{\text{int}} = S_{\text{int}}^{(\mathcal{N})}$ *is a linear combination of traces (4.6), then*

$$\sum_{n=1}^{\mathcal{N}}(E_k - E_l)\frac{\partial^2 \mathcal{Z}_E(J)}{\partial J_{kn}\partial J_{nl}} = \sum_{n=1}^{\mathcal{N}}\Big(J_{ln}\frac{\partial \mathcal{Z}_E(J)}{\partial J_{kn}} - J_{nk}\frac{\partial \mathcal{Z}_E(J)}{\partial J_{nl}}\Big)\;. \tag{7.14}$$

Proof Only (7.14) is to show. For $S_{\text{int}}^{(\mathcal{N})}$ given by (4.6) one has

$$S_{\text{int}}\Big(\Big\{\frac{\partial}{i\partial J_{pq}}\Big\}\Big) = \sum_p \frac{\lambda_p(\mathcal{N})}{i^p p}\sum_{k_1,\dots,k_p=1}^{\mathcal{N}}\frac{\partial^p}{\partial J_{k_1k_2}\cdots\partial J_{k_{p-1}k_p}\partial J_{k_pk_1}}\;.$$

Now both terms in the 2nd line of (7.13) give, when summed over n, the same term

$$\sum_p \frac{\lambda_p(\mathcal{N})}{i^p}\sum_{k_2,\dots,k_p=1}^{\mathcal{N}}\frac{\partial^p}{\partial J_{kk_2}\partial J_{k_2k_3}\cdots\partial J_{k_{p-1}k_p}\partial J_{k_pl}}\mathcal{Z}_E(J)\;.$$

(Without the sum, we have $k_2 \mapsto n$ in the first term, not summed, and $k_p \mapsto n$ in the second term, not summed. Then the difference does not cancel.) □

The *Ward-Takahashi identity* (7.14) was originally proved in [DGMR07] starting from invariance of the matrix Lebesgue measure under unitary transformations. That such transformations are not necessary was only recently observed in [HW18].

Recall from (3.6) that correlation functions are obtained by directional derivatives of the partition function (3.7) for which we have the representation (7.12). A series expansion of exp in (7.12) gives rise to expressions encoded by ribbon graphs on Riemann surfaces with possibly several boundary components and half-edges ending at defects on the boundaries. See Sec. 7.1. As discussed there we recollect the contributions with the same defect structure, which amounts to the same contractions of test functions $J \in \mathcal{A}_*$:

$$\log \frac{\mathcal{Z}_E^{(\mathcal{N})}(J)}{\mathcal{Z}_E^{(\mathcal{N})}(0)} \tag{7.15}$$
$$= \sum_{B=1}^{\infty}\sum_{g=0}^{\infty}\sum_{N_1\leq\cdots\leq N_B}\frac{\mathcal{N}^{2-B-2g}}{S_{N_1\cdots N_B}}\sum_{k_1^1,\dots,k_{N_B}^B}^{(\mathcal{N})} G^{(g)}_{|k_1^1\dots k_{N_1}^1|\dots|k_1^B\dots k_{N_B}^B|}\prod_{\beta=1}^{B}\frac{\mathbb{J}_{k_1^\beta\dots k_{N_\beta}^\beta}}{N_\beta},$$

where $\mathbb{J}_{k_1^\beta\dots k_{N_\beta}^\beta} = \mathrm{i}^{N_\beta}\prod_{i=1}^{N_\beta} J_{k_i^\beta k_{i+1}^\beta}$ with cyclic identification $k_{N_\beta+1}^\beta \equiv k_1^\beta$. The sums over k_i^β are over a finite set determined by $\mathcal{N}$, and $S_{N_1\cdots N_B} = \nu_1!\cdots\nu_s!$ if each ν_j of the N_β coincide. Conversely, the correlation functions $G^{(g)}_{\dots}$ are for pairwise different k_i^β recovered via

$$\sum_{g=0}^{\infty}\mathcal{N}^{-2g}G^{(g)}_{|k_1^1\dots k_{N_1}^1|\dots|k_1^B\dots k_{N_B}^B|} = \frac{1}{\mathcal{N}^{2-B}}\frac{\partial^{N_B}}{\partial\mathbb{J}_{k_{N_B}^B\dots k_1^B}}\cdots\frac{\partial^{N_1}}{\partial\mathbb{J}_{k_{N_1}^1\dots k_1^1}}\log\mathcal{Z}_E^{(\mathcal{N})}(J)\Big|_{J\equiv 0}, \tag{7.16}$$

where $\frac{\partial^{N_\beta}}{\partial\mathbb{J}_{k_{N_\beta}^\beta\dots k_1^\beta}} = (-\mathrm{i})^{N_\beta}\frac{\partial^{N_\beta}}{\partial J_{k_{N_\beta}^\beta k_{N_\beta-1}^\beta}\cdots J_{k_1^\beta k_{N_\beta}^\beta}}$.

Dyson-Schwinger equations result from the interplay between the J-derivatives in (7.16) with the internal J-derivatives in $S_{\mathrm{int}}(\{\frac{\partial}{\mathrm{i}\partial J_{pq}}\})$ according to the representation (7.12) of $\mathcal{Z}_E(J)$. Following an observation in [GW14a], the following programme arises:

Programme 7.7 For QFT models with covariance $C_E(e_{kl},e_{mn}) = \frac{\delta_{kn}\delta_{lm}}{E_k+E_l}$, the interplay of J-derivatives gives rise to expressions known from the Ward-Takahashi identity (7.14). In particular cases, which include the Φ^3 and Φ^4 interactions, the tower of Dyson-Schwinger equations decouples into a closed non-linear equation for the simplest function $G_{\dots}$ and a hierarchy of affine equations for all other functions. The whole model can then (at least in principle) be recursively solved starting from the solution of a single non-linear equation.

This programme succeeded completely for the Φ^3-model (reviewed in Sec. 8) and partially for the Φ^4-model (reviewed in Sec. 9).

8 Exact solution of the Φ^3-model

8.1 Preliminary remarks

It was first stressed in [GS06b] that results about the Kontsevich model can be used to define a quantum field theory on noncommutative Moyal space with Φ^3-interaction and harmonic oscillator covariance (see Sec. 6.1) at critical frequency $\Omega = 1$. By including a linear term proportional to $\mathrm{Tr}(\Phi)$ with carefully adjusted singular coefficient, Grosse and Steinacker were able to renormalise the divergence in the 1-point function. They derived exact formulae for the low-genus one-point function from the intersection numbers computed in [IZ92]. Via quantum equations of motions, higher correlation functions were related to the 1-point function. Shortly later the renormalisation in dimensions $D = 4$ [GS06a] and $D = 6$ [GS08] was also understood. Below we review an extension [GSW17, GSW18, GHW19a] of these techniques based on the Ward-Takahashi identity proved in Corollary 7.6.

Recall the harmonic oscillator Hamiltonian (6.2) which at critical frequency $\Omega = 1$ and in dimension $D \in \{2, 4, 6\}$ reads $H^1(e_{\underline{k}\underline{l}}) = (E_{\underline{k}} + E_{\underline{l}})e_{\underline{k}\underline{l}}$ with $E_{\underline{k}} = \frac{\mu^2}{2} + \frac{D}{\theta} + \frac{4}{\theta}|\underline{k}|$. By $\underline{k} = (k_1, \dots, k_{D/2})$ we understand a $(D/2)$-tuple of natural numbers, of length $|\underline{k}| := k_1 + \dots + k_{D/2}$, which parametrises the matrix bases $(e_{\underline{k}\underline{l}})_{\underline{k},\underline{l}\in\mathbb{N}^{D/2}}$ of the D-dimensional Moyal space (see Sec. 4.3 for $D = 4$). The $E_{\underline{k}}$ will be identified with the labels E_i in the Kontsevich formula (7.8). In fact we can construct QFT models for a more general label function (than resulting from H^1)

$$E = (\tilde{E}_{\underline{k}}\delta_{\underline{k},\underline{l}}), \qquad \tilde{E}_{\underline{k}} := \frac{\tilde{\mu}^2}{2} + \mu^2 e\Big(\frac{|\underline{k}|}{\mu^2 V^{\frac{2}{D}}}\Big), \qquad e(0) \equiv 0\,, \tag{8.1}$$

where $e : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotonously increasing differentiable function. For the covariance of the harmonic oscillator on Moyal space we have $e(x) = x$ independent of D and $V^{\frac{2}{D}} = \frac{\theta}{4}$. The parameter $\mu > 0$ will become the renormalised mass, whereas the bare mass $\tilde{\mu} \equiv \tilde{\mu}(\mathcal{N})$ is a function of $(V, \mathcal{N}, \lambda, \mu)$ identified later. For such label functions $\tilde{E}_{\underline{k}}$ we consider a quantum field theory on a noncommutative geometry, understood as nuclear AF Fréchet algebra $\mathcal{A}$ (see Sec. 3), with generalised matrix basis $(e_{\underline{k}\underline{l}})_{\underline{k},\underline{l}\in\mathbb{N}^{D/2}}$. It is defined by a covariance C_E and an interaction functional which according to Programme 3.8 is parametrised by sequences $\tilde{\mu}, \tilde{Z}, \tilde{\kappa}, \tilde{\nu}, \tilde{\zeta}, \tilde{\lambda}$ in $\mathcal{N}$ which implement the embeddings $\iota_{\mathcal{N}} : \mathcal{A}^{\mathcal{N}} \to \mathcal{A}^{\mathcal{N}+1}$. We have $\mathcal{A}^{\mathcal{N}} = \mathrm{span}(e_{\underline{k}\underline{l}} : \underline{k}, \underline{l} \in \mathbb{N}^{D/2}_{\mathcal{N}})$, where $\mathbb{N}^{D/2}_{\mathcal{N}}$ consists of the $\frac{D}{2}$-tuples $\underline{k}$ with $|\underline{k}| \leq \mathcal{N}$. For the Φ^3-model we choose the covariance

$$C_E^{(\mathcal{N})}(e_{\underline{k}\underline{l}}, e_{\underline{m}\underline{n}}) = \frac{\delta_{\underline{k},\underline{n}}\delta_{\underline{l},\underline{m}}}{V\tilde{Z}(\tilde{E}_{\underline{k}} + \tilde{E}_{\underline{l}})} \tag{8.2}$$

(in which the $\tilde{E}_{\underline{k}}$ also depend on $\mathcal{N}$ via $\tilde{\mu}$) and the interaction functional on $\mathcal{A}'_*$

$$S^{(\mathcal{N})}_{\mathrm{int}}(\Phi) := V\Big(\sum_{\underline{n}\in\mathbb{N}^{D/2}_{\mathcal{N}}} (\tilde{\kappa}+\tilde{\nu}\tilde{E}_{\underline{n}}+\tilde{\zeta}\tilde{E}^2_{\underline{n}})\Phi_{\underline{n}\underline{n}} + \frac{\tilde{\lambda}\tilde{Z}^{\frac{3}{2}}}{3}\sum_{\underline{n},\underline{m},\underline{l}\in\mathbb{N}^{D/2}_{\mathcal{N}}} \Phi_{\underline{n}\underline{m}}\Phi_{\underline{m}\underline{l}}\Phi_{\underline{l}\underline{n}}\Big)\,, \tag{8.3}$$

where $\Phi_{\underline{k}\underline{l}} := \Phi(e_{\underline{k}\underline{l}})$.

8.2 Solution of the planar sector

We return to equation (7.16), adapted to multi-indices $\underline{k}$. Evaluation of the rightmost derivative gives with (7.12) and $\mathcal{F}_{C_E}(J) = \exp\big(-\frac{V}{2}\sum_{\underline{k},\underline{l}\in\mathbb{N}_N^{D/2}} \frac{J_{\underline{k}\underline{l}}J_{\underline{l}\underline{k}}}{\tilde{Z}(\tilde{E}_{\underline{k}}+\tilde{E}_{\underline{l}})}\big)$

$$
\begin{aligned}
(-\mathrm{i})\frac{\partial}{\partial J_{\underline{k}_1^1\underline{k}_{N_1}^1}}\log\mathcal{Z}_E(J) &= \frac{\mathrm{i}VJ_{\underline{k}_{N_1}^1\underline{k}_1^1}}{\tilde{Z}(\tilde{E}_{\underline{k}_1^1}+\tilde{E}_{\underline{k}_{N_1}^1})} - \frac{V(\tilde{\kappa}+\tilde{\nu}\tilde{E}_{\underline{k}_1^1}+\tilde{\zeta}\tilde{E}^2_{\underline{k}_1^1})}{2\tilde{Z}\tilde{E}_{\underline{k}_1^1}}\delta_{\underline{k}_{N_1}^1,\underline{k}_1^1} \\
&\quad + \frac{\tilde{\lambda}\tilde{Z}^{\frac{1}{2}}}{V\mathcal{Z}_E(J)(\tilde{E}_{\underline{k}_1^1}+\tilde{E}_{\underline{k}_{N_1}^1})}\sum_{\underline{n}\in\mathbb{N}_N^{D/2}}\frac{\partial^2}{\partial J_{\underline{k}_{N_1}^1\underline{n}}\partial J_{\underline{n}\underline{k}_1^1}}\mathcal{Z}_E(J)\;.
\end{aligned}
\tag{8.4}
$$

The first line only contributes to $B = 1$ and $N_1 \leq 2$ in (7.16). Inserting (7.15) into the second line and evaluating the remaining derivatives in (7.16) gives exact (non-perturbative) *Dyson-Schwinger equations* between the correlation functions $G^{(g)}_{\dots}$. However, since the last line of (8.4) has one more derivative than the lhs, these equations relate an N-point function to the not yet known $(N+1)$-point function. This would make the Dyson-Schwinger equations rather useless. We are rescued by the Ward-Takahashi identity of Corollary 7.6. There we have to replace $E_k \mapsto V\tilde{Z}\tilde{E}_{\underline{k}}$ to account for the different conventions in (8.2) and Lemma 7.5. We also rescale $J_{kl} \mapsto VJ_{\underline{k}\underline{l}}$. The second line of (8.3) is a trace and does not contribute to the second line of (7.13) when summed over $\underline{n}$. For $|\underline{k}| \neq |\underline{l}|$ we can divide by $\tilde{E}_{\underline{k}} - \tilde{E}_{\underline{l}} \neq 0$ (by the assumptions on $e(x)$ in (8.1)):

$$
\begin{aligned}
\sum_{\underline{n}\in\mathbb{N}_N^{D/2}}\frac{\partial^2\mathcal{Z}_E(J)}{\partial J_{\underline{k}\underline{n}}\partial J_{\underline{n}\underline{l}}} &= \frac{V}{\tilde{Z}(\tilde{E}_{\underline{k}}-\tilde{E}_{\underline{l}})}\sum_{\underline{n}\in\mathbb{N}_N^{D/2}}\Big(J_{\underline{l}\underline{n}}\frac{\partial\mathcal{Z}_E(J)}{\partial J_{\underline{k}\underline{n}}} - J_{\underline{n}\underline{k}}\frac{\partial\mathcal{Z}_E(J)}{\partial J_{\underline{n}\underline{l}}}\Big) \\
&\quad + \frac{V}{\mathrm{i}\tilde{Z}}(\tilde{\nu}+\tilde{\zeta}(\tilde{E}_{\underline{k}}+\tilde{E}_{\underline{l}}))\frac{\partial\mathcal{Z}_E(J)}{\partial J_{\underline{k}\underline{l}}}\;, \qquad \text{for } |\underline{k}|\neq|\underline{l}|\;.
\end{aligned}
\tag{8.5}
$$

The lhs is (assuming $N_1 > 1$; the case $N_1 = 1$ implies $|\underline{k}| = |\underline{l}|$) precisely of the form needed in the second line of (8.4). Starting with the 1-point function $G_{|\underline{k}|}$ which needs a special treatment, one obtains a hierarchy of equations which only depend on data known by induction. Hence, if $G_{|\underline{k}|}$ can be determined, the exact solution of the Φ^3-matricial QFT model is possible.

For $B = 1$, with $N_1 - 1$ further derivatives applied to (8.4), one obtains after insertion of (8.5) and suppression of the upper index $\underline{k}_i^1 \equiv \underline{k}_i$

$$\tilde{Z}(1-\tilde{\lambda}\tilde{Z}^{-\frac{1}{2}}\tilde{\zeta})\Big((\tilde{E}_{\underline{k}_1}+\tilde{E}_{\underline{k}_{N_1}})-\frac{\tilde{\lambda}\tilde{Z}^{-\frac{1}{2}}\tilde{\nu}}{(1-\tilde{\lambda}\tilde{Z}^{-\frac{1}{2}}\tilde{\zeta})}\Big)G^{(g)}_{|\underline{k}_1\dots\underline{k}_{N_1}|}$$
$$=\delta_{g,0}\delta_{N_1,2}+\tilde{\lambda}\tilde{Z}^{\frac{1}{2}}\frac{G^{(g)}_{|\underline{k}_1\dots\underline{k}_{N_1-1}|}-G^{(g)}_{|\underline{k}_2\dots\underline{k}_{N_1}|}}{\tilde{E}_{\underline{k}_1}-\tilde{E}_{\underline{k}_{N_1}}}\;. \tag{8.6}$$

This equation fixes $\tilde{\mu}(\mathcal{N}), \tilde{\lambda}(\mathcal{N}), \tilde{\zeta}(\mathcal{N})$ in terms of $\mu, \lambda, \tilde{Z}(\mathcal{N}), \tilde{\nu}(\mathcal{N})$ to

$$\tilde{Z}(1-\tilde{\lambda}\tilde{Z}^{-\frac{1}{2}}\tilde{\zeta})=1\;,\qquad \tilde{\lambda}\tilde{Z}^{\frac{1}{2}}=\lambda\;,\qquad \tilde{\mu}^2=\mu^2+\frac{\tilde{\lambda}\tilde{Z}^{-\frac{1}{2}}\tilde{\nu}}{(1-\tilde{\lambda}\tilde{Z}^{-\frac{1}{2}}\tilde{\zeta})}\;.$$

The recursion can be solved explicitly [GSW17]:

$$G^{(g)}_{|\underline{k}_1\dots\underline{k}_{N_1}|}=\sum_{i=1}^{N_1}\frac{W^{(g)}_{\underline{k}_i}}{2\lambda}\prod_{j=1,j\neq i}^{N_1}\frac{\lambda}{E^2_{\underline{k}_i}-E^2_{\underline{k}_j}}\;,\qquad \frac{W^{(g)}_{\underline{k}}}{2\lambda}:=G^{(g)}_{|\underline{k}|}+\frac{\delta_{g,0}E_{\underline{k}}}{\lambda}\;, \tag{8.7}$$

where $E_{\underline{k}}:=\tilde{E}_{\underline{k}}-\frac{1}{2}\tilde{\lambda}\tilde{Z}^{\frac{1}{2}}\tilde{\nu}=\mu^2(\frac{1}{2}+e(\frac{|\underline{k}|}{\mu^2V^{2/D}}))$ is the renormalisation of (8.1) which replaces $\tilde{\mu}$ by μ.

The 1-point function is directly obtained from (8.4) at $\underline{k}^1_1=\underline{k}^1_{N_1}\equiv\underline{k}$ and $J\equiv 0$. After renormalisation and insertion of (8.7) one arrives at

$$\sum_{h=0}^{g}W^{(h)}_{\underline{k}}W^{(g-h)}_{\underline{k}}+2\tilde{\nu}\lambda W^{(g)}_{\underline{k}}+\frac{2\lambda^2}{V}\sum_{\underline{n}\in\mathbb{N}^{D/2}_{\mathcal{N}}}\frac{W^{(g)}_{\underline{k}}-W^{(g)}_{\underline{n}}}{E^2_{\underline{k}}-E^2_{\underline{n}}}$$
$$=\Big(\frac{4E^2_{\underline{k}}}{\tilde{Z}}-\tilde{\nu}^2\lambda^2\Big(1+\frac{1}{\tilde{Z}}\Big)-\frac{4\tilde{\kappa}\lambda}{\tilde{Z}}\Big)\delta_{g,0}-4\lambda^2G^{(g-1)}_{|\underline{k}|\underline{k}|}\;. \tag{8.8}$$

The following observation is crucial. It was already employed in [MS91] and brought to perfection in topological recursion [EO07, Eyn16]:

Observation 8.1 For a real parameter c soon to be determined, replacing $4E^2_{\underline{k}}+c\mapsto z^2$ by a complex variable, the equations (8.8) have a continuation $W^{(g)}_{|\underline{k}|}\mapsto W^{(g)}(z)$ which are holomorphic outside the support of $\{(4E^2_{\underline{n}}+c)^{\frac{1}{2}}\}$. All other Dyson-Schwinger equations extend similarly to several complex variables and define holomorphic functions $G^{(g)}(z^1_1,\dots,z^1_{N_1}|\dots|z^B_1,\dots,z^B_{N_B})$ of $z^\beta_i\in\mathbb{C}\setminus\{0\}$, possibly with the exception of diagonals $z^\beta_i=\pm z^\beta_j$. The original matricial correlation functions are recovered from $W^{(g)}_{\underline{k}}=W^{(g)}((4E^2_{\underline{k}}+c)^{\frac{1}{2}})$ and

$$G^{(g)}_{|\underline{k}^1_1\dots\underline{k}^1_{N_1}|\dots|\underline{k}^B_1\dots\underline{k}^B_{N_B}|}$$
$$=G^{(g)}\Big((4E^2_{\underline{k}^1_1}+c)^{\frac{1}{2}},\dots,(4E^2_{\underline{k}^1_{N_1}}+c)^{\frac{1}{2}}\Big|\dots\Big|(4E^2_{\underline{k}^B_1}+c)^{\frac{1}{2}},\dots,(4E^2_{\underline{k}^B_{N_B}}+c)^{\frac{1}{2}}\Big)\;.$$

We pass to mass-dimensionless quantities via multiplication by specified powers of μ [GSW18]. This amounts to choose the mass scale as $\mu = 1$. Also $V = (\frac{\theta}{4})^{D/2}$ is dimensionless from now on. It is convenient to introduce a measure

$$\mathrm{d}\varrho(y) = \frac{8\lambda^2}{V} \sum_{\underline{n}\in\mathbb{N}^{D/2}_{\mathcal{N}}} \delta(y^2 - (4E^2_{\underline{n}} + c))\mathrm{d}y^2 = \frac{8\lambda^2}{V} \sum_{\underline{n}\in\mathbb{N}^{D/2}_{\mathcal{N}}} \delta(y - (4E^2_{\underline{n}} + c)^{\frac{1}{2}})\mathrm{d}y\ . \quad (8.9)$$

The measure has support on $[\sqrt{1+c}, \sqrt{\Lambda^2_{\mathcal{N}} + c}]$, where $\Lambda^2_{\mathcal{N}} = \max(4E^2_{\underline{n}}\ :\ |\underline{n}| = \mathcal{N})$. From now on we drop $\mathcal{N}$ in favour of a dependence of correlation functions on a scale Λ that in the end has to be sent to ∞ by the same renormalisation procedure of Programme 3.8.

After these reparametrisations, eq. (8.8) takes the form

$$\begin{aligned}&\sum_{h=0}^{g} W^{(h)}(z)W^{(g-h)}(z) + 2\tilde{\nu}\lambda W^{(g)}(z) + \int_{\sqrt{1+c}}^{\sqrt{\Lambda^2+c}} \mathrm{d}\varrho(y)\ \frac{W^{(g)}(z) - W^{(g)}(y)}{z^2 - y^2} \\ &= \Big(\frac{z^2 - c}{\tilde{Z}} - \tilde{\nu}^2\lambda^2\Big(1 + \frac{1}{\tilde{Z}}\Big) - \frac{4\tilde{\kappa}\lambda}{\tilde{Z}}\Big)\delta_{g,0} - 4\lambda^2 G^{(g-1)}(z|z)\ . \quad (8.10)\end{aligned}$$

For $g = 0$ one obtains in this way a closed non-linear equation for a sectionally holomorphic function $W^{(0)}(z)$ which (at $\tilde{\kappa} = \tilde{\nu} = 1 - \tilde{Z} = 0$) was solved by Makeenko and Semenoff [MS91] using techniques for boundary values of sectionally holomorphic functions. Alternatively, it can be solved by residue techniques for meromorphic functions [Eyn16]. These methods are easily extended to include $\tilde{\nu}, \tilde{\kappa}, \tilde{Z}$ and give

$$W^{(0)}(z) = \frac{z}{\sqrt{\tilde{Z}}} - \lambda\tilde{\nu} + \frac{1}{2}\int_{\sqrt{1+c}}^{\sqrt{\Lambda^2+c}} \frac{\mathrm{d}\varrho(y)}{y(z+y)}\ , \quad (8.11)$$

$$\text{where} \quad \frac{c}{\tilde{Z}} + \frac{1}{\sqrt{\tilde{Z}}}\int_{\sqrt{1+c}}^{\sqrt{\Lambda^2+c}} \frac{\mathrm{d}\varrho(y)}{y} = -\frac{4\lambda\tilde{\kappa}}{\tilde{Z}} - \frac{\lambda^2\tilde{\nu}^2}{\tilde{Z}}\ . \quad (8.12)$$

We have eventually reached the point where we can describe the renormalisation procedure. It depends on a spectral dimension which characterises the growth rate of $|\underline{n}| \mapsto E_{\underline{n}}$:

Definition 8.2 • *dimension 0*: $\sum_{\underline{n}\in\mathbb{N}^{D/2}} \frac{1}{E_{\underline{n}}}$ converges. No renormalisation is necessary, $\tilde{\kappa} = \tilde{\nu} = \tilde{Z} - 1 = 0$. The finite number c is determined from the consistency equation $c + \int_{\sqrt{1+c}}^{\infty} \frac{\mathrm{d}\varrho(y)}{y} = 0$. This is the case considered in [MS91] and [Eyn16] for the usual Kontsevich model. It is not realised on Moyal space.

- *dimension 2*: $\sum_{\underline{n}\in\mathbb{N}^{D/2}} \frac{1}{E_{\underline{n}}}$ diverges but $\sum_{\underline{n}\in\mathbb{N}^{D/2}} \frac{1}{E^2_{\underline{n}}}$ converges. We can set $\tilde{\nu} = \tilde{Z} - 1 = 0$ and determine $\tilde{\kappa}(c, \Lambda)$ as the solution of (8.12). The finite parameter c translates into a normalisation condition. A natural choice is $G^{(0)}_{|\underline{0}|} = 0$, which by (8.7) translates into $W^{(0)}(\sqrt{1+c}) = 1$. The equation for c is then the limit $\Lambda \to \infty$ of (8.11) at $z = \sqrt{1+c}$.

- *dimension 4*: $\sum_{\underline{n}\in\mathbb{N}^{D/2}} \frac{1}{E_{\underline{n}}^2}$ diverges but $\sum_{\underline{n}\in\mathbb{N}^{D/2}} \frac{1}{E_{\underline{n}}^3}$ converges. We can set $\tilde{Z} = 1$ and determine $\tilde{\nu}(c,\Lambda)$ by the condition that the rhs of (8.11) at $z = \sqrt{1+c}$ equals $1 = W^{(0)}(\sqrt{1+c})$ for any Λ. Then determine $\tilde{\kappa}(c,\Lambda)$ as the solution of (8.12). The finite parameter c is typically obtained from a condition $\frac{\mathrm{d}}{\mathrm{d}E_{\underline{n}}} G^{(0)}_{|\underline{n}|}\big|_{\underline{n}=0} = 0$ which by (8.7) translates into $\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z} W^{(0)}(z)\big|_{z=\sqrt{1+c}} = 1$.
- *dimension 6*: $\sum_{\underline{n}\in\mathbb{N}^{D/2}} \frac{1}{E_{\underline{n}}^3}$ diverges but $\sum_{\underline{n}\in\mathbb{N}^{D/2}} \frac{1}{E_{\underline{n}}^4}$ converges. We determine $\tilde{\nu}(c,\Lambda,\tilde{Z})$ and $\tilde{\kappa}(c,\Lambda,\tilde{Z})$ as for dimension 4 and now fix $\tilde{Z}(c,\Lambda)$ from $\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z} W^{(0)}(z)\big|_{z=\sqrt{1+c}} = 1$. The finite parameter c is typically obtained from another condition $\frac{\mathrm{d}^2}{\mathrm{d}E_{\underline{n}}^2} G^{(0)}_{|\underline{n}|}\big|_{\underline{n}=0} = 0$ with by (8.7) translates into $(\frac{c}{z}\frac{\mathrm{d}}{\mathrm{d}z} + (z^2 - c)\frac{d^2}{dz^2}) W^{(0)}(z)\big|_{z=\sqrt{1+c}} = 0$.
- *dimension* ≥ 8: $\sum_{\underline{n}\in\mathbb{N}^{D/2}} \frac{1}{E_{\underline{n}}^4}$ diverges. No quantum field theory can be achieved in this case.

The normalisation conditions can for $D \in \{0,2,4,6\}$ be summarised to the following equation for the crucial parameter c:

$$(-c)\Big(\frac{2}{1+\sqrt{1+c}}\Big)^{\delta_{D,2}+\delta_{D,4}} = \int_{\sqrt{1+c}}^{\infty} \frac{\mathrm{d}\varrho(y)}{y(\sqrt{1+c}+y)^{D/2}} \;. \tag{8.13}$$

Recalling the prefactor λ^2 in (8.9), the implicit function theorem guarantees a smooth solution $c(\lambda, \{E_{\underline{n}}\})$ of (8.13) inside a disk of radius λ_c.

Remark 8.3 It is remarkable that in this QFT model on noncommutative geometry $\mathcal{A} = \bigcup_N \mathcal{A}^N$ such a non-perturbative renormalisation procedure can be established. Usually one can only renormalise individual (ribbon) graphs with recursive diving into subgraphs [BP57]. This recursive prescription in encoded in a Hopf algebra [Kre98, CK98] and relates to other occurrences of Hopf algebras in noncommutative geometry [CM98]. Here, this diving into subgraphs is completely avoided. One can show [GSW18] that, breaking down these exact formulae into ribbon graphs, there is perfect agreement with the usual BPHZ renormalisation [BP57, Hep66, Zim69], including the handling of overlapping divergences by Zimmermann's forest formula [Zim69].

Remark 8.4 Furthermore, it turns out that in $D = 6$ dimensions and for $\lambda \in \mathbb{R}$ the β-function of the coupling constant λ is *positive* [GSW18]. Nonetheless, there is no triviality problem and the model can rigorously be constructed. We see this as indication that also for realistic quantum field theories (such as QED and the Higgs sector of the standard model) with positive β-function a construction is not completely impossible.

The Dyson-Schwinger equations for higher correlation functions have a simple solution in terms of $1+\cdots+1$-point functions [GSW17]:

$$
\begin{aligned}
&G^{(g)}(z^1_1,\dots,z^1_{N_1}|\dots|z^B_1,\dots,z^B_{N_B}) \qquad (8.14)\\
&=\sum_{i_1=1}^{N_1}\cdots\sum_{i_B=1}^{N_B} G^{(g)}(z^1_{i_1}|\dots|z^B_{i_B})\Big(\prod_{j_1=1,j_1\neq i_1}^{N_1}\frac{4\lambda}{(z^1_{i_1})^2-(z^1_{j_1})^2}\Big)\cdots\Big(\prod_{j_B=1,j_B\neq i_B}^{N_B}\frac{4\lambda}{(z^B_{i_B})^2-(z^B_{j_B})^2}\Big)\,.
\end{aligned}
$$

For $B=1$ on has to replace $G(z_i)$ by $\frac{1}{2\lambda}W^{(g)}(z_i)$, see (8.7). The remaining Dyson-Schwinger equations for the $1+\cdots+1$-point functions all involve the integral operator $\hat{K}_z$ defined by

$$
\hat{K}_z f(z) := (W^{(0)}(z)+\lambda\tilde{\nu})f(z)+\frac{1}{2}\int_{\sqrt{1+c}}^{\sqrt{\Lambda^2+c}} d\varrho(y)\;\frac{f(z)-f(y)}{z^2-y^2}\,. \qquad (8.15)
$$

For instance, the Dyson-Schwinger equation for the planar (i.e. $g=0$) $(1+1)$-point function becomes $\hat{K}_{z_1}G^{(0)}(z_1|z_2)=-\lambda G^{(0)}(z_1,z_2,z_2)$ and has the solution

$$
G^{(0)}(z_1|z_2)=\frac{4\lambda^2}{z_1z_2(z_1+z_2)^2}\,. \qquad (8.16)
$$

Formula (8.16) is the same as the cylinder amplitude [Eyn16, Thm. 6.4.3] in the usual Kontsevich model! This is a clear indication that all topological sectors other than the disk of the Φ^3-QFT model on noncommutative geometries are governed by a universal structure called *topological recursion*. This indication was fully confirmed in [GHW19a]. We give more details in the next subsection. The sector $(g=0, B\geq 3)$ is also accessible by combinatorial techniques [GSW17] which give the following result:

$$
G^{(0)}(z^1|\dots|z^B)=\frac{d^{B-3}}{dt^{B-3}}\Big|_{t=0}\Big(\frac{(-2\lambda)^{3B-4}}{(R(t))^{B-2}\prod_{\beta=1}^{B}((z^\beta)^2-2t)^{\frac{3}{2}}}\Big)\,, \qquad (8.17)
$$

$$
R(t) := \lim_{\Lambda\to\infty}\Big(\frac{1}{\sqrt{\tilde{Z}(\Lambda)}}-\int_{\sqrt{1+c}}^{\sqrt{\Lambda^2+c}}\frac{d\varrho(y)}{y(y+\sqrt{y^2-2t})\sqrt{y^2-2t}}\Big)\,.
$$

The t-differentiation produces a polynomial in $\frac{1}{z_\beta}$, of odd degree in each variable, with coefficients in rational functions of the moments

$$
\varrho_l := \lim_{\Lambda\to\infty}\Big(\frac{\delta_{l,0}}{\sqrt{\tilde{Z}(\Lambda)}}-\frac{1}{2}\int_{\sqrt{1+c}}^{\sqrt{\Lambda^2+c}}\frac{d\varrho(y)}{y^{3+2l}}\Big)\,. \qquad (8.18)
$$

These moments play a key rôle in the solution of the non-planar sector.

8.3 The non-planar sector

The main tool is a differential operator identified in [GHW19a],

$$\hat{\mathrm{A}}^{\dagger g}_{z_1,\dots,z_B} := \sum_{l=0}^{3g+B-4} \Big(- \frac{(3+2l)\varrho_{l+1}}{\varrho_0 z_B^3} + \frac{3+2l}{z_B^{5+2l}} \Big) \frac{\partial}{\partial \varrho_l} + \sum_{i=1}^{B-1} \frac{1}{\varrho_0 z_B^3 z_i} \frac{\partial}{\partial z_i} \ . \qquad (8.19)$$

It is understood to act on Laurent polynomials in $z_2, \dots, z_B$, bounded at ∞, with coefficients in rational functions of the moments ϱ_l defined in (8.18). These differential operators play the rôle of 'boundary creation operators':

Theorem 8.5 ([GHW19a]) *The* $1+\dots+1$*-point function at genus* $g \geq 1$ *is given by*

$$G^{(g)}(z_1|\dots|z_B) = (2\lambda)^{3B-4} \hat{\mathrm{A}}^{\dagger g}_{z_1,\dots,z_B} \big(\hat{\mathrm{A}}^{\dagger g}_{z_1,\dots,z_{B-1}} \big(\cdots \hat{\mathrm{A}}^{\dagger g}_{z_1,z_2} W^{(g)}(z_1) \dots \big) \big) \ , \qquad (8.20)$$

for $z_i \neq 0$.

The proof is lengthy. It consists in checking that taking (8.20) as an ansatz, all Dyson-Schwinger equations for functions with $B \geq 2$ boundary components are identically fulfilled, provided that $W^{(g)}(z_1)$ is an odd Laurent polynomial of z_1, bounded at ∞, which depends only on $\varrho_0, \dots, \varrho_{3g-2}$. The assumptions are later confirmed via solution of (8.21).

Equation (8.10) takes with $\hat{K}_z$ defined in (8.15) for $g \geq 1$ the form

$$\hat{K}_z W^{(g)}(z) = -\frac{1}{2} \sum_{h=1}^{g-1} W^{(h)}(z) W^{(g-h)}(z) - 2\lambda^2 G^{(g-1)}(z|z). \qquad (8.21)$$

We recall $G^{(0)}(z|z) = \frac{\lambda^2}{z^4}$ and $G^{(g-1)}(z|z) = (2\lambda)^2 \hat{\mathrm{A}}^{\dagger g-1}_{z,z} W^{(g-1)}(z)$ for $g \geq 2$. Thus, all $W^{(g)}(z)$ can recursively evaluated if $\hat{K}_z$ has a tractable inverse. This is the case:

Proposition 8.6 *Let* $f(z) = \sum_{k=0}^{\infty} \frac{a_{2k}}{z^{2k}}$ *be an even Laurent series about* $z = 0$ *bounded at* ∞*. Then the inverse of the integral operator* $\hat{K}_z$ *is given by the residue formula*

$$\Big[z^2 \hat{K}_z \frac{1}{z} \Big]^{-1} f(z) = - \operatorname*{Res}_{z' \to 0} \big[K(z,z') \, f(z') \mathrm{d}z' \big] \ , \qquad (8.22)$$

$$\textit{where} \quad K(z,z') := \frac{2}{(W^{(0)}(z') - W^{(0)}(-z'))(z'^2 - z^2)}.$$

The proof can be directly achieved from the series expansion of $K(z,z')$ [GHW19a]. Inspiration for a residue formula (8.22) comes from *topological recursion*:

Remark 8.7 A $(1+1+\dots+1)$-point function of genus g with B boundary components fulfils a universal structure when expressed in terms of $\omega_{g,B}$ defined by

$$\omega_{g,B}(z_1,\dots,z_B) := \Big(\prod_{i=1}^{B} z_i \Big) \Big(G^{(g)}(z_1|\dots|z_B) + 16\lambda^2 \frac{\delta_{g,0}\delta_{2,B}}{(z_1^2 - z_2^2)^2} \Big), \qquad B > 1$$

$$\omega_{g,1}(z) := \frac{z W^{(g)}(z)}{2\lambda} \ .$$

Furthermore, let $y(x)$ be the *spectral curve* defined by $x(z) = z^2$ and

$$y(z) := \frac{W^{(0)}(z)}{2\lambda} = \frac{z}{2\lambda\sqrt{Z}} - \frac{\tilde{\nu}}{2} + \frac{1}{4\lambda}\int_{\sqrt{1+c}}^{\sqrt{1+\Lambda^2}} \frac{\mathrm{d}\varrho(t)}{t(t+z)} .$$

It can be checked that with these definitions, up to trivial redefinitions by powers of 2λ, the theorems proved in topological recursion [Eyn16] apply. These determine all $\omega_{g,B}$ with $2-2g-B<0$ out of the initial data $y(z)$ and $\omega_{0,2}$:

Theorem 8.8 ([Eyn16, Thm. 6.4.4]) *For a subset* $I=\{i_1,\dots,i_{|I|}\}\subset\{1,\dots,B\}$ *let* $z_I:=(z_{i_1},\dots,z_{i_{|I|}})$. *Then for* $2-2g-(1+B)<0$ *the function* $\omega_{g,B+1}(z_0,\dots,z_B)$ *is given by the topological recursion*

$$\begin{aligned}\omega_{g,B+1}(z_0,\dots,z_B) = \operatorname*{Res}_{z\to 0}\Big[K(z_0,z)\,\mathrm{d}z\Big(\omega_{g-1,B+2}(z,-z,z_1,\dots,z_B)\\ + \sideset{}{'}\sum_{\substack{h+h'=g\\ I\uplus I'=\{1,\dots,B\}}} \omega_{h,|I|+1}(z,z_I)\,\omega_{h',|I'|+1}(-z,z_{I'})\Big)\Big],\end{aligned}$$

where $K(z_0,z)=\frac{1}{(z^2-z_0^2)(y(z)-y(-z))}$ *and the sum* $\sum'$ *excludes* $(h,I)=(0,\emptyset)$ *and* $(h,I)=(g,\{1,\dots,B\})$.

Similar topological recursions have been established in various topics, for instance in the one-matrix model [Eyn04], the two-matrix model [CEO06], in the theory of Gromov-Witten invariants [BKMP09] and for hyperbolic volumes of moduli spaces [Mir07].

Proposition (8.6) applied to (8.21) provide with Theorem (8.5) and (8.14) the recursive solution of the planar sector. One can achieve more:

Proposition 8.9 ([GHW19a]) *There is a unique function* F_g *of* $\{\varrho_l\}$ *satisfying* $W^{(g)}(z)=(2\lambda)^4\hat{\mathrm{A}}_z^{\dagger g}F_g(\varrho)$,

$$F_1(\varrho) = -\frac{1}{24}\log\varrho_0, \qquad F_g(\varrho) = \frac{1}{(2-2g)(2\lambda)^4}\sum_{l=0}^{\infty}\operatorname*{Res}_{z\to 0}\Big[\frac{z^{4+2l}\varrho_l}{3+2l}W^{(g)}(z)\mathrm{d}z\Big] .$$

Here we close the circle because the $F_g(\varrho)$ are, after a change of variables, nothing but the restriction to genus g of the generating functions (7.3) of intersection numbers. The change of variables turns out to be $\varrho_0=1-t_0$ and $-(2l+1)!!\varrho_l=t_{l+1}$. This follows essentially from comparison between (8.18) and (7.8) at infinitesimally small λ, i.e. $c=0$, or from a similar relation in topological recursion. Therefore, given the usual generating function of intersection numbers (see [IZ92]),

$$F_g(t_0,t_2,t_3,\dots,t_{3g-2}) := \sum_{(k)} \frac{\langle\tau_2^{k_2}\tau_3^{k_2}\dots\tau_{3g-2}^{k_{3g-2}}\rangle}{(1-t_0)^{2g-2+\sum_i k_i}}\prod_{i=2}^{3g-2}\frac{t_i^{k_i}}{k_i!}, \quad \sum_{i\geq 2}(i-1)k_i = 3g-3, \tag{8.23}$$

we have $F_g(\varrho) := (2\lambda)^{4g-4}F_g(t)\big|_{1-t_0=\varrho_0,\,t_l=-(2l-1)!!\varrho_{l-1}}$, and from there we build $W^{(g)}=(2\lambda)^4\hat{\mathrm{A}}_z^{\dagger g}F_g(\varrho)$ and higher functions via Theorem 8.5.

On the other hand, the solution of equation (8.21) via Proposition 8.6 also permits to derive a formula for F_g. For that it is convenient to collect all genera to $\hat{\mathrm{A}}^\dagger_z := \sum_{g=1}^\infty \hat{\mathrm{A}}^{\dagger,g}_z$ and $\mathcal{Z}^{np}_V := \exp\big(\sum_{g=1}^\infty V^{2-2g} F_g(\varrho)\big)$. Then (8.21) is equivalent to

$$0 = \Big(\frac{2V^2}{(2\lambda)^4}\hat{K}_z \hat{\mathrm{A}}^\dagger_z + \Big(\hat{\mathrm{A}}^\dagger_z + \frac{1}{\varrho_0 z^4}\frac{\partial}{\partial z}\Big)\hat{\mathrm{A}}^\dagger_z + \frac{V^2}{4(2\lambda)^4 z^4}\Big)\mathcal{Z}^{np}_V . \tag{8.24}$$

Inverting $\hat{K}_z \hat{\mathrm{A}}^\dagger_z$ via Proposition 8.6 and Proposition 8.9, and separating the case $g = 1$, the following result can be established:

Theorem 8.10 ([GHW19a]) *The generating function (8.23) of intersection numbers on the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of complex curves of genus g [Wit91, Kon92] is obtained from*

$$\exp\Big(\sum_{g=2}^\infty N^{2-2g} F_g(t)\Big) = \exp\Big(-\frac{1}{N^2}\Delta_t + \frac{F_2(t)}{N^2}\Big)1 , \tag{8.25}$$

where $F_2(t) = \dfrac{7}{240}\cdot\dfrac{t_2^3}{3!T_0^5} + \dfrac{29}{5760}\dfrac{t_2 t_3}{T_0^4} + \dfrac{1}{1152}\dfrac{t_4}{T_0^3}$ *with* $T_0 := (1-t_0)$ *generates the intersection numbers of genus 2 and* $\Delta_t = -\sum_{i,j}\hat{g}^{ij}\partial_i\partial_j - \sum_i \hat{\Gamma}^i \partial_i$ *is a Laplacian on the formal parameters* $t_0, t_2, t_3, \dots$ *given by*

$$\begin{aligned}
\Delta_t := &-\Big(\frac{2t_2^3}{45T_0^3} + \frac{37t_2t_3}{1050T_0^2} + \frac{t_4}{210T_0}\Big)\frac{\partial^2}{\partial t_0^2} - \Big(\frac{2t_2^3}{27T_0^4} + \frac{1097t_2t_3}{12600T_0^3} + \frac{41t_4}{2520T_0^2}\Big)\frac{\partial}{\partial t_0} \\
&- \sum_{k=2}^\infty \Big(\Big(\frac{2t_2^2}{45T_0^3} + \frac{2t_3}{105T_0^2}\Big)t_{k+1} + \frac{t_2\mathcal{R}_{k+1}(t)}{2T_0} + \frac{3\mathcal{R}_{k+2}(t)}{2(3+2k)}\Big)\frac{\partial^2}{\partial t_k \partial t_0} \\
&- \sum_{k,l=2}^\infty \Big(\frac{t_2 t_{k+1} t_{l+1}}{90T_0^2} + \frac{t_{k+1}\mathcal{R}_{l+1}(t)}{4T_0} + \frac{t_{l+1}\mathcal{R}_{k+1}(t)}{4T_0} \\
&\qquad\qquad + \frac{(1{+}2k)!!(1{+}2l)!!\mathcal{R}_{k+l+1}(t)}{4(1{+}2k{+}2l)!!}\Big)\frac{\partial^2}{\partial t_k \partial t_l} \\
&- \sum_{k=2}^\infty \Big(\Big(\frac{19t_2^2}{540T_0^4} + \frac{5t_3}{252T_0^3}\Big)t_{k+1} + \frac{t_2\mathcal{R}_{k+1}(t)}{48T_0^2} + \frac{\mathcal{R}_{k+2}(t)}{16(3+2k)T_0} + \frac{t_2 t_{k+2}}{90T_0^3} \\
&\qquad\qquad + \frac{\mathcal{R}_{k+2}(t)}{2T_0}\Big)\frac{\partial}{\partial t_k}
\end{aligned}$$

with $\mathcal{R}_m(t) := \dfrac{2}{3}\displaystyle\sum_{k=1}^m \frac{(2m{-}1)!!\, kt_{k+1}}{(2k{+}3)!!T_0}\sum_{l=0}^{m-k}\frac{l!}{(m{-}k)!}B_{m-k,l}\Big(\Big\{\frac{j!t_{j+1}}{(2j{+}1)!!T_0}\Big\}_{j=1}^{m-l+1}\Big)$.

The $F_g(t)$ are recursively extracted from $\mathcal{Z}_g(t) := \frac{1}{(g-1)!}(-\Delta_t + F_2(t))^{g-1}1$ and

$$F_g(t) = \mathcal{Z}_g(t) - \frac{1}{(g-1)!}\sum_{k=2}^{g-1} B_{g-1,k}\Big(\{h!F_{h+1}(t)\}_{h=1}^{g-k}\Big)\,. \tag{8.26}$$

Here and in Theorem 8.10, $B_{m,k}(\{x\})$ are the Bell polynomials.

Theorem 8.10 seems to be closely related with $\exp(\sum_{g\geq 0} F_g) = \exp(\hat{W})1$ proved by Alexandrov [Ale11], where $\hat{W} := \frac{2}{3}\sum_{k=1}^{\infty}(k+\frac{1}{2})t_k\hat{L}_{k-1}$ involves the generators $\hat{L}_n$ of the Virasoro algebra. Including N and moving $\exp(N^2F_0+F_1)$ to the other side, Δ_t is in principle obtained via Baker-Campbell-Hausdorff formula from Alexandrov's equation, but evaluating the necessary commutators is not viable.

Theorem 8.10 and eq. (8.26) are easily implemented in any computer algebra system and quickly allow to compute intersection numbers to moderately large g. The result is confirmed by other implementations such as [DSvZ18].

8.4 Summary

The construction of the renormalised Φ^3_D-QFT model on noncommutative geometries of dimension $D \leq 6$ is complete. Given the mass-renormalised sequence $(E_{\underline{n}})$ for the covariance and renormalised coupling constant λ, the planar 1-point function $G^{(g)}(z) = \frac{W^{(0)}(z)-\sqrt{z^2-c}}{2\lambda}$ is described by (8.11) with parameters chosen according to Definition 8.2 and eq. (8.13). It gives rise to planar functions with several boundary components by (8.16), (8.17 and (8.14). The non-planar sector is obtained by the following steps:

1. Compute the free energy $F_g(t)$ via Theorem 8.10 and the note thereafter. Take $F_1 = -\frac{1}{24}\log T_0$ for $g = 1$. Alternatively, start from intersection numbers obtained by other methods (e.g. [DSvZ18]).
2. Change variables to $\varrho_0 = 1 - t_0$ and $\varrho_l = -\frac{t_l+1}{(2l+1)!!}$, where ϱ_l are given by (8.18) for the measure (8.9) and with c implicitly defined by (8.13).
3. Apply to the resulting $F_g(\varrho)$ according to Proposition 8.9 and Theorem 8.5 the boundary creation operators $\hat{\mathrm{A}}^{\dagger g}_{z_1,\dots,z_B}\circ\dots\hat{\mathrm{A}}^{\dagger g}_{z_1,z_2}\circ\hat{\mathrm{A}}^{\dagger g}_{z_1}$ defined in (8.19). Multiply by $(2\lambda)^{4g+3B-4+\delta_{B,1}}$ to obtain $G^{(g)}(z_1|\cdots|z_B)$.
4. Pass to $G^{(g)}(z^1_1\dots z^1_{N_1}|\dots|z^B_1\dots z^B_{N_B})$ via difference quotients (8.14).

Finally, evaluate at $z^\beta_{k_\beta} \mapsto (4E^2_{\underline{p}^\beta_{k_\beta}}+c)^{1/2}$ to obtain $G^{(g)}_{|\underline{p}^1_1\cdots\underline{p}^1_{N_1}|\cdots|\underline{p}^B_1\cdots\underline{p}^B_{N_B}|}$, where $E_{\underline{p}}$ arises by mass-renormalisation from the $\tilde{E}_{\underline{p}}$ in the initial action (8.3) of the model.

There remains a final problem. We have achieved exact formulae for any correlation function at any fixed genus, which corresponds to a convergent sum over amplitudes encoded in infinitely many ribbon graphs. It remains to understand the sum $\sum_{g=0}^{\infty} V^{-2g}G^{(g)}_{\dots}$ over genera in (7.16), which by the steps 1.–4. is derived from the sum $\sum_{g=2}^{\infty} N^{2-2g}F_g(t)$ in (8.25). The generating function F_g contains $p(3g-3) \sim \frac{1}{12\sqrt{3}(g-1)}\exp(\pi\sqrt{2g-2})$ terms, which are too many for ordinary convergence:

? Question 8.11

Is it possible to Borel-sum the series $\sum_{g=1}^{\infty} V^{2-2g} F_g(\varrho)$ for $\varrho_l > 0$? Note that this corresponds to $t_l < 0$ for $l \geq 2$ and $\lambda \in \mathrm{i}\mathbb{R}$ (see (8.9)). One should use asymptotic estimates [MZ15] of intersection numbers or the heat flow of Δ_t given in Theorem 8.10, or a recent estimate by Eynard [Eyn19].

One could also ask whether the metric $\hat{g}$ in $\Delta_t = -\sum_{i,j} \hat{g}^{ij}\partial_i\partial_j - \sum_i \hat{\Gamma}^i \partial_i$ has any significance:

? Question 8.12

Is $\hat{\Gamma}^i$ a Levi-Civita connection for $\hat{g}^{ij}$? Does $\hat{g}^{ij}$ admit a reasonable definition of a volume and a curvature? Is there any relation to the Weil-Petersson volumina which are deeply connected with intersection numbers [AC96, Mir07]?

9 Exact solution of the Φ^4-model

9.1 The planar sector

It would have far-reaching consequences if the Φ^4-model admitted a similar construction as the Φ^3-model. After a decade of work and many failed attempts, such a construction in now in reach. Combining Dyson–Schwinger equations and Ward–Takahashi identity, we derived in [GW09] a closed equation for the planar two-point function of the Φ^4-model. This equation is complicated. A considerable simplification to an angle functions of essentially only one variable was achieved in [GW14a]. In [PW18] the exact solution was found for the important special case of a scaling limit of two-dimensional Moyal space. In [GHW19b] it was understood how to generalise this construction to any covariance of dimension ≤ 4. For finite matrices a representation by rational functions arises. This rationality is strong support for the conjecture that the Φ^4-model is integrable. Below we give a few details. It remains to be seen whether correlation functions at higher (g, B) satisfy any sort of topological recursion [EO07], and to identify the integrable stucture.

The Φ^4-model is defined by the identical covariance $C_E^{(\mathcal{N})}$ as given before in (8.2) with (8.1) but instead of (8.3) by a *quartic* interaction functional

$$S_{\mathrm{int}}^{(\mathcal{N})}(\Phi) := V\frac{\lambda\tilde{Z}^2}{4}\sum_{\underline{k},\underline{l},\underline{m},\underline{n}\in\mathbb{N}_{\mathcal{N}}^{D/2}} \Phi_{\underline{k}\underline{l}}\Phi_{\underline{l}\underline{m}}\Phi_{\underline{m}\underline{n}}\Phi_{\underline{n}\underline{k}}\;, \qquad \Phi_{\underline{k}\underline{l}} := \Phi(e_{\underline{k}\underline{l}})\;. \tag{9.1}$$

By the same techniques as described in Sec. 7.3—Dyson–Schwinger equations combined with Ward–Takahashi identity—exact non-perturbative equations for correlation functions are obtained. Below we give these equations for planar functions with a single boundary component ($g = 0, B = 1$):

Proposition 9.1 ([GW14a])

$$G^{(0)}_{|\underline{k}\underline{l}|} = \frac{1}{\tilde{Z}(\tilde{E}_{\underline{k}} + \tilde{E}_{\underline{l}})} - \frac{\tilde{Z}\lambda}{\tilde{E}_{\underline{k}} + \tilde{E}_{\underline{l}}}\frac{1}{V}\sum_{\underline{n}\in\mathbb{N}^{D/2}_{\mathcal{N}}}\Big(G^{(0)}_{|\underline{k}\underline{l}|}G^{(0)}_{|\underline{k}\underline{n}|} - \frac{G^{(0)}_{|\underline{n}\underline{l}|} - G^{(0)}_{|\underline{k}\underline{l}|}}{\tilde{Z}(\tilde{E}_{\underline{n}} - \tilde{E}_{\underline{k}})}\Big), \tag{9.2}$$

$$G^{(0)}_{|\underline{k}_0\underline{k}_1\dots\underline{k}_{N-1}|} \tag{9.3}$$
$$= (-\lambda)\sum_{l=1}^{\frac{N-2}{2}}\frac{G^{(0)}_{|\underline{k}_0\underline{k}_1\dots\underline{k}_{2l-1}|}G^{(0)}_{|\underline{k}_{2l}\underline{k}_{2l+1}\dots\underline{k}_{N-1}|} - G^{(0)}_{|\underline{k}_{2l}\underline{k}_1\dots\underline{k}_{2l-1}|}G^{(0)}_{|\underline{k}_0\underline{k}_{2l+1}\dots\underline{k}_{N-1}|}}{(E_{\underline{k}_0} - E_{\underline{k}_{2l}})(E_{\underline{k}_1} - E_{\underline{k}_{N-1}})}.$$

The general case including higher-genus contributions can be found in [GW14a]. A manifestly symmetric variant of (9.2) was derived in [PW18].

The first equation (9.2) requires renormalisation (see below), whereas (9.3) is automatically expressed in terms of the renormalised 2-point function $G^{(0)}_{|\underline{k}\underline{l}|}$ and the mass-renormalised sequence $E_{\underline{k}} := \tilde{E}_{\underline{k}} + \frac{\mu^2-\tilde{\mu}^2}{2}$ with $E_{\underline{n}} - E_{\underline{k}} \equiv \tilde{E}_{\underline{n}} - \tilde{E}_{\underline{k}}$. Thus, no renormalisation of the coupling constant λ is necessary, which means that the β-function vanishes identically (provided that the 2-point function can be renormalised). This is the non-perturbative proof of [DGMR07].

Equation (9.3) is the analogue of (8.6). Its explicit solution is a sum of monomials in $G^{(0)}_{|\underline{k}_{2i}\underline{k}_{2j+1}|}$, $\frac{1}{E_{\underline{k}_{2i}}-E_{\underline{k}_{2j}}}$ and $\frac{1}{E_{\underline{k}_{2i+1}}-E_{\underline{k}_{2j+1}}}$. As proved in [dJHW19], these monomials are in one-to-one correspondence with *Catalan tables* of length $\frac{N}{2}$, which are iterations of Catalan tuples.

Definition 9.2 A *Catalan tuple* of length k is a $(k+1)$-tuple $\tilde{t} = (t_0, \dots, t_k)$ with $\sum_{i=0}^k t_i = k$ and $\sum_{i=0}^l t_i > l$ for any $l < k$. We let $|\tilde{t}|$ be the length of $\tilde{t}$.

A *Catalan table* of length k is a $k+1$-tuple $T = \langle\tilde{t}_0, \dots, \tilde{t}_k\rangle$ of Catalan tuples $\tilde{t}_i$ such that $(|\tilde{t}_0| + 1, |\tilde{t}_1|, \dots, |\tilde{t}_1|)$ is itself a Catalan tuple.

There are $C_k = \frac{1}{k+1}\binom{2k}{k}$ different Catalan tuples of length k (whence its name) and $\frac{1}{k+1}\binom{3k+1}{k}$ different Catalan tables T of length k. A Catalan table of length $\frac{N}{2}$ simultaneously encodes a pocket tree for the monomials in $G^{(0)}_{\underline{k}_{2i}\underline{k}_{2j+1}}$, a rooted tree for the monomials in $\frac{1}{E_{\underline{k}_{2i}}-E_{\underline{k}_{2j}}}$ and an opposite tree for the monomials in $\frac{1}{E_{\underline{k}_{2i+1}}-E_{\underline{k}_{2j+1}}}$.

By the same reasoning as before, any solution of (9.2) extends to a sectionally holomorphic function $G^{(0)}(\zeta_1, \zeta_2)$ with $G^{(0)}_{|\underline{k}\underline{l}|} = G^{(0)}(\zeta_1, \zeta_2)\big|_{\zeta_1=\tilde{E}_{\underline{k}}-\tilde{\mu}^2/2, \zeta_2=\tilde{E}_{\underline{l}}-\tilde{\mu}^2/2}$ which satisfies

$$(\tilde{\mu}^2+\zeta_1+\zeta_2)\tilde{Z}G^{(0)}(\zeta_1, \zeta_2) \tag{9.4}$$
$$= 1 - \lambda\int_0^{\Lambda^2} dt\, \varrho(t)\,\Big(\tilde{Z}G^{(0)}(\zeta_1, \zeta_2)\,\tilde{Z}G^{(0)}(\zeta_1, t) - \frac{\tilde{Z}G^{(0)}(t, \zeta_2) - \tilde{Z}G^{(0)}(\zeta_1, \zeta_2)}{(t - \zeta_1)}\Big),$$

where $\varrho(t) := \frac{1}{V}\sum_{\underline{n}\in\mathbb{N}_{\mathcal{N}}^{D/2}} \delta(t-(\tilde{E}_{\underline{n}}-\frac{1}{2}\tilde{\mu}^2))$ and $\tilde{E}_{\underline{n}} \in [\frac{1}{2}\tilde{\mu}^2, \Lambda^2+\frac{1}{2}\tilde{\mu}^2]$ for all $\underline{n}\in\mathbb{N}_{\mathcal{N}}^{D/2}$. Next we temporarily assume that ϱ can be approximated by a Hölder-continuous function. The strategy developed in [GW14a] consists in an ansatz

$$ZG^{(0)}(a,b) = \frac{e^{\mathcal{H}_a^\Lambda[\tau_b(\bullet)]}\sin\tau_b(a)}{\lambda\pi\varrho(a)} = \frac{e^{\mathcal{H}_b^\Lambda[\tau_a(\bullet)]}\sin\tau_a(b)}{\lambda\pi\varrho(b)}\,, \tag{9.5}$$

where the angle function $\tau_a : (0,\Lambda^2)\to[0,\pi]$ for $\lambda>0$ and $\tau_a : (0,\Lambda^2)\to[-\pi,0]$ for $\lambda<0$ remains to be determined. Here,

$$\mathcal{H}_a^\Lambda[f(\bullet)] := \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{[0,\Lambda^2]\setminus[a-\epsilon,a+\epsilon]}\frac{dt\; f(t)}{t-a} = \lim_{\epsilon\to 0}\mathrm{Re}\Big(\frac{1}{\pi}\int_0^{\Lambda^2}\frac{dt\; f(t)}{t-(a+i\epsilon)}\Big) \tag{9.6}$$

denotes the finite Hilbert transform. We go with the ansatz (9.5) into (9.4) at $\zeta_1 = a+i\epsilon$ and $\zeta_2 = b$:

$$\begin{aligned}&\Big(\tilde{\mu}^2+a+b+\lambda\pi\mathcal{H}_a^\Lambda[\varrho(\bullet)]+\frac{1}{\pi}\int_0^{\Lambda^2}dt\; e^{\mathcal{H}_t^\Lambda[\tau_a(\bullet)]}\sin\tau_a(t)\Big)ZG^{(0)}(a,b)\\&= 1+\mathcal{H}_a^\Lambda\big[e^{\mathcal{H}_\bullet^\Lambda[\tau_b]}\sin\tau_b(\bullet)\big]\;.\end{aligned} \tag{9.7}$$

A Hölder-continuous function $\tau : (0,\Lambda^2)\to[0,\pi]$ or $\tau : (0,\Lambda^2)\to[-\pi,0]$ satisfies

$$\begin{aligned}\mathcal{H}_a^\Lambda\big[e^{\mathcal{H}_\bullet^\Lambda[\tau]}\sin\tau(\bullet)\big] &= e^{\mathcal{H}_a^\Lambda[\tau]}\cos\tau(a)-1\,,\\ \int_0^{\Lambda^2}dt\; e^{\pm\mathcal{H}_t^\Lambda[\tau(\bullet)]}\sin\tau(t) &= \int_0^{\Lambda^2}dt\;\tau(t)\;.\end{aligned} \tag{9.8}$$

The first identity appeared in [Tri57], the second one was proved in [PW18]. Inserting both identities into (9.7) gives with (9.5) a consistency relation for the angle function:

$$\tau_a(p) = \arctan\left(\frac{\lambda\pi\varrho(p)}{\tilde{\mu}^2+a+p+\lambda\pi\mathcal{H}_p^\Lambda[\varrho(\bullet)]+\frac{1}{\pi}\int_0^{\Lambda^2}dt\;\tau_p(t)}\right), \tag{9.9}$$

where the arctan-branch in $[0,\pi]$ is selected for $\lambda>0$ and the branch in $[-\pi,0]$ for $\lambda<0$.

The dependence on a in (9.9) is relatively simple so that first attempts focused on the resulting equation for $\tau_0(p)$. This allowed to prove, for the case $\varrho(x)=x$ of 4-dimensional Moyal space with harmonic propagation, existence of a solution [GW16]. Also an interesting phase structure was detected [GW14b], but a solution was missed.

9.2 Exact solution of the planar 2-point function

A breakthrough was achieved in [PW18], where the special case $\varrho(x) = 1$ was solved that describes a scaling limit of the 2-dimensional Moyal space with harmonic propagation:

Theorem 9.3 ([PW18]) *For $\varrho(x) = 1$ and with $\tilde{\mu}^2 = 1 - 2\lambda\log(1+\Lambda^2)$, the consistency equation* (9.9) *has in the limit $\Lambda \to \infty$ for $\lambda > 0$ the solution*

$$\tau_a(p) = \arctan\Big(\frac{\lambda\pi}{a + \lambda\, W_0(\frac{1}{\lambda}\mathrm{e}^{\frac{1+p}{\lambda}}) - \lambda\log\big(\lambda W_0(\frac{1}{\lambda}\mathrm{e}^{\frac{1+p}{\lambda}}) - 1\big)}\Big)\,, \tag{9.10}$$

where W_0 denotes the principal branch of the Lambert function [Lam58, CGH[+]96].

The `HyperInt` package [Pan15] was used to push a perturbative solution of (9.10) far enough to guess the whole perturbation series. The series is resummed by Lagrange-Bürmann formula [Lag70, Bür99] to Lambert-W. The result is confirmed by the residue theorem. The 2-point function $G^{(0)}(a,b)$ is then evaluated from (9.5) via deformation of complex contour integrals (the result will be given below for any ϱ).

Building on [PW18], in [GHW19b] the exact solution of the non-linear equation (9.4) was achieved for *any* density ϱ (which encodes a sequence (E_n) of dimension $D \leq 4$ according to Def. 8.2). Starting point was the observation that the denominator of (9.10) can be written (up to a shift $1+a$) as $-\mathrm{Re}(-f(p) + \lambda\log(-f(p)))$, where $f(p) = \lambda W_0(\frac{1}{\lambda}\mathrm{e}^{\frac{1+p}{\lambda}}) - 1$ solves $1+p = 1+f(p)+\lambda\log(1+f(p)))$. The logarithm is the renormalised Stieltjes transform of the measure $\varrho(t) = 1$. This suggested to try the same combination of reflection $z \leftrightarrow -\mu^2 - z$ with the Stieltjes transform of the given density ϱ. But this was not enough: The general case requires a *deformation of ϱ to an implicitly defined measure function ϱ_λ*:

Definition 9.4 Given a real λ in some open neighbourhood of 0, a scale $\mu^2 > 0$ and a Hölder-continuous function $\varrho : [0,\Lambda^2] \to \mathbb{R}_+$ of dimension $D \in \{0,2,4\}$. Then a function ϱ_λ on $[\nu_\lambda, \Lambda^2_\lambda]$ is implicitly defined by

$$\varrho(t) =: \varrho_\lambda(R_\lambda^{-1}(t))\,, \qquad \Lambda^2_\lambda := R_\lambda^{-1}(\Lambda^2)\,, \qquad \nu_\lambda := R_\lambda^{-1}(0)\,, \tag{9.11}$$

where $R_\lambda : \mathbb{C}\setminus[-\mu^2-\Lambda^2_\lambda, -\mu^2-\nu_\lambda] \to \mathbb{C}$ is defined via the same function ϱ_λ by

$$R_\lambda(z) := z - \lambda(-z)^{\frac{D}{2}} \int_{\nu_\lambda}^{\Lambda^2_\lambda} \frac{\mathrm{d}t\ \varrho_\lambda(t)}{(t+\mu^2)^{\frac{D}{2}}(t+\mu^2+z)}\,. \tag{9.12}$$

The definition is consistent because for $|\lambda|$ small enough, R_λ is a biholomorphic map from the half-plane $\mathrm{Re}(z) > -\frac{\mu^2}{2}$ onto a domain which contains $[0,\Lambda^2]$.

Using the same complex analysis techniques as in [PW18], including Lagrange inversion theorem and Bürmann formula, the following generalisation of Theorem 9.3 can be achieved:

Theorem 9.5 ([GHW19b]) *Let $\varrho : [0,\Lambda^2] \to \mathbb{R}_+$ be a Hölder-continuous measure of dimension $D \in \{0,2,4\}$ and ϱ_λ its deformation according to Definition 9.4 for a real coupling constant λ with $|\lambda| < \big(\int_{\nu_\lambda}^{\Lambda_\lambda^2} dt \frac{\varrho_\lambda(t)}{(t+\mu^2/2)^2} + \delta_{D,4}\int_{\nu_\lambda}^{\Lambda_\lambda^2} dt \frac{\varrho_\lambda(t)}{(t+\mu^2)^2}\big)^{-1}$. Then the consistency equation* (9.9) *for the angle function is solved by*

$$\tau_a(p) = \lim_{\epsilon\to 0} \operatorname{Im}\big(\log(a - R_\lambda(-\mu^2 - R_\lambda^{-1}(p + i\epsilon)))\big)\;, \tag{9.13}$$

where R_λ is defined by (9.12) and $\tilde{\mu}$ is renormalised according to

$$\tilde{\mu}^2 = \mu^2\Big(1 - \lambda\delta_{D,4}\int_{\nu_\lambda}^{\Lambda_\lambda^2} \frac{dt\ \varrho_\lambda(t)}{(t+\mu^2)^2}\Big) - 2\lambda(\delta_{D,2}+\delta_{D,4})\int_{\nu_\lambda}^{\Lambda_\lambda^2} \frac{dt\ \varrho_\lambda(t)}{t+\mu^2}\;. \tag{9.14}$$

A constant measure such as $\varrho(t) = 1$ remains undeformed to $\varrho_\lambda(x) = 1$, and (9.13) reduces for $D = 2$ and $\mu = 1$ to (9.10).

Evaluation of the Hilbert transform of (9.13) yields for (9.5):

Proposition 9.6 ([GHW19b]) *The renormalised planar two-point function of the D-dimensional Φ^4-model is given by*

$$G^{(0)}(a,b) := \frac{(\mu^2)^{\delta_{D,4}}(\mu^2+a+b)\exp(N_\lambda(a,b))}{(\mu^2+b+R_\lambda^{-1}(a))(\mu^2+a+R_\lambda^{-1}(b))}\;, \tag{9.15}$$

where R_λ is built via (9.12) with the deformed measure ϱ_λ defined in (9.11) and

$$\begin{aligned} N_\lambda(a,b) := \frac{1}{2\pi i}\int_{-\infty}^{\infty} dt\, \Bigg\{ &\log\left(\frac{a - R_\lambda(-\frac{\mu^2}{2} - it)}{a - (-\frac{\mu^2}{2} - it)}\right) \frac{d}{dt}\log\left(\frac{b - R_\lambda(-\frac{\mu^2}{2} + it)}{b - (-\frac{\mu^2}{2} + it)}\right) \\ &- \delta_{D,4}\log\left(\frac{R_\lambda(-\frac{\mu^2}{2} - it)}{(-\frac{\mu^2}{2} - it)}\right)\frac{d}{dt}\log\left(\frac{R_\lambda(-\frac{\mu^2}{2} + it)}{(-\frac{\mu^2}{2} + it)}\right)\Bigg\}\;. \end{aligned} \tag{9.16}$$

In $D = 4$ dimensions, $G^{(0)}(a,b)$ is only determined up to a multiplicative constant (the finite part of $\tilde{Z}$) which here is normalised to $G^{(0)}(0,0) = 1$ independently of μ.

Moyal space in dimension $D = 2$ corresponds to $\varrho(x) = 1$ and accordingly $R_\lambda(x) = x + \lambda\log(1+x)$ (when setting $\mu = 1$). The perturbative expansion of $N_\lambda(a,b)$ involves Nielsen's generalised polylogarithms [Nie09] and Riemann zeta values.

Moyal space in dimension $D = 4$ corresponds to $\varrho(t) = t$, which by (9.11) and (9.12) results (for $\Lambda \to \infty$) in

$$\varrho_\lambda(x) = R_\lambda(x) = x - \lambda x^2 \int_0^\infty \frac{dt\ R_\lambda(t)}{(t+\mu^2)^2(t+\mu^2+x)}\;. \tag{9.17}$$

Proposition 9.7 ([GHW19c]) *The Fredholm integral equation (9.17) has the solution*

$$R_\lambda(x) = x\; {}_2F_1\Big(\begin{matrix}\alpha_\lambda,\; 1-\alpha_\lambda\\ 2\end{matrix}\Big| -\frac{x}{\mu^2}\Big), \tag{9.18}$$
$$\textit{where}\quad \alpha_\lambda := \begin{cases} \frac{\arcsin(\lambda\pi)}{\pi} & \textit{for } |\lambda| \leq \frac{1}{\pi},\\ \frac{1}{2} + \mathrm{i}\frac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \textit{for } \lambda \geq \frac{1}{\pi}.\end{cases}$$

Inserting (9.18) into (9.15) provides an integral representation[6] for the planar two-point function, which is *exact* in $\lambda \geq -\frac{1}{\pi}$. Its perturbative expansion involves a particular class of hyperlogarithms in alternating letters $0, -1$ [GHW19c].

Remark 9.8 It is very important that $R_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ given by (9.18) is bijective. This was not expected in the beginning: If ϱ_λ in (9.12) had the same asymptotic behaviour $\varrho_\lambda(x) \propto x$ as $\varrho(x) = x$ for $D = 4$, then R_λ would reach a maximum on $\mathbb{R}_+$ and could not be inverted globally, unless $\lambda = 0$ is trivial. The Φ^4-model on 4-dimensional Moyal space avoids the triviality [Aiz81, Frö82] of the commutative ϕ^4_4-model by a significant modification of the spectral dimension. Defining $D_{\text{spec}}(\rho) = \inf\{p : \int_0^\infty dt\, \frac{\rho(t)}{(1+t)^{p/2}} < \infty\}$, then $D_{\text{spec}}(\varrho_\lambda) = 4 - 2\frac{\arcsin(\lambda\pi)}{\pi}$ for $|\lambda| \leq \frac{1}{\pi}$ but $D_{\text{spec}}(\varrho) = 4$.

The final result (9.15) and (9.16) does not require anymore that ϱ is Hölder-continuous. It also holds for ϱ a finite sum of Dirac measures, and in this case one can even evaluate the remaining integral (9.16):

Theorem 9.9 ([GHW19b]) *Consider the Φ^4-model for $\mathcal{N} \times \mathcal{N}$-matrices in which the covariance is defined by a d-tupel $(E_1, \dots, E_d)$ of positive real numbers, where E_k arises with multiplicity r_k, and $\sum_{k=1}^d r_k = \mathcal{N}$. These data encode a rational function*

$$R(z) := z - \frac{\lambda}{\mathcal{N}} \sum_{k=1}^{d} \frac{\varrho_k}{\varepsilon_k + z}, \tag{9.19}$$

where $\{\varepsilon_k, \varrho_k\}_{k=1,\dots,d}$ are the unique solutions in a neighbourhood of $\lambda = 0$ of

$$E_k = R(\varepsilon_k)\,, \quad r_k = \varrho_k R'(\varepsilon_k) \qquad \textit{with} \quad \lim_{\lambda\to 0} \varepsilon_k = E_k\,, \quad \lim_{\lambda \to 0} \varrho_k = r_k. \tag{9.20}$$

Then the planar two-point function has in an open neighbourhood of $\lambda = 0$ the explicit solution $G^{(0)}_{ab} = \mathcal{G}^{(0)}(\varepsilon_a, \varepsilon_b)$, where $\mathcal{G}^{(0)} : \bar{\mathbb{C}} \times \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is the rational function

$$\mathcal{G}^{(0)}(z,w) = \frac{1 - \dfrac{\lambda}{\mathcal{N}} \displaystyle\sum_{k=1}^{d} \frac{r_k}{(R(z) - R(\varepsilon_k))(R(\varepsilon_k) - R(-w))} \prod_{j=1}^{d} \frac{R(w) - R(-\widehat{\varepsilon_k}^{\,j})}{R(w) - (\varepsilon_j)}}{R(w) - R(-z)} \tag{9.21}$$

[6] The inverse function $R_\lambda^{-1}(x)$ in (9.15) can be combined with $\mathcal{N}_\lambda$ to another integral representation [GHW19b].

in which $z \in \{u, \hat{u}^1, \dots, \hat{u}^d\}$ *is the list of roots of* $R(z) = R(u)$*. The 2-point function is symmetric,* $\mathcal{G}^{(0)}(z,w) = \mathcal{G}^{(0)}(w,z)$*, and defined outside poles located at* $z + w = 0$*, at* $z = \widehat{\varepsilon_k}^m$ *and at* $w = \widehat{\varepsilon_l}^n$*, for* $k, l, m, n = 1, \dots, d$*.*

Theorem 9.9 undeniably establishes that the Φ^4-model is exactly solvable in surprisingly close analogy with the Φ^3-model (i.e. the Kontsevich model). The rationality achieved in (9.21) is overwhelming support for the conjecture that the Φ^4-model is integrable, too, which means it descends from a τ-function satisfying a Hirora equation [Miw82].

The simplest case $E = \frac{\mu^2}{2}$ = const of a single $r_1 = \mathcal{N}$-fold degenerate eigenvalue $E_1 = \frac{\mu^2}{2}$ is already interesting. One finds [GHW19b]

$$G_{11}^{(0)} = \frac{4}{3} \frac{\mu^2 + 2\sqrt{\mu^4 + 12\lambda}}{(\mu^2 + \sqrt{\mu^4 + 12\lambda})^2}. \tag{9.22}$$

It agrees with corresponding formulae in the literature [BIPZ78].

9.3 Outlook

It remains to recursively solve the Dyson–Schwinger equations for $B > 1$. In [GW14a] this problem was already reduced to affine equations for $N_1 + \cdots + N_B$-point functions with all $N_\beta \leq 2$, where $N_\beta = 1$ is much simpler than $N_\beta = 2$. One should start with finite matrices where the initial data of the recursion is known from (9.21). The same change of variables (9.19) brings the equation for the 1 + 1-point function into the form

$$(R(z) - R(-z))\mathcal{G}^{(0)}(z|w) - \frac{\lambda}{\mathcal{N}} \sum_{k=1}^{d} \frac{r_k \mathcal{G}^{(0)}(\varepsilon_k|w)}{R(\varepsilon_k) - R(z)} = \lambda \frac{\mathcal{G}^{(0)}(z,w) - \mathcal{G}^{(0)}(w,w)}{R(z) - R(w)}. \tag{9.23}$$

The lhs agrees exactly with the corresponding operator in topological recursion [EO07, Eyn16], provided that one chooses the *classical spectral curve* $\mathcal{E}(x(z), y(z)) = 0$ as

$$x(z) = R(z)\,, \qquad y(z) = -R(-z). \tag{9.24}$$

But the rhs of (9.23) is completely different. Its poles at $z, w \in \{\widehat{\varepsilon_k}^l\}$ have no counterpart in topological recursion. These poles are expected to proliferate into all functions of higher topology. Moreover, topological recursion assumes that the branched covering $x : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is invariant under the local Galois involution, here $z \mapsto -z$. This invariance does not hold in (9.24). Finally, higher N-point functions arising as inhomogeneity of the recursion are (via the generalisation of (9.3)) non-linear in the basic $N_1 + \cdots + N_B$-point functions with all $N_\beta \leq 2$, whereas in topological recursion the analogous dependence is linear.

Thus, in spite of striking similarities with *the* topological recursion, the recursion of the Φ^4-model differs significantly when looking closer. Solving the Φ^4-recursion from scratch will be a fascinating programme for the next years. The remarkable fact that also the Φ^4-model is exactly solvable raises several questions:

? Question 9.10

- Why are the exact solutions of the Φ^4- and Φ^3-models so similar, whereas their perturbative treatment falls completely apart?
- Do all contributions to correlation functions with topology (g, B) have a significance in topological recursion, or only those with an odd number of defects per boundary component?
- What is the integrable structure of the Φ^4-model? Is the logarithm of the partition function of the Φ^4-model a τ-function for a Hirota equation?
- Is the logarithm of the partition function a series in certain t_i with rational coefficients? If so, do these rational numbers describe some intersection numbers of some characteristic classes on some moduli space?
- Can one identify a Virasoro algebra, or some generalisation, in the Φ^4-model?
- Can the standard model of particle physics learn something from the integrable Φ^4-model? Does the enormous complexity concerning polylogarithms and other transcendental functions in the standard model possibly arise through the perturbative solution of implicitly defined problems similar to Definition 9.4?

10 Osterwalder-Schrader axioms

Strictly speaking, the programme outlined in Sec. 3, completely solved for the Φ^3-model in Sec. 8 and nearly finished for the Φ^4-model in Sec. 9, does not yet produce any quantum field theory. It gives consistent continuum limits of statistical physics models, but not a QFT. For a true quantum field theory a time evolution is necessary. On standard Euclidean space $\mathbb{R}^D$, time evolution is a deep consequence of the Osterwalder-Schrader axioms [OS73, OS75] (see Definition 2.2). The most decisive axiom is *reflection positivity*, a variant of the Hausdorff-Bernstein-Widder theorem:

Theorem 10.1 (Hausdorff-Bernstein-Widder) *Let S be a convex cone in a real vector space X, containing 0. Then for a continuous function $F : S \to \mathbb{R}$ the following are equivalent:*

1. *F is decreasing and positive definite, i.e. $\sum_{i,j=1}^K \overline{c_i} c_j F(t_i + t_j) \geq 0$ for all $c_i, c_j \in \mathbb{C}$ and $t_i, t_j \in S$.*
2. *F is the Laplace transform $F(t) = \int_{X'} d\mu(\lambda)\, e^{-\lambda(t)}$ of a positive measure on X'.*
3. *F is completely monotonic, i.e. $\prod_{i=1}^K (\mathrm{id} - T_{\delta_i}) F \geq 0$ for all $\delta_i \in S$, where $(T_{\delta_i} F)(t) = F(t + \delta_i)$.*

Consequently, F is smooth, and 3. can be replaced by $(-1)^{|n|} f^{(n)}(t) \geq 0$ *for any multi-index n.*

The original publications [Hau23, Ber29, Wid31] prove the theorem for $S = \mathbb{R}_+$. The higher-dimensional version $S = (\mathbb{R}_+)^N$ is given in Bochner's book [Boc55]. It was extended to fairly general Abelian semigroups by Nussbaum [Nus55] and to operators in Hilbert space by several authors, for instance [KL81]. The main feature of the Laplace integral is that it provides a holomorphic extension of F into the tube $S + \mathrm{i}X$. It is this purely imaginary $\mathrm{i}X$ what we refer to as 'time'. Reflection positivity is a challenging topic in mathematics and physics. We refer to [NO18] and the Oberwolfach reports [JNOS17] for more details and for an overview about current research activities.

To develop an Osterwalder-Schrader setup for a (noncommutative) nuclear AF Fréchet algebra $\mathcal{A} = \bigcup_N \mathcal{A}^{\mathcal{A}}$ on which we constructed QFT-models we need a continuous linear map $q_* : \mathcal{A}_* \to C^\infty(U)$ into a vector space of smooth functions. It is tempting to identify q_* with the isomorphism ι_U provided by the Kōmura-Kōmura theorem 3.3. In this case we would need that the image $\iota_U(\mathcal{A}_*)$ is invariant under translations. This is the case for the Moyal algebras in secs. 4.1 and 4.3, but we do not know it in general (see Question 3.4). We briefly review in Sec. 10.1 what is known for the choice $q_* = \iota_{\mathbb{R}^D}$. The lesson will be to proceed differently. We develop first ideas in Sec. 10.2.

10.1 Previous approaches to reflection positivity

Reflection positivity in QFTs on Moyal space has been studied in several contexts. We admit, however, that a satisfactory picture was not yet given. If one had a reflection-positive QFT on ordinary $\mathbb{R}^3$ or $\mathbb{R}^4$, then one can choose to Moyal-deform only a 2-dimensional subspace orthogonal to the time direction. In this case analytic continuation ('Wick rotation') and deformation commute up to an isomorphism of the Minkowskian Moyal algebra [GLLV13]. The restriction to degenerate Moyal space is essential. Without it the continuation of Moyal-deformed Wightman functions leads to Euclidean functions with twists in mass-shell momenta [Bah10].

Another approach proposed in [GW13b] consists in using the isomorphism $\iota_\Theta = \iota_{\mathbb{R}^D}$ to map matrix correlation functions $\langle e_{\underline{k}_1\underline{l}_1} \otimes \cdots \otimes e_{\underline{k}_N\underline{l}_N}\rangle$ defined in (3.6) into $\mathbb{R}^D$-labelled candidate Schwinger functions

$$S_c(x_1,\dots,x_n) \qquad (10.1)$$
$$:= \sum_{\underline{k}_1,\dots,\underline{l}_N \in \mathbb{N}^{D/2}} f_{\underline{k}_1\underline{l}_1}(x_1)\cdots f_{\underline{k}_N\underline{l}_N}(x_N)\frac{(-\mathrm{i})^N}{\mathcal{Z}_C(0)}\frac{\partial^N \mathcal{Z}_C(t_1 e_{\underline{k}_1\underline{l}_1}+\dots+t_N e_{\underline{k}_N\underline{l}_N})}{\partial t_1\cdots\partial t_N}\Big|_{t_i=0},$$

where the $f_{\underline{kl}}$ are extensions of $f^{(\theta)}_{kl}$ defined in (4.3) to $D/2$ components. The function $\mathcal{Z}_C(J)$ is expanded by (7.15). This involves the covariance C defined in (6.7) which essentially relies on the harmonic oscillator Schrödinger operator $H^{\Omega=1} = -\Delta + \frac{4}{\theta^2}x^2$.

The explicit dependence on the position x in this operator makes the candidate Schwinger functions (10.1) not even translation-invariant. However, the dangerous term vanishes in the limit $V = (\frac{\theta}{4})^{D/2} \to \infty$. Note that the limit $V \to \infty$ is highly singular for the matrix basis functions (4.3). It was proved in [GW13b] that the limit $V \to \infty$ of (10.1) is well-defined and gives with a convention $\frac{\delta J_{\underline{mn}}}{\delta J(x)} := \mu^D f_{\underline{mn}}(x)$ the following formula for connected Schwinger functions:

$$\begin{aligned} &S_c(x_1, \dots, x_N) \\ &= \frac{1}{(8\pi)^{\frac{D}{2}}} \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma \in \mathcal{S}_N} \Big(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{dp_\beta}{4\pi^2} \, e^{i\langle p_\beta, \sum_{i=1}^{N_\beta} (-1)^{i-1} x_{\sigma(s_\beta+i)} \rangle} \Big) \\ &\qquad \times G_{|\underbrace{\frac{\|p_1\|^2}{2}, \dots, \frac{\|p_1\|^2}{2}}_{N_1}|\dots|\underbrace{\frac{\|p_B\|^2}{2}, \dots, \frac{\|p_B\|^2}{2}}_{N_B}|} \,. \end{aligned} \tag{10.2}$$

Euclidean invariance is manifest. The most interesting sector is $N_\beta = 2$ in every boundary component. This $(2+\dots+2)$-sector describes the propagation and interaction of B Euclidean 'particles' without any momentum exchange. Absence of momentum transfer is characteristic to integrable models [Mos75, Kul76], but in four dimensions a sign of *triviality* [Aks65]. However, not all assumptions of this triviality proof are satisfied in the models under consideration.

For the Φ^3-model constructed in Sec. 8, the explicit formulae (8.17) and (8.14) admit a direct verification of complete monotonicity (property 3. of Theorem 10.1). For the 2-point function this amounts to prove that $a \mapsto G_{|aa|} \equiv \int_0^\infty \frac{d\rho(m^2)}{a+m^2}$ is a Stieltjes function [GW13b], i.e. the Stieltjes transform of a positive measure $d\rho(m^2)$. Surprisingly, the 2-point function of the Φ^3_D-model is Stieltjes for $D = 4$ and $D = 6$ (and λ real where the partition function is meaningless), *but not for* $D = 2$ or λ purely imaginary [GSW17]. Numerical evidence was given [GW14b] that the same is true for the Φ^4_4-model: The 2-point function is definitely not reflection positive in the stable case $\lambda > 0$, whereas for $\lambda < 0$ positivity seems to hold.

Reflection positivity *cannot be expected to hold* for the whole set of Schwinger functions (10.2) for the Φ^3-model. The reason is the fast decay in $E_{\underline{k}}$ established in (8.17) which contradicts complete monotonicity in Theorem 10.1.

10.2 A proposal

In a sort of outlook we sketch ideas about another approach to the Osterwalder-Schrader axioms in QFTs on noncommutative algebras $\mathcal{A}$. The failure of (10.1) to produce reflection-positive Schwinger functions suggests that the dequantisation $q_* : \mathcal{A}_* \to C^\infty(\mathbb{R}^D)$ *should be different from the Kōmura-Kōmura isomorphism* ι_U.

Instead we propose to build the dequantisation as a *positive map* $(q^*(a^*a))(x) \geq 0$ for all $a \in \mathcal{A}$ and $x \in \mathbb{R}^D$. Adjusting the norm we choose it in the class of $\mathbb{R}^D$-

labelled states $\{\omega_v \;:\; v \in \mathbb{R}^D\}$. There are good reasons for this ansatz. On the Moyal algebra $(\mathcal{S}(\mathbb{R}^2), \star_\theta)$ one can check that $\tilde{\omega}_\gamma(f) := \frac{1}{\pi\gamma\theta}\int_{\mathbb{R}^2} dx\;\, e^{-\frac{|x|^2}{\gamma\theta}} f(x)$ is a state if and only if $\gamma \geq 1$. Pointwise evaluation, a state for the commutative product, would be recovered by $\lim_{\gamma\to 0}\tilde{\omega}_\gamma(f) = f(0)$, but it is not positive for the Moyal product. We argued in the very beginning (Sec. 1.1) that sharp localisation in a QFT is incompatible with gravity. This observation was precisely the reason to introduce QFTs on noncommutative geometries (Sec. 1.2). The smearing in an area $|x|^2 \geq \theta$ via the state $\tilde{\omega}_\gamma$ implements the localisability restrictions, with $\theta = \ell_P^2$ being the Planck area. Moreover, states provide the correct framework to pass from the noncommutative topology encoded in an algebra $\mathcal{A}$ to the metric aspects of noncommutative geometry [Con94]. Given a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, a metric structure is defined on (an appropriate subspace of) states on $\mathcal{A}$ via Connes' distance formula [Con94]

$$\mathrm{dist}(\omega_1, \omega_2) = \sup\{|\omega_1(a) - \omega_2(a)| \;:\; \|[\mathcal{D}, a]\| \leq 1\}\,. \tag{10.3}$$

We therefore propose:

Definition 10.2 Let $\mathcal{A}$ be nuclear AF Fréchet algebra generated by orthonormal matrix bases $\{e_{\underline{k}\underline{l}}\}$. Then renormalised correlation functions $\langle e_{\underline{k}_1\underline{l}_1} \otimes \cdots \otimes e_{\underline{k}_N\underline{l}_N}\rangle$ defined on $\mathcal{A}$ via (3.6) and renormalisation give rise to Schwinger functions by

$$S(v_N) := \sum_{\underline{k}_1,\dots,\underline{l}_N \in \mathbb{N}^{D/2}} \omega_{v_N}(e_{\underline{k}_1\underline{l}_1} \otimes \cdots \otimes e_{\underline{k}_N\underline{l}_N})\langle e_{\underline{k}_1\underline{l}_1} \otimes \cdots \otimes e_{\underline{k}_N\underline{l}_N}\rangle\,, \tag{10.4}$$

where ω_{v_N} is a state on $\mathcal{A}^{\otimes N}$.

We assume that the the Kōmura-Kōmura isomorphism endows $\mathcal{A}$ by an action $\alpha : \mathbb{R}^D \times \mathcal{A} \to \mathcal{A}$ of translations. It is probably not necessary that the $\mathbb{R}^D$-action commutes with multiplication in $\mathcal{A}$. We obtain an $\mathbb{R}^D$-action on the states ω_{v_N} by duality,

$$(\alpha_{t_1,\dots,t_N}\omega_{v_N})(a_1 \otimes \cdots \otimes a_N) := \omega_{v_N}(\alpha_{t_1}a_1 \otimes \cdots \otimes \alpha_{t_N}a_N)\,, \tag{10.5}$$

for $t_i \in \mathbb{R}^D$. Hence, it is enough to specify a single reference state $\omega_{\hat{v}_N}$ which via $\omega_{t_1,\dots,t_N} := \alpha_{t_1,\dots,t_N}\omega_{\hat{v}_N}$ induces an $\mathbb{R}^{ND}$-indexed family of states. This would make the $\mathbb{R}^D$ a universal model of noncommutative geometries defined via the distance formula (10.3). It was shown in [MT13] that for $\mathcal{A}$ the 2D-Moyal algebra one has $\mathrm{dist}(\omega_t, \omega_{t'}) = \|t - t'\|$ for the noncommutative distance (10.3) between any such translates $\omega_t, \omega_{t'}$ of a reference state on the Moyal algebra. In short, everything is consistent.

One of the Osterwalder-Schrader axioms requires translation invariance of the Schwinger functions (10.4), i.e. $S(t_1, \dots, t_N) = S(t_1 + t_0, \dots, t_N + t_0)$ for any $t_0 \in \mathbb{R}^D$. This is not automatic for our proposal, but can be achieved for the Moyal algebra following an observation in [BDFP03]. Namely, the tensor product of Moyal algebras factorises into $\mathcal{A}^{\otimes N} = \mathcal{A} \otimes \mathcal{A}^{\otimes N-1}$, where the first tensor factor describes the centre-of-motion coordinate and the second one depends only on coordinate differences. Let $\iota_c : \mathcal{A}^{\otimes N} \to \mathcal{A} \otimes \mathcal{A}^{\otimes N-1}$ be this factorisation isomorphism, then translation-invariant Schwinger functions can be defined as

$$\begin{aligned} &S(t_1, \dots, t_N) \qquad (10.6)\\ &:= \sum_{\underline{k}_1,\dots,\underline{l}_N \in \mathbb{N}^{D/2}} (\hat{\omega} \otimes \omega_{t_1-t_2,\dots,t_{N-1}-t_N})(\iota_c(e_{\underline{k}_1\underline{l}_1} \otimes \dots \otimes e_{\underline{k}_N\underline{l}_N}))\langle e_{\underline{k}_1\underline{l}_1} \otimes \dots \otimes e_{\underline{k}_N\underline{l}_N}\rangle\,, \end{aligned}$$

where $\hat{\omega}$ averages over the centre of motion and $\omega_{t_1-t_2,\dots,t_{N-1}-t_N}$ is a $\mathbb{R}^{D(N-1)}$-translate of a reference state on $\mathcal{A}^{\otimes N-1}$. The Schwinger functions (10.6) are translation-invariant by construction. It thus remains:

? Question 10.3

Is it possible to find examples of reference states, or even to classify them, for which the free theory with covariance C_E is reflection positive? Does it extend to reflection positivity of Schwinger 2-point functions (10.6) for the moments $\langle e_{\underline{k}_1\underline{l}_1} \otimes e_{\underline{k}_2\underline{l}_2}\rangle$ of the Φ^3 and Φ^4-models?

The dream would be to prove reflection positivity of all Schwinger functions.

Acknowledgements

I thank Alain Connes for the initiative to write such a survey and for constant support and encouragement through the years. I am most grateful to my long-term collaborator Harald Grosse with whom nearly all of the presented results have been achieved. After the two years 2000/2001 together in Vienna, our collaboration has been supported by the Erwin-Schrödinger-Institute, by the Max-Planck-Institute for Mathematics in the Sciences and by the Deutsche Forschungsgemeinschaft (DFG) via the coordinated programmes SFB 478 and SFB 878. Sections 8 and 9 are based on recent results obtained with Harald Grosse, Akifumi Sako, Erik Panzer and Alexander Hock. The survey was finally assembled within the Cluster of Excellence[7] "Mathematics Münster".

References

ABIY02. G. Alexanian, A. P. Balachandran, G. Immirzi, and B. Ydri. Fuzzy CP^2. *J. Geom. Phys.*, 42:28–53, 2002, hep-th/0103023. doi:10.1016/S0393-0440(01)00070-5.

[7] "Gefördert durch die Deutsche Forschungsgemeinschaft (DFG) im Rahmen der Exzellenzstrategie des Bundes und der Länder EXC 2044–390685587, Mathematik Münster: Dynamik–Geometrie–Struktur"

ABK00a. I. Ya. Aref'eva, D. M. Belov, and A. S. Koshelev. A note on UV / IR for noncommutative complex scalar field. 2000, hep-th/0001215.

ABK00b. I. Ya. Aref'eva, D. M. Belov, and A. S. Koshelev. Two loop diagrams in noncommutative ϕ_4^4 theory. *Phys. Lett.*, B476:431–436, 2000, hep-th/9912075. doi:10.1016/S0370-2693(00)00169-6.

AC96. E. Arbarello and M. Cornalba. Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves. *J. Algebraic Geom.*, 5(4):705–749, 1996.

ADJ91. J. Ambjørn, B. Durhuus, and T. Jónsson. Three-dimensional simplicial quantum gravity and generalized matrix models. *Mod. Phys. Lett.*, A6:1133–1146, 1991. doi:10.1142/S0217732391001184.

AGBZ01. L. Alvarez-Gaume, J. L. F. Barbon, and R. Zwicky. Remarks on time space noncommutative field theories. *JHEP*, 05:057, 2001, hep-th/0103069. doi:10.1088/1126-6708/2001/05/057.

AII+00. H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa, and T. Tada. Noncommutative Yang-Mills in IIB matrix model. *Nucl. Phys.*, B565:176–192, 2000, hep-th/9908141. doi:10.1016/S0550-3213(99)00633-1.

Aiz81. M. Aizenman. Proof of the triviality of ϕ_d^4 field theory and some mean field features of Ising models for $d > 4$. *Phys. Rev. Lett.*, 47:1–4, 1981. doi:10.1103/PhysRevLett.47.1.

Aks65. S. Ø. Aks. Proof that scattering implies production in quantum field theory. *J. Math. Phys.*, 6:516–532, 1965. doi:10.1063/1.1704305.

Ale11. A. Alexandrov. Cut-and-join operator representation for Kontsevich-Witten tau-function. *Mod. Phys. Lett.*, A26:2193–2199, 2011, 1009.4887. doi:10.1142/S0217732311036607.

AMNS99. J. Ambjørn, Y. M. Makeenko, J. Nishimura, and R. J. Szabo. Finite N matrix models of noncommutative gauge theory. *JHEP*, 11:029, 1999, hep-th/9911041. doi:10.1088/1126-6708/1999/11/029.

AR95. A. Abdesselam and V. Rivasseau. Trees, forests and jungles: A botanical garden for cluster expansions. In *Constructive physics: Results in field theory, statistical mechanics and condensed matter physics*, volume 446 of *Lect. Notes Phys.*, pages 7–36. 1995, hep-th/9409094. doi:10.1007/3-540-59190-7_20.

ARS99. A. Yu. Alekseev, A. Recknagel, and V. Schomerus. Noncommutative world volume geometries: Branes on SU(2) and fuzzy spheres. *JHEP*, 09:023, 1999, hep-th/9908040. doi:10.1088/1126-6708/1999/09/023.

Ati88. M. Atiyah. Topological quantum field theories. *Inst. Hautes Études Sci. Publ. Math.*, 68:175–186, 1988. URL `http://www.numdam.org/item?id=PMIHES_1988__68__175_0`.

Bah03. D. Bahns. Unitary quantum field theory on the noncommutative Minkowski space. *Fortsch. Phys.*, 51:658–663, 2003, hep-th/0212266. doi:10.1002/prop.200310079.

Bah10. D. Bahns. Schwinger functions in noncommutative quantum field theory. *Annales Henri Poincaré*, 11:1273–1283, 2010, 0908.4537. doi:10.1007/s00023-010-0061-4.

BDD+03. W. Behr, N. G. Deshpande, G. Duplancić, P. Schupp, J. Trampetić, and J. Wess. The $Z \to \gamma\gamma, gg$ decays in the noncommutative standard model. *Eur. Phys. J.*, C29:441–446, 2003, hep-ph/0202121. doi:10.1140/epjc/s2003-01207-4.

BDFP02. D. Bahns, S. Doplicher, K. Fredenhagen, and G. Piacitelli. On the unitarity problem in space-time noncommutative theories. *Phys. Lett.*, B533:178–181, 2002, hep-th/0201222. doi:10.1016/S0370-2693(02)01563-0.

BDFP03. D. Bahns, S. Doplicher, K. Fredenhagen, and G. Piacitelli. Ultraviolet finite quantum field theory on quantum space-time. *Commun. Math. Phys.*, 237:221–241, 2003, hep-th/0301100. doi:10.1007/s00220-003-0857-x.

Ber29. S. Bernstein. Sur les fonctions absolument monotones. *Acta Math.*, 52(1):1–66, 1929. doi:10.1007/BF02547400.

Bet47. H. A. Bethe. The electromagnetic shift of energy levels. *Phys. Rev.*, 72:339–341, 1947. doi:10.1103/PhysRev.72.339.

BFF+78a. F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Deformation theory and quantization. 1. Deformations of symplectic structures. *Annals Phys.*, 111:61, 1978. doi:10.1016/0003-4916(78)90224-5.

BFF+78b. F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Deformation theory and quantization. 2. Physical applications. *Annals Phys.*, 111:111, 1978. doi:10.1016/0003-4916(78)90225-7.

BFG+03. H. Bozkaya, P. Fischer, H. Grosse, M. Pitschmann, V. Putz, M. Schweda, and R. Wulkenhaar. Space-time noncommutative field theories and causality. *Eur. Phys. J.*, C29:133–141, 2003, hep-th/0209253. doi:10.1140/epjc/s2003-01210-9.

BFV03. R. Brunetti, K. Fredenhagen, and R. Verch. The generally covariant locality principle: A new paradigm for local quantum field theory. *Commun. Math. Phys.*, 237:31–68, 2003, math-ph/0112041. doi:10.1007/s00220-003-0815-7.

BG14. J. Ben Geloun. Renormalizable models in rank $d \geq 2$ tensorial group field theory. *Commun. Math. Phys.*, 332:117–188, 2014, 1306.1201. doi:10.1007/s00220-014-2142-6.

BGG+01. A. Bichl, J. Grimstrup, H. Grosse, L. Popp, M. Schweda, and R. Wulkenhaar. Renormalization of the noncommutative photon selfenergy to all orders via Seiberg-Witten map. *JHEP*, 06:013, 2001, hep-th/0104097. doi:10.1088/1126-6708/2001/06/013.

BGK+08. D. N. Blaschke, F. Gieres, E. Kronberger, M. Schweda, and M. Wohlgenannt. Translation-invariant models for non-commutative gauge fields. *J. Phys.*, A41:252002, 2008, 0804.1914. doi:10.1088/1751-8113/41/25/252002.

BGP+02. A. A. Bichl, J. M. Grimstrup, L. Popp, M. Schweda, and R. Wulkenhaar. Perturbative analysis of the Seiberg-Witten map. *Int. J. Mod. Phys.*, A17:2219–2232, 2002, hep-th/0102044. doi:10.1142/S0217751X02010649.

BGR13. J. Ben Geloun and V. Rivasseau. A renormalizable 4-dimensional tensor field theory. *Commun. Math. Phys.*, 318:69–109, 2013, 1111.4997. doi:10.1007/s00220-012-1549-1.

BGRR11. V. Bonzom, R. Gurau, A. Riello, and V. Rivasseau. Critical behavior of colored tensor models in the large N limit. *Nucl. Phys.*, B853:174–195, 2011, 1105.3122. doi:10.1016/j.nuclphysb.2011.07.022.

BHN02. W. Bietenholz, F. Hofheinz, and J. Nishimura. The renormalizability of 2-D Yang-Mills theory on a noncommutative geometry. *JHEP*, 09:009, 2002, hep-th/0203151. doi:10.1088/1126-6708/2002/09/009.

BHN03. W. Bietenholz, F. Hofheinz, and J. Nishimura. Noncommutative field theories beyond perturbation theory. *Fortsch. Phys.*, 51:745–752, 2003, hep-th/0212258. doi:10.1002/prop.200310092.

BIPZ78. E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber. Planar diagrams. *Commun. Math. Phys.*, 59:35–51, 1978. doi:10.1007/BF01614153.

BK87. D. C. Brydges and T. Kennedy. Mayer expansions and the Hamilton-Jacobi equation. *J. Statist. Phys.*, 48(1-2):19–49, 1987. doi:10.1007/BF01010398.

BK90. E. Brezin and V. A. Kazakov. Exactly Solvable Field Theories of Closed Strings. *Phys. Lett.*, B236:144–150, 1990. doi:10.1016/0370-2693(90)90818-Q.

BKMP09. V. Bouchard, A. Klemm, M. Mariño, and S. Pasquetti. Remodeling the B-model. *Commun. Math. Phys.*, 287:117–178, 2009, 0709.1453. doi:10.1007/s00220-008-0620-4.

BKR+10. D. N. Blaschke, E. Kronberger, A. Rofner, M. Schweda, R. I. P. Sedmik, and M. Wohlgenannt. On the problem of renormalizability in non-commutative gauge field models: A critical review. *Fortsch. Phys.*, 58:364–372, 2010, 0908.0467. doi:10.1002/prop.200900102.

Blo15. S. Bloch. Feynman amplitudes in mathematics and physics. 2015, arXiv:1509.00361.

BLS11. D. Buchholz, G. Lechner, and S. J. Summers. Warped convolutions, Rieffel deformations and the construction of quantum field theories. *Commun. Math. Phys.*, 304:95–123, 2011, 1005.2656. doi:10.1007/s00220-010-1137-1.

BM04. G. Benfatto and V. Mastropietro. Ward identities and vanishing of the beta function for $d = 1$ interacting Fermi systems. *J. Statist. Phys.*, 115(1-2):143–184, 2004. doi:10.1023/B:JOSS.0000019825.37968.23.

Boc32. S. Bochner. *Vorlesungen über Fouriersche Integrale*, volume 12 of *Mathematik und ihre Anwendungen in Monographien und Lehrbüchern*. Akademische Verlagsgesellschaft m.b.h., 1932.

Boc55. S. Bochner. *Harmonic analysis and the theory of probability*. University of California Press, Berkeley and Los Angeles, 1955.

BP57. N. N. Bogoliubow and O. S. Parasiuk. Über die Multiplikation der Kausalfunktionen in der Quantentheorie der Felder. *Acta Math.*, 97:227–266, 1957. doi:10.1007/BF02392399.

Bra72. O. Bratteli. Inductive limits of finite dimensional C^*-algebras. *Trans. Amer. Math. Soc.*, 171:195–234, 1972. doi:10.2307/1996380.

Bro13a. D. Broadhurst. Multiple zeta values and modular forms in quantum field theory. In *Computer algebra in quantum field theory*, Texts Monogr. Symbol. Comput., pages 33–73. Springer, Vienna, 2013.

Bro13b. F. Brown. Iterated integrals in quantum field theory. In *Geometric and topological methods for quantum field theory*, pages 188–240. Cambridge Univ. Press, Cambridge, 2013. URL `http://www.ihes.fr/~brown/ColombiaNotes7.pdf`.

BS01. L. Bonora and M. Salizzoni. Renormalization of noncommutative U(N) gauge theories. *Phys. Lett.*, B504:80–88, 2001, hep-th/0011088. doi:10.1016/S0370-2693(01)00281-7.

BS08. D. Buchholz and S. J. Summers. Warped convolutions: A novel tool in the construction of quantum field theories. In *Quantum field theory and beyond: Essays in honor of Wolfhart Zimmermann*, pages 107–121, 2008, 0806.0349. doi:10.1142/9789812833556_0007.

Bür99. H. Bürmann. Essai de calcul fonctionnaire aux constantes ad-libitum. *Mem. Inst. Nat. Sci Arts. Sci. Math. Phys.*, 2:13–17, 1799.

CC97. A. H. Chamseddine and A. Connes. The Spectral action principle. *Commun. Math. Phys.*, 186:731–750, 1997, hep-th/9606001. doi:10.1007/s002200050126.

CCM14. A. H. Chamseddine, A. Connes, and V. Mukhanov. Geometry and the Quantum: Basics. *JHEP*, 12:098, 2014, 1411.0977. doi:10.1007/JHEP12(2014)098.

CCM15. A. H. Chamseddine, A. Connes, and V. Mukhanov. Quanta of Geometry: Noncommutative Aspects. *Phys. Rev. Lett.*, 114(9):091302, 2015, 1409.2471. doi:10.1103/PhysRevLett.114.091302.

CDS98. A. Connes, M. R. Douglas, and A. S. Schwarz. Noncommutative geometry and matrix theory: Compactification on tori. *JHEP*, 02:003, 1998, hep-th/9711162. doi:10.1088/1126-6708/1998/02/003.

CEO06. L. Chekhov, B. Eynard, and N. Orantin. Free energy topological expansion for the 2-matrix model. *JHEP*, 12:053, 2006, math-ph/0603003. doi:10.1088/1126-6708/2006/12/053.

CF00. A. S. Cattaneo and G. Felder. A path integral approach to the Kontsevich quantization formula. *Commun. Math. Phys.*, 212:591–611, 2000, math/9902090. doi:10.1007/s002200000229.

CGH+96. R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth. On the Lambert W function. *Adv. Comput. Math.*, 5:329–359, 1996. doi:10.1007/BF02124750.

CHMS00. S. Cho, R. Hinterding, J. Madore, and H. Steinacker. Finite field theory on noncommutative geometries. *Int. J. Mod. Phys.*, D9:161–199, 2000, hep-th/9903239. doi:10.1142/S0218271800000153.

CJS+02. X. Calmet, B. Jurčo, P. Schupp, J. Wess, and M. Wohlgenannt. The standard model on noncommutative space-time. *Eur. Phys. J.*, C23:363–376, 2002, hep-ph/0111115. doi:10.1007/s100520100873.

CK98. A. Connes and D. Kreimer. Hopf algebras, renormalization and noncommutative geometry. *Commun. Math. Phys.*, 199:203–242, 1998, hep-th/9808042. doi:10.1007/s002200050499.

CM98. A. Connes and H. Moscovici. Hopf algebras, cyclic cohomology and the transverse index theorem. *Comm. Math. Phys.*, 198(1):199–246, 1998, math/9806109. doi:10.1007/s002200050477.

CMS01. C.-S. Chu, J. Madore, and H. Steinacker. Scaling limits of the fuzzy sphere at one loop. *JHEP*, 08:038, 2001, hep-th/0106205. doi:10.1088/1126-6708/2001/08/038.

Con94. A. Connes. *Noncommutative geometry*. Academic Press, Inc., San Diego, CA, 1994.

Con95. A. Connes. Noncommutative geometry and reality. *J. Math. Phys.*, 36:6194–6231, 1995. doi:10.1063/1.531241.

Con96. A. Connes. Gravity coupled with matter and foundation of noncommutative geometry. *Commun. Math. Phys.*, 182:155–176, 1996, hep-th/9603053. doi:10.1007/BF02506388.

CR87. A. Connes and M. A. Rieffel. Yang-Mills for noncommutative two-tori. In *Operator algebras and mathematical physics (Iowa City, Iowa, 1985)*, volume 62 of *Contemp. Math.*, pages 237–266. Amer. Math. Soc., Providence, RI, 1987. doi:10.1090/conm/062/878383.

CR00. I. Chepelev and R. Roiban. Renormalization of quantum field theories on noncommutative $\mathbb{R}^d$. 1. Scalars. *JHEP*, 05:037, 2000, hep-th/9911098. doi:10.1088/1126-6708/2000/05/037.

CR01. I. Chepelev and R. Roiban. Convergence theorem for noncommutative Feynman graphs and renormalization. *JHEP*, 03:001, 2001, hep-th/0008090. doi:10.1088/1126-6708/2001/03/001.

DFGZJ95. P. Di Francesco, P. H. Ginsparg, and J. Zinn-Justin. 2-D gravity and random matrices. *Phys. Rept.*, 254:1–133, 1995, hep-th/9306153. doi:10.1016/0370-1573(94)00084-G.

DFIZ93. P. Di Francesco, C. Itzykson, and J. B. Zuber. Polynomial averages in the Kontsevich model. *Commun. Math. Phys.*, 151:193–219, 1993, hep-th/9206090. doi:10.1007/BF02096753.

DFR95. S. Doplicher, K. Fredenhagen, and J. E. Roberts. The quantum structure of space-time at the Planck scale and quantum fields. *Commun. Math. Phys.*, 172:187–220, 1995, hep-th/0303037. doi:10.1007/BF02104515.

DGMR07. M. Disertori, R. Gurau, J. Magnen, and V. Rivasseau. Vanishing of beta function of noncommutative ϕ^4_4 theory to all orders. *Phys. Lett.*, B649:95–102, 2007, hep-th/0612251. doi:10.1016/j.physletb.2007.04.007.

dGWW07. A. de Goursac, J.-C. Wallet, and R. Wulkenhaar. Noncommutative induced gauge theory. *Eur. Phys. J.*, C51:977–987, 2007, hep-th/0703075. doi:10.1140/epjc/s10052-007-0335-2.

DH98. M. R. Douglas and C. M. Hull. D-branes and the noncommutative torus. *JHEP*, 02:008, 1998, hep-th/9711165. doi:10.1088/1126-6708/1998/02/008.

dJHW19. J. de Jong, A. Hock, and R. Wulkenhaar. Catalan tables and a recursion relation in noncommutative quantum field theory. 2019, 1904.11231.

DN01. M. R. Douglas and N. A. Nekrasov. Noncommutative field theory. *Rev. Mod. Phys.*, 73:977–1029, 2001, hep-th/0106048. doi:10.1103/RevModPhys.73.977.

DPV16. D. D'Ascanio, P. Pisani, and D. V. Vassilevich. Renormalization on noncommutative torus. *Eur. Phys. J.*, C76(4):180, 2016, 1602.01479. doi:10.1140/epjc/s10052-016-4022-z.

DR07. M. Disertori and V. Rivasseau. Two- and three-loops beta function of noncommutative ϕ^4_4 theory. *Eur. Phys. J.*, C50:661–671, 2007, hep-th/0610224. doi:10.1140/epjc/s10052-007-0211-0.

DS90. M. R. Douglas and S. H. Shenker. Strings in less than one dimension. *Nucl. Phys.*, B335:635–654, 1990. doi:10.1016/0550-3213(90)90522-F.

DS03. S. Denk and M. Schweda. Time ordered perturbation theory for nonlocal interactions: Applications to NCQFT. *JHEP*, 09:032, 2003, hep-th/0306101. doi:10.1088/1126-6708/2003/09/032.

DSvZ18. V. Delecroix, J. Schmitt, and J. van Zelm. `admcycles` – a Sage program for calculations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ and computation of cycles of admissible covers, 2018. URL `https://people.math.ethz.ch/~schmittj/admcycles.html`.

EILS08. D. Essouabri, B. Iochum, C. Levy, and A. Sitarz. Spectral action on noncommutative torus. *J. Noncommut. Geom.*, 2(1):53–123, 2008, 0704.0564. doi:10.4171/JNCG/16.

EK82. T. Eguchi and H. Kawai. Reduction of dynamical degrees of freedom in the large N gauge theory. *Phys. Rev. Lett.*, 48:1063, 1982. doi:10.1103/PhysRevLett.48.1063.

Ell76. G. A. Elliott. On the classification of inductive limits of sequences of semisimple finite-dimensional algebras. *J. Algebra*, 38(1):29–44, 1976. doi:10.1016/0021-8693(76)90242-8.

EO07. B. Eynard and N. Orantin. Invariants of algebraic curves and topological expansion. *Commun. Num. Theor. Phys.*, 1:347–452, 2007, math-ph/0702045. doi:10.4310/CNTP.2007.v1.n2.a4.

Eyn04. B. Eynard. Topological expansion for the 1-Hermitian matrix model correlation functions. *JHEP*, 11:031, 2004, hep-th/0407261. doi:10.1088/1126-6708/2004/11/031.

Eyn14. B. Eynard. Invariants of spectral curves and intersection theory of moduli spaces of complex curves. *Commun. Num. Theor. Phys.*, 8:541–588, 2014, 1110.2949. doi:10.4310/CNTP.2014.v8.n3.a4.

Eyn16. B. Eynard. *Counting surfaces*, volume 70 of *Progress in Mathematical Physics*. Birkhäuser/Springer, 2016. doi:10.1007/978-3-7643-8797-6. CRM Aisenstadt chair lectures.

Eyn19. B. Eynard. Large genus behavior of topological recursion. 2019, 1905.11270.

Fey49. R. P. Feynman. Space-time approach to quantum electrodynamics. *Phys. Rev.*, 76:769–789, 1949. doi:10.1103/PhysRev.76.769.

Fil96. T. Filk. Divergencies in a field theory on quantum space. *Phys. Lett.*, B376:53–58, 1996. doi:10.1016/0370-2693(96)00024-X.

FP67. L. D. Faddeev and V. N. Popov. Feynman diagrams for the Yang-Mills field. *Phys. Lett.*, B25:29–30, 1967. doi:10.1016/0370-2693(67)90067-6.

Frö82. J. Fröhlich. On the triviality of $\lambda\phi^4_d$ theories and the approach to the critical point in $d \geq 4$ dimensions. *Nucl. Phys.*, B200:281–296, 1982. doi:10.1016/0550-3213(82)90088-8.

FS09. A. Fischer and R. J. Szabo. Duality covariant quantum field theory on noncommutative Minkowski space. *JHEP*, 02:031, 2009, 0810.1195. doi:10.1088/1126-6708/2009/02/031.

GAO83. A. Gonzalez-Arroyo and M. Okawa. The twisted Eguchi-Kawai model: A reduced model for large N lattice gauge theory. *Phys. Rev.*, D27:2397, 1983. doi:10.1103/PhysRevD.27.2397.

Gar00. M. R. Garousi. Noncommutative world volume interactions on D-branes and Dirac-Born-Infeld action. *Nucl. Phys.*, B579:209–228, 2000, hep-th/9909214. doi:10.1016/S0550-3213(99)00826-3.

Gay03. V. Gayral. The action functional for Moyal planes. *Lett. Math. Phys.*, 65:147–157, 2003, hep-th/0307220. doi:10.1023/B:MATH.0000004380.57824.94.

GBV88. J. M. Gracia-Bondía and J. C. Várilly. Algebras of distributions suitable for phase space quantum mechanics. 1. *J. Math. Phys.*, 29:869–879, 1988. doi:10.1063/1.528200.

GGBI$^+$04. V. Gayral, J. M. Gracia-Bondía, B. Iochum, T. Schücker, and J. C. Várilly. Moyal planes are spectral triples. *Commun. Math. Phys.*, 246:569–623, 2004, hep-th/0307241. doi:10.1007/s00220-004-1057-z.

GHW19a. H. Grosse, A. Hock, and R. Wulkenhaar. A Laplacian to compute intersection numbers on $\overline{\mathcal{M}}_{g,n}$ and correlation functions in NCQFT. 2019, 1903.12526.

GHW19b. H. Grosse, A. Hock, and R. Wulkenhaar. Solution of all quartic matrix models. 2019, 1906.04600.

GHW19c. H. Grosse, A. Hock, and R. Wulkenhaar. Solution of the self-dual Φ^4 QFT-model on four-dimensional Moyal space. 2019, 1908.04543.

GI05. V. Gayral and B. Iochum. The spectral action for Moyal planes. *J. Math. Phys.*, 46:043503, 2005, hep-th/0402147. doi:10.1063/1.1855401.

GJ87. J. Glimm and A. Jaffe. *Quantum physics. A functional integral point of view*. Springer-Verlag, New York, second edition, 1987. doi:10.1007/978-1-4612-4728-9.

GJKW07. V. Gayral, J.-H. Jureit, T. Krajewski, and R. Wulkenhaar. Quantum field theory on projective modules. *J. Noncommut. Geom.*, 1(4):431–496, 2007, hep-th/0612048. doi:10.4171/JNCG/13.

GJS74. J. Glimm, A. Jaffe, and T. Spencer. The Wightman axioms and particle structure in the $\mathcal{P}(\phi)_2$ quantum field model. *Ann. of Math. (2)*, 100:585–632, 1974. doi:10.2307/1970959.

GKP96a. H. Grosse, C. Klimčík, and P. Prešnajder. On finite 4-D quantum field theory in noncommutative geometry. *Commun. Math. Phys.*, 180:429–438, 1996, hep-th/9602115. doi:10.1007/BF02099720.

GKP96b. H. Grosse, C. Klimčík, and P. Prešnajder. Towards finite quantum field theory in noncommutative geometry. *Int. J. Theor. Phys.*, 35:231–244, 1996, hep-th/9505175. doi:10.1007/BF02083810.

GL07. H. Grosse and G. Lechner. Wedge-local quantum fields and noncommutative Minkowski space. *JHEP*, 11:012, 2007, 0706.3992. doi:10.1088/1126-6708/2007/11/012.

GL08. H. Grosse and G. Lechner. Noncommutative deformations of Wightman quantum field theories. *JHEP*, 09:131, 2008, 0808.3459. doi:10.1088/1126-6708/2008/09/131.

GLLV13. H. Grosse, G. Lechner, T. Ludwig, and R. Verch. Wick rotation for quantum field theories on degenerate Moyal space(-time). *J. Math. Phys.*, 54:022307, 2013, 1111.6856. doi:10.1063/1.4790886.

GM90. D. J. Gross and A. A. Migdal. Nonperturbative two-dimensional quantum gravity. *Phys. Rev. Lett.*, 64:127–130, 1990. doi:10.1103/PhysRevLett.64.127.

GM92. H. Grosse and J. Madore. A noncommutative version of the Schwinger model. *Phys. Lett.*, B283:218–222, 1992. doi:10.1016/0370-2693(92)90011-R.

GM00. J. Gomis and T. Mehen. Space-time noncommutative field theories and unitarity. *Nucl. Phys.*, B591:265–276, 2000, hep-th/0005129. doi:10.1016/S0550-3213(00)00525-3.

GM04. A. B. Goncharov and Y. I. Manin. Multiple ζ-motives and moduli spaces $\overline{\mathfrak{M}}_{0,n}$. *Compos. Math.*, 140(1):1–14, 2004. doi:10.1112/S0010437X03000125.

GMRT09. R. Gurau, J. Magnen, V. Rivasseau, and A. Tanasa. A translation-invariant renormalizable non-commutative scalar model. *Commun. Math. Phys.*, 287:275–290, 2009, 0802.0791. doi:10.1007/s00220-008-0658-3.

GMS01. H. Grosse, J. Madore, and H. Steinacker. Field theory on the q deformed fuzzy sphere. 1. *J. Geom. Phys.*, 38:308–342, 2001, hep-th/0005273. doi:10.1016/S0393-0440(00)00068-1.

GMS02. H. Grosse, J. Madore, and H. Steinacker. Field theory on the q deformed fuzzy sphere. 2. Quantization. *J. Geom. Phys.*, 43:205–240, 2002, hep-th/0103164. doi:10.1016/S0393-0440(02)00023-2.

GR07. R. Gurau and V. Rivasseau. Parametric representation of noncommutative field theory. *Commun. Math. Phys.*, 272:811–835, 2007, math-ph/0606030. doi:10.1007/s00220-007-0215-5.

GR11. R. Gurau and V. Rivasseau. The 1/N expansion of colored tensor models in arbitrary dimension. *Europhys. Lett.*, 95(5):50004, 2011, 1101.4182. doi:10.1209/0295-5075/95/50004.

GR12. R. Gurau and J. P. Ryan. Colored tensor models - A review. *SIGMA*, 8:020, 2012, 1109.4812. doi:10.3842/SIGMA.2012.020.

GR15. R. Gurau and V. Rivasseau. The multiscale loop vertex expansion. *Annales Henri Poincaré*, 16(8):1869–1897, 2015, 1312.7226. doi:10.1007/s00023-014-0370-0.

GRM09. R. Gurau, V. Rivasseau, and J. Magnen. Tree quantum field theory. *Annales Henri Poincaré*, 10:867–891, 2009, 0807.4122. doi:10.1007/s00023-009-0002-2.

Gro46. H. J. Groenewold. On the principles of elementary quantum mechanics. *Physica*, 12:405–460, 1946. doi:10.1016/S0031-8914(46)80059-4.

GRS14. R. Gurau, V. Rivasseau, and A. Sfondrini. Renormalization: an advanced overview. 2014, 1401.5003.

GRVT06. R. Gurau, V. Rivasseau, and F. Vignes-Tourneret. Propagators for noncommutative field theories. *Annales Henri Poincaré*, 7:1601–1628, 2006, hep-th/0512071. doi:10.1007/s00023-006-0296-2.

GS99. H. Grosse and A. Strohmaier. Towards a nonperturbative covariant regularization in 4-D quantum field theory. *Lett. Math. Phys.*, 48:163–179, 1999, hep-th/9902138. doi:10.1023/A:1007518622795.

GS01. S. S. Gubser and S. L. Sondhi. Phase structure of noncommutative scalar field theories. *Nucl. Phys.*, B605:395–424, 2001, hep-th/0006119. doi:10.1016/S0550-3213(01)00108-0.

GS06a. H. Grosse and H. Steinacker. A nontrivial solvable noncommutative ϕ^3 model in 4 dimensions. *JHEP*, 08:008, 2006, hep-th/0603052. doi:10.1088/1126-6708/2006/08/008.

GS06b. H. Grosse and H. Steinacker. Renormalization of the noncommutative ϕ^3 model through the Kontsevich model. *Nucl. Phys.*, B746:202–226, 2006, hep-th/0512203. doi:10.1016/j.nuclphysb.2006.04.007.

GS08. H. Grosse and H. Steinacker. Exact renormalization of a noncommutative ϕ^3 model in 6 dimensions. *Adv. Theor. Math. Phys.*, 12(3):605–639, 2008, hep-th/0607235. doi:10.4310/ATMP.2008.v12.n3.a4.

GSW08. H. Grosse, H. Steinacker, and M. Wohlgenannt. Emergent gravity, matrix models and UV/IR mixing. *JHEP*, 04:023, 2008, 0802.0973. doi:10.1088/1126-6708/2008/04/023.

GSW17. H. Grosse, A. Sako, and R. Wulkenhaar. Exact solution of matricial Φ^3_2 quantum field theory. *Nucl. Phys.*, B925:319–347, 2017, 1610.00526. doi:10.1016/j.nuclphysb.2017.10.010.

GSW18. H. Grosse, A. Sako, and R. Wulkenhaar. The Φ^3_4 and Φ^3_6 matricial QFT models have reflection positive two-point function. *Nucl. Phys.*, B926:20–48, 2018, 1612.07584. doi:10.1016/j.nuclphysb.2017.10.022.

Gur11a. R. Gurau. Colored group field theory. *Commun. Math. Phys.*, 304:69–93, 2011, 0907.2582. doi:10.1007/s00220-011-1226-9.

Gur11b. R. Gurau. The 1/N expansion of colored tensor models. *Annales Henri Poincaré*, 12:829–847, 2011, 1011.2726. doi:10.1007/s00023-011-0101-8.

Gur12. R. Gurau. The complete 1/N expansion of colored tensor models in arbitrary dimension. *Annales Henri Poincaré*, 13:399–423, 2012, 1102.5759. doi:10.1007/s00023-011-0118-z.

Gur14. R. Gurau. The $1/N$ expansion of tensor models beyond perturbation theory. *Commun. Math. Phys.*, 330:973–1019, 2014, 1304.2666. doi:10.1007/s00220-014-1907-2.

Gur17. R. Gurau. *Random Tensors*. Oxford University Press, Oxford, 2017.

GW73. D. J. Gross and F. Wilczek. Ultraviolet behavior of nonabelian gauge theories. *Phys. Rev. Lett.*, 30:1343–1346, 1973. doi:10.1103/PhysRevLett.30.1343.

GW02. J. M. Grimstrup and R. Wulkenhaar. Quantisation of θ-expanded noncommutative QED. *Eur. Phys. J.*, C26:139–151, 2002, hep-th/0205153. doi:10.1140/epjc/s2002-01038-9.

GW03. H. Grosse and R. Wulkenhaar. Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^2$ in the matrix base. *JHEP*, 12:019, 2003, hep-th/0307017. doi:10.1088/1126-6708/2003/12/019.

GW04. H. Grosse and R. Wulkenhaar. The β-function in duality covariant noncommutative ϕ^4-theory. *Eur. Phys. J.*, C35:277–282, 2004, hep-th/0402093. doi:10.1140/epjc/s2004-01853-x.

GW05a. H. Grosse and R. Wulkenhaar. Power-counting theorem for non-local matrix models and renormalisation. *Commun. Math. Phys.*, 254:91–127, 2005, hep-th/0305066. doi:10.1007/s00220-004-1238-9.

GW05b. H. Grosse and R. Wulkenhaar. Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base. *Commun. Math. Phys.*, 256:305–374, 2005, hep-th/0401128. doi:10.1007/s00220-004-1285-2.

GW07. H. Grosse and M. Wohlgenannt. Induced gauge theory on a noncommutative space. *Eur. Phys. J.*, C52:435–450, 2007, hep-th/0703169. doi:10.1140/epjc/s10052-007-0369-5.

GW09. H. Grosse and R. Wulkenhaar. Progress in solving a noncommutative quantum field theory in four dimensions. 2009, 0909.1389.

GW12. H. Grosse and R. Wulkenhaar. 8D-spectral triple on 4D-Moyal space and the vacuum of noncommutative gauge theory. *J. Geom. Phys.*, 62:1583–1599, 2012, 0709.0095. doi:10.1016/j.geomphys.2012.03.005.

GW13a. V. Gayral and R. Wulkenhaar. Spectral geometry of the Moyal plane with harmonic propagation. *J. Noncommut. Geom.*, 7:939–979, 2013, 1108.2184. doi:10.4171/JNCG/140.

GW13b. H. Grosse and R. Wulkenhaar. Solvable limits of a 4D noncommutative QFT. 2013, 1306.2816.

GW14a. H. Grosse and R. Wulkenhaar. Self-dual noncommutative ϕ^4-theory in four dimensions is a non-perturbativelyl solvable and non-trivial quantum field theory. *Commun. Math. Phys.*, 329:1069–1130, 2014, 1205.0465. doi:10.1007/s00220-014-1906-3.

GW14b. H. Grosse and R. Wulkenhaar. Solvable 4D noncommutative QFT: phase transitions and quest for reflection positivity. 2014, 1406.7755.

GW16. H. Grosse and R. Wulkenhaar. On the fixed point equation of a solvable 4D QFT model. *Vietnam J. Math.*, 44(1):153–180, 2016, 1505.05161. doi:10.1007/s10013-015-0174-7.

Haa58. R. Haag. Quantum field theories with composite particles and asymptotic conditions. *Phys. Rev.*, 112:669–673, 1958. doi:10.1103/PhysRev.112.669.

Haa96. R. Haag. *Local quantum physics. Fields, particles, algebras*. Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1996. doi:10.1007/978-3-642-61458-3.

Hau23. F. Hausdorff. Momentprobleme für ein endliches Intervall. *Math. Z.*, 16(1):220–248, 1923. doi:10.1007/BF01175684.

Haw99. E. Hawkins. Noncommutative regularization for the practical man. 1999, hep-th/9908052.

Hay00. M. Hayakawa. Perturbative analysis on infrared aspects of noncommutative QED on $\mathbb{R}^4$. *Phys. Lett.*, B478:394–400, 2000, hep-th/9912094. doi:10.1016/S0370-2693(00)00242-2.

Hei38. W. Heisenberg. Die Grenzen der Anwendbarkeit der bisherigen Quantentheorie. *Z. Phys.*, 110:251–266, 1938. doi:10.1007/BF01342872.

Hep66. K. Hepp. Proof of the Bogolyubov-Parasiuk theorem on renormalization. *Commun. Math. Phys.*, 2:301–326, 1966. doi:10.1007/BF01773358.

HK64. R. Haag and D. Kastler. An algebraic approach to quantum field theory. *J. Math. Phys.*, 5:848–861, 1964. doi:10.1063/1.1704187.

HW18. A. Hock and R. Wulkenhaar. Noncommutative 3-colour scalar quantum field theory model in 2D. *Eur. Phys. J.*, C78(7):580, 2018, 1804.06075. doi:10.1140/epjc/s10052-018-6042-3.

IKTW01. S. Iso, Y. Kimura, K. Tanaka, and K. Wakatsuki. Noncommutative gauge theory on fuzzy sphere from matrix model. *Nucl. Phys.*, B604:121–147, 2001, hep-th/0101102. doi:10.1016/S0550-3213(01)00173-0.

ILS09. B. Iochum, C. Levy, and A. Sitarz. Spectral action on $SU_q(2)$. *Commun. Math. Phys.*, 289:107–155, 2009, 0803.1058. doi:10.1007/s00220-009-0810-8.

IMSS11. B. Iochum, T. Masson, T. Schucker, and A. Sitarz. Compact κ-deformation and spectral triples. *Rept. Math. Phys.*, 68:37–64, 2011, 1004.4190. doi:10.1016/S0034-4877(11)60026-8.

IZ92. C. Itzykson and J. B. Zuber. Combinatorics of the modular group. 2. The Kontsevich integrals. *Int. J. Mod. Phys.*, A7:5661–5705, 1992, hep-th/9201001. doi:10.1142/S0217751X92002581.

JNOS17. A. Jaffe, K.-H. Neeb, G. Olafsson, and B. Schlein, editors. *Reflection positivity*, volume 14 of *Oberwolfach Rep.*, 2017. doi:10.4171/OWR/2017/55.

JSSW00. B. Jurčo, S. Schraml, P. Schupp, and J. Wess. Enveloping algebra valued gauge transformations for nonAbelian gauge groups on noncommutative spaces. *Eur. Phys. J.*, C17:521–526, 2000, hep-th/0006246. doi:10.1007/s100520000487.

JSW01. B. Jurčo, P. Schupp, and J. Wess. NonAbelian noncommutative gauge theory via noncommutative extra dimensions. *Nucl. Phys.*, B604:148–180, 2001, hep-th/0102129. doi:10.1016/S0550-3213(01)00191-2.

JW00. A. M. Jaffe and E. Witten. Quantum Yang-Mills theory. 2000. URL `https://www.claymath.org/sites/default/files/yangmills.pdf`.

Kac49. M. Kac. On distributions of certain Wiener functionals. *Trans. Amer. Math. Soc.*, 65:1–13, 1949. doi:10.2307/1990512.

Kit15a. A. Kitaev. A simple model of quantum holography. 1. Talk at KITP, 2015. URL `http://online.kitp.ucsb.edu/online/entangled15/kitaev/`.

Kit15b. A. Kitaev. A simple model of quantum holography. 2. Talk at KITP, 2015. URL `http://online.kitp.ucsb.edu/online/entangled15/kitaev2/`.

KK66. T. Kōmura and Y. Kōmura. Über die Einbettung der nuklearen Räume in $(s)^A$. *Math. Ann.*, 162:284–288, 1966. doi:10.1007/BF01360917.

KL81. A. Klein and L. J. Landau. Construction of a unique selfadjoint generator for a symmetric local semigroup. *J. Funct. Anal.*, 44(2):121–137, 1981. doi:10.1016/0022-1236(81)90007-0.

Kon92. M. Kontsevich. Intersection theory on the moduli space of curves and the matrix Airy function. *Commun. Math. Phys.*, 147:1–23, 1992. doi:10.1007/BF02099526.

Kon03. M. Kontsevich. Deformation quantization of Poisson manifolds. 1. *Lett. Math. Phys.*, 66:157–216, 2003, q-alg/9709040. doi:10.1023/B:MATH.0000027508.00421.bf.

KPT18. I. R. Klebanov, F. Popov, and G. Tarnopolsky. TASI lectures on large N tensor models. *PoS*, TASI2017:004, 2018, 1808.09434. doi:10.22323/1.305.0004.

Kre98. D. Kreimer. On the Hopf algebra structure of perturbative quantum field theories. *Adv. Theor. Math. Phys.*, 2:303–334, 1998, q-alg/9707029. doi:10.4310/ATMP.1998.v2.n2.a4.

Kre00. D. Kreimer. Chen's iterated integral represents the operator product expansion. *Adv. Theor. Math. Phys.*, 3:627–670, 2000, hep-th/9901099. doi:10.4310/ATMP.1999.v3.n3.a7.

KRTW10. T. Krajewski, V. Rivasseau, A. Tanasă, and Z. Wang. Topological graph polynomials and quantum field theory. I. Heat kernel theories. *J. Noncommut. Geom.*, 4(1):29–82, 2010, 0811.0186. doi:10.4171/JNCG/49.

KRVT11. T. Krajewski, V. Rivasseau, and F. Vignes-Tourneret. Topological graph polynomials and quantum field theory, part II. Mehler kernel theories. *Annales Henri Poincaré*, 12:483–545, 2011, 0912.5438. doi:10.1007/s00023-011-0087-2.

KS96. R. Koekoek and R. F. Swarttouw. The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue. 1996, math/9602214.

KS02. A. Konechny and A. S. Schwarz. Introduction to M(atrix) theory and noncommutative geometry. *Phys. Rept.*, 360:353–465, 2002, hep-th/0012145. doi:10.1016/S0370-1573(01)00096-5.

Kul76. P. P. Kulish. Factorization of the classical and quantum S-matrix and conservation laws. *Theor. Math. Phys.*, 26:132, 1976. doi:10.1007/BF01079418. [also in Teor. Mat. Fiz. 26:198, 1976].

KW00. T. Krajewski and R. Wulkenhaar. Perturbative quantum gauge fields on the noncommutative torus. *Int. J. Mod. Phys.*, A15:1011–1030, 2000, hep-th/9903187. doi:10.1142/S0217751X00000495.

Lag70. J. L. Lagrange. Nouvelle méthode pour résoudre des équations littérales par le moyen de séries. *Mém. Acad. Roy. des Sci. et Belles-Lettres de Berlin*, 24, 1770.

LAK54. L. D. Landau, A. A. Abrikosov, and I. M. Khalatnikov. On the removal of infinities in quantum electrodynamics (in russ.). *Dokl. Akad. Nauk SSSR*, 95:497–500, 1954.

Lam58. J. H. Lambert. Observationes variae in mathesin puram. *Acta Helvetica, physico-mathematico-anatomico-botanico-medica*, 3:128–168, 1758.

LLS01. G. Landi, F. Lizzi, and R. J. Szabo. From large N matrices to the noncommutative torus. *Commun. Math. Phys.*, 217:181–201, 2001, hep-th/9912130. doi:10.1007/s002200000356.

LLS04. G. Landi, F. Lizzi, and R. J. Szabo. A new matrix model for noncommutative field theory. *Phys. Lett.*, B578:449–458, 2004, hep-th/0309031. doi:10.1016/j.physletb.2003.10.059.

LPW88. J. M. F. Labastida, M. Pernici, and E. Witten. Topological gravity in two dimensions. *Nucl. Phys.*, B310:611–624, 1988. doi:10.1016/0550-3213(88)90094-6.

LS02a. E. Langmann and R. J. Szabo. Duality in scalar field theory on noncommutative phase spaces. *Phys. Lett.*, B533:168–177, 2002, hep-th/0202039. doi:10.1016/S0370-2693(02)01650-7.

LS02b. Y. Liao and K. Sibold. Time ordered perturbation theory on noncommutative space-time. 2. Unitarity. *Eur. Phys. J.*, C25:479–486, 2002, hep-th/0206011. doi:10.1007/s10052-002-1018-7.

LS02c. Y. Liao and K. Sibold. Time ordered perturbation theory on noncommutative space-time: Basic rules. *Eur. Phys. J.*, C25:469–477, 2002, hep-th/0205269. doi:10.1007/s10052-002-1017-8.

LSZ03. E. Langmann, R. J. Szabo, and K. Zarembo. Exact solution of noncommutative field theory in background magnetic fields. *Phys. Lett.*, B569:95–101, 2003, hep-th/0303082. doi:10.1016/j.physletb.2003.07.020.

LSZ04. E. Langmann, R. J. Szabo, and K. Zarembo. Exact solution of quantum field theory on noncommutative phase spaces. *JHEP*, 01:017, 2004, hep-th/0308043. doi:10.1088/1126-6708/2004/01/017.

LZ04. S. K. Lando and A. K. Zvonkin. *Graphs on surfaces and their applications*, volume 141 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. doi:10.1007/978-3-540-38361-1. With an appendix by Don B. Zagier, Low-Dimensional Topology, II.

Mad92. J. Madore. The fuzzy sphere. *Class. Quant. Grav.*, 9:69–88, 1992. doi:10.1088/0264-9381/9/1/008.

Mad95. J. Madore. *An introduction to noncommutative differential geometry and its physical applications*, volume 206 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995.

Min59. R. A. Minlos. Generalized random processes and their extension in measure. *Trudy Moskov. Mat. Obšč.*, 8:497–518, 1959. URL `http://www.mathnet.ru/php/getFT.phtml?jrnid=mmo&paperid=94&what=fullt&option_lang=eng`.

Mir07. M. Mirzakhani. Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces. *Invent. Math.*, 167(1):179–222, 2007. doi:10.1007/s00222-006-0013-2.

Miw82. T. Miwa. On Hirota's difference equations. *Proc. Japan Acad. Ser. A Math. Sci.*, 58(1):9–12, 1982. URL `http://projecteuclid.org/euclid.pja/1195516178`.

Mos75. J. Moser. Three integrable Hamiltonian systems connnected with isospectral deformations. *Adv. Math.*, 16:197–220, 1975. doi:10.1016/0001-8708(75)90151-6.

Moy49. J. E. Moyal. Quantum mechanics as a statistical theory. *Proc. Cambridge Phil. Soc.*, 45:99–124, 1949. doi:10.1017/S0305004100000487.

MP90. R. C. Myers and V. Periwal. Topological gravity and moduli space. *Nucl. Phys.*, B333:536–550, 1990. doi:10.1016/0550-3213(90)90050-N.

MR08. J. Magnen and V. Rivasseau. Constructive ϕ^4 field theory without tears. *Annales Henri Poincaré*, 9:403–424, 2008, 0706.2457. doi:10.1007/s00023-008-0360-1.

MS91. Yu. Makeenko and G. W. Semenoff. Properties of Hermitean matrix models in an external field. *Mod. Phys. Lett.*, A6:3455–3466, 1991. doi:10.1142/S0217732391003985.

MS16. J. Maldacena and D. Stanford. Remarks on the Sachdev-Ye-Kitaev model. *Phys. Rev.*, D94(10):106002, 2016, 1604.07818. doi:10.1103/PhysRevD.94.106002.

MSR99. C. P. Martín and D. Sánchez-Ruiz. The one loop UV divergent structure of U(1) Yang-Mills theory on noncommutative $\mathbb{R}^4$. *Phys. Rev. Lett.*, 83:476–479, 1999, hep-th/9903077. doi:10.1103/PhysRevLett.83.476.

MSSW00. J. Madore, S. Schraml, P. Schupp, and J. Wess. Gauge theory on noncommutative spaces. *Eur. Phys. J.*, C16:161–167, 2000, hep-th/0001203. doi:10.1007/s100520050012.

MST00. A. Matusis, L. Susskind, and N. Toumbas. The IR / UV connection in the noncommutative gauge theories. *JHEP*, 12:002, 2000, hep-th/0002075. doi:10.1088/1126-6708/2000/12/002.

MT13. P. Martinetti and L. Tomassini. Noncommutative geometry of the Moyal plane: Translation isometries, Connes' distance on coherent states, Pythagoras equality. *Commun. Math. Phys.*, 323:107–141, 2013, 1110.6164. doi:10.1007/s00220-013-1760-8.

MVRS00. S. Minwalla, M. Van Raamsdonk, and N. Seiberg. Noncommutative perturbative dynamics. *JHEP*, 02:020, 2000, hep-th/9912072. doi:10.1088/1126-6708/2000/02/020.

MZ15. M. Mirzakhani and P. Zograf. Towards large genus asymptotics of intersection numbers on moduli spaces of curves. *Geom. Funct. Anal.*, 25(4):1258–1289, 2015, 1112.1151. doi:10.1007/s00039-015-0336-5.

Nel73a. E. Nelson. Construction of quantum fields from Markoff fields. *J. Functional Analysis*, 12:97–112, 1973. doi:10.1016/0022-1236(73)90091-8.

Nel73b. E. Nelson. Quantum fields and Markoff fields. In *Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971)*, pages 413–420. Amer. Math. Soc., Providence, R.I., 1973.

Nie09. N. Nielsen. Der Eulersche Dilogarithmus und seine Verallgemeinerungen. *Nova acta - Kaiserlich Leopoldinisch-Carolinische Deutsche Akademie der Naturforscher*, 90(3):121–212, 1909.

NO18. K.-H. Neeb and G. Ólafsson. *Reflection positivity*, volume 32 of *SpringerBriefs in Mathematical Physics*. Springer, 2018. doi:10.1007/978-3-319-94755-6.

Nus55. A. E. Nussbaum. The Hausdorff-Bernstein-Widder theorem for semi-groups in locally compact Abelian groups. *Duke Math. J.*, 22:573–582, 1955. URL `http://projecteuclid.org/euclid.dmj/1077466543`.

ORZ04. T. Ohl, R. Rückl, and J. Zeiner. Unitarity of time - like noncommutative gauge theories: The violation of Ward identities in time ordered perturbation theory. *Nucl. Phys.*, B676:229–242, 2004, hep-th/0309021. doi:10.1016/j.nuclphysb.2003.10.022.

OS73. K. Osterwalder and R. Schrader. Axioms for Euclidean Green's functions. *Commun. Math. Phys.*, 31:83–112, 1973. doi:10.1007/BF01645738.

OS75. K. Osterwalder and R. Schrader. Axioms for Euclidean Green's functions. 2. *Commun. Math. Phys.*, 42:281–305, 1975. doi:10.1007/BF01608978.

Pan15. E. Panzer. Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals. *Computer Physics Communications*, 188:148–166, 2015, 1403.3385. doi:10.1016/j.cpc.2014.10.019. maintained and available at `https://bitbucket.org/PanzerErik/hyperint`.

Pei33. R. Peierls. Zur Theorie des Diamagnetismus von Leitungselektronen. *Z. Phys.*, 80:763, 1933.

Per07. D. Perrot. Anomalies and noncommutative index theory. In *Geometric and topological methods for quantum field theory*, volume 434 of *Contemp. Math.*, pages 125–160. Amer. Math. Soc., Providence, RI, 2007, hep-th/0603209. doi:10.1090/conm/434/08344.

Pol73. H. D. Politzer. Reliable perturbative results for strong interactions? *Phys. Rev. Lett.*, 30:1346–1349, 1973. doi:10.1103/PhysRevLett.30.1346.

Pol84. J. Polchinski. Renormalization and effective Lagrangians. *Nucl. Phys.*, B231:269–295, 1984. doi:10.1016/0550-3213(84)90287-6.

PW18. E. Panzer and R. Wulkenhaar. Lambert-W solves the noncommutative Φ^4-model. *Commun. Math. Phys.*, (in press), 2018, 1807.02945. doi:10.1007/s00220-019-03592-4.

Rie92. B. Riemann. Ueber die Hypothesen, welche der Geometrie zugrunde liegen. In *Bernhard Riemann's gesammelte mathematische Werke und wissenschaftlicher Nachlaß*, Leipzig, 1892. B. G. Teubner. URL `https://www.emis.de/classics/Riemann/Geom.pdf`.

Rie90. M. A. Rieffel. Noncommutative tori—a case study of noncommutative differentiable manifolds. In *Geometric and topological invariants of elliptic operators (Brunswick,* *ME, 1988)*, volume 105 of *Contemp. Math.*, pages 191–211. Amer. Math. Soc., Providence, RI, 1990. doi:10.1090/conm/105/1047281.

Rie93. M. A. Rieffel. Deformation quantization for actions of $\mathbf{R}^d$. *Mem. Amer. Math. Soc.*, 106(506):x+93, 1993. doi:10.1090/memo/0506.

Rie04. M. A. Rieffel. Gromov-Hausdorff distance for quantum metric spaces. *Mem. Amer. Math. Soc.*, 168(796):viii+91, 2004, math/0011063. doi:10.1090/memo/0796.

Riv91. V. Rivasseau. *From Perturbative to Constructive Renormalization*. Princton University Press, New Jersey, 1991.

Riv07a. V. Rivasseau. Constructive matrix theory. *JHEP*, 09:008, 2007, 0706.1224. doi:10.1088/1126-6708/2007/09/008.

Riv07b. V. Rivasseau. Non-commutative renormalization. In *Quantum Spaces*, volume 53 of *Progress in Mathematical Physics*, pages 19–107. Birkhäuser, Basel, 2007, 0705.0705. doi:10.1007/978-3-7643-8522-4_2.

Riv14. V. Rivasseau. The tensor track, III. *Fortsch. Phys.*, 62:81–107, 2014, 1311.1461. doi:10.1002/prop.201300032.

Rue62. D. Ruelle. On the asymptotic condition in quantum field theory. *Helv. Phys. Acta*, 35:147–163, 1962. doi:10.5169/seals-113272.

RVTW06. V. Rivasseau, F. Vignes-Tourneret, and R. Wulkenhaar. Renormalization of noncommutative ϕ^4-theory by multi-scale analysis. *Commun. Math. Phys.*, 262:565–594, 2006, hep-th/0501036. doi:10.1007/s00220-005-1440-4.

RW12. V. Rivasseau and Z. Wang. Constructive renormalization for Φ^4_2 theory with loop vertex expansion. *J. Math. Phys.*, 53:042302, 2012, 1104.3443. doi:10.1063/1.4705689.

RW15. V. Rivasseau and Z. Wang. Corrected loop vertex expansion for Φ^4_2 theory. *J. Math. Phys.*, 56(6):062301, 2015, 1406.7428. doi:10.1063/1.4922116.

Sch34. E. Schrödinger. Über die Unanwendbarkeit der Geometrie im Kleinen. *Naturwiss.*, 31:518–520, 1934. doi:10.1007/BF01494946.

Sch59. J. Schwinger. Euclidean quantum electrodynamics. *Phys. Rev.*, 115:721–731, 1959. doi:10.1103/PhysRev.115.721.

Sch99. V. Schomerus. D-branes and deformation quantization. *JHEP*, 06:030, 1999, hep-th/9903205. doi:10.1088/1126-6708/1999/06/030.

Seg01. G. Segal. Topological structures in string theory. *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.*, 359(1784):1389–1398, 2001. doi:10.1098/rsta.2001.0841. Topological methods in the physical sciences (London, 2000).

Sim74. B. Simon. *The $P(\phi)_2$ Euclidean (quantum) field theory*. Princeton University Press, Princeton, N.J., 1974. Princeton Series in Physics.

SJ99. M. M. Sheikh-Jabbari. Renormalizability of the supersymmetric Yang-Mills theories on the noncommutative torus. *JHEP*, 06:015, 1999, hep-th/9903107. doi:10.1088/1126-6708/1999/06/015.

Sny47. H. S. Snyder. Quantized space-time. *Phys. Rev.*, 71:38–41, 1947. doi:10.1103/PhysRev.71.38.

SS94. P. Schaller and T. Strobl. Poisson structure induced (topological) field theories. *Mod. Phys. Lett.*, A9:3129–3136, 1994, hep-th/9405110. doi:10.1142/S0217732394002951.

Ste04. H. Steinacker. Quantized gauge theory on the fuzzy sphere as random matrix model. *Nucl. Phys.*, B679:66–98, 2004, hep-th/0307075. doi:10.1016/j.nuclphysb.2003.12.005.

Ste07. H. Steinacker. Emergent gravity from noncommutative gauge theory. *JHEP*, 12:049, 2007, 0708.2426. doi:10.1088/1126-6708/2007/12/049.

Ste10. H. Steinacker. Emergent geometry and gravity from matrix models: an introduction. *Class. Quant. Grav.*, 27:133001, 2010, 1003.4134. doi:10.1088/0264-9381/27/13/133001.

Str67. K. Strebel. On quadratic differentials with closed trajectories and second order poles. *J. Analyse Math.*, 19:373–382, 1967. doi:10.1007/BF02788726.

Str84. K. Strebel. *Quadratic differentials*, volume 5 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 1984. doi:10.1007/978-3-662-02414-0.

SW64. R. Streater and A. Wightman. *PCT, Spin and Statistics, and All that*. W. A. Benjamin Inc., 1964.

SW99. N. Seiberg and E. Witten. String theory and noncommutative geometry. *JHEP*, 09:032, 1999, hep-th/9908142. doi:10.1088/1126-6708/1999/09/032.

SY93. S. Sachdev and J. Ye. Gapless spin fluid ground state in a random, quantum Heisenberg magnet. *Phys. Rev. Lett.*, 70:3339, 1993, cond-mat/9212030. doi:10.1103/PhysRevLett.70.3339.

Sym64. K. Symanzik. Application of functional integrals to Euclidean quantum field theory. In *Proc. Conf. on Theory and Appl. of Analysis in Function Space (Dedham, Mass. 1963)*, pages 197–206. The M.I.T. Press, Cambridge, Mass., 1964.

Sza03. R. J. Szabo. Quantum field theory on noncommutative spaces. *Phys. Rept.*, 378:207–299, 2003, hep-th/0109162. doi:10.1016/S0370-1573(03)00059-0.

tH74. G. 't Hooft. A planar diagram theory for strong interactions. *Nucl. Phys.*, B72:461–473, 1974. doi:10.1016/0550-3213(74)90154-0.

Tri57. F. G. Tricomi. *Integral equations*. Interscience, New York, 1957.

Vas04. D. V. Vassilevich. Noncommutative heat kernel. *Lett. Math. Phys.*, 67:185–194, 2004, hep-th/0310144. doi:10.1023/B:MATH.0000035037.50663.b1.

VGB88. J. C. Várilly and J. M. Gracia-Bondía. Algebras of distributions suitable for phase-space quantum mechanics. 2. Topologies on the Moyal algebra. *J. Math. Phys.*, 29:880–887, 1988. doi:10.1063/1.527984.

VGB99. J. C. Várilly and J. M. Gracia-Bondía. On the ultraviolet behavior of quantum fields over noncommutative manifolds. *Int. J. Mod. Phys.*, A14:1305, 1999, hep-th/9804001. doi:10.1142/S0217751X99000671.

vN32. J. von Neumann. *Mathematische Grundlagen der Quantenmechanik*. Springer-Verlag, Berlin, 1932. URL `http://resolver.sub.uni-goettingen.de/purl?PPN379400774`.

Vog00. D. Vogt. Lectures on Fréchet spaces, 2000. URL `http://www2.math.uni-wuppertal.de/~vogt/vorlesungen/fs.pdf`.

VT07. F. Vignes-Tourneret. Renormalization of the orientable non-commutative Gross-Neveu model. *Annales Henri Poincaré*, 8:427–474, 2007, math-ph/0606069. doi:10.1007/s00023-006-0312-6.

Wan18. Z. Wang. Constructive renormalization of the 2-dimensional Grosse-Wulkenhaar model. *Annales Henri Poincaré*, 19(8):2435–2490, 2018, 1805.06365. doi:10.1007/s00023-018-0688-0.

Wey28. H. Weyl. *Gruppentheorie und Quantenmechanik*. S. Hirzel, Leipzig, 1928.

WG64. A. Wightman and L. Gårding. *Fields as operator-valued distributions in relativistic quantum theory*. Arkiv för fysik. Almqvist & Wiksell, 1964.

Whe55. J. A. Wheeler. Geons. *Phys. Rev.*, 97:511–536, 1955. doi:10.1103/PhysRev.97.511.

Wid31. D. V. Widder. Necessary and sufficient conditions for the representation of a function as a Laplace integral. *Trans. Amer. Math. Soc.*, 33(4):851–892, 1931. doi:10.2307/1989513.

Wig32. E. P. Wigner. On the quantum correction for thermodynamic equilibrium. *Phys. Rev.*, 40:749–760, 1932. doi:10.1103/PhysRev.40.749.

Wig56. A. S. Wightman. Quantum field theory in terms of vacuum expectation values. *Phys. Rev.*, 101:860–866, 1956. doi:10.1103/PhysRev.101.860.

Wil74. K. G. Wilson. Confinement of quarks. *Phys. Rev.*, D10:2445–2459, 1974. doi:10.1103/PhysRevD.10.2445.

Wit88. E. Witten. Topological quantum field theory. *Commun. Math. Phys.*, 117:353, 1988. doi:10.1007/BF01223371.

Wit90. E. Witten. On the structure of the topological phase of two-dimensional gravity. *Nucl. Phys.*, B340:281–332, 1990. doi:10.1016/0550-3213(90)90449-N.

Wit91. E. Witten. Two-dimensional gravity and intersection theory on moduli space. In *Surveys in differential geometry (Cambridge, MA, 1990)*, pages 243–310. Lehigh Univ., Bethlehem, PA, 1991. doi:10.4310/SDG.1990.v1.n1.a5.

Wit92. E. Witten. On the Kontsevich model and other models of two-dimensional gravity. In *Differential geometric methods in theoretical physics. Proceedings, 20th International Conference, New York, 1991*, pages 176–216. World Sci. Publ., River Edge, NJ, 1992.

Wit16. E. Witten. An SYK-like model without disorder. 2016, 1610.09758.

WK74. K. G. Wilson and J. B. Kogut. The renormalization group and the ϵ expansion. *Phys. Rept.*, 12:75–199, 1974. doi:10.1016/0370-1573(74)90023-4.

Wul00. R. Wulkenhaar. Slavnov-Taylor identity in noncommutative geometry. *Int. J. Mod. Phys.*, B14:2503–2509, 2000. doi:10.1142/S0217979200002053.

Wul02. R. Wulkenhaar. Non-renormalizability of θ-expanded noncommutative QED. *JHEP*, 03:024, 2002, hep-th/0112248. doi:10.1088/1126-6708/2002/03/024.

Wul06. R. Wulkenhaar. Field theories on deformed spaces. *J. Geom. Phys.*, 56:108–141, 2006. doi:10.1016/j.geomphys.2005.04.019.

Zah06. J. W. Zahn. *Dispersion relations in quantum electrodynamics on the noncommutative Minkowski space*. PhD thesis, Hamburg U., 2006, 0707.2149. URL `http://www-library.desy.de/cgi-bin/showprep.pl?thesis06-037`.

Zah11. J. Zahn. Divergences in quantum field theory on the noncommutative two-dimensional Minkowski space with Grosse-Wulkenhaar potential. *Annales Henri Poincaré*, 12:777–804, 2011, 1005.0541. doi:10.1007/s00023-011-0089-0.

Zim69. W. Zimmermann. Convergence of Bogolyubov's method of renormalization in momentum space. *Commun. Math. Phys.*, 15:208–234, 1969. doi:10.1007/BF01645676.