Euclidean quantum field theory on commutative and noncommutative spaces

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Summary. I give an introduction to Euclidean quantum field theory from the point of view of statistical physics, with emphasis both on Feynman graphs and on the Wilson-Polchinski approach to renormalisation. In the second part I discuss attempts to renormalise quantum field theories on noncommutative spaces.

1 From classical actions to lattice quantum field theory

1.1 Introduction

Ignoring gravity, space-time is described by Minkowski space given by the metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. In particular, time plays a very different rôle than space. Looking at a classical field theory modelled on Minkowski space, the resulting field equations are hyperbolic ones. The formulation of the associated quantum field theory requires a sophisticated mathematical machinery. The classical reference is [1]. A comprehensive treatment can be found in [2].

From our point of view, it is much easier for a beginner to first study quantum field theory in Euclidean space E_4 given by the metric $g_{\mu\nu} = \delta_{\mu\nu} = \text{diag}(1, 1, 1, 1)$. Euclidean quantum field theory is more than just a bad trick. It has a physical interpretation as a spin system treated in the language of statistical mechanics [3, 4]. Applications to physical models are treated in [5]. There are rigorous theorems which under certain conditions allow to translate quantities computed within Euclidean quantum field theory to the Minkowskian version [6]. Eventually, from a practical point of view, computations of phenomenological relevance are almost exclusively performed in the Euclidean situation, making use of the possibility to translate them into the Minkowskian world. Our presentation of the subject is inspired by [7].

1.2 Classical action functionals

The starting point for both classical and quantum field theories are *action functionals*. We ignore topological questions and regard all fields as smooth (and integrable) functions on the Euclidean space E_4. The most important action functionals are the following ones.

– The real scalar field ϕ:

$$S[\phi] = \int d^4x \Big(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\Big) . \tag{1}$$

Here, m is the mass and λ the coupling constant. We use $\partial_\mu := \frac{\partial}{\partial x^\mu}$ and raise or lower indices with the metric tensor $g^{\mu\nu} = \delta^{\mu\nu}$ or $g_{\mu\nu} = \delta_{\mu\nu}$, such as in $\partial^\mu = g^{\mu\nu}\partial_\nu$. Summation over the same upper and lower greek index from 1 to 4 is self-understood (Einstein's sum convention).

– The Maxwell action for the electromagnetic field $A = \{A_\mu\}_{\mu=1,\dots,4}$ (the photon):

$$S[A] = \int d^4x\ \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} , \qquad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu , \tag{2}$$

where g is the electron charge. That action is invariant under a gauge transformation $A_\mu \mapsto A_\mu + \partial_\mu f$ for any smooth function f.

– The Dirac action for a spinor field (electron) ψ coupled to the electromagnetic field:

$$S[\psi, A] = \int d^4x\ \langle\psi, \mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}A_\mu)\psi\rangle . \tag{3}$$

We regard the electron pointwise as $\psi(x) \in \mathbb{C}^4$ to be multiplied by the traceless (4×4)-matrices γ^μ which satisfy $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}1_{4\times 4}$. By $\langle\ ,\ \rangle$ we understand the scalar product in $\mathbb{C}^4$.

– There are matrix versions $\mathrm{i}A_\mu(x) \in \mathrm{su}(n)$ (gluon field) with $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu]$ of (2) where additionally the matrix trace must be taken. There is also a corresponding $\mathrm{su}(n)$-generalisation of (3).

The importance of action functionals in classical field theory is that they give rise to the equations of motion: A field configuration which satisfies the equation of motion minimises the action functional (Hamilton's principle). For example, to get the equation of motion for the electromagnetic field we vary (2) with respect to A and put the variation to zero:

$$0 = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(S[A+\epsilon\tilde{A}] - S[A]\Big) = \int d^4x\ \frac{1}{g^2}\partial_\mu\tilde{A}_\nu\,F^{\mu\nu} . \tag{4}$$

If $\tilde{A}$ vanishes at infinity we can integrate (4) by parts and obtain, because the variation is zero for any $\tilde{A}$, the Euclidean version of Maxwell's equation in the vacuum $\partial_\mu F^{\mu\nu} = 0$.

1.3 A reminder of thermodynamics

The partition function is for the action what the free energy is for a thermodynamical system. Let us consider a system characterised by discrete energy levels E_i. Since the energy is bounded from below, there will be a ground state E_0 with $E_i \geq E_0$, with equality for $i = 0$ only (we assume the ground state to be non-degenerate). At zero temperature $T = 0$ and in thermodynamical equilibrium the system will be found with an probability $p_0 = 1$ in the ground state. At a temperature $T > 0$, however, there is due to thermal fluctuations some non-vanishing probability p_i to find the system in the energy state E_i. The probability distribution is governed by the entropy[1]

$$\Sigma(p) := -\sum_i p_i \ln p_i \tag{5}$$

and the requirement that the free energy

$$F_E(p) := \sum_i p_i E_i - k_B T \Sigma(p) \tag{6}$$

is minimal in the thermodynamical equilibrium. Here, k_B denotes Boltzmann's constant. The probability distribution $p^{\min}$ which minimises (6) is used to compute *expectation values* in thermodynamic equilibrium, such as the average energy $U = \sum_i p_i^{\min} E_i > E_0$.

1.4 The partition function for discrete actions

Whereas a thermodynamical system is described by its energy levels, a field theory is governed by its action. There is a striking similarity between the energy in thermodynamics and the action in field theory in the sense that the classical configuration is given by the minimum of the energy and the action, respectively. In the same way as the entropy term leads to thermal fluctuations away from the classical configuration if the reference energy $k_B T$ is different from zero, we should expect *quantum fluctuations* away from the classical field configuration if a reference action $\hbar$ is different from zero. Assuming for the moment that in the field theory only discrete actions $S_i \geq S_0$ are realised, we expect the quantum state to be given by the probability distribution $\{p_i\}$ which minimises the "free action"

$$F_S(p) := \sum_i p_i S_i - \hbar \Sigma(p) \, . \tag{7}$$

The entropy is given by (5). In the classical case $\hbar = 0$, the principle of minimising $F_S(p)$ reduces to Hamilton's principle of the minimal action, because $\min_p F_S(p) = S_0$ with $p_i = \delta_{i0}$.

We are going to compute the minimising probability distribution $\{p_i\}$ for $\hbar > 0$. For this purpose let us consider for two probability distributions $\{p_i\}$ and $\{\pi_i\}$ with $\sum_i p_i = \sum_i \pi_i = 1$ the relative entropy

[1] We avoid the standard symbol S for the entropy because S already denotes the action.

$$\Sigma(p|\pi) := -\sum_i p_i \ln \frac{p_i}{\pi_i} \;. \tag{8}$$

One has[2] $\Sigma(p|\pi) \leq 0$ with equality only for $p_i = \pi_i$. Let us consider

$$\pi_i = Z^{-1}\mathrm{e}^{-\frac{S_i}{\hbar}} \;, \qquad Z = \sum_i \mathrm{e}^{-\frac{S_i}{\hbar}} \;. \tag{9}$$

We get

$$F_S(p) = \sum_i p_i S_i + \hbar \sum_i p_i \ln p_i = -\hbar\Sigma(p|\pi) - \hbar \ln Z \geq -\hbar \ln Z \;, \tag{10}$$

with equality for $p_i = \pi_i$ only. Thus, the probability distribution $\{p_i^{\min}\}$ which minimises (7) is the distribution (9) and the minimum is given by $F_S(p^{\min}) = -\hbar \ln Z$.

Taking more and more states with decreasing difference we achieve in the limit a continuous probability density $p(i) \geq 0$ with $\int di\; p(i) = 1$ for the action $S(i)$. In this way we get for the free action

$$\begin{aligned} F_S(p) &= \int di\; p(i)S(i) - \hbar\Sigma(p) \geq F_S(p^{\min}) = -\hbar \ln Z \;, \\ \Sigma(p) &= -\int di\; p(i) \ln p(i) \;, \\ p^{\min}(i) &= Z^{-1}\mathrm{e}^{-\frac{S(i)}{\hbar}} \;, \qquad Z = \int di\; \mathrm{e}^{-\frac{S(i)}{\hbar}} \;. \end{aligned} \tag{11}$$

We first get $i \in \mathbb{R}^+$ but rearranging the indices we can also achieve $i \in \mathbb{R}^n$.

It is tempting now to identify the index i in (11) with the field ϕ in (1). Such an identification requires $\phi \in \mathbb{R}^n$, which we achieve by a lattice approximation to (1).

1.5 Field theory on the lattice

The following steps bring us from Euclidean space to a finite lattice. We first pass to the 4-torus by imposing periodic boundary conditions on the field, $\phi(x_1, x_2, x_3, x_4) = \phi(x_1 + L, x_2, x_3, x_4) = \cdots = \phi(x_1, x_2, x_3, x_4 + L)$. Next we restrict the 4-torus to the sublattice with equidistant spacing $a = L/N$. This lattice has N^4 points. If the field varies slowly, we can approximate

[2] In order to prove (8) we consider the convex function $f(u) = u \ln u$. As such, $f(\sum_i \pi_i u_i) \leq \sum_i \pi_i f(u_i)$. [Write $\pi_1 = \alpha_1$, $\pi_j = \prod_{i=1}^{j-1}(1-\alpha_i)\alpha_j$ for $2 \leq j \leq n$ and $\pi_n = \prod_{i=1}^{n-1}(1-\alpha_i)$, with $0 \leq \alpha_i \leq 1$, and use the definition of convexity $f(\alpha u_1 + (1-\alpha)u_2) \leq \alpha f(u_1) + (1-\alpha) f(u_2)$.] Taking $u_i = p_i/\pi_i$ we get with $\sum_i p_i = 1$ and $f(1) = 0$ the desired inequality $S(p|\pi) \leq 0$. To obtain $f(\sum_i \pi_i u_i) = \sum \pi_i f(u_i)$ we need $u_i = \text{const}$, i.e. $p_i = \pi_i$ due to the normalisation $\sum_i p_i = \sum_i \pi_i = 1$.

it by its values ϕ_q at these lattice points $q \in \mathbb{Z}_N^4$. The partial derivative is approximated by the difference quotient

$$(\partial_\mu \phi)(x) \mapsto \frac{1}{a}(\delta_\mu \phi)_q := \frac{1}{a}\big(\phi_{q+e_\mu} - \phi_q\big) \;, \tag{12}$$

where $q + e_\mu$ denotes the neighboured lattice point of q in the μ^{th} coordinate direction. Now the action (1) can be approximated by

$$S[\phi] = a^4 \sum_{q \in \mathbb{Z}_N^4} \Big(\frac{1}{2} a^{-2} (\delta_\mu \phi)_q (\delta^\mu \phi)_q + \frac{1}{2} m^2 \phi_q^2 + \frac{\lambda}{4!} \phi_q^4\Big) \;. \tag{13}$$

Introducing dimensionless fields and masses $\tilde{\phi}_q := a\phi_q$ and $\tilde{m} := am$ (the coupling constant $\lambda = \tilde{\lambda}$ is already dimensionless), we obtain for the action

$$S[\tilde{\phi}] = \sum_{q \in \mathbb{Z}_N^4} \Big(\frac{1}{2} (\delta_\mu \tilde{\phi})_q (\delta^\mu \tilde{\phi})_q + \frac{1}{2} \tilde{m}^2 \tilde{\phi}_q^2 + \frac{\tilde{\lambda}}{4!} \tilde{\phi}_q^4\Big) \;. \tag{14}$$

We thus have $\tilde{\phi} \in \mathbb{R}^{N^4}$ with components $\tilde{\phi}_q \in \mathbb{R}$, for $q = 1, \dots, N^4$. We can now write the free action and its minimising probability distribution as follows:

$$F_S[p^{\min}] = \int_{\mathbb{R}^{N^4}} d\tilde{\phi}\; p^{\min}(\tilde{\phi}) S[\tilde{\phi}] - \hbar \Sigma(p^{\min}) = -\hbar \ln Z \;, \tag{15}$$

where $d\tilde{\phi} = \prod_{q=1}^{N^4} d\tilde{\phi}_q$ and

$$\Sigma(p^{\min}) = -\int_{\mathbb{R}^{N^4}} d\tilde{\phi}\; p^{\min}(\tilde{\phi}) \ln p^{\min}(\tilde{\phi}) \;, \tag{16}$$

$$p^{\min}(\tilde{\phi}) = Z^{-1} \mathrm{e}^{-S[\tilde{\phi}]/\hbar} \;, \qquad Z = \int_{\mathbb{R}^{N^4}} d\tilde{\phi}\; \mathrm{e}^{-S[\tilde{\phi}]/\hbar} \;. \tag{17}$$

The interesting quantities in quantum field theory are the expectation values of products of fields at n points $q_1, \dots, q_n$ in quantum mechanical equilibrium:

$$\langle \tilde{\phi}_{q_1} \dots \tilde{\phi}_{q_n} \rangle := \int_{\mathbb{R}^{N^4}} d\tilde{\phi}\; p^{\min}(\tilde{\phi})\; \tilde{\phi}_{q_1} \cdots \tilde{\phi}_{q_n} = \frac{\displaystyle\int_{\mathbb{R}^{N^4}} d\tilde{\phi}\; \tilde{\phi}_{q_1} \cdots \tilde{\phi}_{q_n}\; \mathrm{e}^{-S[\tilde{\phi}]/\hbar}}{\displaystyle\int_{\mathbb{R}^{N^4}} d\tilde{\phi}\; \mathrm{e}^{-S[\tilde{\phi}]/\hbar}} \;. \tag{18}$$

These expectation values can also be regarded as the correlation functions between the n fields at lattice sites $q_1, \dots, q_n$. The expectation values are most conveniently organised by the *generating functional*

$$Z[\tilde{j}] = \int_{\mathbb{R}^{N^4}} d\tilde{\phi}\; \mathrm{e}^{-\frac{1}{\hbar}(S[\tilde{\phi}] - \sum_{q \in \mathbb{Z}_N^4} \tilde{\phi}_q \tilde{j}_q)} \;. \tag{19}$$

We thus get

$$\langle\tilde{\phi}_{q_1}\dots\tilde{\phi}_{q_n}\rangle := Z[0]^{-1}\hbar^n \frac{\partial^n Z[\tilde{j}]}{\partial\tilde{j}_{q_1}\dots\partial\tilde{j}_{q_n}}\Big|_{\tilde{j}_q=0} . \qquad (20)$$

Exercise 1. Evaluate $\langle\tilde{\phi}_{q_1}\dots\tilde{\phi}_{q_n}\rangle$ for the classical case $\hbar = 0$. Hint: Insert the minimising probability distribution into the first equation of (18). ◁

At the end we are interested in a continuum field theory. This is, in principle, achieved by a limiting process for the expectation values (20). In the first step we pass to an infinite lattice $\mathbb{Z}^4$ by taking the limit $N \to \infty$. This process is referred to as the *thermodynamic limit*. In the second step we reintroduce the lattice spacing a by inverting the steps leading from (1) to (14). This provides the lattice $(a\mathbb{Z})^4$ embedded into the Euclidean space E_4. The difficulty is then to find an a-dependence of the dimensionless mass $\tilde{m}(a)$ and coupling constant $\tilde{\lambda}(a)$ so that the limit $a \to 0$ (the *continuum limit*) of the expectation values (20) exists. In this process the correlation length ξ of the two-point function, i.e. the inverse physical mass, is kept constant. We thus arrive at well-defined expectation values $\langle\tilde{\phi}(x_1)\dots\tilde{\phi}(x_n)\rangle$ for products of continuum fields at n positions $x_1, \dots, x_n$. Since the limit $a \to 0$ for constant λ means $\frac{\xi}{a} \to \infty$, we can equivalently regard the continuum limit as sending $\xi \to \infty$ on a lattice with constant spacing a. This means that the continuum limit corresponds to a *critical point* (where the correlation length diverges) of a lattice model.

This programme to produce continuum n-point function is called *construction of a quantum field theory*. So far this was successful in two and partly in three dimensions only [8]. Since we are particularly interested in four dimensions, a different (and less rigorous) treatment is required: *perturbative renormalisation*.

2 Field theory in the continuum

2.1 Generating functionals

The idea is to perform the two limits $N \to \infty$ and $a \to 0$ *formally* in the partition function, giving

$$\langle\phi(x_1)\dots\phi(x_n)\rangle = \frac{\int\mathcal{D}\phi\;\phi(x_1)\cdots\phi(x_n)\;\mathrm{e}^{-S[\phi]/\hbar}}{\int\mathcal{D}\phi\;\mathrm{e}^{-S[\phi]/\hbar}} . \qquad (21)$$

Here, the "measure" $\mathcal{D}\phi$ is the formal limit of the measure $a^{N^4}d\tilde{\phi}$ as $N \to \infty$ and $a \to 0$. Again it is useful to introduce a generating functional for the n-point functions (21),

$$Z[j] := \int\mathcal{D}\phi\;\mathrm{e}^{-\frac{1}{\hbar}(S[\phi]-\int d^4x\;\phi(x)j(x))} . \qquad (22)$$

This generating functional has a formal meaning only and it is no surprise that it will produce problems. Using functional derivatives

$$\frac{\delta F[j(y)]}{\delta j(x)} := \lim_{\epsilon\to 0} \frac{1}{\epsilon}\big(F[j(y)+\epsilon\delta(x-y)] - F[j(y)]\big) \tag{23}$$

we can rewrite (21) as

$$\langle\phi(x_1)\dots\phi(x_n)\rangle = Z[0]^{-1}\hbar^n \frac{\delta^n Z[j]}{\delta j(x_1)\dots\delta j(x_n)}\Big|_{j(x)=0} . \tag{24}$$

There are two other important generating functionals derived from Z. The logarithm of Z generates (as we see later) connected n-point functions,

$$W[j] = \hbar \ln Z[j] . \tag{25}$$

By Legendre transformation we obtain the generating functional $\Gamma[\phi_{cl}]$ of one-particle-irreducible (1PI) n-point functions. This construction goes as follows: We first define the classical field ϕ_{cl} via

$$\phi_{cl}(x) := \frac{\delta W[j]}{\delta j(x)} . \tag{26}$$

Then $\Gamma[\phi_{cl}]$, which is also referred to as the effective action, is defined as

$$\Gamma[\phi_{cl}] := \int d^4x\ \phi_{cl}(x) j(x) - W[j] , \tag{27}$$

where $j(x)$ has to be replaced by the inverse solution of (26).

2.2 Perturbative solution

The (perturbative) evaluation of (22) is most conveniently performed in momentum space obtained by Fourier transformation

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4}\ \mathrm{e}^{-\mathrm{i}px}\,\hat\phi(p) , \qquad \hat\phi(p) = \int d^4x\ \mathrm{e}^{\mathrm{i}px}\,\phi(x) . \tag{28}$$

The action (1) reads in momentum space

$$S[\hat\phi] = \int \frac{d^4p}{(2\pi)^4}\ \frac{1}{2}(p^2+m^2)\hat\phi(p)\hat\phi(-p) + S_{\mathrm{int}}[\hat\phi] , \tag{29}$$

$$S_{\mathrm{int}}[\hat\phi] = \frac{\lambda}{4!}\int \Big(\prod_{i=1}^4 \frac{d^4p_i}{(2\pi)^4}\Big)\ (2\pi)^4\delta\Big(\sum_{j=1}^4 p_j\Big)\ \hat\phi(p_1)\hat\phi(p_2)\hat\phi(p_3)\hat\phi(p_4) . \tag{30}$$

For the free scalar field defined by $S_{\mathrm{int}} = 0$ in (29) the generating functional $Z[j]$ is easy to compute:

$$
\begin{aligned}
Z_{\text{free}}[\hat{j}] &:= \int \mathcal{D}\hat{\phi}\; \mathrm{e}^{-\frac{1}{\hbar}\int \frac{d^4p}{(2\pi)^4}\, \left(\frac{1}{2}(p^2+m^2)\hat{\phi}(p)\hat{\phi}(-p) - \hat{\phi}(p)\hat{j}(-p)\right)} \\
&= \int \mathcal{D}\hat{\phi}\; \mathrm{e}^{-\frac{1}{\hbar}\int \frac{d^4p}{(2\pi)^4}\, \left(\frac{1}{2}(p^2+m^2)\hat{\phi}'(p)\hat{\phi}'(-p) - \frac{1}{2}(p^2+m^2)^{-1}\hat{j}(p)\hat{j}(-p)\right)} \\
&= Z[0]\, \mathrm{e}^{\frac{1}{2\hbar}\int \frac{d^4p}{(2\pi)^4}\, (p^2+m^2)^{-1}\hat{j}(p)\hat{j}(-p)}\;, \qquad (31)
\end{aligned}
$$

where we have abbreviated $\hat{\phi}'(p) := \hat{\phi}(p) + (p^2+m^2)^{-1}\hat{j}(p)$ and used the invariance of the measure $\mathcal{D}\hat{\phi} = \mathcal{D}\hat{\phi}'$. The generating functional of free connected n-point functions becomes

$$
W_{\text{free}}[\hat{j}] = W[0] + \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\hat{j}(-p)\frac{1}{(p^2+m^2)}\hat{j}(p)\;, \quad W[0] = \hbar \ln Z[0]\;. \qquad (32)
$$

The momentum space n-point functions are obtained from

$$
(2\pi)^4\delta\Big(\sum_{i=1}^{n} p_i\Big)\, \langle \phi(p_1)\dots\phi(p_n)\rangle = \frac{1}{Z[0]}\frac{\hbar^n \delta^n Z[\hat{j}]}{\delta\hat{j}(-p_1)\dots\delta\hat{j}(-p_n)}\Big|_{\hat{j}(p)=0}\;, \qquad (33)
$$

where the functional derivation in momentum space is defined by

$$
\frac{\delta F[\hat{j}(p)]}{\delta\hat{j}(q)} := \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(F[\hat{j}(p) + \epsilon(2\pi)^4\delta(p-q)] - F[\hat{j}(p)]\Big)\;. \qquad (34)
$$

Exercise 2. Compute the effective action $\Gamma[\hat{\phi}_{cl}] = W[\hat{j}] - \int d^4p\, \hat{\phi}_{cl}(p)\hat{j}(-p)$ for the free scalar field in momentum space. ◁

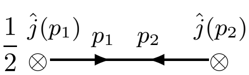
Let us now consider the full interacting ϕ^4-theory with $\lambda \neq 0$ in (30). We get formally

$$
\begin{aligned}
Z[\hat{j}] &:= \int \mathcal{D}\hat{\phi}\; \mathrm{e}^{-\frac{1}{\hbar}S_{\text{int}}[\hat{\phi}(q)] - \frac{1}{\hbar}\int \frac{d^4p}{(2\pi)^4}\, \left(\frac{1}{2}(p^2+m^2)\hat{\phi}(p)\hat{\phi}(-p) - \hat{\phi}(p)\hat{j}(-p)\right)} \\
&= \mathrm{e}^{-\frac{1}{\hbar}S_{\text{int}}\left[\hbar\frac{\delta}{\delta\hat{j}(-q)}\right]}\Big(Z[0]\, \mathrm{e}^{\frac{1}{2\hbar}\int \frac{d^4p}{(2\pi)^4}(p^2+m^2)^{-1}\hat{j}(p)\hat{j}(-p)}\Big)\;. \qquad (35)
\end{aligned}
$$

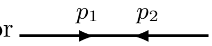
The generating functional for connected n-point function becomes

$$
W[\hat{j}] = \hbar \ln\Big(1 + Z_{\text{free}}[\hat{j}]^{-1}\big(\mathrm{e}^{-\frac{1}{\hbar}S_{\text{int}}\left[\hbar\frac{\delta}{\delta\hat{j}}\right]} - 1\big)Z_{\text{free}}[\hat{j}]\Big) + W_{\text{free}}[\hat{j}]\;. \qquad (36)
$$

It is convenient now to introduce a graphical description for $W[\hat{j}]$. We symbolise the integrand in (32) by

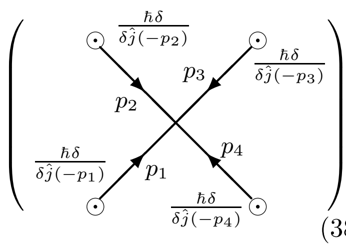
$$
W_{\text{free}}[\hat{j}] = \int \Big(\prod_{i=1}^{2}\frac{d^4p_i}{(2\pi)^4}\Big)(2\pi)^4\delta(p_1+p_2)\Big(\frac{1}{2}
$$


$$
\Big)\;, \qquad (37)
$$

where the propagator 

stands for $(p_1^2+m^2)^{-1}$. The interaction part $S_{\rm int}$ of the action is represented by the vertex

$$S_{\rm int}\Big[\hbar\frac{\delta}{\delta\hat{j}}\Big] = \int\Big(\prod_{i=1}^4 \frac{d^4p_i}{(2\pi)^4}\Big)\,(2\pi)^4\delta\Big(\sum_{j=1}^4 p_j\Big)\left(\text{[vertex diagram]}\right), \tag{38}$$

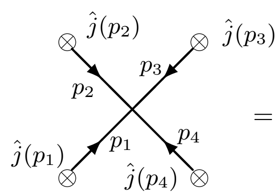


where the cross  stands for $\frac{\lambda}{4!}$. The idea is to expand in (36) both the exponential of (38) and the logarithm $\ln(1+\dots)$ into a Taylor series. In this way one obtains a formal power series in the coupling constant λ with coefficients given by *Feynman graphs*. To obtain a Feynman graph with V vertices one writes V vertices (38) next to each other and evaluates the functional derivations with respect to $\hat{j}(p)$ by their action to the $\hat{j}(q)$ in the exponent given by (37). Integrating out the resulting δ-distributions one arrives at order $V=1$ in λ at

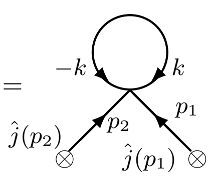
$$\begin{aligned} W[\hat{j}]^{V=1} &= \int\Big(\prod_{i=1}^4 \frac{d^4p_i}{(2\pi)^4}\Big)\,(2\pi)^4\delta(p_1{+}p_2{+}p_3{+}p_4)W^{1,0}_{p_1,p_2,p_3,p_4}[\hat{j}] \\ &\quad + \int\Big(\prod_{i=1}^2 \frac{d^4p_i}{(2\pi)^4}\Big)\,(2\pi)^4\delta(p_1{+}p_2)W^{1,1}_{p_1,p_2}[\hat{j}] + W^{1,2}\,, \end{aligned} \tag{39}$$

where

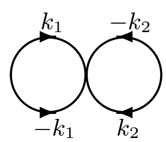
$$W^{1,0}_{p_1,p_2,p_3,p_4}[\hat{j}] = \text{[diagram]} = -\frac{\lambda}{4!}\Big(\prod_{i=1}^4 \frac{\hat{j}(p_i)}{p_i^2+m^2}\Big)\,, \tag{40}$$



$$W^{1,1}_{p_1,p_2}[\hat{j}] = \text{[diagram]} = -\frac{\lambda\hbar}{4}\Big(\prod_{i=1}^2 \frac{\hat{j}(p_i)}{p_i^2+m^2}\Big)\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\,, \tag{41}$$



$$W^{1,2} = \text{[diagram]} = -\frac{\lambda\hbar^2}{8}\int\frac{d^4k_1}{(2\pi)^4}\frac{1}{k_1^2{+}m^2}\int\frac{d^4k_2}{(2\pi)^4}\frac{1}{k_2^2{+}m^2}\,. \tag{42}$$



Exercise 3. Verify (39)–(42). ◁

Exercise 4. Perform the Legendre transformation of the sum of (39) and (32) to $\Gamma[\phi_{cl}]^{V\leq 1}$. Compare the $\hbar$-independent part with (29) and (30). ◁

Exercise 5. Derive the graphical expression for $W[\hat{j}]^{V=2}$. Convince yourself that the resulting graphs are connected. ◁

Exercise 6. Prove that an L-loop graph contributing to $W[\hat{j}]$ leads to a factor $\hbar^L$. Hint: For a connected graph with L loops, I internal lines and V vertices the Euler characteristic reads $\chi = L - I + V = 1$. ◁

We thus deduce the following Feynman rules for the part $W[\hat{j}]^V$ of the generating functional with V vertices in a ϕ^4-model:

1. Draw the V vertices in a plane and connect in all possible ways the valences either with each other or with external sources $\hat{j}(p_i)$ such that the resulting graph is connected. The result is a sum of graphs with certain multiplicities.
2. If the graph has n sources, represent the sources and their attached lines by a factor $\prod_{i=1}^n (p_i^2 + m^2)^{-1} \hat{j}(p_i)$.
3. Represent each internal line connecting vertices by a factor $(q_j^2 + m^2)^{-1}$ and determine the momenta q_j in terms of the external momenta p_i originating from the sources and L independent loop momenta k_l by the requirement that the total momentum flowing into each vertex is zero.
4. Add an integral operator $\int \prod_{l=1}^L \frac{d^4 k_l}{(2\pi)^4}$ for the independent loop momenta and a factor $\int \big(\prod_{i=1}^n \frac{d^4 p_i}{(2\pi)^4}\big) (2\pi)^4 \delta(p_1 + \ldots + p_n)$ for the independent momenta of the sources. Multiply the result by a factor $\frac{(-\lambda)^V \hbar^L}{4! V!}$ for an L-loop graph with V vertices.

2.3 Calculation of simple Feynman graphs

When inserting $Z[\hat{j}] = \exp(W[\hat{j}]/\hbar)$ into (33), the sources $\hat{j}(p_i)$ and the integration operators $\int \frac{d^4 p_i}{(2\pi)^4}$ are removed. It remains the integration over the internal momenta k_l. Due to momentum conservation the integration factorises into integrations over 1PI subgraphs. Let us thus consider an 1PI subgraph with L loops, I internal lines and E external lines. Scaling the independent loop momenta by a factor Λ, the integral will be scaled by a factor Λ^{4L-2I} for $\Lambda \gg 1$. Using Euler's formula $L = I - V + 1$ and the relation $4V = 2I + E$ (a valence of a vertex either attaches to one end of an internal line or to an external line) we get $\Lambda^{4L-2I} = \Lambda^{4-E}$. This means that the integral over the internal momenta of an 1PI graph with $E \leq 4$ external lines will be *divergent*. This divergence is due to the naïve way of performing the $N \to \infty$ and $a \to 0$ limits in the partition function (22).

In some cases (among them is the ϕ^4-model) it is possible to eliminate the divergences in a consistent way by expressing the perturbatively computed n-point functions in terms of a finite number of physically observable quantities. Such a model is called *perturbatively renormalisable*.

Let us describe the removal of divergences for the ϕ^4-model. The first step is to introduce a regulator ε which renders the integrals finite. There are many known possibilities. A common feature of these regularisations is that for dimensional reasons one also has to introduce a mass parameter μ. A very convenient regularisation is *dimensional regularisation* where the integration is performed in $4-2\varepsilon$ dimensions, $d^4k \mapsto \mu^{2\epsilon} d^{4-2\epsilon}k$. With this change of the integration one computes the generating functionals and one requires

$$\Big(\frac{\delta^2 \Gamma[\hat{\phi}_{cl}]}{\delta\hat{\phi}_{cl}(p_1)\delta\hat{\phi}_{cl}(p_2)}\Big|_{\hat{\phi}_{cl}=0}\Big)'_{p_1=p_2=0} = m^2_{\text{phys}}\;, \tag{43}$$

$$\Big(\frac{1}{2}\frac{\partial^2}{\partial p_1^2}\Big(\frac{\delta^2 \Gamma[\hat{\phi}_{cl}]}{\delta\hat{\phi}_{cl}(p_1)\delta\hat{\phi}_{cl}(p_2)}\Big|_{\hat{\phi}_{cl}=0}\Big)'\Big)_{p_1=-p_2=0} = 1\;, \tag{44}$$

$$\Big(\frac{\delta^4 \Gamma[\hat{\phi}_{cl}]}{\delta\hat{\phi}_{cl}(p_1)\delta\hat{\phi}_{cl}(p_2)\delta\hat{\phi}_{cl}(p_3)\delta\hat{\phi}_{cl}(p_4)}\Big|_{\hat{\phi}_{cl}=0}\Big)'_{p_i=0} = \lambda_{\text{phys}}\;. \tag{45}$$

By $(\;)'$ we mean that the factor $(2\pi)^4\delta(p_1+\dots+p_n)$ is omitted.

This means that the parts of the effective action which correspond to the mass, the amplitude of the kinetic term and the coupling constant are *normalised* to their physical values. The original parameters m, g and an additional wavefunction renormalisation factor $\mathcal{Z}$ are expressed in terms of m_{phys}, g_{phys}, ϵ and μ via the normalisation conditions (43)–(45). To prove renormalisability of the ϕ^4-model amounts to show that after that replacement the limit $\varepsilon \to 0$ of the n-point functions exists.

We will discuss in Section 3 another (more efficient) way to prove renormalisability. Here, we only demonstrate the method for a one-loop example. We determine $m[m_{\text{phys}}, g_{\text{phys}}, \epsilon, \mu]$ by computing the integral in (41) in dimensional regularisation.

Exercise 7. Prove that the surface of the sphere $x_1^2 + \dots x_n^2 = 1$ equals $\frac{2\pi^{n/2}}{\Gamma(n/2)}$. Hint: compute $\int d^n x\, \mathrm{e}^{-x^2}$ both in cartesian and radial coordinates. ◃

Using the Schwinger trick $\frac{1}{A^n} = \frac{1}{\Gamma(n)}\int_0^\infty d\alpha\, \alpha^{n-1}\mathrm{e}^{-\alpha A}$ we have in $4-2\varepsilon$ dimensions

$$\begin{aligned}
\int \frac{d^{4-2\varepsilon}k}{(2\pi)^4}\frac{\mu^{2\varepsilon}}{k^2+m^2} &= \frac{2\pi^{2-\varepsilon}\mu^{2\varepsilon}}{(2\pi)^4\Gamma(2-\varepsilon)}\int_0^\infty k^{3-2\varepsilon}dk\int_0^\infty d\alpha\; \mathrm{e}^{-\alpha(k^2+m^2)} \\
&= \frac{\pi^{2-\varepsilon}\mu^{2\varepsilon}}{(2\pi)^4\Gamma(2-\varepsilon)}\int_0^\infty u^{(1-\varepsilon)}du\int_0^\infty d\alpha\; \mathrm{e}^{-\alpha(u+m^2)} \\
&= \frac{\pi^{2-\varepsilon}\mu^{2\varepsilon}}{(2\pi)^4\Gamma(2-\varepsilon)}\Gamma(2-\varepsilon)\int_0^\infty \frac{d\alpha}{\alpha^{2-\epsilon}}\; \mathrm{e}^{-\alpha m^2} \\
&= \frac{m^2}{16\pi^2}\Big(\frac{\mu^2}{\pi m^2}\Big)^{\varepsilon}\Gamma(\varepsilon-1) = -\frac{m^2}{16\pi^2\varepsilon}\Big(\frac{\mu^2}{\pi m^2}\Big)^{\varepsilon}\frac{\Gamma(1+\varepsilon)}{(1-\varepsilon)} \\
&= -\frac{m^2}{16\pi^2\varepsilon} - \frac{m^2}{16\pi^2}\Big(\ln\Big(\frac{\mu^2}{\pi m^2}\Big) + 1 + \psi(1)\Big) + \mathcal{O}(\varepsilon)\;,
\end{aligned} \tag{46}$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$. We have exchanged the order of integrations. If we now reinsert the coupling constant and $\hbar$ and pass to the 1PI function (Exercises 2 and 4) we get from (43)

$$\Big(\frac{\delta^2\Gamma[\hat{\phi}_{cl}]}{\delta\hat{\phi}_{cl}(p_1)\delta\hat{\phi}_{cl}(p_2)}\Big|_{\hat{\phi}_{cl}=0}\Big)'_{p_1=-p_2=0} = m^2 - \frac{m^2\lambda\hbar}{32\pi^2\varepsilon}\Big(\frac{\mu^2}{\pi m^2}\Big)^{\varepsilon}\frac{\Gamma(1+\varepsilon)}{(1-\varepsilon)} + \mathcal{O}(\lambda^2) \equiv m^2_{\text{phys}}\,. \tag{47}$$

Solving the formal power series for m^2 and using $\lambda = \lambda_{\text{phys}} + \mathcal{O}(\lambda^2_{\text{phys}})$ we get

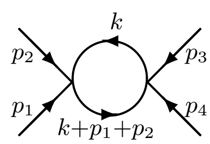
$$m^2(\varepsilon) = m^2_{\text{phys}} + \frac{m^2_{\text{phys}}\lambda_{\text{phys}}\hbar}{32\pi^2\varepsilon}\Big(\frac{\mu^2}{\pi m^2_{\text{phys}}}\Big)^{\varepsilon}\frac{\Gamma(1+\varepsilon)}{(1-\varepsilon)} + \mathcal{O}(\lambda^2_{\text{phys}})\,. \tag{48}$$

In other words, choosing the bare mass $m(\varepsilon)$ according to (48) removes the divergence of the two-point function at first order in λ_{phys}. For the treatment of subdivergences it is more convenient to perform the adjustment of m in two steps: In the first step we choose m such that the singular $\frac{1}{\varepsilon}$-term in (46) is compensated. In the second step we adjust the finite part of m to satisfy (43). Taking the limit $\varepsilon \to 0$ we get instead of (48)

$$m^2 = m^2_{\text{phys}} + \frac{m^2_{\text{phys}}\lambda_{\text{phys}}\hbar}{32\pi^2}\Big(\ln\Big(\frac{\mu^2}{\pi m^2_{\text{phys}}}\Big) + 1 + \psi(1)\Big) + \mathcal{O}(\lambda^2_{\text{phys}})\,. \tag{49}$$

More insight about this method is gained from Exercise 8. One sees that the adjustment of (45) removes the divergences from the four-point function at second order in λ_{phys} not only for zero momenta but for any momenta p_i.

Exercise 8. Compute the integral $\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+m^2)((k+p_1+p_2)^2+m^2)}$ arising in the Feynman graph

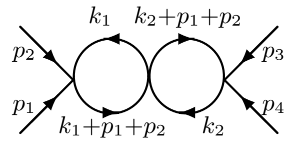


(50)

in dimensional regularisation. Hint: First bring the denominator using the Feynman trick $\frac{1}{AB} = \int_0^1 \frac{dy}{(Ay+B(1-y))^2}$ into the form $(k^2+2kq+r^2)^2$ and then use the Schwinger trick. Next perform the k-integration and finally the α-integration. The y-integral needs not to be computed. ◁

2.4 Treatment of subdivergences

As part of the renormalisation process, the subtraction of divergences $\sim \frac{1}{\varepsilon^i}$ can only be carried out if these divergences appear in the n-point functions (43)–(45) for which we impose normalisation conditions. This means, in particular, that the coefficient of the $\frac{1}{\varepsilon^i}$-terms must not contain logarithms in the momenta. However, if one computes naïvely the integral corresponding to the graph



(51)

as in Exercise 8, one does get logarithms of momenta in front of $\frac{1}{\varepsilon}$. The solution of this problem is a different subtraction of divergences in presence of subdivergences.

The solution of this problem was found by Bogolyubov [1]. A review of the most important renormalisation schemes can be found in [9]. Instead of splitting the integral $I_{\mathcal{G}}$ associated with a Feynman graph $\mathcal{G}$ into convergent and divergent parts, there is a recursive construction of the integral to split. For a Laurent series in ε, let

$$T\Big(\sum_{i=-r}^{\infty} a_i\varepsilon^i\Big) := \sum_{i=-r}^{-1} a_i\varepsilon^i \tag{52}$$

be the projection to the divergent part. Then one defines for a graph $\mathcal{G}$ with disjoint subgraphs $\mathcal{G}_i$ the divergent part as

$$\mathcal{C}_{\mathcal{G}} := -T\Big(\mathcal{I}_{\mathcal{G}} + \sum_{\substack{\{\mathcal{G}_1,\dots,\mathcal{G}_n\}\\ \mathcal{G}_i\in\mathcal{G},\ \mathcal{G}_i\cap\mathcal{G}_j=\emptyset}} \mathcal{C}_{\mathcal{G}_1}\cdots\mathcal{C}_{\mathcal{G}_n}\mathcal{I}_{\mathcal{G}/(\mathcal{G}_1\cup\dots\cup\mathcal{G}_n)}\Big) \tag{53}$$

and the convergent part as

$$\mathcal{R}_{\mathcal{G}} := (1-T)\Big(\mathcal{I}_{\mathcal{G}} + \sum_{\substack{\{\mathcal{G}_1,\dots,\mathcal{G}_n\}\\ \mathcal{G}_i\in\mathcal{G},\ \mathcal{G}_i\cap\mathcal{G}_j=\emptyset}} \mathcal{C}_{\mathcal{G}_1}\cdots\mathcal{C}_{\mathcal{G}_n}\mathcal{I}_{\mathcal{G}/(\mathcal{G}_1\cup\dots\cup\mathcal{G}_n)}\Big)\,. \tag{54}$$

Here, a graph $\mathcal{G}$ is understood a the set of vertices and internal lines, and the sum runs over all sets of disjoint subgraphs. By $\mathcal{G}/(\mathcal{G}_1\cup\dots\cup\mathcal{G}_n)$ we mean the graph obtained by shrinking the subgraphs $\mathcal{G}_1,\dots,\mathcal{G}_n$ in $\mathcal{G}$ to a point.

Exercise 9. Using the result of Exercise 8, compute the integrals $\mathcal{I}_{\mathcal{G}}$ and $\mathcal{R}_{\mathcal{G}}$ for the graph $\mathcal{G}$ given by (51). Hint: The only subgraphs of $\mathcal{G}$ are either the left or the right one-loop subgraphs (50). ◁

There is an explicit solution of the recursion in terms of *forests* found by Zimmermann [10]. The process of passing from $\mathcal{I}_{\mathcal{G}}$ to $\mathcal{C}_{\mathcal{G}}$ and $\mathcal{R}_{\mathcal{G}}$ might seem quite unmotivated. However, as shown by Connes and Kreimer, there is the structure of a Hopf algebra behind the renormalisation process. The subtraction (53), (54) is actually a division of divergences via the antipode of the Hopf algebra, and the splitting (53), (54) is the Birkhoff decomposition solving a Riemann-Hilbert problem [11, 12].

3 Renormalisation by flow equations

3.1 Introduction

We have mentioned that the continuum limit of a lattice field theory corresponds to the critical point of the lattice model. One of the main tools to explore critical points in statistical physics are *renormalisation group* methods. This subject was mainly developed by Wilson [3]. A particular outcome was the understanding of renormalisation in terms of the scaling of effective actions. This idea was further developed by Polchinski to a very efficient proof that the ϕ^4-model is renormalisable to all orders in perturbation theory [13]. We refer to [14] for a textbook on this approach to renormalisation. Whereas renormalisability of the ϕ^4-model can also be proven in the previously presented Feynman graph approach, the superiority of Polchinski's method becomes manifest in the renormalisation problem of noncommutative field theories. We shall therefore present the main ideas of Polchinski's proof, following closely the original article.

3.2 Derivation of the Polchinski equation

The starting point is a reformulation of the generating functional $Z[j]$ introduced in (22). First one brutally removes the modes[3] $\phi(p)$ with $p^2 > 2\Lambda^2$ in the measure $\mathcal{D}\phi$ of the partition function. The crucial idea is to take a *smooth cut-off* distributed over the interval $p^2 = \Lambda^2 \dots 2\Lambda^2$ which allows at a later step to differentiate with respect to the cut-off scale Λ. In this way one obtains a differential equation (the Polchinski equation) which governs the renormalisation flow of the effective action.

To be precise, we choose the cut-off function

$$K\Big(\frac{p^2}{\Lambda^2}\Big) = \begin{cases} 1 & \text{for } p^2 \leq \Lambda^2 \;, \\ 1 - \exp\Big(-\dfrac{\exp\big(-\frac{1}{2\Lambda^2 - p^2}\big)}{p^2 - \Lambda^2}\Big) & \text{for } \Lambda^2 < p^2 < 2\Lambda^2 \;, \\ 0 & \text{for } p^2 \geq 2\Lambda^2 \;. \end{cases} \tag{55}$$

We consider the generating functional

$$\begin{aligned} Z[j,\Lambda] &= \int \mathcal{D}\phi \;\exp\big(-S[\phi,j,\Lambda]\big) \;, \\ S[\phi,j,\Lambda] &:= \int \frac{d^4p}{(2\pi)^4}\,\Big(\frac{1}{2}(p^2+m^2)K^{-1}\Big(\frac{p^2}{\Lambda^2}\Big)\phi(p)\phi(-p) - \phi(p)j(-p)\Big) \\ &\qquad + L[\phi,\Lambda] + C[\Lambda] \;, \end{aligned} \tag{56}$$

[3] In this section we work exclusively in momentum space so that we omit the hat over a Fourier-transformed field for simplicity. We also use natural units $\hbar = 1$.

with $L[0,\Lambda]=0$. Unless $\phi(\pm p)=0$ for $p^2\geq 2\Lambda^2$, we have $S[\phi,j,\Lambda]=+\infty$, which means $Z[j,\Lambda]=0$. In other words, only modes $\phi(p)$ with momenta $p^2<2\Lambda^2$ contribute to $Z[j,\Lambda]$. Now we compute

$$\begin{aligned}&\Big(2\phi(p)K^{-1}(\tfrac{p^2}{\Lambda^2})-(p^2+m^2)^{-1}\frac{\delta S(\phi,j,\Lambda)}{\delta\phi(-p)}\Big)\exp(-S[\phi,j,\Lambda])\\&\qquad=\Big(\phi(p)K^{-1}(\tfrac{p^2}{\Lambda^2})+(p^2+m^2)^{-1}j(p)-(p^2+m^2)^{-1}\frac{\delta L}{\delta\phi(-p)}\Big)\\&\qquad\qquad\times\exp(-S[\phi,j,\Lambda])\,. \qquad (57)\end{aligned}$$

Functional derivation with respect to $\phi(p)$ gives

$$\begin{aligned}&\frac{\delta}{\delta\phi(p)}\Big\{\Big(2\phi(p)K^{-1}(\tfrac{p^2}{\Lambda^2})-(p^2+m^2)^{-1}\frac{\delta S(\phi,j,\Lambda)}{\delta\phi(-p)}\Big)\exp(-S[\phi,j,\Lambda])\Big\}\\&=\Big\{\Big(\phi(p)K^{-1}(\tfrac{p^2}{\Lambda^2})+(p^2+m^2)^{-1}j(p)-(p^2+m^2)^{-1}\frac{\delta L}{\delta\phi(-p)}\Big)\\&\qquad\times\Big(-\phi(-p)(p^2+m^2)K^{-1}(\tfrac{p^2}{\Lambda^2})+j(-p)-\frac{\delta L}{\delta\phi(p)}\Big)\\&\quad+\Big((2\pi)^4\delta(0)K^{-1}(\tfrac{p^2}{\Lambda^2})-(p^2+m^2)^{-1}\frac{\delta^2L}{\delta\phi(-p)\delta\phi(p)}\Big)\Big\}\exp(-S[\phi,j,\Lambda])\,.\end{aligned} \qquad (58)$$

For simplicity we choose $j(p)=0$ for $p^2>\Lambda^2$. This condition is not necessary, but it simplifies the following calculation considerably. We multiply (58) by $\Lambda\frac{\partial K(\frac{p^2}{\Lambda^2})}{\partial\Lambda}$, which is non-zero for $p^2>\Lambda^2$ only, and therefore annihilates $j(\pm p)$. Next, we integrate over d^4p and finally apply the functional integration over $\mathcal{D}\phi$. This yields zero for the lhs of (58), because for each momentum p the derivative $\frac{\delta}{\delta\phi(p)}$ finds an integration over $d\phi(p)$ in the measure $\mathcal{D}\phi$ which by Stokes' theorem gives the value of the term in braces $\{\ \}$ on the lhs of (58) at the boundary $\phi(p)=\pm\infty$. But $\exp(-S[\infty,j,\Lambda])=0$. We thus have

$$\begin{aligned}0=&\int\mathcal{D}\phi\int\frac{d^4p}{(2\pi^4)}\Big\{\phi(p)\phi(-p)(p^2+m^2)\Lambda\frac{\partial K^{-1}(\frac{p^2}{\Lambda^2})}{\partial\Lambda}\\&+(2\pi)^4\delta(0)K^{-1}(\tfrac{p^2}{\Lambda^2})\Lambda\frac{\partial K(\frac{p^2}{\Lambda^2})}{\partial\Lambda}\\&+\frac{1}{p^2+m^2}\Lambda\frac{\partial K(\frac{p^2}{\Lambda^2})}{\partial\Lambda}\Big(\frac{\delta L}{\delta\phi(p)}\frac{\delta L}{\delta\phi(-p)}-\frac{\delta^2L}{\delta\phi(p)\delta\phi(-p)}\Big)\Big\}\exp(-S[\phi,j,\Lambda])\,.\end{aligned} \qquad (59)$$

On the other hand, differentiating $Z[j,\Lambda]$ in (56) with respect to Λ we have

$$\Lambda\frac{\partial Z}{\partial\Lambda} = -\int \mathcal{D}[\phi]\,\Big\{\Lambda\frac{\partial C}{\partial\Lambda} + \Lambda\frac{\partial L}{\partial\Lambda} + \int\frac{d^4p}{(2\pi)^4}\Big(\frac{1}{2}\phi(p)\phi(-p)(p^2+m^2)\Lambda\frac{\partial}{\partial\Lambda}K^{-1}(\tfrac{p^2}{\Lambda^2})\Big)\Big\}\exp(-S[\phi,j,\Lambda])\;. \tag{60}$$

Inserting (59) into (60) we arrive at

$$\Lambda\frac{\partial Z[j,\Lambda]}{\partial\Lambda} = 0 \qquad \text{if} \tag{61}$$

$$\Lambda\frac{\partial C[\Lambda]}{\partial\Lambda} = \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\Big((p^2+m^2)^{-1}\Lambda\frac{\partial K(\frac{p^2}{\Lambda^2})}{\partial\Lambda}\frac{\delta^2 L}{\delta\phi(p)\delta\phi(-p)}\Big|_{\phi=0} - (2\pi)^4\delta(0)\,K^{-1}(\tfrac{p^2}{\Lambda^2})\Lambda\frac{\partial K(\frac{p^2}{\Lambda^2})}{\partial\Lambda}\Big)\;, \tag{62}$$

$$\Lambda\frac{\partial L[\phi,\Lambda]}{\partial\Lambda} = -\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}(p^2+m^2)^{-1}\Lambda\frac{\partial K(\frac{p^2}{\Lambda^2})}{\partial\Lambda}\Big\{\frac{\delta L}{\delta\phi(p)}\frac{\delta L}{\delta\phi(-p)} - \Big[\frac{\delta^2 L}{\delta\phi(p)\delta\phi(-p)}\Big]_\phi\Big\}\;, \tag{63}$$

where $\big[F[\phi]\big]_\phi := F[\phi] - F[0]$.

We learn that the effect of restricting the integration modes in the partition function is undone if the effective action $L[\phi,\Lambda]$ and the vacuum energy $C[\Lambda]$ depend according to (63) and (62) on Λ. This means that instead of the original generating functional $Z[j] = Z[j,\infty]$ we can equally well work with $Z[j,\Lambda]$ for finite Λ. If one computes the Feynman rules from $Z[j,\Lambda]$, one finds that the propagator is given by $K(\frac{p^2}{\Lambda^2})(p^2+m^2)^{-1}$ and the vertices by the expansion coefficients of $L[\phi,\Lambda]$. Since the loop integrations in these Feynman graphs have a finite range, we obtain finite n-point functions if the effective action $L[\phi,\Lambda]$ which evolves from $L[\phi,\infty]$ via the flow (63) is bounded. In other words, the problem to renormalise a quantum field theory is reduced to the proof that the renormalisation flow described by (63) does not produce singularities when starting from appropriate boundary conditions.

3.3 The strategy of renormalisation

If one naïvely integrates (63) from $L[\phi,\infty] = S_{int}[\phi]$ given by (30) down to Λ one will encounter the same divergences as found e.g. in (46) and Exercise 8, which must be removed by the normalisation similar to (43)–(45). The idea is thus to integrate in a first step the differential equation (63) between two scales $\Lambda_R \leq \Lambda \leq \Lambda_0$ where the initial values ρ^0 for L at Λ_0 are adjusted such that the distinguished functions (43)–(45) take given values at Λ_R.

Let us expand L into field monomials,

$$L[\phi,\Lambda,\Lambda_0,\rho^0]=\sum_{m=1}^{\infty}\frac{1}{(2m)!}\int\Big(\prod_{i=1}^{2m}\frac{d^4p_i}{(2\pi)^4}\Big)\,(2\pi)^4\delta(p_1+\dots+p_{2m})\\ \times L_{2m}(p_1,\dots,p_{2m};\Lambda,\Lambda_0,\rho^0)\,\phi(p_1)\cdots\phi(p_{2m})\;,\qquad(64)$$

keeping the symmetry $\phi\mapsto-\phi$. The dependence on the initial conditions ρ_i^0 at $\Lambda=\Lambda_0$ is written explicitly. Thus, (63) becomes an infinite system of coupled differential equations for the amplitudes $L_{2m}(p_1,\dots,p_{2m};\Lambda,\Lambda_0,\rho^0)$ which at $\Lambda=\Lambda_0$ are parametrised by initial conditions ρ_i^0. Anticipating renormalisability we choose the initial condition

$$\begin{aligned}&L[\phi,\Lambda_0,\Lambda_0,\rho^0]\\&\quad=\int\frac{d^4p}{(2\pi)^4}\Big(\frac{1}{2}\rho_1^0\phi(p)\phi(-p)+\frac{1}{2}p^2\rho_2^0\phi(p)\phi(-p)\Big)\\&\quad+\frac{1}{4!}\int\Big(\prod_{i=1}^{4}\frac{d^4p_i}{(2\pi)^4}\Big)\,(2\pi)^4\delta(p_1+\dots+p_4)\,\rho_3^0\,\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\;.\end{aligned}\qquad(65)$$

The evolution of $L[\phi,\Lambda]$ according to (63) will produce more complicated interactions than (65). Among these interactions we distinguish the same Taylor coefficients as in (65):

$$\begin{aligned}\rho_1[\Lambda,\Lambda_0,\rho^0]&:=L_2(0,0;\Lambda,\Lambda_0,\rho^0)\;,\\ \rho_2[\Lambda,\Lambda_0,\rho^0]&:=\frac{1}{2}\frac{\partial^2}{\partial p^2}L_2(p,-p;\Lambda,\Lambda_0,\rho^0)\Big|_{p=0}\;,\\ \rho_3[\Lambda,\Lambda_0,\rho^0]&:=L_4(0,0,0,0;\Lambda,\Lambda_0,\rho^0)\;,\end{aligned}\qquad(66)$$

with

$$\rho_i[\Lambda_0,\Lambda_0,\rho^0]=\rho_i^0\;,\qquad i=1,2,3\;.\qquad(67)$$

At the end we are interested in the limit $\Lambda_0\to\infty$. This limit requires a carefully chosen Λ_0-dependence of the initial data $\rho_i^0[\Lambda_0]$ such that $\rho_i[\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]$ take the given normalised values. We consider the identity

$$\begin{aligned}&L[\phi,\Lambda_R,\Lambda_0',\rho^0[\Lambda_0']]-L[\phi,\Lambda_R,\Lambda_0'',\rho^0[\Lambda_0'']]\\&\equiv\int_{\Lambda_0''}^{\Lambda_0'}\frac{d\Lambda_0}{\Lambda_0}\;\Lambda_0\frac{d}{d\Lambda_0}\Big(L[\phi,\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]\Big)\\&=\int_{\Lambda_0''}^{\Lambda_0'}\frac{d\Lambda_0}{\Lambda_0}\;\Big(\Lambda_0\frac{\partial}{\partial\Lambda_0}L[\phi,\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]\\&\qquad+\sum_{a=1}^{3}\frac{\partial}{\partial\rho_a^0}L[\phi,\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]\Lambda_0\frac{d\rho_a[\Lambda_0]}{d\Lambda_0}\Big)\;.\end{aligned}\qquad(68)$$

On the other hand, we express the fact that $\rho_a[\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]$ is kept fixed:

$$0 = d\rho_a[\Lambda_R, \Lambda_0, \rho^0[\Lambda_0]] = \frac{\partial \rho_a[\Lambda_R, \Lambda_0, \rho^0[\Lambda_0]]}{\partial \Lambda_0} d\Lambda_0 + \sum_{b=1}^{3} \frac{\partial \rho_a[\Lambda_R, \Lambda_0, \rho^0[\Lambda_0]]}{\partial \rho_b^0} d\rho_b^0[\Lambda_0] \;. \tag{69}$$

To be precise, we choose

$$\rho_1[\Lambda_R, \Lambda_0, \rho^0[\Lambda_0]] = 0 \;, \quad \rho_2[\Lambda_R, \Lambda_0, \rho^0[\Lambda_0]] = 0 \;, \quad \rho_3[\Lambda_R, \Lambda_0, \rho^0[\Lambda_0]] = \lambda \;. \tag{70}$$

Assuming that we can invert the (3×3)-matrix $\frac{\partial \rho_a}{\partial \rho_b^0}$, which is always possible in perturbation theory, we can rewrite (69) as

$$\frac{d\rho_j^0[\Lambda_0]}{d\Lambda_0} = -\sum_{i=1}^{3} \frac{\partial \rho_j^0}{\partial \rho_i[\Lambda_R, \Lambda_0, \rho^0[\Lambda_0]]} \frac{\partial \rho_i[\Lambda_R, \Lambda_0, \rho^0[\Lambda_0]]}{\partial \Lambda_0} \;. \tag{71}$$

Inserting this result into (68) we get

$$L[\phi, \Lambda_R, \Lambda_0', \rho^0[\Lambda_0']] - L[\phi, \Lambda_R, \Lambda_0'', \rho^0[\Lambda_0'']] = \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0} \, V[\phi, \Lambda_R, \Lambda_0, \rho^0[\Lambda_0]] \;, \tag{72}$$

where

$$V[\phi, \Lambda, \Lambda_0, \rho^0[\Lambda_0]] := \Lambda_0 \frac{\partial L[\phi, \Lambda, \Lambda_0, \rho^0]}{\partial \Lambda_0} - \sum_{b=1}^{3} \frac{\partial L[\Lambda, \Lambda_0, \rho^0[\Lambda_0]]}{\partial \rho_b[\Lambda, \Lambda_0, \rho^0[\Lambda_0]]} \, \Lambda_0 \frac{\partial \rho_b[\Lambda, \Lambda_0, \rho^0[\Lambda_0]]}{\partial \Lambda_0} \;, \tag{73}$$

$$\frac{\partial L[\Lambda, \Lambda_0, \rho^0[\Lambda_0]]}{\partial \rho_b[\Lambda, \Lambda_0, \rho^0[\Lambda_0]]} := \sum_{a=1}^{3} \frac{\partial L[\Lambda, \Lambda_0, \rho^0[\Lambda_0]]}{\partial \rho_a^0} \frac{\partial \rho_a^0}{\partial \rho_b[\Lambda, \Lambda_0, \rho^0[\Lambda_0]]} \;. \tag{74}$$

The function V is linear in L and therefore in L_{2m} and its Taylor coefficients. Therefore, the projection of $V[\phi, \Lambda, \Lambda_0, \rho^0]$ to the initial field monomials as in (66) vanishes identically for all Λ, which means that V filters out power-counting divergent part of the effective action. It remains to show that the other coefficients of V have a Λ_0-dependence which lead to a convergence of (72) in the limit $\Lambda_0' \to \infty$.

For that purpose we compute the Λ-scaling of V. We define

$$M[V] := -\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} (p^2+m^2)^{-1} \Lambda \frac{\partial K(\frac{p^2}{\Lambda^2})}{\partial \Lambda} \Big\{ 2 \frac{\delta L}{\delta \phi(p)} \frac{\delta V}{\delta \phi(-p)} - \Big[\frac{\delta^2 V}{\delta \phi(p) \delta \phi(-p)} \Big]_\phi \Big\} \tag{75}$$

and expand M into field monomials,

$$M[V] = \sum_{m=1}^{\infty} \frac{1}{(2m)!} \int \Big(\prod_{i=1}^{2m} \frac{d^4p_i}{(2\pi)^4} \Big) (2\pi)^4 \delta(p_1 + \dots + p_{2m}) \\ \times \mathcal{M}_{2m}[V](p_1, \dots, p_{2m}; \Lambda, \Lambda_0, \rho^0)\, \phi(p_1) \cdots \phi(p_{2m}) \;. \quad (76)$$

As before we distinguish the coefficients

$$\begin{aligned} M_1[V] &:= \mathcal{M}_2[V](0,0;\Lambda,\Lambda_0,\rho^0) \;, \\ M_2[V] &:= \frac{1}{2}\frac{\partial^2}{\partial p^2} \mathcal{M}_2[V](p,-p;\Lambda,\Lambda_0,\rho^0)\Big|_{p=0} \;, \\ M_3[V] &:= \mathcal{M}_4[V](0,0,0,0;\Lambda,\Lambda_0,\rho^0) \;. \end{aligned} \quad (77)$$

Then one finds

$$\Lambda \frac{\partial V}{\partial \Lambda} = M[V] - \sum_{b=1}^{3} \frac{\partial L}{\partial \rho_b} M_b[V] \;, \quad (78)$$

$$\Lambda \frac{\partial}{\partial \Lambda} \Big(\frac{\partial L}{\partial \rho_b} \Big) = M\Big[\frac{\partial L}{\partial \rho_b}\Big] - \sum_{a=1}^{3} \frac{\partial L}{\partial \rho_a} M_a \Big[\frac{\partial L}{\partial \rho_b}\Big] \;, \quad (79)$$

where $M[\frac{\partial L}{\partial \rho_b}]$ arises from $M[V]$ by replacing V in (75) and (77) by $\frac{\partial L}{\partial \rho_b}$.

Exercise 10. Prove (78) and (79). Hint: First differentiate (73) with respect to Λ, taking into account the identity $(a^{-1})' = -(a^{-1})a'(a^{-1})$ in the $\rho[\Lambda]$-part. The Λ-derivatives act on derivatives of L or ρ with respect to Λ_0 or ρ^0. Since the derivatives commute, represent the second derivatives of L by the result of the differentiation of (63) with respect to Λ_0 or ρ^0 and the second derivatives of ρ by the projection similar to (77). Using the linearity of $M[?]$ and the ϕ-independence of the ρ-coefficients everything reassembles to (78). The proof of (79) is similar. ◁

By estimating V using the differential equation (78) and knowledge of the estimations of L and $\frac{\partial L}{\partial \rho_b}$ obtained by solving (63) and (79) before we can via (72) control the limit $\lim_{\Lambda_0 \to \infty} L[\phi, \Lambda_R, \Lambda_0, \rho^0[\Lambda_0]]$.

3.4 Perturbative solution of the flow equations

The evolution of the functions L, $\frac{\partial L}{\partial \rho_b}$ and V is essentially determined by the mass dimensions. We therefore absorb the dimensionality in an appropriate power of Λ. Expanding the functions also into a power series in the coupling constant we define

$$
\begin{aligned}
&L[\phi,\varLambda,\varLambda_0,\rho^0[\varLambda_0]]\\
&=\sum_{m=1}^{\infty}\frac{1}{(2m)!}\int\Big(\prod_{i=1}^{2m}\frac{d^4p_i}{(2\pi)^4}\Big)\,(2\pi)^4\delta(p_1+\dots+p_{2m})\\
&\qquad\times\varLambda^{4-2m}\Big(\sum_{r=1}^{\infty}\lambda^r A^{(r)}_{2m}(p_1,\dots,p_{2m};\varLambda,\varLambda_0,\rho^0)\Big)\,\phi(p_1)\cdots\phi(p_{2m})\;, \qquad (80)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial L}{\partial\rho_a}[\phi,\varLambda,\varLambda_0,\rho^0[\varLambda_0]]\\
&=\sum_{m=1}^{\infty}\frac{1}{(2m)!}\int\Big(\prod_{i=1}^{2m}\frac{d^4p_i}{(2\pi)^4}\Big)\,(2\pi)^4\delta(p_1+\dots+p_{2m})\\
&\qquad\times\varLambda^{4-2m-2\delta_{a1}}\Big(\sum_{r=1}^{\infty}\lambda^r B^{a(r)}_{2m}(p_1,\dots,p_{2m};\varLambda,\varLambda_0,\rho^0)\Big)\,\phi(p_1)\cdots\phi(p_{2m})\;, \qquad (81)
\end{aligned}
$$

$$
\begin{aligned}
&V[\phi,\varLambda,\varLambda_0,\rho^0[\varLambda_0]]\\
&=\sum_{m=1}^{\infty}\frac{1}{(2m)!}\int\Big(\prod_{i=1}^{2m}\frac{d^4p_i}{(2\pi)^4}\Big)\,(2\pi)^4\delta(p_1+\dots+p_{2m})\\
&\qquad\times\varLambda^{4-2m}\Big(\sum_{r=1}^{\infty}\lambda^r V^{(r)}_{2m}(p_1,\dots,p_{2m};\varLambda,\varLambda_0,\rho^0)\Big)\,\phi(p_1)\cdots\phi(p_{2m})\;. \qquad (82)
\end{aligned}
$$

The functions $A^{(r)}_{2m}, B^{a(r)}_{2m}, V^{(r)}_{2m}$ are dimensionless. Inserting these definitions into (63), (79) and (78) one gets

$$
\begin{aligned}
&\Big(\varLambda\frac{\partial}{\partial\varLambda}+4-2m\Big)A^{(r)}_{2m}(p_1,\dots,p_{2m};\varLambda)\\
&=\Big\{-\frac{1}{2}\sum_{l=1}^{m}\sum_{s=1}^{t-1}Q(P,\varLambda,m^2)A^{(r-s)}_{2l}(p_1,\dots,p_{2l-1},-P;\varLambda)\\
&\qquad\qquad\times A^{(s)}_{2m-2l+2}(p_{2l},\dots,p_{2m},P;\varLambda)+\Big(\binom{2m}{2l-1}-1\Big)\text{ perm.}\Big\}\\
&+\frac{1}{2}\int\frac{d^4p}{(2\pi\varLambda)^4}\;Q(p,\varLambda,m^2)A^{(r)}_{2m+2}(p_1,\dots,p_{2m},p,-p;\varLambda)\;, \qquad (83)
\end{aligned}
$$

$$
\begin{aligned}
&\Big(\varLambda\frac{\partial}{\partial\varLambda}+4-2m-2\delta_{a1}\Big)\big(B^{a(r)}_{2m}(p_1,\dots,p_{2m};\varLambda)\big)\\
&=\Big\{-\sum_{l=1}^{m}\sum_{s=0}^{r-1}Q(P,\varLambda,m^2)A^{(r-s)}_{2l}(p_1,\dots,p_{2l-1},-P;\varLambda)\\
&\qquad\qquad\times B^{a(s)}_{2m-2l+2}(p_{2l},\dots,p_{2m},P;\varLambda)+\Big(\binom{2m}{2l-1}-1\Big)\text{ perm.}\Big\}\\
&+\frac{1}{2}\int\frac{d^4p}{(2\pi\varLambda)^4}\;Q(p,\varLambda,m^2)\,B^{a(r)}_{2m+2}(p_1,\dots,p_{2m},p,-p;\varLambda)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{s=0}^{r} B_{2m}^{1(r-s)}(p_1,\dots,p_{2m};\Lambda)\int\frac{d^4q}{(2\pi\Lambda)^4}\,Q(q,\Lambda,m^2)\,B_4^{a(s)}(0,0,q,-q;\Lambda)\\
&-\frac{\Lambda^2}{2}\sum_{s=0}^{r} B_{2m}^{2(r-s)}(p_1,\dots,p_{2m};\Lambda)\\
&\qquad\times\int\frac{d^4q}{(2\pi\Lambda)^4}\,Q(q,\Lambda,m^2)\,\frac{\partial^2 B_4^{a(s)}(p,-p,q,-q;\Lambda)}{\partial p^2}\Big|_{p=0}\\
&-\frac{1}{2}\sum_{s=0}^{r} B_{2m}^{3(r-s)}(p_1,\dots,p_{2m};\Lambda)\\
&\qquad\times\int\frac{d^4q}{(2\pi\Lambda)^4}\,Q(q,\Lambda,m^2)\,B_6^{a(s)}(0,0,0,0,q,-q;\Lambda)\;, \qquad (84)
\end{aligned}
$$

$$
\begin{aligned}
&\Big(\Lambda\frac{\partial}{\partial\Lambda}+4-2m\Big)\big(V_{2m}^{(r)}(p_1,\dots,p_{2m};\Lambda)\big)\\
&=\Big\{-\sum_{l=1}^{m}\sum_{s=1}^{r-1} Q(P,\Lambda,m^2)A_{2l}^{(r-s)}(p_1,\dots,p_{2l-1},-P;\Lambda)\\
&\qquad\times V_{2m-2l+2}^{(s)}(p_{2l},\dots,p_{2m},P;\Lambda)+\Big(\binom{2m}{2l-1}-1\Big)\text{ perm.}\Big\}\\
&+\frac{1}{2}\int\frac{d^4p}{(2\pi\Lambda)^4}Q(p,\Lambda,m^2)\;V_{2m+2}^{(r)}(p_1,\dots,p_{2m},p,-p;\Lambda)\\
&-\frac{1}{2}\sum_{s=1}^{r} B_{2m}^{1(r-s)}(p_1,\dots,p_{2m};\Lambda)\int\frac{d^4q}{(2\pi\Lambda)^4}\,Q(q,\Lambda,m^2)\,V_4^{(s)}(0,0,q,-q;\Lambda)\\
&-\frac{\Lambda^2}{2}\sum_{s=1}^{r} B_{2m}^{2(r-s)}(p_1,\dots,p_{2m};\Lambda)\\
&\qquad\times\int\frac{d^4q}{(2\pi\Lambda)^4}\,Q(q,\Lambda,m^2)\,\frac{\partial^2 V_4^{(s)}(p,-p,q,-q;\Lambda)}{\partial p^2}\Big|_{p=0}\\
&-\frac{1}{2}\sum_{s=1}^{r} B_{2m}^{3(r-s)}(p_1,\dots,p_{2m};\Lambda)\\
&\qquad\times\int\frac{d^4q}{(2\pi\Lambda)^4}\,Q(q,\Lambda,m^2)\,V_6^{(s)}(0,0,0,0,q,-q;\Lambda)\;, \qquad (85)
\end{aligned}
$$

where $Q(p,\Lambda,m^2)=\frac{1}{p^2+m^2}\Lambda^3\frac{\partial K(\frac{p^2}{\Lambda^2})}{\partial\Lambda}$ and $P:=p_1+\dots+p_{2l-1}$. There are $\binom{2m}{2l-1}$ possibilities to assign $2l{-}1$ of the $2m$ momenta to the first function and the remaining ones to the second function.

Exercise 11. Prove (83), (84) and (85). Hint: In the last two equations a trilinear term in the functions disappears because at vanishing external momenta these functions are connected by $Q(0,\Lambda,m^2)=0$. (The support of $Q(p,\Lambda,m^2)$ is $\Lambda^2<p^2<2\Lambda^2$.) ◁

Due to the grading in the coupling constant, the differential equations (83), (84) and (85) allow us to recursively compute the functions $A^{(r)}_{2m}, B^{a(r)}_{2m}, V^{(r)}_{2m}$ starting from $A^{(1)}_4 = 1$. The concrete form is not necessary for the renormalisation proof. All we need are the norms

$$\|f\| \equiv \|f(p_1,\dots,p_m;\Lambda)\| := \max_{p_i^2 \leq 2\Lambda^2} |f(p_1,\dots,p_m;\Lambda)| \tag{86}$$

of these functions. The norms are computed in terms of $A^{(1)}_4 = 1$ and the bounds

$$\int \frac{d^4p}{(2\pi)^4} |Q(p,\Lambda,m^2)| \leq C\Lambda^4 \;, \qquad \Big|\frac{\partial^n}{\partial p^n} Q(p,\Lambda,m^2)\Big| \leq D_n \Lambda^{-n} \;, \tag{87}$$

for the propagator Q, for some constants C, D_n. Due to momentum conservation we also need a symbol $\partial^\mu_{i,j} := \frac{\partial}{\partial p_{i\mu}} - \frac{\partial}{\partial p_{j\mu}}$ for the independent momentum derivatives.

Exercise 12. Verify (87). ◁

Now we can derive the estimations for the functions $A^{(r)}_{2m}, B^{a(r)}_{2m}, V^{(r)}_{2m}$.

Lemma 1.

$$\|\partial^{\mu_1}_{i_1,j_1} \dots \partial^{\mu_d}_{i_d,j_d} A^{(r)}_{2m}(p_1,\dots,p_{2m};\Lambda)\| \begin{cases} \leq \Lambda^{-d} P^{2r-m}\big[\ln \frac{\Lambda_0}{\Lambda_R}\big] & \textit{for } r{+}1 \geq m \;, \\ = 0 & \textit{for } r{+}1 < m \;, \end{cases} \tag{88}$$

where $P^n[x]$ stands for a polynomial in x of degree n.

Remarks on the proof. The condition $A^{(r)}_{2m} \equiv 0$ for $r+1 < m$ is actually an additional requirement which guarantees a graphical interpretation of the A-functions: a connected graph with r vertices has at most $2r+2$ external legs. The Lemma is true for $m = 2$ and $r = 1$. Integrating the differential equation (83) either from Λ_0 down to Λ or from Λ_R up to Λ one obtains by induction upward in the number r of vertices and for given r downward in the number $2m$ of external legs one of the estimations

$$\begin{aligned} &\|\partial^{\mu_1}_{i_1,j_1} \dots \partial^{\mu_d}_{i_d,j_d} A^{(r)}_{2m}(p_1,\dots,p_{2m};\Lambda)\| \\ &\qquad \leq \|\partial^{\mu_1}_{i_1,j_1} \dots \partial^{\mu_d}_{i_d,j_d} A^{(r)}_{2m}(p_1,\dots,p_{2m};\Lambda_0)\| \\ &\qquad + \Lambda^{2m+d-4} \int_\Lambda^{\Lambda_0} d\Lambda' \Lambda'^{\,3-2m-d} P^{2r-m-1}\big[\ln \frac{\Lambda_0}{\Lambda_R}\big] \end{aligned} \tag{89}$$

or

$$\begin{aligned} &\|\partial^{\mu_1}_{i_1,j_1} \dots \partial^{\mu_d}_{i_d,j_d} A^{(r)}_{2m}(p_1,\dots,p_{2m};\Lambda)\| \\ &\qquad \leq \|\partial^{\mu_1}_{i_1,j_1} \dots \partial^{\mu_d}_{i_d,j_d} A^{(r)}_{2m}(p_1,\dots,p_{2m};\Lambda_R)\| \\ &\qquad + \Lambda^{2m+d-4} \int_{\Lambda_R}^{\Lambda} d\Lambda' \Lambda'^{\,3-2m-d} P^{2r-m-1}\big[\ln \frac{\Lambda_0}{\Lambda_R}\big] \;. \end{aligned} \tag{90}$$

For $m \geq 3$ we use (89) and the initial condition $A^{(r)}_{2m}(p_1,\dots,p_{2m};\Lambda_0) = 0$ to prove (88). In the same way we prove the (88) for $m = 2, d \geq 1$. Then one uses (90) with the initial condition $A^{(r)}_4(0,0,0,0;\Lambda_R) = \delta^{r1}$ (which normalises the physical coupling constant at Λ_R to λ) to obtain $\|A^{(r)}_4(0,0,0,0;\Lambda)\| \leq P^{2r-2}\big[\ln\frac{\Lambda_0}{\Lambda_R}\big]$. The total four-point function is then reconstructed from Taylor's theorem

$$\begin{aligned}
&A^{(2)}_4(p_1,p_2,p_3,p_4;\Lambda)\\
&= A^{(r)}_4(0,0,0,0;\Lambda)\\
&+ \sum_{i,j=1}^{3} p_{\mu,i}\, p_{\nu,j} \int_0^1 d\xi\,(1{-}\xi)\,\partial'^{\mu}_{i,4}\partial'^{\nu}_{j,4} A^{(r)}_4(p'_1,\dots,p'_4;\Lambda)\big|_{p'_k=\xi p_k} . \qquad (91)
\end{aligned}$$

The first derivative of $A^{(r)}_4$ at zero momentum vanishes. We thus get (88) for $m = 2$. The extension to $m = 1$ is similar, taking into account the initial conditions for ρ_1 and ρ_2 at Λ_0. The detailed proof is left as an exercise. □

Lemma 2.

$$\big\|\partial^{\mu_1}_{i_1,j_1}\dots\partial^{\mu_d}_{i_d,j_d} B^{b(r)}_{2m}(p_1,\dots,p_{2m};\Lambda)\big\| \begin{cases} \leq \Lambda^{-d} P^{2r-m+1+\delta^{b3}}\big[\ln\frac{\Lambda_0}{\Lambda_R}\big] & \\ \qquad\qquad \textit{for } r{+}2 \geq m\,, & \\ = 0 \qquad\qquad \textit{for } r{+}2 < m\,. \end{cases} \qquad (92)$$

Remarks on the proof. The first step is to derive the boundary condition

$$B^{a(r)}_{2m}(p_1,\dots,p_{2m};\Lambda_0) = \delta^{r0}\Big(\delta_{m1}\delta^{b1} + \frac{p_1^2}{\Lambda_0^2}\delta_{m1}\delta^{b2} + \delta_{m2}\delta^{b3}\Big)\,, \qquad (93)$$

and the starting point $r = 0$ in (92) of the induction by explicitly evaluating (74) for the initial data (65) at lowest order in the coupling constant. The further proof is similar to that of (88), with integration according to (89), apart from the problem of terms with $s = 0$ and $s = r$ in the last three lines of (84). For $m \geq 3$ there is a problem with the third to last line only, which fortunately appears for $a = 3$ only. One thus proves first (92) for $a = 1, 2$ and using this result one repeats to proof for $a = 3$. Then one passes to $m = 2, d \geq 1$ and again excludes $a = 3$ to be processed later. The full function $B^{a(r)}_4$ is reconstructed from Taylor's theorem. The treatment of $m = 1$ is similar. □

Lemma 3.

$$\|\partial^{\mu_1}_{i_1,j_1}\dots\partial^{\mu_d}_{i_d,j_d} V^{(r)}_{2m}(p_1,\dots,p_{2m};\Lambda)\| \begin{cases} \leq \Lambda^{-d}\big(\frac{\Lambda^2}{\Lambda_0^2}\big) P^{2r-m}\big[\ln\frac{\Lambda_0}{\Lambda_R}\big] & \\ \qquad\qquad \textit{for } r{+}1 \geq m\,, & \\ = 0 \qquad\qquad \textit{for } r{+}1 < m\,. \end{cases} \qquad (94)$$

Remarks on the proof. At $\Lambda = \Lambda_0$ one has

$$V[\phi,\Lambda_0,\Lambda_0,\rho^0] := \Lambda_0 \frac{\partial L[\phi,\Lambda,\Lambda_0,\rho^0]}{\partial \Lambda_0}\Big|_{\Lambda=\Lambda_0} - \sum_{a=1}^{3} \frac{\partial L[\phi,\Lambda_0,\Lambda_0,\rho^0]}{\partial \rho^0_a} \Lambda_0 \frac{\partial \rho_a[\Lambda,\Lambda_0,\rho^0]}{\partial \Lambda_0}\Big|_{\Lambda=\Lambda_0} . \qquad (95)$$

The result is zero for the distinguished coefficients (66). For all other interaction coefficients $\perp \rho$ we have $L[\phi,\Lambda_0,\Lambda_0,\rho^0]\big|_{\perp\rho} \equiv 0$ independent of Λ_0 and ρ^0. This means

$$\begin{aligned} 0 &= \Lambda_0 \frac{\partial}{\partial \Lambda_0} L[\phi,\Lambda_0,\Lambda_0,\rho^0]\big|_{\perp\rho} \\ &= \Lambda_0 \frac{\partial}{\partial \Lambda_0} L[\phi,\Lambda,\Lambda_0,\rho^0]\big|_{\perp\rho}\Big|_{\Lambda=\Lambda_0} + \Lambda \frac{\partial}{\partial \Lambda} L[\phi\Lambda,\Lambda_0,\rho^0]\big|_{\perp\rho}\Big|_{\Lambda=\Lambda_0} . \end{aligned} \qquad (96)$$

Using (88) one gets for $m \leq r{+}1$

$$\|\partial^{\mu_1}_{i_1,j_1} \dots \partial^{\mu_d}_{i_d,j_d} V^{(r)}_{2m}(p_1,\dots,p_{2m};\Lambda_0)\big|_{\perp\rho}\| \leq \Lambda_0^{-d} P^{2r-m}\Big[\ln \frac{\Lambda_0}{\Lambda_R}\Big] . \qquad (97)$$

Next one sees that for the first non-vanishing function $V^{(2)}_6(p_1,\dots,p_6;\Lambda)$ the rhs of (85) is zero so that $\Lambda^{-2}V^{(2)}_6(p_1,\dots,p_6;\Lambda) = \text{const}$. With the initial condition (97) one obtains

$$\|\partial^{\mu_1}_{i_1,j_1} \dots \partial^{\mu_d}_{i_d,j_d} V^{(2)}_{6}(p_1,\dots,p_{2m};\Lambda)\big|\| \leq \frac{\Lambda^2}{\Lambda_0^2} \Lambda^{-d} P^{1}\Big[\ln \frac{\Lambda_0}{\Lambda_R}\Big] . \qquad (98)$$

Since (85) is a linear differential equation, the factor $\frac{\Lambda^2}{\Lambda_0^2}$ first appearing in (98) survives in all $V\big|_{\perp\rho}$ coefficients. □

Theorem 1. *The limit*

$$\lim_{\Lambda_0\to\infty} L^{(r)}_{2m}(p_1,\dots,p_{2m};\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]) := L^{(r)}_{2m}(p_1,\dots,p_{2m};\Lambda_R,\infty)$$

exists (order by order in perturbation theory) and satisfies

$$\begin{aligned} &\big\|L^{(r)}_{2m}(p_1,\dots,p_{2m};\Lambda_R,\infty) - L^{(r)}_{2m}(p_1,\dots,p_{2m};\Lambda_R,\Lambda_0,\rho^0[\Lambda_0])\big\| \\ &\qquad \begin{cases} \leq \Lambda_R^{4-2m}\big(\frac{\Lambda_R^2}{\Lambda_0^2}\big) P^{2r-m}\big[\ln \frac{\Lambda_R}{\Lambda_0}\big] & \textit{for } r{+}1 \geq m\ , \\ = 0 & \textit{for } r{+}1 < m\ . \end{cases} \end{aligned} \qquad (99)$$

Remarks on the proof. We reinsert the dimensional factors $L^{(r)}_{2m}[\Lambda_R] = \Lambda_R^{4-2m} A^{(r)}_{2m}[\Lambda_R]$. The existence of the limit and its property (99) follow from (94) inserted into (72) and Cauchy's criterion. □

Let us summarise what we have achieved. A quantum field theory is determined by an initial (classical) action S which gives rise to generating functionals $Z[j]$ for the n-point functions. Performing the integration of the generating functional gives meaningless results. Thus, one has to introduce a regularisation parameter ϵ and a mass scale μ and to fine-tune the initial action $S[\epsilon]$ in such a way that the limit $\epsilon \to 0$ for the n-point functions exists in perturbation theory. These renormalised n-point functions will now depend on μ. One possible regularisation is the momentum cut-off $p^2 \leq 2\Lambda_0^2$, where $\Lambda_0 = \mu/\epsilon$. Thus one would take a Λ_0-dependent initial action determined by the requirement that for n-point functions the limit $\Lambda_0 \to \infty$ exists in perturbation theory.

Here, a different approach is taken. We compare the cut-off theory at Λ_0 with another theory cut-off at Λ_R and require that the generating functionals of both theories coincide. This leads to a certain evolution of the initial interaction of the theory at Λ_0 to that of the other theory at Λ_R. The evolution is described by the Polchinski equation which is integrated from Λ_0 down to Λ_R. Integrating the differential equation requires the specification of initial conditions. It seems natural to take the given classical action at Λ_0 as initial condition. However, renormalisation requires a fine-tuning of the initial action at Λ_0, which gives certain parts of the initial condition at Λ_0 in terms of their normalised values at Λ_R.

In this way we obtain an effective action $L[\Lambda_R, \Lambda_0]$ for the theory at Λ_R which still depends on the initial cut-off scale Λ_0. At the end we want to send Λ_0 to ∞. It is then a rather long proof that the limit $\lim_{\Lambda_0 \to \infty} L[\Lambda_R, \Lambda_0]$ exists (in perturbation theory). However, the proof is technically very simple. All one needs are dimensional analysis and brutal majorisations, there is no need to evaluate Feynman graphs and to discuss overlapping divergences. One thus obtains a generating functional for *renormalised* n-point functions in which the propagator $(p^2 + m^2)^{-1} K(\frac{p^2}{\Lambda_R^2})$ cuts off momenta bigger than $\sqrt{2}\Lambda_R$. One still has to evaluate Feynman graphs in order to obtain the n-point functions. However, the loop momenta through the propagators are bounded so that there are no divergences any more in these n-point functions. The vertices in these Feynman graphs are given by the expansion coefficients of the effective action $L[\Lambda_R, \infty]$. In some sense, the effective action is obtained by integrating out in the partition function the fields with momenta bigger than $\sqrt{2}\Lambda_R$, avoiding the divergences by the mixed boundary conditions for the flow equation.

4 Quantum field theory on noncommutative geometries

4.1 Motivation

We have learned how the entropy term leads to quantum fluctuations about action functionals on Euclidean space E_4 and how to construct renormalised

n-point functions. For suitably chosen action functionals one achieves a remarkable agreement of up to 10^{-11} between theoretical predictions derived from these n-point functions and experimental data. This shows that quantum field theories are very successful.

Unfortunately, this concept is inconsistent when taking gravity into account. The problem can not be cured by just developing the quantum field theory on a Riemannian manifold with general metric $g_{\mu\nu}$. The true problem is that combining the fundamental principles of both general relativity and quantum mechanics one concludes that space(-time) cannot be a differentiable manifold [15]. To the best of our knowledge, such a possibility was first discussed in [16].

To make this transparent, let us ask how we explore technically the geometry of space(-time). The building blocks of a manifold are the *points* labelled by coordinates $\{x^\mu\}$ in a given chart. Points enter quantum field theory via the *values* of the fields at the point labelled by $\{x^\mu\}$. This observation provides a way to "visualise" the points: we have to prepare a distribution of matter which is sharply localised around $\{x^\mu\}$. For a perfect visualisation we need a δ-distribution of the matter field. This is physically not possible, but one would think that a δ-distribution could be arbitrarily well approximated. However, that is not the case, there are limits of localisability long before the δ-distribution is reached.

Let us assume that there is a matter distribution which is believed to have two separated peaks within a space-time region R of diameter d. How do we test this conjecture? We perform a scattering experiment in the hope of finding interferences which tell us about the internal structure in the region R. We clearly need test particles of de Broglie wave length $\lambda = \frac{\hbar c}{E} \lesssim \mathrm{d}$, otherwise we can only resolve a single peak. For $\lambda \to 0$ the gravitational field of the test particles becomes important. The gravitational field created by an energy E can be measured in terms of the Schwarzschild radius

$$r_s = \frac{2G_N E}{c^4} = \frac{2G_N \hbar}{\lambda c^3} \gtrsim \frac{2G_N \hbar}{\mathrm{d} c^3}\;, \tag{100}$$

where G_N is Newton's constant. If the Schwarzschild radius r_s becomes larger than the radius $\frac{\mathrm{d}}{2}$, the inner structure of the region R can no longer be resolved (it is behind the horizon). Thus, $\frac{\mathrm{d}}{2} \geq r_s$ leads to the condition

$$\frac{\mathrm{d}}{2} \gtrsim \ell_P := \sqrt{\frac{G_N \hbar}{c^3}}\;, \tag{101}$$

which means that the Planck length ℓ_P is the fundamental length scale below of which length measurements become meaningless. Space-time cannot be a manifold.

Since geometric concepts are indispensable in physics, we need a replacement for the space-time manifold which still has a geometric interpretation. Quantum physics tells us that whenever there are measurement limits we have to describe the situation by non-commuting operators on a Hilbert space. Fortunately for physics, mathematicians have developed a generalisation of geometry, baptised noncommutative geometry [17], which is perfectly designed for our purpose. However, in physics we need more than just a better geometry: We need renormalisable quantum field theories modelled on such a noncommutative geometry.

Remarkably, it turned out to be very difficult to renormalise quantum field theories even on the simplest noncommutative spaces [18]. It would be a wrong conclusion, however, that this problem singles out the standard commutative geometry as the only one compatible with quantum field theory. The problem tells us that we are still at the very beginning of *understanding* quantum field theory. Thus, apart from curing the contradiction between gravity and quantum physics, in doing quantum field theory on noncommutative geometries we learn a lot about quantum field theory itself.

4.2 The noncommutative $\mathbb{R}^D$

The simplest noncommutative generalisation of Euclidean space is the so-called noncommutative $\mathbb{R}^D$. Although this space arises naturally in a certain limit of string theory [19], we should not expect it to be a good model for nature. In particular, the noncommutative $\mathbb{R}^D$ does not allow for gravity. For us the main purpose of this space is to develop an understanding of quantum field theory which has a broader range of applicability.

The noncommutative $\mathbb{R}^D$, $D = 2, 4, 6, \dots$, is defined as the algebra $\mathbb{R}^D_\theta$ which as a vector space is given by the space $\mathcal{S}(\mathbb{R}^D)$ of (complex-valued) Schwartz class functions of rapid decay, equipped with the multiplication rule

$$(a \star b)(x) = \int \frac{d^D k}{(2\pi)^D} \int d^D y\, a(x+\tfrac{1}{2}\theta{\cdot}k)\, b(x+y)\, \mathrm{e}^{\mathrm{i}k\cdot y} \;, \qquad (102)$$
$$(\theta{\cdot}k)^\mu = \theta^{\mu\nu} k_\nu \;, \quad k{\cdot}y = k_\mu y^\mu \;, \quad \theta^{\mu\nu} = -\theta^{\nu\mu} \;.$$

The entries $\theta^{\mu\nu}$ in (102) have the dimension of an area. The physical interpretation is $\|\theta\| \approx \ell_P^2$. Much information about the noncommutative $\mathbb{R}^D$ can be found in [20].

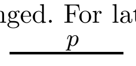
Exercise 13. Prove associativity $((a \star b) \star c)(x) = (a \star (b \star c))(x)$ of (102). Show that the product (102) is noncommutative, $a \star b \neq b \star a$ and that complex conjugation is an involution, $\overline{a \star b} = \overline{b} \star \overline{a}$. Show $\int d^D x\, (a \star b)(x) = \int d^D x\; a(x) b(x)$. Verify that partial derivatives are derivations, $\partial_\mu(a \star b) = (\partial_\mu a) \star b + a \star (\partial_\mu b)$. Hint: One often needs the identity $\int \frac{d^D k}{(2\pi)^D}\, \mathrm{e}^{\mathrm{i}k\cdot(x-y)} = \delta(x-y)$. ◁

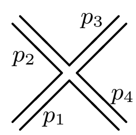
Exercise 14. The multiplier algebra $\mathcal{M}(\mathbb{R}^D_\theta)$ consists of the distributions f which satisfy $f \star a \in \mathbb{R}^D_\theta$ and $a \star f \in \mathbb{R}^D_\theta$, for all $a \in \mathbb{R}^D_\theta$ and the same $\star$-product (102). Verify that for the coordinate functions $y^\mu \in \mathcal{M}(\mathbb{R}^D_\theta)$, $y^\mu(x) := x^\mu$, one has $\big([y^\mu, y^\nu]_\star \star a\big)(x) := \big(y^\mu \star (y^\nu \star a) - y^\nu \star (y^\mu \star a)\big)(x) = \mathrm{i}\theta^{\mu\nu} a(x)$. ◁

4.3 Field theory on noncommutative $\mathbb{R}^D$

A field theory is defined by an action functional. We obtain action functionals on $\mathbb{R}^D_\theta$ by replacing in action functionals on E_D the ordinary product of functions by the $\star$-product. For example, the noncommutative ϕ^4-action is given by

$$S[\phi] := \int d^Dx\Big(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{1}{2}m^2\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)\,. \tag{103}$$

As described in Section 1, the entropy term leads to quantum fluctuations away from the minimum of (103). Expectation values are governed by the probability distribution which minimises the free action. A convenient way to organise the expectation values is the generating functional $Z[j]$ which is perturbatively solved by Feynman graphs. Due to $\int d^Dx\;(a\star b)(x) = \int d^Dx\;a(x)b(x)$, see Exercise 13, the propagator in momentum space is unchanged. For later purpose it is, however, convenient to write it as a double line,  $= (p^2+m^2)^{-1}$. The novelty are phase factors in the vertices, which we also write in double line notation,



$$= \frac{\lambda}{4!}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\sum_{i<j}p_i^\mu p_j^\nu\theta_{\mu\nu}}\,. \tag{104}$$

Exercise 15. Derive the Feynman rule (104) by repeating the steps leading to (38) for the interaction term in (103). ◁

The double line notation reflects the fact that the vertex (104) is invariant only under cyclic permutations of the legs (using momentum conservation). The resulting Feynman graphs are *ribbon graphs* which depend crucially on how the valences of the vertices are connected. For *planar graphs* the total phase factor of the integrand is independent of internal momenta, whereas *non-planar graphs* have a total phase factor which involves internal momenta. Planar graphs are integrated as usual. The resulting phase factor is precisely of the form of the original two-point function or vertex (104) so that the divergence can be removed via the normalisation conditions (43)–(45). Non-planar graphs require a separate treatment.

According to [21] there is a closed formula for the integral associated to a noncommutative Feynman graph in terms of the intersection matrices I, J, K which encode the phase factors and the incidence matrix $\mathcal{E}$. We give an orientation to each inner line l and let k_l be the momentum flowing through the line l. For each vertex v we define[4]

[4] We assume that tadpoles (a line starting and ending at the same vertex) are absent.

$$\mathcal{E}_{vl} = \begin{cases} 1 & \text{if } l \text{ leaves from } v\,, \\ -1 & \text{if } l \text{ arrives at } v\,, \\ 0 & \text{if } l \text{ is not attached to } v\,. \end{cases} \tag{105}$$

We let P_v be the total external momentum flowing into the vertex v. Restricting ourselves to 4 dimensions, an 1PI Feynman graph $\mathcal{G}$ with I internal lines and V vertices gives rise to the integral

$$\begin{aligned} \mathcal{I}_{\mathcal{G}} = \int \prod_{l=1}^{I} \frac{d^4k_l}{(k_l^2+m^2)} \prod_{v=1}^{V} (2\pi)^4 \delta\Big(P_v - \sum_{l=1}^{l} \mathcal{E}_{vl} k_l\Big) \\ \times \exp \mathrm{i}\theta_{\mu\nu}\Big(\sum_{m,n=1}^{I} I^{mn} k_m^\mu k_n^\nu + \sum_{m=1}^{I}\sum_{v=1}^{V} J^{mv} k_m^\mu P_v^\nu + \sum_{v,w=1}^{V} K^{vw} P_v^\mu P_w^\nu \Big)\,. \end{aligned} \tag{106}$$

One can show that $I^{mn}, J^{mv}, K^{vw} \in \{1,-1,0\}$ after use of momentum conservation [22]. In terms of Schwinger parameters, this integral is evaluated to

$$\begin{aligned} \mathcal{I}_{\mathcal{G}} = (2\pi)^4 \delta\Big(\sum_{v=1}^{V} P_v\Big) \frac{1}{16^I \pi^{2L}} \exp\Big(\mathrm{i}\theta_{\mu\nu} \sum_{v,w=1}^{V} K^{vw} P_v^\mu P_w^\nu\Big) \\ \times \int_0^\infty \prod_{l=1}^{I} d\alpha_l \, \frac{\mathrm{e}^{-\sum_{l=1}^{I} \alpha_l m^2}}{\sqrt{\det \mathcal{A} \det \mathcal{B}}} \exp\Big(-\frac{1}{4} (J\tilde{P})^T \mathcal{A}^{-1} (J\tilde{P}) \\ + \frac{1}{4} \big(\bar{\mathcal{E}} \mathcal{A}^{-1} (J\tilde{P}) + 2\mathrm{i}P'\big)^T \mathcal{B}^{-1} \big(\bar{\mathcal{E}} \mathcal{A}^{-1} (J\tilde{P}) + 2\mathrm{i}P'\big)\Big)\,, \end{aligned} \tag{107}$$

where

$$\begin{aligned} \mathcal{A}^{mn}_{\mu\nu} &:= \alpha_m \delta^{mn} \delta_{\mu\nu} - \mathrm{i} I^{mn} \theta_{\mu\nu}\,, \qquad & (J\tilde{P})^m_\mu &:= \sum_{v=1}^{V} J^{mv} \theta_{\mu\nu} P_v^\nu\,, \\ \bar{\mathcal{E}}^{\bar{v}l} &:= \mathcal{E}_{\bar{v}l}\,, & P'^{\bar{v}}_\mu &:= P^\mu_{\bar{v}} \quad \text{for } \bar{v} = 1,\dots,V-1\,, \\ \mathcal{B}^{\bar{v}\bar{w}}_{\mu\nu} &:= \sum_{m,n=1}^{I} \bar{\mathcal{E}}^{\bar{v}m} (\mathcal{A}^{-1})^{\mu\nu}_{mn} \bar{\mathcal{E}}^{\bar{w}n}\,. \end{aligned} \tag{108}$$

The formula (107) is referred to as the parametric integral representation of a noncommutative Feynman graph.

Exercise 16. Verify (107). Hint: First introduce Schwinger parameters and the identity $\delta(q_v) = \int d^4y_v \, \mathrm{e}^{\mathrm{i}y_v q_v}$ for each vertex in (106). Complete the squares in k and perform the Gaußian k-integrations. Write $y_{\bar{v}} = y_V + z_{\bar{v}}$ for $\bar{v} = 1,\dots,V-1$ and notice that $\sum_{v=1}^{V} y_v \mathcal{E}_{vl} = \sum_{\bar{v}=1}^{V-1} z_{\bar{v}} \bar{\mathcal{E}}^{\bar{v}l}$. Then perform the y_V-integration, complete the squares for $z_{\bar{v}}$ and finally evaluate the Gaußian z_v-integrations. ◁

Possible divergences of (107) show up in the $\alpha_i \to 0$ behaviour. In order to analyse them one reparametrises the integration domain in (107), similar to the usual procedure described in [2]. For each sector

$$\alpha_{\pi_1} \leq \alpha_{\pi_2} \leq \cdots \leq \alpha_{\pi_I} \qquad \text{related to a permutation } \pi \text{ of } 1,\dots,I \tag{109}$$

one defines $\alpha_{\pi_i} = \prod_{j=i}^{I} \beta_j^2$, with $0 \leq \beta_I < \infty$ and $0 \leq \beta_j \leq 1$ for $j \neq I$. The leading contribution for small β_j has a topological interpretation.

A ribbon graph can be drawn on a genus-g Riemann surface with possibly several holes to which the external legs are attached [21, 23]. We say more on ribbon graphs on Riemann surfaces in Section 5. We explain, in particular, how a ribbon graph $\mathcal{G}$ defines a Riemann surface. On such a Riemann surface one considers *cycles*, i.e. equivalence classes of closed paths which cannot be contracted to a point. Actually one also factorises with respect to commutants, i.e. one considers the path $aba^{-1}b^{-1}$ involving two cycles a, b as trivial. We let $c_{\mathcal{G}}(\mathcal{G}_i)$ be the number of non-trivial cycles of the ribbon graph $\mathcal{G}$ wrapped by the subgraph $\mathcal{G}_i$. Next, there may exist external lines m, n such that the graph obtained by connecting m, n has to be drawn on a Riemann surface of genus $g_{mn} > g$. If this happens one declares an index $j(\mathcal{G}) = 1$, otherwise $j(\mathcal{G}) = 0$. The index extends to subgraphs by defining $j_{\mathcal{G}}(\mathcal{G}_i) = 1$ if there are external lines m, n of $\mathcal{G}$ which are already attached to $\mathcal{G}_i$ so that the line connecting m, n wraps a cycle of the additional genus $g \to g_{mn}$ of $\mathcal{G}$.

Now we can formulate the relation between the parametric integral representation and the topology of the ribbon graph. Each sector (109) of the α-parameters defines a sequence of (possibly disconnected) subgraphs $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset \mathcal{G}_I = \mathcal{G}$, where $\mathcal{G}_i$ is made of the i double-lines $\pi_1, \dots, \pi_i$ and the vertices to which these lines are attached. If $\mathcal{G}_i$ forms L_i loops it has a power-counting degree of divergence $\omega_i = 4L_i - 2i$. Then one has

$$\begin{aligned} \mathcal{I}_{\mathcal{G}} = (2\pi)^4 \delta\Big(\sum_{v=1}^{V} P_v\Big) \frac{1}{8^I \pi^{2L} (\det\theta)^g} \exp\Big(\mathrm{i}\theta_{\mu\nu} \sum_{v,w=1}^{V} K^{vw} P_v^\mu P_w^\nu\Big) \\ \times \int_0^\infty \frac{d\beta_I \, \mathrm{e}^{-\beta_I^2 m^2}}{\beta_I^{1+\omega_I - 4c_{\mathcal{G}}(\mathcal{G})}} \int_0^1 \Big(\prod_{i=1}^{I-1} \frac{d\beta_i}{\beta_i^{1+\omega_i-4c_{\mathcal{G}}(\mathcal{G}_i)}}\Big) \\ \times \exp\Big(-f(\pi,P) \prod_{i=1}^{I} \frac{1}{\beta^{2j_{\mathcal{G}}(\mathcal{G}_i)}}\Big)\Big(1 + \mathcal{O}(\beta^2)\Big) , \end{aligned} \tag{110}$$

where $f(\pi, P) \geq 0$, with equality for exceptional momenta. The (very complicated) proof of (110) was given by Chepelev and Roiban [21, 23]. In order to obtain a finite integral $\mathcal{I}_{\mathcal{G}}$, we obviously need

1. $\omega_i - 4c_{\mathcal{G}}(\mathcal{G}_i) < 0$ for all i if $j(\mathcal{G}) = 0$ or $j(\mathcal{G}) = 1$ but the external momenta are exceptional,

2. $\omega_i - 4c_{\mathcal{G}}(\mathcal{G}_i) < 0$ or $j_{\mathcal{G}}(\mathcal{G}_i) = 1$ for all i if $j(\mathcal{G}) = 1$ and the external momenta are non-exceptional.

There are two types of divergences for which these conditions are violated.

First let the non-planarity be due to internal lines only, $j(\mathcal{G}) = 0$. Since the total graph $\mathcal{G}$ is non-planar, one has $c_{\mathcal{G}}(\mathcal{G}) > 0$ and therefore no superficial divergence. However, there might exist subgraphs $\mathcal{G}_i$ related to a sector of integration (109) where $\omega_i - 4c_{\mathcal{G}}(\mathcal{G}_i) \geq 0$. The standard example is a subgraph consisting of three or more *disconnected* loops wrapping the same handle of the Riemann surface. In this case the integral (107) does not exist unless one introduces a regulator. The problem is that such a subdivergence may appear in graphs with an arbitrary number of external lines. In the commutative theory this also happens, but there we renormalise already the subdivergence via the procedure described in Section 2.4. This procedure is based on normalisation conditions, which can only be imposed for *local* divergences. Since a non-planar graph wrapping a handle of a Riemann surface is clearly a non-local object (it cannot be reduced to a point, i.e. a counterterm vertex), it is not possible in the noncommutative case to remove that subdivergence. We are thus forced to use normalisation conditions for the total graph, but as the problem is independent of the number of external legs of the total graph, we finally need an infinite number of normalisation conditions. Hence, the model is not renormalisable in the standard way. The proposal to treat this problem is a reordering of the perturbation series [18].

The second class of problems is found in graphs where the non-planarity is at least partly due to the external legs, $j(\mathcal{G}) = 1$. This means that there is no way to remove possible divergences in these graphs by normalisation conditions. Fortunately, these graphs are superficially finite as long as the external momenta are non-exceptional. Subdivergences are supposed to be treated by a resummation. However, since the non-exceptional external momenta can become arbitrarily close to exceptional ones, these graphs are unbounded: For every $\delta > 0$ one finds non-exceptional momenta such that $\big|\langle\phi(x_1)\dots\phi(x_n)\rangle\big| > \delta^{-1}$. We present in Section 5 a different approach which solves all these problems.

5 Renormalisation group approach to noncommutative scalar models

5.1 Introduction

We have seen that quantum field theories on noncommutative $\mathbb{R}^D$ are not renormalisable by standard Feynman graph evaluations. One may speculate that the origin of this problem is the too naïve way one performs the continuum limit. A way to treat the limit more carefully is the use of flow equations. We can therefore hope that applying Polchinski's method to the noncommutative ϕ^4-model we are able to prove renormalisability to all orders. There is, however, a serious problem of the momentum space proof. We have to guarantee that planar graphs only appear in the distinguished interaction coefficients for which we fix the boundary condition at Λ_R. Non-planar graphs have phase factors which involve inner momenta. Polchinski's method consists in taking norms of the interaction coefficients, and these norms ignore possible phase factors. Thus, we would find that boundary conditions for non-planar graphs at Λ_R are required. Since there is an infinite number of different non-planar structures, the model is not renormalisable in this way. A more careful examination of the phase factors is also not possible because the cut-off integrals prevent the Gaußian integration required for the parametric integral representation (107).

Fortunately, there is a matrix representation of the noncommutative $\mathbb{R}^D$ where the $\star$-product becomes a simple product of infinite matrices. The price for this simplification is that the propagator becomes complicated, but the difficulties can be overcome.

5.2 Matrix representation

For simplicity we restrict ourselves to the noncommutative $\mathbb{R}^2$. There exists a matrix base $\{f_{mn}(x)\}_{m,n\in\mathbb{N}}$ of the noncommutative $\mathbb{R}^2$ which satisfies

$$(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x) \;, \qquad \int d^2x \; f_{mn}(x) = 2\pi\theta_1 \;, \tag{111}$$

where $\theta_1 := \theta_{12} = -\theta_{21}$. In terms of radial coordinates $x_1 = \rho\cos\varphi,\ x_2 = \rho\sin\varphi$ one has

$$f_{mn}(\rho,\varphi) = 2(-1)^m \mathrm{e}^{\mathrm{i}(n-m)\varphi} \sqrt{\tfrac{m!}{n!}} \Big(\sqrt{\tfrac{2\rho^2}{\theta_1}}\Big)^{n-m} L_m^{n-m}\Big(\tfrac{2\rho^2}{\theta_1}\Big)\, \mathrm{e}^{-\frac{\rho^2}{\theta_1}} \;, \tag{112}$$

where $L_n^\alpha(z)$ are the Laguerre polynomials. The matrix representation was also used to obtain exactly solvable noncommutative quantum field theories [26, 27].

Exercise 17. Prove (111). [If you have a table of special functions you can also prove (112)]. Hint: First define $f_{00}(x_1,x_2) := 2\,\mathrm{e}^{-\frac{(x_1^2+x_2^2)}{\theta_1}}$ and check $f_{00}\star f_{00} = f_{00}$. Define creation and annihilation operators $a = \frac{1}{\sqrt{2}}(x_1+\mathrm{i}x_2)$ and $\bar{a} = \frac{1}{\sqrt{2}}(x_1-\mathrm{i}x_2)$ and the corresponding derivatives $\frac{\partial}{\partial a} = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} - \mathrm{i}\frac{\partial}{\partial x_2})$ and $\frac{\partial}{\partial \bar{a}} = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_2})$. Derive general rules for $a\star f,\ f\star a,\ \bar{a}\star f,\ f\star\bar{a}$ and prove that $f_{mn} = \frac{1}{\sqrt{m!n!\theta^{m+n}}}\bar{a}^{\star m}\star f_{00}\star a^{\star n}$ with $b^{\star n} := b\star b^{\star(n-1)}$ satisfies (111). In order to obtain (112) one has to resolve the $\star$-product in favour of an ordinary product, pass to radial coordinates and compare the result with the definition of Laguerre polynomials. ◁

Now we can write down the noncommutative ϕ^4-action in the matrix base by expanding the field as $\phi(x) = \sum_{m,n\in\mathbb{N}} \phi_{mn} f_{mn}(x)$. It turns out, however, that in order to prove renormalisability we have to consider a more general action than (103) at the initial scale Λ_0. This action is obtained by adding a harmonic oscillator potential to the standard noncommutative ϕ^4-action:

$$\begin{aligned}S[\phi] :=& \int d^2x \Big(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{1}{2}\Omega^2(\tilde{x}^\mu\phi)\star(\tilde{x}_\mu\phi) + \frac{1}{2}\mu_0^2\phi\star\phi \\ &\qquad + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x) \\ =& 2\pi\theta_1 \sum_{m,n,k,l}\Big(\frac{1}{2}G_{mn;kl}\phi_{mn}\phi_{kl} + \frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\Big)\,, \end{aligned} \tag{113}$$

where $\tilde{x}^\mu := 2(\theta^{-1})^{\mu\nu}x_\nu$ and

$$G_{mn;kl} := \int \frac{d^2x}{2\pi\theta_1}\,\Big(\partial_\mu f_{mn}\star\partial^\mu f_{kl} + \Omega^2(\tilde{x}^\mu f_{mn})\star(\tilde{x}_\mu f_{kl}) + \mu_0^2 f_{mn}\star f_{kl}\Big)\,. \tag{114}$$

We view Ω as a regulator and refer to the action (113) as describing a regularised ϕ^4-model. One finds

$$\begin{aligned}G_{mn;kl} =& \big(\mu_0^2+(n+m+1)\mu^2\big)\delta_{nk}\delta_{ml} \\ &- \mu^2\sqrt{\omega\,(n+1)(m+1)}\,\delta_{n+1,k}\delta_{m+1,l} - \mu^2\sqrt{\omega\,nm}\,\delta_{n-1,k}\delta_{m-1,l}\,,\end{aligned} \tag{115}$$

where $\mu^2 = \frac{2(1+\Omega^2)}{\theta_1}$ and $\sqrt{\omega} = \frac{1-\Omega^2}{1+\Omega^2}$, with $-1 < \sqrt{\omega} \leq 1$.

Exercise 18. Prove (115) using the formulae derived in Exercise 17. $\triangleleft$

The kinetic matrix $G_{mn;kl}$ has the important property that $G_{mn;kl} = 0$ unless $m+k = n+l$. The same relation is induced for the propagator $\Delta_{nm;lk}$ defined by $\sum_{k,l=0}^{\infty} G_{mn;kl}\Delta_{lk;sr} = \sum_{k,l=0}^{\infty}\Delta_{nm;lk}G_{kl;rs} = \delta_{mr}\delta_{ns}$:

$$\begin{aligned}\Delta_{mn;kl} =& \frac{\delta_{m+k,n+l}}{(1+\sqrt{1-\omega})\mu^2}\sum_{v=\frac{|m-l|}{2}}^{\frac{\min(m+l,k+n)}{2}} B\big(\tfrac{1}{2}+\tfrac{\mu_0^2}{2\sqrt{1-\omega}\,\mu^2}+\tfrac{1}{2}(m+k)-v,\,1+2v\big) \\ &\times\sqrt{\binom{n}{v+\frac{n-k}{2}}\binom{k}{v+\frac{k-n}{2}}\binom{m}{v+\frac{m-l}{2}}\binom{l}{v+\frac{l-m}{2}}}\Big(\frac{(1-\sqrt{1-\omega})^2}{\omega}\Big)^v \\ &\times {}_2F_1\left(\begin{matrix}1+2v\,,\ \frac{1}{2}+\frac{\mu_0^2}{2\sqrt{1-\omega}\,\mu^2}-\frac{1}{2}(m+k)+v \\ \frac{3}{2}+\frac{\mu_0^2}{2\sqrt{1-\omega}\,\mu^2}+\frac{1}{2}(m+k)+v\end{matrix}\,\middle|\,\frac{(1-\sqrt{1-\omega})^2}{\omega}\right)\,.\end{aligned} \tag{116}$$

Here, $B(a,b)$ is the Beta-function and $F\big({a,b \atop c};z\big)$ the hypergeometric function. The derivation of (116), which is performed in [28], involves Meixner polynomials [29] in a crucial way. We recall that in the momentum space version of the ϕ^4-model, the interactions contain oscillating phase factors which make a renormalisation by flow equations impossible. Here we use an adapted base which eliminates the phase factors from the interaction. We see from (116) that these oscillations do not reappear in the propagator. Note that all matrix elements $\Delta_{nm;lk}$ are non-zero for $m+k = n+l$.

5.3 The Polchinski equation for matrix models

Introducing a cut-off for the matrix indices

$$\Delta^K_{nm;lk}(\Lambda) = K\big(\tfrac{m\mu^2}{\Lambda^2}\big)K\big(\tfrac{n\mu^2}{\Lambda^2}\big)K\big(\tfrac{k\mu^2}{\Lambda^2}\big)K\big(\tfrac{l\mu^2}{\Lambda^2}\big)\Delta_{nm;lk}\;, \tag{117}$$

for the same function K as in (55), one can derive in analogy to (63) the Polchinski equation in the matrix base of $\mathbb{R}^2_\theta$:

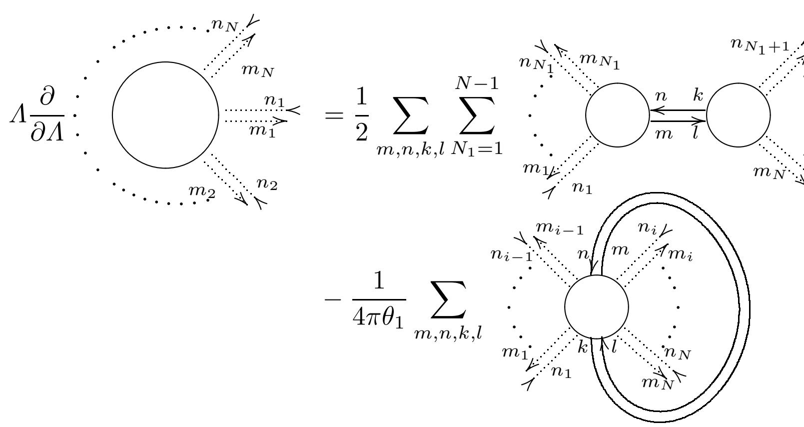
$$\Lambda\frac{\partial L[\phi,\Lambda]}{\partial\Lambda} = \sum_{m,n,k,l}\frac{1}{2}\Lambda\frac{\partial\Delta^K_{nm;lk}(\Lambda)}{\partial\Lambda}\bigg(\frac{\partial L[\phi,\Lambda]}{\partial\phi_{mn}}\frac{\partial L[\phi,\Lambda]}{\partial\phi_{kl}} - \frac{1}{2\pi\theta_1}\Big[\frac{\partial^2 L[\phi,\Lambda]}{\partial\phi_{mn}\,\partial\phi_{kl}}\Big]_\phi\bigg)\;. \tag{118}$$

Again, the differential equation (118) ensures (together with easier differential equations for functions such as C) that the partition function $Z[J,\Lambda]$ is actually independent of the cut-off Λ. This means that we can equally well evaluate the partition function for finite Λ, where it leads to Feynman graphs with vertices given by the Taylor expansion coefficients $A^{(V)}_{m_1n_1;\dots;m_Nn_N}$ in

$$L[\phi,\Lambda] = \lambda\sum_{V=1}^\infty\big(2\pi\theta_1\lambda\big)^{V-1}\sum_{N=2}^\infty\frac{1}{N!}\sum_{m_i,n_i}A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\phi_{m_1n_1}\cdots\phi_{m_Nn_N}\;. \tag{119}$$

These vertices are connected with each other by internal lines $\Delta^K_{nm;lk}(\Lambda)$ and to sources j_{kl} by external lines $\Delta^K_{nm;lk}(\Lambda_0)$. Since the summation variables are cut-off in the propagator $\Delta^K_{nm;lk}(\Lambda)$, loop summations are finite, provided that the interaction coefficients $A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ are finite.

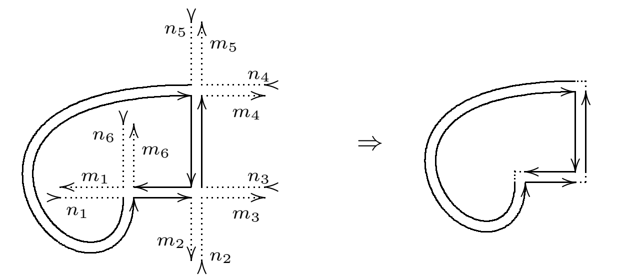
Inserting the expansion (119) into (118) and restricting to the part with N external legs we get the graphical expression



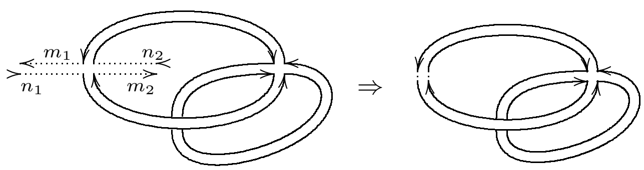
(120)

Combinatorial factors are not shown and symmetrisation in all indices m_in_i has to be performed. On the rhs of (120) the two valences mn and kl of subgraphs are connected to the ends of a *ribbon* which symbolises the differentiated propagator $\overset{n \qquad k}{\underset{m \qquad l}{\overleftarrow{\underrightarrow{\qquad\quad}}}} = \Lambda\frac{\partial}{\partial\Lambda}\Delta^K_{nm;lk}$. We see that for the simple fact that the fields ϕ_{mn} carry two indices, the effective action is expanded into ribbon graphs.

In the expansion of L there will occur very complicated ribbon graphs with crossings of lines which cannot be drawn any more in a plane. A general ribbon graph can, however, be drawn on a *Riemann surface* of some *genus* g. In fact, a ribbon graph *defines* the Riemann surfaces topologically through the *Euler characteristic* χ. We have to regard here the external lines of the ribbon graph as amputated (or closed), which means to directly connect the single lines m_i with n_i for each external leg $m_i n_i$. A few examples may help to understand this procedure:



$\tilde{L} = 2$
$I = 3$
$V = 3$
$g = 0$
$B = 2$
$N = 6$



$\tilde{L} = 1$
$I = 3$
$V = 2$
$g = 1$
$B = 1$
$N = 2$
(121)

The genus is computed from the number $\tilde{L}$ of single-line loops, the number I of internal (double) lines and the number V of vertices of the graph according to Euler's formula $\chi = 2 - 2g = \tilde{L} - I + V$. The number B of boundary components of a ribbon graph is the number of those loops which carry at least one external leg. There can be several possibilities to draw the graph and its Riemann surface, but $\tilde{L}, I, V, B$ and thus g remain unchanged. Indeed, the Polchinski equation (118) interpreted as in (120) tells us which external legs of the vertices are connected. It is completely irrelevant how the ribbons are drawn between these legs. In particular, there is no distinction between overcrossings and undercrossings.

We expect that non-planar ribbon graphs with $g > 0$ and/or $B > 1$ behave differently under the renormalisation flow than planar graphs having $B = 1$ and $g = 0$. This suggests to introduce a further grading in g, B in the interactions coefficients $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}$. Technically, our strategy is to apply the summations in (120) either to the propagator or the subgraph only and to maximise the other object over the summation indices. For that purpose one has to introduce further characterisations of a ribbon graph which disappear at the end, see [24].

5.4 ϕ^4-theory on noncommutative $\mathbb{R}^2$

First one estimates the A-functions by solving (118) perturbatively:

Lemma 4. *The homogeneous parts $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}$ of the coefficients of the effective action describing a regularised ϕ^4-theory on $\mathbb{R}^2_\theta$ in the matrix base are for $2 \leq N \leq 2V{+}2$ and $\sum_{i=1}^N (m_i - n_i) = 0$ bounded by*

$$\begin{aligned}&\big|A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\omega,\rho_0]\big| \\ &\quad \leq \Big(\frac{\Lambda^2}{\mu^2}\Big)^{2-V-B-2g}\Big(\frac{1}{\sqrt{1-\omega}}\Big)^{3V-\frac{N}{2}+B+2g-2} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,. \qquad (122)\end{aligned}$$

We have $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N} \equiv 0$ for $N > 2V{+}2$ or $\sum_{i=1}^N (m_i - n_i) \neq 0$.

The proof of (122) for general matrix models by induction goes over 20 pages! The formula specific for the ϕ^4-model on $\mathbb{R}^2_\theta$ follows from the asymptotic behaviour of the cut-off propagator (117,116) and a certain index summation, see [24, 25].

We see from (122) that the only divergent function is

$$\begin{aligned}A^{(1,1,0)}_{m_1n_1;m_2n_2} &= A^{(1,1,0)}_{00;00}\delta_{m_1n_2}\delta_{m_2n_1} \\ &\quad + \Big(A^{(1,1,0)}_{m_1n_1;m_2n_2}[\Lambda,\Lambda_0,\rho^0] - A^{(1,1,0)}_{00;00}\delta_{m_1n_2}\delta_{m_2n_1}\Big)\,, \qquad (123)\end{aligned}$$

which is split into the distinguished divergent function

$$\rho[\Lambda,\Lambda_0,\rho^0] := A^{(1,1,0)}_{00;00}[\Lambda,\Lambda_0,\rho^0] \qquad (124)$$

for which we impose the boundary condition $\rho[\Lambda_R,\Lambda_0,\rho^0] = 0$ and a convergent part with boundary condition at Λ_0.

One remarks that the limit $\omega \to 1$ in (122) is singular. In fact the estimation for $\omega = 1$ with an optimal choice of the ρ-coefficients (different than (124)!) would be

$$\begin{aligned}&\sum_{\mathcal{E}^s}\big|A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,1,\rho^0]\big| \\ &\quad \leq \Big(\frac{\Lambda}{\mu}\Big)^{V-\frac{N}{2}-B-2g+2}\Big(\frac{\mu}{\mu_0}\Big)^{3V-\frac{N}{2}+B+2g-2} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,. \qquad (125)\end{aligned}$$

Since the exponent of Λ can be arbitrarily large, there would be an infinite number of divergent interaction coefficients, which means that the ϕ^4-model is not renormalisable when keeping $\omega = 1$.

The limit $\Lambda_0 \to \infty$ is now governed by an identity like (72) and a ρ-subtracted function like (73) for which one has a differential equation like (78). It is then not difficult to see that the regularised ϕ^4-model with $\omega < 1$ is renormalisable. It turns out that one can even prove more [25]: On can endow the parameter ω for the oscillator frequency with an Λ_0-dependence so that in the limit $\Lambda_0 \to \infty$ one obtains a standard ϕ^4-model without the oscillator term:

Theorem 2. *The ϕ^4-model on $\mathbb{R}^2_\theta$ is (order by order in the coupling constant) renormalisable in the matrix base by adjusting the bare mass $\Lambda_0^2\rho[\Lambda_0]$ to give $A^{(1,1,0)}_{m_1n_1;m_2n_2}[\Lambda_R] = 0$ and by performing the limit $\Lambda_0 \to \infty$ along the path of regulated models characterised by $\omega[\Lambda_0] = 1 - (1 + \ln \frac{\Lambda_0}{\Lambda_R})^{-2}$. The limit $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty] := \lim_{\Lambda_0\to\infty} A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\Lambda_0,\omega[\Lambda_0],\rho^0[\Lambda_0]]$ of the expansion coefficients of the effective action $L[\phi,\Lambda_R,\Lambda_0,\omega[\Lambda_0],\rho^0[\Lambda_0]]$ exists and satisfies*

$$\Big|\lambda\big(2\pi\theta_1\lambda\big)^{V-1} A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty] - \big(2\pi\theta_1\lambda\big)^{V-1} A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}\big[\Lambda_R,\Lambda_0,1-\tfrac{1}{(1+\ln\frac{\Lambda_0}{\Lambda_R})^2},\rho^0\big]\Big| \leq \frac{\Lambda_R^4}{\Lambda_0^2}\Big(\frac{\lambda}{\Lambda_R^2}\Big)^V \Big(\frac{\mu^2(1+\ln\frac{\Lambda_0}{\Lambda_R})}{\Lambda_R^2}\Big)^{B+2g-1} P^{5V-N-1}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,. \tag{126}$$

In this way we have proven that the real ϕ^4-model on $\mathbb{R}^2_\theta$ is perturbatively renormalisable when formulated in the matrix base. This proof was not simply a generalisation of Polchinski's original proof to the noncommutative case. The naïve procedure would be to take the standard ϕ^4-action at the initial scale Λ_0, with Λ_0-dependent bare mass to be adjusted such that at Λ_R it is scaled down to the renormalised mass. Unfortunately, this does not work. In the limit $\Lambda_0 \to \infty$ one obtains an unbounded power-counting degree of divergence for the ribbon graphs. The solution is the observation that the cut-off action at Λ_0 is (due to the cut-off) not translation invariant. We are therefore free to break the translational symmetry of the action at Λ_0 even more by adding a harmonic oscillator potential for the fields ϕ. There exists a Λ_0-dependence of the oscillator frequency Ω with $\lim_{\Lambda_0\to\infty}\Omega = 0$ such that the effective action at Λ_R is convergent (and thus bounded) order by order in the coupling constant in the limit $\Lambda_0 \to \infty$. This means that the partition function of the original (translation-invariant) ϕ^4-model without cut-off and with suitable divergent bare mass can equally well be solved by Feynman graphs with propagators cut-off at Λ_R and vertices given by the bounded expansion coefficients of the effective action at Λ_R. Hence, this model is renormalisable, and in contrast to the naïve Feynman graph approach in momentum space [23] there is no problem with exceptional configurations. Whereas the treatment of the oscillator potential is easy in the matrix base, a similar procedure in momentum space will face enormous difficulties. This makes clear that the adaptation of Polchinski's renormalisation programme is the preferred method for noncommutative field theories.

5.5 ϕ^4-theory on noncommutative $\mathbb{R}^4$

The renormalisation of ϕ^4-theory on $\mathbb{R}^4_\theta$ in the matrix base is performed in an analogous way. We choose a coordinate system in which $\theta_1 = \theta_{12} = -\theta_{21}$ and $\theta_2 = \theta_{34} = -\theta_{43}$ are the only non-vanishing components of θ. Moreover, we assume $\theta_1 = \theta_2$ for simplicity. Then we expand the scalar field according to $\phi(x) = \sum_{m_1,n_1,m_2,n_2\in\mathbb{N}} \phi_{\substack{m_1\, n_1\\ m_2\, n_2}} f_{m_1n_1}(x_1,x_2) f_{m_2n_2}(x_3,x_4)$. The action (113) with integration over $\mathbb{R}^4$ leads then to a kinetic term generalising (115) and a propagator generalising (116). Using estimates on the asymptotic behaviour of that propagator one proves the four-dimensional generalisation of Lemma 4 on the power-counting degree of the N-point functions. For $\omega < 1$ one finds that all non-planar graphs ($B > 1$ and/or $g > 0$) and all graphs with $N \geq 6$ external legs are convergent.

The remaining infinitely many planar two- and four-point functions have to be split into a divergent ρ-part and a convergent complement. Using some sort of locality for the propagator (116) one proves that

$$
\begin{aligned}
A^{\text{planar}}_{\substack{m_1\, n_1;\, k_1\, l_1\\ m_2\, n_2;\, k_2\, l_2}} &- A^{\text{planar}}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0}} \delta_{m_1l_1}\delta_{n_1k_1}\delta_{m_2l_2}\delta_{n_2k_2} \\
&- m_1\Big(A^{\text{planar}}_{\substack{1\,0;\,0\,1\\ 0\,0;\,0\,0}} - A^{\text{planar}}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0}}\Big)\delta_{m_1l_1}\delta_{n_1k_1}\delta_{m_2l_2}\delta_{n_2k_2} \\
&- m_2\Big(A^{\text{planar}}_{\substack{0\,0;\,0\,0\\ 1\,0;\,0\,1}} - A^{\text{planar}}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0}}\Big)\delta_{m_1l_1}\delta_{n_1k_1}\delta_{m_2l_2}\delta_{n_2k_2} \\
&- A^{\text{planar}}_{\substack{1\,1;\,0\,0\\ 0\,0;\,0\,0}}\big(\sqrt{(m_1{+}1)(n_1{+}1)}\delta_{m_1+1,l_1}\delta_{n_1+1,k_1}\delta_{m_2l_2}\delta_{n_2k_2} \\
&\qquad\qquad + \sqrt{m_1n_1}\delta_{m_1-1,l_1}\delta_{n_1-1,k_1}\delta_{m_2l_2}\delta_{n_2k_2}\big) \\
&- A^{\text{planar}}_{\substack{0\,0;\,0\,0\\ 1\,1;\,0\,0}}\big(\sqrt{(m_2{+}1)(n_2{+}1)}\delta_{m_2+1,l_2}\delta_{n_2+1,k_2}\delta_{m_1l_1}\delta_{n_1k_1} \\
&\qquad\qquad + \sqrt{m_2n_2}\delta_{m_2-1,2_1}\delta_{n_2-1,k_2}\delta_{m_1l_1}\delta_{n_1k_1}\big)\,, \qquad (127)
\end{aligned}
$$

$$
A^{\text{planar}}_{\substack{m_1\, n_1;\ldots;\, m_4\, n_4\\ m_1'\, n_1';\ldots;\, m_4'\, n_4'}} - A^{\text{planar}}_{\substack{0\,0;\ldots;\,0\,0\\ 0\,0;\ldots;\,0\,0}}\big(\tfrac{1}{6}\delta_{\substack{n_1\, m_2\\ n_1'\, m_2'}}\delta_{\substack{n_2\, m_3\\ n_2'\, m_4'}}\delta_{\substack{n_3\, m_4\\ n_3'\, m_4'}}\delta_{\substack{n_4\, m_1\\ n_4'\, m_1'}} + 5\text{ perm's}\big)\,, \qquad (128)
$$

are convergent functions, thus identifying

$$
\begin{aligned}
\rho_1 &:= A^{\text{planar}}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0}}\,, \\
\rho_2 &:= A^{\text{planar}}_{\substack{1\,0;\,0\,1\\ 0\,0;\,0\,0}} - A^{\text{planar}}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0}} = A^{\text{planar}}_{\substack{0\,0;\,0\,0\\ 1\,0;\,0\,1}} - A^{\text{planar}}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0}}\,, \\
\rho_0 &:= A^{\text{planar}}_{\substack{1\,1;\,0\,0\\ 0\,0;\,0\,0}} = A^{\text{planar}}_{\substack{0\,0;\,0\,0\\ 1\,1;\,0\,0}} \\
\rho_3 &:= A^{\text{planar}}_{\substack{0\,0;\,0\,0;\,0\,0;\,0\,0\\ 0\,0;\,0\,0;\,0\,0;\,0\,0}} \qquad (129)
\end{aligned}
$$

as the distinguished divergent ρ-functions. Details are given in [28].

The function ρ_0 has no commutative analogue in (66). Due to (127) it corresponds to a normalisation condition for the frequency parameter ω in (115). This means that in contrast to the two-dimensional case we cannot remove the oscillator potential with the limit $\Lambda_0 \to \infty$. In other words, the oscillator potential in (113) is a necessary companionship to the $\star$-product interaction. This observation is in agreement with the UV/IR-entanglement first observed in [18]. Whereas the UV/IR-problem prevents the renormalisation of ϕ^4-theory on $\mathbb{R}^4_\theta$ in momentum space [23], we have found a self-consistent solution of the problem by providing the unique (due to properties of the Meixner polynomials) renormalisable extension of the action. We remark that the diagonalisation of the free action via the Meixner polynomials leads to discrete momenta as the only difference to the commutative case. The inverse of such a momentum quantum can be interpreted as the size of the (finite!) universe, as it is seriously discussed in cosmology [30].

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