Renormalisation of noncommutative ϕ^4_4-theory to all orders

— Habilitationsschrift —

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Summary

The renormalisation of noncommutative quantum field theories was an open problem for a long time due to the mixing of ultraviolet and infrared divergences. In this Habilitation thesis, I prove that the real ϕ^4-model on the four-dimensional Euclidean Moyal plane is renormalisable to all orders in perturbation theory. It turns out that—compared with the commutative case—the bare action of relevant and marginal couplings contains necessarily an additional term: an harmonic oscillator potential for the free scalar field action. This entails a modified dispersion relation for the free theory, which becomes important at large distances (UV/IR-entanglement).

First, I represent the ϕ^4-action in the harmonic oscillator base of the Moyal plane, where the action describes a matrix model the kinetic term of which is neither constant nor diagonal. I derive a closed formula for the resulting propagator, using Meixner polynomials in an essential way.

Then, I develop the renormalisation group approach for dynamical matrix models, the core of which is a flow equation for the effective action. The renormalisation proof is now reduced to the verification that the flow equation—a non-linear first-order differential equation—admits a regular solution which depends on finitely many initial data. In the perturbative regime, the flow equation is solved by ribbon graphs drawn on Riemann surfaces. I prove a general power-counting theorem which relates the power-counting behaviour of ribbon graphs to their topology and to two scaling dimensions of the cut-off propagator.

For the model under consideration, I determine these scaling dimensions by numerical methods. As a result, only planar graphs with two or four external legs can be relevant or marginal. These graphs are labelled by an infinite number of matrix indices. I prove the existence of a discrete Taylor expansion, which decomposes the (infinite number of) planar two- and four-leg graphs into a linear combination of four relevant or marginal base functions and an irrelevant remainder. These four universal base functions have the same index dependence as the original action in matrix formulation, which implies the renormalisability of the model. Moreover, I prove that the effective action converges quadratically in the inverse scale of the bare interactions.

Additionally, I compute the one-loop β-functions of the four-dimensional noncommutative ϕ^4-model with oscillator term. The β-function for the coupling constant is non-negative and vanishes for those frequency of the oscillator potential where the action is invariant under a duality transformation which exchanges positions and momenta.

Finally, I prove that ϕ^4-theory on the two-dimensional Moyal plane is super-renormalisable, where the one-loop planar two-leg graph is the only one which is marginal. The proof requires again an oscillator potential which, however, can be switched off at the end by adjusting it to the inverse logarithm of the scale of the bare interactions.

Zusammenfassung

Die Renormierung nichtkommutativer Quantenfeldtheorien war wegen der Mischung von Ultraviolett- und Infrarotdivergenzen über lange Zeit ein offenes Problem. In dieser Habilitationsschrift beweise ich, daß das reelle ϕ^4-Modell auf der vierdimensionalen Euklidischen Moyal-Ebene zu allen Ordnungen der Störungstheorie renormierbar ist. Es stellt sich heraus, daß—verglichen mit dem kommutativen Fall—die nackte Wirkung der relevanten und marginalen Kopplungen einen Zusatzterm besitzt, welcher durch ein Oszillatorpotential für die Wirkung des freien Feldes beschrieben wird. Dieses führt zu modifizierten Dispersionsrelationen für die freie Theorie, welche bei großen Abständen bedeutsam werden (UV/IR-Mischung).

Zunächst stelle ich die ϕ^4-Wirkung in der harmonischen Oszillatorbasis der Moyal-Ebene dar. In dieser Basis beschreibt die Wirkung ein Matrixmodell, dessen kinetischer Teil weder diagonal noch konstant ist. Ich leite eine geschlossene Formel für den resultierenden Propagator her, wobei Meixner-Polynome eine wesentliche Rolle spielen.

Danach entwickle ich den Renormierungsgruppenzugang für dynamische Matrixmodelle, dessen Kernstück eine Flußgleichung für die effektive Wirkung ist. Der Renormierungsbeweis reduziert sich nun auf die Überprüfung, daß die Flußgleichung—eine nichtlineare Differentialgleichung erster Ordnung—eine reguläre Lösung besitzt, die nur von endlich vielen Anfangsdaten abhängt. Die störungstheoretische Lösung der Flußgleichung ist durch Bandgraphen gegeben, welche auf einer Riemannschen Fläche dargestellt werden. Ich beweise ein Theorem, welches das allgemeine Potenzabzählverhalten eines Bandgraphen zu dessen Topologie und zu zwei Skalendimensionen des abgeschnittenen Propagators in Verbindung setzt.

Für das betrachtete Modell bestimme ich die Skalendimensionen mittels numerischer Methoden. Es stellt sich im Ergebnis heraus, daß die planaren Graphen mit zwei oder vier äußeren Beinen die einzigen sind, welche relevant oder marginal sind. Diese Graphen sind durch unendlich viele Matrixindizes charakterisiert. Ich beweise die Existenz einer diskreten Taylor-Entwicklung, welche diese (unendlich vielen) planaren Graphen mit zwei oder vier äußeren Beinen in eine Linearkombination von vier relevanten oder marginalen Basisfunktionen sowie einen irrelevanten Rest zerlegt. Diese vier universellen Basisfunktionen haben dieselbe Indexabhängigkeit wie die Ausgangswirkung in der Matrixformulierung, woraus schließlich die Renormierbarkeit des Modells folgt. Weiters beweise ich, daß die effektive Wirkung quadratisch in der inversen Skala der nackten Wechselwirkungen konvergiert.

Zusätzlich berechne ich in Einschleifennäherung die β-Funktionen des nichtkommutativen ϕ^4-Modells mit Oszillatorterm. Die β-Funktion der Kopplungskonstante ist nichtnegativ und verschwindet für jene Frequenz des Oszillatorpotentials, für die die Wirkung invariant unter einer Dualitätstransformation zwischen Orten und Impulsen wird.

Schließlich beweise ich, daß die nichtkommutative ϕ^4-Theorie in zwei Dimensionen superrenormierbar ist, wobei der planare Einschleifengraph mit zwei äußeren Beinen der einzige marginale Graph ist. Der Beweis erfordert wieder ein Oszillatorpotential, welches am Ende jedoch ausgeschaltet werden kann, indem man es proportional zum inversen Logarithmus der Skala der nackten Wirkung ansetzt.

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To
Patricia, Aline & Amélie

1 Introduction

1.1 Motivation

Half a century of high energy physics has drawn the following picture of the microscopic world: There are matter fields and carriers of interactions between them. Four different types of interactions are known: electromagnetic, weak and strong interactions as well as gravity. The traditional mathematical language to describe these structures of physics is that of fibre bundles (see e.g. [Nak90]). The base manifold M of these bundles is a four-dimensional metric space with line element $ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$. Matter fields ψ are sections of a vector bundle over M. The carriers of electromagnetic, weak and strong interactions are described by connection one-forms A of $U(1)$, $SU(2)$ and $SU(3)$ principal fibre bundles, respectively. Gravity is the determination of the metric g by the one-forms A and sections ψ, and vice-versa.

The dynamics of (A, ψ, g) is governed by an action functional $S[A, \psi, g]$, which yields the equations of motions when varied with respect to A, ψ, g. The complete action functional for the phenomenologically most successful model, the standard model of particle physics, consists of numerous individual pieces when expressed in terms of (A, ψ, g).

Next, there is a clever calculus, called *quantum field theory*, which as the input takes the action functional S and as the output returns *numbers*. There is another (much more expensive) source of numbers: experiments. There is a remarkable agreement[1] of up to 10^{-11} between corresponding numbers calculated by quantum field theory and those coming from experiment. This tells us two things: The action functional (here: of the standard model) is very well chosen and, in particular, quantum field theory is an extraordinarily successful calculus.

However, this can only be an approximation: Taking gravity (i.e. the dynamics of the space-time manifold) into account, quantum field theory is ill-defined. To see this, let us recall how we measure technically the geometry of space-time. The building blocks of a manifold are the *points* labelled by coordinates $\{x^\mu\}$ in a given chart. Points enter quantum field theory via the sections $\psi(x)$ and $A(x)$, i.e. the *values* of the fields at the point labelled by $\{x^\mu\}$. This observation provides a way to "visualise" the points: We have to prepare a distribution of matter which is sharply localised about $\{x^\mu\}$. For a perfect visualisation we need a δ-distribution of the matter field. This is physically not possible, but one would think that a δ-distribution could be arbitrarily well approximated. However, that is not the case, there are limits of localisability long before the δ-distribution is reached [DFR95].

Let us assume that there is a distribution of matter which is supposed to have two separated peaks within a space-time region R of diameter d. How do we test this conjecture? We perform a scattering experiment in the hope to find interferences which tell us about the internal structure in the region R. We clearly need test particles of de Broglie wave length $\lambda = \frac{\hbar c}{E} \lesssim$ d, otherwise we observe a single peak even if there is a double peak. For $\lambda \to 0$ the gravitational field of the test particles becomes important. The gravitational

[1] There are of course experimental data which so far could not be derived from first principles, such as the energy spectrum of hadrons.

field created by an energy E can be measured in terms of the Schwarzschild radius

$$r_s = \frac{2GE}{c^4} = \frac{2G\hbar}{\lambda c^3} \gtrsim \frac{2G\hbar}{\mathrm{d}c^3}\,, \tag{1.1}$$

where G is Newton's constant. If the Schwarzschild radius r_s becomes larger than the radius $\frac{\mathrm{d}}{2}$, the inner structure of the region R can no longer be resolved (it is behind the horizon). Consequently, we have to require $\frac{\mathrm{d}}{2} \geq r_s$, which implies the condition

$$\frac{\mathrm{d}}{2} \gtrsim \ell_P := \sqrt{\frac{G\hbar}{c^3}}\,. \tag{1.2}$$

Hence, the Planck length ℓ_P is the fundamental length scale below of which length measurements become operational meaningless [DFR95]. Space-time cannot be a manifold.

What does this mean for quantum field theory? It means that we cannot trust traditional quantum field theories like the (quantum) standard model because they rely on *non-existing information* about the short-distance structure of physics which is implicitly used in the loop calculations.

There exist a few proposals about how to replace the space-time manifold, notably string theory and quantum gravity. For deep background information I refer to Rovelli's beautiful dialogue [Rov03]. I refrain from further commenting these two religions, because the subject of this Habilitation thesis is a third one.

We know from quantum mechanics that any measurement uncertainty (enforced by principles of Nature and not due to lack of experimental skills) goes hand in hand with noncommutativity. In particular, commutation relations between coordinate operators which yield the localisation requirement (1.2) have been identified in [DFR95]. However, space-time is more than just a copy of quantum mechanical phase space. It is the arena for field theory. Thus, apart from only describing the algebra of space-time operators, we have to realise the geometric world of gauge fields, fermions, differential calculi, Dirac operators and action functionals associated with this algebra. Fortunately for us, the relevant mathematical framework—*noncommutative geometry*—has been developed, foremost by Alain Connes [Con94, Con00]. Related monographs are [Mad00, Lan97, Vár97, GBVF01].

Noncommutative geometry is the reformulation of geometry in an algebraic and functional-analytic language, in this way permitting an enormous generalisation. Today, noncommutative geometry is well-established and indispensable in mathematics. In physics, the most important achievement of noncommutative geometry is to overcome the distinction between *continuous* and *discrete* spaces, in the same way as quantum mechanics washed away the discrepancy between waves and particles.

This achievement is particularly visible in the *standard model* of particle physics. The standard model was proposed around 1970 as the conglomerate of the *electroweak model* of Glashow, Salam and Weinberg [Gla61, Sal68, Wei67], including the (at first sight artificial) *Higgs sector* [Hig64, EB64, Kib67] to give enormous masses to the (at that time conjectured) W- and Z-bosons, and the independent *quantum chromodynamics* [FGM73, GW73, Pol73] to describe the strong interactions. At that time, few people would have expected that this ugly standard model survives the experiments of the following 30 years.

As a matter of fact, the standard model is not only natural but also rather unique[2] [ISS03] from the point of view of noncommutative geometry: It is a spectral triple [Con95]. Actually, the standard model inspired Connes to discover the axioms [Con96] of spectral triples. In particular, the language in which spectral triples are formulated is very close to field theory: Besides the algebra $\mathcal{A}$ represented on a Hilbert space $\mathcal{H}$ (which alone are only good for measure theory), to describe *metric spaces with spin structure* one also needs a Dirac operator $\mathcal{D}$, the chirality γ^5 and the charge conjugation J. For the proof of Connes' theorem [Con96] that commutative spectral triples are spin manifolds, see [Ren01, GBVF01]. All finite spectral triples are known [PS98, Kra98a].

As already underlined, noncommutative geometry evaporates the distinction between continuous and discrete spaces. For the standard model, the relevant geometry is that of the two-sheeted universe [CL91], i.e. two copies (one for left-handed and one for right-handed fermions) of the four-dimensional space separated from each other by the de Broglie wavelength of the Higgs boson. It is a discrete Kaluza-Klein geometry [MW02] with discrete fibre consisting of two points. Writing down gauge theory on such a disconnected space, the component of the gauge field in the discrete direction is a scalar, the Higgs field, and the corresponding part of the noncommutative Yang-Mills action gives the Higgs potential responsible for spontaneous symmetry breaking. Moreover, the Yukawa coupling of the Higgs with the fermions is nothing but the restriction of the minimal coupling of the gauge fields with the fermions to the discrete direction. In fact, the geometrical insight goes much deeper. For instance, the spectral triple description *enforces* the following (in the language of Yang-Mills-Higgs models unrelated) features [CIS99]:

- weak interactions break parity maximally,
- weak interactions suffer spontaneous breakdown,
- strong interactions do not break parity,
- strong interactions do not suffer spontaneous breakdown.

I refer to [IKS95, MGBV98] for details about the noncommutative geometrical construction of the standard model and to [GB02] for a historical review.

Eventually, noncommutative geometry achieved via the spectral action principle [CC97] a true unification of the standard model with general relativity on the level of classical field theories. Kinematically, Yang-Mills fields, Higgs fields and gravitons are all regarded as *fluctuations* of the free Dirac operator [Con96]. The spectral action

$$S = \operatorname{trace} \chi\Big(z\frac{\mathcal{D}^2}{\Lambda^2}\Big) \,, \tag{1.3}$$

(which is the weighted sum of the eigenvalues of $\mathcal{D}^2$ up to the cut-off Λ^2) of the single fluctuated Dirac operator $\mathcal{D}$ gives the complete bosonic action of the standard model, the Einstein-Hilbert action (with cosmological constant) and an additional Weyl action term in one stroke [CC97]. See also [IKS97]. The parameter z in (1.3) is the "noncommutative coupling constant" [CIS99]. Assuming the spectral action (1.3) to produce the bare action

[2] It is not possible to formulate grand unified theories with spectral triples. It is possible to stay within the spirit of noncommutative geometry, but one has to relax the tight connection with other parts of mathematics. See [CFF92, CF94, Wul99] for examples of noncommutative GUTs.

at the (grand unification) energy scale Λ, the renormalisation flow based on the one-loop β-functions leads to a Higgs mass of $182 \dots 201\,\mathrm{GeV}$ [CIS99].

Of course, the unification of the standard model with general relativity via the spectral action is of limited value as long as it is not achieved at the level of quantum field theory. On the other hand, the arguments of [DFR95] make clear that this will not be possible with almost commutative geometries (products of commutative geometries with matrices). Space-time has to be noncommutative itself. The complete problem of a gravitational dynamics of the noncommutative space-time being too difficult to treat, the first step is to consider field theory on noncommutative spaces with fixed background. The most natural candidate is the Moyal plane [Gro46, Moy49], which was identified as a solution of the uncertainty conditions for coordinate operators [DFR95]. The (D-dimensional) Moyal plane is characterised by the *non-local* $\star$-product

$$(a \star b)(x) := \int d^D y \frac{d^D k}{(2\pi)^D} a\big(x + \tfrac{1}{2}\theta \cdot k\big) b(x+y)\, \mathrm{e}^{iky} , \qquad \theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R} , \tag{1.4}$$

which is associative but not commutative. The Moyal plane is a spectral triple [GGBI$^+$04] and the spectral action has been computed [Vas04, GI04]. Other interesting noncommutative spectral triples are the noncommutative torus [Con80, Rie81, Rie90], the Connes-Landi spheres [CL01] and the (mostly spherical) examples found by Connes and Dubois-Violette [CDV02]. Remarkably, the noncommutative torus is relevant for the compactification of M-theory [CDS98] and the Moyal plane arises as a limiting case of type IIA string theory [DH98, SW99].

It is not difficult to write down classical action functionals on noncommutative spaces (the first example was Yang-Mills on the noncommutative torus [CR87]), but it is not clear that quantum field theories [BS59, VW76, IZ80] can be defined consistently[3]. As locality is so important in quantum field theory [EG73], it is perfectly possible that quantum field theories are implicitly built upon the assumption that the action functional has to live on a (commutative) manifold. It was, therefore, an important step to prove that Yang-Mills theory is one-loop renormalisable on the Moyal plane and on the noncommutative torus [MSR99, SJ99, KW00]. This means that these models are divergent [Fil96], but the one-loop divergences are absorbable in a multiplicative renormalisation of the initial action such that the Ward identities are fulfilled.

In this line of success, it was somewhat surprising when Minwalla, Van Raamsdonk and Seiberg [MVRS00] pointed out that there is a new type of infrared-like divergences which makes the renormalisation of scalar field theories on the Moyal plane very unlikely. To get an idea about the problem one has to compute the non-planar one-loop two-point function resulting from the noncommutative ϕ^4-action. The corresponding integral is finite, but behaves $\sim (\theta p)^{-2}$ for small momenta p of the two-point function. The finiteness is important, because the p-dependence of the non-planar graph has no counterpart in the original ϕ^4-action, and thus (if divergent) cannot be absorbed by multiplicative renormalisation. However, if one inserts the non-planar graph declared as finite as a subgraph into a bigger graph, one easily builds examples (with an arbitrary number of external legs) where the $\sim p^{-2}$ behaviour leads to non-integrable integrals at small inner momenta. This is the so-called *UV/IR-mixing* problem [MVRS00].

[3]This refers to infinite-dimensional quantum field theories. There is no problem with finite-dimensional examples [GKP96b, GKP96a].

The heuristic argumentation can be made exact: Using a sophisticated mathematical machinery, Chepelev and Roiban have proven a power-counting theorem [CR00, CR01] which relates the power-counting degree of divergence to the topology of *ribbon graphs*. The rough summary of the power-counting theorem is that noncommutative field theories with quadratic divergences become meaningless beyond a certain loop order[4]. For example, in the real noncommutative ϕ^4-model there exist (in four dimensions) three-loop graphs which cannot be integrated.

Thus, to prove renormalisability of quantum field theories on the four-dimensional Moyal plane is an enormous challenge. One has to circumvent the power-counting theorem of Chepelev and Roiban [CR01], but at the same time respect the physical insight in the UV/IR-mixing mechanism. This is subject of my Habilitation thesis.

1.2 Renormalisation group approach to noncommutative field theories

In this Habilitation thesis I prove that the real noncommutative ϕ^4-model is *renormalisable to all orders* in four dimensions—a work based on a very pleasant collaboration with Harald Grosse. The proof is contained in the two articles [GW03a, GW04b]. A summary was given in [GW04c]. The one-loop β-function is evaluated in [GW04a]. The proof of the two-dimensional case is given in [GW03b].

At first sight, a renormalisability proof of ϕ^4-theory on the four-dimensional Moyal plane seems to be in grave contradiction with [MVRS00, CR00, CR01]. However, this is not the case. In fact, the results of these papers are reconfirmed, it is only that their message is taken serious. The message of the UV/IR-entanglement is that

> *noncommutativity relevant at very short distances modifies—whether we like it or not—the physics of the model at very large distances.*

The required modification is, to the best of my knowledge, unique: It is given by an harmonic oscillator potential for the free field action. The following theorem is proven in the thesis:

Theorem 1 *The quantum field theory associated with the action*

$$S = \int d^4x \Big(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi) \star (\tilde{x}^\mu\phi) + \frac{\mu_0^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi\Big)(x) \,, \tag{1.5}$$

for $\tilde{x}_\mu := 2(\theta^{-1})_{\mu\nu}x^\nu$, ϕ-real, Euclidean metric, is perturbatively renormalisable to all orders in λ.

The action (1.5) is covariant with respect to a remarkable *duality* between position space and momentum space [LS02a]: Under the exchange of position and momentum (i.e. not the Fourier transformation),

$$p_\mu \leftrightarrow \tilde{x}_\mu \,, \qquad \hat{\phi}(p) \leftrightarrow \pi^2\sqrt{|\det\theta|}\,\phi(x) \,, \tag{1.6}$$

[4] There exist proposals to resum the perturbation series, see [MVRS00, CR01], but there is no complete proof that this is consistent to all orders.

together with $\hat{\phi}(p_a) = \int d^4x\, \mathrm{e}^{(-1)^a \mathrm{i} p_{a,\mu} x_a^\mu} \phi(x_a)$ for a being a cyclic label, one has

$$S\big[\phi; \mu_0, \lambda, \Omega\big] \mapsto \Omega^2 S\Big[\phi; \frac{\mu_0}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}\Big] . \tag{1.7}$$

Of course, we cannot treat the quantum field theory associated with the action (1.5) in momentum space[5]. Fortunately, there is a matrix representation [BM49] of the Moyal plane, where the $\star$-product becomes a simple product of infinite matrices and where the duality between positions and momenta is manifest. The matrix representation plays an important rôle in the proof that the Moyal plane is a spectral triple [GGBI$^+$04]. It is also crucial for the exact solution of quantum field theories [Lan03, LSZ03, LSZ04] on noncommutative phase space.

In the traditional Feynman graph approach the value of the integral associated to non-planar graphs is not unique, because one exchanges the order of integrations in integrals which are not absolutely convergent. To avoid this problem one should use a renormalisation scheme where the various limiting processes are better controlled.

The preferred method is the use of *flow equations*. The idea goes back to Wilson [WK74]. It was then used by Polchinski [Pol84] to give a very efficient renormalisability proof for commutative ϕ^4-theory. Several improvements to the proof have been made in [KKS92]. Later, the method has been applied, for instance, to massless ϕ^4-theory [KK94], to QED [KK96] and to spontaneously broken Yang-Mills theory [KM00]. For an introduction to that renormalisation method, see [Sal99].

Applying Polchinski's method to the noncommutative ϕ^4-model there is, however, a serious problem in momentum space. We have to guarantee that planar graphs only appear in the distinguished interaction coefficients for which we fix the boundary conditions at the renormalisation scale Λ_R. Non-planar graphs have phase factors which involve inner momenta. Polchinski's method consists in taking norms of the interaction coefficients, and these norms ignore possible phase factors. Thus, we would find that boundary conditions for non-planar graphs at Λ_R are required. Since there is an infinite number of different non-planar structures, the model is not renormalisable in this way[6]. A more careful examination of the phase factors is also not possible, because the cut-off integrals prevent the Gaußian integration required for the parametric integral representation [CR00, CR01]. In conclusion, I believe it is extremely difficult (if not impossible) to use the exact renormalisation group equation for noncommutative field theories in momentum space. The best one can hope is to restrict oneself to limiting cases where e.g. the non-planar graphs are suppressed [BGI02, BGI03]. Even this restricted model has rich topological features.

In other words, we have to avoid the momentum space formulation of the noncom-

[5]It should be possible to use the Mehler formula for momentum space computations, see (7.5) on page 88, although this is probably not so easy.

[6]There are already some attempts [GP01] to use Polchinski's method to renormalise noncommutative field theories. I have, however, severe reservations on the method and results. The main argument in [GP01] is that the Polchinski equation is a one-loop equation so that the authors simply compute an integral having exactly one loop. It is, however, not true that nothing new happens at higher loop order. For instance, all one-loop graphs can be drawn on a genus-zero Riemann surface. The entire complexity of Riemann surfaces of higher genus as discussed by Chepelev and Roiban [CR00, CR01] shows up at higher loop order and is completely ignored by the authors of [GP01]. As I will demonstrate in the Habilitation thesis, the same discussion of Riemann surfaces is necessary in the renormalisation group approach, too.

mutative ϕ^4-model. I have already stressed that the matrix base[7] of the Moyal plane is convenient to realise both the partial derivatives and the coordinate multiplication in the classical action (1.5). In the matrix base, the interaction part $\int d^Dx(\phi \star \phi \star \phi \star \phi)$ in (1.5) becomes simply $\mathrm{tr}(\phi^4)$, where ϕ is now an infinite matrix (with entries of rapid decay). Thus, we get rid of the oscillating phase factors—the first condition to apply the renormalisation group techniques. The price for the simplification of the interaction is that the kinetic matrix, or rather its inverse, the propagator, will become very complicated. However, in Polchinski's approach the propagator is anyway made complicated when multiplying it with the smooth cut-off function. Indeed, all difficulties can be overcome.

The renormalisation proof is very technical. I do not claim that it is the most efficient one. However, it was for us (Harald Grosse and me), for the time being, the only possible way. There are several "miracles" without which the proof had failed. The first is that the propagator is complicated but numerically accessible. We had thus convinced ourselves that the propagator has such an asymptotic behaviour that all non-planar graphs and all graphs with $N > 4$ external legs are irrelevant according to our general power-counting theorem for dynamical matrix models [GW03a]. However, this still leaves an infinite number of planar two- or four- point functions which would be relevant or marginal according to [GW03a]. In the first versions of [GW03a] we had, therefore, to propose some consistency relations inspired by [Zim85] in order to get a meaningful theory.

Miraculously, all this was not necessary. We had found numerically that the propagator has some universal locality properties suggesting that the infinite number of relevant / marginal planar two- or four- point functions can be decomposed into four relevant / marginal base interactions and an irrelevant remainder. Of course, there must exist a reason for such a coincidence, and the reason are orthogonal polynomials. In our case, it means that the kinetic matrix corresponding to the free action (3.28) written in the matrix base of the Moyal plane is diagonalised by orthogonal Meixner polynomials [Mei34][8]. Then, having a closed solution for the free theory in the preferred base of the interaction, the desired local and asymptotic behaviour of the propagator can be derived.

I stress, however, that some of the corresponding estimations of Section 5.3 are, so far, verified numerically only. There is no doubt that the estimations are correct, but for the purists I have to formulate the result as follows: The quantum field theory corresponding to the action (1.5) is renormalisable to all orders provided that the estimations given in Section 5.3 hold. Already this weaker result is a considerable progress, because the elimination of the last possible doubt amounts to verify properties of hypergeometric functions.

Noncommutative ϕ^4-theory in two dimensions is different. One also needs the harmonic oscillator potential of (1.5) in all intermediate steps of the renormalisation proof, but at the end it can be switched off with the removal of the cut-off. This is in agreement with

[7]For another matrix realisation of the Moyal plane and its treatment by renormalisation group methods, see [Nic03].

[8]In our renormalisation proof [GW03b] of the two-dimensional noncommutative ϕ^4-model we had originally termed these polynomials "deformed Laguerre polynomials", which we had only constructed via their recursion relation. The closed formula was not known to us. Thus, I am especially grateful to Stefan Schraml who has provided us first with [MR91], from which we got the information that we were using Meixner polynomials, and then with the encyclopaedia [KS96] of orthogonal polynomials, which was the key to complete the renormalisation proof.

the common belief that the UV/IR-mixing problem can be cured in models with only logarithmic divergences.

1.3 Organisation of the Habilitation thesis

The Habilitation thesis is divided into three parts:

- In the main part, consisting of Sections 3–6 and Appendices B–F, I prove that the duality-covariant noncommutative ϕ^4-model is renormalisable to all orders and convergent in the limit of removed cut-off. This part is based on [GW03a, GW04b]. At the end of this Section I make a few comments on the strategy of the proof.
- In the introductory part, placed into Section 2 before the main part, I give a summary of the ideas and techniques used in the renormalisation proof. The essential formulae of the main part are presented, but without any proofs and partly with simplified notation. This part is based on [GW04c]. Moreover, I give in Appendix A a limited historical overview about field theories on noncommutative spaces and attempts of their renormalisation.
- In a supplementary part, consisting of Appendices G–H, I apply the results of the main part to two interesting exercises:
  - In Appendix G, I compute the one-loop β-functions of the duality-covariant noncommutative ϕ^4_4-model, based on the identification of relevant and marginal graphs achieved in the main part. This computation follows [GW04a].
  - In Appendix H, I prove that the two-dimensional noncommutative ϕ^4-model is renormalisable to all orders. The proof uses the general framework given in Sections 3 and 4 and refers to some formulae of Sections 5 and 6. The proof is inspired by [GW03b]. It is, however, considerably streamlined thanks to orthogonal polynomials which had not been identified yet at the time of writing of [GW03b].

Section 7 contains the conclusion and gives an outlook to subsequent activities. These applications demonstrate that the Habilitation thesis, which in the first instance solves a longstanding technical problem concerning the renormalisation of noncommutative ϕ^4-theory, provides new insight into noncommutative field theories in general. Moreover, there are potential applications of the developed methods and obtained results to other areas of quantum field theory.

Figure 1 shows the dependency of the various sections of the Habilitation thesis. The central results are contained in Sections 5.4, 6.5 and H.2.

I would like to add a few comments on the strategy in the main part. The first step is to rewrite the ϕ^4-action (1.5) in the harmonic oscillator base of the Moyal plane, see (3.42) and (3.45). The free theory is solved by the propagator (3.49), which I compute in Appendix B.3 using Meixner polynomials in an essential way. The propagator is represented by a finite sum which enables a fast numerical evaluation. Unfortunately, I can offer analytic estimations only in a few special cases.

The propagator is so complicated that a direct calculation of Feynman graphs is not practicable. Therefore, I employ the renormalisation method based on flow equations [Pol84, KKS92], which I adapt in Section 4 to non-local (dynamical) matrix models. The modification $K[\Lambda]$ of the weights of the matrix indices in the kinetic term is undone in

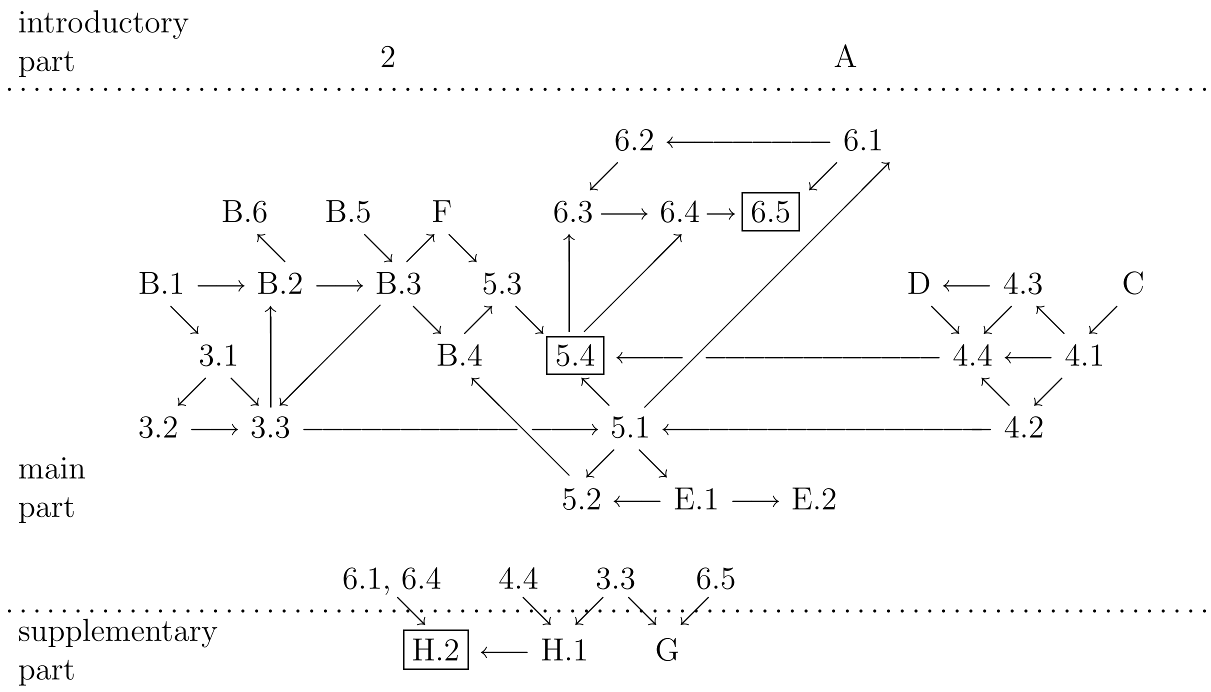


Figure 1: Flow chart of the Habilitation thesis.

the partition function by a careful adaptation of the effective action $L[\phi, \Lambda]$, which is described by the matrix Polchinski equation (4.15). For a modification given by a cut-off function $K[\Lambda]$, renormalisation of the model amounts to prove that the matrix Polchinski equation (4.15) admits a regular solution which depends on a finite number of initial data.

In a perturbative expansion, the matrix Polchinski equation is solved by ribbon graphs drawn on Riemann surfaces, see Section 4.3. Then, I prove in Appendix D the Power-counting Theorem 10 which relates the general power-counting behaviour of a ribbon graph to its topology and to two scaling exponents of the cut-off propagator. In this way, regular scaling dimensions guarantee the existence of a regular solution of the matrix Polchinski equation.

According to Appendix F, the model under consideration is indeed characterised by regular scaling dimensions. However, the general proof involves an infinite number of initial conditions, which is physically not acceptable. Therefore, the challenge is to prove the reduction to a finite number of initial data for the renormalisation flow.

The answer is the integration procedure given in Definition 12, Section 5.1, which entails mixed boundary conditions for certain planar two- and four-point functions. The idea is to introduce four types of reference graphs with vanishing external indices and to split the integration of the Polchinski equation for the distinguished two- and four-point graphs into an integration of the difference to the reference graphs and a different integration of the reference graphs themselves. The difference between original graph and reference graph is further reduced to differences of propagators, which I call "composite propagators". See Section 5.2.

The proof of the power-counting estimations for the interaction coefficients (Proposition 13 in Section 5.4) requires the following extensions of the general case treated in

Section 4:

- I prove that graphs where the index jumps along the trajectory between incoming and outgoing indices are suppressed. This leaves 1PI planar four-point functions with constant index along the trajectory and 1PI planar two-point functions with (in total) at most two index jumps along the trajectories as the only graphs which are marginal or relevant.
- For these types of graphs I prove that the leading relevant/marginal contribution is captured by reference graphs with vanishing external indices, whereas the difference to the reference graphs is irrelevant. This is the discrete analogue of the BPHZ Taylor subtraction of the expansion coefficients to lowest order in the external momenta.

Thus, Proposition 13 provides bounds for the interaction coefficients of the effective action at a scale $\Lambda \in [\Lambda_R, \Lambda_0]$. Here, Λ_R is the renormalisation scale where the four reference graphs are normalised, and Λ_0 is the initial scale for the integration which has to be sent to ∞ in order to scale away possible initial conditions for the irrelevant functions. The estimations of Proposition 13 are actually independent of Λ_0 so that the limit $\Lambda_0 \to \infty$ can be taken. This already ensures the renormalisation of the model.

However, one would also like to know whether the interaction coefficients converge in the limit $\Lambda_0 \to \infty$ and if so, with which rate. That analysis is performed in Section 6 which culminates in Theorem 16, confirming convergence with a rate Λ_0^{-2}.

Figure 2 explains the relations between the main steps of the proof. The central results are the power-counting behaviour of Proposition 13 and the convergence theorem (Theorem 16). Note that the numerical estimations for the propagator influence the entire chain of the proof.

I would like to finish the Introduction with a TEXnical remark. The Habilitation thesis contains numerous cross references. Thanks to the hyperref package, it is very convenient to jump to a cited equation, reference or section and then back to the place of reading. Moreover, I have equipped the Bibliography on page 168 with links to the eprint arXiv and to the SPIRES database. Of course, these convenient features are only available in the electronic version of the Habilitation thesis. Therefore, I would like to encourage the reader of a printed copy to ask me for the electronic files.

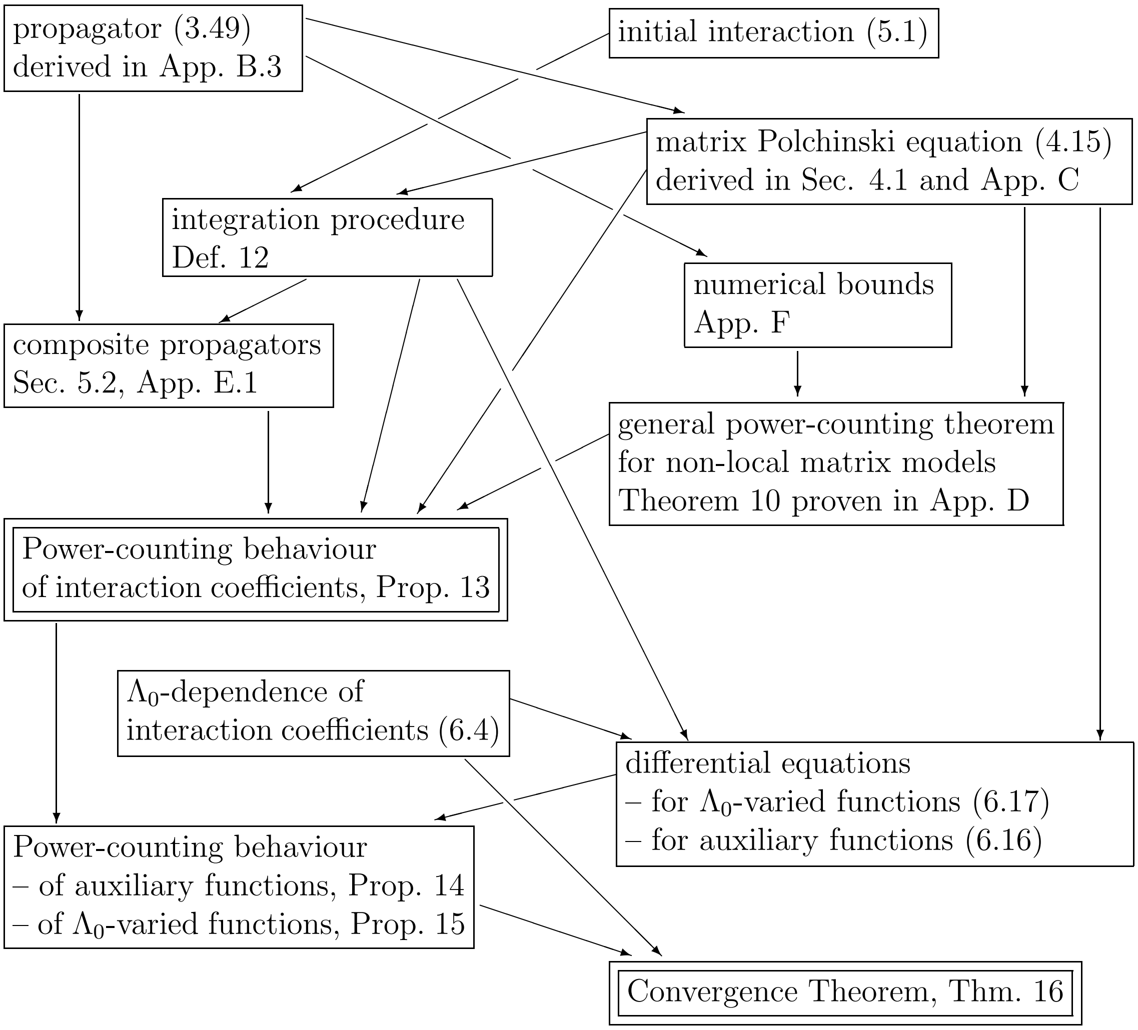


Figure 2: Flow chart of the renormalisation proof.

2 Summary of ideas and techniques

As the renormalisation proof of Theorem 1 is quite long and technical, I give in this section an overview about the main ideas and techniques. This is basically the contents of the letter [GW04c] I have written with Harald Grosse.

2.1 Reformulation as a dynamical matrix model

As mentioned before, the explicit x-dependence of the action (1.5) forces us to work in the matrix base of the Moyal plane. We choose a coordinate frame where $\theta = \theta_{12} = -\theta_{21} = \theta_{34} = -\theta_{43}$ are the only non-vanishing θ-components. We expand the fields according to $\phi(x) = \sum_{m^1,m^2,n^1,n^2\in\mathbb{N}} \phi_{\substack{m^1\\m^2}\substack{n^1\\n^2}} b_{\substack{m^1\\m^2}\substack{n^1\\n^2}}(x)$ where $b_{\substack{m^1\\m^2}\substack{n^1\\n^2}}(x) = f_{m^1n^1}(x_1,x_2) f_{m^2n^2}(x_3,x_4)$, with

$$f_{m^1n^1}(x_1,x_2) = \frac{(x_1{-}\mathrm{i}x_2)^{\star m^1}}{\sqrt{m^1!(2\theta)^{m^1}}} \star \Big(2\mathrm{e}^{-\frac{1}{\theta}(x_1^2+x_2^2)}\Big) \star \frac{(x_1{+}\mathrm{i}x_2)^{\star n^1}}{\sqrt{n^1!(2\theta)^{n^1}}}\;, \tag{2.1}$$

$$(b_{mn}\star b_{kl})(x) = \delta_{nk} b_{ml}(x)\;, \qquad \int d^4x\, b_{mn}(x) = (2\pi\theta)^2\,\delta_{mn}\;. \tag{2.2}$$

Due to (2.2) the non-local $\star$-product interaction becomes a simple matrix product, at the price of rather complicated kinetic terms and propagators. We obtain for the action (1.5)

$$S = (2\pi\theta)^2 \sum_{m,n,k,l\in\mathbb{N}^2} \Big(\frac{1}{2}\phi_{mn}G_{mn;kl}\phi_{kl} + \frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\Big)\;, \tag{2.3}$$

$$\begin{aligned}
G_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}} &= \big(\mu_0^2 + \tfrac{2+2\Omega^2}{\theta}(m^1{+}n^1{+}m^2{+}n^2{+}2)\big)\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2}\\
&\quad - \tfrac{2-2\Omega^2}{\theta}\big(\sqrt{k^1l^1}\,\delta_{n^1+1,k^1}\delta_{m^1+1,l^1} + \sqrt{m^1n^1}\,\delta_{n^1-1,k^1}\delta_{m^1-1,l^1}\big)\delta_{n^2k^2}\delta_{m^2l^2}\\
&\quad - \tfrac{2-2\Omega^2}{\theta}\big(\sqrt{k^2l^2}\,\delta_{n^2+1,k^2}\delta_{m^2+1,l^2} + \sqrt{m^2n^2}\,\delta_{n^2-1,k^2}\delta_{m^2-1,l^2}\big)\delta_{n^1k^1}\delta_{m^1l^1}\;.
\end{aligned} \tag{2.4}$$

We have $G_{mn;kl} = 0$ unless $m{+}k = n{+}l$, which is due to the $SO(2)\times SO(2)$-invariance of the action.

We are interested in a perturbative solution of the quantum field theory about the free theory, the solution of which is given by the propagator $\Delta_{mn;kl}$, i.e. the inverse of $G_{mn;kl}$. In a first step we diagonalise the kinetic matrix:

$$G_{\substack{m^1\\m^2}\substack{m^1+\alpha^1\\m^2+\alpha^2};\substack{l^1+\alpha^1\\l^2+\alpha^2}\substack{l^1\\l^2}} = \sum_{y^1,y^2=0}^{\infty} U^{(\alpha^1)}_{m^1y^1}U^{(\alpha^2)}_{m^2y^2}\big(\mu_0^2 + \tfrac{4\Omega}{\theta}(2y^1{+}2y^2{+}\alpha^1{+}\alpha^2{+}2)\big)U^{(\alpha^1)}_{y^1l^1}U^{(\alpha^2)}_{y^2l^2}\;, \tag{2.5}$$

$$U^{(\alpha)}_{ny} = \sqrt{\binom{\alpha+n}{n}\binom{\alpha+y}{y}}\,\Big(\frac{1-\Omega}{1+\Omega}\Big)^{n+y}\Big(\frac{2\sqrt{\Omega}}{1+\Omega}\Big)^{\alpha+1}\,{}_2F_1\Big(\begin{matrix}-n,-y\\1+\alpha\end{matrix}\Big|\frac{4\Omega}{(1+\Omega)^2}\Big)\;.$$

For fixed α, the kinetic matrix is in both components a Jacobi matrix (a certain tridiagonal band matrix) [MR91]. The diagonalisation of that band matrix yields the recursion relation for (orthogonal) Meixner polynomials $M_n(y;\beta,c) = {}_2F_1\Big(\begin{smallmatrix}-n,-y\\\beta\end{smallmatrix}\Big|1{-}c\Big)$, see [KS96]. The corresponding equidistant eigenvalues are those of the harmonic oscillator. To compute the propagator we have to invert the eigenvalues $\big(\mu_0^2 + \frac{4\Omega}{\theta}(2y^1{+}2y^2{+}\alpha^1{+}\alpha^2{+}2)\big)$ in

(2.5). Using the identity

$$\sum_{y=0}^{\infty} \frac{(\alpha+y)!}{y!\alpha!} a^y \, {}_2F_1\Big(\begin{matrix} -m, -y \\ 1+\alpha \end{matrix}\Big| b\Big) \, {}_2F_1\Big(\begin{matrix} -l, -y \\ 1+\alpha \end{matrix}\Big| b\Big) = \frac{(1-(1-b)a)^{m+l}}{(1-a)^{\alpha+m+l+1}} \, {}_2F_1\Big(\begin{matrix} -m\,, \ -l \\ 1+\alpha \end{matrix}\Big| \frac{ab^2}{(1-(1-b)a)^2}\Big)\,, \qquad |a|<1\,, \tag{2.6}$$

which can be regarded as the heart of the renormalisation proof, we arrive at

$$\begin{aligned}
&\Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}} \\
&= \frac{\theta}{2(1+\Omega)^2} \sum_{v^1=\frac{|m^1-l^1|}{2}}^{\frac{m^1+l^1}{2}} \sum_{v^2=\frac{|m^2-l^2|}{2}}^{\frac{m^2+l^2}{2}} B\big(1+\tfrac{\mu_0^2\theta}{8\Omega}+\tfrac{1}{2}(m^1+k^1+m^2+k^2)-v^1-v^2, 1+2v^1+2v^2\big) \\
&\times {}_2F_1\Bigg(\begin{matrix} 1+2v^1+2v^2\,, \ \frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2 \\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2 \end{matrix}\Bigg| \frac{(1-\Omega)^2}{(1+\Omega)^2}\Bigg) \Big(\frac{1-\Omega}{1+\Omega}\Big)^{2v^1+2v^2} \\
&\times \prod_{i=1}^{2} \delta_{m^i+k^i,n^i+l^i} \sqrt{\binom{n^i}{v^i+\frac{n^i-k^i}{2}}\binom{k^i}{v^i+\frac{k^i-n^i}{2}}\binom{m^i}{v^i+\frac{m^i-l^i}{2}}\binom{l^i}{v^i+\frac{l^i-m^i}{2}}}\,.
\end{aligned} \tag{2.7}$$

One should appreciate here that the sum in (2.7) is finite, i.e. we succeeded to solve the free theory with respect to the preferred base of the interaction. The explicit solution enables a fast numerical evaluation of the propagator, which is necessary to determine the asymptotic behaviour of the propagator for large indices. In few cases I can evaluate the sum exactly:

- $0 \le \Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(\mu_0) < \Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(0)$

  This means that we can ignore the mass μ_0 in our estimations for $\Omega > 0$.

- $\Delta_{\substack{m\\0}\substack{m\\0};\substack{m\\0}\substack{m\\0}}(0) = \frac{\theta}{2(1+\Omega)^2(m+1)} \, {}_2F_1\Big(\begin{matrix} 1,-m \\ m+2 \end{matrix}\Big| \frac{(1-\Omega)^2}{(1+\Omega)^2}\Big) \sim \frac{\theta/8}{\Omega(m+1)+\sqrt{\frac{4}{\pi}(m+1)}}$

  There is a discontinuity in the asymptotic behaviour of the propagator at $\Omega = 0$. For $\Omega = 0$ there is a long-range correlation which decays only very slowly with $\frac{1}{\sqrt{m}}$. This is the origin of the UV/IR-mixing. For $\Omega > 0$ the correlation decays with $\frac{1}{m}$ which guarantees a good power-counting behaviour of the model with $\Omega > 0$. The asymptotic behaviour provides the easy part of the renormalisation proof.

- $\Delta_{\substack{m^1\\m^2}\substack{m^1\\m^2};\substack{0\\0}\substack{0\\0}}(0) = \frac{\theta}{2(1+\Omega)^2(m^1+m^2+1)} \big(\frac{1-\Omega}{1+\Omega}\big)^{m^1+m^2}$

  This property controls the non-locality. The model is non-local in the sense that there is a correlation $\Delta_{mn;kl}$ for arbitrarily large $\|m-l\|$. However, that correlation is exponentially suppressed, preserving some sort of quasi-locality. This provides the tricky part of the renormalisation proof.

2.2 The Polchinski equation

It is, in principle, possible to proceed with the discussion of Feynman graphs built with the propagator (2.7) according to Zimmermann's forest formula [Zim69]. But the complexity of the arising graphs (compare (2.7) with the simple $\frac{1}{k^2+m^2}$ of commutative field theories) requires a more sophisticated approach: the renormalisation by flow equations. The idea goes back to Wilson [WK74] and was further developed by Polchinski to an efficient renormalisation proof of commutative ϕ^4-theory [Pol84].

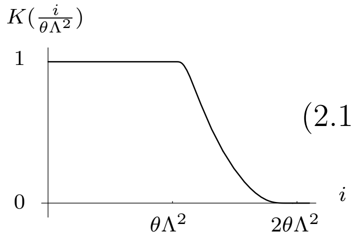
The starting point is the definition of the quantum field theory by the cut-off partition function

$$Z[J,\Lambda] = \int \Big(\prod_{a,b} d\phi_{ab}\Big)\, \exp\big(-S[\phi,J,\Lambda]\big)\;, \tag{2.8}$$

$$\begin{aligned} S[\phi,J,\Lambda] = (2\pi\theta)^2\Big(&\sum_{m,n,k,l} \frac{1}{2}\phi_{mn} G^K_{mn;kl}(\Lambda)\,\phi_{kl} + L[\phi,\Lambda] + C[\Lambda] \\ &+ \sum_{m,n,k,l} \phi_{mn}F_{mn;kl}[\Lambda]J_{kl} + \sum_{m,n,k,l}\frac{1}{2}J_{mn}E_{mn;kl}[\Lambda]J_{kl}\Big)\;. \end{aligned} \tag{2.9}$$

The most important pieces here are the cut-off kinetic term

$$G^K_{\substack{m^1\, n^1\,;\,k^1\, l^1\\ m^2\, n^2\,;\,k^2\, l^2}}(\Lambda) := \Bigg(\prod_{\substack{i\,\in m^1,\,m^2,\,n^1,\,n^2,\\ k^1,\,k^2,\,l^1,\,l^2}} K^{-1}\big(\tfrac{i}{\theta\Lambda^2}\big)\Bigg) G_{\substack{m^1\, n^1\,;\,k^1\, l^1\\ m^2\, n^2\,;\,k^2\, l^2}}\;, \tag{2.10}$$



where the weight of the matrix indices is altered according to a *smooth* cut-off function[9] K, and the *effective action* $L[\phi,\Lambda]$ which compensates the effect of the cut-off. We are interested in the limit $\Lambda\to\infty$, where the cut-off goes away, $\lim_{\Lambda\to\infty} K(\frac{i}{\theta\Lambda^2}) = 1$. Thus, we would formally obtain the original model for $\Lambda=\infty$ and $L[\phi,\infty] = \frac{\lambda}{4!}\sum_{m,n,k,l}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}$, $C[\infty]=0$, $E_{mn;kl}[\infty]=0$, $F_{mn;kl}[\infty]=\delta_{nk}\delta_{ml}$. However, $\Lambda=\infty$ is difficult to obtain due to the appearance of divergences, which require compensating counterterms in $L[\phi]$.

The genial idea of the renormalisation group approach is to require instead the *independence* of the partition function from the cut-off, $\Lambda\frac{\partial}{\partial\Lambda}Z[J,\Lambda]=0$. Working out the details one arrives, in particular, at the Polchinski equation for matrix models

$$\Lambda\frac{\partial L[\phi,\Lambda]}{\partial\Lambda} = \sum_{m,n,k,l}\frac{1}{2}\Lambda\frac{\partial\Delta^K_{nm;lk}(\Lambda)}{\partial\Lambda}\Big(\frac{\partial L[\phi,\Lambda]}{\partial\phi_{mn}}\frac{\partial L[\phi,\Lambda]}{\partial\phi_{kl}} - \frac{1}{(2\pi\theta)^2}\frac{\partial^2 L[\phi,\Lambda]}{\partial\phi_{mn}\,\partial\phi_{kl}}\Big)\;, \tag{2.11}$$

where $\Delta^K_{nm;lk}(\Lambda) := \Big(\prod_{i\in m^1,m^2,\dots,l^1,l^2} K(\frac{i}{\theta\Lambda^2})\Big)\Delta_{nm;lk}$. To obtain (2.11) it is important to realise finite matrices via a smooth function K. There are other differential equations for

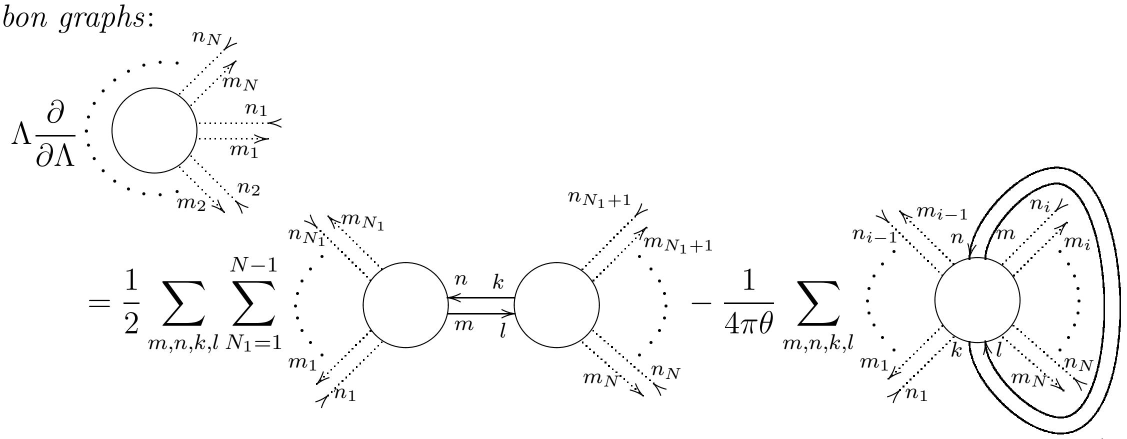
[9] I understand the cut-off as a limiting process $\epsilon\to 0$ in $K^{-1}(\frac{i}{\theta\Lambda^2}) = \frac{1}{\epsilon}$ for $i\geq 2\theta\Lambda^2$. In the limit, the partition function (2.8) vanishes unless $\phi_{\substack{m^1\, n^1\\ m^2\, n^2}} = 0$ if $\max(m^1,m^2,n^1,n^2)\geq 2\theta\Lambda^2$, thus implementing a cut-off of the measure $\prod_{a,b} d\phi_{ab}$ in (2.8). All other formulae involve $K(\frac{i}{\theta\Lambda^2})$.

the functions C, E, F in (2.9) which, however, are trivial to integrate. The true difficulties are contained in the non-linear differential equation (2.11).

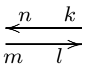
The Polchinski equation has a non-perturbative meaning, but to solve it we need, for the time being, a power series ansatz:

$$L[\phi,\Lambda] = \sum_{V=1}^{\infty} \lambda^V \sum_{N=2}^{2V+2} \frac{(2\pi\theta)^{\frac{N}{2}-2}}{N!} \sum_{m_i,n_i\in\mathbb{N}^2} A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\phi_{m_1n_1}\cdots\phi_{m_Nn_N}\;. \tag{2.12}$$

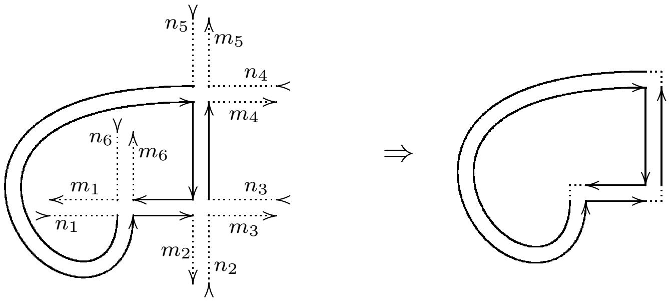
Then, the differential equation (2.11) provides an explicit recursive solution for the coefficients $A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ which, because the fields ϕ_{mn} carry two indices, is represented by *ribbon graphs*:



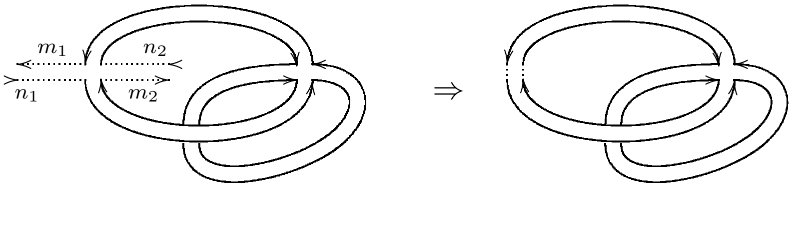
(2.13)

An internal double line symbolises the propagator $Q_{mn;kl}(\Lambda) := \frac{1}{2\pi\theta}\Lambda\frac{\partial}{\partial\Lambda}\Delta^K_{mn;kl}(\Lambda) =$  .

Clearly, in this way we produce very complicated ribbon graphs which cannot be drawn any more in a plane. Ribbon graphs define a Riemann surface on which they can be drawn. The Riemann surface is characterised by its genus g computable via the Euler characteristic of the graph, $g = 1 - \frac{1}{2}(\tilde{L} - I + V)$, and the number B of holes. Here, $\tilde{L}$ is the number of single-line loops if we close the external lines of the graph, I is the number of double-line propagators and V the number of vertices. The number B of holes coincides with the number of single-line loops which carry external legs. A few examples might help to understand the closure of external lines and the resulting topological data:



$$\tilde{L}=2 \quad g=0 \qquad I=3 \quad B=2 \qquad V=3 \quad N=6 \tag{2.14}$$

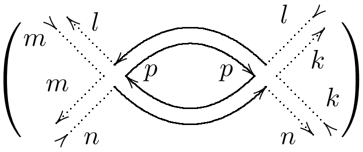


$$\tilde{L}=1 \quad g=1 \qquad I=3 \quad B=1 \qquad V=2 \quad N=2 \tag{2.15}$$

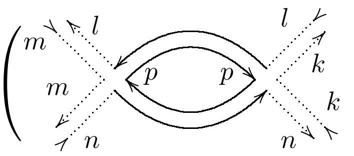
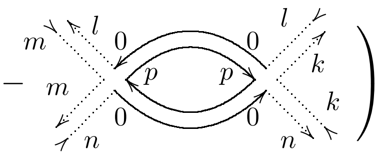
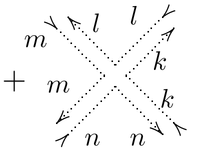
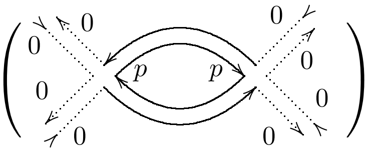
According to the topology we label the expansion coefficients of the effective action by $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}$.

2.3 Integration procedure of the Polchinski equation

The integration procedure of the Polchinski equation represents the entire magic of renormalisation. Suppose we want to evaluate the planar one-particle irreducible four-point function with two vertices, $A^{(2,1,0)\mathrm{1PI}}_{m_1n_1;\dots;m_Nn_N}$. The Polchinski equation (2.13) provides the Λ-derivative of that function:

$$\Lambda\frac{\partial}{\partial\Lambda}A^{(2,1,0)\mathrm{1PI}}_{mn;nk;kl;lm}[\Lambda] = \sum_{p\in\mathbb{N}^2}\Bigg($$

$$\Bigg)[\Lambda] + \text{ permutations }. \qquad (2.16)$$

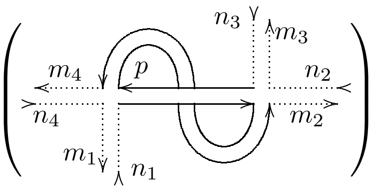
We consider the special case with constant indices on the trajectories. The first guess would be to perform the Λ-integration of (2.16) from some initial scale Λ_0 (sent to ∞ at the end) down to Λ. However, this choice of integration leads to $A^{(2,1,0)\mathrm{1PI}}_{mn;nk;kl;lm}[\Lambda] \sim \ln\frac{\Lambda_0}{\Lambda}$, which diverges when we remove the cut-off $\Lambda_0 \to \infty$. Following Polchinski [Pol84] we understand renormalisation as the change of the boundary condition for the integration. Thus, the idea would be to introduce a renormalisation scale Λ_R so that we would integrate (2.16) from Λ_R up to Λ. Then, $A^{(2,1,0)\mathrm{1PI}}_{mn;nk;kl;lm}[\Lambda] \sim \ln\frac{\Lambda}{\Lambda_R}$, and there would be no problem any more sending $\Lambda_0 \to \infty$. However, since there is an *infinite number* of matrix indices and there is no symmetry which could relate the amplitudes for different indices, that integration procedure entails an infinite number of initial conditions $A^{(2,1,0)\mathrm{1PI}}_{mn;nk;kl;lm}[\Lambda_R]$. These initial conditions correspond to normalisation experiments, and clearly a model requiring an infinite number of normalisation experiments has no physical meaning. Thus, to have a renormalisable model, we can only afford a finite number of integrations from Λ_R up to Λ. The discussion shows that the correct integration procedure is something like

$$A^{(2,1,0)\mathrm{1PI}}_{mn;nk;kl;lm}[\Lambda] = -\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\sum_{p\in\mathbb{N}^2}\Bigg($$

$$-$$

$$\Bigg)[\Lambda']$$
$$+$$

$$\Bigg[\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\sum_{p\in\mathbb{N}^2}\Bigg($$

$$\Bigg)[\Lambda'] + A^{(2,1,0)\mathrm{1PI}}_{00;00;00;00}[\Lambda_R]\Bigg]. \qquad (2.17)$$

The second graph in the first line of the rhs and the graph in brackets in the last line are identical, because only the indices on the propagators determine the value of the graph. Moreover, the vertex in the last line in front of the bracket equals 1. Thus, differentiating (2.17) with respect to Λ we obtain indeed (2.16). As a further check one can consider (2.17) for $m=n=k=l=0$. Finally, the independence of $A^{(2,1,0)\mathrm{1PI}}_{mn;nk;kl;lm}[\Lambda_0]$ on the indices m,n,k,l is built-in. This property is, for $\Lambda_0\to\infty$, dynamically generated by the model.

There is a similar Λ_0-Λ_R-mixed integration procedure for the planar 1PI two-point functions $A^{(V,1,0)\mathrm{1PI}}_{{m^1 \atop m^2}{n^1 \atop n^2};{n^1 \atop n^2}{m^1 \atop m^2}}$, $A^{(V,1,0)\mathrm{1PI}}_{{m^1+1 \atop m^2}{n^1+1 \atop n^2};{n^1 \atop n^2}{m^1 \atop m^2}}$ and $A^{(V,1,0)\mathrm{1PI}}_{{m^1 \atop m^2+1}{n^1 \atop n^2+1};{n^1 \atop n^2}{m^1 \atop m^2}}$. These involve in total three different sub-integrations from Λ_R up to Λ. All other graphs are integrated from Λ_0 down to Λ, e.g.

$$A^{(2,2,0)1\mathrm{PI}}_{m_1n_1;\dots;m_4n_4}[\Lambda] = -\int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg(\quad \Bigg)[\Lambda'] \,. \tag{2.18}$$



2.4 The power-counting estimation

Proposition 2 *The previous integration procedure yields*

$$\big|A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \leq \big(\sqrt{\theta}\Lambda\big)^{(4-N)+4(1-B-2g)}\; P^{4V-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big]\; P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,, \tag{2.19}$$

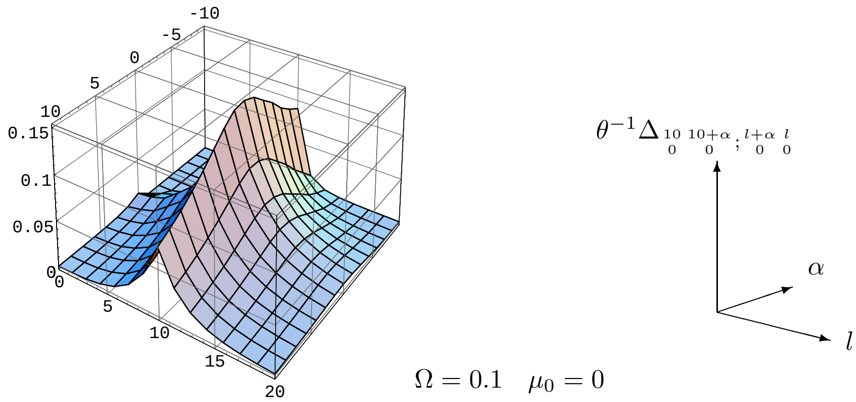
where $P^q[X]$ stands for a polynomial of degree q in X. The notation $\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}$ stands for the set of ratios $\frac{m_1^1}{\theta\Lambda^2}, \frac{m_1^2}{\theta\Lambda^2}, \dots, \frac{n_N^2}{\theta\Lambda^2}$.

Idea of the proof. The cut-off propagator $Q_{mn;kl}(\Lambda)$ contains both an UV and an IR cut-off, $Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(\Lambda) \neq 0$ only for $\theta\Lambda^2 < \max(m^1,\dots,l^2) < 2\theta\Lambda^2$. The global maximum of the propagator $\Delta_{mn;kl}$ is at $m=n=k=l=\substack{0\\0}$. If Λ increases, at least one of the indices of $Q_{mn;kl}$ must increase as well, resulting in a decrease of $\big|Q_{mn;kl}(\Lambda)\big|$ with Λ. If we normalise the volume of the support of $Q_{mn;kl}(\Lambda)$ with respect to a single index to $\theta^2\Lambda^4$ (corresponding to a four-dimensional model), then

$$|Q_{mn;kl}(\Lambda)| < \frac{C_0}{\Omega\theta\Lambda^2}\delta_{m+k,n+l}\,. \tag{2.20}$$

Thus, the propagator and the volume of a loop summation have the same power-counting dimensions as a commutative ϕ^4-model in momentum space, giving the total power-counting degree $4-N$ for an N-point function.

This is (more or less, see below) correct for planar graphs. The scaling behaviour of non-planar graphs is considerably improved by the *anisotropy* (or quasi-locality) of the propagator:



(2.21)

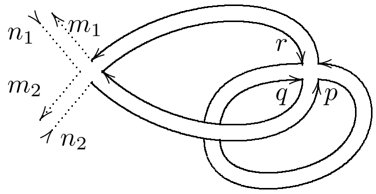
$\Omega = 0.1 \quad \mu_0 = 0$

As a consequence, for given index m of the propagator $Q_{mn;kl}(\Lambda) = \overset{n\quad k}{\underset{m\quad l}{\rightleftarrows}}$, the contribution to a graph is strongly suppressed unless the other index l on the trajectory through m is close to m. Thus, the sum over l for given m converges and does not alter (apart from a factor Ω^{-1}) the power-counting behaviour of (2.20):

$$\sum_{l\in\mathbb{N}^2} \Big(\max_{n,k} |Q_{mn;kl}(\Lambda)| \Big) < \frac{C_1}{\theta\Omega^2\Lambda^2} \,. \tag{2.22}$$

In a non-planar graph like the one in (2.18), the index n_3—fixed as an external index—localises the summation index $p \approx n_3$. Thus, we save one volume factor $\theta^2\Lambda^4$ compared with a true loop summation as in (2.17). In general, each hole in the Riemann surface saves one volume factor, and each handle even saves two: In the genus-1 graph

$$\sum_{p,q,r\in\mathbb{N}^2} \tag{2.23}$$



n_2 is fixed as an external index, and the quasi-locality (2.21) implies $n_2 \approx p \approx q \approx r$. Thus, instead of the two loops of a corresponding line graph, the non-planar ribbon graph (2.23) does not require any volume factor in the power-counting estimation.

A more careful analysis of (2.7) shows that also planar graphs get suppressed with $\big|Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(\Lambda)\big| < \frac{C_2}{\Omega\theta\Lambda^2}\prod_{i=1}^2 \big(\frac{\max(m^i,l^i)+1}{\theta\Lambda^2}\big)^{\frac{|m^i-l^i|}{2}}$, for $m^i \leq n^i$, if the index along a trajectory jumps. This leaves the functions $A^{(V,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}$, $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$, $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$ and $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1\\m^2+1}\substack{n^1\\n^2+1};\substack{n^1\\n^2}\substack{m^1\\m^2}}$ as the only relevant or marginal ones. In these functions one has to use a discrete version of the Taylor expansion,

$$\Big|Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}(\Lambda) - Q_{\substack{0\\0}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{0\\0}}(\Lambda)\Big| < \frac{C_3}{\Omega\theta\Lambda^2}\Big(\frac{\max(m^1,m^2)}{\theta\Lambda^2}\Big) \,, \tag{2.24}$$

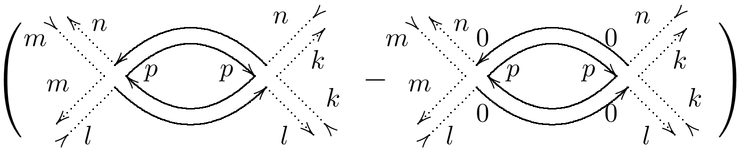
$$\begin{aligned}\Big|Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}(\Lambda) - Q_{\substack{0\\0}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{0\\0}}(\Lambda) - m^1\big(Q_{\substack{1\\0}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{1\\0}}(\Lambda) - Q_{\substack{0\\0}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{0\\0}}(\Lambda)\big)&\\ - m^2\big(Q_{\substack{0\\1}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{0\\1}}(\Lambda) - Q_{\substack{0\\0}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{0\\0}}(\Lambda)\big)\Big| &< \frac{C_4}{\Omega\theta\Lambda^2}\Big(\frac{\max(m^1,m^2)}{\theta\Lambda^2}\Big)^2 \,,\end{aligned} \tag{2.25}$$

$$\Big|Q_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}(\Lambda) - \sqrt{m^1{+}1}\,Q_{\substack{1\\0}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{0\\0}}(\Lambda)\Big| < \frac{C_5}{\Omega\theta\Lambda^2}\Big(\frac{\max(m^1,m^2)}{\theta\Lambda^2}\Big)^{\frac{3}{2}} \,. \tag{2.26}$$

These estimations are traced back to the Meixner polynomials. The factor $\sqrt{m^1+1}$ in (2.26) is particularly remarkable. Any other Taylor subtraction (e.g. with prefactors $\sqrt{m^1}$ or $\sqrt{m^1{+}2}$) would kill the renormalisation proof.

These discrete Taylor subtractions are used in the integration from Λ_0 down to Λ in prescriptions like (2.17):

$$-\int_{\Lambda}^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\sum_{p\in\mathbb{N}^2}\Bigg($$



$$\Bigg)[\Lambda']$$

$$= \int_{\Lambda}^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\int_{\Lambda'}^{\Lambda_0}\frac{d\Lambda''}{\Lambda''}\sum_{p\in\mathbb{N}^2}\Big((Q_{np;pn}-Q_{0p;p0})(\Lambda')Q_{lp;pl}(\Lambda'') + Q_{0p;p0}(\Lambda')(Q_{lp;pl}-Q_{0p;p0})(\Lambda'')\Big) \sim \frac{C(\|n\|+\|l\|)}{\theta\Omega^2\Lambda^2}\,. \qquad (2.27)$$

Factors like $\frac{\|n\|}{\theta\Lambda^2}$ and $\frac{\|l\|}{\theta\Lambda^2}$ in (2.27) are responsible for the appearance of the polynomial $P^{4V-N}\big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\big]$ in (2.19). □

Thus, decomposing (similar as in the BPHZ subtraction) in planar 2- and 4-point functions the propagators into reference propagators at zero-indices and an irrelevant part, we have

$$\begin{aligned}
A^{(V,1,0)}_{\substack{m^1\,n^1;\,k^1\,l^1\\ m^2\,n^2;\,k^2\,l^2}} &= \Big\{A^{(V,1,0)}_{\substack{00;00\\00;00}} + m^1\Big(A^{(V,1,0)}_{\substack{10;01\\00;00}} - A^{(V,1,0)}_{\substack{00;00\\00;00}}\Big) + n^1\Big(A^{(V,1,0)}_{\substack{01;10\\00;00}} - A^{(V,1,0)}_{\substack{00;00\\00;00}}\Big)\\
&\quad + m^2\Big(A^{(V,1,0)}_{\substack{00;00\\10;01}} - A^{(V,1,0)}_{\substack{00;00\\00;00}}\Big) + n^2\Big(A^{(V,1,0)}_{\substack{00;00\\01;10}} - A^{(V,1,0)}_{\substack{00;00\\00;00}}\Big)\Big\}\delta_{m^1l^1}\delta_{n^1k^1}\delta_{m^2l^2}\delta_{n^2k^2}\\
&\quad + A^{(V,1,0)}_{\substack{11;00\\00;00}}\big(\sqrt{k^1l^1}\delta_{m^1+1,l^1}\delta_{n^1+1,k^1}\delta_{m^2l^2}\delta_{n^2k^2} + \sqrt{m^1n^1}\delta_{m^1-1,l^1}\delta_{n^1-1,k^1}\delta_{m^2l^2}\delta_{n^2k^2}\big)\\
&\quad + A^{(V,1,0)}_{\substack{00;00\\11;00}}\big(\sqrt{k^2l^2}\delta_{m^2+1,l^2}\delta_{n^2+1,k^2}\delta_{m^1l^1}\delta_{n^1k^1} + \sqrt{m^2n^2}\delta_{m^2-1,2^1}\delta_{n^2-1,k^2}\delta_{m^1l^1}\delta_{n^1k^1}\big)\\
&\quad + \text{ irrelevant part}\,, \qquad (2.28)
\end{aligned}$$

$$A^{(V,1,0)}_{m_1n_1;\dots;m_4n_4} = A^{(V,1,0)}_{00;\dots;00}\Big(\tfrac{1}{6}\delta_{n_1m_2}\delta_{n_2m_3}\delta_{n_3m_4}\delta_{n_4m_1} + 5\text{ perms}\Big) + \text{ irrelevant part}\,. \qquad (2.29)$$

We conclude that there are four independent relevant/marginal interaction coefficients:

$$\begin{aligned}
\rho_1[\Lambda] &= A^{(V,1,0)}_{\substack{00;00\\00;00}}[\Lambda]\,, & \rho_2[\Lambda] &= A^{(V,1,0)}_{\substack{10;01\\00;00}} - A^{(V,1,0)}_{\substack{00;00\\00;00}}[\Lambda]\,,\\
\rho_3[\Lambda] &= A^{(V,1,0)}_{\substack{11;00\\00;00}}[\Lambda] = A^{(V,1,0)}_{\substack{00;00\\11;00}}[\Lambda]\,, & \rho_4[\Lambda] &= A^{(V,1,0)}_{\substack{00;00;00;00\\00;00;00;00}}[\Lambda]\,. \qquad (2.30)
\end{aligned}$$

At $\Lambda = \Lambda_0$ we recover the same index structure as in the initial action (2.3), (2.4), identifying $\rho_a[\Lambda_0] \equiv \rho_a^0$ as functions of the parameters $\mu_0, \theta, \Omega, \lambda$. Together with the Λ_0-independence of Proposition 2, this already ensures the renormalisation of the model [KKS92]. However, we would also like to control the limit $\Lambda_0 \to \infty$.

2.5 Removal of the cut-off

For given data Λ_0, ρ_a^0, the integration of the Polchinski equation yields the coefficients $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho_a^0]$ and thus, via (2.30), $\rho_b[\Lambda,\Lambda_0,\rho_a^0]$. Now, according to Section 2.3, in particular (2.17), we keep $\rho_b[\Lambda_R,\Lambda_0,\rho_a^0]$ *constant* when varying Λ_0. This leads to the identity

$$L[\phi,\Lambda_R,\Lambda_0',\rho^0[\Lambda_0']] - L[\phi,\Lambda_R,\Lambda_0'',\rho^0[\Lambda_0'']] = \int_{\Lambda_0''}^{\Lambda_0'}\frac{d\Lambda_0}{\Lambda_0}\,R[\phi,\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]\,, \qquad (2.31)$$

$$R[\phi,\Lambda,\Lambda_0,\rho^0] := \Lambda_0 \frac{\partial L[\phi,\Lambda,\Lambda_0,\rho^0]}{\partial \Lambda_0} - \sum_{b=1}^{4} H^b[\Lambda,\Lambda_0,\rho^0]\; \Lambda_0 \frac{\partial \rho_b[\Lambda,\Lambda_0,\rho^0]}{\partial \Lambda_0}\;, \tag{2.32}$$

$$H^b[\Lambda,\Lambda_0,\rho^0] := \sum_{a=1}^{4} \frac{\partial L[\phi,\Lambda,\Lambda_0,\rho^0]}{\partial \rho_a^0} \frac{\partial \rho_a^0}{\partial \rho_b[\Lambda,\Lambda_0,\rho^0]}\;. \tag{2.33}$$

From (2.11) one derives flow equations for the coefficients of R and H^a:

$$\Lambda \frac{\partial R}{\partial \Lambda} = M[L,R] - \sum_{a=1}^{4} H^a M_a[L,R]\;, \quad \Lambda \frac{\partial H^a}{\partial \Lambda} = M[L,H^a] - \sum_{b=1}^{4} H^b M_b[L,H^a]\;, \tag{2.34}$$

for certain functions M, M_a which are linear in the second argument. We only have initial conditions at Λ_0 for these coefficients, thus the integration must always be performed from Λ_0 down to Λ. Fortunately, there are (by construction) remarkable cancellations in the rhs of (2.34) so that relevant contributions never appear. One proves

Proposition 3

$$\begin{aligned}
&\big|H^{a(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho_0]\big| \\
&\leq \big(\sqrt{\theta}\Lambda\big)^{(4-N-2\delta^{a1})+4(1-B-2g)} P^{4V-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big]\; P^{2V+1+\delta^{a4}-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;, \qquad (2.35)\\
&\big|R^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho_0]\big| \\
&\leq \Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)\big(\sqrt{\theta}\Lambda\big)^{(4-N)+4(1-B-2g)}\; P^{4V-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big]\; P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\;. \qquad (2.36)
\end{aligned}$$

I give the main ideas of the proof of (2.36). First, $R^{(1,1,0)}_{m_1n_1;\dots;m_4n_4} \equiv 0$, because the ϕ^4-vertex is scale-independent, which leads to a vanishing coefficient according to (2.32). Then, as $R^{(1,1,0)}_{m_1n_1;\dots;m_4n_4}$ appears in each term on the rhs of the first differential equation (2.34) for the 2-vertex six-point function and the 1-vertex two-point function, the coefficients $R^{(2,1,0)}_{m_1n_1;\dots;m_6n_6}, R^{(1,1,0)}_{m_1n_1;m_2n_2}$ and $R^{(1,2,0)}_{m_1n_1;m_2n_2}$ are Λ-independent. Next, one derives e.g.

$$R^{(2,1,0)}_{m_1n_1;\dots;m_6n_6}[\Lambda_0,\Lambda_0,\rho^0] = -\Big(\Lambda\frac{\partial}{\partial\Lambda} A^{(2,1,0)}_{m_1n_1;\dots;m_6n_6}[\Lambda,\Lambda_0,\rho^0]\Big)_{\Lambda=\Lambda_0} \sim \frac{C}{\theta\Lambda_0^2}\;, \tag{2.37}$$

where the scaling behaviour follows from (2.19). Since the first differential equation (2.34) is linear in R and relevant coefficients are projected away, the relative factor $\frac{\Lambda^2}{\Lambda_0^2}$ between $|A[\Lambda]|$ and $|R[\Lambda]|$ which first appears in (2.37) and similarly in $R^{(1,1,0)}_{m_1n_1;m_2n_2}, R^{(1,2,0)}_{m_1n_1;m_2n_2}$ survives to all R-coefficients. By integration of (2.31) we thus obtain

Theorem 4 *The duality-covariant noncommutative ϕ^4-model is (order by order in the coupling constant) renormalisable*

- *by an adjustment of the initial coefficients $\rho_a^0[\Lambda_0]$ to give renormalised constant couplings $\rho_a^R = \rho_a[\Lambda_R,\Lambda_0,\rho_b^0[\Lambda_0]]$, and*
- *by the corresponding integration of the flow equations.*

The limit $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty] := \lim_{\Lambda_0\to\infty} A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]$ *of the expansion coefficients of the effective action* $L[\phi,\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]$ *exists and satisfies*

$$\begin{aligned}&\Big|(2\pi\theta)^{\frac{N}{2}-2}A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty]-(2\pi\theta)^{\frac{N}{2}-2}A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\Lambda_0,\rho^0]\Big|\\&\qquad\le\frac{\Lambda_R^{6-N}}{\Lambda_0^2}\Big(\frac{1}{\theta^2\Lambda_R^4}\Big)^{B+2g-1}P^{4V-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda_R^2}\Big]\,P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\;. \qquad (2.38)\end{aligned}$$

2.6 Renormalisation group equation

Knowing the relevant/marginal couplings, we can compute Feynman graphs with sharp matrix cut-off $\mathcal{N}$. The most important question concerns the β-functions appearing in the renormalisation group equation, which describe the cut-off dependence of the expansion coefficients $\Gamma_{m_1n_1;\dots;m_Nn_N}$ of the effective action when imposing normalisation conditions for the relevant and marginal couplings. We have

$$\lim_{\mathcal{N}\to\infty}\Big(\mathcal{N}\frac{\partial}{\partial\mathcal{N}}+N\gamma+\mu_0^2\beta_{\mu_0}\frac{\partial}{\partial\mu_0^2}+\beta_\lambda\frac{\partial}{\partial\lambda}+\beta_\Omega\frac{\partial}{\partial\Omega}\Big)\Gamma_{m_1n_1;\dots;m_Nn_N}[\mu_0,\lambda,\Omega,\mathcal{N}]=0\;, \qquad (2.39)$$

where

$$\begin{aligned}\beta_\lambda&=\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\lambda[\mu_{\rm phys},\lambda_{\rm phys},\Omega_{\rm phys},\mathcal{N}]\Big)\;, &\beta_\Omega&=\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\Omega[\mu_{\rm phys},\lambda_{\rm phys},\Omega_{\rm phys},\mathcal{N}]\Big)\;,\\ \beta_{\mu_0}&=\frac{\mathcal{N}}{\mu_0^2}\frac{\partial}{\partial\mathcal{N}}\Big(\mu_0^2[\mu_{\rm phys},\lambda_{\rm phys},\Omega_{\rm phys},\mathcal{N}]\Big)\;, &\gamma&=\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\ln\mathcal{Z}[\mu_{\rm phys},\lambda_{\rm phys},\Omega_{\rm phys},\mathcal{N}]\Big)\;.\end{aligned} \qquad (2.40)$$

Here, $\mathcal{Z}$ is the wavefunction renormalisation. To one-loop order we find

$$\begin{aligned}\beta_\lambda&=\frac{\lambda_{\rm phys}^2}{48\pi^2}\frac{(1{-}\Omega_{\rm phys}^2)}{(1{+}\Omega_{\rm phys}^2)^3}\;, &\beta_\Omega&=\frac{\lambda_{\rm phys}\Omega_{\rm phys}}{96\pi^2}\frac{(1{-}\Omega_{\rm phys}^2)}{(1{+}\Omega_{\rm phys}^2)^3}\;,\\ \beta_{\mu_0}&=-\frac{\lambda_{\rm phys}\Big(4\mathcal{N}\ln(2)+\frac{(8+\theta\mu_{\rm phys}^2)\Omega_{\rm phys}^2}{(1+\Omega_{\rm phys}^2)^2}\Big)}{48\pi^2\theta\mu_{\rm phys}^2(1{+}\Omega_{\rm phys}^2)}\;, &\gamma&=\frac{\lambda_{\rm phys}}{96\pi^2}\frac{\Omega_{\rm phys}^2}{(1{+}\Omega_{\rm phys}^2)^3}\;.\end{aligned} \qquad (2.41)$$

There are two remarkable special cases. First, for $\Omega=1$, which corresponds to a self-dual model according to (1.7), we have $\beta_\lambda=\beta_\Omega=0$. This is true to all orders for β_Ω and conjectured for β_λ due to the resemblance of the duality-invariant theory with the exactly solvable models discussed in [LSZ04]. Second, β_Ω also vanishes in the limit $\Omega\to0$, which defines the standard noncommutative ϕ^4-quantum field theory. Thus, the limit $\Omega\to0$ exists at least at the one-loop level.

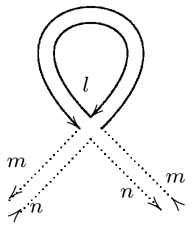
2.7 The two-dimensional case

Repeating the renormalisation group analysis for the two-dimensional duality-covariant ϕ^4-action, one obtains the following power-counting estimation for the expansion coefficients of the effective action:

$$\begin{aligned}\big|A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big|_{D=2} \leq \Big(\frac{1}{\theta\Lambda^2}\Big)^{(V-1)+(B+2g-1)} \Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}-2+B+2g} \\ \times P^{V-\frac{N}{2}-B-2g+2}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big]\, P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,. \end{aligned} \tag{2.42}$$

The only marginal graphs are the one-loop planar two-point graphs

$$\Lambda\frac{\partial}{\partial\Lambda}A^{(1,1,0)}_{mn;nm}[\Lambda] = \left(\sum_{l=0}^{\infty}\right)[\Lambda] + \big(m \leftrightarrow n\big)\,. \tag{2.43}$$



Due to the matrix indices these graphs still represent an infinite number of divergent graphs. Again, the leading divergence is captured by the reference graph with $m = n = 0$ in (2.43) whereas the difference between (2.43) and the reference graph is irrelevant.

It is important to notice that we do not need a normalisation condition for the oscillator frequency. This makes it possible to use the harmonic oscillator potential as a regulator. The estimation (2.42) is obtained by integrating the Polchinski equation for a fixed scale Λ_0 and a fixed frequency Ω. It turns out that relating Ω to Λ_0 according to $\Omega[\Lambda_0] = \big(1+\ln\frac{\Lambda_0}{\Lambda_R}\big)^{-1}$, the limit $\Lambda_0 \to \infty$ still exists. In this way, the standard noncommutative ϕ^4-theory (without oscillator term) is constructed as the limit of a sequence of duality-covariant ϕ^4-models which converges with Λ_0^{-2}.

3 The duality-covariant noncommutative ϕ^4-model

3.1 The noncommutative $\mathbb{R}^D$

The noncommutative $\mathbb{R}^D$, $D = 2, 4, 6, \dots$, also referred to as the D-dimensional Moyal [Moy49] plane, is defined as the algebra $\mathbb{R}^D_\theta$ which as a vector space is given by the space $\mathcal{S}(\mathbb{R}^D)$ of (complex-valued) Schwartz class functions of rapid decay, equipped with the multiplication rule [GBV88]

$$
\begin{gathered}
(a \star b)(x) = \int \frac{d^D k}{(2\pi)^D} \int d^D y \; a(x+\tfrac{1}{2}\theta{\cdot}k)\, b(x+y)\, \mathrm{e}^{\mathrm{i}k\cdot y} \;, \\
(\theta{\cdot}k)^\mu = \theta^{\mu\nu} k_\nu \;, \quad k{\cdot}y = k_\mu y^\mu \;, \quad \theta^{\mu\nu} = -\theta^{\nu\mu} \;.
\end{gathered}
\tag{3.1}
$$

The entries $\theta^{\mu\nu}$ in (3.1) have the dimension of an area. Generalisations of (3.1) to deformations of C^*-algebras are considered in [Rie93]. For some historical remarks, see Appendix A.1.

Using the identity $\int \frac{d^D k}{(2\pi)^D}\, \mathrm{e}^{\mathrm{i}k\cdot(x-y)} = \delta(x-y)$ it is not difficult to prove that the $\star$-product (3.1) is associative $((a \star b) \star c)(x) = (a \star (b \star c))(x)$ and non-commutative, $a \star b \neq b \star a$. Moreover, complex conjugation is an involution, $\overline{a \star b} = \overline{b} \star \overline{a}$. One has the important property

$$
\int d^D x \; (a \star b)(x) = \int d^D x \; a(x) b(x) \;. \tag{3.2}
$$

Partial derivatives are derivations, $\partial_\mu (a \star b) = (\partial_\mu a) \star b + a \star (\partial_\mu b)$. For various proofs (such as in [GGBI+04]) one needs the fact that for each $f \in \mathbb{R}^D_\theta$ there exist $f_1, f_2 \in \mathbb{R}^D_\theta$ with $f = f_1 \star f_2$, see [GBV88].

There is a (unfortunately more popular) different version of the $\star$-product,

$$
(a \star b)(x) = \exp\Big(\mathrm{i}\theta^{\mu\nu} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial z^\nu}\Big) a(y) b(z)\Big|_{y=z=x} \;, \tag{3.3}
$$

which is obtained by the following steps from (3.1):

- Taylor expansion of $a(x + \frac{1}{2}\theta{\cdot}k)$ about $k = 0$
- repeated representation of $k_\mu \mathrm{e}^{\mathrm{i}k\cdot y} = -\mathrm{i}\frac{\partial}{\partial y^\mu} \mathrm{e}^{\mathrm{i}k\cdot y}$
- integration by parts in y
- k-integration yielding $\int \frac{d^D k}{(2\pi)^D} \mathrm{e}^{\mathrm{i}k\cdot y} = \delta(y)$
- y-integration

Of course, as the Taylor expansion is involved, at least one of the functions a, b has to be analytic. Actually, the formula (3.3) is an *asymptotic expansion* of the $\star$-product (3.1) which becomes exact under the conditions given in [EGBV89]. I would like to stress that the most important property concerning physics is the *non-locality* of the $\star$-product (3.1), not its non-commutativity. To the value of $a \star b$ at the point x there contribute individual values of the functions a, b far away from x. This non-locality is hidden in (3.3): At first sight it seems to be local, as only the derivatives of a, b at x contribute to $(a \star b)(x)$. However, the point is that analyticity is required, where the information about a functions is not localised at all.

A third version of the $\star$-product which is particularly useful for field theory in momentum space is obtained by expressing on the rhs of (3.1) the functions by their Fourier transformation[10]. This yields

$$(a\star b)(x) = \int \frac{d^D p}{(2\pi)^D}\, \mathrm{e}^{-\mathrm{i}px} \int \frac{d^D q}{(2\pi)^D}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^{\mu\nu}p_\mu q_\nu}\, \hat{a}(p{-}q)\hat{b}(q)\;. \tag{3.4}$$

Being a non-compact space, the noncommutative $\mathbb{R}^D$ cannot have a unit. For various reasons, the restriction of the $\star$-product to Schwartz class functions should be relaxed. That extension to tempered distribution was performed in [GBV88]. A good summary is the appendix of [GBLMV02]. Since (3.1) is smooth, for T being a tempered distribution and $f,g\in\mathcal{S}(\mathbb{R}^D)$ one defines the product $T\star f$ via

$$\langle T\star f, g\rangle := \langle T, f\star g\rangle\;, \tag{3.5}$$

and similarly for $f\star T$. Both $T\star f$ and $f\star T$ are smooth functions [GBV88], but not necessarily of Schwartz class. The set of those T for which $T\star f$ is of Schwartz class is the left multiplier algebra $M_L(\mathbb{R}^D_\theta)$, and similarly for $M_R(\mathbb{R}^D_\theta)$ (which is different). Then, the *Moyal algebra* is defined as $M(\mathbb{R}^D_\theta) := M_L(\mathbb{R}^D_\theta)\cap M_R(\mathbb{R}^D_\theta)$. It is a unital algebra (in fact the largest compactification of $\mathbb{R}^D_\theta$) and contains also the coordinate functions x^μ and the "plane waves" $\mathrm{e}^{\mathrm{i}p_\mu x^\mu}$. In fact, the famous commutation relation $[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu}$ holds in $M(\mathbb{R}^D_\theta)$ and not in $\mathbb{R}^D_\theta$. The Moyal algebra is huge so that for practical purposes appropriate subalgebras must be considered [GBV88, GGBI+04]. There are several surprises on $M(\mathbb{R}^D_\theta)$: For instance, the Dirac δ-distribution belongs to $M(\mathbb{R}^D_\theta)$, with $\delta\star\delta = \frac{2^D}{\det\theta}1$. On the other hand, $\mathrm{e}^{\frac{2\mathrm{i}}{a}x^1x^2}\in M(\mathbb{R}^2_\theta)$ iff $|a|\neq\theta_1$, $\theta := \theta^{12} = -\theta^{21}$. This proves, by the way, that for different θ the Moyal algebras $M(\mathbb{R}^D_\theta)$ are different.

Most calculations on the Moyal plane simplify considerably in an adapted coordinate frame. For our purpose we loose nothing in placing ourselves into a coordinate system in which θ has in D dimensions the form

$$\theta_{\mu\nu} = \begin{pmatrix} \boldsymbol{\theta}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{\theta}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{\theta}_{\frac{D}{2}} \end{pmatrix}\;, \qquad \boldsymbol{\theta}_i = \begin{pmatrix} 0 & \theta_i \\ -\theta_i & 0 \end{pmatrix}\;. \tag{3.6}$$

Traditionally, physicists expand the algebra $\mathbb{R}^D_\theta$ into the Weyl basis (plane waves) $\mathrm{e}^{\mathrm{i}p_\mu x^\mu}$, which has the advantage that the resulting computations are similar to the usual treatment of commutative field theories in momentum space. For both mathematical investigations (see e.g. [GBV88, GGBI+04]) and our renormalisability proof it is, however, much more convenient to use the harmonic oscillator basis given by the eigentransitions of the Hamiltonian $H=\frac{1}{2}x_\mu x^\mu$. Thus, in $D=2$ dimensions, we have

$$H\star f_{m^1n^1} = \theta_1(m^1+\tfrac{1}{2})f_{m^1n^1}\;, \qquad f_{m^1n^1}\star H = \theta_1(n^1+\tfrac{1}{2})f_{m^1n^1}\;. \tag{3.7}$$

[10] I use the convention that $f(x)=\int\frac{d^Dp}{(2\pi)^D}\,\mathrm{e}^{-\mathrm{i}px}\hat{f}(p)$ and $\hat{f}(p)=\int d^Dx\,\mathrm{e}^{\mathrm{i}px}f(x)$.

These eigentransitions are given by

$$f_{m^1n^1} = \frac{2}{\sqrt{n^1!m^1!\,\theta_1^{m^1+n^1}}}\,\bar{a}^{\star m}\star \mathrm{e}^{-\frac{2H}{\theta_1}}\star a^{\star n}\;, \tag{3.8}$$

where $a=\frac{1}{\sqrt{2}}(x_1+\mathrm{i}x_2)$ and $\bar{a}=\frac{1}{\sqrt{2}}(x_1-\mathrm{i}x_2)$. I derive these (and other useful) formulae in Appendix B.1 following the presentation in [GBV88]. In particular, the $f_{m^1n^1}$ are given by Laguerre polynomials in radial direction and Fourier modes in angular direction, see (B.12), and correspond to the transition between levels of the harmonic oscillator as first derived in [BM49].

Further, we note that the $f_{m^1n^1}$ are also the common eigenfunctions of the Landau Hamiltonian

$$H_L^{\pm} = \frac{1}{2}(\mathrm{i}\partial_\mu \pm A_\mu)(\mathrm{i}\partial^\mu \pm A^\mu)\;, \qquad A_\mu = \frac{1}{2}B_{\mu\nu}x^\nu\;. \tag{3.9}$$

If $B_{\mu\nu}=4(\theta^{-1})_{\mu\nu}$, and thus $B:=\frac{4}{\theta_1}$, one has

$$H_L^{+} f_{m^1n^1} = B(m^1+\tfrac{1}{2})f_{m^1n^1}\;, \qquad\qquad H_L^{-} f_{m^1n^1} = B(n^1+\tfrac{1}{2})f_{m^1n^1}\;. \tag{3.10}$$

Thus, the harmonic oscillator basis has the additional merit of diagonalising the Landau Hamiltonian. This observation was the starting point of various exact solutions of quantum field theories on noncommutative phase space [Lan03, LSZ03, LSZ04].

Due to the choice (3.6), the D-dimensional generalisation of the harmonic oscillator base of $\mathbb{R}^D_\theta$ is

$$b_{mn}(x) = f_{m^1n^1}(x_1,x_2)\,f_{m^2n^2}(x_3,x_4)\dots f_{m^{D/2}n^{D/2}}(x_{D-1},x_D)\;, \tag{3.11}$$

$$\mathbb{N}^{\frac{D}{2}} \ni m \equiv \begin{matrix} m^1\\ m^2\\ \vdots\\ m^{D/2}\end{matrix}\;, \qquad \mathbb{N}^{\frac{D}{2}} \ni n \equiv \begin{matrix} n^1\\ n^2\\ \vdots\\ n^{D/2}\end{matrix}\;. \tag{3.12}$$

The identification (3.12) will frequently be used in this Habilitation thesis, in particular in $D=4$ dimensions. As we derive in Appendix B.1, this base satisfies

$$(b_{mn}\star b_{kl})(x) = \delta_{nk}b_{ml}(x)\;, \qquad\qquad \int d^Dx\, b_{mn} = (2\pi)^{D/2}\sqrt{\det\theta}\,\delta_{mn}\;. \tag{3.13}$$

Thus, the $f_{m^1n^1}$ behave like infinite standard matrices with entry 1 at the intersection of the $(m^1+1)^{\text{th}}$ row with the $(l^1+1)^{\text{th}}$ column, and with entry 0 everywhere else. In fact, the decomposition

$$\mathbb{R}^2_\theta \ni a(x) = \sum_{m^1,n^1=0}^{\infty} a_{m^1n^1}f_{m^1n^1}(x) \tag{3.14}$$

defines a Fréchet algebra isomorphism between $\mathbb{R}^2_\theta$ and the matrix algebra of rapidly decreasing double sequences $\{a_{m^1n^1}\}$ for which

$$r_k(a) := \Big(\sum_{m^1,n^1=0}^{\infty} \theta_1^{2k}(m^1+\tfrac{1}{2})^k(n^1+\tfrac{1}{2})^k|a_{m^1n^1}|^2\Big)^{\frac{1}{2}} \tag{3.15}$$

is finite for all $k \in \mathbb{N}$, see [GBV88].

For more information about the noncommutative $\mathbb{R}^D$ I refer to [GBV88, VGB88, GGBI$^+$04].

The Moyal plane is closely related to the noncommutative torus, which is the best-studied noncommutative space [Con80, Rie81]. A basis for the algebra $\mathbb{T}^D_\theta$ of the noncommutative D-torus is given by unitarities U^p labelled by $p = \{p_\mu\} \in \mathbb{Z}^D$, with $U^p(U^p)^* = (U^p)^* U^p = 1$. The multiplication is defined by

$$U^p U^q = \mathrm{e}^{\mathrm{i}\pi\theta^{\mu\nu}p_\mu q_\nu} U^{p+q} \;, \qquad \mu,\nu = 1,\dots,D \;, \quad \theta^{\mu\nu} = -\theta^{\nu\mu} \in \mathbb{R} \;. \tag{3.16}$$

Elements $a \in \mathbb{T}^d_\theta$ have the following form:

$$a = \sum_{p\in\mathbb{Z}^d} a_p U^p \;, \qquad a_p \in \mathbb{C} \;, \quad \|p\|^n |a_p| \to 0 \text{ for } \|p\| \to \infty \;. \tag{3.17}$$

If $\theta^{\mu\nu} \notin \mathbb{Q}$ (irrational case) one can define partial derivatives

$$\partial_\mu U^p := -\mathrm{i} p_\mu U^p \;, \tag{3.18}$$

which satisfy the Leibniz rule and Stokes' law with respect to the integral

$$\int a = a_0 \;, \tag{3.19}$$

where a is given by (3.17).

An excellent presentation of the noncommutative torus was given by Rieffel [Rie90].

3.2 Field theory on noncommutative $\mathbb{R}^D$

A field theory is defined by an action functional. The most natural action from the point of view of noncommutative geometry is $U(N)$ Yang-Mills theory in four dimensions:

$$S_{\mathrm{YM}}[A] = \int d^4x \,\mathrm{tr}_{M_N(\mathbb{C})}\Big(\frac{1}{4g^2} F_{\mu\nu} \star F^{\mu\nu}\Big) \;, \tag{3.20}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}(A_\mu \star A_\nu - A_\nu \star A_\mu) \;, \qquad A_\mu = A_\mu^* \in \mathbb{R}^4_\theta \otimes M_N(\mathbb{C}) \;. \tag{3.21}$$

This action arises from the Connes-Lott action functional [Gay03] and the spectral action principle [Vas04, GI04] as well as in the zero-slope limit of string theory [SW99]. For quantum field theory it has to be extended—as usual—by the ghost sector:

$$S_{\mathrm{gf}} = \int d^4x \,\mathrm{tr}_{M_N(\mathbb{C})}\Big(s\Big\{\bar{c} \star \partial_\mu A^\mu + \frac{\alpha}{2}\bar{c} \star B + \rho^\mu \star A_\mu + \sigma \star c\Big\}\Big) \;, \tag{3.22}$$

where α is the gauge parameter. The components of $\bar{c}, c, \rho^\mu$ are anticommuting fields and the graded BRST differential s [BRS76] (which commutes with ∂_μ) is defined by

$$\begin{aligned} sA_\mu &= \partial_\mu c - \mathrm{i}(A_\mu \star c - c \star A_\mu) \;, & sc &= \mathrm{i} c \star c \;, \\ s\bar{c} &= B \;, & sB &= s\rho^\mu = s\sigma = 0 \;. \end{aligned} \tag{3.23}$$

The external fields ρ^μ and σ are the Batalin-Vilkovisky antifields [BV81] relative to A_μ and c, respectively. For $N = 1$, one-loop renormalisability of the quantum field theory associated with the action $S_{\rm YM} + S_{\rm gf}$ was proven in [MSR99].

Note that each $\boldsymbol{\theta}_i$ in (3.6) is invariant under two-dimensional rotations. This means that action functionals which involve the $\star$-product like (3.20) are invariant under the subgroup $\big(SO(2)\big)^{\frac{D}{2}}$ of the D-dimensional rotation group $SO(D)$.

The Yang-Mills action (3.20) suggests that action functionals for field theories on $\mathbb{R}^D_\theta$ are simply obtained by replacing the ordinary (commutative) product of functions on Euclidean space by the $\star$-product (3.1). This procedure leads to the following action for noncommutative ϕ^4-theory:

$$S[\phi] := \int d^Dx \Big(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{1}{2}m^2\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x)\;. \tag{3.24}$$

It must be stressed, however, that this is a formal procedure and that—in contrast to the Yang-Mills action (3.20)—the scalar field action (3.24) does not directly follow from noncommutative geometry or the scaling limit of string theory [SW99]. In fact, I am going to prove that it has to be extended according to Theorem 1 on page 5.

It was pointed out by Langmann and Szabo [LS02a] that the $\star$-product interaction is (up to rescaling) invariant under a duality transformation between positions and momenta. Indeed, using a modified Fourier transformation $\hat\phi(p_a) = \int d^4x\ \mathrm{e}^{(-1)^a\mathrm{i}p_{a,\mu}x_a^\mu}\phi(x_a)$, where the subscript a refers to the cyclic order in the $\star$-product, one obtains from the definitions (3.1) and (3.4) and the reality $\phi(x) = \overline{\phi(x)}$ the representation

$$\begin{aligned} S_{\rm int}[\phi;\lambda] &= \int d^4x\ \frac{\lambda}{4!}(\phi\star\phi\star\phi\star\phi)(x) \\ &= \int\Big(\prod_{a=1}^4 d^4x_a\Big)\ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\, V(x_1,x_2,x_3,x_4) \qquad (3.25\text{a})\\ &= \int\Big(\prod_{a=1}^4 \frac{d^4p_a}{(2\pi)^4}\Big)\ \hat\phi(p_1)\hat\phi(p_2)\hat\phi(p_3)\hat\phi(p_4)\, \hat V(p_1,p_2,p_3,p_4)\;, \qquad (3.25\text{b})\end{aligned}$$

with

$$\begin{aligned}\hat V(p_1,p_2,p_3,p_4) &= \frac{\lambda}{4!}(2\pi)^4\delta^4(p_1{-}p_2{+}p_3{-}p_4)\ \cos\Big(\tfrac{1}{2}\theta^{\mu\nu}(p_{1,\mu}p_{2,\nu}+p_{3,\mu}p_{4,\nu})\Big)\;, \qquad (3.26\text{a})\\ V(x_1,x_2,x_3,x_4) &= \frac{\lambda}{4!}\frac{1}{\pi^4\det\theta}\,\delta^4(x_1{-}x_2{+}x_3{-}x_4)\ \cos\Big(2(\theta^{-1})_{\mu\nu}(x_1^\mu x_2^\nu + x_3^\mu x_4^\nu)\Big)\;. \qquad (3.26\text{b})\end{aligned}$$

Thus, the replacements

$$\hat\phi(p) \leftrightarrow \pi^2\sqrt{|\det\theta|}\ \phi(x)\;, \qquad\qquad p_\mu \leftrightarrow \tilde x_\mu := 2(\theta^{-1})_{\mu\nu}x^\nu\;, \tag{3.27}$$

exchange the a,b-versions of (3.25) and (3.26).

On the other hand, the usual free scalar field action given by $\lambda = 0$ in (3.24) is not invariant under that duality transformation. In order to achieve this we have to extend the free scalar field action by a harmonic oscillator potential:

$$S_{\rm free}[\phi;\mu,\Omega_0] = \int d^4x\ \Big(\frac{1}{2}(\partial_\mu\phi)\star(\partial^\mu\phi) + \frac{\Omega^2}{2}(\tilde x_\mu\phi)\star(\tilde x^\mu\phi) + \frac{\mu_0^2}{2}\phi\star\phi\Big)(x)\;. \tag{3.28}$$

Of course, the oscillator potential breaks translation invariance. For complex scalar fields φ of electric charge Ω, another possibility is given by a constant external magnetic field $B_{\mu\nu} = 4(\theta^{-1})_{\mu\nu}$ via the covariant derivative $D_\mu\varphi := \partial_\mu\varphi + \mathrm{i}\Omega A_\mu\varphi$, with $A_\mu = \frac{1}{2}B_{\mu\nu}x^\nu$:

$$S^B_{\text{free}}[\varphi;\mu_0,\Omega] = \int d^4x\,\Big(\frac{1}{2}(D_\mu\varphi)^*\star(D^\mu\varphi) + \frac{\mu_0^2}{2}\varphi^*\star\varphi\Big)(x)\;. \tag{3.29}$$

Adding the interaction term $S_{\text{int}}[\varphi;\lambda] = \frac{\lambda}{4!}\int d^4x\,\varphi\star\varphi^*\star\varphi\star\varphi^*$, the quantum field theory associated with the magnetic field action (3.29) was analysed and for $\Omega = 1$ exactly solved in [LSZ03, LSZ04]. Note that

$$S_{\text{free}}[\phi_1;\mu_0,\Omega] + S_{\text{free}}[\phi_2;\mu_0,\Omega] = \frac{1}{2}S^B_{\text{free}}[\phi_1{+}\mathrm{i}\phi_2;\mu_0,\Omega] + \frac{1}{2}S^B_{\text{free}}[\phi_1{+}\mathrm{i}\phi_2;\mu_0,-\Omega]\;. \tag{3.30}$$

The interaction mixes ϕ_1,ϕ_2, though.

Now, under the transformation (3.27) one has for the total action (1.5)

$$S\big[\phi;\mu_0,\lambda,\Omega\big] \mapsto \Omega^2 S\Big[\phi;\frac{\mu_0}{\Omega},\frac{\lambda}{\Omega^2},\frac{1}{\Omega}\Big]\;, \tag{3.31}$$

and accordingly for $S^B_{\text{free}}[\varphi,\Omega]+S_{\text{int}}[\varphi,\lambda]$. In the special case $\Omega = 1$ the action $S[\phi;\mu_0,\lambda,1]$ is invariant under the duality (3.27) and can be written as a standard matrix model. This was exploited in [LSZ03, LSZ04].

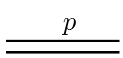
Before starting to analyse the duality-covariant model I would like to make some comments on the quantum field theory associated with the action (3.24). As usual, the Euclidean quantum field theory is (formally) defined via the partition function,

$$Z[J] := \int \mathcal{D}[\phi]\,\mathrm{e}^{-S[\phi]-\int d^Dx\,J(x)\phi(x)}\;. \tag{3.32}$$

We suppose here that the fields are expanded in the Weyl basis $\phi(x) = \int \frac{d^Dp}{(2\pi)^D}\,\phi(p)\mathrm{e}^{\mathrm{i}px}$, where $\phi(p)$ are commuting amplitudes of rapid decay in $\|p\|$ and $\mathrm{e}^{\mathrm{i}px}$ is the base of an appropriate subalgebra of the Moyal algebra $M(\mathbb{R}^D_\theta)$ (see page 24). Then, the "measure" of the functional integration is formally defined as $\mathcal{D}[\phi] = \prod_{p\in\mathbb{R}^D} d\phi(p)$.

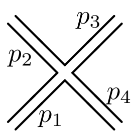
As usual, the integral (3.32) is solved perturbatively about the solution of the free theory given by $\lambda = 0$. The solution is conveniently organised by *Feynman graphs* built according to *Feynman rules* out of propagators and vertices. For the noncommutative scalar field action (3.24), the representation (3.4) leads to the following rules:

- Due to (3.2), the propagator is unchanged compared with commutative ϕ^4-theory, but for later purpose written in double line notation:



$$= \frac{1}{p^2+m^2}\;. \tag{3.33}$$

- The vertices receive phase factors [Fil96] which depend on the cyclic order of the legs:



$$= \frac{\lambda}{4!}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\sum_{i<j}p_i^\mu p_j^\nu\theta_{\mu\nu}}\;. \tag{3.34}$$

There is momentum conservation $p_1+p_2+p_3+p_4 = 0$ at each vertex (due to translation invariance of (3.24)).

The double line notation reflects the fact that the vertex (3.34) is invariant only under cyclic permutations of the legs (using momentum conservation). The resulting Feynman graphs are *ribbon graphs* [Haw99, CR00] which depend crucially on how the valences of the vertices are connected. For *planar graphs* the total phase factor of the integrand is independent of internal momenta, whereas *non-planar graphs* have a total phase factor which involves internal momenta. Planar graphs are integrated as usual and give (up to symmetry factors) the same divergences as commutative ϕ^4-theory [Fil96]. One would remove these divergences as usual by appropriate normalisation conditions for physical correlation functions. Non-planar graphs require a separate treatment.

I refrain from evaluating the standard one-loop graphs which is done in hundreds of papers. Instead, I review the main ideas of the remarkable power-counting analysis of Chepelev and Roiban [CR00, CR01]. First, there is a closed formula for the integral associated to a noncommutative Feynman graph in terms of the intersection matrices I, J, K which encode the phase factors and the incidence matrix $\mathcal{E}$. We give an orientation to each inner line l and let k_l be the momentum flowing through the line l. For each vertex v we define[11]

$$\mathcal{E}_{vl} = \begin{cases} 1 & \text{if } l \text{ leaves from } v\ , \\ -1 & \text{if } l \text{ arrives at } v\ , \\ 0 & \text{if } l \text{ is not attached to } v\ . \end{cases} \tag{3.35}$$

We let P_v be the total external momentum flowing into the vertex v. Restricting ourselves to 4 dimensions, an 1PI (one-particle irreducible) Feynman graph $\mathcal{G}$ with I internal lines and V vertices gives rise to the integral

$$\begin{aligned} \mathcal{I}_{\mathcal{G}}(P) = \int \prod_{l=1}^{I} \frac{d^4k_l}{(k_l^2+m^2)} \prod_{v=1}^{V} (2\pi)^4 \delta\Big(P_v - \sum_{l=1}^{l} \mathcal{E}_{vl} k_l\Big) \\ \times \exp \mathrm{i}\theta_{\mu\nu} \Big(\sum_{m,n=1}^{I} I^{mn} k_m^\mu k_n^\nu + \sum_{m=1}^{I} \sum_{v=1}^{V} J^{mv} k_m^\mu P_v^\nu + \sum_{v,w=1}^{V} K^{vw} P_v^\mu P_w^\nu \Big)\ . \end{aligned} \tag{3.36}$$

One can show that $I^{mn}, J^{mv}, K^{vw} \in \{1, -1, 0\}$ after use of momentum conservation [Fil96].

Next, one introduces Schwinger parameters $\frac{1}{k^2+m^2} = \int d\alpha\, \mathrm{e}^{-\alpha(k^2+m^2)}$ and the identity $(2\pi)^4\delta(q_v) = \int d^4y_v\ \mathrm{e}^{\mathrm{i}y_v q_v}$ for each vertex in (3.36), then complete the squares in k and performs the Gaußian k-integrations[12]. Writing $y_{\bar{v}} = y_V + z_{\bar{v}}$ for $\bar{v} = 1, \dots, V-1$ one has $\sum_{v=1}^{V} y_v \mathcal{E}_{vl} = \sum_{\bar{v}=1}^{V-1} z_{\bar{v}} \bar{\mathcal{E}}^{\bar{v}l}$. The y_V-integration yields the overall momentum conservation. It remains to complete the squares for $z_{\bar{v}}$ and finally to evaluate the Gaußian

[11]We assume that tadpoles (a line starting and ending at the same vertex) are absent. In the final formula they can be taken into account [CR01].

[12]This means that the order of integrations is exchanged in an integral which is in general not absolutely convergent. Thus, the result (3.37) is based on a certain limiting procedure, which is not necessarily unique. That leaves the possibility of circumventing the UV/IR-problems arising from (3.37) by different limiting procedures. This observation was the starting point for my work.

z_v-integrations. The result is [CR00]

$$\begin{aligned}\mathcal{I}_{\mathcal{G}}(P) = (2\pi)^4\delta\Big(\sum_{v=1}^{V} P_v\Big)\frac{1}{16^I\pi^{2L}}\exp\Big(\mathrm{i}\theta_{\mu\nu}\sum_{v,w=1}^{V} K^{vw}P_v^\mu P_w^\nu\Big) \\ \times\int_0^\infty \prod_{l=1}^{I} d\alpha_l\,\frac{\mathrm{e}^{-\sum_{l=1}^{I}\alpha_l m^2}}{\sqrt{\det\mathcal{A}\det\mathcal{B}}}\,\exp\Big(-\frac{1}{4}(J\tilde{P})^T\mathcal{A}^{-1}(J\tilde{P}) \\ +\frac{1}{4}\big(\bar{\mathcal{E}}\mathcal{A}^{-1}(J\tilde{P})+2\mathrm{i}P'\big)^T\mathcal{B}^{-1}\big(\bar{\mathcal{E}}\mathcal{A}^{-1}(J\tilde{P})+2\mathrm{i}P'\big)\Big)\;,\end{aligned} \tag{3.37}$$

where

$$\begin{aligned}\mathcal{A}^{mn}_{\mu\nu} &:= \alpha_m\delta^{mn}\delta_{\mu\nu} - \mathrm{i}I^{mn}\theta_{\mu\nu}\;, & (J\tilde{P})^m_\mu &:= \sum_{v=1}^{V} J^{mv}\theta_{\mu\nu}P_v^\nu\;, \\ \bar{\mathcal{E}}^{\bar{v}l} &:= \mathcal{E}_{\bar{v}l}\quad\text{for } \bar{v}=1,\dots,V{-}1\;, & P'^{\bar{v}}_\mu &:= P^\mu_{\bar{v}}\quad\text{for } \bar{v}=1,\dots,V{-}1\;, \\ \mathcal{B}^{\bar{v}\bar{w}}_{\mu\nu} &:= \sum_{m,n=1}^{I}\bar{\mathcal{E}}^{\bar{v}m}(\mathcal{A}^{-1})^{\mu\nu}_{mn}\bar{\mathcal{E}}^{\bar{w}n}\;.\end{aligned} \tag{3.38}$$

The formula (3.37) is referred to as the parametric integral representation of a noncommutative Feynman graph. See also [MVRS00]. Actually, [CR01] treats a more general case where also derivative couplings are admitted.

Possible divergences of (3.37) show up in the $\alpha_i \to 0$ behaviour[13]. In order to analyse them one reparametrises the integration domain in (3.37), similar to the usual procedure described in [IZ80]. For each Hepp sector [Hep66]

$$\alpha_{\pi_1} \leq \alpha_{\pi_2} \leq \cdots \leq \alpha_{\pi_I} \qquad \text{related to a permutation } \pi \text{ of } 1,\dots,I \tag{3.39}$$

one defines $\alpha_{\pi_i} = \prod_{j=i}^{I}\beta_j^2$, with $0 \leq \beta_I < \infty$ and $0 \leq \beta_j \leq 1$ for $j \neq I$. The leading contribution for small β_j has a topological interpretation.

A ribbon graph can be drawn on a genus-g Riemann surface with possibly several holes to which the external legs are attached [CR00, CR01]. I will say more on ribbon graphs on Riemann surfaces in Section 4.3 starting on page 42. I will explain, in particular, how a ribbon graph $\mathcal{G}$ defines a Riemann surface. On such a Riemann surface one considers *cycles*, i.e. equivalence classes of closed paths which cannot be contracted to a point. According to homological algebra [Spr81], one actually factorises with respect to commutants, i.e. one considers the path $aba^{-1}b^{-1}$ involving two cycles a, b as trivial. We let $c_{\mathcal{G}}(\mathcal{G}_i)$ be the number of non-trivial cycles of the ribbon graph $\mathcal{G}$ wrapped by the subgraph $\mathcal{G}_i$. Next, there may exist external lines m, n such that the graph obtained by connecting m, n has to be drawn on a Riemann surface of genus $g_{mn} > g$. If this happens one declares an index $j(\mathcal{G}) = 1$, otherwise $j(\mathcal{G}) = 0$. The index extends to subgraphs by defining $j_{\mathcal{G}}(\mathcal{G}_i) = 1$ if there are external lines m, n of $\mathcal{G}$ which are already attached to $\mathcal{G}_i$ so that the line connecting m, n wraps a cycle of the additional genus $g \to g_{mn}$ of $\mathcal{G}$.

Now we can formulate the relation between the parametric integral representation and the topology of the ribbon graph. Each sector (3.39) of the α-parameters defines

[13] The mass term regularises the $\alpha \to \infty$ behaviour of (3.37). It should be possible to proceed accordingly for massless models using Lowenstein's trick of auxiliary masses [Low76].

a sequence of (possibly disconnected) subgraphs $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset \mathcal{G}_I = \mathcal{G}$, where $\mathcal{G}_i$ is made of the i double-lines $\pi_1, \dots, \pi_i$ and the vertices to which these lines are attached. If $\mathcal{G}_i$ forms L_i loops it has a power-counting degree of divergence $\omega_i = 4L_i - 2i$. Using sophisticated mathematical techniques on determinants (e.g. Cauchy-Binet theorem and Jacobi ratio theorem), Chepelev and Roiban have derived in [CR01] the following leading contribution to the integral:

$$\begin{aligned}\mathcal{I}_{\mathcal{G}}(P) = (2\pi)^4 \delta\Big(\sum_{v=1}^{V} P_v\Big) \frac{1}{8^I \pi^{2L} (\det\theta)^g} \exp\Big(\mathrm{i}\theta_{\mu\nu} \sum_{v,w=1}^{V} K^{vw} P_v^\mu P_w^\nu\Big) \\ \times \sum_{\text{Hepp sectors}} \int_0^\infty \frac{d\beta_I\, \mathrm{e}^{-\beta_I^2 m^2}}{\beta_I^{1+\omega_I - 4c_{\mathcal{G}}(\mathcal{G})}} \int_0^1 \Big(\prod_{i=1}^{I-1} \frac{d\beta_i}{\beta_i^{1+\omega_i - 4c_{\mathcal{G}}(\mathcal{G}_i)}}\Big) \\ \times \exp\Big(-f_\pi(P) \prod_{i=1}^{I} \frac{1}{\beta^{2j_{\mathcal{G}}(\mathcal{G}_i)}}\Big) \Big(1 + \mathcal{O}(\beta^2)\Big)\,, \qquad (3.40)\end{aligned}$$

where $f_\pi(P) \geq 0$, with equality for *exceptional momenta*. In order to obtain a finite integral $\mathcal{I}_{\mathcal{G}}$, one obviously needs

1. $\omega_i - 4c_{\mathcal{G}}(\mathcal{G}_i) < 0$ for all i if $j(\mathcal{G}) = 0$ or $j(\mathcal{G}) = 1$ but the external momenta are exceptional, or

2. $\omega_i - 4c_{\mathcal{G}}(\mathcal{G}_i) < 0$ or $j_{\mathcal{G}}(\mathcal{G}_i) = 1$ for all i if $j(\mathcal{G}) = 1$ and the external momenta are non-exceptional.

There are two types of divergences where these conditions are violated.

First let the non-planarity be due to internal lines only, $j(\mathcal{G}) = 0$. Since the total graph $\mathcal{G}$ is non-planar, one has $c_{\mathcal{G}}(\mathcal{G}) > 0$ and therefore no superficial divergence. However, there might exist subgraphs $\mathcal{G}_i$ related to a Hepp sector of integration (3.39) where $\omega_i - 4c_{\mathcal{G}}(\mathcal{G}_i) \geq 0$. Such a situation requires *disconnected*[14] loops wrapping the same handle of the Riemann surface. In this case the integral (3.37) does not exist unless one introduces a regulator. The problem is that such a subdivergence may appear in graphs with an arbitrary number of external lines. In the commutative theory this also happens, but there one renormalise already the subdivergence. This procedure is based on normalisation conditions, which can only be imposed for *local* divergences. Since a non-planar graph wrapping a handle of a Riemann surface is clearly a non-local object (it cannot be reduced to a point, i.e. a counterterm vertex), it is not possible in the noncommutative case to remove that subdivergence. We are thus forced to use normalisation conditions for the total graph, but as the problem is independent of the number of external legs of the total graph, we finally need an infinite number of normalisation conditions. Hence, the model is not renormalisable in the standard way. This is the UV/IR-mixing problem.

[14] I have the impression that the problem with disconnected graphs as discovered by Chepelev and Roiban is completely ignored in the recent literature. Therefore, I have to stress the following: In renormalisation schemes for noncommutative quantum field theories which are based on the forest formula, it is not possible to restrict oneself to connected graphs. The reason is that, in contrast to the commutative situation, disconnected subgraphs can be coupled in the noncommutative case via the topology of the Riemann surface defined by the total graph.

The proposal to treat this problem is a reordering of the perturbation series [MVRS00], but a complete proof is missing[15]. Clearly, the problem is absent in theories with only logarithmic divergences.

The second class of problems is found in graphs where the non-planarity is at least partly due to the external legs, $j(\mathcal{G}) = 1$. This means that there is no way to remove possible divergences in these graphs by normalisation conditions. Fortunately, these graphs are superficially finite as long as the external momenta are non-exceptional. Subdivergences are supposed to be treated by a resummation. However, since the non-exceptional external momenta can become arbitrarily close to exceptional ones, these graphs are unbounded: For every $\delta > 0$ one finds non-exceptional momenta $\{p_n\}$ such that $\big|\langle\phi(p_1)\dots\phi(p_n)\rangle\big| > \frac{1}{\delta}$. This problem also arises in models with only logarithmic divergences.

3.3 The duality-covariant ϕ^4-action in the matrix base

I now return to the duality-covariant ϕ^4-model given by the action (1.5) on page 5. As already mentioned, we have to proceed in the matrix base (3.11) of $\mathbb{R}^D_\theta$, which means that the fields are expanded according to

$$\phi(x) = \sum_{m,n\in\mathbb{N}^{\frac{D}{2}}} \phi_{mn} b_{mn}(x) \;. \tag{3.41}$$

This has the advantage that the $\star$-product (3.1) is represented by a product (3.13) of infinite matrices and that according to (B.4) the multiplication by x^ρ is easy to realise. On the other hand, the kinetic term (3.28) of that action becomes very complicated. We can thus rewrite the action (1.5) as follows:

$$S[\phi;\mu_0,\lambda,\Omega] = (2\pi)^{\frac{D}{2}}\sqrt{\det\theta} \sum_{m,n,k,l\in\mathbb{N}^{\frac{D}{2}}} \Big(\frac{1}{2}G_{mn;kl}\phi_{mn}\phi_{kl} + \frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\Big) \;, \tag{3.42}$$

where

$$G_{mn;kl} := \int \frac{d^Dx}{(2\pi)^{\frac{D}{2}}\sqrt{\det\theta}} \Big(\partial_\mu b_{mn}\star\partial^\mu b_{kl} + \Omega^2(\tilde{x}_\mu b_{mn})\star(\tilde{x}^\mu b_{kl}) + \mu_0^2\, b_{mn}\star b_{kl}\Big)(x) \;. \tag{3.43}$$

I cite the two-dimensional result from (B.11) in Appendix B.1:

$$\begin{aligned} G_{m^1n^1;k^1l^1} &= \Big(\mu_0^2 + \frac{2(1{+}\Omega^2)}{\theta_1}(m^1{+}n^1{+}1)\Big)\delta_{n^1k^1}\delta_{m^1l^1} \\ &\quad - \frac{2(1{-}\Omega^2)}{\theta_1}\big(\sqrt{(n^1{+}1)(m^1{+}1)}\delta_{n^1+1,k^1}\delta_{m^1+1,l^1} + \sqrt{n^1m^1}\delta_{n^1-1,k^1}\delta_{m^1-1,l^1}\big) \;. \end{aligned} \tag{3.44}$$

[15] I conjecture that the result of such a reordering and resummation procedure would be equivalent to the duality-covariant ϕ^4-action, but I cannot prove this idea.

The self-dual case $\Omega = 1$ is particularly simple. Due to (3.11) it is not difficult to generalise this result to four dimensions:

$$
\begin{aligned}
&G_{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{k^1 \\ k^2}\substack{l^1 \\ l^2}} \\
&= \Big(\mu_0^2 + \frac{2}{\theta_1}(1+\Omega^2)(n^1+m^1+1) + \frac{2}{\theta_2}(1+\Omega^2)(n^2+m^2+1)\Big)\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2} \\
&- \frac{2}{\theta_1}(1-\Omega^2)\big(\sqrt{(n^1+1)(m^1+1)}\,\delta_{n^1+1,k^1}\delta_{m^1+1,l^1} + \sqrt{n^1m^1}\,\delta_{n^1-1,k^1}\delta_{m^1-1,l^1}\big)\delta_{n^2k^2}\delta_{m^2l^2} \\
&- \frac{2}{\theta_2}(1-\Omega^2)\big(\sqrt{(n^2+1)(m^2+1)}\,\delta_{n^2+1,k^2}\delta_{m^2+1,l^2} + \sqrt{n^2m^2}\,\delta_{n^2-1,k^2}\delta_{m^2-1,l^2}\big)\delta_{n^1k^1}\delta_{m^1l^1}\;.
\end{aligned}
\tag{3.45}
$$

The D-dimensional generalisation is obvious.

The quantum field theory is defined by the partition function

$$
Z[J] = \int \Big(\prod_{a,b\in\mathbb{N}^{\frac{D}{2}}} d\phi_{ab}\Big)\, \exp\Big(-S[\phi] - (2\pi)^{\frac{D}{2}}\sqrt{\det\theta} \sum_{m,n\in\mathbb{N}^{\frac{D}{2}}} \phi_{mn}J_{nm}\Big)\;. \tag{3.46}
$$

The measure used in (3.46) differs by the Jacobian $\big|\det \frac{\partial\phi(p)}{\partial\phi_{mn}}\big| = \big|\det \tilde{f}_{mn}(p,\psi)\big|$ from the measure of the partition function (3.32), where $\tilde{f}_{mn}(p,\psi)$ is evaluated in (B.15). As the Jacobian is field-independent, it can safely be ignored. For instance, raising it to the action à la Faddeev-Popov [FP67], we obtain a free ghost sector which decouples from the fields ϕ_{mn}.

For the free theory defined by $\lambda = 0$ in (3.42), the solution of (3.46) is given by

$$
Z[J]\big|_{\lambda=0} = Z[0]\,\exp\Big((2\pi)^{\frac{D}{2}}\sqrt{\det\theta} \sum_{m,n,k,l\in\mathbb{N}^{\frac{D}{2}}} \frac{1}{2}J_{mn}\Delta_{mn;kl}J_{kl}\Big)\;, \tag{3.47}
$$

where the propagator Δ is defined as the inverse of the kinetic matrix G:

$$
\sum_{k,l\in\mathbb{N}^{\frac{D}{2}}} G_{mn;kl}\Delta_{lk;sr} = \sum_{k,l\in\mathbb{N}^{\frac{D}{2}}} \Delta_{nm;lk}G_{kl;rs} = \delta_{mr}\delta_{ns}\;. \tag{3.48}
$$

I compute the propagator in $D = 4$ dimensions in Appendix B.3, after diagonalising the kinetic matrix (3.45) in Appendix B.2. For $\theta_1 = \theta_2 \equiv \theta$ the result is

$$
\begin{aligned}
&\Delta_{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{k^1 \\ k^2}\substack{l^1 \\ l^2}} = \frac{\theta}{2(1+\Omega)^2}\delta_{m^1+k^1,n^1+l^1}\delta_{m^2+k^2,n^2+l^2} \\
&\times \sum_{v^1=\frac{|m^1-l^1|}{2}}^{\frac{\min(m^1+l^1,n^1+k^1)}{2}} \;\sum_{v^2=\frac{|m^2-l^2|}{2}}^{\frac{\min(m^2+l^2,n^2+k^2)}{2}} B\big(1+\tfrac{\mu_0^2\theta}{8\Omega}+\tfrac{1}{2}(m^1+m^2+k^1+k^2)-v^1-v^2, 1+2v^1+2v^2\big) \\
&\times {}_2F_1\bigg(\genfrac{}{}{0pt}{}{1+2v^1+2v^2\,,\; \frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(m^1+m^2+k^1+k^2)+v^1+v^2}{2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(m^1+m^2+k^1+k^2)+v^1+v^2}\bigg|\frac{(1-\Omega)^2}{(1+\Omega)^2}\bigg) \\
&\times \prod_{i=1}^{2}\sqrt{\binom{n^i}{v^i+\frac{n^i-k^i}{2}}\binom{k^i}{v^i+\frac{k^i-n^i}{2}}\binom{m^i}{v^i+\frac{m^i-l^i}{2}}\binom{l^i}{v^i+\frac{l^i-m^i}{2}}}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{2v^i}\;.
\end{aligned}
\tag{3.49}
$$

Here, $B(a,b)$ is the Beta-function and ${}_2F_1\big(\genfrac{}{}{0pt}{}{a,b}{c}\big|z\big)$ the hypergeometric function. In (3.49) one should appreciate the finiteness of the sum, i.e. we have obtained a closed solution of the partition function of the free theory ($\lambda = 0$) with respect to the preferred base of the interaction. The D-dimensional generalisation is not difficult.

The usual procedure would be to solve the interacting theory perturbatively:

$$Z[J] = Z[0] \exp\Big(-V\Big[\frac{\partial}{\partial J}\Big]\Big) \exp\Big((2\pi)^{\frac{D}{2}}\sqrt{\det\theta} \sum_{m,n,k,l\in\mathbb{N}^{\frac{D}{2}}} \frac{1}{2} J_{mn}\Delta_{mn;kl}J_{kl}\Big) ,$$
$$V\Big[\frac{\partial}{\partial J}\Big] := \frac{\lambda}{4!((2\pi)^{\frac{D}{2}}\sqrt{\det\theta})^3} \sum_{m,n,k,l\in\mathbb{N}^{\frac{D}{2}}} \frac{\partial^4}{\partial J_{ml}\,\partial J_{lk}\,\partial J_{kn}\,\partial J_{nm}} . \tag{3.50}$$

It is convenient to pass to the generating functional of connected Green's functions, $W[J] = \ln Z[J]$:

$$W[J] = \ln Z[0] + W_{\text{free}}[J] + \ln\bigg(1 + \mathrm{e}^{-W_{\text{free}}[J]}\Big(\exp\Big(-V\Big[\frac{\partial}{\partial J}\Big]\Big) - 1\Big)\mathrm{e}^{W_{\text{free}}[J]}\bigg) ,$$
$$W_{\text{free}}[J] := (2\pi)^{\frac{D}{2}}\sqrt{\det\theta} \sum_{m,n,k,l\in\mathbb{N}^{\frac{D}{2}}} \frac{1}{2} J_{mn}\Delta_{mn;kl}J_{kl} . \tag{3.51}$$

In order to obtain the expansion in λ one has to expand $\ln(1+x)$ as a power series in x and $\exp(V)$ as a power series in V. By Legendre transformation we pass to the generating functional of one-particle irreducible Green's functions:

$$\Gamma[\phi^{c\ell}] := (2\pi)^{\frac{D}{2}}\sqrt{\det\theta} \sum_{m,n\in\mathbb{N}^{\frac{D}{2}}} \phi^{c\ell}_{mn}J_{nm} - W[J] , \tag{3.52}$$

where J has to be replaced by the inverse solution of

$$\phi^{c\ell}_{mn} := \frac{1}{(2\pi)^{\frac{D}{2}}\sqrt{\det\theta}} \frac{\partial W[J]}{\partial J_{nm}} . \tag{3.53}$$

In principle, it should be possible to renormalise the action functional (3.53) by standard Feynman graph computations (involving loop *sums*) combined with an appropriate generalisation of the forest formula [Zim69]. However, the complicated structure of the propagator (3.49) makes it extremely difficult to proceed as in momentum space. Therefore, we shall use an adapted version of Polchinski's renormalisation proof based on flow equations for effective actions. This approach is conceptually much easier, in particular, there is no need to discuss overlapping divergences.

I will return to (3.53) in Section G when computing the one-loop β-function of the four-dimensional model.

4 Flow equations for non-local matrix models

According to the previous remarks, in particular those of Section 1.2, I am going to renormalise the duality-covariant noncommutative ϕ^4-model by means of flow equations. This requires an adaptation of the Wilson-Polchinski approach [WK74, Pol84] to matrices. In view of a future application to other examples, I find it convenient to develop the matrix formulation of the Wilson-Polchinski programme in a general context. Indeed, many noncommutative field theories have a matrix formulation. I think of fuzzy spaces [Mad92, GKP96b, GKP96a] and q-deformed models [GMS01, GMS02].

I derive in Section 4.1 the matrix version of the Polchinski equation which describes how the effective action $L[\phi,\Lambda]$ has to be adjusted with a variation of the cut-off scale Λ in order to make the partition function Λ-independent. The perturbative solution of the Polchinski equation is given by ribbon graphs as introduced in Section 4.3. In Section 4.4, I formulate the power-counting theorem for the effective action $L[\phi,\Lambda]$ obtained by solving (better: estimating) the Polchinski equation perturbatively. The very long proof is delegated to Appendix D.

We will see that the power-counting degree of divergence of a ribbon graph depends on the topological data of the graph and on two scaling dimensions of the cut-off propagator. In this way, suitable scaling dimensions provide a simple criterion to decide whether a non-local matrix model has the chance to be renormalisable or not. However, having the right scaling dimensions is not sufficient for the renormalisability of a model, because a divergent interaction is parametrised by an infinite number of matrix indices. Thus, a renormalisable model needs further structures which relate these infinitely many interaction coefficients to a finite number of base couplings. In case of the duality-covariant noncommutative ϕ^4-model, these additional structures are induced by properties of orthogonal polynomials.

My derivation and solution of the matrix Polchinski equation combines the original ideas of [Pol84] with some of the improvements made in [KKS92]. In particular, I follow [KKS92] to obtain Λ_0-independent estimations for the interaction coefficients. Another suggestion of [KKS92], the removal of the restrictions to the external parameters (here: the range of the matrix indices), will be important in Section 5. For an introduction into the classical techniques of renormalisation by flow equations (in momentum space) I refer to the monograph [Sal99].

4.1 The exact renormalisation group equation

We consider a ϕ^4-matrix model with a general (non-diagonal) kinetic term,

$$S[\phi] = \mathcal{V}_D\Big(\sum_{m,n,k,l}\frac{1}{2}G_{mn;kl}\phi_{mn}\phi_{kl} + \sum_{m,n,k,l}\frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\Big)\;, \tag{4.1}$$

where $m,n,k,l \in \mathbb{N}^q$. For the noncommutative $\mathbb{R}^D$, D even, we have $q = \frac{D}{2}$. The factor $\mathcal{V}_D$ is the volume of an elementary cell. The choice of ϕ^4 is no restriction but for us the most natural one because we are interested in four-dimensional models. Standard matrix models are given by

$$q = 1\;, \qquad G_{mn;kl} = \frac{1}{\mu_0^2}\delta_{ml}\delta_{nk}\;. \tag{4.2}$$

For reviews on matrix models and their applications I refer to [Dij91, DFGZJ95]. The idea to apply renormalisation group techniques to matrix models is also not new [BZJ92]. The difference of our approach is that we will not demand that the action can be written as a trace of a polynomial in the field, that is, we allow for matrix-valued kinetic terms. The only restriction we are imposing is

$$G_{mn;kl} = 0 \qquad \text{unless } m + k = n + l \;. \tag{4.3}$$

The restriction (4.3) is due to the fact that the action comes from a trace. It is verified for the noncommutative $\mathbb{R}^D$ due to the $\big(SO(2)\big)^{\frac{D}{2}}$-symmetry of both the interaction and the kinetic term. The kinetic matrix $G_{mn;kl}$ contains the entire information about the differential calculus, including the underlying (Riemannian) geometry, and the masses of the model. More important than the kinetic matrix G will be its inverse, the propagator Δ defined by

$$\sum_{k,l} G_{mn;kl}\Delta_{lk;sr} = \sum_{k,l} \Delta_{nm;lk}G_{kl;rs} = \delta_{mr}\delta_{ns} \;. \tag{4.4}$$

Due to (4.3) we have the same index restrictions for the propagator:

$$\Delta_{nm;lk} = 0 \qquad \text{unless } m + k = n + l \;. \tag{4.5}$$

Let us introduce a notion of locality:

Definition 5 *A matrix model is called* <u>*local*</u> *if* $\Delta_{nm;lk} = \Delta(m,n)\delta_{ml}\delta_{nk}$ *for some function* $\Delta(m,n)$*, otherwise* <u>*non-local*</u>*.*

We add sources J to the action (4.1) and define a (Euclidean) quantum field theory by the generating functional (partition function)

$$Z[J] = \int \mathcal{D}[\phi]\, \exp\big(-S[\phi] - \mathcal{V}_D \sum_{m,n} \phi_{mn}J_{nm}\big)\;, \qquad \mathcal{D}[\phi] = \prod_{m,n} d\phi_{mn}\;. \tag{4.6}$$

According to Polchinski's derivation of the exact renormalisation group equation we now consider a (at first sight) different problem than (4.6). Via a cut-off function $K[m,\Lambda]$, which is smooth in Λ and satisfies $K[m,\infty] = 1$, we modify the weight of a matrix index m as a function of a certain *scale* Λ:

$$Z[J,\Lambda] = \int \mathcal{D}[\phi]\, \exp\big(-S[\phi,J,\Lambda]\big)\;, \tag{4.7}$$

$$\begin{aligned} S[\phi,J,\Lambda] = \mathcal{V}_D\Big(&\sum_{m,n,k,l} \frac{1}{2}\phi_{mn}G^K_{mn;kl}(\Lambda)\,\phi_{kl} \\ &+ \sum_{m,n,k,l} \phi_{mn}F_{mn;kl}[\Lambda]J_{kl} + \sum_{m,n,k,l}\frac{1}{2}J_{mn}E_{mn;kl}[\Lambda]J_{kl} + L[\phi,\Lambda] + C[\Lambda]\Big)\;, \end{aligned} \tag{4.8}$$

$$G^K_{mn;kl}(\Lambda) := \Big(\prod_{i\in m,n,k,l} K[i,\Lambda]^{-1}\Big)G_{mn;kl}\;, \tag{4.9}$$

with $L[0,\Lambda]=0$. Accordingly, we define

$$\Delta^K_{nm;lk}(\Lambda)=\Big(\prod_{i\in m,n,k,l}K[i,\Lambda]\Big)\Delta_{nm;lk}\;. \tag{4.10}$$

For indices $m=(m^1,\dots,m^{\frac{D}{2}})\in\mathbb{N}^{\frac{D}{2}}$ we would write the cut-off function as a product $K[m,\Lambda]=\prod_{i=1}^{\frac{D}{2}}K\Big(\frac{m^i}{(\mathcal{V}_D)^{\frac{2}{D}}\Lambda^2}\Big)$ where $K(x)$ is a smooth function on $\mathbb{R}^+$ with $K(x)=1$ for $0\leq x\leq 1$ and $K(x)=\epsilon$ for $x\geq 2$. In the limit $\epsilon\to 0$, the partition function (2.8) vanishes unless $\phi_{mn}=0$ for $\max_i(m^i,n^i)\geq 2(\mathcal{V}_D)^{\frac{2}{D}}\Lambda^2$, thus implementing a cut-off of the measure $\mathcal{D}[\phi]=\prod_{a,b}d\phi_{ab}$ in (2.8). All other formulae involve positive powers of $K\Big(\frac{m^i}{(\mathcal{V}_D)^{\frac{2}{D}}\Lambda^2}\Big)$ which multiply through the cut-off propagator (5.3) the appearing matrix indices. In the limit $\epsilon\to 0$, $K[m,\Lambda]$ has finite support in m so that all infinite-sized matrices are reduced to finite ones.

The function $C[\Lambda]$ is the vacuum energy and the matrices E and F, which are not necessary in the commutative case, must be introduced because the propagator Δ is non-local. It is, in general, not possible to separate the support of the sources J from the support of the Λ-variation of K. Due to $K[m,\infty]=1$ we formally obtain (4.6) for $\Lambda\to\infty$ in (4.7) if we set

$$L[\phi,\infty]=\sum_{m,n,k,l}\frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\;,\quad C[\infty]=0\;,\quad E_{mn;kl}[\infty]=0\;,\quad F_{mn;kl}[\infty]=\delta_{ml}\delta_{nk}\;. \tag{4.11}$$

However, we shall expect divergences in the partition function which require a renormalisation, i.e. additional (divergent) counterterms in $L[\phi,\infty]$. In the Feynman graph solution of the partition function one carefully adapts these counterterms so that all divergences disappear. If such an adaptation is possible with a *finite number* of *local* counterterms, the model is considered as perturbatively renormalisable.

Following Polchinski [Pol84] we proceed differently to prove renormalisability. We first ask ourselves how to choose L,C,E,F in order to make $Z[J,\Lambda]$ *independent* of Λ. For this purpose I derive in Appendix C starting on page 113 the following identity:

$$\begin{aligned}0=\int\mathcal{D}[\phi]\,\mathcal{V}_D\Big(&\sum_{k,l,m,n}\frac{1}{2}\phi_{mn}\frac{\partial G^K_{mn;kl}(\Lambda)}{\partial\Lambda}\phi_{kl} &(4.12)\\ &+\sum_{k,l,m,n}\frac{1}{2}\frac{\partial\Delta^K_{lk;nm}(\Lambda)}{\partial\Lambda}\Big(\frac{\partial L[\phi,\Lambda]}{\partial\phi_{kl}}\frac{\partial L[\phi,\Lambda]}{\partial\phi_{mn}}-\frac{1}{\mathcal{V}_D}\frac{\partial^2 L[\phi,\Lambda]}{\partial\phi_{kl}\,\partial\phi_{mn}}\Big)\\ &-\sum_{k,l,m,n,r,s,t,u}\frac{1}{2}J_{tu}F^T_{tu;kl}[\Lambda]\frac{\partial\Delta^K_{lk;nm}(\Lambda)}{\partial\Lambda}F_{mn;rs}[\Lambda]J_{rs}\\ &-\sum_{k,l,m,n,r,s,t,u}\phi_{rs}G^K_{rs;kl}\frac{\partial\Delta^K_{lk;nm}(\Lambda)}{\partial\Lambda}F_{mn;tu}[\Lambda]J_{tu}+\frac{1}{\mathcal{V}_D}\sum_{m,n}\frac{\partial}{\partial\Lambda}\ln\big(K[m,\Lambda]K[n,\Lambda]\big)\Big)\\ &\times\exp(-S[\phi,J,\Lambda])\;. &(4.13)\end{aligned}$$

On the other hand, we differentiate (4.7) with respect to Λ, compare it with (4.13) and conclude

$$\frac{\partial}{\partial\Lambda} Z[J,\Lambda] = 0 \qquad \text{iff} \tag{4.14}$$

$$\Lambda \frac{\partial L[\phi,\Lambda]}{\partial\Lambda} = \sum_{m,n,k,l} \frac{1}{2} \Lambda \frac{\partial \Delta^K_{nm;lk}(\Lambda)}{\partial\Lambda} \Big(\frac{\partial L[\phi,\Lambda]}{\partial\phi_{mn}} \frac{\partial L[\phi,\Lambda]}{\partial\phi_{kl}} - \frac{1}{\mathcal{V}_D} \Big[\frac{\partial^2 L[\phi,\Lambda]}{\partial\phi_{mn}\,\partial\phi_{kl}} \Big]_\phi \Big) , \tag{4.15}$$

$$\Lambda \frac{\partial F_{mn;kl}[\Lambda]}{\partial\Lambda} = - \sum_{m',n',k',l'} G^K_{mn;m'n'}(\Lambda)\, \Lambda \frac{\partial \Delta^K_{n'm';l'k'}(\Lambda)}{\partial\Lambda} F_{k'l';kl}[\Lambda] , \tag{4.16}$$

$$\Lambda \frac{\partial E_{mn;kl}[\Lambda]}{\partial\Lambda} = - \sum_{m',n',k',l'} F^T_{mn;m'n'}[\Lambda]\, \Lambda \frac{\partial \Delta^K_{n'm';l'k'}(\Lambda)}{\partial\Lambda} F_{k'l';kl}[\Lambda] , \tag{4.17}$$

$$\begin{aligned} \Lambda \frac{\partial C[\Lambda]}{\partial\Lambda} &= \frac{1}{\mathcal{V}_D} \sum_{m,n} \Lambda \frac{\partial}{\partial\Lambda} \ln \big(K[m,\Lambda] K[n,\Lambda] \big) \\ &\quad - \frac{1}{2\mathcal{V}_D} \sum_{m,n,k,l} \Lambda \frac{\partial \Delta^K_{nm;lk}(\Lambda)}{\partial\Lambda} \frac{\partial^2 L[\phi,\Lambda]}{\partial\phi_{mn}\,\partial\phi_{kl}} \Big|_{\phi=0} , \end{aligned} \tag{4.18}$$

where $\big[f[\phi]\big]_\phi := f[\phi] - f[0]$. Naïvely we would integrate (4.15)–(4.18) for the initial conditions (4.11). Technically, this would be achieved by imposing the conditions (4.11) not at $\Lambda = \infty$ but at some finite scale $\Lambda = \Lambda_0$, followed by taking the limit $\Lambda_0 \to \infty$. This is easily done for (4.16)–(4.18):

$$F_{mn;kl}[\Lambda] = \sum_{m',n'} G^K_{mn;m'n'}(\Lambda) \Delta^K_{n'm';kl}(\Lambda_0) , \tag{4.19}$$

$$E_{mn;kl}[\Lambda] = \sum_{m',n',k',l'} \Delta^K_{mn;m'n'}(\Lambda_0) \Big(G^K_{n'm';l'k'}(\Lambda) - G^K_{n'm';l'k'}(\Lambda_0) \Big) \Delta^K_{k'l';kl}(\Lambda_0) , \tag{4.20}$$

$$\begin{aligned} C[\Lambda] &= \frac{2}{\mathcal{V}_D} \ln \Big(\prod_m K[m,\Lambda] K^{-1}[m,\Lambda_0] \Big) \\ &\quad + \frac{1}{2\mathcal{V}_D} \int_\Lambda^{\Lambda_0} d\Lambda' \sum_{m,n,k,l} \frac{\partial \Delta^K_{nm;lk}(\Lambda')}{\partial\Lambda'} \frac{\partial^2 L[\phi,\Lambda']}{\partial\phi_{mn}\,\partial\phi_{kl}} \Big|_{\phi=0} . \end{aligned} \tag{4.21}$$

At $\Lambda = \Lambda_0$ the functions F, E, C become independent of Λ_0 and satisfy, in particular, (4.11) in the limit $\Lambda_0 \to \infty$.

The differential equation (4.15), referred to as the Polchinski equation for matrix models, is of a different type than (4.16)–(4.17): it is a non-linear differential equation. Its integration is highly non-trivial. Before passing to the integration procedure, I would like to derive the physical interpretation of the effective action. For this purpose we insert the solutions (4.19) and (4.20) into the cut-off partition function (4.7). The vacuum energy $C[\Lambda]$ can be ignored, because it does not contribute to correlation functions. We thus obtain

$$
\begin{aligned}
Z[J,\Lambda,\Lambda_0] = \int \mathcal{D}[\phi]\; \exp\Big(&- \mathcal{V}_D \Big(\sum_{m,n,k,l} \frac{1}{2}\phi_{mn} G^K_{mn;kl}(\Lambda)\phi_{kl} \\
&+ \sum_{m,n,k,l,r,s,t,u} \frac{1}{2} J_{mn}\Delta^K_{nm;lk}(\Lambda_0)\big(G^K_{kl;rs}(\Lambda) - G^K_{kl;rs}(\Lambda_0)\big)\Delta^K_{sr;ut}(\Lambda_0) J_{tu} \\
&+ \sum_{m,n,k,l,r,s} \phi_{mn} G^K_{mn;kl}(\Lambda)\Delta^K_{lk;sr}(\Lambda_0) J_{rs} + L[\phi,\Lambda,\Lambda_0]\Big)\Big) \\
= \exp\Big(- \mathcal{V}_D & \sum_{m,n,k,l,r,s,t,u} \frac{1}{2} J_{mn}\Delta^K_{nm;lk}(\Lambda_0)\big(G^K_{kl;rs}(\Lambda) - G^K_{kl;rs}(\Lambda_0)\big)\Delta^K_{st;ut}(\Lambda_0) J_{tu}\Big) \\
\times \exp\Big(- \mathcal{V}_D & L[\phi,\Lambda,\Lambda_0]\Big|_{\phi_{mn}=\sum_{k,l,r,s} \Delta^K_{nm;lk}(\Lambda) G^K_{kl;rs}(\Lambda_0)\frac{\partial}{\partial J_{rs}}}\Big) \\
&\cdot \exp\Big(\mathcal{V}_D \sum_{m,n,k,l,r,s,t,u} \frac{1}{2} J_{mn}\Delta^K_{nm;lk}(\Lambda_0) G^K_{kl;rs}(\Lambda)\Delta^K_{sr;ut}(\Lambda_0) J_{tu}\Big)\; Z_0[\Lambda,\Lambda_0]\;,
\end{aligned}
\tag{4.22}
$$

where

$$
\begin{aligned}
Z_0[\Lambda,\Lambda_0] = \int \mathcal{D}[\phi]\; \exp\Big(- \mathcal{V}_D \sum_{m,n,k,l} \frac{1}{2}\big(\phi_{mn} + \sum_{r,s} J_{rs}\Delta^K_{sr;nm}(\Lambda_0)\big) G^K_{mn;kl}(\Lambda) & \\
\times \big(\phi_{kl} + \sum_{t,u} \Delta^K_{lk;ut}(\Lambda_0) J_{tu}\big)\Big)\;. &
\end{aligned}
\tag{4.23}
$$

Thus, Z is expanded into a series of Feynman graphs with vertices given by the Taylor expansion coefficients

$$
L_{m_1n_1;\dots;m_Nn_N}[\Lambda] := \frac{1}{N!}\Big(\frac{\partial^N L[\phi,\Lambda]}{\partial\phi_{m_1n_1}\,\partial\phi_{m_2n_2}\,\dots\partial\phi_{m_Nn_N}}\Big)_{\phi=0}
\tag{4.24}
$$

connected with each other by internal lines $\Delta^K(\Lambda)$ and to sources J by external lines $\Delta^K(\Lambda_0)$. We choose $K[m,\Lambda]$ to be a cut-off function, which means that K has finite support in m for finite Λ. Then, for finite Λ, the summation variables in the above Feynman graphs are via the propagator $\Delta^K(\Lambda)$ restricted to a finite set. Thus, loop summations are finite, provided that the interaction coefficients $L_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ are bounded. In other words, for the renormalisation of a non-local matrix model it is necessary to prove that the differential equation (4.15) admits a regular solution. As pointed out in the introduction to Section 4, to obtain a physically reasonable quantum field theory one has additionally to prove that there is a regular solution of (4.15) which depends on a *finite number of initial conditions* only. This requirement is difficult to fulfil because there is, a priori, an infinite number of degrees of freedom given by the Taylor expansion coefficients (4.24). This is the reason for the fact that renormalisable (four-dimensional) quantum field theories are rare.

4.2 On the integration of the Polchinski equation

We are going to integrate (4.15) between a certain renormalisation scale Λ_R and the initial scale Λ_0. We assume that $L_{m_1n_1;\dots;m_Nn_N}$ can be decomposed into parts $L^{(i)}_{m_1n_1;\dots;m_Nn_N}$ which for $\Lambda_R \leq \Lambda \leq \Lambda_0$ scale homogeneously:

$$\Big|\Lambda\frac{\partial L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]}{\partial\Lambda}\Big| \leq \Lambda^{r_i}\; P^{q_i}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;. \tag{4.25}$$

Here, $P^q[X] \geq 0$ stands for some polynomial of degree q in $X \geq 0$. Clearly, $P^q[X]$, for $X \geq 0$, can be further bound by a polynomial with non-negative coefficients. As usual we define

Definition 6 *Homogeneous parts $L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ in (4.25) with $r_i > 0$ are called* *relevant, with $r_i < 0$ irrelevant and with $r_i = 0$ marginal.*

There are two possibilities for the integration, either from Λ_0 down to Λ or from Λ_R up to Λ, corresponding to the identities

$$L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda] = L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0] - \int_{\Lambda}^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\Big(\Lambda'\frac{\partial}{\partial\Lambda'}L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda']\Big) \tag{4.26a}$$

$$= L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R] + \int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\Big(\Lambda'\frac{\partial}{\partial\Lambda'}L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda']\Big)\;. \tag{4.26b}$$

One has [GR00, §2.722]

$$\int dx\, x^{r-1}\Big(\ln\frac{x}{x_R}\Big)^q = \begin{cases} \dfrac{(-1)^q q!}{r^{q+1}}x^r\displaystyle\sum_{j=0}^{q}\frac{\big(-r\ln\frac{x}{x_R}\big)^j}{j!} + \text{const} & \text{for } r \neq 0\;, \\ \dfrac{1}{q{+}1}\Big(\ln\dfrac{x}{x_R}\Big)^{q+1} + \text{const} & \text{for } r = 0\;. \end{cases} \tag{4.27}$$

At the end we are interested in the limit $\Lambda_0 \to \infty$. This requires that positive powers of Λ_0 must be avoided in the estimations. For $r_i < 0$ we we can safely take the direction (4.26a) of integration and then, because all coefficients are positive, the limit $\Lambda_0 \to \infty$ in the integral of (4.26a). Thus,

$$\begin{aligned} \big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| &\leq \big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0]\big| + \int_{\Lambda}^{\infty}\frac{d\Lambda'}{\Lambda'}\Big|\Lambda'\frac{\partial}{\partial\Lambda'}L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda']\Big| \\ &\leq \big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0]\big| + \Lambda^{-|r_i|}P^{q_i}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;, \qquad \text{for } r_i < 0\;. \end{aligned} \tag{4.28}$$

Here, P^{q_i} is a new polynomial of degree q_i with non-negative coefficients. Now, the limit $\Lambda_0 \to \infty$ carried out later requires that $\big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0]\big|$ in (4.28) is bounded, i.e. $\big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0]\big| < \frac{C}{\Lambda_0^{s_i}}$, with $s_i > 0$. As the resulting estimation (4.28) is further iterated, s_i must be sufficiently large. We do not investigate this question in detail and simply note that it is safe to require

$$\big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0]\big| < \Lambda_0^{-|r_i|}P^{q_i}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\;, \qquad \text{for } r_i < 0\;, \tag{4.29}$$

for the boundary condition.

In the other case $r_i \geq 0$, the integration direction (4.26a) will produce divergences in $\Lambda_0 \to \infty$. Thus, we have to choose the other direction (4.26b). The integration (4.27) produces alternating signs, but these can be ignored in the maximisation. The only contribution from the lower bound Λ_R in the integral of (4.26b) is the term with $j = 0$ in (4.27). There, we can obviously ignore it in the difference $\Lambda^r - \Lambda_R^r$. We thus obtain from (4.27) the estimation

$$\big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \leq \big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R]\big| + \begin{cases} \Lambda^{r_i} P^{q_i}\Big[\ln \frac{\Lambda}{\Lambda_R}\Big] & \text{for } r_i > 0 \\ P^{q_i+1}\Big[\ln \frac{\Lambda}{\Lambda_R}\Big] & \text{for } r_i = 0\,. \end{cases} \tag{4.30}$$

The reduction from $P\big[\ln \frac{\Lambda_0}{\Lambda_R}\big]$ in Polchinski's original work [Pol84] to $P\big[\ln \frac{\Lambda}{\Lambda_R}\big]$ is due to [KKS92]. We can summarise these considerations as follows:

Definition/Lemma 7 *Let $\big|\Lambda\frac{\partial}{\partial\Lambda} L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big|$ be bounded by (4.25),*

$$\Big|\Lambda \frac{\partial L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]}{\partial\Lambda}\Big| \leq \Lambda^{r_i} P^{q_i}\Big[\ln \frac{\Lambda}{\Lambda_R}\Big]\,. \tag{4.31}$$

The integration of (4.31) is for irrelevant interactions performed from Λ_0 down to Λ starting from an initial condition bounded by $\big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0]\big| < \Lambda_0^{-|r_i|} P^{q_i}\big[\ln \frac{\Lambda_0}{\Lambda_R}\big]$. For relevant and marginal interactions we have to integrate (4.31) from Λ_R up to Λ, starting from an initial condition $L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R] < \infty$. Under these conventions we have

$$\big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \leq \Lambda^{r_i} P^{q_i+1}\Big[\ln \frac{\Lambda}{\Lambda_R}\Big]\,. \tag{4.32}$$

A few comments:

- The stability (4.31) versus (4.32) of the estimation will be very useful in the iteration process.
- Integrations according to the direction (4.26b), which entail an initial condition $L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R]$, are expensive for renormalisation, because each such condition (even the choice $L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R] = 0$) corresponds to a normalisation experiment. In order to have a meaningful theory, there has to be only a finite number of required normalisation experiments. Initial data at Λ_0 do not correspond to normalisation conditions, because the interaction at $\Lambda_0 \to \infty$ is experimentally not accessible. Moreover, unless artificially kept alive[16], an irrelevant coupling scales away for $\Lambda_0 \to \infty$ via its own dynamics. The property $\lim_{\Lambda_0\to\infty} L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0] = 0$ for an irrelevant coupling is, therefore, a result and no condition.
- There might be cases where the direction (4.26b) for $r_i < 0$ gives convergence for $\Lambda_0 \to \infty$ nevertheless. This corresponds to the over-subtractions [Zim73] in the BPHZ renormalisation scheme. We shall not exploit this possibility.

[16] An example of an irrelevant coupling which remains present for $\Lambda_0 \to \infty$ is the initial ϕ^4-interaction in two-dimensional models, see Appendix H.

Unless there are further correlations between functions with different indices, specifying $L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R]$ means to impose an *infinite number* of normalisation conditions (because of $m_i, n_i \in \mathbb{N}^{D/2}$). Hence, a non-local matrix model with relevant and/or marginal interactions can only be renormalisable if some additional structures exist which relate all divergent functions to a finite number of relevant/marginal base interactions. Such a distinguished property depends crucially on the model. Presumably, the class of models where such a reduction is possible is rather small.

When we return to the duality-covariant ϕ^4-model in Section 5 on page 50, these reductions will be identified and taken into account. Here, we will restrict ourselves to find the general power-counting behaviour of a non-local matrix model which limits the class of divergent functions among which the reduction has to be studied in detail. We will find that—under very general conditions on the propagator—all non-planar graphs (as defined below) are irrelevant. Such a result is already an enormous gain[17] for the detailed investigation of a model.

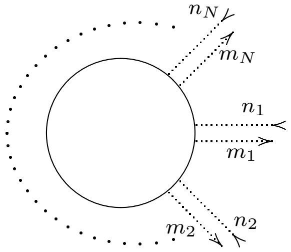
Thus, our strategy is to integrate the Polchinski equation (4.15) perturbatively between two scales Λ_R and Λ_0 for a self-determined choice of the boundary condition according to Definition/Lemma 7. The resulting normalisation condition for relevant and marginal interactions will not be the correct choice for a renormalisable model. Nevertheless, the resulting estimation (4.32) is compatible with a more careful treatment. As we will see for the example of the duality-covariant ϕ^4-model in Section 5, we can replace

- almost all of the relevant functions with bound $\frac{\Lambda^2}{\mu^2} P^q[\ln \frac{\Lambda}{\Lambda_R}]$ in (4.32) by irrelevant functions with bound $\big(\max(m_1, n_1, \dots, m_N, n_N)\big)^2 \frac{\mu^2}{\Lambda^2} P^q[\ln \frac{\Lambda}{\Lambda_R}]$, and
- almost all marginal functions with bound $P^q[\ln \frac{\Lambda}{\Lambda_R}]$ in (4.32) by irrelevant functions with bound $\max(m_1, n_1, \dots, m_N, n_N) \frac{\mu^2}{\Lambda^2} P^q[\ln \frac{\Lambda}{\Lambda_R}]$,

for some reference scale μ.

4.3 Ribbon graphs and their topologies

We can symbolise the expansion coefficients $L_{m_1n_1;\dots;m_Nn_N}$ as

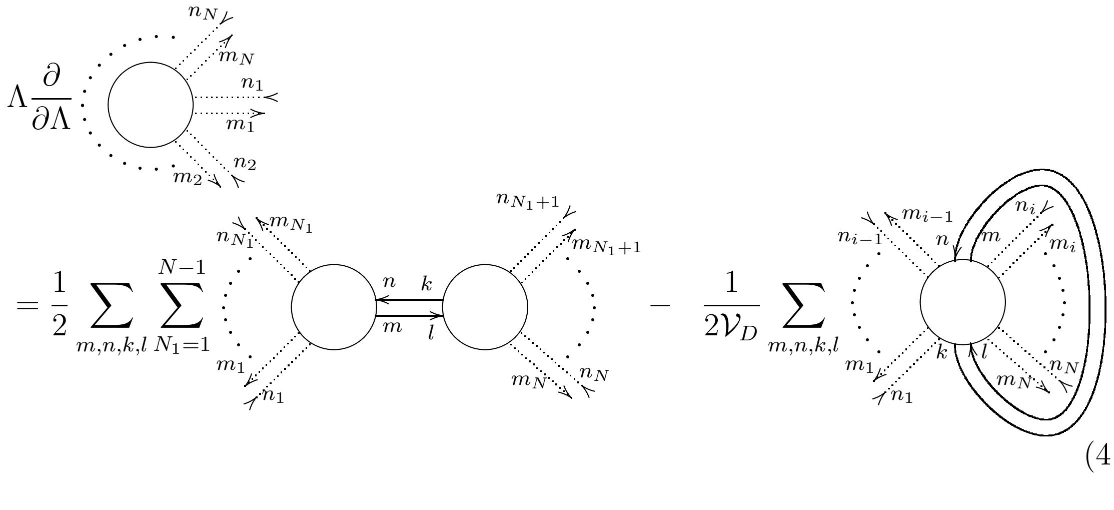


$$+ (N! - 1) \text{ permutations of } \{m_i n_i\} \,. \tag{4.33}$$

The big circle stands for a possibly very complex interior and the outer (dotted) double lines stand for the valences produced by differentiation (4.24) with respect to the N fields $\phi_{m_in_i}$. The arrows are merely added for bookkeeping purposes in the proof of the power-counting theorem. Since we work with real fields, i.e. $\phi_{mn} = \overline{\phi_{nm}}$, the expansion coefficients $L_{m_1n_1;\dots;m_Nn_N}$ have to be unoriented. The situation is different for complex fields where $\phi \neq \phi^*$ leads to an orientation of the lines. In this case we would draw both arrows at the double line either incoming or outgoing.

[17] We recall [MVRS00] that non-planar graphs produce the trouble in noncommutative quantum field theories in momentum space.

The graphical interpretation of the matrix Polchinski equation (4.15) is found when differentiating it with respect to the fields $\phi_{m_i n_i}$:



(4.34)

Combinatorial factors are not shown and a symmetrisation in all indices $m_i n_i$ has to be performed. On the rhs of (4.34) the two valences mn and kl of the subgraphs are connected to the ends of a *ribbon* which symbolises the differentiated propagator $\underset{m \quad l}{\overset{n \quad k}{\rightleftarrows}} = \Lambda \frac{\partial}{\partial \Lambda} \Delta^K_{nm;lk}$. For local matrix models in the sense of Definition 5 we can regard the ribbon as a product of single lines with interaction given by $\Delta(m,n)$. For non-local matrix models there is an exchange of indices within the entire ribbon.

We can regard (4.15) as a formal construction scheme for $L[\phi,\Lambda]$ if we introduce a grading $L[\phi,\Lambda] = \sum_{V=1}^{\infty} \lambda^V L^{(V)}[\phi,\Lambda]$ and additionally impose a cut-off in N for $V=1$, i.e

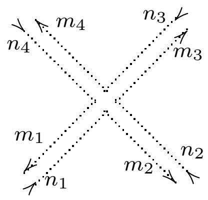
$$L^{(1)}_{m_1 n_1;\dots;m_N n_N}[\Lambda] = 0 \qquad \text{for } N > N_0 \,. \tag{4.35}$$

In order to obtain a ϕ^4-model we choose $N_0 = 4$ and the grading as the degree V in the coupling constant λ. We conclude from (4.15) that $L^{(1)}_{m_1 n_1;\dots;m_4 n_4}$ is independent of Λ so that it is identified with the original $(\lambda/4!)\phi^4$-interaction in (4.1):

$$\begin{aligned} L^{(1)}_{m_1 n_1;m_2 n_2;m_3 n_3;m_4 n_4}[\Lambda] = \frac{1}{4!\,6}\Big(& \delta_{n_1 m_2}\delta_{n_2 m_3}\delta_{n_3 m_4}\delta_{n_4 m_1} + \delta_{n_1 m_3}\delta_{n_3 m_4}\delta_{n_4 m_2}\delta_{n_2 m_1} \\ & + \delta_{n_1 m_4}\delta_{n_4 m_2}\delta_{n_2 m_3}\delta_{n_3 m_1} + \delta_{n_1 m_4}\delta_{n_4 m_3}\delta_{n_3 m_2}\delta_{n_2 m_1} \\ & + \delta_{n_1 m_3}\delta_{n_3 m_2}\delta_{n_2 m_4}\delta_{n_4 m_1} + \delta_{n_1 m_2}\delta_{n_2 m_4}\delta_{n_4 m_3}\delta_{n_3 m_1}\Big)\,. \end{aligned} \tag{4.36}$$

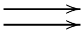
To the first term on the rhs of (4.36) we associate the graph

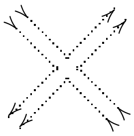
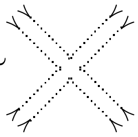
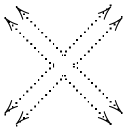
$$\delta_{n_1 m_2}\delta_{n_2 m_3}\delta_{n_3 m_4}\delta_{n_4 m_1} =$$



(4.37)

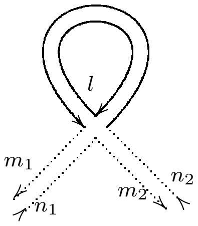
The graphs for the other five terms are obtained by permutation of indices.

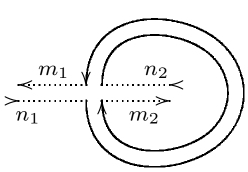
As mentioned before, a complex ϕ^4-model would be given by oriented propagators  and examples for vertices are

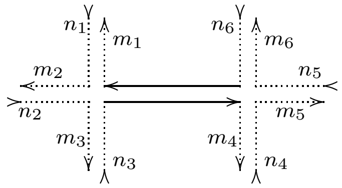
$\phi\phi^*\phi\phi^* \sim$  $\phi\phi\phi\phi \sim$  $\phi^*\phi^*\phi^*\phi^* \sim$  (4.38)

The consequence is that many graphs of the real ϕ^4-model are now excluded. We can thus obtain the complex ϕ^4-model from the real one by deleting the impossible graphs.

The iteration of (4.34) with starting point (4.37) leads to *ribbon graphs*. The first examples of the iteration are

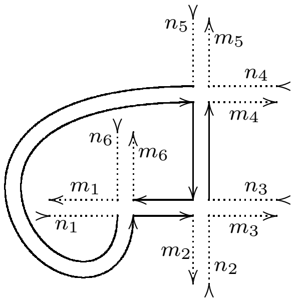


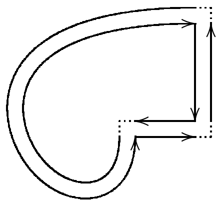




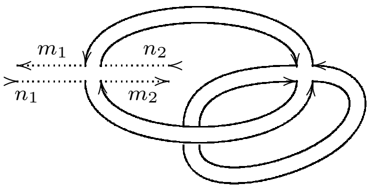
(4.39)

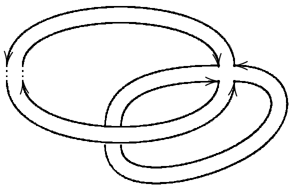
We can obviously build very complicated ribbon graphs with crossings of lines which cannot be drawn any more in a plane. A general ribbon graph can, however, be drawn on a *Riemann surface* of some *genus* g. In fact, a ribbon graph *defines* the Riemann surfaces topologically through the *Euler characteristic* χ. We have to regard here the external lines of the ribbon graph as amputated (or closed), which means to directly connect the single lines m_i with n_i for each external leg $m_i n_i$. A few examples may help to understand this procedure:



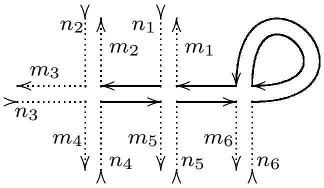
$\Rightarrow$ 

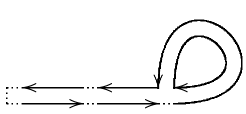
$$\begin{array}{ll} \tilde{L}=2 & B=2 \\ I=3 & N=6 \\ V=3 & V^e=3 \\ g=0 & \iota=0 \end{array} \tag{4.40}$$



$\Rightarrow$ 

$$\begin{array}{ll} \tilde{L}=1 & B=1 \\ I=3 & N=2 \\ V=2 & V^e=1 \\ g=1 & \iota=1 \end{array} \tag{4.41}$$



$\Rightarrow$ 

$$\begin{array}{ll} \tilde{L}=2 & B=1 \\ I=3 & N=6 \\ V=3 & V^e=3 \\ g=0 & \iota=0 \end{array} \tag{4.42}$$

The genus is computed from the number $\tilde{L}$ of single-line loops of the closed graph, the number I of internal (double) lines and the number V of vertices of the graph via

$$\chi = 2 - 2g = \tilde{L} - I + V \ . \tag{4.43}$$

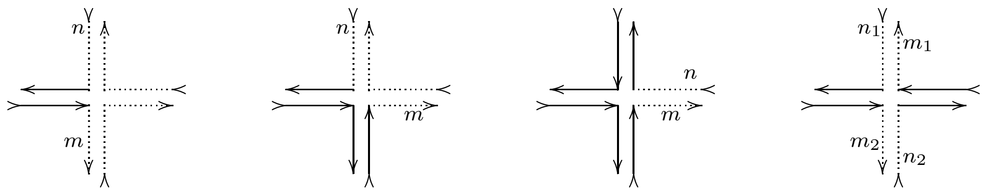
There can be several possibilities to draw the graph and its Riemann surface, but $\tilde{L}, I, V$ and thus g remain unchanged. Indeed, the Polchinski equation (4.15) interpreted as in (4.34) tells us which external legs of the vertices are connected. It is completely irrelevant how the ribbons are drawn between these legs. In particular, there is no distinction between overcrossings and undercrossings.

There are two types of loops in (amputated) ribbon graphs:

- Some of them carry at least one external leg. They are called *boundary components* or holes of the Riemann surface. Their number is B.
- Some of them do not carry any external leg. They are called *inner loops.* Their number is $\tilde{L}_0 = \tilde{L} - B$.

Boundary components consist of a concatenation of *trajectories* from an incoming index n_i to an outgoing index m_j. In the example (4.40) the inner boundary component consists of the single trajectory $\overrightarrow{n_1 m_6}$ whereas the outer boundary component is made of two trajectories $\overrightarrow{n_3 m_4}$ and $\overrightarrow{n_5 m_2}$. We let $\mathfrak{o}[n_j]$ be the outgoing index to n_j and $\mathfrak{i}[m_j]$ be the incoming index to m_j.

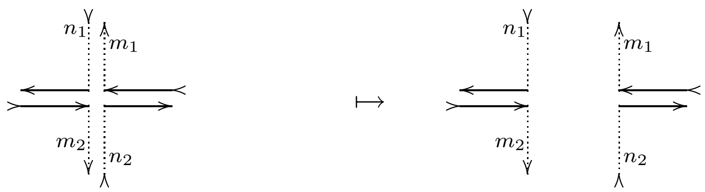
I have to introduce a few additional notations for ribbon graphs. An *external vertex* is a vertex which has at least one external leg. I denote by V^e the total number of external vertices. For the arrangement of external legs at an external vertex there are the following possibilities:



(4.44)

I call the first three types of external vertices *simple vertices.* They provide one starting point and one end point of trajectories through a ribbon graph. The fourth vertex in (4.44) is called *composed vertex.* It has two starting points and two end points of trajectories.

A composed vertex can be decomposed by pulling the two propagators with attached external lines apart:



(4.45)

In this way a given graph with composed vertices is decomposed into S *segments.* The external vertices of the segments are either true external vertices or the halves of a composed vertex. If composed vertices occur in loops, their decomposition does not always increase the number of segments. We need the following

Definition 8 *The segmentation index ι of a graph is the maximal number of decompositions of composed vertices which keep the graph connected.*

It follows immediately that if V^c is the number of composed vertices of a graph and S the number of segments obtained by decomposing all composed vertices we have

$$\iota = V^c - S + 1 \ . \tag{4.46}$$

In order to evaluate $L_{m_1 n_1;\dots;m_N n_N}[\Lambda]$ by connection and contraction of subgraphs according to (4.34) we need estimations for *index summations* of ribbon graphs. Namely, our strategy is to apply the summations in (4.34) either to the propagator or the subgraph only and to maximise the other object over the summation indices. We agree to fix all starting points of trajectories and sum over the end points of trajectories. However, due to (4.5) and (4.36) not all summations are independent: The sum of outgoing indices equals for each segment the sum of incoming indices. Since there are $V^e + V^c$ (end points of) trajectories in a ribbon graph, there are

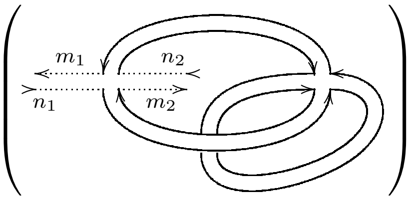
$$s \leq V^e + V^c - S = V^e + \iota - 1 \tag{4.47}$$

independent index summations. The inequality (4.47) also holds for the restriction to each segment if V^e includes the number of halves of composed vertices which belong to the segment. We let $\mathcal{E}^s$ be the set of s end points of trajectories in a graph over which we are going to sum, keeping the starting points of these trajectories fixed. We define

$$\sum_{\mathcal{E}^s} \equiv \sum_{m_1} \cdots \sum_{m_s} \Big|_{\mathfrak{i}[m_j]=\text{const}} \qquad \text{if } \mathcal{E}^s = \{m_1, m_2, \dots, m_s\}\ . \tag{4.48}$$

Taking the example of the graph (4.40), we can due to $V^e + \iota = 3$ apply up to two index summations, i.e. a summation over at most two of the end points of trajectories m_2, m_4, m_6, where the corresponding incoming indices $\mathfrak{i}[m_2] = n_5$, $\mathfrak{i}[m_4] = n_3$ and $\mathfrak{i}[m_6] = n_1$ are kept fixed. For the example of the graph (4.41) we can due to $V^e + \iota = 2$ apply at most one index summation, either over m_1 for fixed $\mathfrak{i}[m_1] = n_2$ or over $\mathfrak{i}[m_2] = n_1$. For $\mathcal{E}^1 = \{m_2\}$ we would consider

$$\sum_{m_2} \Bigg(\quad \Bigg)_{n_1=\text{const}} \ . \tag{4.49}$$



Note that for given n_2 the other outgoing index is determined to $m_1 = n_1 + n_2 - m_2$ through index conservation at propagators (4.5) and vertices (4.37). It is part of the proof to show that the index summation (4.49) is bounded independently of the incoming indices n_1, n_2.

4.4 Formulation of the power-counting theorem

We first have to transform the matrix Polchinski equation (4.15) into a dimensionless form. It is important here that in the class of models we consider there is always a dimensionful parameter,

$$\mu = \big(\mathcal{V}_D\big)^{-\frac{1}{D}}\ , \tag{4.50}$$

which instead of Λ can be used to absorb the mass dimensions. The effective action $L[\phi, \Lambda]$ has total mass dimension D, a field ϕ has dimension $\frac{D-2}{2}$ and the dimension of the coupling constant for the $\lambda\phi^4$ interaction is $4-D$. We thus decompose $L[\phi,\Lambda]$ according to the number of fields and the order in the coupling constant:

$$L[\phi,\Lambda]=\sum_{V=1}^{\infty}\sum_{N=2}^{2V+2}\frac{1}{N!}\sum_{m_i,n_i}\Big(\frac{\lambda}{\mu^{4-D}}\Big)^V\mu^D A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\Big(\frac{\phi_{m_1n_1}}{\mu^{\frac{D-2}{2}}}\Big)\cdots\Big(\frac{\phi_{m_Nn_N}}{\mu^{\frac{D-2}{2}}}\Big)\,. \quad (4.51)$$

The functions $A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ are assumed to be symmetric in their indices m_in_i. Inserted into (4.15) we get

$$\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda}A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\\
&=\sum_{m,n,k,l}\frac{1}{2}Q_{nm;lk}(\Lambda)\Big\{\sum_{N_1=2}^{N}\sum_{V_1=1}^{V-1}A^{(V_1)}_{m_1n_1;\dots;m_{N_1-1}n_{N_1-1};mn}[\Lambda]A^{(V-V_1)}_{m_{N_1}n_{N_1};\dots;m_Nn_N;kl}[\Lambda]\\
&\qquad\qquad+\Big(\binom{N}{N_1-1}-1\Big)\text{ permutations}\\
&\qquad\qquad-A^{(V)}_{m_1n_1;\dots;m_Nn_N;mn;kl}[\Lambda]\Big\}\,,
\end{aligned} \quad (4.52)$$

where

$$Q_{nm;lk}(\Lambda):=\mu^2\Lambda\frac{\partial}{\partial\Lambda}\Delta^K_{nm;lk}(\Lambda)\,. \quad (4.53)$$

The permutations refer to the possibilities to choose N_1-1 of the pairs of indices $m_1n_1,\dots,m_Nn_N$ which label the external legs of the first A-function.

The cut-off function K in (4.10) has to be chosen such that for finite Λ there is a finite number of indices m,n,k,l with $Q_{nm;lk}(\Lambda)\neq 0$. By suitable normalisation we can achieve that the volume of the support of $Q_{nm;lk}(\Lambda)$ with respect to a chosen index scales as Λ^D:

$$\sum_m \operatorname{sign}|K[m;\Lambda]|\leq C_D\Big(\frac{\Lambda}{\mu}\Big)^D\,, \quad (4.54)$$

for some constant C_D independent of Λ. For such a normalisation we define two exponents δ_0,δ_1 by

$$\max_{m,n,k,l}|Q_{nm;lk}(\Lambda)|\leq C_0\Big(\frac{\mu}{\Lambda}\Big)^{\delta_0}\delta_{m+k,n+l}\,, \quad (4.55)$$

$$\max_n\Big(\sum_k\Big(\max_{m,l}|Q_{nm;lk}(\Lambda)|\Big)\Big)\leq C_1\Big(\frac{\mu}{\Lambda}\Big)^{\delta_1}\,. \quad (4.56)$$

In (4.56) the index n is kept constant for the summation over k. It is convenient to encode the dimension D in a further exponent δ_2 which describes the product of (4.54) with (4.55):

$$\max_{m,n,k,l}|Q_{nm;lk}(\Lambda)|\sum_m\operatorname{sign}\big(|K[m,\Lambda]|\big)\leq C_2\Big(\frac{\Lambda}{\mu}\Big)^{\delta_2}\,. \quad (4.57)$$

We have obviously $C_2=C_DC_0$ and $\delta_2=D-\delta_0$.

Definition 9 *A non-local matrix model defined by the cut-off propagator $Q_{nm;kl}$ given by (4.10) and (4.53) and the normalisation (4.54) of the cut-off scale Λ is called regular if $\delta_0 = \delta_1 = 2$, otherwise anomalous.*

The three exponents $\delta_0, \delta_1, \delta_2$ play an essential rôle in the power-counting theorem which yields the Λ-scaling of a homogeneous part $A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ of the interaction coefficients

$$A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda] = \sum_{1\leq V^e\leq V} \sum_{1\leq B\leq N} \sum_{0\leq g\leq 1+\frac{V}{2}-\frac{N}{4}-\frac{B}{2}} \sum_{0\leq\iota\leq B-1} A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\Big|_{2\leq N\leq 2V+2} . \tag{4.58}$$

It is important that the sums over the graphical (topological) data V^e, B, g, ι in (4.58) are finite. We are going to prove

Theorem 10 *The homogeneous parts $A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ of the coefficients of the effective action describing a ϕ^4-matrix model with initial interaction (4.36) and cut-off propagator characterised by the three exponents $\delta_0, \delta_1, \delta_2$ are for $2 \leq N \leq 2V{+}2$ and $\sum_{i=1}^N (m_i{-}n_i) = 0$ bounded by*

$$\begin{aligned}&\sum_{\mathcal{E}^s} \big|A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \\ &\quad\leq \Big(\frac{\Lambda}{\mu}\Big)^{\delta_2(V-\frac{N}{2}+2-2g-B)} \Big(\frac{\mu}{\Lambda}\Big)^{\delta_1(V-V^e-\iota+2g+B-1+s)} \Big(\frac{\mu}{\Lambda}\Big)^{\delta_0(V^e+\iota-1-s)} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big] ,\end{aligned} \tag{4.59}$$

provided that for all $V'{<}V$, $2 \leq N' \leq 2V'{+}2$ and $V'{=}V$, $N{+}2 \leq N' \leq 2V{+}2$ the initial conditions for $\left\{\begin{array}{c}\text{relevant / marginal}\\ \text{irrelevant}\end{array}\right\}$ $A^{(V',V'^e,B',g',\iota')}_{m'_1n'_1;\dots;m'_{N'}n'_{N'}}[\Lambda]$ are imposed at $\left\{\begin{array}{c}\Lambda_R\\ \Lambda_0\end{array}\right\}$, respectively, according to Definition/Lemma 7. The bound (4.59) is independent of the unsummed indices $m_i, n_i \notin \mathcal{E}^s$. We have $A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda] \equiv 0$ for $N > 2V{+}2$ or $\sum_{i=1}^N (m_i{-}n_i) \neq 0$.

I remark that $\tilde{L}_0 = V - \frac{N}{2} + 2 - 2g - B$ is the number of inner loops of a graph. The (very long) proof of this theorem is delegated to Appendix D between pages 116 and 135.

The power-counting estimation (4.59) does not make any reference to the initial scale Λ_0 so that we can safely take the limit $\Lambda_0 \to \infty$ [KKS92]. In this way we have constructed a regular solution of the Polchinski equation (4.15) associated with the non-local matrix model. However, this solution remains useless unless it can be achieved by a *finite number* of integrations from Λ_R to Λ depending on a finite number of initial conditions at Λ_R. I refer to the remarks following Definition/Lemma 7. A first step would be to achieve regular scaling dimensions:

Corollary 11 *For regular matrix models according to Definition 9 we have independently of the segmentation index and the numbers of external vertices and index summations*

$$\sum_{\mathcal{E}^s} \big|A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \leq \Big(\frac{\Lambda}{\mu}\Big)^{\omega-D(2g+B-1)} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big] , \tag{4.60}$$

where $\omega = D + V(D-4) - N\frac{D-2}{2}$ is the classical power-counting degree of divergence.

We have derived the relation (4.60) with respect to the classical power-counting degree of divergence only for ϕ^4-matrix models, but it is plausible that it also holds for more general interactions.

The power-counting theorem (Theorem 10) provides a necessary condition for renormalisability: The two scaling exponents δ_0, δ_1 of the cut-off propagator have to be large enough relative to the dimension of the underlying space. We will see during the next section[18] that for the usual noncommutative ϕ^4-model given by the action (3.24) on page 27 these exponents equal $\delta_0 = 1$ and $\delta_1 = 0$. The weak decay $\sim \Lambda^{-1}$ of the propagator leads to divergences in $\Lambda \sim \Lambda_0 \to \infty$ of arbitrarily high degree. The appearance of unbounded degrees of divergences in field theories on noncommutative $\mathbb{R}^4$ is often related to the UV/IR-mixing [MVRS00]. We learn from Theorem 10 that similar effects will show up in any matrix model in which the propagator decays too slowly with Λ. This means that the correlation between distant modes is too strong, i.e. the model is too non-local.

Adding a harmonic oscillator potential to the action one achieves $\delta_0 = \delta_1 = 2$, which thanks to Theorem 10 provides the first part of the renormalisation proof. The more difficult part consists in proving that the infinitely many boundary conditions at Λ_R used in Theorem 10 can actually be related to finitely many base interactions.

[18] This is the case $\Omega = 0$ in (5.19) and (5.20) on page 57.

5 Renormalisation group analysis of the duality-covariant noncommutative ϕ^4_4-model

I have developed in Section 4 the Wilson-Polchinski renormalisation programme for non-local matrix models where the kinetic term (Taylor coefficient matrix of the two-point function) is neither constant nor diagonal. Introducing a cut-off in the measure $\prod_{m,n} d\phi_{mn}$ of the partition function Z, the resulting effect is undone by adjusting the effective action $L[\phi]$ (and other terms which are easy to evaluate). If the cut-off function is a smooth function of the cut-off scale Λ, the adjustment of $L[\phi,\Lambda]$ is described by the differential equation (4.15) on page 38. Integrating (4.15) perturbatively between the initial scale Λ_0 and the renormalisation scale $\Lambda_R \ll \Lambda_0$, I have derived Theorem 10 on page 48 which describes the power-counting behaviour of the expansion coefficients of $L[\phi,\Lambda]$. The power-counting degree is given by topological data of ribbon graphs and two scaling exponents of the (summed and differentiated) cut-off propagator. This power-counting theorem is model independent, but it relied on boundary conditions for the integrations which do not correspond to a physically meaningful model.

I will show in this Section that the four-dimensional duality-covariant noncommutative ϕ^4-theory given by the action (1.5) on page 5 admits an improved power-counting behaviour which only relies on a finite number of physical boundary conditions for the integration.

5.1 Integration of the matrix Polchinski equation

To some extent, finding a renormalisable model is a matter of trial and error: The model is defined by the set of relevant and marginal interactions as it comes out of the power-counting theorem, but on the other hand this output is used as the input to derive the power-counting theorem. To say it differently: One has to be lucky to make the right ansatz for the initial interaction which is then reconfirmed by the power-counting theorem as the set of relevant and marginal interactions. I am going to prove that the following ansatz for the initial interaction of a $(D=4)$-dimensional model is such a lucky choice:

$$
\begin{aligned}
L[\phi,\Lambda_0,\Lambda_0,\rho^0] = \sum_{m^1,m^2,n^1,n^2\in\mathbb{N}} \frac{1}{2\pi\theta}\Big(\frac{1}{2}\big(\rho^0_1+(n^1+m^1+n^2+m^2)\rho^0_2\big)\phi_{\substack{m^1 n^1\\ m^2 n^2}}\phi_{\substack{n^1 m^1\\ n^2 m^2}} \\
-\rho^0_3\big(\sqrt{n^1m^1}\,\phi_{\substack{m^1 n^1\\ m^2 n^2}}\phi_{\substack{n^1-1\; m^1-1\\ n^2\; m^2}} + \sqrt{n^2m^2}\,\phi_{\substack{m^1 n^1\\ m^2 n^2}}\phi_{\substack{n^1\; m^1\\ n^2-1\; m^2-1}}\big)\Big) \\
+\sum_{m^1,m^2,n^1,n^2,k^1,k^2,l^1,l^2\in\mathbb{N}} \frac{1}{4!}\rho^0_4\,\phi_{\substack{m^1 n^1\\ m^2 n^2}}\phi_{\substack{n^1 k^1\\ n^2 k^2}}\phi_{\substack{k^1 l^1\\ k^2 l^2}}\phi_{\substack{l^1 m^1\\ l^2 m^2}}\;. \qquad (5.1)
\end{aligned}
$$

For simplicity, I impose a symmetry between the upper and lower component, which could be relaxed by taking different ρ-coefficients in front of m^i+n^i and $\sqrt{m^in^i}$. Accordingly, I choose the same weights in the noncommutativity matrix, $\theta_1=\theta_2\equiv\theta$.

The differential equation (4.15) is non-perturbatively defined. However, we shall solve it perturbatively as a formal power series in a coupling constant λ which later on (equation (5.14) on page 55) will be related to a normalisation condition at $\Lambda=\Lambda_R$. We thus consider the expansion (4.51) on page 47 where we put $D=4$ and

$$
\mathcal{V}_4 = (2\pi\theta)^2\;, \qquad (5.2)
$$

identified by comparison of (4.1) on page 35 with (3.42) on page 32. According to (4.50) we have $\mu^{-1} = \sqrt{2\pi\theta}$. We choose

$$
\begin{aligned}
&K[{}^{m^1}_{m^2}, \Lambda] := K\Big(\frac{m^1}{\theta\Lambda^2}\Big) K\Big(\frac{m^2}{\theta\Lambda^2}\Big)\;,\\
\Rightarrow \quad &\Delta^K_{{}^{m^1}_{m^2}\,{}^{n^1}_{n^2};{}^{k^1}_{k^2}\,{}^{l^1}_{l^2}}(\Lambda) = \Bigg(\prod_{\substack{i\,\in m^1,\,m^2,\,n^1,\,n^2,\\ k^1,\,k^2,\,l^1,\,l^2}} K\Big(\frac{i}{\theta\Lambda^2}\Big)\Bigg) \Delta_{{}^{m^1}_{m^2}\,{}^{n^1}_{n^2};{}^{k^1}_{k^2}\,{}^{l^1}_{l^2}}\;,
\end{aligned}
\tag{5.3}
$$

where $K(x)$ is a smooth monotonous cut-off function[19] with $K(x) = 1$ for $x \leq 1$ and $K(x) = 0$ for $x \geq 2$. With all these specifications, the normalised Polchinski equation (4.52) on page 47 takes the form

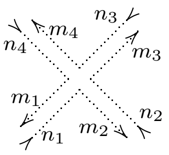
$$
\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda} A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda, \Lambda_0, \rho^0]\\
&\quad = \sum_{N_1=2}^{N} \sum_{V_1=1}^{V-1} \sum_{m,n,k,l\in\mathbb{N}^2} \frac{1}{2} Q_{nm;lk}(\Lambda) A^{(V_1)}_{m_1n_1;\dots;m_{N_1-1}n_{N_1-1};mn}[\Lambda] A^{(V-V_1)}_{m_{N_1}n_{N_1};\dots;m_Nn_N;kl}[\Lambda]\\
&\qquad\qquad\qquad\qquad\qquad\qquad + \Big(\binom{N}{N_1-1} - 1\Big) \text{ permutations}\\
&\quad - \sum_{m,n,k,l\in\mathbb{N}^2} \frac{1}{2} Q_{nm;lk}(\Lambda) A^{(V)}_{m_1n_1;\dots;m_Nn_N;mn;kl}[\Lambda]\;,
\end{aligned}
\tag{5.4}
$$

with

$$
Q_{nm;lk}(\Lambda) := \frac{1}{2\pi\theta} \Lambda \frac{\partial \Delta^K_{nm;lk}(\Lambda)}{\partial\Lambda}\;. \tag{5.5}
$$

I am going to compute the functions $A^{(V)}_{m_1n_1;\dots;m_Nm_N}$ by iteratively integrating the Polchinski equation (5.4) starting from boundary conditions either at Λ_R or at Λ_0. The right choice of the integration direction is an art: The boundary condition influences crucially the estimation, which in turn justifies or discards the original choice of the boundary condition. At the end of numerous trial-and-error experiments with the boundary condition, one convinces oneself that the procedure described in Definition 12 below is—up to finite re-normalisations discussed later—the unique possibility[20] to renormalise the model.

I recall from Section 4.3 that the matrix Polchinski equation (5.4) is solved by *ribbon graphs* drawn on a Riemann surface of uniquely determined *genus* g and uniquely determined number B of *boundary components* (holes). The ribbons are made of double-line propagators $\overset{n \quad k}{\underset{m \quad l}{\rightleftarrows}} = Q_{mn;kl}(\Lambda)$ attached to vertices



$$
= \delta_{n_1m_2}\delta_{n_2m_3}\delta_{n_3m_4}\delta_{n_4m_1}\;. \tag{5.6}
$$

[19] A possible plot is shown in (2.10) on page 14.

[20] I "only" prove that the method works, not its uniqueness. The reader who doubts uniqueness of the integration procedure is invited to attempt a different way.

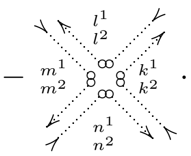
A graph γ is produced via a certain *history* of contractions of (in each step) either two smaller subgraphs (with fewer vertices) or a self-contraction of a subgraph with two additional external legs. At a given order V of vertices there are finitely many N-leg graphs (distinguished by their topology and the permutation of external indices) contributing the part $A^{(V)\gamma}_{m_1n_1;\dots;m_Nm_N}$ to a function $A^{(V)}_{m_1n_1;\dots;m_Nm_N}$. A ribbon graph is called one-particle irreducible (1PI) if it remains connected when removing a single propagator. The first term on the rhs of the Polchinski equation (5.4) leads always to one-particle reducible graphs, because it is left disconnected when removing the propagator $Q_{nm;lk}$ in (5.4).

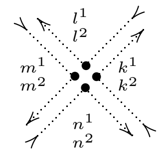
According to the detailed properties a graph γ we define the following recursive procedure (starting with the vertex (5.6) which does not have any subgraphs) to integrate the matrix Polchinski equation (5.4):

Definition 12 *We consider generalised*[21] *ribbon graphs γ which result via a history of contractions of subgraphs which at each contraction step have already been integrated according to the rules given below.*

1. *Let γ be a planar ($B = 1$, $g = 0$) one-particle irreducible graph with $N = 4$ external legs, where the index along each of its trajectories is constant (this includes the two external indices of a trajectory and the chain of indices at contracting inner vertices in between them). Then, the contribution $A^{(V)\gamma}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{k^1\\k^2};\substack{k^1\\k^2}\substack{l^1\\l^2};\substack{l^1\\l^2}\substack{m^1\\m^2}}(\Lambda)$ (using the natural cyclic order of legs of a planar graph) of γ to the effective action is integrated as follows:*

$$
\begin{aligned}
&A^{(V)\gamma}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{k^1\\k^2};\substack{k^1\\k^2}\substack{l^1\\l^2};\substack{l^1\\l^2}\substack{m^1\\m^2}}[\Lambda]\\
&:= -\int_{\Lambda}^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\Bigg\{\widehat{\Lambda'\frac{\partial}{\partial\Lambda'}A^{(V)\gamma}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{k^1\\k^2};\substack{k^1\\k^2}\substack{l^1\\l^2};\substack{l^1\\l^2}\substack{m^1\\m^2}}[\Lambda']}
\end{aligned}
$$

$$-$$  $$\cdot\Bigg[\widehat{\Lambda'\frac{\partial}{\partial\Lambda'}A^{(V)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda']}\Bigg]\Bigg\}$$

$$+$$  $$\Bigg[\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\Big(\widehat{\Lambda'\frac{\partial}{\partial\Lambda'}A^{(V)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda']}\Big)+A^{(V)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_R]\Bigg]\,. \qquad (5.7)$$

Here (and in the sequel), the wide hat over the Λ'-derivative of an A^γ-function indicates that the rhs of the matrix Polchinski equation (5.4) has to be inserted. The two vertices in the third and fourth lines of (5.7) are identical (both are equal to 1). The four-leg graph in the third line of (5.7) indicates that the graph corresponding to the function in brackets right of it has to be inserted into the holes. The result[22] *is a graph with the same topology as the function in the second line, but different indices on inner trajectories. The graph in the fourth line of (5.7) is identical to the original vertex (5.6). The different symbol shall remind us that in the analytic expression for subgraphs containing the vertex of the last line in (5.7) we have to insert the value in brackets right of it.*

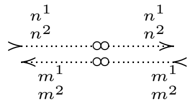
[21] This refers to graphs with composite propagators as defined in Section 5.2 on page 55.

[22] See (2.17) for an example.

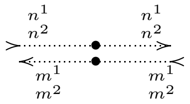
Remark: I use here (and in all other cases below) the convention (its consistency will be shown later) that at $\Lambda = \Lambda_0$ *the contribution to the initial four-point function is independent of the external indices,* $A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda_0] = A^{(V)\gamma}_{\substack{m^1\,n^1\\m^2\,n^2};\substack{n^1\,k^1\\n^2\,k^2};\substack{k^1\,l^1\\k^2\,l^2};\substack{l^1\,m^1\\l^2\,m^2}}[\Lambda_0]$*. This is not really necessary, we could admit* $A^{(V)\gamma}_{\substack{m^1\,n^1\\m^2\,n^2};\substack{n^1\,k^1\\n^2\,k^2};\substack{k^1\,l^1\\k^2\,l^2};\substack{l^1\,m^1\\l^2\,m^2}}[\Lambda_0] - A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda_0] = C_{\substack{m^1\,n^1\\m^2\,n^2};\substack{n^1\,k^1\\n^2\,k^2};\substack{k^1\,l^1\\k^2\,l^2};\substack{l^1\,m^1\\l^2\,m^2}}[\Lambda_0]$ *with* $\big|C_{\substack{m^1\,n^1\\m^2\,n^2};\substack{n^1\,k^1\\n^2\,k^2};\substack{k^1\,l^1\\k^2\,l^2};\substack{l^1\,m^1\\l^2\,m^2}}[\Lambda_0]\big| \leq \frac{\mathrm{const}}{\theta\Lambda_0^2}$.

2. *Let* γ *be a planar* ($B = 1$, $g = 0$) *1PI graph with* $N = 2$ *external legs, where either the index is constant along each trajectory, or one component of the index jumps*[23] *once by* ± 1 *and back on one of the trajectories, whereas the index along the possible other trajectory remains constant. Then, the contribution* $A^{(V)\gamma}_{\substack{m^1\,n^1\\m^2\,n^2};\substack{n^1\,m^1\\n^2\,m^2}}[\Lambda]$ *of* γ *to the effective action is integrated as follows:*

$$
\begin{aligned}
&A^{(V)\gamma}_{\substack{m^1\,n^1\\m^2\,n^2};\substack{n^1\,m^1\\n^2\,m^2}}[\Lambda] \\
&:= -\int_\Lambda^{\Lambda_0} \frac{d\Lambda'}{\Lambda'}\, \Bigg\{ \Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{m^1\,n^1\\m^2\,n^2};\substack{n^1\,m^1\\n^2\,m^2}}}(\Lambda')
\end{aligned}
$$

$-$ 

$$
\begin{aligned}
&\cdot\Bigg[\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}}[\Lambda'] + m^1\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{1\,0\\0\,0};\substack{0\,1\\0\,0}}}[\Lambda'] - \Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}}[\Lambda']\Big) \\
&\qquad + m^2\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0\\1\,0};\substack{0\,0\\0\,1}}}[\Lambda'] - \Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}}[\Lambda']\Big) \\
&\qquad + n^1\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,1\\0\,0};\substack{1\,0\\0\,0}}}[\Lambda'] - \Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}}[\Lambda']\Big) \\
&\qquad + n^2\;\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0\\0\,1};\substack{0\,0\\1\,0}}}[\Lambda'] - \Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}}[\Lambda']\Big)\Bigg]\Bigg\}
\end{aligned}
$$

$+$ 

$$
\begin{aligned}
&\Bigg[\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\,\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}}[\Lambda']\Big) + A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda_R] \\
&+ m^1\Big(\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\,\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{1\,0\\0\,0};\substack{0\,1\\0\,0}}}[\Lambda'] - \Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}}[\Lambda']\Big) + A^{(V)\gamma}_{\substack{1\,0\\0\,0};\substack{0\,1\\0\,0}}[\Lambda_R] - A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda_R]\Big) \\
&+ m^2\Big(\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\,\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0\\1\,0};\substack{0\,0\\0\,1}}}[\Lambda'] - \Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}}[\Lambda']\Big) + A^{(V)\gamma}_{\substack{0\,0\\1\,0};\substack{0\,0\\0\,1}}[\Lambda_R] - A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda_R]\Big) \\
&+ n^1\Big(\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\,\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,1\\0\,0};\substack{1\,0\\0\,0}}}[\Lambda'] - \Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}}[\Lambda']\Big) + A^{(V)\gamma}_{\substack{0\,1\\0\,0};\substack{1\,0\\0\,0}}[\Lambda_R] - A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda_R]\Big) \\
&+ n^2\Big(\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\,\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0\\0\,1};\substack{0\,0\\1\,0}}}[\Lambda'] - \Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}}[\Lambda']\Big) + A^{(V)\gamma}_{\substack{0\,0\\0\,1};\substack{0\,0\\1\,0}}[\Lambda_R] - A^{(V)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda_R]\Big)\Bigg].
\end{aligned}
\tag{5.8}
$$

[23] A jump forward and backward means the following: Let $k_1, \dots, k_{a-1}$ be the sequence of indices at inner vertices on the considered trajectory $\overrightarrow{nm}$, in correct order between n and m. Then, for either $r = 1$ or $r = 2$ we require $n^r = k_i^r = m^r$ for all $i \in [1, p{-}1] \cup [q, a{-}1]$ and $k_i^r = n^r \pm 1$ (fixed sign) for all $i \in [p, q{-}1]$. The cases $p = 1$, $q = p{+}1$ and $q = a$ are admitted. The other index component is constant along the trajectory.

3. *Let γ be a planar ($B=1$, $g=0$) 1PI graph having $N=2$ external legs with external indices $m_1n_1;m_2n_2 = {}^{m^1\pm1}_{m^2}{}^{n^1\pm1}_{n^2};{}^{n^1}_{n^2}{}^{m^1}_{m^2}$ (equal sign) or $m_1n_1;m_2n_2 = {}^{m^1}_{m^2\pm1}{}^{n^1}_{n^2\pm1};{}^{n^1}_{n^2}{}^{m^1}_{m^2}$, with a single[24] jump in the index component of each trajectory. Under these conditions the contribution of γ to the effective action is integrated as follows:*

$$
\begin{aligned}
&A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{m^1+1}{m^2}\genfrac{}{}{0pt}{}{n^1+1}{n^2};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}[\Lambda] \\
&:= -\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\Bigg\{\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{m^1+1}{m^2}\genfrac{}{}{0pt}{}{n^1+1}{n^2};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}[\Lambda']} \\
&\qquad\qquad -\sqrt{(m^1{+}1)(n^1{+}1)}\;
\begin{smallmatrix} \genfrac{}{}{0pt}{}{n^1+1}{n^2} & & \genfrac{}{}{0pt}{}{n^1}{n^2} \\ \succ\cdots\cdots & \rhd\!\rhd & \cdots\cdots\succ \\ \prec\cdots\cdots & \rhd\!\rhd & \cdots\cdots\prec \\ \genfrac{}{}{0pt}{}{m^1+1}{m^2} & & \genfrac{}{}{0pt}{}{m^1}{m^2} \end{smallmatrix}
\cdot\Bigg[\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{1}{0}\genfrac{}{}{0pt}{}{1}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda']}\Bigg]\Bigg\} \\
&+\sqrt{(m^1{+}1)(n^1{+}1)}\;
\begin{smallmatrix} \genfrac{}{}{0pt}{}{n^1+1}{n^2} & & \genfrac{}{}{0pt}{}{n^1}{n^2} \\ \succ\cdots\cdots & \blacktriangleright & \cdots\cdots\succ \\ \prec\cdots\cdots & \blacktriangleright & \cdots\cdots\prec \\ \genfrac{}{}{0pt}{}{m^1+1}{m^2} & & \genfrac{}{}{0pt}{}{m^1}{m^2} \end{smallmatrix}
\;\Bigg[\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{1}{0}\genfrac{}{}{0pt}{}{1}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda']}\Big)+A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{1}{0}\genfrac{}{}{0pt}{}{1}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda_R]\Bigg]\,,
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
&A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{m^1}{m^2+1}\genfrac{}{}{0pt}{}{n^1}{n^2+1};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}[\Lambda] \\
&:= -\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\Bigg\{\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{m^1}{m^2+1}\genfrac{}{}{0pt}{}{n^1}{n^2+1};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}[\Lambda']} \\
&\qquad\qquad -\sqrt{(m^2{+}1)(n^2{+}1)}\;
\begin{smallmatrix} \genfrac{}{}{0pt}{}{n^1}{n^2+1} & & \genfrac{}{}{0pt}{}{n^1}{n^2} \\ \succ\cdots\cdots & \rhd\!\rhd & \cdots\cdots\succ \\ \prec\cdots\cdots & \rhd\!\rhd & \cdots\cdots\prec \\ \genfrac{}{}{0pt}{}{m^1}{m^2+1} & & \genfrac{}{}{0pt}{}{m^1}{m^2} \end{smallmatrix}
\cdot\Bigg[\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{0}{1}\genfrac{}{}{0pt}{}{0}{1};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda']}\Bigg]\Bigg\} \\
&+\sqrt{(m^2{+}1)(n^2{+}1)}\;
\begin{smallmatrix} \genfrac{}{}{0pt}{}{n^1}{n^2+1} & & \genfrac{}{}{0pt}{}{n^1}{n^2} \\ \succ\cdots\cdots & \blacktriangleright & \cdots\cdots\succ \\ \prec\cdots\cdots & \blacktriangleright & \cdots\cdots\prec \\ \genfrac{}{}{0pt}{}{m^1}{m^2+1} & & \genfrac{}{}{0pt}{}{m^1}{m^2} \end{smallmatrix}
\;\Bigg[\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{0}{1}\genfrac{}{}{0pt}{}{0}{1};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda']}\Big)+A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{0}{1}\genfrac{}{}{0pt}{}{0}{1};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda_R]\Bigg]\,.
\end{aligned}
\tag{5.10}
$$

4. *Let γ be any other type of graph. This includes non-planar graphs ($B>1$ and/or $g>0$), graphs with $N\geq 6$ external legs, one-particle reducible graphs, four-point graphs with non-constant index along at least one trajectory and two-point graphs where the integrated absolute value of the jump along the trajectories is bigger than 2. Then, the contribution of γ to the effective action is integrated as follows:*

$$
A^{(V)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda] := -\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda']}\Big)\,. \tag{5.11}
$$

The integration procedure identifies the following distinguished functions $\rho_a[\Lambda,\Lambda_0,\rho^0]$:

$$
\rho_1[\Lambda,\Lambda_0,\rho^0] := \sum_{\gamma\text{ as in Def. 12.2}} A^{\gamma}_{\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda,\Lambda_0,\rho^0]\,, \tag{5.12a}
$$

$$
\rho_2[\Lambda,\Lambda,\rho^0] := \sum_{\gamma\text{ as in Def. 12.2}} \Big(A^{\gamma}_{\genfrac{}{}{0pt}{}{1}{0}\genfrac{}{}{0pt}{}{0}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{1}{0}}[\Lambda,\Lambda_0,\rho^0] - A^{\gamma}_{\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda,\Lambda_0,\rho^0]\Big)\,, \tag{5.12b}
$$

[24]In case of the index arrangement $m_1n_1;m_2n_2 = {}^{m^1+1}_{m^2}{}^{n^1+1}_{n^2};{}^{n^1}_{n^2}{}^{m^1}_{m^2}$ and sequences $k_1,\dots,k_{a-1}$ ($l_1,\dots,l_{b-1}$) of indices at inner vertices on the trajectory $\overrightarrow{n_1m_2}$ ($\overrightarrow{n_2m_1}$) this means that there exist labels p,q with $n^1{+}1 = k^1_i$ for all $i\in[1,p{-}1]$, $n^1=k^1_i$ for all $i\in[p,a{-}1]$ and $n^2=k^2_i$ for all $i\in[1,a{-}1]$ on one trajectory and $m^1{+}1 = l^1_j$ for all $j\in[q,b{-}1]$, $m^1=k^1_j$ for all $j\in[1,q{-}1]$ and $m^2=k^2_j$ for all $j\in[1,b{-}1]$ on the other trajectory. The cases $p\in\{1,a\}$ and $q\in\{1,b\}$ are admitted.

$$\rho_3[\Lambda,\Lambda_0,\rho^0] := \sum_{\gamma \text{ as in Def. 12.3}} \Big(- A^{\gamma}_{\substack{1\,1\\0\,0};\substack{0\,0\\0\,0}}[\Lambda,\Lambda_0,\rho^0]\Big)\;, \tag{5.12c}$$

$$\rho_4[\Lambda,\Lambda_0,\rho^0] := \sum_{\gamma \text{ as in Def. 12.1}} A^{\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda,\Lambda_0,\rho^0]\;. \tag{5.12d}$$

This identification uses the symmetry properties of the A-functions when summed over all contributing graphs. It follows from Definition 12 and (5.1) that

$$\rho_a[\Lambda_0,\Lambda_0,\rho^0] \equiv \rho^0_a\;, \qquad a=1,\dots,4\;. \tag{5.13}$$

As part of the renormalisation strategy encoded in Definition 12, the coefficients (5.12) are kept constant at $\Lambda=\Lambda_R$. We define

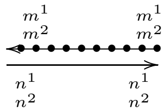
$$\rho_a[\Lambda_R,\Lambda_0,\rho^0] = 0 \qquad \text{for } a=1,2,3\;, \qquad\qquad \rho_4[\Lambda_R,\Lambda_0,\rho^0] = \lambda\;. \tag{5.14}$$

The normalisation (5.14) for ρ_1,ρ_2,ρ_3 identifies $\Delta^K_{nm;lk}(\Lambda_R)$ as the cut-off propagator related to the normalised two-point function at Λ_R. This entails a normalisation of the mass μ_0, the oscillator frequency Ω and the amplitude of the fields ϕ_{mn}. The normalisation condition for $\rho_4[\Lambda_R,\Lambda_0,\rho^0]$ defines the coupling constant used in the expansion (4.51) on page 47.

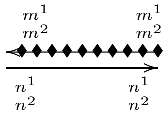
5.2 Ribbon graphs with composite propagators

It is convenient to write the linear combination of the functions in braces { } in (5.7)–(5.10) as a (non-unique) linear combination of graphs in which we find at least one of the following composite propagators:

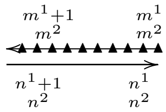
$$\mathcal{Q}^{(0)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}} := Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}} - Q_{\substack{0\\0}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{0\\0}} = \tag{5.15a}$$



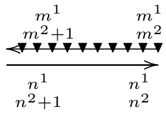
$$\mathcal{Q}^{(1)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}} := \mathcal{Q}^{(0)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}} - m^1\mathcal{Q}^{(0)}_{\substack{1\\0}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{1\\0}} - m^2\mathcal{Q}^{(0)}_{\substack{0\\1}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{0\\1}} = \tag{5.15b}$$



$$\mathcal{Q}^{(+\frac{1}{2})}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}} := Q_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}} - \sqrt{m^1+1}\,Q_{\substack{1\\0}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{0\\0}} = \tag{5.15c}$$



$$\mathcal{Q}^{(-\frac{1}{2})}_{\substack{m^1\\m^2+1}\substack{n^1\\n^2+1};\substack{n^1\\n^2}\substack{m^1\\m^2}} := Q_{\substack{m^1\\m^2+1}\substack{n^1\\n^2+1};\substack{n^1\\n^2}\substack{m^1\\m^2}} - \sqrt{m^2+1}\,Q_{\substack{0\\1}\substack{n^1\\n^2+1};\substack{n^1\\n^2}\substack{0\\0}} = \tag{5.15d}$$



To obtain the linear combination we recall how the graph γ under consideration is produced via a history of contractions and integrations of subgraphs. For a history a-b-…-n (a first) we have

$$\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda} A^{(V)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda] \\
&= \sum_{m_a,n_a,k_a,l_a,\dots,m_n,n_n,k_n,l_n} \int_{(\Lambda)_A}^{(\Lambda)_B} \frac{d\Lambda_n}{\Lambda_n} \int_{(\Lambda_n)_A}^{(\Lambda_n)_B} \frac{d\Lambda_{n-1}}{\Lambda_{n-1}} \cdots \int_{(\Lambda_b)_A}^{(\Lambda_b)_B} \frac{d\Lambda_a}{\Lambda_a} \\
&\qquad\times Q_{m_nn_n;k_nl_n}(\Lambda_n)\dots Q_{m_bn_b;k_bl_b}(\Lambda_b) Q_{m_an_a;k_al_a}(\Lambda_a)\, V^{m_an_ak_al_a\dots m_nn_nk_nl_n}_{m_1n_1\dots m_Nn_N}\;,
\end{aligned} \tag{5.16}$$

where $V^{m_a n_a k_a l_a \dots m_n n_n k_n l_n}_{m_1 n_1 \dots m_N n_N}$ is the vertex operator and either $(\Lambda_i)_A = \Lambda_i, (\Lambda_i)_B = \Lambda_0$ or $(\Lambda_i)_A = \Lambda_R, (\Lambda_i)_B = \Lambda_i$. Note that in (5.16) there is one integration less than the number of propagators. The graph $\Lambda\frac{\partial}{\partial\Lambda}A^{(V)\gamma}_{00;\dots;00}[\Lambda]$ is obtained via the same procedure (including the choice of the integration direction), except that we use the vertex operator $V^{m_a n_a k_a l_a \dots m_n n_n k_n l_n}_{00\dots 00}$. This means that all propagator indices which are not determined by the external indices are the same. Therefore, we can factor out in the difference of graphs all completely inner propagators and the integration operations.

We first consider the difference in (5.7). Since γ is one-particle irreducible with constant index on each trajectory, we get for a certain permutation π ensuring the history of integrations

$$
\begin{aligned}
&\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{mn;nk;kl;lm}}[\Lambda'] - \Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{00;00;00;00}}[\Lambda'] \\
&= \dots\Big\{\prod_{i=1}^{a} Q_{m_{\pi_i}k_{\pi_i};k_{\pi_i}m_{\pi_i}}(\Lambda_{\pi_i}) - \prod_{i=1}^{a} Q_{0k_{\pi_i};k_{\pi_i}0}(\Lambda_{\pi_i})\Big\} \\
&= \dots\Big\{\sum_{b=1}^{a}\Big(\prod_{i=1}^{b-1} Q_{0k_{\pi_i};k_{\pi_i}0}(\Lambda_{\pi_i})\Big)\mathcal{Q}^{(0)}_{m_{\pi_b}k_{\pi_b};k_{\pi_b}m_{\pi_b}}(\Lambda_{\pi_b}) \\
&\qquad\qquad\times\Big(\prod_{j=b+1}^{a} Q_{m_{\pi(j)}k_{\pi(j)};k_{\pi(j)}m_{\pi(j)}}(\Lambda_{\pi(j)})\Big)\Big\}\;, \qquad (5.17)
\end{aligned}
$$

where γ contains a propagators with external indices and $m_{\pi_i} \in \{m, n, k, l\}$. The parts of the analytic expression common to both $\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{mn;nk;kl;lm}}[\Lambda']$ and $\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{00;00;00;00}}[\Lambda']$ are symbolised by the dots. The k_{π_i} are inner indices. We thus learn that the difference of graphs appearing in the braces in (5.7) can be written as a sum of graphs each one having one composite propagator (5.15a). Of course, the identity (5.17) is nothing but a generalisation of $a^n - b^n = \sum_{k=0}^{n-1} b^k(a-b)a^{n-k-1}$. There are similar identities for the differences appearing in (5.8)–(5.10). I delegate their derivation to Appendix E.1. I show in Appendix E.2 how the difference operation works for a concrete example of a two-leg graph.

5.3 Bounds for the cut-off propagator

Differentiating the cut-off propagator (5.3) with respect to Λ and recalling from (2.10) that the cut-off function $K(x)$ is constant unless $x \in [1, 2]$, we notice that for our choice $\theta_1 = \theta_2 \equiv \theta$ the indices are restricted as follows:

$$
\Lambda\frac{\partial\Delta^K_{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{k^1 \\ k^2}\substack{l^1 \\ l^2}}(\Lambda)}{\partial\Lambda} = 0 \qquad \text{unless} \quad \theta\Lambda^2 \le \max(m^1, m^2, n^1, n^2, k^1, k^2, l^1, l^2) \le 2\theta\Lambda^2\;. \tag{5.18}
$$

In particular, the volume of the support of the differentiated cut-off propagator (5.18) with respect to a single index $m, n, k, l \in \mathbb{N}^2$ equals $4\theta^2\Lambda^4$, in agreement with (4.54) on page 47 for a $(D = 4)$-dimensional model.

I determine in Appendix F on page 145 the Λ-dependence of the maximised propagator $\Delta^{\mathcal{C}}_{mn;kl}$, which is the application of the sharp cut-off realising the condition (5.18) to the propagator, for selected values of $\mathcal{C} = \theta\Lambda^2$ and Ω, which is extremely well reproduced by (F.2) and (F.5). We thus obtain for the maximum of (5.5)

$$\begin{aligned}\max_{m,n,k,l} \big|Q_{mn;kl}(\Lambda)\big| &\leq \frac{1}{2\pi\theta}\,(32\max_x |K'(x)|)\,\max_{m,n,k,l}\big|\Delta^{\mathcal{C}}_{mn;kl}\big|_{\mathcal{C}=\Lambda^2\theta} \\ &\leq \begin{cases} \dfrac{C_0}{\Omega\Lambda^2\theta}\,\delta_{m+k,n+l} & \text{for } \Omega > 0\;, \\[2ex] \dfrac{C_0}{\sqrt{\Lambda^2\theta}}\,\delta_{m+k,n+l} & \text{for } \Omega = 0\;, \end{cases}\end{aligned} \tag{5.19}$$

where $C_0 = C_0' \frac{40}{3\pi} \max_x |K'(x)|$. The constant $C_0' \gtrapprox 1$ corrects the fact that (F.2) holds asymptotically only. Next, from (F.3) and (F.6) we obtain

$$\begin{aligned}\max_m \sum_l \max_{n,k} \big|Q_{mn;kl}(\Lambda)\big| &\leq \frac{1}{2\pi\theta}(32\max_x |K'(x)|)\,\max_m \sum_l \max_{n,k}\big|\Delta^{\mathcal{C}}_{mn;kl}\big|_{\mathcal{C}=\Lambda^2\theta} \\ &\leq \begin{cases} \dfrac{C_1}{\Omega^2\theta\Lambda^2} & \text{for } \Omega > 0\;, \\[2ex] \dfrac{3C_1}{\theta\mu_0^2} & \text{for } \Omega = 0\;, \end{cases}\end{aligned} \tag{5.20}$$

where $C_1 = 48C_1'/(7\pi)\max_x |K'(x)|$. The product of (5.19) by the volume $4\theta^2\Lambda^4$ of the support of the cut-off propagator with respect to a single index leads to the following bound:

$$\sum_m \Big(\max_{n,k,l}\big|Q_{mn;kl}(\Lambda)\big|\Big) \leq \begin{cases} 4C_0 \dfrac{\theta\Lambda^2}{\Omega} & \text{for } \Omega > 0\;, \\[2ex] 4C_0\,\big(\sqrt{\theta}\Lambda\big)^3 & \text{for } \Omega = 0\;. \end{cases} \tag{5.21}$$

According to (B.49) on page 108 there is the following refinement of the estimation (5.19):

$$\Big|Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1-a^1\\n^2-a^2}\substack{m^1-a^1\\m^2-a^2}}(\Lambda)\Big|_{a^r\geq 0,\,m^r\leq n^r} \leq C_{a^1,a^2}\Big(\frac{m^1}{\theta\Lambda^2}\Big)^{\frac{a_1}{2}}\Big(\frac{m^2}{\theta\Lambda^2}\Big)^{\frac{a_2}{2}}\frac{1}{\Omega\theta\Lambda^2}\;. \tag{5.22}$$

This property will imply that graphs with big total jump along the trajectories are suppressed, provided that the indices on the trajectory are "small". However, there is a potential danger from the presence of completely inner vertices, where the index summation runs over "large" indices as well. Fortunately, according to (F.4) this case can be controlled by the following property of the propagator:

$$\Bigg(\sum_{\substack{l\,\in\,\mathbb{N}^2\\ \|m-l\|_1\,\geq\,5}} \max_{k,n\in\mathbb{N}^2}\big|Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(\Lambda)\big|\Bigg) \leq C_4\Big(\frac{\|m\|_\infty+1}{\theta\Lambda^2}\Big)^2\frac{1}{\Omega^2\theta\Lambda^2}\;, \tag{5.23}$$

where we have defined the following norms:

$$\|m-l\|_1 := \sum_{r=1}^{2}|m^r-l^r|\;, \qquad \|m\|_\infty := \max(m^1,m^2) \qquad \text{if } m = {\textstyle\substack{m^1\\m^2}}\;, \quad l = {\textstyle\substack{l^1\\l^2}}\;. \tag{5.24}$$

Moreover, we define

$$\|m_1n_1;\dots;m_Nn_N\|_\infty := \max_{i=1,\dots,N}\big(\|m_i\|_\infty,\|n_i\|_\infty\big)\;. \tag{5.25}$$

Finally, we need estimations for the composite propagators (5.15), page 55, and (E.7), page 140:

$$\Big|\mathcal{Q}^{(0)}_{\genfrac{}{}{0pt}{}{m^1}{m^2}\genfrac{}{}{0pt}{}{n^1}{n^2};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}(\Lambda)\Big| \leq C_5\frac{\|m\|_\infty}{\theta\Lambda^2}\frac{1}{\Omega\theta\Lambda^2}\;, \tag{5.26}$$

$$\Big|\mathcal{Q}^{(1)}_{\genfrac{}{}{0pt}{}{m^1}{m^2}\genfrac{}{}{0pt}{}{n^1}{n^2};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}(\Lambda)\Big| \leq C_6\Big(\frac{\|m\|_\infty}{\theta\Lambda}\Big)^2\frac{1}{\Omega\theta\Lambda^2}\;, \tag{5.27}$$

$$\Big|\mathcal{Q}^{(+\frac{1}{2})}_{\genfrac{}{}{0pt}{}{m^1+1}{m^2}\genfrac{}{}{0pt}{}{n^1+1}{n^2};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}(\Lambda)\Big| \leq C_7\left(\frac{\big\|\genfrac{}{}{0pt}{}{m^1}{m^2}\genfrac{}{}{0pt}{}{n^1}{n^2};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{m^1}{m^2}\big\|_\infty}{\theta\Lambda^2}\right)^{\frac{3}{2}}\frac{1}{\Omega\theta\Lambda^2}\;. \tag{5.28}$$

These estimations follow from (B.51) and (B.53) on page 109.

5.4 The power-counting estimation

Now, I am going to determine the power-counting behaviour of the duality-covariant noncommutative ϕ^4-model, generalising Theorem 10 on page 48. The generalisation concerns 1PI planar graphs and their subgraphs. I refer to Section 4.3 on page 42 for the definition of the segmentation index ι, of a trajectory and of an index summation. A subgraph of a planar graph has necessarily genus $g = 0$ and an even number of legs on each boundary component. We distinguish one boundary component of the subgraph which after a sequence of contractions will be part of the unique boundary component of an 1PI planar graph. For a trajectory $\overrightarrow{nm}$ on the distinguished boundary component, which passes through the indices $k_1,\dots,k_a$ when going from n to $m=\mathfrak{o}[n]$, I define the total jump as

$$\langle\overrightarrow{nm}\rangle := \|n-k_1\|_1 + \Big(\sum_{c=1}^{a-1}\|k_c-k_{c+1}\|_1\Big) + \|k_a-m\|_1\;. \tag{5.29}$$

Clearly, the jump is additive: if we connect two trajectories $\overrightarrow{nm}$ and $\overrightarrow{mm'}$ to a new trajectory $\overrightarrow{nm'}$, then $\langle\overrightarrow{nm'}\rangle = \langle\overrightarrow{nm}\rangle + \langle\overrightarrow{mm'}\rangle$. We let T be a set of trajectories $\overrightarrow{n_j\mathfrak{o}[n_j]}$ on the distinguished boundary component for which we measure the total jump. By definition, the end points of a trajectory in T cannot belong to $\mathcal{E}^s$.

Moreover, we consider a second set T' of t' trajectories $\overrightarrow{n_j\mathfrak{o}[n_j]}$ of the distinguished boundary component where one of the end points m_j or n_j is kept fixed and the other end point is summed over. However, we require the summation to run over $\langle\overrightarrow{n_j\mathfrak{o}[n_j]}\rangle \geq 5$ only, see (5.23). We let $\sum_{\mathcal{E}^{t'}}$ be the corresponding summation operator.

Additionally, I have to introduce a new notation in order to control

- the behaviour for large indices and given Λ,
- the behaviour for given indices and large Λ

For this purpose we let $P^a_b\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big]$ denote a function of the indices $m_1, n_1, \dots, m_N, n_N$ and the scale Λ which is bounded as follows:

$$
\begin{aligned}
0 \le P^a_b\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] &\le \begin{cases} C_aM^a & \text{for } M\ge 1\,,\\ C_bM^b & \text{for } M \le 1\,,\end{cases} \\
M := \max_{m_i,n_i\notin\mathcal{E}^s,\mathcal{E}^{t'}} &\Big(\frac{m^r_1+1}{2\theta\Lambda^2}, \frac{n^r_1+1}{2\theta\Lambda^2}, \dots, \frac{n^r_N+1}{2\theta\Lambda^2}\Big)\,,
\end{aligned}
\tag{5.30}
$$

for some constants C_a, C_b. The maximisation over the indices m^r_i, n^r_i excludes the summation indices $\mathcal{E}'^t$. By definition,

$$
P^{a-a'}_{b+b'}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] \le P^a_b\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big]\,,
\tag{5.31}
$$

for $0 \le a' \le a$ and $b' \ge 0$, assuming appropriate C_a, C_b. Moreover,

$$
P^{a_1}_{b_1}\Big[\frac{m_1n_1;\dots;m_{N_1}n_{N_2}}{\theta\Lambda^2}\Big]P^{a_2}_{b_2}\Big[\frac{m_{N_1+1}n_{N_1+1};\dots;m_Nn_N}{\theta\Lambda^2}\Big] \le P^{a_1+a_2}_{b_1+b_2}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big]\,.
\tag{5.32}
$$

I am going to prove:

Proposition 13 *Let γ be a ribbon graph having N external legs, V vertices, V^e external vertices and segmentation index ι, which is drawn on a genus-g Riemann surface with B boundary components. We require the graph γ to be constructed via a history of subgraphs and an integration procedure according to Definition 12 on page 52. Then, the contribution $A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}$ of γ to the expansion coefficient of the effective action describing a duality-covariant ϕ^4-theory on $\mathbb{R}^4_\theta$ in the matrix base is bounded as follows:*

1. If γ is as in Definition 12.1, we have

$$
\begin{aligned}
&\Big|A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{k^1\\k^2};\substack{k^1\\k^2}\substack{l^1\\l^2};\substack{l^1\\l^2}\substack{m^1\\m^2}}[\Lambda,\Lambda_0,\rho_0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda,\Lambda_0,\rho_0]\Big| \\
&\qquad\le P^{4V-N}_1\left[\frac{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{k^1\\k^2};\substack{k^1\\k^2}\substack{l^1\\l^2};\substack{l^1\\l^2}\substack{m^1\\m^2}}{\theta\Lambda^2}\right]\Big(\frac{1}{\Omega}\Big)^{3V-2-V^e} P^{2V-2}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,,
\end{aligned}
\tag{5.33a}
$$

$$
\Big|A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda,\Lambda_0,\rho_0]\Big| \le \Big(\frac{1}{\Omega}\Big)^{3V-2-V^e} P^{2V-2}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,.
\tag{5.33b}
$$

2. If γ is as in Definition 12.2, we have

$$
\begin{aligned}
\Big|A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda,\Lambda_0,\rho_0] &- A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda,\Lambda_0,\rho_0] \\
&- m^1\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{1\\0}\substack{0\\0};\substack{0\\0}\substack{1\\0}}[\Lambda,\Lambda_0,\rho_0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda,\Lambda_0,\rho_0]\big) \\
&- n^1\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{1\\0};\substack{1\\0}\substack{0\\0}}[\Lambda,\Lambda_0,\rho_0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda,\Lambda_0,\rho_0]\big) \\
&- m^2\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\1}\substack{0\\0};\substack{0\\0}\substack{0\\1}}[\Lambda,\Lambda_0,\rho_0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda,\Lambda_0,\rho_0]\big) \\
&- n^2\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\1};\substack{0\\1}\substack{0\\0}}[\Lambda,\Lambda_0,\rho_0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda,\Lambda_0,\rho_0]\big)\Big| \\
&\le (\theta\Lambda^2)\, P^{4V-N}_2\left[\frac{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}{\theta\Lambda^2}\right]\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e} P^{2V-1}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,,
\end{aligned}
\tag{5.34a}
$$

$$\Big|A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\,;0\,0\\0\,0\,;0\,0}}[\Lambda,\Lambda_0,\rho_0]\Big| \le (\theta\Lambda^2)\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e} P^{2V-1}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,, \tag{5.34b}$$

$$\Big|A^{(V,V^e,1,0,0)\gamma}_{\substack{1\,0\,;0\,1\\0\,0\,;0\,0}}[\Lambda,\Lambda_0,\rho_0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\,;0\,0\\0\,0\,;0\,0}}[\Lambda,\Lambda_0,\rho_0]\Big| \le \Big(\frac{1}{\Omega}\Big)^{3V-1-V^e} P^{2V-1}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,. \tag{5.34c}$$

3. If γ is as in Definition 12.3, we have

$$\begin{aligned}
&\Big|A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1+1\;n^1+1\,;\,n^1\;m^1\\m^2\quad n^2\;;\;n^2\;m^2}}[\Lambda,\Lambda_0,\rho_0] - \sqrt{(m^1{+}1)(n^1{+}1)}\,A^{(V,V^e,1,0,0)\gamma}_{\substack{1\,1\,;0\,0\\0\,0\,;0\,0}}[\Lambda,\Lambda_0,\rho_0]\Big| \\
&\qquad\le (\theta\Lambda^2)\,P_2^{4V-N}\Bigg[\frac{\substack{m^1+1\;n^1+1\,;\,n^1\;m^1\\m^2\quad n^2\;;\;n^2\;m^2}}{\theta\Lambda^2}\Bigg]\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e} P^{2V-1}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,,
\end{aligned}\tag{5.35a}$$

$$\Big|A^{(V,V^e,1,0,0)\gamma}_{\substack{1\,1\,;0\,0\\0\,0\,;0\,0}}[\Lambda,\Lambda_0,\rho_0]\Big| \le \Big(\frac{1}{\Omega}\Big)^{3V-1-V^e} P^{2V-1}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,. \tag{5.35b}$$

4. If γ is a subgraph of an 1PI planar graph with a selected set T of trajectories on one distinguished boundary component and a second set T' of summed trajectories on that boundary component, we have

$$\begin{aligned}
&\sum_{\mathcal{E}^s}\sum_{\mathcal{E}^{t'}}\big|A^{(V,V^e,B,0,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho_0]\big| \\
&\quad\le \big(\theta\Lambda^2\big)^{(2-\frac{N}{2})+2(1-B)} P^{4V-N}_{\left(2t'+\sum_{\overrightarrow{n_j\circ[n_j]}\in T}\min(2,\frac{1}{2}\langle\overrightarrow{n_j\circ[n_j]}\rangle)\right)}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] \\
&\quad\times\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}-1+B-V^e-\iota+s+t'} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,.
\end{aligned}\tag{5.36}$$

5. If γ is a non-planar graph, we have

$$\begin{aligned}
\sum_{\mathcal{E}^s}\big|A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho_0]\big| &\le \big(\theta\Lambda^2\big)^{(2-\frac{N}{2})+2(1-B-2g)} P_0^{4V-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] \\
&\quad\times\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}-1+B+2g-V^e-\iota+s} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,.
\end{aligned}\tag{5.37}$$

Proof. We prove the Proposition by induction upward in the vertex order V and for given V downward in the number N of external legs.

5. We start with with the proof for non-planar graphs, noticing that due to (5.31) the estimations (5.33), (5.34), (5.35) and (5.36) can be further bounded by (5.37). The proof of (5.37) reduces to the proof of Theorem 10 given in Appendix D, where we have to take for $\big(\frac{\mu}{\Lambda}\big)^{\delta_0}$ $\big(\frac{\mu}{\Lambda}\big)^{\delta_1}$ and $\big(\frac{\Lambda}{\mu}\big)^{\delta_2}$ the estimations (5.19), (5.20) and (5.21) with both their Λ- and Ω-dependence. Independent of the factor (5.30), the non-planarity of the graph guarantees the irrelevance of the corresponding function so that the integration according to Definition 12 agrees with the procedure of Theorem 10. The dependence on $\frac{m_i^r}{\theta\Lambda^2}, \frac{n_i^r}{\theta\Lambda^2}$ through (5.30) is preserved in its structure, because for $\omega>0$ we have

$$\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\frac{C}{\Lambda'^\omega}P_b^a\Big[\frac{m}{\theta\Lambda'^2}\Big] \le \int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\frac{C}{\Lambda'^\omega}C_b\Big(\frac{m+1}{2\theta\Lambda'^2}\Big)^b \le \frac{1}{\omega+2b}\frac{C}{\Lambda^\omega}C_b\Big(\frac{m+1}{2\theta\Lambda^2}\Big)^b \tag{5.38a}$$

for $m{+}1 \leq 2\theta\Lambda^2$ and

$$\int_\Lambda^{\Lambda_0} \frac{d\Lambda'}{\Lambda'}\frac{C}{\Lambda'^\omega}P^a_b\Big[\frac{m}{\theta\Lambda'^2}\Big] \leq \int_{\sqrt{\frac{m+1}{2\theta}}}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'}\frac{C}{\Lambda'^\omega}C_b\Big(\frac{m{+}1}{2\theta\Lambda'^2}\Big)^b + \int_\Lambda^{\sqrt{\frac{m+1}{2\theta}}} \frac{d\Lambda'}{\Lambda'}\frac{C}{\Lambda'^\omega}C_a\Big(\frac{m{+}1}{2\theta\Lambda'^2}\Big)^a$$
$$\leq \frac{1}{\omega{+}2a}\frac{C}{\Lambda^\omega}C_a\Big(\frac{m{+}1}{2\theta\Lambda^2}\Big)^a + \frac{(\omega{+}2a)C_b-(\omega{+}2b)C_a}{(\omega{+}2a)(\omega{+}2b)}\frac{C}{\big(\frac{m+1}{2\theta}\big)^{\frac{\omega}{2}}} \tag{5.38b}$$

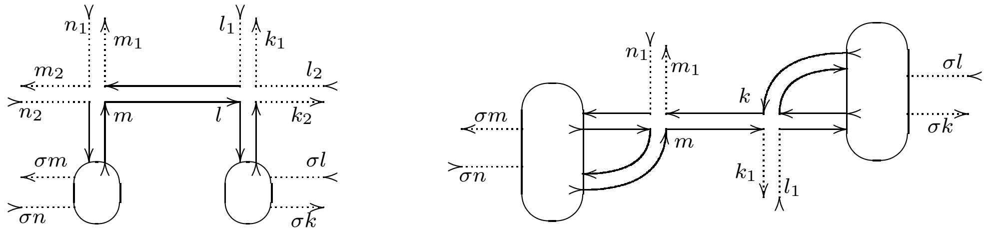
for $m{+}1 \geq 2\theta\Lambda^2$. For $(\omega{+}2b)C_a > (\omega{+}2a)C_b$ we can omit the last term in the second line of (5.38b), and for $(\omega{+}2b)C_a < (\omega{+}2a)C_b$ we estimate it by $\frac{(\omega+2a)C_b-(\omega+2b)C_a}{C_a(\omega+2b)}$ times the first term. Taking a polynomial in $\ln\frac{\Lambda}{\Lambda_R}$ into account, the spirit of (5.38) is unchanged due to (4.27).

The proof given in Appendix D uses the bounds (5.19) and (5.20) of the propagator, which does not add factors $\frac{m}{\theta\Lambda^2}$. Since two legs of the subgraph(s) are contracted, the total a-degree of (5.30) becomes $4V-N-2$, which due to (5.31) can be regarded as degree $4V-N$, too.

4. The proof of (5.36) is essentially a repetition of the proof of (5.37) (and thus of the proof of Theorem 10), with particular care when contracting trajectories on the distinguished boundary component. The verification of the exponents of $(\theta\Lambda^2)$, $\frac{1}{\Omega}$ and $\ln\frac{\Lambda}{\Lambda_R}$ in (5.36) is identical to the proof of (5.37). We can thus restrict ourselves in verifying the a,b-degrees of the factor (5.30).

   We first consider the contraction of two smaller graphs γ_1 (left subgraph) and γ_2 (right subgraph) to the total graph γ.

4.1. We first assume additionally that all indices of the contracting propagator are determined (this is the case for $V_1^e{+}V_2^e = V^e$ and $\iota_1{+}\iota_2 = \iota$), e.g.



(5.39)

As a subgraph of an 1PI planar graph, at most one side ml or m_1l_1 (mk_1 or m_1k) of the contracting propagator $Q_{m_1m;ll_1}$ ($Q_{m_1m;k_1k}$) can belong to a trajectory in T.

In the left graph of (5.39) let us assume that the side $\overrightarrow{ml}$ connects two trajectories $\overrightarrow{\mathfrak{i}[m]m} \in T_1$ and $\overrightarrow{l\mathfrak{o}[l]} \in T_2$ to a new trajectory $\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]} \in T$. The proof for the small-Λ degree $a = 4V-N$ in (5.36) is immediate, because the contraction reduces the number of external legs by 2 and we are free to estimate the contracting propagator by its global maximisation (5.19). Concerning the large-Λ degree b, there is nothing to prove if already $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle + \langle\overrightarrow{l\mathfrak{o}[l]}\rangle \geq 4$. For $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle + \langle\overrightarrow{l\mathfrak{o}[l]}\rangle < 4$ we use the refined estimation (5.22) for the contracting propagator, which gives a relative factor $M^{\frac{1}{2}\langle\overrightarrow{lm}\rangle}$ compared with (5.19), where $M = \max\big(\frac{\|m\|_\infty+1}{2\theta\Lambda^2}, \frac{\|l\|_\infty+1}{2\theta\Lambda^2}\big)$. Now, the result follows from (5.29). Because of $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle + \langle\overrightarrow{l\mathfrak{o}[l]}\rangle < 4$, the indices m^r, l^r from the propagator can be estimated by $\mathfrak{i}[m]^r$ and $\mathfrak{o}[l]^r$. If the resulting jump leads to $\frac{1}{2}\langle\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]}\rangle > 2$, we use (5.31) to reduce it to 2. In this way we can guarantee that the b-degree does not exceed the a-degree. Alternatively, if $\langle\overrightarrow{lm}\rangle \geq 5$ we can avoid a huge $\langle\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]}\rangle$ by estimating the contracting propagator via (5.23) instead of[25] (5.22), because a single propagator $|Q_{mm_1;l_1l}|$ is clearly smaller than the entire sum over $\|m-l\|_1 \geq 5$.

If $\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]} \in T'$, then the sum over $\mathfrak{o}[l]$ with $\langle\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]}\rangle \geq 5$ can be estimated by the combined sum

- over a finite number of combinations of m, l with $\max(\langle\overrightarrow{\mathfrak{i}[m]m}\rangle, \langle\overrightarrow{ml}\rangle, \langle\overrightarrow{l\mathfrak{o}[l]}\rangle) \leq 4$, which via (5.22) and the induction hypotheses relative to T_1, T_2 contributes a factor $M^{\langle\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]}\rangle}$ to (5.36), where $M = \max\Big(\frac{\mathfrak{o}[l]^r+1}{2\theta\Lambda^2}, \frac{\mathfrak{i}[m]^r+1}{2\theta\Lambda^2}, \frac{l^r+1}{2\theta\Lambda^2}, \frac{m^r+1}{2\theta\Lambda^2}\Big)$. We use (5.31) to reduce the b-degree from $\frac{1}{2}\langle\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]}\rangle$ to 2.
- over m via the induction hypothesis relative to $\overrightarrow{\mathfrak{i}[m]m} \in T_1'$, combined with the usual maximisation (5.19) of the contracting propagator and an estimation of γ_2 where $\overrightarrow{l\mathfrak{o}[l]} \notin T_2, T_2'$,
- over l for fixed $m \approx \mathfrak{i}[m]$, via (5.23), taking $\overrightarrow{\mathfrak{i}[m]m} \notin T_1, T_1'$ and $\overrightarrow{l\mathfrak{o}[l]} \notin T_2, T_2'$.
- over $\mathfrak{o}[l]$ for fixed m and l, with $\mathfrak{i}[m] \approx m \approx l$, via the induction hypothesis relative to $\overrightarrow{l\mathfrak{o}[l]} \in T_2'$, the bound (5.19) of the propagator and $\overrightarrow{\mathfrak{i}[m]m} \notin T_1, T_1'$.

A summation over $\mathfrak{i}[m]$ with given $\mathfrak{o}[l]$ is analogous.

In conclusion, we have proven that the integrand for the graph γ is bounded by (5.36). Since we are dealing with a $N \geq 6$-point function, the total Λ-exponent is negative. Using (5.38) we thus obtain the same bound (5.36) after integration from Λ_0 down to Λ. If $\overrightarrow{m_1l_1} \in T$ or $\overrightarrow{m_1l_1} \in T'$ we get (5.36) directly from (5.22) or (5.23).

The discussion of the right graph in (5.39) is similar, showing that the integrand is bounded by (5.36). As long as the integrand is irrelevant (i.e. the total Λ-exponent is negative), we get (5.36) after Λ-integration, too. However, γ might have two legs only with $\langle\overrightarrow{i(m)m}\rangle + \langle\overrightarrow{lo(l)}\rangle \leq 2$. In this case the integrand is marginal or relevant, but according to Definition 12.4 we nonetheless integrate from Λ_0 down to Λ. We have to take into account that the cut-off propagator at the scale Λ vanishes for $\Lambda^2 \geq \|m_1m;k_1k\|_\infty/\theta$. Assuming two relevant two-leg subgraphs γ_1, γ_2 bounded by $\theta\Lambda'^2$ times a polynomial in $\ln\frac{\Lambda'}{\Lambda_R}$ each, we have

$$
\begin{aligned}
&\int_\Lambda^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \big|Q_{m_1m;k_1k}(\Lambda')\, A^{\gamma_1}[\Lambda']\, A^{\gamma_2}[\Lambda']\big| \\
&\qquad \leq \frac{C_0}{\Omega} \int_\Lambda^{\sqrt{\|m_1m;k_1k\|_\infty/\theta}} \frac{d\Lambda'}{\Lambda'} (\theta\Lambda'^2) P^{2V-2}\Big[\ln\frac{\Lambda'}{\Lambda_R}\Big]
\end{aligned}
$$

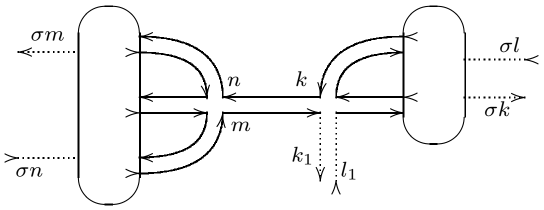
[25]In this case there is an additional factor $\frac{1}{\Omega}$ in (5.20) compared with (5.19). It is plausible that this is due to the summation, which we do not need here. However, I do not prove a corresponding formula without summation. In order to be on a safe side, one could replace Ω in the final estimation (6.56) by Ω^2. Since Ω is finite anyway, there is no change of the final result. I therefore ignore the discrepancy in $\frac{1}{\Omega}$.

$$\leq \frac{C_0}{\Omega}\|m_1m;k_1k\|_\infty\, P^{2V-2}\Big[\ln\Big(\frac{\|m_1m;k_1k\|_\infty}{\theta\Lambda_R^2}\Big)^{\frac{1}{2}}\Big]$$
$$\leq \frac{C_0}{\Omega}(\theta\Lambda^2)P_1^2\Big[\frac{m_1m;k_1k}{2\theta\Lambda^2}\Big]P^{2V-2}\Big[\frac{\Lambda}{\Lambda_R}\Big]\,. \tag{5.40}$$

Here, I have inserted the estimation (5.19) for the propagator, restricted to its support. In the logarithm I have expanded $\ln\sqrt{\frac{m}{\theta\Lambda_R^2}} = \ln\sqrt{\frac{m}{\theta\Lambda^2}} + \ln\frac{\Lambda}{\Lambda_R}$ and estimated $\big(\ln\sqrt{\frac{m}{\theta\Lambda^2}}\big)^q < c\frac{m}{2\theta\Lambda^2}$. Thus, the small-$\Lambda$ degree a of the total graph is increased by 2 over the sum of the small-Λ degrees of the subgraphs (taken $= 0$ here), in agreement with (5.36). The estimation for the logarithm is not necessary for the large-Λ degree b in (5.30). Using (5.31) we could reduce that degree to $b = 0$. I would like to underline that the integration of 1PR graphs is one of the sources for the factor (5.30) in the power-counting theorem. Taking the factors (5.30) in the bounds for the subgraphs γ_i into account, the formula modifies accordingly. We confirm (5.36) in any case.

It is clear that all other possibilities with determined propagator indices as discussed in Appendix D.1 are treated similarly.

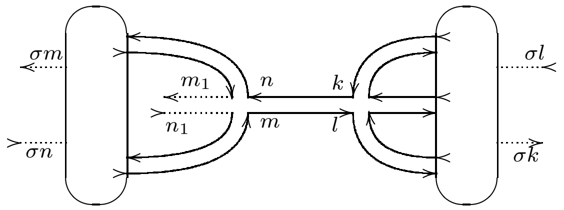
4.2. Next, let one index of the contracting propagator be an undetermined summation index, e.g.



(5.41)

Let $\overrightarrow{\mathfrak{i}[k]\mathfrak{o}[n]} \in T$. Then, k is determined by the external indices of γ_2. There is nothing to prove for $\langle\overrightarrow{\mathfrak{i}[k]k}\rangle \geq 4$. For $\langle\overrightarrow{\mathfrak{i}[k]k}\rangle < 4$ we partitionate the sum over n into $\langle\overrightarrow{nk}\rangle \leq 4$, where each term yields the integrand (5.36) as before in the case of determined indices (5.39), and the sum over $\langle\overrightarrow{nk}\rangle \geq 5$, which yields the desired factor in (5.36) via (5.23) and the similarity $k \approx \mathfrak{i}[k]$ of indices. As a subgraph of a planar graph, $m \neq \mathfrak{o}[n]$ in γ_1, so that a possible k_1-summation can be transferred to m. If $\overrightarrow{\mathfrak{i}[k]\mathfrak{o}[n]} \in T'$ then in the same way as for (5.39) the summation splits into the four possibilities related to the pieces $\overrightarrow{n\mathfrak{o}[n]}$, $\overrightarrow{kn}$ and $\overrightarrow{\mathfrak{i}[k]k}$, which yield the integrand (5.36) via the induction hypotheses for the subgraphs and via (5.22) or (5.23). The Λ-integration yields (5.36) via (5.38) if the integrand is irrelevant, whereas we have to perform similar considerations as in (5.40) if the integrand is relevant or marginal.

4.3. The discussion of graphs with two summation indices on the contracting propagator, such as in



(5.42)

is similar. Note that the planarity requirement implies $m \neq \mathfrak{o}[n]$ and $l \neq \mathfrak{i}[k]$.

4.4. Next, we look at self-contractions of the same vertex of a graph. Among the examples discussed in Section D.2 there are only two possibilities which can appear in subgraphs of planar graphs:

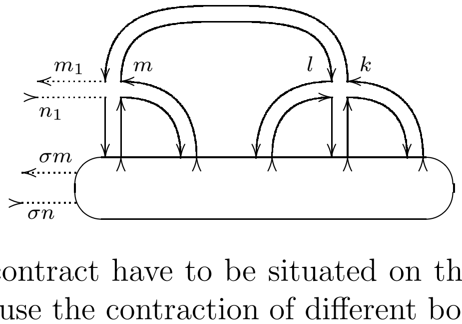
(5.43)

There is nothing to prove for the left graph in (5.43). To verify the large-Λ degree b relative to the right graph, we partitionate the sum over n into

- $\langle\overrightarrow{\mathfrak{i}[n]n}\rangle \leq 4$ and $\langle\overrightarrow{n\mathfrak{o}[n]}\rangle \leq 4$, where each term yields (5.36) via the induction hypothesis for the trajectories $\overrightarrow{\mathfrak{i}[n]n} \in T_1$ and $\overrightarrow{n\mathfrak{o}[n]} \in T_1$ of the subgraph (in the same way as for the examples with determined propagator indices),
- $\langle\overrightarrow{\mathfrak{i}[n]n}\rangle \leq 4$ and $\langle\overrightarrow{n\mathfrak{o}[n]}\rangle \geq 5$, for which the induction hypothesis for $\overrightarrow{\mathfrak{i}[n]n} \notin T_1, T_1'$ and $\overrightarrow{n\mathfrak{o}[n]} \in T_1'$, together with $\mathfrak{i}[n] \approx n$, gives a contribution of 2 to the b-degree in (5.30), and
- $\langle\overrightarrow{\mathfrak{i}[n]n}\rangle \geq 5$, which via the induction hypothesis for $\overrightarrow{\mathfrak{i}[n]n} \in T_1'$ and $\overrightarrow{n\mathfrak{o}[n]} \notin T_1, T_1'$ gives a contribution of 2 to the b-degree in (5.30).

The case $\overrightarrow{\mathfrak{i}[n]\mathfrak{o}[n]} \in T'$ is similar to discuss. At the end we always arrive at the integrand (5.36). If it is irrelevant, the integration from Λ_0 down to Λ yields (5.36) according to (5.38). If the integrand is marginal/relevant and γ is one-particle reducible, then the indices of the propagator contracting 1PI subgraphs are of the same order as the incoming and outgoing indices of the trajectories through the propagator (otherwise the 1PI subgraphs are irrelevant). Now, a procedure similar to (5.40) yields (5.36) after integration from Λ_0 down to Λ, too. If γ is 1PI and marginal or relevant, it is actually of the type 1–3 of Definition 12 and will be discussed below.

4.5. Finally, there will be self-contractions of different vertices of a subgraph, such as in



(5.44)

The vertices to contract have to be situated on the same (distinguished) boundary component, because the contraction of different boundary components increases the genus and for contractions of other boundary components the proof is immediate. Only the large-Λ degree b is questionable.

Let $\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]} \in T$, with $\mathfrak{i}[m] \neq \mathfrak{o}[l]$ due to planarity. According to (D.47), m is regarded as a summation index. As before we split that sum over m into a piece with $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle \leq 4$, which yields the b-degree of the integrand (5.36) term by term via the induction hypothesis relative to $\overrightarrow{\mathfrak{i}[m]m}, \overrightarrow{l\mathfrak{o}[l]} \in T_1$ and (5.22) for the contracting propagator, and a piece with $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle \geq 5$, which gives (5.36) via the induction hypothesis relative to $\overrightarrow{\mathfrak{i}[m]m} \in T_1'$ and $\overrightarrow{l\mathfrak{o}[l]} \notin T_1, T_1'$. If $\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]} \in T'$ the sum over $\mathfrak{o}[l]$ with $\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]} \geq 5$ is estimated by a finite number of combinations of m, l with $\max(\langle\overrightarrow{\mathfrak{i}[m]m}\rangle, \langle\overrightarrow{ml}\rangle, \langle\overrightarrow{l\mathfrak{o}[l]}\rangle) \leq 4$, which yields the integrand (5.36) via the induction hypothesis for T_1 and (5.22), and the sum over index combinations

- $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle \leq 4$, $\langle\overrightarrow{ml}\rangle \leq 4$, $\langle\overrightarrow{l\mathfrak{o}[l]}\rangle \geq 5$
- $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle \leq 4$, $\langle\overrightarrow{ml}\rangle \geq 5$
- $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle \geq 5$

which is controlled by the induction hypothesis relative to T_1' or (5.23), together with the similarity of trajectory indices at those parts where the jumps is bounded by 4. The case where $\overrightarrow{\mathfrak{i}[k]m_1} \in T$ or $\overrightarrow{\mathfrak{i}[k]m_1} \in T'$ is easier to treat. We thus arrive in any case at the estimation (5.36) for the integrand of γ, which leads to the same estimation (5.36) for γ itself according to the considerations at the end of 4.4. If γ is of type 1–3 of Definition 12 we will treat it below.

The discussion of all other possible self-contractions as listed in Appendix D.3 is similar.

This finishes the part 4 of the proof of Proposition 13.

1. Now, we consider 1PI planar 4-leg graphs γ with constant index on each trajectory. If the external indices are zero, we get (5.33b) directly from Theorem 10, because the integration direction used in the proof given in Appendix D agrees with Definition 12.1.

   For non-zero external indices we decompose the difference (5.33a) according to (5.17) on page 56 into graphs with composite propagators (5.15a), page 55, bounded by (5.26) on page 58. The composite propagators appear on one of the trajectories of γ, and as such already on the trajectory of a sequence of subgraphs of γ, starting with some minimal subgraph γ_0. The composite propagator is the contracting propagator for γ_0. Now, the integrand of the minimal subgraph γ_0 with composite propagator is bounded by a factor $C_5' \frac{\|m\|}{\Lambda^2\theta}$ times the integrand of the would-be graph γ_0 with ordinary propagator, where m is the index at the trajectory under consideration. If γ_0 is irrelevant, the factor $C_5' \frac{\|m\|}{\Lambda^2\theta}$ of the integrand survives according to (5.38) to the subgraph γ_0 itself. Otherwise, if γ_0 is relevant or marginal, it is decomposed according to 1–3 of Definition 12. Here, the last lines of (5.7)–(5.10) are independent of the external index m so that in the difference relative to the composite propagator these last lines of (5.7)–(5.10) cancel identically. There remains the first part of (5.7)–(5.10), which is integrated from Λ_0 downward and which is irrelevant by induction. Thus, (5.38) applies in this case, too, saving the factor $C_5' \frac{\|m\|}{\Lambda^2\theta}$ to γ_0 in any case. This factor thus appears in the integrand of the subgraph of γ next larger than γ_0. By iteration of the procedure we obtain the additional factor $C_5' \frac{\|m\|}{\Lambda^2\theta}$ in the integrand of the total graph γ with composite propagators, the Λ-degree of which being thus reduced by 2 compared with the original graph γ. Since γ itself is a marginal graph according to the general power-counting behaviour (5.37), the graph with composite propagator is irrelevant and according to Definition 12.1 to be integrated from Λ_0 down to Λ. This explains (5.33a).

3. Similarly, we conclude from the proof of (5.36) that the integrands of graphs γ according to Definition 12.3 are marginal. In particular, we immediately confirm (5.35b). For non-zero external indices we decompose the difference (5.35a) according to (E.1) into graphs either with composite propagators (5.15a) bounded by (5.26) or with composite propagators (5.15c)/(5.15d) bounded by (5.28). In such a graph there are—apart from usual propagators with bound (5.19)/(5.20)—two propagators with $a^1+a^2=1$ in (5.22) and a composite propagator with bound (5.26), or one propagator with $a^1+a^2=1$ in (5.22) and one composite propagator with bound (5.28). In both cases we get a total factor $\frac{\|\frac{m^1+1}{m^2}\frac{n^1+1}{n^2}\|^2_\infty}{(\theta\Lambda^2)^2}$ compared with a general planar two-point graph (5.37). The detailed discussion of the subgraphs is similar as under 1.

2. Finally, we have to discuss graphs γ according to Definition 12.2. We first consider the case that γ has constant index on each trajectory. It is then clear from the proof of (5.37) that (in particular) at vanishing indices the graph γ is relevant, which is expressed by (5.34b). Next, the difference (5.34c) of graphs can as in (5.17) be written as a sum of graphs with one composite propagator (5.15a), the bound of which is given by (5.26). After the treatment of subgraphs as described under 1, the integrand of each term in the linear combination is marginal. According to Definition 12.2 we have to integrate these terms from Λ_R up to Λ which according to the general procedure of Definition/Lemma 7 leads to (5.34c). Finally, according to (E.3) and (E.4), the linear combination constituting the lhs of (5.34a) results in a linear combination of graphs with either one propagator (5.15b) with bound (5.27), or with two propagators (5.15a) with bound (5.26). A similar discussion as under 1 then leads to (5.34a).

   The second case is when one index component jumps once on a trajectory and back. According to the proof of (5.36) the integrand of γ at vanishing external indices is marginal. We regard it nevertheless as relevant using the inequality $1 \leq (\theta\Lambda^2)(\theta\Lambda_R^2)^{-1}$, where $(\theta\Lambda_R^2)^{-1}$ is some number kept constant in our renormalisation procedure. We now obtain (5.34b). Similarly, the integrand relative to the difference (5.34c) would be irrelevant, but is considered as marginal via the same trick. Finally, the linear combination constituting the lhs of (5.34a) is according to (E.3) and (E.6)–(E.9) a linear combination of graphs having either two propagators with $a^1+a^2=1$ in (5.22) and a composite propagator with bound (5.26), or one propagator with $a^1+a^2=1$ in (5.22) and one composite propagator (5.15c)/(5.15d) with bound (5.28). The discussion as before would lead to an increased large-Λ degrees P_3^{4V-N} instead of P_2^{4V-N} in (5.34a), which can be reduced to P_2^{4V-N} according to (5.31).

This finishes the proof of Proposition 13. □

It is now important to realise that the estimations (5.33)–(5.37) of Proposition 13 do not make any reference to the initial scale Λ_0. Therefore, the estimations (5.33)–(5.37), which give finite bounds for the interaction coefficients with finite external indices, also hold in the limit $\Lambda_0 \to \infty$ [KKS92]. This is the renormalisation of the duality-covariant noncommutative ϕ^4-model.

In numerical computations the limit $\Lambda_0 \to \infty$ is difficult to realise. Taking instead a large but finite Λ_0, it is then important to estimate the error and the rate of convergence as Λ_0 approaches ∞. This type of estimations is the subject of the next section.

I finish this section with a remark on the freedom of normalisation conditions. One of the most important steps in the proof is the integration procedure for the matrix Polchinski equation given in Definition 12. For presentational reasons I have chosen the smallest possible set of graphs to be integrated from Λ_R upward. This can easily be generalised. We could admit in (5.7) any planar 1PI four-point graphs for which the incoming index of each trajectory is equal to the outgoing index on that trajectory, but with arbitrary jump along the trajectory. There is no change of the estimation (5.33a), because (according to Proposition 13.4) $\Lambda' \frac{\partial}{\partial \Lambda'} A^{(V,V^e,\widehat{1,0,0)\gamma}}_{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{n^1 \\ n^2}\substack{k^1 \\ k^2};\substack{k^1 \\ k^2}\substack{l^1 \\ l^2};\substack{l^1 \\ l^2}\substack{m^1 \\ m^2}}[\Lambda']$ is already irrelevant for these graphs, so is the difference in braces in (5.7). Moreover, integrating such an irrelevant graph according to the last line of (5.7) from Λ_R upward we obtain a bound $\frac{1}{(\Lambda_R^2\theta)} P^{2V-2}[\ln \frac{\Lambda}{\Lambda_R}]$, which agrees with (5.33b), because $\Lambda_R^2\theta$ is finite. Similarly, we can relax the conditions on the jump along the trajectory in (5.8)–(5.10). We would then define the $\rho_a[\Lambda, \Lambda_0, \rho^0]$-functions in (5.12) for that enlarged set of graphs γ.

In a second generalisation we could admit one-particle reducible graphs in 1–3 of Definition 12 and even non-planar graphs with the same condition on the external indices as in 1–3 of Definition 12. Since there is no difference in the power-counting behaviour between non-planar graphs and planar graphs with large jump, the discussion is as before. However, the convergence theorem developed in the next section cannot be adapted in an easy way to normalisation conditions involving non-planar graphs.

In summary, the proposed generalisations constitute different normalisation conditions of the same duality-covariant ϕ^4-model. Passing from one normalisation to another one is a finite re-normalisation. The invariant characterisation of our model is its definition via four independent normalisation conditions for the ρ-functions so that at large scales the effective action approaches (5.1).

6 The convergence theorem

In this section I prove the *convergence* of the coefficients of the effective action in the limit $\Lambda_0 \to \infty$, relative to the integration procedure given in Definition 12. This is a stronger result than the power-counting estimation of Proposition 13, which e.g. would be compatible with bounded oscillations. Additionally, I identify the rate of convergence of the interaction coefficients.

6.1 The Λ_0-dependence of the effective action

We have to control the Λ_0-dependence which enters the effective action via the integration procedure of Definition 12. There is an explicit dependence via the integration domain of irrelevant graphs and an implicit dependence through the normalisation (5.14), which requires a carefully adapted Λ_0-dependence of ρ_a^0. For fixed $\Lambda = \Lambda_R$ but variable Λ_0 we consider the identity

$$\begin{aligned} L[\phi,\Lambda_R,\Lambda_0',\rho^0[\Lambda_0']] - L[\phi,\Lambda_R,\Lambda_0'',\rho^0[\Lambda_0'']] &\equiv \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0}\Big(\Lambda_0 \frac{d}{d\Lambda_0} L[\phi,\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]\Big) \\ &= \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0}\Big(\Lambda_0 \frac{\partial L[\phi,\Lambda_R,\Lambda_0,\rho^0]}{\partial\Lambda_0} + \sum_{a=1}^{4} \Lambda_0 \frac{d\rho_a^0}{d\Lambda_0}\frac{\partial L[\phi,\Lambda_R,\Lambda_0,\rho^0]}{\partial\rho_a^0}\Big) . \end{aligned} \tag{6.1}$$

The model is defined by fixing the boundary condition for ρ_b at Λ_R, i.e. by keeping $\rho_b[\Lambda_R,\Lambda_0,\rho^0] = \text{constant}$:

$$0 = d\rho_b[\Lambda_R,\Lambda_0,\rho^0] = \frac{\partial\rho_b[\Lambda_R,\Lambda_0,\rho^0]}{\partial\Lambda_0} d\Lambda_0 + \sum_{a=1}^{4} \frac{\partial\rho_b[\Lambda_R,\Lambda_0,\rho^0]}{\partial\rho_a^0}\frac{d\rho_a^0}{d\Lambda_0} d\Lambda_0 . \tag{6.2}$$

Assuming that we can invert the matrix $\frac{\partial\rho_b[\Lambda_R,\Lambda_0,\rho^0]}{\partial\rho_a^0}$, which is possible in perturbation theory, we get

$$\frac{d\rho_a^0}{d\Lambda_0} = -\sum_{b=1}^{4} \frac{\partial\rho_a^0}{\partial\rho_b[\Lambda_R,\Lambda_0,\rho^0]}\frac{\partial\rho_b[\Lambda_R,\Lambda_0,\rho^0]}{\partial\Lambda_0} . \tag{6.3}$$

Inserting (6.3) into (6.1) we obtain

$$L[\phi,\Lambda_R,\Lambda_0',\rho^0[\Lambda_0']] - L[\phi,\Lambda_R,\Lambda_0'',\rho^0[\Lambda_0'']] = \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0} R[\phi,\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]] , \tag{6.4}$$

with

$$\begin{aligned} R[\phi,\Lambda,\Lambda_0,\rho^0] := {}& \Lambda_0 \frac{\partial L[\phi,\Lambda,\Lambda_0,\rho^0]}{\partial\Lambda_0} \\ & - \sum_{a,b=1}^{4} \frac{\partial L[\phi,\Lambda,\Lambda_0,\rho^0]}{\partial\rho_a^0}\frac{\partial\rho_a^0}{\partial\rho_b[\Lambda,\Lambda_0,\rho^0]}\Lambda_0\frac{\partial\rho_b[\Lambda,\Lambda_0,\rho^0]}{\partial\Lambda_0} . \end{aligned} \tag{6.5}$$

Following [Pol84] we differentiate (6.5) with respect to Λ:

$$\Lambda\frac{\partial R}{\partial\Lambda} = \Lambda_0\frac{\partial}{\partial\Lambda_0}\Big(\Lambda\frac{\partial L}{\partial\Lambda}\Big) - \sum_{a,b=1}^{4}\frac{\partial}{\partial\rho_a^0}\Big(\Lambda\frac{\partial L}{\partial\Lambda}\Big)\frac{\partial\rho_a^0}{\partial\rho_b}\Lambda_0\frac{\partial\rho_b}{\partial\Lambda_0}$$
$$+\sum_{a,b,c,d=1}^{4}\frac{\partial L}{\partial\rho_a^0}\frac{\partial\rho_a^0}{\partial\rho_b}\frac{\partial}{\partial\rho_c^0}\Big(\Lambda\frac{\partial\rho_b}{\partial\Lambda}\Big)\frac{\partial\rho_c^0}{\partial\rho_d}\Lambda_0\frac{\partial\rho_d}{\partial\Lambda_0} - \sum_{a,b=1}^{4}\frac{\partial L}{\partial\rho_a^0}\frac{\partial\rho_a^0}{\partial\rho_b}\Lambda_0\frac{\partial}{\partial\Lambda_0}\Big(\Lambda\frac{\partial\rho_b}{\partial\Lambda}\Big)\ . \tag{6.6}$$

I have omitted the dependencies for simplicity and made use of the fact that the derivatives with respect to $\Lambda, \Lambda_0, \rho^0$ commute. Using (4.15), with $\mathcal{V}_4 = (2\pi\theta)^2$, we compute the terms on the rhs of (6.6):

$$\begin{aligned}
&\Lambda_0\frac{\partial}{\partial\Lambda_0}\Big(\Lambda\frac{\partial L[\phi,\Lambda,\Lambda_0,\rho^0]}{\partial\Lambda}\Big)\\
&\quad= \sum_{m,n,k,l}\frac{1}{2}\Lambda\frac{\partial\Delta^K_{nm;lk}(\Lambda)}{\partial\Lambda}\bigg(2\frac{\partial L[\phi,\Lambda,\Lambda_0,\rho^0]}{\partial\phi_{mn}}\frac{\partial}{\partial\phi_{kl}}\Big(\Lambda_0\frac{\partial}{\partial\Lambda_0}L[\phi,\Lambda,\Lambda_0,\rho^0]\Big)\\
&\qquad\qquad - \frac{1}{(2\pi\theta)^2}\Big[\frac{\partial^2}{\partial\phi_{mn}\,\partial\phi_{kl}}\Big(\Lambda_0\frac{\partial}{\partial\Lambda_0}L[\phi,\Lambda,\Lambda_0,\rho^0]\Big)\Big]_\phi\bigg)\\
&\quad\equiv M\Big[L,\Lambda_0\frac{\partial L}{\partial\Lambda_0}\Big]\ .
\end{aligned}\tag{6.7}$$

Similarly, we have

$$\frac{\partial}{\partial\rho_a^0}\Big(\Lambda\frac{\partial L[\phi,\Lambda,\Lambda_0,\rho^0]}{\partial\Lambda}\Big) = M\Big[L,\frac{\partial L}{\partial\rho_a^0}\Big]\ . \tag{6.8}$$

For (6.6) we also need the function $\Lambda_0\frac{\partial}{\partial\Lambda_0}\big(\Lambda\frac{\partial\rho_b}{\partial\Lambda}\big)$, which is obtained from (6.7) by first expanding L on the lhs according to (4.51) and by further choosing the indices at the A-coefficients according to (5.12). Applying these operations to the rhs of (6.7), we obtain for $U\mapsto\Lambda_0\frac{\partial L}{\partial\Lambda_0}$ or $U\mapsto\frac{\partial L}{\partial\rho_a^0}$ the expansions

$$M[L,U] = \sum_{N=2}^{\infty}\sum_{m_i,n_i\in\mathbb{N}^2}\frac{1}{N!}M_{m_1n_1;\dots;m_Nn_N}[L,U]\phi_{m_1n_1}\cdots\phi_{m_Nn_N} \tag{6.9}$$

and the projections

$$M_1[L,U] := \sum_{\gamma\text{ as in Def. 12.2}} M^{\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[L,U]\ , \tag{6.10a}$$

$$M_2[L,U] := \sum_{\gamma\text{ as in Def. 12.2}} \Big(M^{\gamma}_{\substack{1\,0\\0\,0};\substack{0\,1\\0\,0}}[L,U] - M^{\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[L,U]\Big)\ , \tag{6.10b}$$

$$M_3[M,U] := \sum_{\gamma\text{ as in Def. 12.3}} \Big(-M^{\gamma}_{\substack{1\,1\\0\,0};\substack{0\,0\\0\,0}}[L,U]\Big)\ , \tag{6.10c}$$

$$M_4[L,U] := \sum_{\gamma\text{ as in Def. 12.1}} M^{\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[L,U]\ . \tag{6.10d}$$

Since the graphs γ in (6.10) are one-particle irreducible, only the third line of (6.7) can contribute[26] to M_a. Using (6.7), (6.8) and (6.10) as well as the linearity of $M[L, U]$ in the second argument we can rewrite (6.6) as

$$\Lambda \frac{\partial R}{\partial \Lambda} = M[L, R] - \sum_{a=1}^{4} \frac{\partial L}{\partial \rho_a} M_a[L, R] \;, \tag{6.11}$$

where the function

$$\frac{\partial L}{\partial \rho_a}[\Lambda, \Lambda_0, \rho^0] := \sum_{b=1}^{4} \frac{\partial L[\Lambda, \Lambda_0, \rho^0]}{\partial \rho_b^0} \frac{\partial \rho_b^0}{\partial \rho_a[\Lambda, \Lambda_0, \rho^0]} \tag{6.12}$$

scales according to

$$\Lambda \frac{\partial}{\partial \Lambda}\Big(\frac{\partial L}{\partial \rho_a}\Big) = M\Big[L, \frac{\partial L}{\partial \rho_a}\Big] - \sum_{b=1}^{4} \frac{\partial L}{\partial \rho_b} M_b\Big[L, \frac{\partial L}{\partial \rho_a}\Big] \;, \tag{6.13}$$

as a similar calculation shows.

Next, we also expand (6.5) and (6.12) as power series in the coupling constant:

$$\frac{\partial L}{\partial \rho_a}[\phi, \Lambda, \Lambda_0, \rho^0] = \sum_{V=0}^{\infty} \lambda^V \sum_{N=2}^{2V+4} \frac{(2\pi\theta)^{\frac{N}{2}-2}}{N!} \sum_{m_i, n_i} H^{a(V)}_{m_1 n_1; \dots; m_N n_N}[\Lambda, \Lambda_0, \rho^0] \phi_{m_1 n_1} \cdots \phi_{m_N n_N} \;, \tag{6.14}$$

$$R[\phi, \Lambda, \Lambda_0, \rho^0] = \sum_{V=1}^{\infty} \lambda^V \sum_{N=2}^{2V+2} \frac{(2\pi\theta)^{\frac{N}{2}-2}}{N!} \sum_{m_i, n_i} R^{(V)}_{m_1 n_1; \dots; m_N n_N}[\Lambda, \Lambda_0, \rho^0] \phi_{m_1 n_1} \cdots \phi_{m_N n_N} \;. \tag{6.15}$$

The differential equations (6.13) and (6.11) can now with (6.9) and (6.10) be written as

$$\begin{aligned}
&\Lambda \frac{\partial}{\partial \Lambda} H^{a(V)}_{m_1 n_1; \dots; m_N n_N}[\Lambda, \Lambda_0, \rho^0] \\
&= \Big\{ \sum_{N_1=2}^{N} \sum_{V_1=1}^{V} \sum_{m,n,k,l} Q_{nm;lk}(\Lambda) A^{(V_1)}_{m_1 n_1; \dots; m_{N_1-1} n_{N_1-1}; mn}[\Lambda] H^{a(V-V_1)}_{m_{N_1} n_{N_1}; \dots; m_N n_N; kl}[\Lambda] \\
&\qquad + \Big(\binom{N}{N_1 - 1} - 1\Big) \text{ permutations } \Big\} - \sum_{m,n,k,l} \frac{1}{2} Q_{nm;lk}(\Lambda) H^{a(V)}_{m_1 n_1; \dots; m_N n_N; mn; kl}[\Lambda] \\
&- \sum_{V_1=0}^{V} H^{1(V-V_1)}_{m_1 n_1; \dots; m_N n_N}[\Lambda] \Big\{ -\frac{1}{2} \sum_{m,n,k,l} Q_{nm;lk}(\Lambda) H^{a(V_1)}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}; mn; kl}[\Lambda] \Big\}_{\text{[Def. 12.2]}} \\
&- \sum_{V_1=0}^{V} H^{2(V-V_1)}_{m_1 n_1; \dots; m_N n_N}[\Lambda] \Big\{ -\frac{1}{2} \sum_{m,n,k,l} Q_{nm;lk}(\Lambda) \Big(H^{a(V_1)}_{\substack{1\,0\\0\,0};\substack{0\,1\\0\,0}; mn; kl}[\Lambda] - H^{a(V_1)}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}; mn; kl}[\Lambda] \Big) \Big\}_{\text{[Def. 12.2]}}
\end{aligned}$$

[26] If one-particle reducible graphs are included in the normalisation conditions as discussed at the end of Section 5.4, also the second line of (6.7) must be taken into account.

$$
\begin{aligned}
&+\sum_{V_1=0}^{V} H^{3(V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\Big\{-\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) H^{a(V_1)}_{{}^{1\,1}_{0\,0};{}^{0\,0}_{0\,0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 12.3}]}\\
&-\sum_{V_1=1}^{V} H^{4(V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\Big\{-\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) H^{a(V_1)}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 12.1}]}\,, \qquad (6.16)
\end{aligned}
$$

$$
\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda} R^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho^0]\\
&=\Big\{\sum_{N_1=2}^{N}\sum_{V_1=1}^{V-1}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) A^{(V_1)}_{m_1n_1;\dots;m_{N_1-1}n_{N_1-1};mn}[\Lambda] R^{(V-V_1)}_{m_{N_1}n_{N_1};\dots;m_Nn_N;kl}[\Lambda]\\
&\quad+\Big(\binom{N}{N_1-1}-1\Big)\text{ permutations }\Big\}-\sum_{m,n,k,l}\frac{1}{2} Q_{nm;lk}(\Lambda) R^{(V)}_{m_1n_1;\dots;m_Nn_N;mn;kl}[\Lambda]\\
&-\sum_{V_1=1}^{V} H^{1(V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\Big\{-\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) R^{(V_1)}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 12.2}]}\\
&-\sum_{V_1=1}^{V} H^{2(V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\Big\{-\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda)\Big(R^{(V_1)}_{{}^{1\,0}_{0\,0};{}^{0\,1}_{0\,0};mn;kl}[\Lambda]-R^{(V_1)}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};mn;kl}[\Lambda]\Big)\Big\}_{[\text{Def. 12.2}]}\\
&+\sum_{V_1=1}^{V} H^{3(V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\Big\{-\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) R^{(V_1)}_{{}^{1\,1}_{0\,0};{}^{0\,0}_{0\,0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 12.3}]}\\
&-\sum_{V_1=1}^{V} H^{4(V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\Big\{-\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) R^{(V_1)}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 12.1}]}\,. \qquad (6.17)
\end{aligned}
$$

I have used several times symmetry properties of the expansion coefficients and of the propagator and the fact that to the 1PI projections (6.10) only the last line of (6.9) can contribute. By $\{\dots\}_{[\text{Def. 12.?}]}$ I understand the restriction to H-graphs and R-graphs, respectively, which satisfy the index criteria on the trajectories as given in Definition 12. The H-graphs will be constructed later in Section 6.2. The R-graphs are in their structure identical to the previously considered graphs for the A-functions, but have a different meaning. See Section 6.4.

6.2 Initial data and graphs for the auxiliary functions

Next, I derive the bounds for the H-functions. Inserting (5.1) into the definition (6.12) and expanding it according to (6.14) we obtain immediately the initial condition at $\Lambda=\Lambda_0$:

$$
\begin{aligned}
&H^{1(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\rho^0]=\delta_{N2}\delta^{V0}\delta_{n_1m_2}\delta_{n_2m_1}\,, \qquad (6.18)\\
&H^{2(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\rho^0]=\delta_{N2}\delta^{V0}(m_1^1+n_1^1+m_1^2+n_1^2)\delta_{n_1m_2}\delta_{n_2m_1}\,, \qquad (6.19)\\
&H^{3(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\rho^0] \qquad (6.20)\\
&\qquad=-\delta_{N2}\delta^{V0}\Big(\Big(\sqrt{n_1^1m_1^1}\,\delta_{n_1^1,m_2^1+1}\delta_{n_2^1+1,m_1^1}+\sqrt{n_2^1m_2^1}\,\delta_{n_1^1,m_2^1-1}\delta_{n_2^1-1,m_1^1}\Big)\delta_{n_1^2,m_2^2}\delta_{n_2^2,m_1^2}\\
&\qquad+\Big(\sqrt{n_1^2m_1^2}\,\delta_{n_1^2,m_2^2+1}\delta_{n_2^2+1,m_1^2}+\sqrt{n_2^2m_2^2}\,\delta_{n_1^2,m_2^2-1}\delta_{n_2^2-1,m_1^2}\Big)\delta_{n_1^1,m_2^1}\delta_{n_2^1,m_1^1}\Big)\,, \qquad (6.21)
\end{aligned}
$$

$$H^{4(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\rho^0] = \delta_{N4}\delta^{V0}\Big(\frac{1}{6}\delta_{n_1m_2}\delta_{n_2m_3}\delta_{n_3m_4}\delta_{n_4m_1} + \text{ 5 permutations}\Big)\;. \quad (6.22)$$

I first compute $H^{a(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda]$ for $a \in \{1,2,3\}$. Since there is no 6-point function at order 0 in V, the differential equation (6.16) reduces to

$$\begin{aligned}
&\Lambda\frac{\partial H^{a(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda]}{\partial\Lambda} \\
&= \sum_{b=1}^{3}\sum_{m,n,k,l,m',n',k',l'} C_b^{m'n';k'l'} H^{b(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda]\Big(Q_{nm;lk}(\Lambda)H^{a(0)}_{m'n';k'l';mn;kl}[\Lambda]\Big)\;, \qquad (6.23)
\end{aligned}$$

for certain matrices $C_b^{m'n';k'l'}$. The solution is due to (5.5) given by

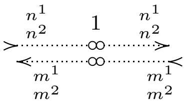
$$\begin{aligned}
&H^{a(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda] = H^{a(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda_0] \\
&+\sum_{b=1}^{3}\sum_{m,n,k,l,m',n',k',l'} C_b^{m'n';k'l'} H^{b(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda_0]\big(\Delta^K_{nm;lk}(\Lambda)-\Delta^K_{nm;lk}(\Lambda_0)\big)H^{a(0)}_{m'n';k'l';mn;kl}[\Lambda_0] \\
&+\sum_{b=1}^{3}\sum_{m,n,k,l,m',n',k',l'} C_b^{m'n';k'l'} H^{b(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda_0] \\
&\quad\times\Big(\big(\Delta^K_{nm;lk}(\Lambda)-\Delta^K_{nm;lk}(\Lambda_0)\big)\sum_{b'=1}^{3}\sum_{m'',n'',k'',l'',m''',n''',k''',l'''} C_{b'}^{m'''n''';k'''l'''} H^{b'(0)}_{m'n';k'l';mn;kl}[\Lambda_0]\Big) \\
&\qquad\times\Big(\big(\Delta^K_{n''m'';l''k''}(\Lambda)-\Delta^K_{n''m'';l''k''}(\Lambda_0)\big)H^{a(0)}_{m'''n''';k'''l''';m''n'';k''l''}[\Lambda_0]\Big) \\
&\quad+\dots\;. \qquad (6.24)
\end{aligned}$$

With the initial conditions (6.18)–(6.21) we get

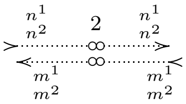
$$H^{a(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda,\Lambda_0,\rho^0] \equiv 0 \qquad \text{for } a\in\{1,2,3\}\;. \qquad (6.25)$$

Inserting (6.25) into (6.16) we see that $H^{a(0)}_{m_1n_1;m_2n_2}$ for $a \in \{1,2,3\}$ and $H^{4(0)}_{m_1n_1;\dots;m_4n_4}$ are constant, which means that the relations (6.18)–(6.22) hold actually at any value Λ and not only at $\Lambda = \Lambda_0$.

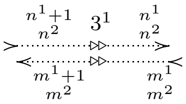
We need a graphical notation for the H-functions. I represent the base functions (6.18)–(6.22), valid for any Λ, as follows:

$$H^{1(0)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda] =$$ 

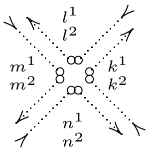
(6.26)

$$H^{2(0)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda] =$$ 

(6.27)

$$H^{3(0)}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda] =$$ 

(6.28)

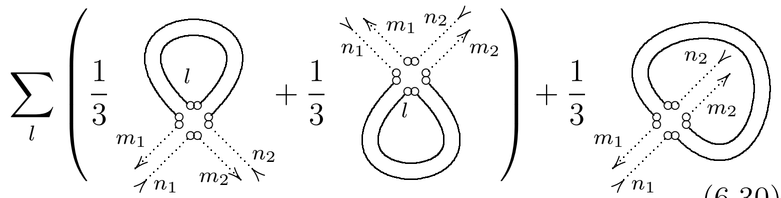
$$H^{4(0)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{k^1\\k^2};\substack{k^1\\k^2}\substack{l^1\\l^2};\substack{l^1\\l^2}\substack{m^1\\m^2}}[\Lambda] = \frac{1}{6}$$ 

$+$ 5 permutations . (6.29)

The special vertices stand for some sort of hole into which we can insert planar two- or four-point functions at vanishing external indices. However, the graph remains connected at these holes, in particular, there is index conservation at the hole ∞ and a jump by ${}^{1}_{0}$ or ${}^{0}_{1}$ at the hole $\triangleright\!\triangleright$.

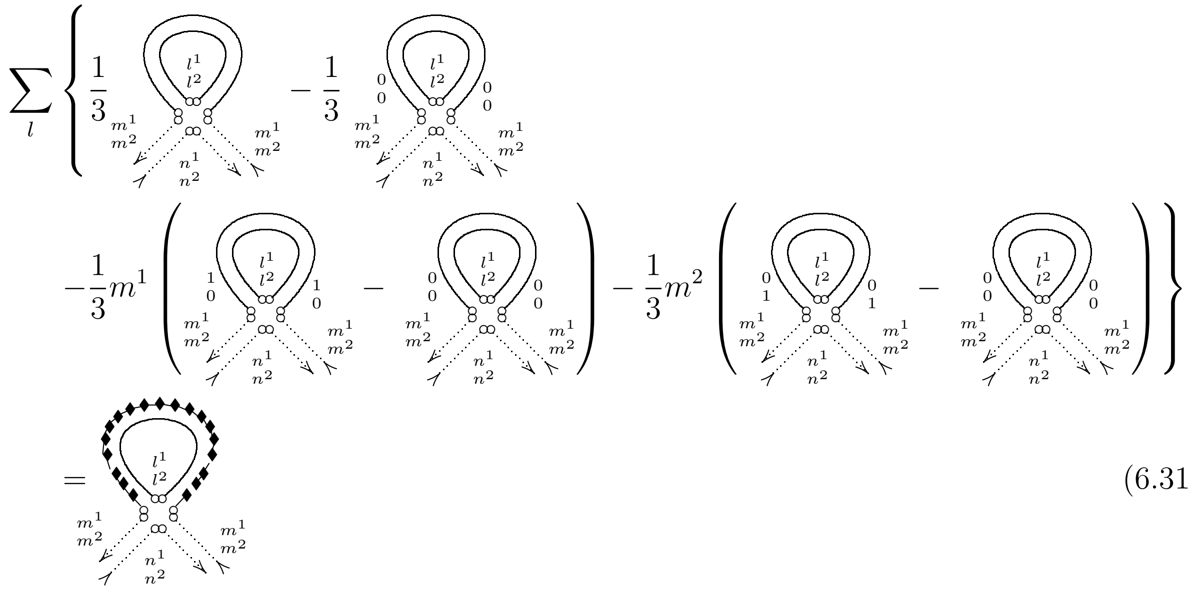
By repeated contraction with A-graphs and self-contractions we build out of (6.26)–(6.29) more complicated graphs with holes. For instance, there is a planar and a non-planar self-contraction of (6.29):

$$\sum_{m,n,k,l} Q_{mn;kl} H^{4(0)}_{m_1n_1;m_2n_2;mn;kl} =$$



(6.30)

These contractions correspond (with a factor $-\frac{1}{2}$) to last term in the third line of (6.16). We also have to subtract (again up to the factor $-\frac{1}{2}$) the fourth to last lines of (6.16). For instance, the fourth line amounts to insert the *planar graphs* of (6.30) with $m_1 = n_1 = m_2 = n_2 = {}^{0}_{0}$ into (6.26). The total contribution corresponding to the first graph in (6.30), with $m_1 = n_2 = {}^{m^1}_{m^2}$ and $n_1 = m_2 = {}^{n^1}_{n^2}$, reads



(6.31)

The second graph in the first line of (6.31) corresponds to the fourth line of (6.16). The second line of (6.31) represents the fifth line of (6.16), undoing the symmetry properties of the upper and lower component used in (6.16). The difference of graphs corresponding to the ${}^{n^1}_{n^2}$ component vanishes, because the value of the graph is independent of ${}^{n^1}_{n^2}$. There is no planar contribution from the last two lines of (6.16). In total, we get the projection (5.27) to the irrelevant part of the graph. The same procedure leads to the irrelevant part of the second graph in (6.30).

With these considerations, the differential equation (6.16) takes for $N = 2$ and $a = 4$

the form

$$\Lambda\frac{\partial}{\partial\Lambda}H^{4(0)}_{m_1n_1;m_2n_2}[\Lambda] = -\frac{1}{6}\delta_{n_1m_2}\delta_{n_2m_1}\Big(\sum_{l\in\mathbb{N}^2}\mathcal{Q}^{(1)}_{n_1l;ln_1}(\Lambda)+\sum_{l\in\mathbb{N}^2}\mathcal{Q}^{(1)}_{n_2l;ln_2}(\Lambda)\Big)$$
$$-\frac{1}{6}Q_{m_1n_1;m_2n_2}(\Lambda)\;. \tag{6.32}$$

The first line comes from the planar graphs in (6.30) and the subtraction terms according to (6.31), whereas the second line of (6.32) is obtained from the last (non-planar) graph in (6.30). Using the initial condition (6.22) at $\Lambda=\Lambda_0$, the bounds (5.19) and (5.27) combined with the volume factor $(C_2\theta\Lambda^2)^2$ for the l-summation we get

$$|H^{4(0)}_{m_1n_1;m_2n_2}[\Lambda]| \leq \frac{C\big(\|m_1\|^2_\infty+\|n_1\|^2_\infty\big)}{\Omega\theta\Lambda^2}\,\delta_{n_1m_2}\delta_{n_2m_1}+\frac{C_0}{\Omega\theta\Lambda^2}\,\delta_{n_1m_2}\delta_{n_2m_1}\;. \tag{6.33}$$

It is extremely important here that the irrelevant projection $\mathcal{Q}^{(1)}_{nl;ln}$ and not the propagator $Q_{nl;ln}$ itself appears in the first line of (6.32).

6.3 The power-counting behaviour of the auxiliary functions

The example suggests that similar cancellations of relevant and marginal parts appear in general, too. Thus, we expect all H-functions to be irrelevant. This is indeed the case:

Proposition 14 *Let γ be a ribbon graph with holes having N external legs, V vertices, V^e external vertices and segmentation index ι, which is drawn on a genus-g Riemann surface with B boundary components. Then, the contribution $H^{a(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}$ of γ to the expansion coefficient of the auxiliary function of a duality-covariant ϕ^4-theory on $\mathbb{R}^4_\theta$ in the matrix base is bounded as follows:*

1. *For γ according to Definition 12.1 we have*

$$\Big|\sum_{\gamma\text{ as in Def. 12.1}} H^{a(V,V^e,1,0,0)\gamma}_{{m^1\atop m^2}{n^1\atop n^2};{n^1\atop n^2}{k^1\atop k^2};{k^1\atop k^2}{l^1\atop l^2};{l^1\atop l^2}{m^1\atop m^2}}[\Lambda,\Lambda_0,\rho_0]\Big|$$
$$\leq \big(\theta\Lambda^2\big)^{-\delta^{1a}}P^{4V-2+2\delta^{a4}}_{1-\delta^{V0}}\left[\frac{{m^1\atop m^2}{n^1\atop n^2};{n^1\atop n^2}{k^1\atop k^2};{k^1\atop k^2}{l^1\atop l^2};{l^1\atop l^2}{m^1\atop m^2}}{\theta\Lambda^2}\right]\Big(\frac{1}{\Omega}\Big)^{3V-1+\delta^{a4}-V^e}P^{2V-1+\delta^{a4}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;, \tag{6.34}$$

where all vertices on the trajectories contribute to V^e.

2. *For γ according to Definition 12.2 we have*

$$\Big|\sum_{\gamma\text{ as in Def. 12.2}} H^{a(V,V^e,1,0,0)\gamma}_{{m^1\atop m^2}{n^1\atop n^2};{n^1\atop n^2}{m^1\atop m^2}}[\Lambda,\Lambda_0,\rho_0]\Big|$$
$$\leq \big(\theta\Lambda^2\big)^{1-\delta^{1a}}P^{4V+2\delta^{a4}}_{2-\delta_{V0}(2\delta^{a1}+\delta^{a2})}\left[\frac{{m^1\atop m^2}{n^1\atop n^2};{n^1\atop n^2}{m^1\atop m^2}}{\theta\Lambda^2}\right]\Big(\frac{1}{\Omega}\Big)^{3V+\delta^{a4}-V^e}P^{2V+\delta^{a4}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;, \tag{6.35}$$

where all vertices on the trajectories contribute to V^e.

3. *For γ according to Definition 12.3 we have*

$$\Big|\sum_{\gamma \text{ as in Def. 12.3}} H^{a(V,V^e,1,0,0)}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda,\Lambda_0,\rho_0]\Big|$$
$$\leq \big(\theta\Lambda^2\big)^{1-\delta^{1a}} P^{4V+2\delta^{a4}}_{2-\delta_{V0}}\left[\frac{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}{\theta\Lambda^2}\right]\Big(\frac{1}{\Omega}\Big)^{3V+\delta^{a4}-V^e} P^{2V+\delta^{a4}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big], \tag{6.36}$$

where all vertices on the trajectories contribute to V^e.

4. *If γ is a subgraph of an 1PI planar graph with a selected set T of trajectories on one distinguished boundary component and a second set T' of summed trajectories on that boundary component, we have*

$$\sum_{\mathcal{E}^s}\sum_{\mathcal{E}^{t'}}\big|H^{a(V,V^e,B,0,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho_0]\big|$$
$$\leq \big(\theta\Lambda^2\big)^{(2-\delta^{1a}-\frac{N}{2})+2(1-B)} P^{4V+2+2\delta^{a4}-N}_{\big(2t'+\sum_{\overrightarrow{n_j\circ[n_j]}\in T}\min(2,\frac{1}{2}\langle\overrightarrow{n_j\circ[n_j]}\rangle)\big)}\left[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\right]$$
$$\times\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}+\delta^{a4}+B-V^e-\iota+s+t'} P^{2V+1+\delta^{a4}-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]. \tag{6.37}$$

The number of summations is now restricted by $s+t'\leq V^e{+}\iota$.

5. *If γ is a non-planar graph or a graph with $N>4$ external legs, we have*

$$\sum_{\mathcal{E}^s}\big|H^{a(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho_0]\big|$$
$$\leq \big(\theta\Lambda^2\big)^{(2-\delta^{a1}-\frac{N}{2})+2(1-B-2g)} P^{4V+2+2\delta^{a4}-N}_{0}\left[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\right]$$
$$\times\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}+\delta^{a4}+B+2g-V^e-\iota+s} P^{2V+1+\delta^{a4}-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]. \tag{6.38}$$

The number of summations is now restricted by $s\leq V^e{+}\iota$.

Proof. The Proposition will be proven by induction upward in the number V of vertices and for given V downward in the number N of external legs.

5. Taking (5.31) into account, the estimations (6.34)–(6.37) are further bound by (6.38). In particular, the inequality (6.38) correctly reproduces the bounds for $V=0$ derived in Section 6.2. By comparison with (5.37), the estimation (6.38) follows immediately for the *H-linear* parts on the rhs of (6.16) which contribute to the integrand of $H^{a(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$. Since planar two- and four-point functions are preliminarily excluded, the Λ-integration (from Λ_0 down to Λ) confirms (6.38) for those contributions which arise from H-linear terms on the rhs of (6.16) that are non-planar or have $N>4$ external legs.

   We now consider in the H-bilinear part on the rhs of (6.16) the contributions of non-planar graphs or graphs with $N>4$ external legs. We start with the fourth line in

(6.16), with the first term being a non-planar H-function which (apart from the number of vertices and the hole label a) has the same topological data as the total H-graph to estimate. From the induction hypothesis it is clear that the term in braces $\{\ \}$ is bounded by the planar unsummed version $(B_1 = 1, g_1 = 0, \iota_1 = 0, s_1 = 0)$ of (6.37), with $N_1 = 2$ and $T = T' = \emptyset$, and with a reduction of the degree of the polynomial in $\ln \frac{\Lambda}{\Lambda_R}$ by 1:

$$\begin{aligned}
&\Big|\Big\{ -\frac{1}{2} \sum_{m,n,k,l} Q_{nm;lk}(\Lambda) H^{a(V_1)}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};mn;kl}[\Lambda]\Big\}_{\text{Def. 12.2}}\Big| \\
&\qquad \leq \big(\theta\Lambda^2\big)^{(1-\delta^{a1})} \Big(\frac{1}{\Omega}\Big)^{3V_1+\delta^{a4}-V_1^e} P^{2V_1-1+\delta^{a4}}\Big[\ln \frac{\Lambda}{\Lambda_R}\Big]\,, \qquad (6.39a) \\
&\sum_{\mathcal{E}^s} \big|H^{1(V-V_1,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \\
&\quad \leq \big(\theta\Lambda^2\big)^{(1-\frac{N}{2})+2(1-B-2g)} P_0^{4(V-V_1)+2-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] \\
&\qquad \times \Big(\frac{1}{\Omega}\Big)^{3(V-V_1)-\frac{N}{2}+B+2g-V^e-\iota+s} P^{2(V-V_1)+1-\frac{N}{2}}\Big[\ln \frac{\Lambda}{\Lambda_R}\Big]\,. \qquad (6.39b)
\end{aligned}$$

We can ignore the term $P^a_b[\ ]$, see (5.30), in (6.39a) because the external indices of that part are zero. In the first step we exclude $a = 4$ so that the sum over V_1 in (6.16) starts due to (6.18)–(6.21) at $V_1 = 1$. For $V_1 = V$ there is a contribution to (6.39b) with $N = 2$ only, where (6.39a) can be regarded as known by induction. Since the factor $(\frac{1}{\Omega})^{-V_1^e}$ can safely be absorbed in the polynomial $P[\ln \frac{\Lambda}{\Lambda_R}]$, the product of (6.39a) and (6.39b) confirms the bound (6.38) for the integrand under consideration, preliminarily for $a \neq 4$. In the next step we repeat the argumentation for $a = 4$, where (6.39b), with $V_1 = 0$, is known from the first step.

Second, we consider the fifth line in (6.16). The difference of functions in braces $\{\ \}$ involves graphs with constant index along the trajectories. We have seen in Section 5.2 that such a difference can be written as a sum of graphs each having a composite propagator (5.26) at a trajectory. As such the $(\theta\Lambda^2)$-degree of the part in braces $\{\ \}$ is reduced[27] by 1 compared with planar analogues of (6.38) for $N = 2$. The difference of functions in braces $\{\ \}$ involves also graphs where the index along one of the trajectories jumps once by ${}^1_0$ or ${}^0_1$ and back. For these graphs we conclude from (5.22) (and the fact that the maximal index along the trajectory is 2) that the $(\theta\Lambda^2)$-degree of the part in braces $\{\ \}$ is also reduced by 1:

$$\begin{aligned}
&\Big|\Big\{ -\frac{1}{2} \sum_{m,n,k,l} Q_{nm;lk}(\Lambda)\Big(H^{a(V_1)}_{{}^{1\,0}_{0\,0};{}^{0\,1}_{0\,0};mn;kl}[\Lambda] - H^{a(V_1)}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};mn;kl}[\Lambda]\Big)\Big\}_{[\text{Def. 12.2}]}\Big| \\
&\qquad \leq \big(\theta\Lambda^2\big)^{(-\delta^{a1})} \Big(\frac{1}{\Omega}\Big)^{3V_1+\delta^{a4}(1-V_1^e)} P^{2V_1-1+\delta^{a4}}\Big[\ln \frac{\Lambda}{\Lambda_R}\Big]\,, \qquad (6.40a)
\end{aligned}$$

[27]The origin of the reduction is the term $P^a_b[\ ]$ introduced in (5.30), with $b = 1$ in presence of a composite propagator (5.26). The argument in the brackets of $P^a_b[\ ]$ is the ratio of the maximal external index to the reference scale $\theta\Lambda^2$. Since the maximal index along the trajectory is 1, we can globally estimate in this case $P^a_1[\ ]$ by a constant times $(\theta\Lambda^2)^{-1}$.

$$\begin{aligned}
&\sum_{\mathcal{E}^s} \big|H^{2(V-V_1,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \\
&\quad\leq \big(\theta\Lambda^2\big)^{(2-\frac{N}{2})+2(1-B-2g)} P_0^{4(V-V_1)+2-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] \\
&\qquad\times \Big(\frac{1}{\Omega}\Big)^{3(V-V_1)-\frac{N}{2}+1+B+2g-V^e-\iota+s} P^{2(V-V_1)+1-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,. \qquad (6.40b)
\end{aligned}$$

Again we have to exclude $a=4$ in the first step, which then confirms the bound (6.38) for the integrand under consideration. In the second step we repeat the argumentation for $a=4$.

Third, the discussion of the sixth line of (6.16) is completely similar, because there the index on each trajectory jumps once by ${}^{1}_{0}$ or ${}^{0}_{1}$. This leads again to a reduction by 1 of the $(\theta\Lambda^2)$-degree of the part in braces $\{\;\}$ compared with planar analogues of (6.38) for $N=2$.

Finally, the part in braces in the last line of (6.16) can be estimated by a planar $N=4$ version of (6.38), again with a reduction by 1 of the degree of $P[\ln\frac{\Lambda}{\Lambda_R}]$:

$$\begin{aligned}
&\Big|\Big\{-\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) H^{a(V_1)}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 12.1}]}\Big| \\
&\leq \big(\theta\Lambda^2\big)^{-\delta^{a1}}\Big(\frac{1}{\Omega}\Big)^{3V_1-1+\delta^{a4}-V_1^e} P^{2V_1-2+\delta^{a4}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,, \qquad (6.41a) \\
&\sum_{\mathcal{E}^s} \big|H^{4(V-V_1,B,g,V^e,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \\
&\leq \big(\theta\Lambda^2\big)^{(2-\frac{N}{2})+2(1-B-2g)} P_0^{4(V-V_1)+4-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] \\
&\qquad\times \Big(\frac{1}{\Omega}\Big)^{3(V-V_1)-\frac{N}{2}+1+B+2g-V^e-\iota+s} P^{2(V-V_1)+2-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,. \qquad (6.41b)
\end{aligned}$$

We confirm again the bound (6.38) for the integrand under consideration.

Since the total H-graph is by assumption non-planar or has $N>4$ external legs, the integrand (6.38) is irrelevant so that their integration from Λ_0 down to Λ (and use of the initial conditions (6.18)–(6.22)) yields the same bound (6.38) for the graph, too.

4. According to Section 6.2, the inequality (6.37) is correct for $V=0$. By comparison with (5.36), the estimation (6.37) follows immediately for the *H-linear* parts on the rhs of (6.16) which contribute to the integrand of $H^{a(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$. Excluding planar two- and four-point functions with constant index on the trajectory or with limited jump according to 1–3 of Definition 12, the Λ-integration confirms (6.37) for those contributions which arise from H-linear terms on the rhs of (6.16) that correspond to subgraphs of planar graphs (subject to the above restrictions). The proof of (6.37) for the H-bilinear terms in (6.16) is completely analogous to the non-planar case. We only have to replace (6.39b), (6.40b) and (6.41b) by the adapted version of (6.37). In particular, the distinguished trajectory with its subsets T, T' of indices comes exclusively from the (6.37)-analogues of (6.39b), (6.40b) and (6.41b) and not from the terms in braces in (6.16).

1. We first consider $a \neq 4$. Then, according to (6.18)–(6.21) we need $V \geq 1$ in order to have a non-vanishing contribution to (6.34). Since according to Definition 12.1 the index along each trajectory of the (planar) graph γ is constant, we have

$$H^{a(V,V^e,1,0,0)\gamma}_{m_1n_1;m_2n_2;m_3n_3;m_4n_4}[\Lambda] = \frac{1}{6}H^{a(V,V^e,1,0,0)}_{m_1m_2;m_2m_3;m_3m_4;m_4n_1}[\Lambda] + 5 \text{ permutations}\,. \tag{6.42}$$

Then, using (6.18)–(6.22) and the fact that γ is 1PI, the differential equation (6.16) reduces to

$$\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda}\Big(\sum_{\gamma \text{ as in Def. 12.1}} H^{a(V,V^e,1,0,0)\gamma}_{m_1n_1;m_2n_2;m_3n_3;m_4n_4}[\Lambda,\Lambda_0,\rho^0]\Big)_{a\neq 4} \\
&= \Big(-\frac{1}{12}\Big\{\sum_{m,n,k,l} Q_{nm;lk}(\Lambda)\Big(H^{a(V,V^e,B,0,\iota)}_{m_1m_2;m_2m_3;m_3m_4;m_4n_1;mn;kl}[\Lambda] - H^{a(V,V^e,B,0,\iota)}_{00;00;00;00;mn;kl}[\Lambda]\Big)\Big\}_{[\text{Def. } 12.1]} \\
&\qquad + 5 \text{ permutations}\Big) + \text{the } 4^{\text{th}} \text{ to last lines of (6.16) with } \sum_{V_1=0}^{V} \mapsto \sum_{V_1=1}^{V-1}\,.
\end{aligned} \tag{6.43}$$

Here, the term $H^{a(V,V^e,B,0,\iota)}_{00;00;00;00;mn;kl}[\Lambda]$ in the second line of (6.43) comes from the ($V_1{=}V$)-contribution of the last line in (6.16), together with (6.22). In the same way as in Section 5.2 we conclude that the second line of (6.43) can be written as a linear combination of graphs having a composite propagator (5.15a) on one of the trajectories. As such we have to replace the bound (5.20) relative to the contribution of an ordinary propagator by (5.26). For the total graph this amounts to multiply the corresponding estimation (6.37) of ordinary H-graphs with $N = 4$ by a factor $\frac{\max\|m_i\|}{\theta\Lambda^2}$, which yields the subscript 1 of the part $P_1^{4V-2+2\delta^{a4}}[\;]$ of the integrand (6.34), for the time being restricted to the second line of (6.43). Since the resulting integrand is irrelevant, we also obtain (6.34) after Λ-integration from Λ_0 down to Λ. Clearly, this is the only contribution for $V = 1$ so that (6.34) is proven for $V = 1$ and $a \neq 4$.

In the second step we use this result to extend the proof to $V = 1$ and $a = 4$. Now, the differential equation (6.16) reduces to the second line of (6.43), with $a = 4$, and the fourth and fifth lines of (6.16) with $V = 1$ and $V_1 = 0$. There is no contribution from the sixth line of (6.16) for $V_1 = 0$, because the part in braces would be non-planar, which is excluded in Definition 12.3. Inserting (6.22) we obtain the composite propagator (5.15b) in the part in braces $\{\;\}$ of the fifth line of (6.16). Together with (6.34) for $V = 1$ and $a \neq 4$ already proven we verify the integrand (6.34) for $V = 1$ and $a = 4$. After Λ-integration we thus obtain (6.34) for $V = 1$ and any a.

This allows us to use (6.34) as induction hypothesis for the remaining contributions in the last line of (6.43). This is similar to the procedure in 5, we only have to replace (6.39b), (6.40b) and (6.41b) by the according parametrisation of (6.34). We thus prove (6.34) to all orders.

2. We first consider $a \neq 4$. Then, according to (6.18)–(6.21) only terms with $V_1 \geq 1$ contribute to (6.16). Using (6.18)–(6.22) and the fact that γ is 1PI, the differential

equation (6.16) reduces to

$$
\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda}\Big(\sum_{\gamma \text{ as in Def. 12.2}} H^{a(V,V^e,1,0,0)\gamma}_{\substack{m^1\,n^1;\,n^1\,m^1\\ m^2\,n^2;\,n^2\,m^2}}[\Lambda,\Lambda_0,\rho^0]\Big)_{a\neq 4}\\
&= -\frac{1}{2}\Big\{\sum_{m,n,k,l} Q_{nm;lk}(\Lambda)\Big(H^{a(V,V^e,B,0,\iota)}_{\substack{m^1\,n^1;\,n^1\,m^1\\ m^2\,n^2;\,n^2\,m^2};mn;kl}[\Lambda] - H^{a(V,V^e,B,0,\iota)}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0};mn;kl}[\Lambda]\\
&\qquad - m^1\big(H^{a(V,V^e,B,0,\iota)}_{\substack{1\,0;\,0\,1\\ 0\,0;\,0\,0};mn;kl}[\Lambda] - H^{a(V,V^e,B,0,\iota)}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0};mn;kl}[\Lambda]\big)\\
&\qquad - n^1\big(H^{a(V,V^e,B,0,\iota)}_{\substack{0\,1;\,1\,0\\ 0\,0;\,0\,0};mn;kl}[\Lambda] - H^{a(V,V^e,B,0,\iota)}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0};mn;kl}[\Lambda]\big)\\
&\qquad - m^2\big(H^{a(V,V^e,B,0,\iota)}_{\substack{0\,0;\,0\,0\\ 1\,0;\,0\,1};mn;kl}[\Lambda] - H^{a(V,V^e,B,0,\iota)}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0};mn;kl}[\Lambda]\big)\\
&\qquad - n^2\big(H^{a(V,V^e,B,0,\iota)}_{\substack{0\,0;\,0\,0\\ 0\,1;\,1\,0};mn;kl}[\Lambda] - H^{a(V,V^e,B,0,\iota)}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0};mn;kl}[\Lambda]\big)\Big)\Big\}_{[\text{Def. 12.2}]} &\text{(6.44a)}\\
&- H^{4(0)}_{\substack{m^1\,n^1;\,n^1\,m^1\\ m^2\,n^2;\,n^2\,m^2}}[\Lambda]\Big\{-\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) H^{a(V)}_{\substack{0\,0;\,0\,0;\,0\,0;\,0\,0\\ 0\,0;\,0\,0;\,0\,0;\,0\,0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 12.1}]} &\text{(6.44b)}\\
&+ \text{the } 4^{\text{th}} \text{ to last lines of (6.16) with } \sum_{V_1=0}^{V} \mapsto \sum_{V_1=1}^{V-1}\,. &\text{(6.44c)}
\end{aligned}
$$

If the graphs have constant indices along the trajectories, we conclude in the same way as in Appendix E.1 that the part (6.44a) can be written as a linear combination of graphs having either a composite propagator (5.15b) or two composite propagators (5.15a) on the trajectories. As such we have to replace the bound (5.20) relative to the contribution of an ordinary propagator by (5.27) or twice (5.20) by (5.26). For the total graph this amounts to multiply the corresponding estimation (6.37) of ordinary H-graphs with $N = 2$ by a factor $\big(\frac{\max(m^r,n^r)}{\theta\Lambda^2}\big)^2$, which yields the subscript 2 of the part $P_2^{4V+2\delta^{a4}}[\;]$ of the integrand (6.35), for the time being restricted to the part (6.44a). For graphs with index jump in Definition 12.2 we obtain according to Appendix E.1 the same improvement by $\big(\frac{\max(m^r,n^r)}{\theta\Lambda^2}\big)^2$. Next, the product of (6.33) with (6.41a) gives for (6.44b) the same bound (6.35) for the integrand. Since the resulting integrand is irrelevant, we also obtain (6.35) after Λ-integration. Clearly, this is the only contribution for $V = 1$ so that (6.35) is proven for $V = 1$ and $a \neq 4$.

In the second step we use this result to extend the proof to $V = 1$ and $a = 4$. Now, the differential equation (6.16) reduces to the sum of (6.44a) and (6.44b), with $a = 4$, and the fourth and fifth lines of (6.16) with $V = 1$ and $V_1 = 0$. There is again no contribution of the sixth line of (6.16) for $V_1 = 0$. Inserting (6.22) we obtain the composite propagators (5.15a) in the fifth line of (6.16), which together with (6.35) for $V = 1$ and $a \neq 4$ already proven verifies the integrand (6.35) for $V = 1$ and $a = 4$. After Λ-integration we thus obtain (6.35) for $V = 1$ and any a.

This allows us to use (6.35) as induction hypothesis for the remaining contributions (6.44c) for $V > 1$. This is similar to the procedure in 5, we only have to replace (6.39b), (6.40b) and (6.41b) by the according parametrisation of (6.35). We thus prove (6.35) to all orders.

3. The proof of (6.36) is performed along the same lines as the proof of (6.34) and (6.35). There is one factor $\frac{\max(m^r,n^r)}{\theta\Lambda^2}$ from $\langle\overrightarrow{n_1\mathfrak{o}[n_1]}\rangle+\langle\overrightarrow{n_2\mathfrak{o}[n_2]}\rangle = 2$ in (6.37) and a second factor from the composite propagator (5.26) or (5.28) appearing according to Appendix E.1 in the $(V_1 = V)$-contribution to (6.16).

This finishes the proof of Proposition 14. □

6.4 The power-counting behaviour of the Λ_0-varied functions

The estimations in Propositions 13 and 14 allow us to estimate the R-functions by integrating the differential equation (6.17). Again, the R-functions are expanded in terms of ribbon graphs. Let us look at R-ribbon graphs of the type described in Definition 12.1. Since $\sum_{\gamma \text{ as in Def. 12.1}} A^{(V)\gamma}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0}}[\Lambda,\Lambda_0,\rho^0] \equiv \rho_4[\Lambda,\Lambda_0,\rho^0]$, we can rewrite the expansion coefficients of (6.5) as follows:

$$
\begin{aligned}
&\sum_{\gamma \text{ as in Def. 12.1}} R^{(V)\gamma}_{{}^{m^1}_{m^2}{}^{n^1}_{n^2};{}^{n^1}_{n^2}{}^{k^1}_{k^2};{}^{k^1}_{k^2}{}^{l^1}_{l^2};{}^{l^1}_{l^2}{}^{m^1}_{m^2}}[\Lambda,\Lambda_0,\rho^0] \\
&= \Lambda_0\frac{\partial}{\partial\Lambda_0}\sum_{\gamma \text{ as in Def. 12.1}}\Big(A^{(V)\gamma}_{{}^{m^1}_{m^2}{}^{n^1}_{n^2};{}^{n^1}_{n^2}{}^{k^1}_{k^2};{}^{k^1}_{k^2}{}^{l^1}_{l^2};{}^{l^1}_{l^2}{}^{m^1}_{m^2}}[\Lambda,\Lambda_0,\rho^0] - A^{(V)\gamma}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0}}[\Lambda,\Lambda_0,\rho^0]\Big) \\
&- \sum_{a,b=1}^{4}\frac{\partial}{\partial\rho^0_a}\sum_{\gamma \text{ as in Def. 12.1}}\Big(A^{(V)\gamma}_{{}^{m^1}_{m^2}{}^{n^1}_{n^2};{}^{n^1}_{n^2}{}^{k^1}_{k^2};{}^{k^1}_{k^2}{}^{l^1}_{l^2};{}^{l^1}_{l^2}{}^{m^1}_{m^2}}[\Lambda,\Lambda_0,\rho^0] - A^{(V)\gamma}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0}}[\Lambda,\Lambda_0,\rho^0]\Big) \\
&\qquad\qquad\times\frac{\partial\rho^0_a}{\partial\rho_b[\Lambda,\Lambda_0,\rho^0]}\Lambda_0\frac{\partial}{\partial\Lambda_0}\rho_b[\Lambda,\Lambda_0,\rho^0]\,.
\end{aligned}
\tag{6.45}
$$

This means that (by construction) only the (Λ_0,ρ^0)-derivatives of the projection to the irrelevant part (5.33a) of the planar four-point function contributes to R. Similarly, only the (Λ_0,ρ^0)-derivatives of the irrelevant parts (5.34a) and (5.35a) of the planar two-point function contribute to R. According to the initial condition (5.1), these projections and the other functions given in Definition 12.4 vanish at $\Lambda=\Lambda_0$ *independently* of Λ_0 or ρ^0_a:

$$
\begin{aligned}
0 &= \Lambda_0\frac{\partial}{\partial\Lambda_0}\Big(A^{(V,V^e,1,0,0)\gamma}_{{}^{m^1}_{m^2}{}^{n^1}_{n^2};{}^{n^1}_{n^2}{}^{k^1}_{k^2};{}^{k^1}_{k^2}{}^{l^1}_{l^2};{}^{l^1}_{l^2}{}^{m^1}_{m^2}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0}}[\Lambda_0,\Lambda_0,\rho^0]\Big)\Big|_{\gamma \text{ as in Def. 12.1}} \\
&= \frac{\partial}{\partial\rho^0_a}\Big(A^{(V,V^e,1,0,0)\gamma}_{{}^{m^1}_{m^2}{}^{n^1}_{n^2};{}^{n^1}_{n^2}{}^{k^1}_{k^2};{}^{k^1}_{k^2}{}^{l^1}_{l^2};{}^{l^1}_{l^2}{}^{m^1}_{m^2}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0}}[\Lambda_0,\Lambda_0,\rho^0]\Big)\Big|_{\gamma \text{ as in Def. 12.1}}\,,
\end{aligned}
\tag{6.46a}
$$

$$
\begin{aligned}
0 = \Lambda_0\frac{\partial}{\partial\Lambda_0}\Big(&A^{(V,V^e,1,0,0)\gamma}_{{}^{m^1}_{m^2}{}^{n^1}_{n^2};{}^{n^1}_{n^2}{}^{m^1}_{m^2}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0}}[\Lambda_0,\Lambda_0,\rho^0] \\
&- m^1\big(A^{(V,V^e,1,0,0)\gamma}_{{}^{1\,0}_{0\,0};{}^{0\,1}_{0\,0}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0}}[\Lambda_0,\Lambda_0,\rho^0]\big) \\
&- n^1\big(A^{(V,V^e,1,0,0)\gamma}_{{}^{0\,1}_{0\,0};{}^{1\,0}_{0\,0}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0}}[\Lambda_0,\Lambda_0,\rho^0]\big) \\
&- m^2\big(A^{(V,V^e,1,0,0)\gamma}_{{}^{0\,0}_{1\,0};{}^{0\,0}_{0\,1}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0}}[\Lambda_0,\Lambda_0,\rho^0]\big) \\
&- n^2\big(A^{(V,V^e,1,0,0)\gamma}_{{}^{0\,0}_{0\,1};{}^{0\,0}_{1\,0}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0}}[\Lambda_0,\Lambda_0,\rho^0]\big)\Big)\Big|_{\gamma \text{ as in Def. 12.2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\partial}{\partial \rho^0_a}\Big(A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1\, n^1;\, n^1\, m^1\\ m^2\, n^2;\, n^2\, m^2}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0}}[\Lambda_0,\Lambda_0,\rho^0] \\
&\qquad - m^1\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{1\,0;\,0\,1\\ 0\,0;\,0\,0}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0}}[\Lambda_0,\Lambda_0,\rho^0]\big) \\
&\qquad - n^1\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,1;\,1\,0\\ 0\,0;\,0\,0}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0}}[\Lambda_0,\Lambda_0,\rho^0]\big) \\
&\qquad - m^2\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0;\,0\,0\\ 1\,0;\,0\,1}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0}}[\Lambda_0,\Lambda_0,\rho^0]\big) \\
&\qquad - n^2\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0;\,0\,0\\ 0\,1;\,1\,0}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0}}[\Lambda_0,\Lambda_0,\rho^0]\big)\Big)\Big|_{\gamma \text{ as in Def. 12.2}}, \qquad (6.46\text{b})\\
0 &= \Lambda_0 \frac{\partial}{\partial \Lambda_0}\Big(A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1+1\, n^1+1;\, n^1\, m^1\\ m^2\, n^2;\, n^2\, m^2}}[\Lambda_0,\Lambda_0,\rho^0] \\
&\qquad - \sqrt{(m^1+1)(n^1+1)}\, A^{(V,V^e,1,0,0)\gamma}_{\substack{1\,1;\,0\,0\\ 0\,0;\,0\,0}}[\Lambda_0,\Lambda_0,\rho^0]\Big)\Big|_{\gamma \text{ as in Def. 12.3}} \\
&= \frac{\partial}{\partial \rho^0_a}\Big(A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1+1\, n^1+1;\, n^1\, m^1\\ m^2\, n^2;\, n^2\, m^2}}[\Lambda_0,\Lambda_0,\rho^0] \\
&\qquad - \sqrt{(m^1+1)(n^1+1)}\, A^{(V,V^e,1,0,0)\gamma}_{\substack{1\,1;\,0\,0\\ 0\,0;\,0\,0}}[\Lambda_0,\Lambda_0,\rho^0]\Big)\Big|_{\gamma \text{ as in Def. 12.3}}, \qquad (6.46\text{c})\\
0 &= \Lambda_0 \frac{\partial}{\partial \Lambda_0} A^{(V,V^e,B,g,\iota)\gamma}_{m_1 n_1;\dots;m_N n_N}[\Lambda_0,\Lambda_0,\rho^0]\Big|_{\gamma \text{ as in Def. 12.4}} \\
&= \frac{\partial}{\partial \rho^0_a} A^{(V,V^e,B,g,\iota)\gamma}_{m_1 n_1;\dots;m_N n_N}[\Lambda_0,\Lambda_0,\rho^0]\Big|_{\gamma \text{ as in Def. 12.4}}. \qquad (6.46\text{d})
\end{aligned}
$$

The Λ_0-derivative at $\Lambda = \Lambda_0$ has to be considered with care:

$$
\begin{aligned}
0 &= \Lambda_0 \frac{\partial}{\partial \Lambda_0} A^{(V,V^e,B,g,\iota)\gamma}_{m_1 n_1;\dots;m_N n_N}[\Lambda_0,\Lambda_0,\rho^0] \\
&= \Big(\Lambda \frac{\partial}{\partial \Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{m_1 n_1;\dots;m_N n_N}[\Lambda,\Lambda_0,\rho^0]\Big)_{\Lambda=\Lambda_0} + \Big(\Lambda_0 \frac{\partial}{\partial \Lambda_0} A^{(V,V^e,B,g,\iota)\gamma}_{m_1 n_1;\dots;m_N n_N}[\Lambda,\Lambda_0,\rho^0]\Big)_{\Lambda=\Lambda_0},
\end{aligned}
\tag{6.47}
$$

and similarly for (6.46a)–(6.46c). Inserting (6.45), (6.46), (6.47) and according formulae into the Taylor expansion of (6.5) we thus have

$$
\begin{aligned}
&\sum_{\gamma \text{ as in Def. 12.1}} R^{(V,V^e,1,0,0)\gamma}_{\substack{m^1\, n^1;\, n^1\, k^1;\, k^1\, l^1;\, l^1\, m^1\\ m^2\, n^2;\, n^2\, k^2;\, k^2\, l^2;\, l^2\, m^2}}[\Lambda_0,\Lambda_0,\rho^0] \\
&= - \sum_{\gamma \text{ as in Def. 12.1}} \Big(\Lambda \frac{\partial}{\partial \Lambda}\Big(A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1\, n^1;\, n^1\, k^1;\, k^1\, l^1;\, l^1\, m^1\\ m^2\, n^2;\, n^2\, k^2;\, k^2\, l^2;\, l^2\, m^2}}[\Lambda,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0;\,0\,0;\,0\,0;\,0\,0\\ 0\,0;\,0\,0;\,0\,0;\,0\,0}}[\Lambda,\Lambda_0,\rho^0]\Big)\Big)_{\Lambda=\Lambda_0},
\end{aligned}
\tag{6.48a}
$$

$$
\begin{aligned}
&\sum_{\gamma \text{ as in Def. 12.2}} R^{(V,V^e,1,0,0)\gamma}_{\substack{m^1\, n^1;\, n^1\, m^1\\ m^2\, n^2;\, n^2\, m^2}}[\Lambda_0,\Lambda_0,\rho^0] \\
&= - \sum_{\gamma \text{ as in Def. 12.2}} \Big(\Lambda \frac{\partial}{\partial \Lambda}\Big(A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1\, n^1;\, n^1\, m^1\\ m^2\, n^2;\, n^2\, m^2}}[\Lambda,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0}}[\Lambda,\Lambda_0,\rho^0] \\
&\qquad - m^1\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{1\,0;\,0\,1\\ 0\,0;\,0\,0}}[\Lambda,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0}}[\Lambda,\Lambda_0,\rho^0]\big) \\
&\qquad - n^1\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,1;\,1\,0\\ 0\,0;\,0\,0}}[\Lambda,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0;\,0\,0\\ 0\,0;\,0\,0}}[\Lambda,\Lambda_0,\rho^0]\big)
\end{aligned}
$$

$$
\begin{aligned}
&- m^2\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\\1\,0};\substack{0\,0\\0\,1}}[\Lambda,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda,\Lambda_0,\rho^0]\big)\\
&- n^2\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\\0\,1};\substack{0\,0\\1\,0}}[\Lambda,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda,\Lambda_0,\rho^0]\big)\Big)\Big)_{\Lambda=\Lambda_0}\;, \qquad (6.48b)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{\gamma\text{ as in Def. 12.3}} R^{(V,2,1,0,0)\gamma}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda_0,\Lambda_0,\rho^0]\\
&= -\sum_{\gamma\text{ as in Def. 12.3}}\Big(\Lambda\frac{\partial}{\partial\Lambda}\Big(A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda,\Lambda_0,\rho^0]\\
&\qquad - \sqrt{(m^1{+}1)(n^1{+}1)}\,A^{(V,V^e,1,0,0)\gamma}_{\substack{1\,1\\0\,0};\substack{0\,0\\0\,0}}[\Lambda,\Lambda_0,\rho^0]\Big)\Big)_{\Lambda=\Lambda_0}\;, \qquad (6.48c)
\end{aligned}
$$

$$
\begin{aligned}
&R^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\rho^0]\Big|_{\gamma\text{ as in Def. 12.4}}\\
&= -\Big(\Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho^0]\Big|_{\gamma\text{ as in Def. 12.4}}\Big)_{\Lambda=\Lambda_0}\;. \qquad (6.48d)
\end{aligned}
$$

In particular,

$$
R^{(1,1,1,0,0)}_{m_1n_1;\dots;m_4n_4}[\Lambda,\Lambda_0,\rho^0] \equiv 0\;. \qquad (6.49)
$$

We first get (6.49) at $\Lambda=\Lambda_0$ from (6.48a). Since the rhs of (6.17) vanishes for $V=1$ and $N=4$, we conclude (6.49) for any Λ.

Proposition 15 *Let γ be an R-ribbon graph having N external legs, V vertices, V^e external vertices and segmentation index ι, which is drawn on a genus-g Riemann surface with B boundary components. Then, the contribution $R^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}$ of γ to the expansion coefficient of the Λ_0-varied effective action describing a duality-covariant ϕ^4-theory on $\mathbb{R}^4_\theta$ in the matrix base is bounded as follows:*

1. *If γ is of the type described under 1–3 of Definition 12, we have*

$$
\begin{aligned}
&\Big|\sum_{\gamma\text{ as in Def. 12.1}} R^{(V,V^e,1,0,0)\gamma}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{k^1\\k^2};\substack{k^1\\k^2}\substack{l^1\\l^2};\substack{l^1\\l^2}\substack{m^1\\m^2}}[\Lambda,\Lambda_0,\rho_0]\Big|\\
&\qquad\le\Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)P_1^{4V-4}\Big[\frac{\substack{m^1\,n^1\\m^2\,n^2};\substack{n^1\,k^1\\n^2\,k^2};\substack{k^1\,l^1\\k^2\,l^2};\substack{l^1\,m^1\\l^2\,m^2}}{\theta\Lambda^2}\Big]\Big(\frac{1}{\Omega}\Big)^{3V-2-V^e}P^{2V-2}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\;, \qquad (6.50)\\
&\Big|\sum_{\gamma\text{ as in Def. 12.2}} R^{(V,V^e,1,0,0)\gamma}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda,\Lambda_0,\rho_0]\Big|\\
&\qquad\le\Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)(\theta\Lambda^2)P_2^{4V-2}\Big[\frac{\substack{m^1\,n^1\\m^2\,n^2};\substack{n^1\,m^1\\n^2\,m^2}}{\theta\Lambda^2}\Big]\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e}P^{2V-1}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\;, \qquad (6.51)\\
&\Big|\sum_{\gamma\text{ as in Def. 12.3}} R^{(V,V^e,1,0,0)}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda,\Lambda_0,\rho_0]\Big|\\
&\qquad\le\Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)(\theta\Lambda^2)P_2^{4V-2}\Big[\frac{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\,m^1\\n^2\,m^2}}{\theta\Lambda^2}\Big]\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e}P^{2V-1}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\;, \qquad (6.52)
\end{aligned}
$$

2. *If γ is a subgraph of an 1PI planar graph with a selected set T of trajectories on one distinguished boundary component and a second set T' of summed trajectories on that boundary component, we have*

$$\sum_{\mathcal{E}^s}\sum_{\mathcal{E}^{t'}}\big|R^{(V,V^e,B,0,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho_0]\big| \\ \leq \Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)\big(\theta\Lambda^2\big)^{(2-\frac{N}{2})+2(1-B)} P^{4V-N}_{\big(2t'+\sum_{\overrightarrow{n_j\mathfrak{o}[n_j]}\in T}\min(2,\frac{1}{2}\langle\overrightarrow{n_j\mathfrak{o}[n_j]}\rangle)\big)}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] \\ \times\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}-1+B+2g-V^e-\iota+s+t'} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,. \tag{6.53}$$

3. *If γ is a non-planar graph or a graph with $N\geq 6$ external legs, we have*

$$\sum_{\mathcal{E}^s}\big|R^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho_0]\big| \leq \Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)\big(\theta\Lambda^2\big)^{(2-\frac{N}{2})+2(1-B-2g)} P_0^{4V-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] \\ \times\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}-1+B+2g-V^e-\iota+s} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,. \tag{6.54}$$

We have $R^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}\equiv 0$ for $N>2V+2$ or $\sum_{i=1}^N(m_i-n_i)\neq 0$.

Proof. Inserting the estimations of Proposition 13 into (6.48) we confirm Proposition 15 for $\Lambda=\Lambda_0$, which serves as initial condition for the Λ-integration of (6.17). This entails the polynomial in $\ln\frac{\Lambda_0}{\Lambda_R}$ instead of $\ln\frac{\Lambda}{\Lambda_R}$ appearing in Propositions 13 and 14. Accordingly, when using Propositions 13 and 14 as the input for (6.17), we will further bound these estimations by replacing $\ln\frac{\Lambda}{\Lambda_R}$ by $\ln\frac{\Lambda_0}{\Lambda_R}$.

Due to (6.49) the rhs of (6.17) vanishes for $N=2, V=1$ and for $N=6, V=2$. This means that the corresponding R-functions are constant in Λ so that the Proposition holds for $R^{(1,1,1,0,0)}_{m_1n_1;m_2n_2}[\Lambda]$, $R^{(1,1,2,0,1)}_{m_1n_1;m_2n_2}[\Lambda]$ and $R^{(2,2,1,0,0)}_{m_1n_1;\dots;m_2n_2}[\Lambda]$. Since (6.17) is a linear differential equation, the factor $\frac{\Lambda^2}{\Lambda_0^2}$ relative to the estimation of the A-functions of Proposition 13, first appearing in $R^{(1,1,1,0,0)}_{m_1n_1;m_2n_2}[\Lambda]$, $R^{(1,1,2,0,1)}_{m_1n_1;m_2n_2}[\Lambda]$ and $R^{(2,2,1,0,0)}_{m_1n_1;\dots;m_2n_2}[\Lambda]$, survives to more complicated graphs, provided that none of the R-functions is relevant in Λ.

For graphs according to Definition 12.4, the first two lines on the rhs of (6.17) yield in the same way as in the proof of (5.37) on page 60 the integrand (6.54), with the degree of the polynomial in $\ln\frac{\Lambda_0}{\Lambda_R}$ lowered by 1. Since under the given conditions an A-graph would be irrelevant, an R-graph with the additional factor $\frac{\Lambda^2}{\Lambda_0^2}$ is relevant or marginal. Thus, the Λ-integration of the first two lines on the rhs of (6.17) can be estimated by the integrand and a factor $P^1[\ln\frac{\Lambda_0}{\Lambda_R}]$, in agreement with (6.54). In the same way we verify (6.53) for the first two lines on the rhs of (6.17).

In the remaining lines of (6.17) we get by induction the following estimation:

$$\Big|\Big\{\sum_{m,n,k,l} Q_{nm;lk}(\Lambda)R^{(V_1)}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 12.2}]}\Big| \\ \leq\Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)\big(\theta\Lambda^2\big)\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e} P^{2V-2}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,, \tag{6.55a}$$

$$\Big|\Big\{\sum_{m,n,k,l} Q_{nm;lk}(\Lambda)\Big(R^{(V_1)}_{\substack{1\,0\\0\,0};\substack{0\,1\\0\,0};mn;kl}[\Lambda]-R^{(V_1)}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};mn;kl}[\Lambda]\Big)\Big\}_{[\text{Def. 12.2}]}\Big|$$
$$\leq\Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e}P^{2V-2}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,, \tag{6.55b}$$
$$\Big|\Big\{\sum_{m,n,k,l} Q_{nm;lk}(\Lambda)R^{(V_1)}_{\substack{1\,1\\0\,0};\substack{0\,0\\0\,0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 12.3}]}\Big|$$
$$\leq\Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e}P^{2V-2}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,, \tag{6.55c}$$
$$\Big|\Big\{\sum_{m,n,k,l} Q_{nm;lk}(\Lambda)R^{(V_1)}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 12.1}]}\Big|$$
$$\leq\Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)\Big(\frac{1}{\Omega}\Big)^{3V-2-V^e}P^{2V-3}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,. \tag{6.55d}$$

These estimations are obtained in a similar way as (6.39a), (6.40a) and (6.41a). In particular, the improvement by $(\theta\Lambda^2)^{-1}$ in (6.55b) is due to the difference of graphs which according to Section 5.2 yield a composite propagator (5.15a). To obtain (6.55c) we have to use (6.53) with $\langle\overrightarrow{n_1 o(n_1)}\rangle+\langle\overrightarrow{n_2 o(n_2)}\rangle=2$, which for the graphs under consideration is known by induction.

Multiplying (6.55) by versions of Proposition 14 according to (6.17), for $V_1<V$, we obtain again (6.54) or (6.53), with the degree of the polynomial in $\ln\frac{\Lambda_0}{\Lambda_R}$ lowered by 1, for the integrand. Then, the Λ-integration proves (6.54) and (6.53).

For graphs as in 1–3 of Definition 12 one shows in the same way as in the proof of 1–3 of Proposition 14 that the last term in the third line of (6.17) and the $(V_1=V)$-terms in the remaining lines project to the irrelevant part of these R-functions, i.e. lead to (6.50)–(6.52). This was already clear from (6.45). For the remaining $(V_1<V)$-terms in the fourth to last lines of (6.17) we obtain (6.50)–(6.52) from (6.55) and (6.34)–(6.36). This finishes the proof. □

6.5 Finishing the convergence and renormalisation theorem

We return now to the starting point of the entire estimation procedure—the identity (6.4). We put $\Lambda=\Lambda_R$ in Proposition 15 and perform the Λ_0-integration in (6.4):

Theorem 16 *The ϕ^4-model on $\mathbb{R}^4_\theta$ is (order by order in the coupling constant) renormalisable in the matrix base by adjusting the coefficients $\rho^0_a[\Lambda_0]$ defined in (5.13) and (5.12) of the initial interaction (5.1) to give (5.14) and by integrating the the Polchinski equation according to Definition 12.*

The limit $A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty]:=\lim_{\Lambda_0\to\infty}A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]$ of the expansion coefficients of the effective action $L[\phi,\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]$, see (4.51), exists and satisfies

$$\Big|(2\pi\theta)^{\frac{N}{2}-2}A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty]-(2\pi\theta)^{\frac{N}{2}-2}A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\Lambda_0,\rho^0]\Big|$$
$$\leq\frac{\Lambda_R^{6-N}}{\Lambda_0^2}\Big(\frac{1}{\Omega\theta\Lambda_R^2}\Big)^{2(B+2g-1)}P_0^{4V-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda_R^2}\Big]\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}-V^e-\iota}P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,. \tag{6.56}$$

Proof. The existence of the limit and its property (6.56) are a consequence of Cauchy's criterion applied to (6.4) after insertion of Proposition 15 taken at $\Lambda = \Lambda_R$. We have also used (5.31) in Proposition 15.1. Note that $\int \frac{dx}{x^3} P^q[\ln x] = \frac{1}{x^2} P'^q[\ln x]$, see (4.27). □

The proof of the renormalisation theorem 16 is the main result of the Habilitation thesis. I will summarise and discuss these achievements in Section 7.

One can now address several other questions which depend on or are related to the renormalisation proof. For instance, it is very interesting to compute the β-function of the duality-covariant ϕ^4-model. I provide this calculation in Appendix G starting on page 149. It turns out that the one-loop β-function for the coupling constant remains non-negative. The self-dual case $\Omega = 1$ has particular features. Moreover, I find that the limit $\Omega \to 0$ exists at the one-loop level. I interpret this result as related to the fact that the UV/IR-mixing in momentum space becomes problematic only at higher loop order.

Another interesting exercise is the renormalisation of noncommutative ϕ^4-theory in two dimensions, which I perform in Appendix H starting on page 156. The renormalisation proof is much simpler in two dimensions. In fact, I can prove more: It is possible to couple the frequency Ω to the initial scale Λ_0 so that in two dimensions the limit $\Omega \to 0$ exists as a perturbatively renormalisable quantum field theory. This seems to be related to the folklore that the UV/IR-mixing is not a problem in two dimensions.

Finally, a few comments on the limit $\theta \to 0$. In the developed approach, θ defines the reference size of an elementary cell in the Moyal plane. All dimensionful quantities, in particular the energy scale Λ, are measured in units of (appropriate powers of) θ. In the final result of Theorem 16, these mass dimensions are restored. Then, we learn from (6.56) that a finite θ regularises the non-planar graphs. This means that for given Λ_0 and Λ_R the limit $\theta \to 0$ cannot be taken.

On the other hand, there could be a chance to let θ depend on Λ_0 in the same way as in the two-dimensional case treated in Appendix H the oscillator frequency Ω was switched off with the limit $\Lambda_0 \to \infty$. However, this does not work. The point is that taking in (6.7) on page 69 instead of the Λ_0-derivative the θ-derivative, there is now a contribution from the θ-dependence of the propagator. This leads in the analogue of the differential equation (6.11) to a term bilinear in L. Looking at the proof of Proposition 15, we see that this L-bilinear term will remove the factor Λ_0^{-2}.

Thus, the limit $\theta \to 0$ is singular. This is not surprising. In the limit $\theta \to 0$ the distinction between planar and non-planar graphs disappears (which is immediately clear in momentum space). Then, non-planar two- and four-point functions should yield the same divergent values as their planar analogues. Whereas the bare divergences in the planar sector are avoided by the mixed boundary conditions in 1-3 of Definition 12, the naïve initial condition in Definition 12.4 for non-planar graphs leaves the bare divergences in the limit $\theta \to 0$.

7 Conclusion

In this Habilitation thesis I have proven that the real Euclidean ϕ^4-model on the four-dimensional Moyal plane is renormalisable to all orders in perturbation theory. The proof is based on the sequence of articles [GW03a, GW04b, GW04c] and the additional material in [GW03b, GW04a] on the two-dimensional case and the β-function.

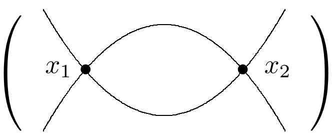
My Habilitation thesis solves, first of all, a longstanding technical problem concerning the renormalisation of noncommutative ϕ^4-theory. On the other hand, the proof produced several by-products which are valuable on their own. This includes the adaptation of the renormalisation scheme based on flow equations to dynamical matrix models and the identification of a deep connection between local and global aspects in noncommutative quantum field theories.

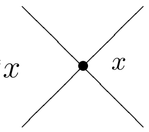
The bare action of relevant and marginal couplings of the model is parametrised by four (divergent) quantities which require normalisation to the experimental data at a physical renormalisation scale. The corresponding physical parameters which determine the model are the mass, the field amplitude (to be normalised to 1), the coupling constant and—in addition to the commutative version—the frequency of a harmonic oscillator potential.

The crucial decision was to work in the matrix base of the Moyal plane, which avoids the oscillating phase factors of the Weyl basis. I was able to derive a closed solution of the free theory in the matrix base, see (3.49) on page 33. This solution was of enormous importance during the renormalisation proof. To the best of my knowledge, the solution (3.49) and its limit (B.63) for vanishing oscillator potential were not known before.

The next achievement was the development of the renormalisation group approach for non-local (dynamical) matrix models. Its importance goes beyond the present renormalisation proof. Many noncommutative algebras have a matrix representation, whereas the possibility of Fourier modes is a rather exceptional feature of the Moyal plane. Thus, the tools developed here make the superior efficiency of renormalisation by flow equations [Pol84] available for more general noncommutative spaces. In particular, the very general power-counting theorem (Theorem 10 on page 48) gives a first criterion whether a field theory on a noncommutative space has the chance to be renormalisable or not. It is remarkable that the language of ribbon graphs drawn on Riemann surfaces is required both in the proof of our power-counting theorem for non-local matrix models and in the momentum space approach to field theories on the Moyal plane [CR00, CR01]. I hope that these techniques will prove fruitful for other investigations on matrix models.

For the renormalisation proof it was important to add the harmonic oscillator potential to the standard noncommutative ϕ^4-action. From the point of view of traditional renormalisation theory it seems very surprising that such a brutal x-dependence yields a renormalisable model. The renormalisation of the duality-covariant noncommutative ϕ^4-model teaches us a lesson which, although fairly obvious, was completely ignored before: A Feynman graph in a perturbative quantum field theory is a *non-local* object in the sense that it is made of several local vertices connected over the distance via propagators. Renormalisation requires that the singular part (in the sense of the forest formula) of such a non-local graph is *proportional to a local counterterm vertex*. For example,

$\int d^4x_1\, d^4x_2 \Big($


$\Big)_{\text{singular part}} \sim \int d^4x$


(7.1)

In noncommutative quantum field theory we are fully aware of the delicate situation when x_1, x_2 come close to each other, where the singular part of the loop has to reproduce the noncommutative multiplication encoded in the vertex. However, we did not take into account that also the opposite case where x_1, x_2 are far away from each other has to reproduce the noncommutative multiplication. This is, in fact, a constraint on the topology of the underlying space. Thus, renormalisation necessarily entangles the local and the global properties of the model. In this light, the separation of the infrared and ultraviolet regimes in a commutative quantum field theory was just an enormous chance. Unfortunately, the resulting prejudice obscured the view and had us look for a renormalisation of field theories on an asymptotically Euclidean Moyal plane. This did not work [MVRS00], and after the previous remarks we should not be surprised about that.

I have had the luck to identify the right topology which belongs to the $\star$-product: the harmonic oscillator potential. Although originally introduced for computational reasons[28], this is not a bad trick but a true physical effect. It is the self-consistent solution of the UV/IR-mixing problem found in the traditional noncommutative ϕ^4-model in momentum space. It implements the duality (see also [LS02a]) that *noncommutativity relevant at short distances goes hand in hand with a modified structure of the space relevant at large distances.*

At fist sight, such a modified structure of space at very large distances seems to be in contradiction with experimental data. But this is not true. Neither position space nor momentum space are the adapted frames to interpret the model. In the spirit of noncommutative geometry [Con96], the model is invariantly characterised by the *spectrum* of the Schrödinger operator relative to the free theory,

$$H\psi = \frac{1}{2}(p^2 + \mu_0^2)\psi \;, \qquad H = -\frac{1}{2}\partial_\mu\partial^\mu + \frac{\Omega^2}{2}\tilde{x}_\mu\tilde{x}^\mu + \frac{\mu_0^2}{2} \;. \tag{7.2}$$

As this operator describes the quantum mechanics of the harmonic oscillator, we obtain[29] with the definition of $\tilde{x}$ the quantisation of momenta

$$p^2 = \big(m^1{+}n^1{+}m^2{+}n^2{+}2\big)\frac{4\Omega}{\theta} \;, \qquad m^i, n^i \in \mathbb{N} \;. \tag{7.3}$$

These quantised momenta give us the impression of a finite universe. Indeed, there is some evidence of a non-trivial topology of the universe in the angular power spectrum of the cosmic microwave background [Lev02] measured by the WMAP satellite [B$^+$03]. There is currently a debate about which topological model gives the best fit of the WMAP data, see e.g. [LWR$^+$03, ALST04]. Other WMAP constraints imply that the extension of the universe is larger than $L_u = 24\,\mathrm{Gpc} = 7.4 \times 10^{28}\,\mathrm{cm}$ [CSSK04]. Assuming $\theta = \ell_P^2$,

[28] It was clear very early that the propagator computed from the standard noncommutative ϕ^4-action does not have the scaling dimensions of a renormalisable model. In the matrix base of the Moyal plane, the standard Laplace operator is a tri-diagonal band matrix. The main diagonal behaves nicely, but the two adjacent diagonals are "too big" and compensate the desired behaviour of the main diagonal. Making the adjacent diagonals "smaller" one preserves the properties of the main diagonal and obtains the good scaling dimensions required for a renormalisable model. The deformation of the adjacent diagonals corresponds to the inclusion of a harmonic oscillator potential in the free field action.

[29] The relation (7.3) can also be obtained from comparison of (B.22) with (B.31) and (B.32).

where $\ell_P = 1.6 \times 10^{-33}\,\mathrm{cm}$ is the Planck length (1.2), we would obtain from (7.3) the incredibly small value of $\Omega = \frac{\theta}{4L_u^2} < 10^{-124}$. It is clear that for typical momenta on earth, the discretisation of momenta is not visible.

There is no reason to believe that the spectrum of the Schrödinger operator (7.2) gives a good account for the WMAP data. But the renormalisation of the duality-covariant noncommutative ϕ^4-model prepares us to accept that there is a deep connection between cosmology and noncommutative quantum field theory via the renormalisability constraint. If Nature abhors the infinitesimal small (in form of noncommutativity of the space), it cannot admit the infinitely large.

Several further activities are conceivable.

- *Analytic bounds for the asymptotic behaviour of the propagator.* It is somewhat disappointing that the powerful renormalisation proof for the duality-covariant noncommutative ϕ^4-model relies on a couple of numerical verifications. There is no doubt that these estimations are correct, but the situation is not satisfactory. I have tried hard to prove these estimations analytically, but for the time being without success. The problem is the opposite behaviour of the various terms in (B.39). The binomials become huge for $u^i \approx \frac{1}{2}\max(m^i, l^i)$ whereas the hypergeometric function is minimal for these values and maximal for small and large u^i. Clearly, the more tractable equation (B.37) shows the same behaviour. One should probably keep the hypergeometric function in (B.36) (which is also a Meixner polynomial) unexpanded and use appropriate asymptotic expansions of these special functions. This is a recent topic in mathematics [Tem03, JW98].

- *The Mehler formula.* Although the harmonic oscillator potential breaks translation invariance (in form of quantised momenta), the Schrödinger operator can be exploited in momentum space using the Mehler kernel [Sim79]
$$\Big(p^2+m^2-\frac{\partial^2}{\partial p^2}\Big)K_M(p,q) = \delta(p-q)\;. \tag{7.4}$$
After some calculus one can represent the partition function associated with the bilinear part of the action (3.42) as follows:
$$\begin{aligned}Z_{\text{free}} &= Z[0]\exp\Big(\frac{1}{2}\int\frac{d^4p\,d^4q}{(2\pi)^4}J(-p)K_M(p,q)J(q)\Big)\;,\\ K_M(p,q) &= \frac{\theta^3}{2\pi^2\Omega^3}\int_0^1 dz\,\frac{z^{1+\frac{\mu_0^2\theta}{4\Omega}}}{(1-z^2)^2}\exp\Big(-\frac{\theta}{8\Omega}\frac{(1-z)}{(1+z)}(p+q)^2-\frac{\theta}{8\Omega}\frac{(1+z)}{(1-z)}(p-q)^2\Big)\;.\end{aligned} \tag{7.5}$$
Thus, the Mehler kernel gives the Feynman rule for the propagator in momentum space. The rules for the vertices are unchanged and are given by (3.34). It would be interesting to compute simple graphs and eventually to repeat the power-counting analysis of Chepelev and Roiban [CR00, CR01] for the propagator (7.5). The difficulty is the loss of momentum conservation over the propagator. This means that (7.5) actually replaces $\frac{\delta(p-q)}{p^2+\mu_0^2}$. In particular, there is a divergence in (7.5) for $p=q$. One has to separate the true divergences from the shadow of momentum conservation at $p=q$.

- *The duality-covariant noncommutative ϕ^4-model in the spectral base.* It is natural to expand the action (3.42) into the eigenfunctions of the Schrödinger operator (7.2). Then, the propagator reads $\frac{1}{p^2+\mu_0^2}$, with p^2 now being discrete. In contrast to the Mehler formula, there is now momentum conservation over the propagator. In order to obtain the vertices we have to shift the unitary matrices $U_m^{(\alpha)}$ appearing in (B.22) from the kinetic matrix or the propagator into the vertex. This involves a summation over the matrix labels m, l, α at the vertex, leaving the eigenvalues $x = p^2$ as the dynamical variables. It will be interesting to study the properties of these dressed (physical) vertices. In particular, special functions often have locality properties in the sense that the integral over a product of them vanishes unless certain parameter conditions are satisfied. I expect such relations for Meixner polynomials, too. This would correspond to some fuzzy momentum conservation at the vertices.

- *Gauge theories.* The renormalisation of noncommutative ϕ^4-theory is a remarkable result, but for phenomenological reasons it would be much more important to renormalise gauge theories on noncommutative spaces. Moreover, gauge theories arise naturally in noncommutative geometry from fluctuations of a Dirac operator [Con96]. There are two natural candidates for a four-dimensional Dirac operator which carry a germ of an oscillator potential:

$$\mathcal{D}' = \mathrm{i}\gamma^\mu\partial_\mu + 2\mathrm{i}\Omega'\gamma^5\gamma^\mu x_\mu \;, \qquad \mathcal{D} = \mathrm{i}\gamma^\mu\partial_\mu + 2\mathrm{i}\Omega\gamma^5\gamma^\mu(\theta^{-1})_{\mu\nu}x^\nu \;. \tag{7.6}$$

Both $\mathcal{D}'$ and $\mathcal{D}$ are formally self-adjoint and have the required charge-conjugation property $J\mathcal{D}^{(\prime)}J^{-1} = \mathcal{D}^{(\prime)}$, where $J\psi = \gamma^0\gamma^2\bar{\psi}$, with $\bar{\psi}$ denoting the complex conjugated spinor. Conjugation by unitary elements of the Hilbert space and generalisation of the resulting pure gauge to a general gauge yields the fluctuated Dirac operator

$$\mathcal{D}'_A = \mathcal{D} + (\gamma^\mu + \mathrm{i}\Omega'\gamma^5\gamma_\kappa\theta^{\kappa\mu})\rho(A_\mu) + (-\gamma^\mu + \mathrm{i}\Omega\gamma^5\gamma_\kappa\theta^{\kappa\mu})J\rho(A_\mu)J^{-1} \;, \tag{7.7}$$

and similarly when starting from $\mathcal{D}$. Here, $\rho(A_\mu)\psi := A_\mu \star \psi$ stands for the representation given by left $\star$-multiplication. It is interesting that both the momentum part ∂_μ and the position part x_μ in (7.6) generate the same gauge potential $A_\mu = \overline{A_\mu}$. Another unitary conjugation gives the usual gauge transformation,

$$\big(\rho(u)J\rho(u)J^{-1}\big)\mathcal{D}'_A\big(\rho(u^*)J\rho(u^*)J^{-1}\big) = \mathcal{D}'_{A^u} \;, \quad A^u_\mu = \mathrm{i}u \star \partial_\mu u^* + u \star A_\mu \star u^* \;. \tag{7.8}$$

for u being an unitary element of an appropriate subalgebra of the Moyal algebra.

By construction, the spectral action $\mathrm{trace}\big(\chi\big(\mathcal{D}'_A/\Lambda\big)^2\big)$, see (1.3), is invariant under gauge transformations $A_\mu \mapsto A^u_\mu$. The free Laplacian $\mathcal{D}'^2$ has a discrete spectrum so that there is no need to smear it with a function of compact support as in [GI04]. The actual computation of the spectral action remains a big challenge. The renormalisation by flow equations as in [KK96, KM00] is a formidable task.

A faster approach would be to construct an action for a noncommutative $U(1)$ gauge field directly from "covariant coordinates" $X_\mu = \theta^{-1}_{\mu\nu}x^\nu + A_\mu$, see [MSSW00]. Then, one can consider the following action functional:

$$
\begin{aligned}
S &= \int d^4x \Big(\alpha_1 [X_\mu, X_\nu]_\star \star [X^\nu, X^\mu]_\star + \alpha_2 X_\mu \star X^\mu \star X_\nu \star X^\nu \Big) \\
&= \int d^4x \Big(\alpha_1 F_{\mu\nu} \star F^{\mu\nu} + \alpha_1 (\theta^{-1})_{\mu\nu} (\theta^{-1})^{\mu\nu} + \tfrac{1}{16} \alpha_2 (\tilde{x}_\mu \tilde{x}^\mu)^2 \\
&\qquad + \alpha_2 \Big((\tilde{x}^\mu A_\mu)^2 + (A_\mu \star A^\mu)^2 + \tfrac{1}{2} (\tilde{x}_\mu \tilde{x}^\mu)(A_\nu \star A^\nu) + 2 (\tilde{x}^\mu A_\mu)(A_\nu \star A^\nu) \\
&\qquad\quad + \tfrac{1}{2} (\tilde{x}_\mu \tilde{x}^\mu)(\tilde{x}^\nu A_\nu) \Big) \Big) , \qquad (7.9)
\end{aligned}
$$

where $F_{\mu\nu}$ is the field strength defined in (3.20) on page 26. The A-linear term in the last line has to be eliminated by a shift $A_\mu = A'_\mu + a_\mu(x)$, which in turn leads to the gauge transformation $A'_\mu \mapsto u \star \big[\mathrm{i}\partial_\mu + a_\mu(x), u^* \big]_\star + u \star A'_\mu \star u^*$. The resulting action contains an explicit x-dependence in the A'-bilinear part similar to the duality-covariant noncommutative ϕ^4-action. Working out the details is an interesting exercise.

On the other hand, one should try to include the harmonic oscillator potential merely as a computational trick—like the auxiliary mass in Lowenstein's approach [Low76] to massless models—which at the end should be consistently removed as in the noncommutative ϕ^4_2-model discussed in Appendix H.

- *A Higgs-like model.* It was noticed by David Broadhurst that the quantity $\frac{\Omega^2}{\lambda}$ is according to (G.24) and (G.25), page 153, stable against one-loop corrections of the duality-covariant noncommutative ϕ^4_4-model. This combination $\frac{\Omega^2}{\lambda}$ describes the location of the minimum of the potential $-\frac{1}{2}\Omega^2 \tilde{x}^2 \phi^2 + \frac{\lambda}{4!}\phi^4$ which arises from the massless duality-covariant noncommutative ϕ^4-action after changing the sign of the oscillator potential. Thus, it would be interesting to look for non-trivial solutions of the corresponding classical field equation

$$
0 = -\partial_\mu \partial^\mu \phi - \Omega^2 \tilde{x}^2 \phi + \mu_0^2 \phi + \frac{\lambda}{6} \phi^3 \qquad (7.10)
$$

and to attempt a quantisation about such a solution.

- *Renormalisation of other models.* The derivation of the propagator in Appendices B.2 and B.3, which was related to the orthogonal Meixner polynomials, is easily generalised to a large class of special functions. The spectral representation (B.22) of the kinetic matrix and the orthogonality relation (B.23) can be written abstractly as an integration over the spectrum σ with measure $d\mu(\sigma)$:

$$
G_{ab} := \int d\mu(\sigma) U_a(\sigma)(\mu_0^2 + \sigma) U_b(\sigma) \ , \quad \Delta_{ab} := \int d\mu(\sigma) U_a(\sigma)(\mu_0^2 + \sigma)^{-1} U_b(\sigma) \qquad (7.11)
$$

and

$$
\delta_{\overrightarrow{ab}} \delta_{\overrightarrow{ba}} = \int d\mu(\sigma) U_a(\sigma) U_b(\sigma) \ , \qquad \delta(\sigma - \sigma') = \sum_a U_a(\sigma) U_b(\sigma') \ . \qquad (7.12)
$$

The corresponding vertices of a ϕ^n-matrix model would be

$$
V_{a_1 \dots a_n} := \frac{\lambda}{n!} \delta_{\overrightarrow{a_1, a_2}} \cdots \delta_{\overrightarrow{a_{n-1}, a_n}} \delta_{\overrightarrow{a_n, a_1}} + \text{ permutations} \ . \qquad (7.13)
$$

In our example, the matrix indices a, b stand for pairs of elements of $\mathbb{N}^{\frac{D}{2}}$ so that we need oriented δ-functions $\delta_{\overrightarrow{mn;kl}} := \delta_{nk} \neq \delta_{\overrightarrow{kl;mn}} = \delta_{ml}$. To each solution of (7.12) one can associate a dynamical matrix model for which one should investigate the renormalisation along the presented lines. In particular, this includes the classical orthogonal polynomials and their q-analogues [KS96]. So far I have only studied the case of Meixner polynomials (which is the duality-covariant noncommutative ϕ^4-model, see Appendix B.2) and of Laguerre polynomials (which is the standard noncommutative ϕ^4-model, see Appendix B.6, where the renormalisation fails). Of particular interest is the case of truncated Legendre polynomials, which is related to the fuzzy sphere [GKP96b].

- *Investigations beyond the perturbation series.* The renormalisation of the duality-covariant noncommutative ϕ^4-model performed here was based on a perturbative expansion of the effective action. This is not satisfactory. A direct non-perturbative solution of the flow equations would be too ambitious. Therefore, the optimal method would consist in a (Borel) resummation of the perturbation series in the spirit of *constructive renormalisation* [Riv91, Riv00]. See also [Sim74, GJ87].

  There is probably not much hope for the model under consideration, because the one-loop β-function is positive, see Appendix G. However, the sign of the β-function can formally be changed by taking the "wrong sign" of the coupling constant. Moreover, we have seen that planar graphs only require a renormalisation, thus making perfect contact with the investigations in [tH82a, Riv84]. It remains to work out the details.

  In general, as the naïve resummation is obstructed by the requirement to renormalise divergent graphs [Riv91] and, on the other hand, in (renormalisable) noncommutative field theories planar graphs only can be divergent, these models seem to be well-suited for constructive techniques [tH82b]. The most interesting model to start with is probably the noncommutative version of the Gross-Neveu model [GN74] in two dimensions—to be built around the Dirac operator in (7.6)—the commutative version of which was constructed in [DR00].

  Moreover, the duality-covariant noncommutative ϕ^4-model is characterised by *countably many degrees of freedom*—in the same way as models on lattices [Kog79]. There could be a chance to extend some of the powerful tools available for lattice models (see e.g. [Bal88]) to noncommutative spaces, too.

- *Minkowskian signature of the metric.* Since we were working on a Euclidean space, the duality-covariant noncommutative ϕ^4-theory is rather a model for a spin system than a quantum field theory. It is probably a bad idea to convert the results by Wick rotation to Minkowski space [BDFP02]. In particular, the Osterwalder-Schrader axioms [OS73] are unlikely to satisfy without adaptation. Therefore, a direct Minkowskian approach following [DFR95, BDFP02] is necessary.

  Unfortunately, the required breaking of Lorentz invariance (at least in intermediate steps) and its effect to the phase factors in momentum space have prevented so far any conclusions valid to all orders for these models. I am optimistic that the use of an adapted (matrix) base where the oscillating phases disappear will turn out very useful for quantum field theories on noncommutative Minkowski space. Probably, one should keep the Weyl base for the time direction in order to make use of the Cauchy integrals. However, there is no need of the Weyl base for the space directions, and an adapted base for them should improve the calculus considerably.

- *Relations to string theory.* Gauge theory on the Moyal plane arises in the zero-slope limit of string theory in presence of a Neveu-Schwarz B-field [DH98, SW99]. As I argued that renormalisation requires an appropriate structure of the space at very large distances, the question arises whether the oscillator potential has a counterpart in string theory. I am not an expert of string theory to make a qualified comment, but it is tempting[30] to relate the oscillator potential to the maximally supersymmetric pp-wave background metric of type IIB string theory found in [BFOHP02],

$$ds^2 = 2x^+x^- - 4\lambda^2 \sum_{i=1}^{8} (x^i)^2 (dx^-)^2 + \sum_{i=1}^{8} (dx^i)^2 , \qquad dx^{\pm} = \frac{1}{\sqrt{2}}(dx^9 \pm dx^{10}) , \quad (7.14)$$

which solves Einstein's equations for an energy-momentum tensor relative to the 5-form field strength

$$F_5 = \lambda dx^- \big(dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8\big) . \qquad (7.15)$$

It was suggested in [BMN02] that this background is related to the large-N limit of $U(N)$ $\mathcal{N} = 4$ super Yang-Mills theory.

[30]This was pointed out by Giulio Bonelli.

A Some historical comments

There is such a vast literature on quantum field theories over noncommutative spaces that a comprehensive overview is outside the scope of this Habilitation thesis. I give a personal incomplete selection and apologise to those authors which feel that their work is not adequately highlighted.

A.1 Commutation relations for space-time coordinates

To the best of my knowledge, the possibility that geometry looses its meaning in quantum physics was first[31] considered by Schrödinger [Sch34]. On the other hand[32], Heisenberg suggested to use coordinate uncertainty relations to ameliorate the short-distance singularities in the quantum theory of fields. His idea (which appeared later [Hei38]) inspired Peierls in the treatment of electrons in a strong external magnetic field [Pei33]. Via Pauli and Oppenheimer the idea came to Snyder who was the first to write down uncertainty relations between coordinates [Sny47].

The philosophy that a fuzzy structure [Mad92] of space-time regularises quantum field theories was revived in noncommutative geometry [Mad00] where the UV-regularisation is automatic [GKP96b, GKP96a, GS99]. See also [Ydr01]. Another construction of finite quantum field theories on noncommutative spaces is based on point-splitting via tensor products [CHMS00, BDFP03].

The uncertainty relations for coordinates were rediscovered by Doplicher, Fredenhagen and Roberts [DFR95] as a means to avoid gravitational collapse when localising events with extreme precision. According to [DFR95], the coordinate uncertainties Δx^μ have to satisfy $\Delta x^0(\Delta x^1 + \Delta x^2 + \Delta x^3) \geq \ell_P^2$ and $\Delta x^1\Delta x^2 + \Delta x^2\Delta x^3 + \Delta x^3\Delta x^1 \geq \ell_P^2$, where $\ell_P = \sqrt{\frac{G\hbar}{c^3}}$ is the Planck length. These uncertainty relations are induced by coordinate operators $\hat{x}^\mu = (\hat{x}^\mu)^*$ under the condition

$$[[\hat{x}^\mu, \hat{x}^\nu], \hat{x}^\rho] = 0\;, \tag{A.1}$$

$$[\hat{x}_\mu, \hat{x}_\nu][\hat{x}^\mu, \hat{x}^\nu] = 0\;, \qquad \Big(\frac{1}{8}[\hat{x}^\mu, \hat{x}^\nu][\hat{x}^\rho, \hat{x}^\sigma]\epsilon_{\mu\nu\rho\sigma}\Big)^2 = \ell_P^8\;. \tag{A.2}$$

The equation (A.1) qualifies the resulting algebra as a special Moyal plane, see Section 3.1 on page 23. Moreover, in [DFR95] first steps are taken towards a perturbative quantum field theory on the resulting (Minkowskian) quantum space-time.

The Moyal product has its origin in quantum mechanics, in particular in Weyl's operator calculus [Wey28]. Wigner introduced the useful concept of the phase space distribution function [Wig32]. Then, Groenewold [Gro46] and Moyal [Moy49] showed that quantum mechanics can be formulated on classical phase space using the *twisted product* concept. In particular, Moyal proposed the "sine-Poisson bracket" (nowadays called Moyal bracket), which is the analogue of the quantum mechanical commutation relations. The twisted product was extended from Schwartz class functions to (appropriate) tempered distributions by Gracia-Bondía and Várilly [GBV88]. The programme of

[31] Actually, Riemann himself speculated in his famous Habilitationsvortrag [Rie92] about the possibility that the hypotheses of geometry lose their validity in the infinitesimal small.

[32] These historical remarks are extracted from [Jac03].

Groenewold and Moyal culminated in the axiomatic approach of *deformation quantisation* [BFF$^+$78a, BFF$^+$78b]. The problem to lift a given Poisson structure to an associative $\star$-product was solved by Kontsevich [Kon97]. Cattaneo and Felder [CF00] found a physical derivation of Kontsevich's formula in terms of a path integral quantisation of a Poisson sigma model [SS94].

The belief that space-time noncommutativity cures the ultraviolet divergences of quantum field theory was smashed by Filk [Fil96] who showed that the planar graphs of a field theory on the Moyal plane[33] are identical to the commutative theory (and thus have the same divergences). An achievement in [Fil96] which turned out to be important for later work was the definition of the *intersection matrix* of a graph which is read-off from its reduction to a rosette. In [VGB99] the persistence of divergences was rephrased in the framework of noncommutative geometry, based on the general definition of a dimension and the noncommutative formulation of external field quantisation. See also [CDP00].

A.2 Field theory on the noncommutative torus

The first noncommutative space where field theory has been studied was the noncommutative torus [CR87]. It became popular for field theorists when Connes, Douglas and Schwarz proposed to compactify M-theory on the noncommutative torus [CDS98]. M-theory lives in higher dimensions so that some of them must be compactified to give a realistic model. Compactifying on a noncommutative instead of a commutative torus amounts to turn on a constant background 3-form C. An alternative interpretation based on D-branes on tori in presence of a Neveu-Schwarz B-field was given by Douglas and Hull [DH98]. Similar effects are obtained in boundary conformal field theory [Sch99].

Later, the appearance of noncommutative field theory in the zero-slope limit of type IIA string theory was thoroughly investigated by Seiberg and Witten [SW99]. Moreover, using the results of [NS98] about instantons on noncommutative $\mathbb{R}^4$, Seiberg and Witten argued that there is an equivalence between the Yang-Mills theories on standard $\mathbb{R}^4$ and noncommutative $\mathbb{R}^4$, which I comment on in Section A.4.

It should be mentioned that matrix theories were studied long before M-theory was proposed, and that these matrix theories did contain certain noncommutative features. In the large-N limit of two-dimensional $SU(N)$ lattice gauge theory, the number of degrees of freedom is reduced and corresponds to a zero-dimensional model [EK82]. The restriction to two dimensions can be relaxed [GAO83] by twisted boundary conditions [tH81] so that the equivalence between lattice $SU(N)$ gauge theory for large N and the twisted Eguchi-Kawai model holds in any dimensions [GAKA83]. Here, the action can be rewritten in terms of noncommuting matrix derivatives $[\Gamma^{(j)}, \ .\ ]$, with $[\Gamma^{(2j)}, \Gamma^{(2j+1)}] = -2\pi\mathrm{i}/N$.

The paper [CDS98] inspired many activities on the interface between string/M-theory and noncommutative geometry (I come back to that in Section A.3). Among others the question was raised whether Yang-Mills theory on the noncommutative torus is renormalisable. See also [Dou99]. We have confirmed one-loop renormalisability in [KW00]: Using ζ-function techniques and cocycle identities we have extracted the pole parts related to the

[33] Filk's model refers to [DFR95] but is formulated in the $\star$-product formalism. It is certainly inspired by the twisted Eguchi-Kawai model [GAO83, GAKA83] which I discuss in Section A.2.

Feynman graphs and proved that they can be removed by multiplicative renormalisation of the initial action. In particular, the Ward identities are satisfied. See also [Kra98b].

Based on ideas developed in [AII+00] on type IIB matrix models, it was shown in [AMNS99] that, imposing a natural constraint for the (finite) matrices, the twisted Eguchi-Kawai construction [GAO83] can be generalised to noncommutative Yang-Mills theory on a toroidal lattice. The appearing gauge-invariant operators are the analogues of Wilson loops [Wil74]. This formulation enabled numerical simulations [BHN02, BHN03] of the various limiting procedures which confirmed conjectures [GS01] about striped and disordered patterns in the phase diagram of spontaneously broken noncommutative ϕ^4-theory. On the other hand, the limit $N \to \infty$ of the matrix size is mathematically delicate [LLS01]. To deal with that problem, a new formulation [LLS04b, LLS04a] of matrix models approximating field theories on the noncommutative torus has been proposed which is based on noncommutative solitons [GMS00].

There are also other noncommutative spaces which arise as limiting cases of string theory [ARS99].

A.3 Renormalisation of noncommutative quantum field theories

With the motivation of the Moyal plane in [DFR95], the proof that UV-divergences in quantum field theories persist [Fil96], and the relationship of the noncommutative torus to M-theory [CDS98] and the noncommutative $\mathbb{R}^D$ to type IIA string theory [DH98, NS98], time was ready in 1998 to investigate the renormalisation of quantum field theories on the noncommutative torus and the noncommutative $\mathbb{R}^D$. It is, therefore, not surprising that this question was addressed by different groups at about the same time [MSR99, SJ99, KW00].

Martín and Sánchez-Ruiz [MSR99] investigated U(1) Yang-Mills theory on the noncommutative $\mathbb{R}^4$ at the one-loop level. They found that all one-loop pole terms of this model in dimensional regularisation[34] can be removed by multiplicative renormalisation (minimal subtraction) in a way preserving the BRST symmetry. This is completely analogous to the situation on the noncommutative 4-torus [KW00][35]. Shortly later there appeared also an investigation of $(2+1)$-dimensional super-Yang-Mills theory with the two-dimensional space being the noncommutative torus [SJ99].

The paper [SW99] of Seiberg and Witten from August 1999 made the interface between string theory and noncommutative geometry extremely popular. Thousands of papers on this subject appeared, making it impossible to give an adequate overview. I restrict myself to the renormalisation question and refer to the following reviews for further information:

- by Konechny and Schwarz with focus on compactifications of M-theory on noncommutative tori [KS02a] as well as on instantons and solitons on noncommutative $\mathbb{R}^D$ [KS02b],

[34] There is of course a problem extending θ to complex dimensions, this is however discussed in [MSR99].

[35] Our work was ready in autumn 1998 for the Ph.D. thesis [Kra98b] of T. Krajewski (defended in December 1998) and as such known in the community of the noncommutative standard model. We did not publish it immediately because we looked for some results at higher loop order. Fortunate for us, a complex i was missing in the first version of [MSR99], leading to the opposite conclusion about the asymptotic freedom. In an e-mail exchange we pointed out the sign error and communicated our calculations, which thus found their way into the final version of [MSR99] the next day.

- by Douglas and Nekrasov [DN01] and by Szabo [Sza03], both with focus on field theory on noncommutative spaces in relation to string theory,
- by Aref'eva, Belov, Giryavets, Koshelev and Medvedev [ABG$^+$01] with focus on string field theory.

A systematic analysis of field theories on noncommutative $\mathbb{R}^D$, to any loop order, was first performed by Chepelev and Roiban [CR00]. The essential technique is the representation of Feynman graphs as ribbon graphs [tH74] (for noncommutative field theories first suggested in [Haw99]), drawn on an (oriented) Riemann surface with boundary, to which the external legs of the graph are attached. Using sophisticated mathematical tools, Chepelev and Roiban were able to relate the power-counting behaviour to the topology of the graph (I review the main ideas in Section 3.2). Their first conclusion was that a noncommutative field theory is renormalisable iff its commutative counterpart is renormalisable. Then, the authors of [MVRS00] gave a counter-example (see below). It turned out that this problem was simply overlooked in the first version of [CR00], with the power-counting analysis of being correct. A refined proof of the power-counting theorem was given in [CR01].

By computing the non-planar one-loop graphs explicitly, Minwalla, Van Raamsdonk and Seiberg pointed out a serious problem in the renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ and ϕ^3-theory on noncommutative $\mathbb{R}^6$ [MVRS00]. Non-planar graphs are regulated by the phase factors in the $\star$-product (3.4), but only if the external momenta of the graph are non-exceptional. As a matter of fact, inserting non-planar graphs (declared as regular) as subgraphs into bigger graphs, external momenta of the subgraph are internal momenta for the total graph. As such, exceptional external momenta for the subgraph are realised in the loop integration, resulting in an divergent integral for the total graph. The problem is independent of the number of external legs of the total graph so that the divergences cannot be removed by usual UV-subtractions. Instead, it was proposed in [MVRS00] to reorder the perturbation series (more details of this idea are given in [CR01]). The procedure is promising, but a renormalisation proof based on the resummation of non-planar graphs is still missing.

Anyway, the problem discovered in [MVRS00] made the subject of noncommutative field theories extremely popular. In the following months, an enormous number of articles doing (mostly) one-loop computations of all kind of models appeared. I do not want to give an overview about these activities and mention only a few papers: the two-loop calculation of ϕ^4-theory [ABK00b]; the renormalisation of complex $\phi \star \phi^* \star \phi \star \phi^*$ theory [ABK00a], later explained by a topological analysis [CR01]; computations in noncommutative QED [Hay99]; the calculation of noncommutative $U(1)$ Yang-Mills theory [MST00], with an outlook to super-Yang-Mills theory; the one-loop analysis of noncommutative $U(N)$ Yang-Mills theory [BS01].

All these achievements are overshadowed by the power-counting theorem of Chepelev and Roiban [CR01] which decides the renormalisability question of (massive, Euclidean) quantum field theories on noncommutative $\mathbb{R}^D$ to all orders. Roughly speaking, quantum field theories with only logarithmic divergences are renormalisable[36] on noncommutative $\mathbb{R}^D$. Still, the 1PI Green's functions do not exist pointwise (at exceptional momenta)

[36]The reason is that logarithms are integrable, see [GKW00] for an explicit construction of the estimations.

so that multiplication with IR-smoothening test functions is necessary. Except for some exceptional cases such as the $\phi\star\phi^*\star\phi\star\phi^*$ interaction, models with quadratic divergences are not perturbatively renormalisable.

The only way to circumvent the power-counting theorem of [CR01] is a different limiting procedure of the loop calculations. Namely, in intermediate steps one changes the order of integrations of integrals which are not absolutely convergent. One possibility is the use [BGP$^+$02] of the Seiberg-Witten map [SW99], which I review in Section A.4 (and which does not help either [Wul02]). Another strategy could be the double-scaling limit of matrix models, see e.g. [LSZ04, Ste04, LLS04b] or the construction as limiting cases of the fuzzy sphere [CMS01, VY03, Ydr04].

A further possibility is a more careful way of performing the limits in the spirit of Wilson [WK74] and Polchinski [Pol84]. Early attempts [GP01, Sar02] did not notice the new effects in higher-genus graphs of noncommutative field theories, which are not visible in one-loop calculations. A rigorous treatment exists for the large-θ limit [BGI02, BGI03]. Eventually, the Wilson-Polchinski programme for noncommutative ϕ^4-theory was realised in the series of papers [GW03a, GW03b, GW04c] I have written with Harald Grosse. The main ideas are summarised in [GW04b]. These articles are the basis of the present Habilitation thesis. We achieved the remarkable balance of proving renormalisability of the ϕ^4-model to all orders and reconfirming the UV/IR-duality of [MVRS00].

A.4 θ-expanded field theories

In their famous paper on type IIA string theory in presence of a Neveu-Schwarz B-field [SW99], Seiberg and Witten noticed that passing to the zero-slope limit in two different regularisation schemes (point-splitting and Pauli-Villars) gave rise to a Yang-Mills theory either on noncommutative or on commutative $\mathbb{R}^D$. Since the regularisation scheme cannot matter, Seiberg and Witten argued that both theories must be gauge-equivalent. More general, under an infinitesimal transformation of θ, which can be related to deformation quantisation as in [SW99] or simply to a coordinate rotation [BGG$^+$02], one has to require that gauge-invariant quantities remain gauge-invariant. This requirement leads to the Seiberg-Witten differential equation

$$\frac{dA_\mu}{d\theta_{\rho\sigma}} = -\frac{1}{8}\big\{A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\big\}_\star + \frac{1}{8}\big\{A_\sigma, \partial_\rho A_\mu + F_{\rho\mu}\big\}_\star\,, \tag{A.3}$$

where $\{a,b\}_\star = a\star b + b\star a$.

The differential equation (A.3) is usually solved by integrating it from an initial condition $A^{(0)}$ at $\theta=0$ in the spirit of deformation quantisation [BFF$^+$78a, BFF$^+$78b]. Then, A becomes a formal power series in θ and the initial condition $A^{(0)}$. The solution depends on the path of integration, but the difference between paths is a field redefinition [AK99]. The solution to all orders in θ and lowest order in $A^{(0)}$ was given in [Gar00]. A generating functional for the complete solution of (A.3) was derived in [JSW01]. The Seiberg-Witten approach was popularised in [JSSW00] where it was argued that this is the only way to obtain a finite number of degrees of freedom in non-Abelian noncommutative Yang-Mills theory.

Inserting the solution of the Seiberg-Witten differential equation (A.3) into the noncommutative Yang-Mills action $\int d^Dx F_{\mu\nu}F^{\mu\nu}$ leads to the so-called θ-expanded field theories. It must be stressed, however, that unless a complete solution to all orders in θ and $A^{(0)}$ is known (which is not the case), the θ-expansion of the noncommutative Yang-Mills action describes a *local* field theory. As such, θ-expanded field theories loose the interesting features of the original field theory on the Moyal plane.

The quantum field theoretical treatment of θ-expanded field theories was initiated in [BGP+02]. We have shown that the one-loop divergences to the θ-expanded Maxwell action in second order in θ are gauge-invariant, independent of a linear or a non-linear gauge fixing and independent of the gauge parameter. There is no UV/IR-problem in that approach. We have shown in [BGG+01] that these one-loop divergences can be removed by a field redefinition related to the freedom in the Seiberg-Witten map. In fact, the superficial divergences in the photon self-energy are field redefinitions to all orders in θ and any loop order [BGG+01]. However, I have shown in [Wul02] that θ-expanded field theories are not renormalisable concerning more complicated graphs than the self-energy. On the other hand, I have found in [Wul02] striking evidence for new symmetries in the θ-expanded action which eliminates several divergences expected from the counting of allowed divergences modulo field redefinitions. Finally, we have shown in [GW02] that the use of the θ-expanded $\star$-product (3.3) without application of the Seiberg-Witten map leads (up to field redefinitions) to exactly the same result. Thus, the Seiberg-Witten map is merely an unphysical (but convenient) change of variables [KOS61].

Recently, phenomenological investigations of θ-expanded field theories became popular [CJS+02, BDD+03]. However, quantitative statements are questionable because in presence of a new field $\theta^{\mu\nu}$, many new terms in the action are not only possible but in fact required by renormalisability [GW02] or the desire to cure the UV/IR-problem [Sla03]. Moreover, deformed spaces are too rigid to be a realistic model [Haw02].

A.5 Noncommutative space-time

I have to stress that all mentioned contributions refer to a Euclidean space and a definition of the quantum field theory via the partition function (the Euclidean analogue of the path integral). It was pointed out in [BDFP02] that a simple Wick rotation does *not* give a meaningful theory on Minkowskian space-time, first of all because unitarity is lost [GM00, AGBZ01, CLZ02]. The original proposal [DFR95] of a quantum field theory on noncommutative space-time stayed withing the Minkowskian framework, but later work started from Feynman graphs, the admissibility of a Wick rotation taken (erroneously) for granted. To obtain a consistent Minkowskian quantum field theory, it was proposed in [BDFP02] to iteratively solve the field equations à la Yang-Feldman [YF50]. See also [Bah03]. Other possibilities are a functional formalism for the S-matrix [RY03] and time-ordered perturbation theory [LS02b, LS02c]. See also [BFG+03, DS03]. Unfortunately, the resulting Feynman rules become so complicated that apart from tadpole-like diagrams [BFG+03] it seems impossible to perform perturbative calculations in time-ordered perturbation theory. Moreover, it seems impossible to preserve Ward identities [ORZ04].

On the other hand, the rôle of time in noncommutative geometry is not completely clear. Time should be established around the ideas presented in [CR94]. For general approaches to Minkowskian noncommutative spaces I refer to [Haw97, Str01, KP02]. There is a recent proposal [PV04] to combine spectral geometry with local covariant quantum field theory.

B Matrix formulation of the Moyal plane

B.1 The matrix basis of $\mathbb{R}^2_\theta$

This section contains supplementary material for the treatment of the Moyal plane in Section 3.1. My presentation of the harmonic oscillator base of the Moyal plane follows closely [GBV88], with some notational adaptations. The starting point is the observation that the Gaußian

$$f_{00}(x) = 2\mathrm{e}^{-\frac{1}{\theta_1}(x_1^2+x_2^2)}\;, \tag{B.1}$$

with $\theta_1 \equiv \theta^{12} = -\theta^{21} > 0$, is an idempotent,

$$(f_{00} \star f_{00})(x) = 4\int d^2y \int \frac{d^2k}{(2\pi)^2}\, \mathrm{e}^{-\frac{1}{\theta_1}(2x^2+y^2+2x\cdot y+x\cdot\theta\cdot k+\frac{1}{4}\theta_1^2k^2)+\mathrm{i}k\cdot y} = f_{00}(x)\;. \tag{B.2}$$

We consider creation and annihilation operators

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}(x_1+\mathrm{i}x_2)\;, & \bar{a} &= \frac{1}{\sqrt{2}}(x_1-\mathrm{i}x_2)\;,\\ \frac{\partial}{\partial a} &= \frac{1}{\sqrt{2}}(\partial_1-\mathrm{i}\partial_2)\;, & \frac{\partial}{\partial \bar{a}} &= \frac{1}{\sqrt{2}}(\partial_1+\mathrm{i}\partial_2)\;. \end{aligned} \tag{B.3}$$

For any $f \in \mathbb{R}^2_\theta$ we have

$$\begin{aligned} (a\star f)(x) &= a(x)f(x) + \frac{\theta_1}{2}\frac{\partial f}{\partial \bar{a}}(x)\;, & (f\star a)(x) &= a(x)f(x) - \frac{\theta_1}{2}\frac{\partial f}{\partial \bar{a}}(x)\;,\\ (\bar{a}\star f)(x) &= \bar{a}(x)f(x) - \frac{\theta_1}{2}\frac{\partial f}{\partial a}(x)\;, & (f\star \bar{a})(x) &= \bar{a}(x)f(x) + \frac{\theta_1}{2}\frac{\partial f}{\partial a}(x)\;. \end{aligned} \tag{B.4}$$

This implies $\bar{a}^{\star m} \star f_{00} = 2^m\bar{a}^m f_{00}$, $f_{00}\star a^{\star n} = 2^n a^n f_{00}$ and

$$\begin{aligned} a\star\bar{a}^{\star m}\star f_{00} &= \begin{cases} m\theta_1(\bar{a}^{\star(m-1)}\star f_{00}) & \text{for } m\geq 1\\ 0 & \text{for } m=0\end{cases}\\ f_{00}\star a^{\star n}\star\bar{a} &= \begin{cases} n\theta_1(f_{00}\star a^{\star(n-1)}) & \text{for } n\geq 1\\ 0 & \text{for } n=0\end{cases} \end{aligned} \tag{B.5}$$

where $a^{\star n} = a\star a\star\cdots\star a$ (n factors) and similarly for $\bar{a}^{\star m}$. Now, defining

$$\begin{aligned} f_{mn} &:= \frac{1}{\sqrt{n!m!\,\theta_1^{m+n}}}\,\bar{a}^{\star m}\star f_{00}\star a^{\star n} \qquad\qquad (B.6)\\ &= \frac{1}{\sqrt{n!m!\,\theta_1^{m+n}}}\sum_{k=0}^{\min(m,n)}(-1)^k\binom{m}{k}\binom{n}{k}\,k!\,2^{m+n-2k}\,\theta_1^k\,\bar{a}^{m-k}\,a^{n-k}f_{00}\;, \end{aligned}$$

(the second line is proven by induction) it follows from (B.5) and (B.2) that

$$(f_{mn}\star f_{kl})(x) = \delta_{nk}f_{ml}(x)\;. \tag{B.7}$$

The multiplication rule (B.7) identifies the $\star$-product with the ordinary matrix product:

$$\begin{aligned} a(x) &= \sum_{m,n=0}^{\infty} a_{mn} f_{mn}(x) , & b(x) &= \sum_{m,n=0}^{\infty} b_{mn} f_{mn}(x) \\ \Rightarrow \quad (a\star b)(x) &= \sum_{m,n=0}^{\infty} (ab)_{mn} f_{mn}(x) , & (ab)_{mn} &= \sum_{k=0}^{\infty} a_{mk} b_{kn} . \end{aligned} \tag{B.8}$$

In order to describe elements of $\mathbb{R}^2_\theta$, the sequences $\{a_{mn}\}$ must be of rapid decay [GBV88]:

$$\sum_{m,n=0}^{\infty} a_{mn} f_{mn} \in \mathbb{R}^2_\theta \qquad \text{iff} \qquad \sum_{m,n=0}^{\infty} \big((2m{+}1)^{2k}(2n{+}1)^{2k}|a_{mn}|^2\big)^{\frac{1}{2}} < \infty \quad \text{for all } k . \tag{B.9}$$

Finally, using (B.2), the trace property of the integral and (B.5) we compute

$$\begin{aligned} \int d^2x\, f_{mn}(x) &= \frac{1}{\sqrt{m!n!\,\theta_1^{m+n}}} \int d^2x\, \big(\bar{a}^{\star m} \star f_{00} \star f_{00} \star a^{\star n}\big)(x) = \delta_{mn} \int d^2x f_{00}(x) \\ &= 2\pi\theta_1 \delta_{mn} . \end{aligned} \tag{B.10}$$

Using (B.3)–(B.7) as well as (B.10) we can now compute the kinetic matrix $G_{mn;kl}$ given in (3.43) on page 32 in the two-dimensional case:

$$\begin{aligned} G_{mn;kl} &:= \int \frac{d^2x}{2\pi\theta_1} \Big(\big(\partial_1 f_{mn} \star \partial_1 f_{kl} + \partial_2 f_{mn} \star \partial_2 f_{kl} \\ &\qquad + \frac{4\Omega^2}{\theta_1^2} \big((x_1 f_{mn}) \star (x_1 f_{kl}) + (x_2 f_{mn}) \star (x_2 f_{kl})\big) + \mu_0^2\, f_{mn} \star f_{kl} \Big) \\ &= \int \frac{d^2x}{2\pi\theta_1} \Big(\frac{1{+}\Omega^2}{\theta_1^2} f_{mn} \star (a \star \bar{a} + \bar{a} \star a) \star f_{kl} + \frac{1{+}\Omega^2}{\theta_1^2} f_{kl} \star (a \star \bar{a} + \bar{a} \star a) \star f_{mn} \\ &\qquad - \frac{2(1{-}\Omega^2)}{\theta_1^2} f_{mn} \star a \star f_{kl} \star \bar{a} - \frac{2(1{-}\Omega^2)}{\theta_1^2} f_{kl} \star a \star f_{mn} \star \bar{a} + \mu_0^2\, f_{mn} \star f_{kl} \Big) \\ &= \Big(\mu_0^2 + \frac{2(1{+}\Omega^2)}{\theta_1}(m{+}n{+}1)\Big) \delta_{nk}\delta_{ml} \\ &\qquad - \frac{2(1{-}\Omega^2)}{\theta_1} \sqrt{(n{+}1)(m{+}1)}\, \delta_{n+1,k}\delta_{m+1,l} - \frac{2(1{-}\Omega^2)}{\theta_1} \sqrt{nm}\, \delta_{n-1,k}\delta_{m-1,l} . \end{aligned} \tag{B.11}$$

The functions f_{mn} with $m, n < \mathcal{N}$ provide a cut-off both in position space and momentum space. Passing to radial coordinates $x_1 = \rho\cos\varphi$, $x_2 = \rho\sin\varphi$ we can compare (B.6) with the expansion of Laguerre polynomials [GR00, §8.970.1]:

$$f_{mn}(\rho,\varphi) = 2(-1)^m \sqrt{\tfrac{m!}{n!}}\, \mathrm{e}^{\mathrm{i}\varphi(n-m)} \Big(\sqrt{\tfrac{2}{\theta_1}}\rho\Big)^{n-m} L_m^{n-m}\big(\tfrac{2}{\theta_1}\rho^2\big)\, \mathrm{e}^{-\frac{\rho^2}{\theta_1}} . \tag{B.12}$$

The derivation of (B.12) assumed $m \leq n$, but this restriction can actually be relaxed due to the identity

$$L_{m+\alpha}^{-\alpha}(z) = \frac{m!}{(m{+}\alpha)!} (-1)^\alpha z^\alpha L_m^\alpha(z) . \tag{B.13}$$

The function $L^\alpha_m(z) z^{\alpha/2} \mathrm{e}^{-z/2}$ is rapidly decreasing beyond the last maximum $(z^\alpha_m)_{\max}$. One finds numerically $(z^\alpha_m)_{\max} < 2\alpha + 4m$ and thus the radial cut-off

$$\rho_{\max} \approx \sqrt{2\theta_1 \mathcal{N}} \qquad \text{for } m,n < \mathcal{N}\ . \tag{B.14}$$

On the other hand, for $p_1 = -p\sin\psi$, $p_2 = p\cos\psi$ we compute with (B.12), [GR00, §8.411.1] and [GR00, §7.421.5]

$$\begin{aligned}
\tilde{f}_{mn}(p,\psi) &:= \int_0^\infty \rho\, d\rho \int_0^{2\pi} d\varphi\ \mathrm{e}^{\mathrm{i}p\rho\sin(\varphi-\psi)} f_{mn}(\rho,\varphi) \\
&= 4\pi(-1)^n \sqrt{\tfrac{m!}{n!}} \mathrm{e}^{\mathrm{i}\psi(n-m)} \int_0^\infty \rho d\rho \left(\sqrt{\tfrac{2}{\theta_1}}\rho\right)^{n-m} L^{n-m}_m(\tfrac{2}{\theta_1}\rho^2) J_{n-m}(\rho p)\, \mathrm{e}^{-\frac{\rho^2}{\theta_1}} \\
&= 2\pi\theta_1 \sqrt{\tfrac{m!}{n!}} \mathrm{e}^{\mathrm{i}(\psi+\pi)(n-m)} \left(\sqrt{\tfrac{\theta_1}{2}}\, p\right)^{n-m} L^{n-m}_m(\tfrac{\theta_1}{2}p^2) \mathrm{e}^{-\frac{\theta_1}{4}p^2}\ .
\end{aligned} \tag{B.15}$$

We thus have

$$p_{\max} \approx \sqrt{\frac{8\mathcal{N}}{\theta_1}} \qquad \text{for } m,n < \mathcal{N}\ . \tag{B.16}$$

Finally, I solve the eigenvalue problem (3.7). We compute in radial coordinates

$$\begin{aligned}
\Big(\frac{1}{2}(x_1^2+x_2^2)\Big) \star f &= \Big\{ a\bar{a} + \frac{\theta}{2}\Big(\bar{a}\frac{\partial}{\partial\bar{a}} - a\frac{\partial}{\partial a}\Big) - \frac{\theta^2}{4}\frac{\partial^2}{\partial a\partial\bar{a}}\Big\} f(x_1,x_2) \\
&= \Big(\frac{\rho^2}{2} + \frac{\mathrm{i}\theta}{2}\frac{\partial}{\partial\varphi} - \frac{\theta^2}{8}\Big(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\Big)\Big) f(\rho,\varphi)\ .
\end{aligned} \tag{B.17}$$

The ansatz $f(\rho,\varphi) = \mathrm{e}^{\mathrm{i}\alpha\varphi}\rho^\alpha \mathrm{e}^{-\frac{\rho^2}{\theta}} g(\rho)$ leads to

$$\Big(\frac{1}{2}(x_1^2+x_2^2)\Big) \star f = -\frac{\theta^2}{8}\mathrm{e}^{\mathrm{i}\alpha\varphi}\rho^\alpha \mathrm{e}^{-\frac{\rho^2}{\theta}}\Big(g''(\rho) + \frac{(2\alpha{+}1)}{\rho}g'(\rho) - \frac{4}{\theta}\rho g'(\rho) - \frac{4}{\theta}g(\rho)\Big)\ . \tag{B.18}$$

Now, we put $g(\rho) = h(z)$, with $z = \frac{2\rho^2}{\theta}$, and obtain

$$\Big(\frac{1}{2}(x_1^2+x_2^2)\Big) \star f = -\theta \mathrm{e}^{\mathrm{i}\alpha\varphi}\rho^\alpha \mathrm{e}^{-\frac{\rho^2}{\theta}}\Big(zh''(z) + (\alpha{+}1{-}z)h'(z) - \frac{1}{2}h(z)\Big)\ . \tag{B.19}$$

In order to get an eigenvalue problem we have to require $zh''(z) + (\alpha{+}1{-}z)h'(z) = -mh(z)$, the solution of which is according to [GR00, §8.979] given by Laguerre polynomials:

$$\begin{aligned}
&\Big(\frac{1}{2}(x_1^2+x_2^2)\Big) \star f^{(\alpha)}_m = \theta\Big(m+\frac{1}{2}\Big) f^{(\alpha)}_m\ , \\
&\qquad f^{(\alpha)}_m(\rho,\varphi) = 2(-1)^m \sqrt{\frac{m!}{\Gamma(m{+}\alpha{+}1)}}\mathrm{e}^{\mathrm{i}\alpha\varphi}\Big(\frac{2\rho^2}{\theta}\Big)^{\frac{\alpha}{2}} \mathrm{e}^{-\frac{\rho^2}{\theta}} L^\alpha_m\Big(\frac{2\rho^2}{\theta}\Big)\ .
\end{aligned} \tag{B.20}$$

The prefactors ensure $\frac{1}{2\pi\theta}\int_0^\infty \rho d\rho \int_0^{2\pi} d\varphi\ |f^{(\alpha)}_m|^2 = 1$. The eigenvalue problem for right $\star$-multiplication by $\frac{1}{2}(x_1^2{+}x_2^2)$ is obtained by complex conjugation of (B.20) and replacement

$\alpha \mapsto -\beta$:

$$
\begin{aligned}
f_n^{(\beta)} \star \Big(\frac{1}{2}(x_1^2+x_2^2)\Big) &= \theta\Big(n+\frac{1}{2}\Big) f_n^{(\beta)} \;, \\
f_n^{(\beta)}(\rho,\varphi) &= 2(-1)^n \sqrt{\frac{n!}{\Gamma(n-\beta+1)}} \mathrm{e}^{\mathrm{i}\beta\varphi} \Big(\frac{2\rho^2}{\theta}\Big)^{-\frac{\beta}{2}} \mathrm{e}^{-\frac{\rho^2}{\theta}} L_n^{-\beta}\Big(\frac{2\rho^2}{\theta}\Big) \;. \qquad \text{(B.21)}
\end{aligned}
$$

Using (B.13) we identify $\beta = n-m$ and conclude that left and right $\star$-multiplication by $\frac{1}{2}(x_1^2+x_2^2)$ have common eigentransitions (B.12) with eigenvalues given by (3.7).

B.2 Diagonalisation of the kinetic matrix via Meixner polynomials

Our goal is to diagonalise the (four-dimensional) kinetic matrix $G_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}$ given in (3.45) on page 33, making use of the index conservation $\alpha^r = n^r - m^r = k^r - l^r$. For $\alpha^r \geq 0$ we thus look for a representation

$$
\begin{aligned}
G_{\substack{m^1\\m^2}\substack{m^1+\alpha^1\\m^2+\alpha^2};\substack{l^1+\alpha^1\\l^2+\alpha^2}\substack{l^1\\l^2}} &= \sum_{i^1,i^2} U^{(\alpha^1)}_{m^1 i^1} U^{(\alpha^2)}_{m^2 i^2} \big(\tfrac{2}{\theta_1} v_{i^1} + \tfrac{2}{\theta_2} v_{i^2} + \mu_0^2\big) U^{(\alpha^1)}_{i^1 l^1} U^{(\alpha^2)}_{i^2 l^2} \;, \qquad \text{(B.22)}\\
\delta_{ml} &= \sum_i U^{(\alpha)}_{mi} U^{(\alpha)}_{il} \;. \qquad \text{(B.23)}
\end{aligned}
$$

The sum over i^1, i^2 would be an integration for continuous eigenvalues v_{i^r}. Comparing this ansatz with (3.45) we obtain, eliminating i in favour of v, the recurrence relation

$$
\begin{aligned}
(1-\Omega^2)\sqrt{m(\alpha+m)}U^{(\alpha)}_{m-1}(v) &+ \big(v - (1+\Omega^2)(\alpha+1+2m)\big)U^{(\alpha)}_m(v) \\
&+ (1-\Omega^2)\sqrt{(m+1)(\alpha+m+1)}U^{(\alpha)}_{m+1}(v) = 0 \qquad \text{(B.24)}
\end{aligned}
$$

to determine $U^{(\alpha)}_m(v)$ and v. We consider the case $\Omega > 0$ and repeat the analysis for $\Omega = 0$ in Appendix B.6. In order to make contact with standard formulae we put

$$
U^{(\alpha)}_m(v) = f^{(\alpha)}(v)\,\frac{1}{\tau^m}\sqrt{\frac{(\alpha+m)!}{m!}} V^{(\alpha)}_m(v) \;, \qquad v = \nu x + \rho \;. \qquad \text{(B.25)}
$$

We obtain after division by $f^{(\alpha)}(v)\sqrt{\frac{(\alpha+m)!}{m!}}$

$$
\begin{aligned}
-\frac{\nu}{\tau(1-\Omega^2)} x V^{(\alpha)}_m(\nu x+\rho) &= m V^{(\alpha)}_{m-1}(\nu x+\rho) - \frac{(1+\Omega^2)(\alpha+1+2m)-\rho}{\tau(1-\Omega^2)} V^{(\alpha)}_m(\nu x+\rho) \\
&+ \frac{1}{\tau^2}(\alpha+m+1)V^{(\alpha)}_{m+1}(\nu x+\rho) \;. \qquad \text{(B.26)}
\end{aligned}
$$

Now, we put

$$
1+\alpha = \beta \;, \quad \frac{1}{\tau^2} = c \;, \quad \frac{2(1+\Omega^2)}{\tau(1-\Omega^2)} = 1+c \;, \quad \frac{(1+\Omega^2)\beta - \rho}{\tau(1-\Omega^2)} = \beta c \;, \quad \frac{\nu}{(1-\Omega^2)\tau} = 1-c \quad \text{(B.27)}
$$

and

$$V_n^{(\alpha)}(\nu x+\rho) = M_n(x;\beta,c) \;, \tag{B.28}$$

which yields the recursion relation for the Meixner polynomials [KS96]:

$$\begin{aligned}(c-1)xM_m(x;\beta,c) &= c(m+\beta)M_{m+1}(x;\beta,c)\\ &\quad - (m+(m+\beta)c)M_m(x;\beta,c) + mM_{m-1}(x;\beta,c) \;.\end{aligned} \tag{B.29}$$

The solution of (B.27) is

$$\tau = \frac{(1\pm\Omega)^2}{1-\Omega^2} \equiv \frac{1\pm\Omega}{1\mp\Omega} \;, \qquad c = \frac{(1\mp\Omega)^2}{(1\pm\Omega)^2} \;, \qquad \nu = \pm 4\Omega \;, \qquad \rho = \pm 2\Omega(1+\alpha) \;. \tag{B.30}$$

We have to chose the upper sign, because the eigenvalues v are positive. We thus obtain

$$\begin{aligned}U_m^{(\alpha)}(v_x) &= f^{(\alpha)}(x)\sqrt{\frac{(\alpha+n)!}{n!}}\Big(\frac{1-\Omega}{1+\Omega}\Big)^m M_m\Big(x;1+\alpha,\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big) \;,\\ v_x &= 2\Omega(2x+\alpha+1) \;.\end{aligned} \tag{B.31}$$

The function $f^{(\alpha)}(x)$ is identified by comparison of (B.23) with the orthogonality relation of Meixner polynomials [KS96],

$$\sum_{x=0}^{\infty} \frac{\Gamma(\beta+x)c^x}{\Gamma(\beta)x!} M_m(x;\beta,c)M_n(x;\beta,c) = \frac{c^{-n}n!\Gamma(\beta)}{\Gamma(\beta+n)(1-c)^\beta}\delta_{mn} \;. \tag{B.32}$$

The result is

$$U_m^{(\alpha)}(v_x) = \sqrt{\binom{\alpha+m}{m}\binom{\alpha+x}{x}}\Big(\frac{2\sqrt{\Omega}}{1+\Omega}\Big)^{\alpha+1}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{m+x} M_m\Big(x;1+\alpha,\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big) \;. \tag{B.33}$$

The Meixner polynomials can be represented by hypergeometric functions [KS96],

$$M_m\Big(x;1+\alpha,\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big) = {}_2F_1\Big(\begin{matrix}-m,-x\\ 1+\alpha\end{matrix}\Big|-\frac{4\Omega}{(1-\Omega)^2}\Big) \;. \tag{B.34}$$

This shows that the matrices $U_{ml}^{(\alpha)}$ in (B.22) and (B.23) are symmetric in the lower indices.

B.3 Evaluation of the propagator

Now, we return to the computation of the propagator, which is obtained by sandwiching the inverse eigenvalues $(\frac{2}{\theta_1}v_{i^1} + \frac{2}{\theta_2}v_{i^2} + \mu_0^2)$ between the unitary matrices $U^{(\alpha)}$. With (B.31) and the use of Schwinger's trick $\frac{1}{A} = \int_0^\infty dt\, \mathrm{e}^{-tA}$ we have for $\theta_1 = \theta_2 = \theta$

$$
\begin{aligned}
&\Delta_{\substack{m^1 \\ m^2}\substack{m^1+\alpha^1 \\ m^2+\alpha^2};\substack{l^1+\alpha^1 \\ l^2+\alpha^2}\substack{l^1 \\ l^2}} \\
&= \frac{\theta}{8\Omega}\int_0^\infty dt \sum_{x^1,x^2=0}^\infty \mathrm{e}^{-\frac{t}{4\Omega}(v_{x^1}+v_{x^2}+\theta\mu_0^2/2)} U_{m^1}^{(\alpha^1)}(v_{x^1}) U_{m^2}^{(\alpha^2)}(v_{x^2}) U_{l^1}^{(\alpha^1)}(v_{x^1}) U_{l^2}^{(\alpha^2)}(v_{x^2}) \\
&= \frac{\theta}{8\Omega}\int_0^\infty dt\; \mathrm{e}^{-t(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2))} \prod_{i=1}^2 \Bigg\{ \sqrt{\binom{\alpha^i+m^i}{m^i}\binom{\alpha^i+l^i}{l^i}} \Big(\frac{4\Omega}{(1+\Omega)^2}\Big)^{\alpha^i+1} \Big(\frac{1-\Omega}{1+\Omega}\Big)^{m^i+l^i} \\
&\times \sum_{x^i=0}^\infty \frac{(\alpha^i+x^i)!}{x^i!\alpha^i!} \Big(\frac{\mathrm{e}^{-t}(1-\Omega)^2}{(1+\Omega)^2}\Big)^{x^i} {}_2F_1\Big(\begin{matrix} -m^i, -x^i \\ 1+\alpha^i \end{matrix}\Big| -\frac{4\Omega}{(1-\Omega)^2}\Big) {}_2F_1\Big(\begin{matrix} -l^i, -x^i \\ 1+\alpha^i \end{matrix}\Big| -\frac{4\Omega}{(1-\Omega)^2}\Big) \Bigg\} .
\end{aligned}
\tag{B.35}
$$

We use the following identity for hypergeometric functions,

$$
\begin{aligned}
&\sum_{x=0}^\infty \frac{(\alpha+x)!}{x!\alpha!} a^x\, {}_2F_1\Big(\begin{matrix} -m, -x \\ 1+\alpha \end{matrix}\Big| b\Big) {}_2F_1\Big(\begin{matrix} -l, -x \\ 1+\alpha \end{matrix}\Big| b\Big) \\
&\qquad = \frac{(1-(1-b)a)^{m+l}}{(1-a)^{\alpha+m+l+1}} {}_2F_1\Big(\begin{matrix} -m\,,\ -l \\ 1+\alpha \end{matrix}\Big| \frac{ab^2}{(1-(1-b)a)^2}\Big) , \qquad |a| < 1 .
\end{aligned}
\tag{B.36}
$$

I give in Section B.5 the proof of (B.36), which is crucial for the solution of the free theory. The identity (B.36) is probably known, but I did not find any reference. We insert the rhs of (B.36), expanded as a finite sum, into (B.35), where we also put $z = \mathrm{e}^{-t}$:

$$
\begin{aligned}
&\Delta_{\substack{m^1 \\ m^2}\substack{m^1+\alpha^1 \\ m^2+\alpha^2};\substack{l^1+\alpha^1 \\ l^2+\alpha^2}\substack{l^1 \\ l^2}} \\
&= \frac{\theta}{8\Omega} \sum_{u^1=0}^{\min(m^1,l^1)} \sum_{u^2=0}^{\min(m^2,l^2)} \int_0^1 dz\, \frac{z^{\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+u^1+u^2} (1-z)^{m^1+m^2+l^1+l^2-2u^1-2u^2}}{\big(1-\frac{(1-\Omega)^2}{(1+\Omega)^2}z\big)^{\alpha^1+\alpha^2+m^1+m^2+l^1+l^2+2}} \\
&\times \prod_{i=1}^2 \Bigg\{ \Big(\frac{4\Omega}{(1+\Omega)^2}\Big)^{\alpha^i+2u^i+1} \Big(\frac{1-\Omega}{1+\Omega}\Big)^{m^i+l^i-2u^i} \frac{\sqrt{m^i!(\alpha^i+m^i)!l^i!(\alpha^i+l^i)!}}{(m^i-u^i)!(l^i-u^i)!(\alpha^i+u^i)!u^i!} \Bigg\} .
\end{aligned}
\tag{B.37}
$$

This formula tells us the important property

$$
0 \leq \Delta_{\substack{m^1 \\ m^2}\substack{m^1+\alpha^1 \\ m^2+\alpha^2};\substack{l^1+\alpha^1 \\ l^2+\alpha^2}\substack{l^1 \\ l^2}} \leq \Delta_{\substack{m^1 \\ m^2}\substack{m^1+\alpha^1 \\ m^2+\alpha^2};\substack{l^1+\alpha^1 \\ l^2+\alpha^2}\substack{l^1 \\ l^2}}\Big|_{\mu_0^2=0} ,
\tag{B.38}
$$

i.e. all matrix elements of the propagator are positive and majorised by the massless matrix elements. The representation (B.37) seems to be the most convenient one for analytical estimations of the propagator. The strategy[37] would be to divide the integration domain into slices and to maximise the individual z-dependent terms of the integrand over the slice, followed by resummation. However, this procedure is not so easy, and so far I cannot offer the result.

[37] I am grateful to Vincent Rivasseau for this idea, which is inspired by techniques used in constructive renormalisation [Riv91].

The z-integration in (B.37) leads according to [GR00, §9.111] again to a hypergeometric function:

$$\begin{aligned}
&\Delta_{\substack{m^1\ m^1+\alpha^1\\ m^2\ m^2+\alpha^2};\substack{l^1+\alpha^1\ l^1\\ l^2+\alpha^2\ l^2}}\\
&= \frac{\theta}{8\Omega}\sum_{u^1=0}^{\min(m^1,l^1)}\sum_{u^2=0}^{\min(m^2,l^2)} \frac{\Gamma\big(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+u^1+u^2\big)(m^1+m^2+l^1+l^2-2u^1-2u^2)!}{\Gamma\big(2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2+l^1+l^2-u^1-u^2\big)}\\
&\times {}_2F_1\left(\begin{matrix}1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+u^1+u^2\,,\ 2+m^1+m^2+l^1+l^2+\alpha^1+\alpha^2\\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2+l^1+l^2-u^1-u^2\end{matrix}\,\middle|\,\frac{(1-\Omega)^2}{(1+\Omega)^2}\right)\\
&\times\prod_{i=1}^{2}\left\{\Big(\frac{4\Omega}{(1+\Omega)^2}\Big)^{\alpha^i+2u^i+1}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{m^i+l^i-2u^i}\frac{\sqrt{m^i!(\alpha^i+m^i)!l^i!(\alpha^i+l^i)!}}{(m^i-u^i)!(l^i-u^i)!(\alpha^i+u^i)!u^i!}\right\}\\
&= \frac{\theta}{2(1+\Omega)^2}\sum_{u^1=0}^{\min(m^1,l^1)}\sum_{u^2=0}^{\min(m^2,l^2)}\prod_{i=1}^{2}\sqrt{\binom{\alpha^i+m^i}{\alpha^i+u^i}\binom{\alpha^i+l^i}{\alpha^i+u^i}\binom{m^i}{u^i}\binom{l^i}{u^i}}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{m^i+l^i-2u^i}\\
&\times B\big(1+\tfrac{\mu_0^2\theta}{8\Omega}+\tfrac{1}{2}(\alpha^1+\alpha^2)+u^1+u^2, 1+m^1+m^2+l^1+l^2-2u^1-2u^2\big)\\
&\times {}_2F_1\left(\begin{matrix}1+m^1+m^2+l^1+l^2-2u^1-2u^2\,,\ \frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(\alpha^1+\alpha^2)-u^1-u^2\\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2+l^1+l^2-u^1-u^2\end{matrix}\,\middle|\,\frac{(1-\Omega)^2}{(1+\Omega)^2}\right). \qquad \text{(B.39)}
\end{aligned}$$

I have used [GR00, §9.131.1] to obtain the last line. The form (B.39) will be useful for the evaluation of special cases and of the asymptotic behaviour. In the main part, for presentational purposes, α^i is eliminated in favour of k^i, n^i and the summation variable $v^i := m^i + l^i - 2u^i$ is used. The final result is given in (3.49) on page 33.

For $\mu_0 = 0$ the sum over u^i can be evaluated exactly in a few cases. First, for $l^i = 0$ we also have $u^i = 0$. If additionally $\alpha^i = 0$ we get

$$\Delta_{\substack{m^1\ m^1\\ m^2\ m^2};\substack{0\ 0\\ 0\ 0}}\Big|_{\mu_0=0} = \frac{\theta}{2(1+\Omega)^2(1+m^1+m^2)}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{m^1+m^2}. \qquad \text{(B.40)}$$

One should notice here the exponential decay for $\Omega > 0$. It can be seen numerically that this is a general feature of the propagator: Given m^i and α^i, the maximum of the propagator is attained at $l^i = m^i$. Moreover, the decay with $|l^i - m^i|$ is exponentially so that the sum

$$\sum_{l^1,l^2}\Delta_{\substack{m^1\ m^1+\alpha^1\\ m^2\ m^2+\alpha^2};\substack{l^1+\alpha^1\ l^1\\ l^2+\alpha^2\ l^2}} \qquad \text{(B.41)}$$

converges. I confirm this argumentation numerically in (F.3).

It turns out numerically that the maximum of the propagator for indices restricted by $\mathcal{C} \leq \max(m^1, m^2, n^1, n^2, k^1, k^2, l^1, l^2) \leq 2\mathcal{C}$ is found in the subclass $\Delta_{\substack{m^1\ n^1\\ 0\ \ 0};\substack{n^1\ m^1\\ 0\ \ 0}}$ of propagators. Coincidently, the computation in case of $m^2 = l^2 = \alpha^2 = 0$ simplifies considerably. If additionally $m^1 = n^1$ we obtain a closed result:

$$\begin{aligned}
\Delta_{\substack{m\ m\\ 0\ 0};\substack{m\ m\\ 0\ 0}} &= \frac{\theta}{2(1+\Omega)^2}\sum_{u=0}^{m}\frac{(m!)^2(2u)!}{(m-u)!(u!)^2(1+m+u)!}\,{}_2F_1\left(\begin{matrix}1+2u\,,\ u-m\\ 2+m+u\end{matrix}\,\middle|\,\frac{(1-\Omega)^2}{(1+\Omega)^2}\right)\Big(\frac{1-\Omega}{1+\Omega}\Big)^{2u}\\
&= \frac{\theta}{2(1+\Omega)^2}\sum_{u=0}^{m}\sum_{s=0}^{m-u}(-1)^s\frac{(m!)^2(2u+s)!}{(m-u-s)!(u!)^2(1+m+u+s)!s!}\Big(\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)^{u+s}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\theta}{2(1{+}\Omega)^2} \sum_{u=0}^{m} \sum_{r=u}^{m} (-1)^{u+r} \frac{(m!)^2 (r{+}u)!}{(m{-}r)!(u!)^2(1{+}m{+}r)!(r{-}u)!} \Big(\frac{(1{-}\Omega)^2}{(1{+}\Omega)^2}\Big)^r \\
&= \frac{\theta}{2(1{+}\Omega)^2} \sum_{r=0}^{m} (-1)^r \frac{(m!)^2}{(m{-}r)!(1{+}m{+}r)!}\; {}_2F_1\Big(\begin{matrix} r{+}1\,, \; -r \\ 1 \end{matrix}\Big| 1\Big) \Big(\frac{(1{-}\Omega)^2}{(1{+}\Omega)^2}\Big)^r \\
&= \frac{\theta}{2(1{+}\Omega)^2} \sum_{r=0}^{m} \frac{(m!)^2}{(m{-}r)!(1{+}m{+}r)!} \Big(\frac{(1{-}\Omega)^2}{(1{+}\Omega)^2}\Big)^r \\
&= \frac{\theta}{2(1{+}\Omega)^2(m{+}1)}\; {}_2F_1\Big(\begin{matrix} 1\,, \; -m \\ m{+}2 \end{matrix}\Big| -\frac{(1{-}\Omega)^2}{(1{+}\Omega)^2}\Big) \\
&\sim \begin{cases} \dfrac{\theta}{8\Omega(m{+}1)} & \text{for } \Omega > 0\,,\; m \gg 1\,, \\[2ex] \dfrac{\sqrt{\pi}\theta}{4\sqrt{m{+}\frac{3}{4}}} & \text{for } \Omega = 0\,,\; m \gg 1\,. \end{cases}
\end{aligned}
\tag{B.42}
$$

We see a crucial difference in the asymptotic behaviour for $\Omega > 0$ versus $\Omega = 0$. The slow decay with $m^{-\frac{1}{2}}$ of the propagator is responsible for the non-renormalisability of the ϕ^4-model in case of $\Omega = 0$. The numerical result (F.2) on page 145 shows that the maximum of the propagator for indices restricted by $\mathcal{C} \leq \max(m^1, m^2, n^1, n^2, k^1, k^2, l^1, l^2) \leq 2\mathcal{C}$ is very close to the result (B.42), for $m = \mathcal{C}$. For $\Omega = 0$ the maximum is exactly given by the 6$^{\text{th}}$ line of (B.42).

B.4 Asymptotic behaviour of the propagator for large α^i

I consider various limiting cases of the propagator, making use of the asymptotic expansion (Stirling's formula) of the Γ-function,

$$
\Gamma(n{+}1) \sim \Big(\frac{n}{\mathrm{e}}\Big)^n \sqrt{2\pi(n + \tfrac{1}{6})} + \mathcal{O}(n^{-2})\,. \tag{B.43}
$$

This implies

$$
\frac{\Gamma(n{+}1{+}a)}{\Gamma(n{+}1{+}b)} \sim n^{a-b}\Big(1 + \frac{(a{-}b)(a{+}b{+}1)}{2n} + \mathcal{O}(n^{-2})\Big)\,. \tag{B.44}
$$

I rewrite the propagator (B.39) in a manner where the large-α^i behaviour is easier to discuss:

$$
\begin{aligned}
&\Delta_{\substack{m^1 \\ m^2} \substack{m^1+\alpha^1 \\ m^2+\alpha^2} ; \substack{l^1+\alpha^1 \\ l^2+\alpha^2} \substack{l^1 \\ l^2}} \\
&= \sum_{u^1=0}^{\min(m^1,l^1)} \sum_{u^2=0}^{\min(m^2,l^2)} \frac{\theta}{2(1{+}\Omega)^2(1{+}\frac{\mu_0^2\theta}{8\Omega}{+}\frac{1}{2}(\alpha^1{+}\alpha^2){+}m^1{+}m^2{+}l^1{+}l^2{-}u^1{-}u^2)} \\
&\times \frac{(m^1{+}m^2{+}l^1{+}l^2{-}2u^1{-}2u^2)!\sqrt{m^1!l^1!m^2!l^2!}}{(m^1{-}u^1)!(l^1{-}u^1)!u^1!(m^2{-}u^2)!(l^2{-}u^2)!u^2!} \Big(\frac{1{-}\Omega}{1{+}\Omega}\Big)^{m^1+m^2+l^1+l^2-2u^1-2u^2}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big\{\frac{\Gamma\big(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+u^1+u^2\big)}{\Gamma\big(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2+l^1+l^2-u^1-u^2\big)}\\
&\qquad\qquad\times\sqrt{\frac{(\alpha^1+m^1)!(\alpha^1+l^1)!}{(\alpha^1+u^1)!(\alpha^1+u^1)!}}\sqrt{\frac{(\alpha^2+m^2)!(\alpha^2+l^2)!}{(\alpha^2+u^2)!(\alpha^2+u^2)!}}\Big\}\\
&\times {}_2F_1\bigg(\genfrac{}{}{0pt}{}{1+m^1+m^2+l^1+l^2-2u^1-2u^2\,,\;\frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(\alpha^1+\alpha^2)-u^1-u^2}{2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2+l^1+l^2-u^1-u^2}\bigg|\frac{(1-\Omega)^2}{(1+\Omega)^2}\bigg)\,. \qquad \text{(B.45)}
\end{aligned}
$$

We assume $\frac{1}{2}(\alpha_1+\alpha_2)\geq\max(\frac{\mu_0^2\theta}{8\Omega},m,l)$. The term in braces $\{\ \}$ in (B.45) behaves like

$$
\begin{aligned}
\{\dots\}&\sim\big(\tfrac{1}{2}(\alpha^1+\alpha^2)\big)^{2u^1+2u^2-m^1-m^2-l^2-l^2}(\alpha^1)^{\frac{1}{2}(m^1+l^1-2u^1)}(\alpha^2)^{\frac{1}{2}(m^2+l^2-2u^1)}\\
&\times\Big(1+\frac{(2u^1+2u^2-m^1-m^2-l^2-l^2)(m^1+m^2+l^1+l^2+\frac{\mu_0^2\theta}{4\Omega}+1)}{(\alpha^1+\alpha^2)}+\mathcal{O}\big((\alpha^1+\alpha^2)^{-2}\big)\Big)\\
&\times\Big(1+\frac{(m^1-u^1)(m^1+u^1+1)}{4\alpha^1}+\frac{(l^1-u^1)(l^1+u^1+1)}{4\alpha^1}+\mathcal{O}\big((\alpha^1)^{-2}\big)\Big)\\
&\times\Big(1+\frac{(m^2-u^2)(m^2+u^2+1)}{4\alpha^2}+\frac{(l^2-u^2)(l^2+u^2+1)}{4\alpha^1}+\mathcal{O}\big((\alpha^2)^{-2}\big)\Big)\,. \qquad \text{(B.46)}
\end{aligned}
$$

We look for the maximum of the propagator under the condition $\mathcal{C}\leq\max(\alpha^1,\alpha^2)\leq 2\mathcal{C}$. Defining $s^i=m^i+l^i-2u^i$, the dominating term in (B.46) is

$$
\frac{(\alpha^1)^{\frac{s^1}{2}}(\alpha^2)^{\frac{s^2}{2}}}{\big(\frac{1}{2}(\alpha^1+\alpha^2)\big)^{s^1+s^2}}\bigg|_{\mathcal{C}\leq\max(\alpha^1,\alpha^2)\leq2\mathcal{C}}\leq\frac{\max\Big(\frac{(\frac{s^1}{s^1+2s^2})^{\frac{s^1}{2}}}{(\frac{s^1+s^2}{s^1+2s^2})^{s^1+s^2}},\frac{(\frac{s^2}{s^2+2s^1})^{\frac{s^2}{2}}}{(\frac{s^1+s^2}{s^2+2s^1})^{s^1+s^2}}\Big)}{\mathcal{C}^{\frac{s^1+s^2}{2}}}\,. \qquad \text{(B.47)}
$$

The maximum is attained at $(\alpha^1,\alpha^2)=(\frac{s^1\mathcal{C}}{s^1+2s^2},\mathcal{C})$ for $s^1\leq s^2$ and at $(\alpha^1,\alpha^2)=(\mathcal{C},\frac{s^2\mathcal{C}}{s^2+2s^1})$ for $s^1\geq s^2$. Thus, the leading contribution to the propagator will come from the summation index $u^i=\min(m^i,l^i)$.

Next, I evaluate the leading contribution of the hypergeometric function:

$$
\begin{aligned}
&{}_2F_1\bigg(\genfrac{}{}{0pt}{}{1+m^1+m^2+l^1+l^2-2u^1-2u^2\,,\;\frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(\alpha^1+\alpha^2)-u^1-u^2}{2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2+l^1+l^2-u^1-u^2}\bigg|\frac{(1-\Omega)^2}{(1+\Omega)^2}\bigg)\\
&\sim\sum_{k=0}^{\infty}\frac{(m^1+m^2+l^1+l^2-2u^1-2u^2+k)!}{(m^1+m^2+l^1+l^2-2u^1-2u^2)!}\frac{\big(-\frac{(1-\Omega)^2}{(1+\Omega)^2}\big)^k}{k!}\Big(1+\frac{k(2u^1+2u^2-\frac{\mu_0^2\theta}{4\Omega}+1-k)}{\alpha^1+\alpha^2}\\
&\qquad\qquad-\frac{k(3+2m^1+2m^2+2l^1+2l^2-2u^1-2u^2+\frac{\mu_0^2\theta}{4\Omega}+k)}{\alpha^1+\alpha^2}+\mathcal{O}\big((\alpha^1+\alpha^2)^{-2}\big)\Big)\\
&=\sum_{k=0}^{\infty}\frac{(s+k)!}{s!}\Big(1-\frac{2k(1+s+\frac{\mu_0^2\theta}{4\Omega}+k)}{\alpha^1+\alpha^2}\Big)\frac{\big(-\frac{(1-\Omega)^2}{(1+\Omega)^2}\big)^k}{k!}\bigg|_{s=(m^1+m^2+l^1+l^2-2u^1-2u^2)}\\
&=\Big(\frac{(1+\Omega)^2}{2(1+\Omega^2)}\Big)^{1+s}\Bigg(1+\frac{\frac{(1-\Omega)^2}{1+\Omega^2}(1+s)}{(\alpha^1+\alpha^2)}\Big(1+\frac{\mu_0^2\theta}{4\Omega}+\frac{s}{2}+(s+2)\frac{\Omega}{(1+\Omega^2)}\Big)\Bigg)\bigg|_{s=(m^1+m^2+l^1+l^2-2u^1-2u^2)}\\
&\qquad+\mathcal{O}\big((\alpha^1+\alpha^2)^{-2}\big)\,. \qquad \text{(B.48)}
\end{aligned}
$$

Assuming $s^1 \leq s^2$, we obtain from (B.43), (B.47) and (B.48) the following leading contribution to the propagator (B.45)

$$\begin{aligned}
&\Delta_{\substack{m^1 \\ m^2}\substack{m^1+\alpha^1 \\ m^2+\alpha^2};\substack{l^1+\alpha^1 \\ l^2+\alpha^2}\substack{l^1 \\ l^2}}\Big|_{\max(m^1,m^2,l^1,l^2)\ll\mathcal{C}\leq\max(\alpha^1,\alpha^2)\leq 2\mathcal{C}} \\
&= \frac{\theta\big(\max(m^1,l^1)\big)^{\frac{s^1}{2}}\big(\max(m^2,l^2)\big)^{\frac{s^2}{2}}}{(1+\Omega)^2\mathcal{C}^{1+\frac{s^1+s^2}{2}}}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{s^1+s^2}\Big(\frac{(1+\Omega)^2}{2(1+\Omega^2)}\Big)^{1+s^1+s^2} \\
&\qquad\times\frac{(s^1+s^2)^{s^1+s^2}\sqrt{2\pi(s^1+s^2)}}{(s^1)^{s^1}(s^2)^{s^2}2\pi\sqrt{s^1s^2}}\,\frac{\big(\frac{s^1}{s^1+2s^2}\big)^{\frac{s^1}{2}}}{\big(\frac{s^1+s^2}{s^1+2s^2}\big)^{1+s^1+s^2}}\Big(1+\mathcal{O}(\mathcal{C}^{-1})\Big)\Big|_{s^i:=|m^i-l^i|}\;. \qquad (B.49)
\end{aligned}$$

The denominator comes from $\sqrt{\frac{m!}{l!}} \leq m^{\frac{m-l}{2}}$ for $m \geq l$. The estimation (B.49) is the explanation of (5.22) on page 57.

Let us now look at propagators with $m^i = l^i$ and $m^i \ll \mathcal{C} \leq \max(\alpha^1,\alpha^2) \leq 2\mathcal{C}$:

$$\begin{aligned}
&\Delta_{\substack{m^1 \\ m^2}\substack{m^1+\alpha^1 \\ m^2+\alpha^2};\substack{m^1+\alpha^1 \\ m^2+\alpha^2}\substack{m^1 \\ m^2}} \\
&= \frac{\theta}{2(1+\Omega)^2\big(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2\big)}\bigg(\frac{(1+\Omega)^2}{2(1+\Omega^2)}+\frac{\big(\frac{1-\Omega^2}{1+\Omega^2}\big)^2}{2(\alpha_1+\alpha_2)}\Big(1+\frac{\mu_0^2\theta}{4\Omega}+\frac{2\Omega}{(1+\Omega^2)}\Big)\bigg) \\
&+\frac{\theta}{2(1+\Omega)^2\big(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2+1\big)}\Big(\frac{1-\Omega^2}{1+\Omega^2}\Big)^2\frac{(1+\Omega)^2}{1+\Omega^2}\,\frac{m^1\alpha^1+m^2\alpha^2}{(\alpha^1+\alpha^2)^2} \\
&+\mathcal{O}\big((\alpha^1+\alpha^2)^{-3}\big)\;. \qquad (B.50)
\end{aligned}$$

This means

$$\begin{aligned}
&\Delta_{\substack{m_1 \\ m_2}\substack{m_1+\alpha_1 \\ m_2+\alpha_2};\substack{m_1+\alpha_1 \\ m_2+\alpha_2}\substack{m_1 \\ m_2}} - \Delta_{\substack{0 \\ 0}\substack{m^1+\alpha_1 \\ m^2+\alpha_2};\substack{m^1+\alpha_1 \\ m^2+\alpha_2}\substack{0 \\ 0}} \\
&= -(m_1+m_2)\frac{\theta}{8(1+\Omega^2)\big(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2+m^1+m^2)\big)^2} \\
&+\frac{\theta}{2(1+\Omega^2)\big(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2+m^1+m^2)\big)}\Big(\frac{1-\Omega^2}{1+\Omega^2}\Big)^2\frac{m^1(\alpha^1+m^1)+m^2(\alpha^2+m^2)}{(\alpha^1+\alpha^2+m^1+m^2)^2} \\
&+\mathcal{O}\Big(\frac{1}{(\alpha^1+\alpha^2+m^1+m^2)}\frac{(m^1+m^2)^2}{(\alpha^1+\alpha^2+m^1+m^2)^2}\Big) \\
&= m_1\Big(\Delta_{\substack{1 \\ 0}\substack{m_1+\alpha_1 \\ m_2+\alpha_2};\substack{m_1+\alpha_1 \\ m_2+\alpha_2}\substack{1 \\ 0}} - \Delta_{\substack{0 \\ 0}\substack{m_1+\alpha_1 \\ m_2+\alpha_2};\substack{m_1+\alpha_1 \\ m_2+\alpha_2}\substack{0 \\ 0}}\Big) + m_2\Big(\Delta_{\substack{0 \\ 1}\substack{m_1+\alpha_1 \\ m_2+\alpha_2};\substack{m_1+\alpha_1 \\ m_2+\alpha_2}\substack{0 \\ 1}} - \Delta_{\substack{0 \\ 0}\substack{m_1+\alpha_1 \\ m_2+\alpha_2};\substack{m_1+\alpha_1 \\ m_2+\alpha_2}\substack{0 \\ 0}}\Big) \\
&+\mathcal{O}\Big(\frac{1}{(\alpha^1+\alpha^2+m^1+m^2)}\frac{(m^1+m^2)^2}{(\alpha^1+\alpha^2+m^1+m^2)^2}\Big)\;. \qquad (B.51)
\end{aligned}$$

The second and third line of (B.51) explains the estimation (5.26) on page 58. Clearly, the next term in the expansion is of the order $\frac{(m^1+m^2)^2}{(\alpha^1+\alpha^2+m^1+m^2)^3}$, which explains the estimation (5.27).

For $m_1 = l_1 + 1$ and $m_2 = l_2$ we have

$$
\begin{aligned}
&\Delta_{\substack{l_1+1 \\ l_2}\substack{l_1+1+\alpha_1 \\ l_2+\alpha_2};\substack{l_1+\alpha_1 \\ l_2+\alpha_2}\substack{l_1 \\ l_2}} \\
&= \frac{\theta}{2(1+\Omega)^2\big(\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+l^1+l^2+2\big)}\frac{1-\Omega^2}{1+\Omega^2}\frac{\sqrt{(l^1+1)(l^1+\alpha^1+1)}}{\alpha^1+\alpha^2}\Big(1+\mathcal{O}\big((\alpha_1+\alpha_2)^{-1}\big)\Big)\,.
\end{aligned}
\tag{B.52}
$$

This yields

$$
\begin{aligned}
&\Delta_{\substack{l_1+1 \\ l_2}\substack{l_1+1+\alpha_1 \\ l_2+\alpha_2};\substack{l_1+\alpha_1 \\ l_2+\alpha_2}\substack{l_1 \\ l_2}} - \sqrt{l_1+1}\,\Delta_{\substack{1 \\ 0}\substack{l_1+1+\alpha_1 \\ l_2+\alpha_2};\substack{l_1+\alpha_1 \\ l_2+\alpha_2}\substack{0 \\ 0}} \\
&\qquad = \mathcal{O}\bigg(\frac{1}{(\alpha^1+\alpha^2+l^1+l^2)}\sqrt{\frac{l^1+1}{\alpha^1+\alpha^2+l^1+l^2}}\frac{(l^1+1)}{(\alpha^1+\alpha^2+l^1+l^2)}\bigg)\,,
\end{aligned}
\tag{B.53}
$$

which explains the estimation (5.28). Similarly, we have

$$
\Delta_{\substack{1 \\ 1}\substack{1+\alpha_1 \\ 1+\alpha_2};\substack{\alpha_1 \\ 1+\alpha_2}\substack{0 \\ 1}} - \Delta_{\substack{1 \\ 0}\substack{1+\alpha_1 \\ 1+\alpha_2};\substack{\alpha_1 \\ 1+\alpha_2}\substack{0 \\ 0}} = \mathcal{O}\bigg(\frac{\theta\sqrt{\alpha^1+1}}{(\alpha^1+\alpha^2+1)^3}\bigg)\,,
\tag{B.54}
$$

which shows that the norm of (E.7) on page 140 is of the same order as (5.28).

B.5 An identity for hypergeometric functions

For terminating hypergeometric series ($m, l \in \mathbb{N}$) I compute the sum in the last line of (B.35):

$$
\begin{aligned}
&\sum_{x=0}^{\infty} \frac{(\alpha+x)!}{x!\alpha!}a^x\,{}_2F_1\Big(\begin{matrix}-m,-x \\ 1+\alpha\end{matrix}\Big|b\Big)\,{}_2F_1\Big(\begin{matrix}-l,-x \\ 1+\alpha\end{matrix}\Big|b\Big) \\
&= \sum_{x=0}^{\infty}\sum_{r=0}^{\min(x,m)}\sum_{s=0}^{\min(x,l)} \frac{(\alpha+x)!}{x!\alpha!}a^x\frac{m!x!\alpha!}{(m-r)!(x-r)!(\alpha+r)!r!}b^r\frac{l!x!\alpha!}{(m-s)!(x-s)!(\alpha+s)!s!}b^s \\
&= \sum_{r=0}^{m}\sum_{s=0}^{l}\sum_{x=\max(r,s)}^{\infty}(\alpha+x)!x!\alpha!m!l!a^x\frac{b^{r+s}}{(m-r)!(x-r)!(\alpha+r)!r!(l-s)!(x-s)!(\alpha+s)!s!} \\
&= \sum_{r=0}^{m}\sum_{s=0}^{l}\frac{\alpha!m!l!}{(m-r)!(\alpha+r)!r!(m-s)!(\alpha+s)!s!}a^{\max(r,s)}b^{r+s} \\
&\qquad\times\sum_{y=0}^{\infty}\frac{(\alpha+y+\max(r,s))!(y+\max(r,s))!}{(y+|r-s|)!y!}a^y \\
&= \sum_{r=0}^{m}\sum_{s=0}^{l}\frac{\alpha!m!l!}{(m-r)!(\alpha+r)!r!(l-s)!(\alpha+s)!s!}a^{\max(r,s)}b^{r+s} \\
&\qquad\times\frac{(\alpha+\max(r,s))!(\max(r,s))!}{(|r-s|)!}\,{}_2F_1\Big(\begin{matrix}\alpha+\max(r,s)+1\,,\ \max(r,s)+1 \\ |r-s|+1\end{matrix}\Big|a\Big)
\end{aligned}
$$

$$
\begin{aligned}
&=^* \sum_{r=0}^{m}\sum_{s=0}^{l} \frac{\alpha!m!l!}{(m-r)!(\alpha+r)!r!(l-s)!(\alpha+s)!s!}\frac{a^{\max(r,s)}\,b^{r+s}}{(1-a)^{\alpha+r+s+1}} \\
&\qquad\times \frac{(\alpha+\max(r,s))!(\max(r,s))!}{(|r-s|)!}\,{}_2F_1\Big(\begin{matrix}-\min(\alpha+r,\alpha+s)\,,\;-\min(r,s)\\ |r-s|+1\end{matrix}\Big|a\Big) \\
&= \sum_{r=0}^{m}\sum_{s=0}^{l} \frac{\alpha!m!l!}{(m-r)!(\alpha+r)!r!(l-s)!(\alpha+s)!s!}\frac{a^{\max(r,s)}\,b^{r+s}}{(1-a)^{\alpha+r+s+1}} \\
&\qquad\times \sum_{u'=0}^{\min(r,s)} \frac{(\alpha+\max(r,s))!(\max(r,s))!(\alpha+\min(r,s))!(\min(r,s))!}{(|r-s|+u')!(\min(r,s)-u')!(\alpha+\min(r,s)-u')!u'!}a^{u'} \\
&= \sum_{r=0}^{m}\sum_{s=0}^{l}\sum_{u=0}^{\min(r,s)} \frac{\alpha!m!l!}{(m-r)!(r-u)!(l-s)!(s-u)!(\alpha+u)!u!}\frac{a^{r+s-u}\,b^{r+s}}{(1-a)^{\alpha+r+s+1}} \\
&= \sum_{u=0}^{\min(m,l)}\sum_{r=u}^{m}\sum_{s=u}^{l} \frac{\alpha!m!l!}{(m-r)!(r-u)!(l-s)!(s-u)!(\alpha+u)!u!}\frac{a^{r+s-u}\,b^{r+s}}{(1-a)^{\alpha+r+s+1}} \\
&= \sum_{u=0}^{\min(m,l)} \frac{\alpha!m!l!}{(m-u)!(l-u)!(\alpha+u)!u!}\Big(\frac{ab}{1-a}\Big)^{2u}\Big(1+\frac{ab}{1-a}\Big)^{m+l-2u}\frac{1}{a^u(1-a)^{\alpha+1}} \\
&= \frac{(1-a+ab)^{m+l}}{(1-a)^{m+l+\alpha+1}}\,{}_2F_1\Big(\begin{matrix}-m\,,\;-l\\ \alpha+1\end{matrix}\Big|\frac{ab^2}{(1-a+ab)^2}\Big)\;. \qquad\text{(B.55)}
\end{aligned}
$$

In the step denoted by $=^*$ I have used [GR00, §9.131.1]. All other transformations should be self-explaining.

B.6 The propagator for $\Omega = 0$

I repeat the calculation of Appendix B.2 for $\Omega = 0$. The starting point is (B.22) and (B.23), where the continuous spectrum of the Laplace operator indicates that the sum over i^r is in fact an integration. Instead of (B.25) we make the ansatz

$$U_m^{(\alpha)}(v) = f^{(\alpha)}(v)\sqrt{\frac{m!}{(\alpha+m)!}}L_m^\alpha(v)\;, \qquad\text{(B.56)}$$

so that (B.24) reads after division by $f^{(\alpha)}(v)\sqrt{\frac{m!}{(\alpha+m)!}}$

$$(\alpha+m)L_{m-1}^\alpha(v) + \big(v-(\alpha+1+2m)\big)L_m^\alpha(v) + (m+1)L_{m+1}^\alpha(v) = 0\;. \qquad\text{(B.57)}$$

This is indeed the recursion relation [GR00, §9.971.6] of Laguerre polynomials $L_m^\alpha(v)$. The orthogonality relation for Laguerre polynomials [GR00, §8.904] implies

$$f^{(\alpha)}(v) = v^{\frac{\alpha}{2}}\mathrm{e}^{-\frac{v}{2}} \qquad\text{(B.58)}$$

and an integration over the spectrum from 0 to ∞. We thus obtain for the propagator

$$
\begin{aligned}
\Delta^{(\Omega=0)}_{\substack{m^1\\m^2}\substack{m^1+\alpha^1\\m^2+\alpha^2};\substack{l^1+\alpha^1\\l^2+\alpha^2}\substack{l^1\\l^2}} &= \sqrt{\frac{m^1!l^1!}{(m^1+\alpha^1)!(l^1+\alpha^1)!}\frac{m^2!l^2!}{(m^2+\alpha^2)!(l^2+\alpha^2)!}} \\
&\times\int_0^\infty dy_1dy_2\,\mathrm{e}^{-y_1-y_2}y_1^{\alpha^1}y_2^{\alpha^2}\frac{L_{m^1}^{\alpha^1}(y_1)L_{l^1}^{\alpha^1}(y_1)L_{m^2}^{\alpha^2}(y_2)L_{l^2}^{\alpha^2}(y_2)}{\frac{2}{\theta}y_1+\frac{2}{\theta}y_1+\mu_0^2}\;. \qquad\text{(B.59)}
\end{aligned}
$$

We introduce a Schwinger parameter and perform the y_i-integrations using [GR00, §7.414.4]:

$$\begin{aligned}
&\Delta^{(\Omega=0)}_{\substack{m^1 \\ m^2}\substack{m^1+\alpha^1 \\ m^2+\alpha^2};\substack{l^1+\alpha^1 \\ l^2+\alpha^2}\substack{l^1 \\ l^2}} \\
&= \frac{\theta}{2}\int_0^\infty dt\, \mathrm{e}^{-\frac{\mu_0^2\theta}{2}t}\prod_{i=1}^2 \int_0^\infty dy_i \sqrt{\frac{m^i!l^i!}{(m^i{+}\alpha^i)!(l^i{+}\alpha^i)!}}\mathrm{e}^{-y_i(1+t)} y_i^{\alpha^i} L^{\alpha^i}_{m^i}(y_i)L^{\alpha^i}_{l^i}(y_i) \\
&= \frac{\theta}{2}\int_0^\infty dt\, \mathrm{e}^{-\frac{\mu_0^2\theta}{2}t}\prod_{i=1}^2 \frac{(m^i{+}l^i{+}\alpha^i)!}{\sqrt{m^i!l^i!(m^i{+}\alpha^i)!(l^i{+}\alpha^i)!}}\frac{t^{m^i+l^i}}{(1{+}t)^{m^i+l^i+\alpha^i+1}}\, {}_2F_1\left(\begin{matrix} -m^i,\, -l^i \\ -m^i{-}l^i{-}\alpha^i\end{matrix}\middle| 1{-}\frac{1}{t^2}\right) .
\end{aligned} \tag{B.60}$$

The argument of the hypergeometric function in (B.60) is inconvenient. Expanding it as a finite sum and the argument as a polynomial in t^{-2} it is straightforward to derive for non-negative integers m, l the identity

$$\begin{aligned}
{}_2F_1\left(\begin{matrix} -m,\, -l \\ -m{-}l{-}\alpha\end{matrix}\middle| 1-\frac{1}{t^2}\right) &= \frac{\Gamma(m{+}\alpha{+}1)\Gamma(l{+}\alpha{+}1)}{\Gamma(\alpha{+}1)\Gamma(m{+}l{+}\alpha{+}1)}\, {}_2F_1\left(\begin{matrix} -m,\, -l \\ \alpha{+}1\end{matrix}\middle| \frac{1}{t^2}\right) \\
&\equiv \frac{\Gamma(m{+}\alpha{+}1)\Gamma(l{+}\alpha{+}1)}{\Gamma(m{+}l{+}\alpha{+}1)}\sum_{u=0}^{\min(m,l)} \frac{m!l!\, t^{-2u}}{u!(m{-}u)!(l{-}u)!\Gamma(\alpha{+}u{+}1)} .
\end{aligned} \tag{B.61}$$

We insert (B.61) into (B.60) and express the result in terms of the confluent hypergeometric function $\Psi(\alpha, \gamma, z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty dt\, \mathrm{e}^{-zt}t^{\alpha-1}(1{+}t)^{\gamma-\alpha-1}$, see [GR00, §9.211.4]:

$$\begin{aligned}
&\Delta^{(\Omega=0)}_{\substack{m^1 \\ m^2}\substack{m^1+\alpha^1 \\ m^2+\alpha^2};\substack{l^1+\alpha^1 \\ l^2+\alpha^2}\substack{l^1 \\ l^2}} \\
&= \frac{\theta}{2}\sum_{u^1=0}^{\min(m^1,l^1)}\sum_{u^2=0}^{\min(m^2,l^2)} (m^1{+}m^2{+}l^1{+}l^2{-}2u^1{-}2u^2)!\left(\prod_{i=1}^2 \frac{\sqrt{m^i!l^i!(m^i{+}\alpha^i)!(l^i{+}\alpha^i)!}}{(m^i{-}u^i)!(l^i{-}u^i)!(\alpha^i{+}u^i)!u^i!}\right) \\
&\qquad \times \Psi\left(m^1{+}m^2{+}l^1{+}l^2{-}2u^1{-}2u^2{+}1, -\alpha^1{-}\alpha^2{-}2u^1{-}2u^2, \frac{\mu_0^2\theta}{2}\right) .
\end{aligned} \tag{B.62}$$

Now, we reinsert $n^i = m^i{+}\alpha^i$, $k^i = l^i{+}\alpha^i$, put $u^i = \frac{1}{2}(m^i + l^i) - v^i$ and restore the symmetry in $m \leftrightarrow n$, $l \leftrightarrow k$:

$$\begin{aligned}
&\Delta^{(\Omega=0)}_{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{k^1 \\ k^2}\substack{l^1 \\ l^2}} = \frac{\theta}{2}\delta_{m^1+k^1,n^1+l^1}\delta_{m^2+k^2,n^2+l^2} \\
&\times \sum_{v^1=\frac{|m^1-l^1|}{2}}^{\frac{\min(m^1+l^1,n^1+k^1)}{2}}\sum_{v^2=\frac{|m^2-l^2|}{2}}^{\frac{\min(m^2+l^2,n^2+k^2)}{2}} \left(\prod_{i=1}^2\sqrt{\binom{n^i}{v^i{+}\frac{n^i-k^i}{2}}\binom{k^i}{v^i{+}\frac{k^i-n^i}{2}}\binom{m^i}{v^i{+}\frac{m^i-l^i}{2}}\binom{l^i}{v^i{+}\frac{l^i-m^i}{2}}}\right) \\
&\qquad \times (2v^1{+}2v^2)!\,\Psi\left(2v^1{+}2v^2{+}1, 2v^1{+}2v^2{-}m^1{-}m^2{-}k^1{-}k^2, \frac{\mu_0^2\theta}{2}\right) .
\end{aligned} \tag{B.63}$$

Putting first $\mu_0 = 0$ and then $\Omega = 0$ in (3.49) and using [GR00, §9.122.1] we obtain after comparison with (B.63) the remarkable commutativity of the limits

$$
\begin{aligned}
\lim_{\mu_0\to 0} \Delta^{(\Omega=0)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}} &\equiv \lim_{\Omega\to 0}\Big(\lim_{\mu_0\to 0}\Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}\Big)\\
&= \frac{\theta}{2}\delta_{m^1+k^1,n^1+l^1}\delta_{m^2+k^2,n^2+l^2}\\
&\times \sum_{v^1=\frac{|m^1-l^1|}{2}}^{\frac{\min(m^1+l^1,n^1+k^1)}{2}} \sum_{v^2=\frac{|m^2-l^2|}{2}}^{\frac{\min(m^2+l^2,n^2+k^2)}{2}} B\big(2v^1{+}2v^2{+}1, m^1{+}m^2{+}k^1{+}k^2{-}2v^1{-}2v^2{+}1\big)\\
&\times \prod_{i=1}^2 \sqrt{\binom{n^i}{v^i{+}\frac{n^i-k^i}{2}}\binom{k^i}{v^i{+}\frac{k^i-n^i}{2}}\binom{m^i}{v^i{+}\frac{m^i-l^i}{2}}\binom{l^i}{v^i{+}\frac{l^i-m^i}{2}}}\;. \qquad \text{(B.64)}
\end{aligned}
$$

C Towards the derivation of the Polchinski equation

I provide here an auxiliary calculation used in Section 4.1. In order to derive the Polchinski equation we first differentiate the action $S[\phi, J, \Lambda]$ given in (4.8) on page 36 with respect to ϕ, abbreviating $K_{mn}(\Lambda) := K[m, \Lambda]K[n, \Lambda]$:

$$\begin{aligned}
&\frac{\partial \exp(-S[\phi, J, \Lambda])}{\partial \phi_{mn}} \\
&= -\mathcal{V}_D \Big(\sum_{r,s} K^{-1}_{mn}(\Lambda) G_{mn;rs} K^{-1}_{rs}(\Lambda)\phi_{rs} + \sum_{r,s} F_{mn;rs}[\Lambda] J_{rs} + \frac{\partial L[\phi, \Lambda]}{\partial \phi_{mn}} \Big) \exp(-S[\phi, J, \Lambda]) \;.
\end{aligned} \tag{C.1}$$

We have used the symmetry $G_{mn;kl} = G_{kl;mn}$ which is obvious from the definition of G. For the same reason the propagator (4.4) is symmetric as well, $\Delta_{nm;lk} = \Delta_{lk;nm}$. In the first step we change the relative sign between $G_{mn;rs}$ and $\frac{\partial L}{\partial \phi_{mn}}$ in (C.1) and contract the result with $\Delta_{lk;nm} K_{mn}(\Lambda)$. Recalling the definition of the propagator (4.4) we obtain

$$\begin{aligned}
&\Big(2\mathcal{V}_D K^{-1}_{kl}(\Lambda)\phi_{kl} + 2\mathcal{V}_D \sum_{m,n,r,s} \Delta_{lk;nm} K_{mn}(\Lambda) F_{mn;rs}[\Lambda] J_{rs} \\
&\qquad\qquad + \sum_{m,n} \Delta_{lk;nm} K_{mn}(\Lambda) \frac{\partial}{\partial \phi_{mn}} \Big) \exp(-S[\phi, J, \Lambda]) \\
&= \mathcal{V}_D \Big(K^{-1}_{kl}(\Lambda)\phi_{kl} + \sum_{m,n,r,s} \Delta_{lk;nm} K_{mn}(\Lambda) F_{mn;rs}[\Lambda] J_{rs} - \sum_{m,n} \Delta_{lk;nm} K_{mn}(\Lambda) \frac{\partial L[\phi, \Lambda]}{\partial \phi_{mn}} \Big) \\
&\qquad \times \exp(-S[\phi, J, \Lambda]) \;.
\end{aligned} \tag{C.2}$$

A second identity is obtained by contracting the rewritten equation (C.1) instead with $\Delta_{lk;nm} \frac{\partial}{\partial \Lambda} K_{mn}(\Lambda)$:

$$\begin{aligned}
&\Big(2\mathcal{V}_D \sum_{m,n,r,s} \Big(\Delta_{lk;nm} K^{-1}_{mn}(\Lambda) \frac{\partial K_{mn}(\Lambda)}{\partial \Lambda} G_{mn;rs} K^{-1}_{rs}(\Lambda)\phi_{rs} + \Delta_{lk;nm} \frac{\partial K_{mn}(\Lambda)}{\partial \Lambda} F_{mn;rs}[\Lambda] J_{rs} \Big) \\
&\qquad\qquad + \sum_{m,n} \Delta_{lk;nm} \frac{\partial K_{mn}(\Lambda)}{\partial \Lambda} \frac{\partial}{\partial \phi_{mn}} \Big) \exp(-S[\phi, J, \Lambda]) \\
&= \mathcal{V}_D \Big(\sum_{m,n,r,s} \Delta_{lk;nm} K^{-1}_{mn}(\Lambda) \frac{\partial K_{mn}(\Lambda)}{\partial \Lambda} G_{mn;rs} K^{-1}_{rs}(\Lambda)\phi_{rs} \\
&\qquad + \sum_{m,n,r,s} \Delta_{lk;nm} \frac{\partial K_{mn}(\Lambda)}{\partial \Lambda} F_{mn;rs}[\Lambda] J_{rs} - \sum_{m,n} \Delta_{lk;nm} \frac{\partial K_{mn}(\Lambda)}{\partial \Lambda} \frac{\partial L[\phi, \Lambda]}{\partial \phi_{mn}} \Big) \\
&\qquad \times \exp(-S[\phi, J, \Lambda]) \;.
\end{aligned} \tag{C.3}$$

Next, we differentiate (C.2) with respect to ϕ_{kl}, multiply it by $\frac{\partial K_{kl}(\Lambda)}{\partial \Lambda}$, and sum over k, l:

$$\begin{aligned}
&\sum_{k,l} \frac{\partial K_{kl}(\Lambda)}{\partial \Lambda} \frac{\partial}{\partial \phi_{kl}} \Big(\Big(2\mathcal{V}_D K^{-1}_{kl}(\Lambda)\phi_{kl} + 2\mathcal{V}_D \sum_{m,n,r,s} \Delta_{lk;nm} K_{mn}(\Lambda) F_{mn;rs}[\Lambda] J_{rs} \\
&\qquad\qquad + \sum_{m,n} \Delta_{lk;nm} K_{mn}(\Lambda) \frac{\partial}{\partial \phi_{mn}} \Big) \exp(-S[\phi, J, \Lambda]) \Big)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{V}_D\Big(\sum_{k,l} \frac{\partial K_{kl}(\Lambda)}{\partial \Lambda} K^{-1}_{kl}(\Lambda) - \sum_{m,n,k,l} \frac{\partial K_{kl}(\Lambda)}{\partial \Lambda} \Delta_{lk;nm} K_{mn}(\Lambda) \frac{\partial^2 L[\phi,\Lambda]}{\partial \phi_{kl}\,\partial \phi_{mn}} \\
&- \mathcal{V}_D \sum_{k,l} \Big(\sum_{t,u} \phi_{tu} K^{-1}_{tu}(\Lambda) G_{tu;kl} K^{-1}_{kl}(\Lambda) + \sum_{t,u} J_{tu} F^T_{tu;kl}[\Lambda] + \frac{\partial L[\phi,\Lambda]}{\partial \phi_{kl}} \Big) \frac{\partial K_{kl}(\Lambda)}{\partial \Lambda} \\
&\qquad \times \Big(K^{-1}_{kl}(\Lambda)\phi_{kl} + \sum_{m,n,r,s} \Delta_{lk;nm} K_{mn}(\Lambda) F_{mn;rs}[\Lambda] J_{rs} - \sum_{m,n} \Delta_{lk;nm} K_{mn}(\Lambda) \frac{\partial L[\phi,\Lambda]}{\partial \phi_{mn}} \Big)\Big) \\
&\qquad \times \exp(-S[\phi,J,\Lambda]) \\
&= (\mathcal{V}_D)^2 \Big(\sum_{k,l,t,u} \phi_{tu} K^{-1}_{tu}(\Lambda) G_{tu;kl} \frac{\partial K^{-1}_{kl}(\Lambda)}{\partial \Lambda} \phi_{kl} + \frac{1}{\mathcal{V}_D} \sum_{k,l} \frac{\partial K_{kl}(\Lambda)}{\partial \Lambda} K^{-1}_{kl}(\Lambda) \\
&\quad + \sum_{k,l,m,n} \frac{\partial K_{kl}(\Lambda)}{\partial \Lambda} \Delta_{lk;nm} K_{mn}(\Lambda) \Big(\frac{\partial L[\phi,\Lambda]}{\partial \phi_{kl}} \frac{\partial L[\phi,\Lambda]}{\partial \phi_{mn}} - \frac{1}{\mathcal{V}_D} \frac{\partial^2 L[\phi,\Lambda]}{\partial \phi_{kl}\,\partial \phi_{mn}} \Big) \\
&\quad - \sum_{k,l,m,n,r,s,t,u} J_{tu} F^T_{tu;kl}[\Lambda] \frac{\partial K_{kl}(\Lambda)}{\partial \Lambda} \Delta_{lk;nm} K_{mn}(\Lambda) F_{mn;rs}[\Lambda] J_{rs} \\
&\quad - \sum_{k,l,m,n,r,s} \Big(\sum_{t,u} \phi_{tu} K^{-1}_{tu}(\Lambda) G_{tu;kl} K^{-1}_{kl}(\Lambda) + \frac{\partial L[\phi,\Lambda]}{\partial \phi_{kl}} \Big) \frac{\partial K_{kl}(\Lambda)}{\partial \Lambda} \Delta_{lk;nm} K_{mn}(\Lambda) F_{mn;rs}[\Lambda] J_{rs} \\
&\quad - \sum_{k,l,t,u} \Big(\phi_{kl} K^{-1}_{kl}(\Lambda) - \sum_{m,n} \frac{\partial L[\phi,\Lambda]}{\partial \phi_{mn}} K_{mn}(\Lambda) \Delta_{nm;lk} \Big) \frac{\partial K_{kl}(\Lambda)}{\partial \Lambda} F_{kl;tu}[\Lambda] J_{tu} \\
&\quad + \Big(\sum_{k,l,m,n,t,u} \phi_{tu} K^{-1}_{tu} G_{tu;kl} K^{-1}_{kl}(\Lambda) \frac{\partial K_{kl}(\Lambda)}{\partial \Lambda} \Delta_{lk;nm} K_{mn}(\Lambda) \frac{\partial L[\phi,\Lambda]}{\partial \phi_{mn}} \\
&\qquad - \sum_{k,l} \phi_{kl} K^{-1}_{kl}(\Lambda) \frac{\partial K_{kl}(\Lambda)}{\partial \Lambda} \frac{\partial L[\phi,\Lambda]}{\partial \phi_{kl}} \Big) \Big) \exp(-S[\phi,J,\Lambda]) \,. \qquad \text{(C.4)}
\end{aligned}
$$

On the other hand, differentiating (C.3) with respect to ϕ_{kl}, multiplying by $K_{kl}(\Lambda)$ and summing over k,l we obtain

$$
\begin{aligned}
&\sum_{k,l} K_{kl}(\Lambda) \frac{\partial}{\partial \phi_{kl}} \Big(\Big(2\mathcal{V}_D \sum_{m,n,r,s} \Delta_{lk;nm} K^{-1}_{mn}(\Lambda) \frac{\partial K_{mn}(\Lambda)}{\partial \Lambda} G_{mn;rs} K^{-1}_{rs}(\Lambda) \phi_{rs} \\
&\qquad + 2\mathcal{V}_D \sum_{m,n,r,s} \Delta_{lk;nm} \frac{\partial K_{mn}(\Lambda)}{\partial \Lambda} F_{mn;rs}[\Lambda] J_{rs} + \sum_{m,n} \Delta_{lk;nm} \frac{\partial K_{mn}(\Lambda)}{\partial \Lambda} \frac{\partial}{\partial \phi_{mn}} \Big) \\
&\qquad \times \exp(-S[\phi,J,\Lambda]) \Big) \\
&= \mathcal{V}_D \Big(\sum_{m,n} K^{-1}_{mn}(\Lambda) \frac{\partial K_{mn}(\Lambda)}{\partial \Lambda} - \sum_{k,l,m,n} K_{kl}(\Lambda) \Delta_{lk;nm} \frac{\partial K_{mn}(\Lambda)}{\partial \Lambda} \frac{\partial^2 L[\phi,\Lambda]}{\partial \phi_{kl}\,\partial \phi_{mn}} \\
&- \mathcal{V}_D \sum_{k,l} \Big(\sum_{t,u} \phi_{tu} K^{-1}_{tu}(\Lambda) G_{tu;kl} K^{-1}_{kl}(\Lambda) + \sum_{t,u} J_{tu} F^T_{tu;kl}[\Lambda] + \frac{\partial L[\phi,\Lambda]}{\partial \phi_{kl}} \Big) K_{kl}(\Lambda) \\
&\quad \times \Big(\sum_{m,n,r,s} \Delta_{lk;nm} K^{-1}_{mn}(\Lambda) \frac{\partial K_{mn}(\Lambda)}{\partial \Lambda} G_{mn;rs} K^{-1}_{rs}(\Lambda) \phi_{rs} + \sum_{m,n,r,s} \Delta_{lk;nm} \frac{\partial K_{mn}(\Lambda)}{\partial \Lambda} F_{mn;rs}[\Lambda] J_{rs} \\
&\qquad - \sum_{m,n} \Delta_{lk;nm} \frac{\partial K_{mn}(\Lambda)}{\partial \Lambda} \frac{\partial L[\phi,\Lambda]}{\partial \phi_{mn}} \Big) \Big) \exp(-S[\phi,J,\Lambda])
\end{aligned}
$$

$$
\begin{aligned}
&= (\mathcal{V}_D)^2\Big(\sum_{m,n,r,s}\phi_{mn}\frac{\partial K^{-1}_{mn}(\Lambda)}{\partial\Lambda}G_{mn;rs}K^{-1}_{rs}(\Lambda)\phi_{rs}+\frac{1}{\mathcal{V}_D}\sum_{p,q}\frac{\partial K_{mn}(\Lambda)}{\partial\Lambda}K^{-1}_{mn}(\Lambda)\\
&\quad+\sum_{k,l,m,n}K_{kl}(\Lambda)\Delta_{lk;nm}\frac{\partial K_{mn}(\Lambda)}{\partial\Lambda}\Big(\frac{\partial L[\phi,\Lambda]}{\partial\phi_{kl}}\frac{\partial L[\phi,\Lambda]}{\partial\phi_{mn}}-\frac{1}{\mathcal{V}_D}\frac{\partial^2 L[\phi,\Lambda]}{\partial\phi_{kl}\,\partial\phi_{mn}}\Big)\\
&\quad-\sum_{k,l,m,n,r,s,t,u}J_{tu}F^T_{tu;kl}[\Lambda]K_{kl}(\Lambda)\Delta_{lk;nm}\frac{\partial K_{mn}(\Lambda)}{\partial\Lambda}F_{nm;rs}[\Lambda]J_{rs}\\
&\quad-\sum_{k,l,m,n,t,u}\Big(\sum_{r,s}\phi_{rs}K^{-1}_{rs}(\Lambda)G_{rs;mn}K^{-1}_{mn}(\Lambda)-\frac{\partial L[\phi,\Lambda]}{\partial\phi_{mn}}\Big)\frac{\partial K_{mn}(\Lambda)}{\partial\Lambda}\Delta_{nm;lk}K_{kl}(\Lambda)F_{kl;tu}[\Lambda]J_{tu}\\
&\quad-\sum_{m,n,r,s}\Big(\phi_{mn}K^{-1}_{mn}(\Lambda)+\sum_{k,l}\frac{\partial L[\phi,\Lambda]}{\partial\phi_{kl}}K_{kl}(\Lambda)\Delta_{lk;nm}\Big)\frac{\partial K_{mn}(\Lambda)}{\partial\Lambda}F_{mn;rs}[\Lambda]J_{rs}\\
&\quad-\Big(\sum_{k,l,m,n,r,s}\phi_{rs}K^{-1}_{rs}G_{rs;mn}K^{-1}_{mn}(\Lambda)\frac{\partial K_{mn}(\Lambda)}{\partial\Lambda}\Delta_{nm;lk}K_{kl}\frac{\partial L[\phi,\Lambda]}{\partial\phi_{kl}}\\
&\qquad-\sum_{m,n}\phi_{mn}K^{-1}_{mn}(\Lambda)\frac{\partial K_{mn}(\Lambda)}{\partial\Lambda}\frac{\partial L[\phi,\Lambda]}{\partial\phi_{mn}}\Big)\Big)\exp(-S[\phi,J,\Lambda])\;.
\end{aligned}
\tag{C.5}
$$

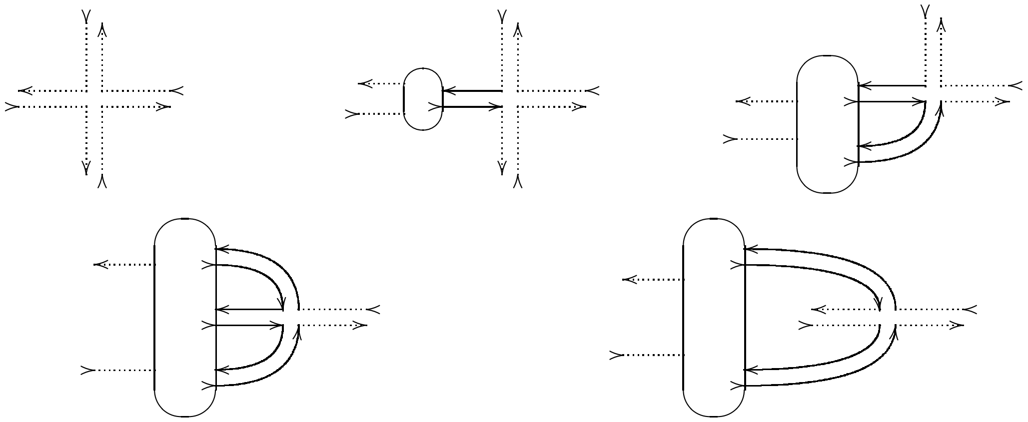
Recalling the definitions (4.9) of $G^K_{mn;kl}(\Lambda)$ on page 36 and (4.10) of $\Delta^K_{nm;lk}(\Lambda)$ we can write down the sum of (C.4) and (C.5), divided by $2\mathcal{V}_D$, in the form

$$
\begin{aligned}
&\frac{1}{2\mathcal{V}_D}\sum_{k,l,m,n}\frac{\partial}{\partial\phi_{kl}}\frac{\partial\Delta^K_{lk;nm}(\Lambda)}{\partial\Lambda}\\
&\qquad\times\Big(\Big(2\mathcal{V}_D\sum_{r,s}\big(G^K_{mn;rs}\phi_{rs}+F_{mn;rs}J_{rs}\big)+\frac{\partial}{\partial\phi_{mn}}\Big)\exp(-S[\phi,J,\Lambda])\Big)\\
&=\mathcal{V}_D\Big(\sum_{k,l,m,n}\frac{1}{2}\phi_{mn}\frac{\partial G^K_{mn;kl}(\Lambda)}{\partial\Lambda}\phi_{kl}-\sum_{k,l,m,n,r,s,t,u}\frac{1}{2}J_{tu}F^T_{tu;kl}[\Lambda]\frac{\partial\Delta^K_{lk;nm}(\Lambda)}{\partial\Lambda}F_{mn;rs}[\Lambda]J_{vw}\\
&\quad+\sum_{k,l,m,n}\frac{1}{2}\frac{\partial\Delta^K_{lk;nm}(\Lambda)}{\partial\Lambda}\Big(\frac{\partial L[\phi,\Lambda]}{\partial\phi_{kl}}\frac{\partial L[\phi,\Lambda]}{\partial\phi_{mn}}-\frac{1}{\mathcal{V}_D}\frac{\partial^2L[\phi,\Lambda]}{\partial\phi_{kl}\,\partial\phi_{mn}}\Big)\\
&\quad-\sum_{k,l,m,n,r,s,t,u}\phi_{rs}G^K_{rs;kl}\frac{\partial\Delta^K_{lk;nm}(\Lambda)}{\partial\Lambda}F_{mn;tu}[\Lambda]J_{tu}+\frac{1}{\mathcal{V}_D}\sum_{m,n}\frac{\partial}{\partial\Lambda}\big(\ln K_{mn}(\Lambda)\big)\Big)\\
&\qquad\times\exp(-S[\phi,J,\Lambda])\;.
\end{aligned}
\tag{C.6}
$$

Now, we apply to (C.6) the path integral operator $\int\mathcal{D}[\phi]=\int_{-\infty}^{\infty}\prod_{a,b}d\phi_{ab}$. This yields zero for the lhs of (C.6), because for every k,l we have a suitable integral over ϕ_{kl} which annihilates the derivative with respect to ϕ_{kl}. The result is the identity (4.13) in the main part, which was to show.

D Proof of the power-counting theorem

I provide here the proof of Theorem 10 on page 48, which is quite long and technical. The proof amounts to study all possible connections of two external legs of either different graphs or the same graph. It will be essential how the legs to connect are situated with respect to the remaining part of the graph. There are the following arrangements of the external legs at the distinguished vertex one or two of which we are going to connect:



(D.1)

A big oval stands for other parts of the graph the specification of which is not necessary for the proof. Dotted lines entering and leaving the oval stand for the set of all external legs different from the external legs of the distinguished vertex to contract. If two or three internal lines are connected to the oval this does not necessarily mean that these two lines are part of an inner loop.

I am going to integrate the matrix Polchinski equation (4.52), page 47, by induction upward in V and for constant V downward in N. Due to the grading (V, N), the differential equation (4.52) is actually constructive. I consider in Section D.1 the connection of two smaller graphs of (V_1, N_1) and (V_2, N_2) vertices and external legs and in Sections D.2 and D.3 the self-contraction of a graph with $(V_1 = V,\ N_1 = N + 2)$ vertices and external legs. These graphs are further characterised by V_i^e, B_i, g_i, ι_i external vertices, boundary components, genera and segmentation indices, respectively. Since the sums in (4.58) and the number of arrangements of legs in (D.1) are finite, it is sufficient to regard the contraction of subgraphs individually. That is, we consider individual subgraphs γ_1, γ_2 the contraction of which produces an individual graph γ. We also ignore the problem of making the graphs symmetric in the indices $m_i n_i$ of the external legs. At the very end we project the sum of graphs γ to homogeneous degree (V, V^e, B, g, ι). To these homogeneous parts there contributes according to (4.58) a finite number of contractions of γ_i. We thus get the bound (4.59) on page 48 if we can prove it for any individual contraction.

The Theorem is certainly correct for the initial ϕ^4-interaction (4.36) which due to (4.51) gives $|A^{(1,1,1,0,0)}_{m_1 n_1;\dots;m_4 n_4}[\Lambda]| \leq 1$.

D.1 Tree-contractions of two subgraphs

I start with the first term on the rhs of (4.52), page 47, which describes the connection of two smaller subgraphs γ_1, γ_2 of V_1, V_2 vertices and N_1, N_2 external legs via a propagator.

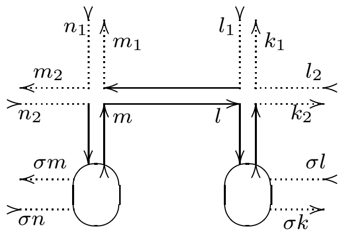
The total graph γ for a tree-contraction has

$$V = V_1{+}V_2 \text{ vertices}\,, \qquad\qquad N = N_1{+}N_2{-}2 \text{ external legs}\,,$$
$$I = I_1{+}I_2{+}1 \text{ propagators}\,, \qquad\qquad \tilde{L} = \tilde{L}_1{+}\tilde{L}_2{-}1 \text{ loops}\,, \tag{D.2}$$

because two loops of the subgraphs are merged to a new loop in the total graph. It follows from (4.43) that for tree-contractions we always have additivity of the genus,

$$g = g_1 + g_2\,. \tag{D.3}$$

As an example for a contraction between graphs in the first line of (D.1) let us consider



(D.4)

where σm and σn stand for the set of all other outgoing and incoming indices via external legs at the remaining part of the left subgraph γ_1 and similarly for σk and σl for the right subgraph γ_2. The two boundary components to which the contracted vertices belong are joint in the total graph, i.e. $B = B_1 + B_2 - 1$. Moreover, we obviously have $V^e = V_1^e + V_2^e$ and $\iota = \iota_1 + \iota_2$. The graph (D.4) determines the Λ-scaling

$$\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;m_2n_2;\sigma m\,\sigma n;\sigma k\,\sigma l;k_2l_2;k_1l_1}[\Lambda] \\
&= \frac{1}{2}\sum_{m,l} A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{m_1n_1;m_2n_2;\sigma m\,\sigma n;mm_1}[\Lambda]\, Q_{m_1m;ll_1}(\Lambda)\, A^{(V_2,V_2^e,B_2,g_2,\iota_2)\gamma_2}_{l_1l;\sigma k\,\sigma l;k_2l_2;k_1l_1}[\Lambda]\,.
\end{aligned} \tag{D.5}$$

Due to the conservation of the total amount of indices in γ_1 and γ_2 by induction hypothesis (4.59), both

$$m = \sigma n - \sigma m + n_2 \qquad \text{and} \qquad l = \sigma k - \sigma l + k_2 \tag{D.6}$$

are completely fixed by the other external indices so that from the sum over m and l there survives a single term only. Then, because of the relation $m_1 + l = m + l_1$ from the propagator $Q_{m_1m;ll_1}(\Lambda)$, see (4.55), it follows that the total amount of indices for $A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;m_2n_2;\sigma m\,\sigma n;\sigma k\,\sigma l;k_2l_2;k_1l_1}$ is conserved as well.

Let $\bar{V}_i^e$ and $\bar{\iota}_i$ be the numbers of external vertices and segmentation indices on the segments of the subgraphs γ_i on which the contracted vertices are situated. The induction hypothesis (4.59) gives us the bound if these segments carry $\bar{s}_i \leq \bar{V}_i^e + \bar{\iota}_i - 1$ index summations. The new segment of the total graph γ created by connecting the boundary components of γ_i carries $\bar{V}_1^e + \bar{V}_2^e$ external vertices and $\bar{\iota}_1 + \bar{\iota}_2$ segmentation indices and therefore admits up to $\bar{s}_1 + \bar{s}_2 + 1$ index summations. In (D.4) that additional index summation will be the m_1-summation.

Due to (4.47) (for segments) on page 46 there has to be an external leg on each segment the outgoing index of which is not allowed to be summed. If on the γ_2-part of the contracted segment there is an unsummed external leg, we can choose m as that particular index in γ_1. In this case we take in the propagator the maximum over m, l and sum the part γ_2 for given l over those indices which belong to $\mathcal{E}^s$. The result is bounded independently of l and all other incoming indices. Next, we sum over the indices in $\mathcal{E}^s$ which belong to γ_1, regarding m as an unsummed index. There is the possibility of an m_1-summation applied to the propagator in the last step, with l_1 kept fixed, for which the bound is given by (4.56) on page 47. In this case we therefore get

$$\begin{aligned}
&\sum_{\mathcal{E}^s,\ \bar{s}_2\leq \bar{V}_2^e+\bar{\iota}_2-1,\ m_1\in\mathcal{E}^s} \Big|\Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;m_2n_2;\sigma m\,\sigma n;\sigma k\,\sigma l;k_2l_2;k_1l_1}[\Lambda]\Big| \\
&\qquad\leq \frac{1}{2}\Big(\sum_{\mathcal{E}_1^{s_1}}\big|A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{m_1n_1;m_2n_2;\sigma m\,\sigma n;mm_1}[\Lambda]\big|\Big)\Big(\max_{l_1}\sum_{m_1}\max_{m,l}\big|Q_{m_1m;ll_1}(\Lambda)\big|\Big) \\
&\qquad\qquad\times\Big(\sum_{\mathcal{E}_2^{s_2}}\big|A^{(V_2,V_2^e,B_2,g_2,\iota_2)\gamma_2}_{l_1l;\sigma k\,\sigma l;k_2l_2;k_1l_1}[\Lambda]\big|\Big) \\
&\qquad\leq \frac{1}{2}C_1\Big(\frac{\Lambda}{\mu}\Big)^{\delta_2(V-\frac{N+2}{2}+4-2g-(B+1))}\Big(\frac{\mu}{\Lambda}\Big)^{\delta_1(1+V-V^e-\iota+2g+(B+1)-2+(s-1))} \\
&\qquad\qquad\times\Big(\frac{\mu}{\Lambda}\Big)^{\delta_0(V^e+\iota-2-(s-1))}P^{2V-\frac{N+2}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;. \qquad\qquad (\mathrm{D}.7)
\end{aligned}$$

We have used the induction hypothesis (4.59) for the subgraphs as well as (4.56) for the propagator and have inserted $N_1+N_2=N+2$, $V_1+V_2=V$, $V_1^e+V_2^e=V^e$, $\iota_1+\iota_2=\iota$, $B_1+B_2=B+1$, $g_1+g_2=g$ and $s_1+s_2=s-1$, because there is an additional summation over m_1 which belongs to $\mathcal{E}^s$ but not to $\mathcal{E}_i^{s_i}$. If $m_1\notin\mathcal{E}^s$ we take instead the unsummed propagator and replace in (D.7) one factor (4.56) by (4.55) as well as $(s-1)$ by s. The total exponents of γ remain unchanged.

Next, let there be no unsummed external leg on the contracted segment of γ_2 viewed from γ. Now, we cannot directly use the induction hypothesis. On the other hand, for a given index configuration of γ_2 and the propagator, the index k_2 is not an independent summation index:

$$k_2 = l+\sigma l-\sigma k = m-m_1+l_1+\sigma l-\sigma k\;. \qquad (\mathrm{D}.8)$$

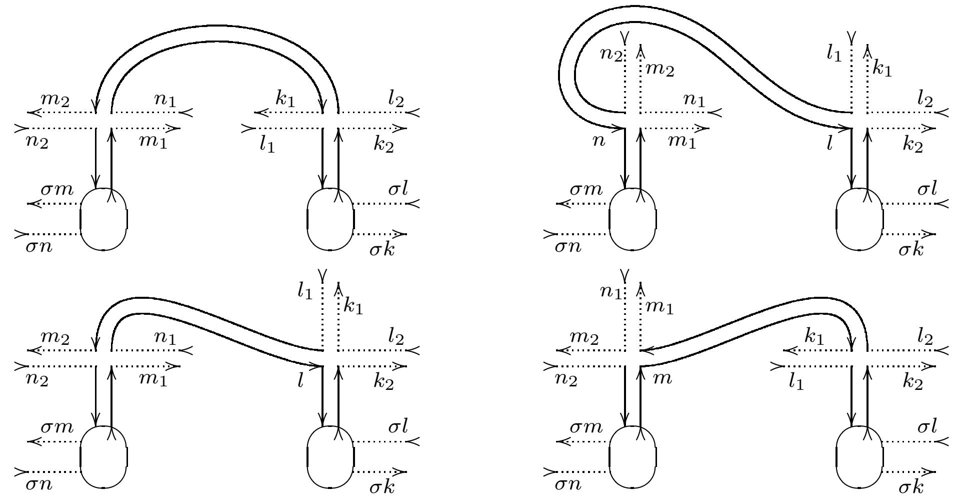
See also (D.6). If $m_1\in\mathcal{E}^s$ there must be an unsummed outgoing index on the contracted segment of γ_1. We can thus realise the k_2-summation as a summation over m in γ_1 for fixed index configuration of γ_2 and m_1, l_1. This m-summation is applied together summation over the γ_1-indices of $\mathcal{E}^s$ to γ_1 as the first step, taking again the maximum of the propagator over m, l. In the second step we sum over the restriction of $\mathcal{E}^s$ to γ_2 and the propagator. It is obvious that the estimation (D.7) remains unchanged, in particular, $s_1+s_2=(s_1+1)+(s_2-1)=s-1$. If m_1 is the only unsummed index we realise the k_2-summation as a summation of the propagator over l. Here, one has to take into account that the subgraph γ_2 is bounded independently of the incoming index l. Again we get the same exponents as in (D.7).

We can summarise (D.7) and its discussed modification to

$$\sum_{\mathcal{E}^s} \Big| \Lambda \frac{\partial}{\partial \Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{m_1 n_1; m_2 n_2; \sigma m\, \sigma n; \sigma k\, \sigma l; k_2 l_2; k_1 l_1}[\Lambda] \Big| \\ \leq \Big(\frac{\Lambda}{\mu}\Big)^{\delta_2(V-\frac{N}{2}+2-2g-B)} \Big(\frac{\mu}{\Lambda}\Big)^{\delta_1(V-V^e-\iota+2g+B-1+s)} \Big(\frac{\mu}{\Lambda}\Big)^{\delta_0(V^e+\iota-1-s)} P^{2V-\frac{N}{2}-1}\Big[\ln \frac{\Lambda}{\Lambda_R}\Big] . \tag{D.9}$$

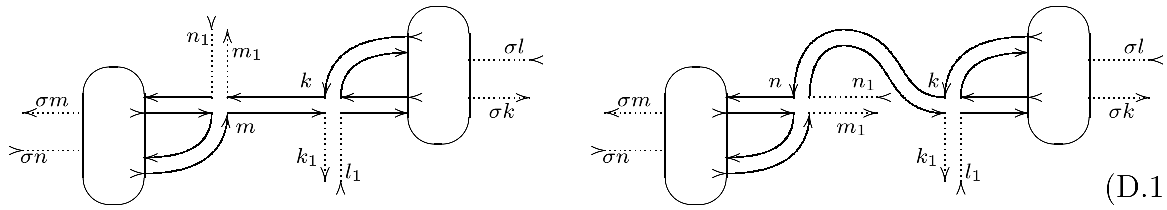
For the choice of the boundary conditions according to Definition/Lemma 7, the Λ-integration increases (again according to Definition/Lemma 7) the degree of the polynomial in $\ln \frac{\Lambda}{\Lambda_R}$ by 1. Hence, we have extended (4.59) to a bigger degree V for contractions of type (D.4). In particular, the bound is independent of the incoming indices n_i, l_i (by induction starting with (4.56), which represents the third graph in (4.39)).

The verification of (4.59) for any contraction between graphs of the first line in (D.1) is performed in a similar manner. Taking the same subgraphs as in (D.4), but with a contraction of other legs, the discussion is in fact a little easier because there are no trajectories going through both subgraphs:



(D.10)

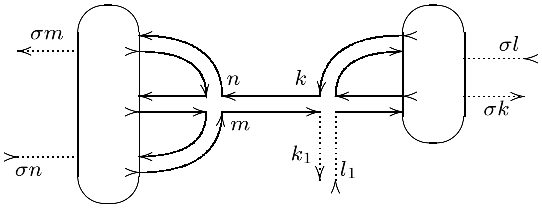
The contractions



(D.11)

are treated in the same way. The point is that the summation indices of the propagator (m, k on the left and n, k on the right in (D.11)) are fixed by index conservation for the subgraphs. In the same way one also discusses any contraction between the second and third graph in (D.1).

Let us now contract the left graph in the second line of (D.1) with any graph of the first line of (D.1), e.g.



(D.12)

The number of boundary components is reduced by 1, giving $B_1+B_2=B+1$. We clearly have $\iota=\iota_1+\iota_2$, but there is now one external vertex less on which we can apply an index summation, $V^e=V^e_1+V^e_2-1$. At the same time we need the index summation from the subgraph, because in the Λ-scaling

$$\Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,g,\iota)\gamma}_{k_1l_1;\sigma k\,\sigma l;\sigma m\,\sigma n}[\Lambda]=\frac{1}{2}\sum_{m,n,k}A^{(V_1,V^e_1,B_1,g_1,\iota_1)\gamma_1}_{mn;\sigma m\,\sigma n}[\Lambda]\,Q_{nm;k_1k}(\Lambda)\,A^{(V_2,V^e_2,B_2,g_2,\iota_2)\gamma_2}_{kk_1;k_1l_1;\sigma k\,\sigma l}[\Lambda] \tag{D.13}$$

there is now one undetermined summation index:

$$k=l_1+\sigma l-\sigma k\;,\qquad m(n)=n+\sigma n-\sigma m\;. \tag{D.14}$$

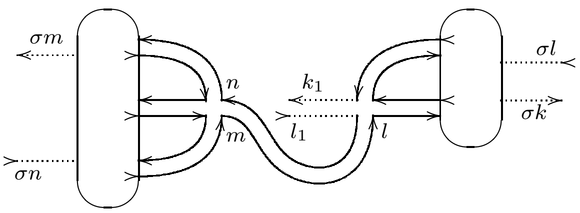
First, let there be an additional unsummed external leg on the segment of m,n in γ_1. Then, the induction hypothesis (4.59) gives the bound for a summation over m. We thus fix n,k and all indices of γ_2 in the first step and realise a possible k_1-summation due to $k_1=m+k-n$ as an m-summation, which is applied together with the summation over the γ_1-indices of $\mathcal{E}^s$, after maximising the propagator over m,k_1. The result is independent of n. We thus restrict the n-summation to the propagator, see (4.56), and apply the remaining $\mathcal{E}^s$-summations to γ_2, where k remains unsummed. We have $s_1+s_2=s$ and get the estimation

$$\begin{aligned}
&\sum_{\mathcal{E}^s\ni k_1,\;\bar{s}_1\leq\bar{V}^e_1+\bar{\iota}_1-2}\Big|\Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,g,\iota)\gamma}_{k_1l_1;\sigma k\,\sigma l;\sigma m\,\sigma n}[\Lambda]\Big|\\
&\qquad\leq\frac{1}{2}\Big(\sum_{m,\mathcal{E}^{s_1}_1}\big|A^{(V_1,V^e_1,B_1,g_1,\iota_1)\gamma_1}_{mn;\sigma m\,\sigma n}[\Lambda]\big|\Big)\Big(\max_k\sum_n\max_{m,k_1}\big|Q_{nm;k_1k}(\Lambda)\big|\Big)\\
&\qquad\qquad\times\Big(\sum_{\mathcal{E}^{s_2}_2\not\ni k_1}\big|A^{(V_2,V^e_2,B_2,g_2,\iota_2)\gamma_2}_{kk_1;k_1l_1;\sigma k\,\sigma l}[\Lambda]\big|\Big)\\
&\qquad\leq\frac{1}{2}C_1\Big(\frac{\Lambda}{\mu}\Big)^{\delta_2(V-\frac{N+2}{2}+4-2g-(B+1))}\Big(\frac{\mu}{\Lambda}\Big)^{\delta_1(1+V-(V^e+1)-\iota+2g+(B+1)-2+s)}\\
&\qquad\qquad\times\Big(\frac{\mu}{\Lambda}\Big)^{\delta_0((V^e+1)+\iota-2-s)}P^{2V-\frac{N+2}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;.
\end{aligned} \tag{D.15}$$

If $k_1\notin\mathcal{E}^s$ we do not need the m-summation on γ_1. Again we have $s=s_1+s_2$ and (D.15) remains unchanged. Here, we may allow for index summations at all other external legs on the segment of m,n in γ_1.

If there is no unsummed external leg on the segment of m,n in γ_1, we must realise the k_1-summation as follows: We proceed as before up to the step where we sum the propagator for given k over n. For each term in this sum we have $k_1 = k + \sigma n - \sigma m$. We thus achieve a different k_1 if for given $\sigma m, \sigma n$ we start from a different k. Since the result of the summations over γ_1 and the propagator is independent of k, see (4.56), we realise the k_1-summation as a sum over k restricted to γ_2. We now get the same exponents as in (D.15) also for this case. According to Definition/Lemma 7, the Λ-integration extends for contractions of type (D.12) the bound (4.59) to a bigger order V.

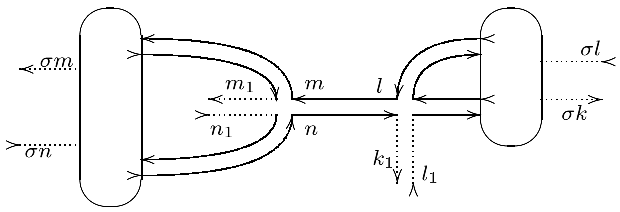
The contraction of the other leg of the right vertex



(D.16)

is easier to discuss because the k_1-summation is directly applied to γ_2. Taking the second vertex of the first line of (D.1) instead, we have two contractions which are identical to (D.12) and (D.16) and a third one with contractions as in the first and last graphs of (D.10) where γ_1 and γ_2 form different segments in γ. This case is much easier because there is no trajectory involving both subgraphs.

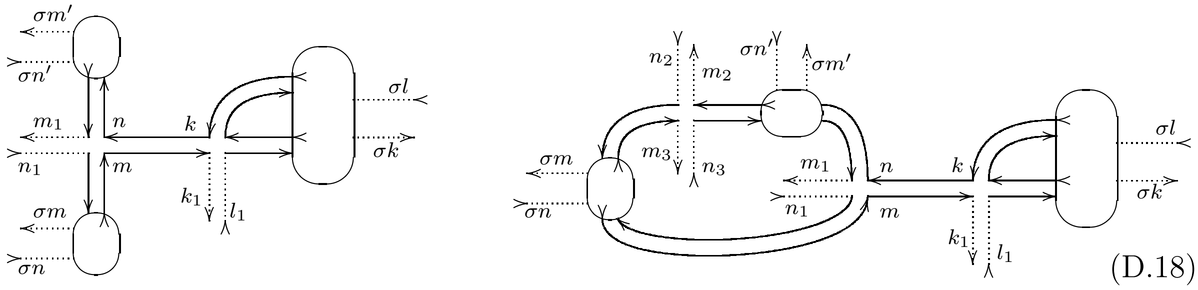
Contracting instead the last vertex of the second line of (D.1) we obtain the same estimates if the two propagators between the vertex and the oval belong to the same segment:



(D.17)

The only modification to (D.15) and its variants is to replace $(V^e + 1)$ by V and ι by $(\iota + 1)$, because the total number of external vertices is unchanged whereas the total segmentation index is reduced by 1.

If we contract the second vertex of the last line in (D.1) in such a way that the contracted indices m, n belong to different segments of γ_1, e.g.

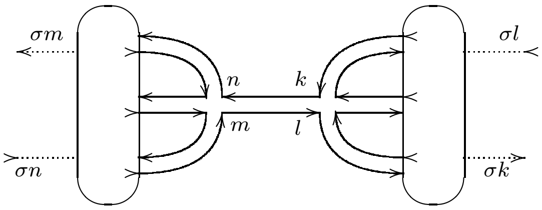


(D.18)

they are actually determined by index conservation for the segments. The entire discussion of these examples is therefore similar to the graph (D.4) with bound (D.7) and its modifications. Note that we have $V^e = V_1^e + V_2^e$ and $\iota = \iota_1 + \iota_2$ in (D.18).

Accordingly, we can replace in all previous examples a vertex of the first line of (D.1) by the composed vertex under the condition that the two contracted trajectories at the composed vertex belong to different segments.

It remains to study the contraction



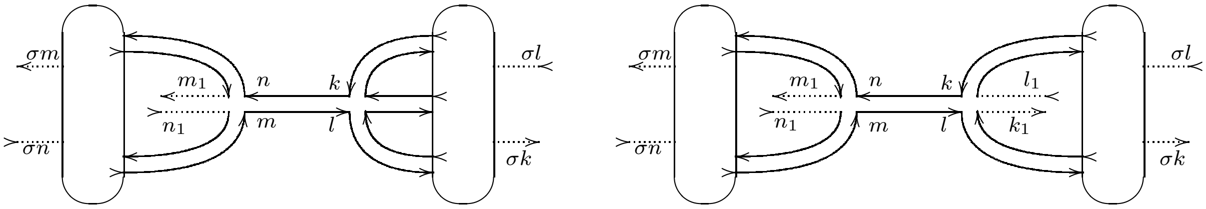
(D.19)

where two contraction indices (m or n and k or l) are undetermined. We have $V^e = V_1^e+V_2^e-2$ and $\iota = \iota_1+\iota_2$. We first assume that at least one of the boundary components of γ_i to contract carries more than one external vertex. In this case we have $B = B_1+B_2-1$. There has to be at least one unsummed external vertex on the segment, say on γ_2. We fix the indices of γ_2 as well as n in the first step, take in the propagator the maximum over m, l and sum over the γ_1-indices of $\mathcal{E}^s$. Here, m can be regarded as an unsummed index. We take the maximum of γ_1 over n so that the n-summation restricts to the propagator only. We take in the summed propagator the maximum over k so that the remaining k-summation is applied together with the summation over the γ_2-indices of $\mathcal{E}^s$. We thus need $s_1 + s_2 = s + 1$ summations and the bound (4.56) for the propagator:

$$
\begin{aligned}
&\sum_{\mathcal{E}^s}\Big|\Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,g,\iota)\gamma}_{\sigma k\,\sigma l;\sigma m\,\sigma n}[\Lambda]\Big| \\
&\quad\leq \frac{1}{2}\Big(\max_n \sum_{\mathcal{E}_1^{s_1}}\big|A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{mn;\sigma m\,\sigma n}[\Lambda]\big|\Big)\Big(\max_k\sum_n \max_{m,l}\big|Q_{nm;lk}(\Lambda)\big|\Big) \\
&\qquad\qquad\times\Big(\sum_{k,\mathcal{E}_2^{s_2}}\big|A^{(V_2,V_2^e,B_2,g_2,\iota_2)\gamma_2}_{kl;\sigma k\,\sigma l}[\Lambda]\big|\Big) \\
&\quad\leq \frac{1}{2}C_1\Big(\frac{\Lambda}{\mu}\Big)^{\delta_2(V-\frac{N+2}{2}+4-2g-(B+1))}\Big(\frac{\mu}{\Lambda}\Big)^{\delta_1(1+V-(V^e+2)-\iota+2g+(B+1)-2+(s+1))} \\
&\qquad\qquad\times\Big(\frac{\mu}{\Lambda}\Big)^{\delta_0((V^e+2)+\iota-2-(s+1))}P^{2V-\frac{N+2}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,.
\end{aligned}
\tag{D.20}
$$

Finally, we have to consider the case where the only external vertices of both boundary components of γ_i to contract are just the contracted vertices. In this case the contraction removes these two boundary components at expense of a completely inner loop, giving $B = B_1 + B_2 - 2$. The differences $n - m$ and $k - l$ are fixed by the remaining indices of γ_i. For given m we may thus take the maximum of γ_2 over l and realise the l-summation as a summation (4.56) over the propagator. We thus exhaust all differences $m - l$. In order to exhaust all values of m we take the maximum of γ_1 over m, n and multiply the result by a volume factor (4.54). We thus replace in (D.20) $(s + 1) \mapsto s$ and $(B + 1) \mapsto (B + 2)$, and combine one factor (4.55) and a volume factor (4.54) to (4.57). We thus get the same total exponents as in (4.59) so that the Λ-integration extends (4.59) to a bigger order V for all contractions represented by (D.19).

The contractions



(D.21)

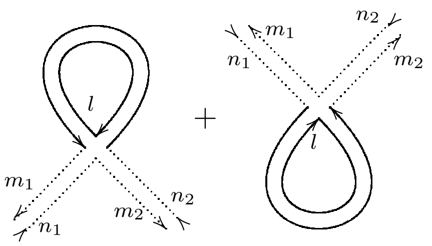
are treated in the same way as (D.19), now with the two unknown summation indices taken into account by a reduction of $V^e + \iota = (V_1^e + \iota_1) + (V_2^e + \iota_2) - 2$. In particular, there is also the situation where m, n and k, l are the only external legs of their boundary components before the contraction. In this case the number of boundary components drops by 2, which requires a volume factor in order to realise the sum over the starting point of the inner loop.

Thus, (4.59) is proven for any contractions produced by the first (bilinear) term on the rhs of (4.52).

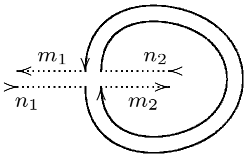
D.2 Loop-contractions at the same vertex

It remains to verify the scaling formula (4.59) for the second term (the last line) on the rhs of the matrix Polchinski equation (4.52), which describes self-contractions of graphs. The graphical data for the subgraph will obtain a subscript 1, such as the number of external vertices V_1^e, the segmentation index ι_1 and the set $\mathcal{E}_1^{s_1}$ of summation indices. We always have $V_1 = V$ and $N_1 = N + 2$. We first consider contractions of external lines at the same vertex, for which we have the possibilities shown in (D.1).

The very first vertex leads to two different self-contractions:



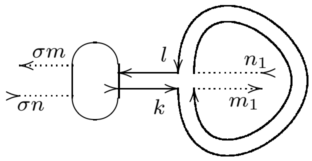
$$\Big|\Lambda\frac{\partial}{\partial\Lambda}A^{(1,1,1,0,0)}_{m_1n_1;m_2n_2}[\Lambda]\Big| = \sum_l |Q_{m_1l;ln_2}(\Lambda)|\,\delta_{n_1m_2} + \sum_l |Q_{ln_1;m_2l}(\Lambda)|\,\delta_{n_2m_1}\;, \qquad \text{(D.22)}$$



$$\Big|\Lambda\frac{\partial}{\partial\Lambda}A^{(1,1,2,0,1)}_{m_1n_1;m_2n_2}[\Lambda]\Big| = |Q_{m_1n_2;m_2n_1}(\Lambda)|\;. \qquad \text{(D.23)}$$

For the planar contraction (D.22) we estimate the l-summation by a volume factor so that we obtain (4.59) from (4.57). For the non-planar graph (D.23) we obtain (4.59) for $s = 0$ directly from (4.55). According to (4.47) we can apply one index summation which yields (4.59) via (4.56).

For the second graph in the first line of (D.1) we first investigate the contraction



(D.24)

The number of loops of the amputated graph is increased by 1, $\tilde{L} = \tilde{L}_1 + 1$, so that due to (4.43) on page 44 and $I = I_1 + 1$ we get $g = g_1$. The graph (D.24) determines the Λ-variation

$$\Lambda \frac{\partial}{\partial \Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{m_1 n_1;\sigma m\,\sigma n}[\Lambda] = -\frac{1}{2} \sum_{k,l} Q_{m_1 k; l n_1}(\Lambda)\, A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{m_1 n_1; n_1 l; \sigma m\,\sigma n; k m_1}[\Lambda]\;, \tag{D.25}$$

with one of the indices k, l being undetermined. First, let there be at least one further external leg on the same boundary component as l, k. In this case the number of boundary components is increased by 1, $B = B_1 + 1$. If there is an unsummed index on the segment of k, l we can realise the k-summation in γ as a summation in γ_1 after taking in the propagator the maximum over k, l. We thus have $s_1 = s + 1$ and consequently

$$\begin{aligned}
&\sum_{\mathcal{E}^s,\, m_1 \notin \mathcal{E}^s} \Big| \Lambda \frac{\partial}{\partial \Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{m_1 n_1;\sigma m\,\sigma n}[\Lambda] \Big| \\
&\qquad \leq \frac{1}{2} \Big(\max_{k,l} \big| Q_{m_1 k; l n_1}(\Lambda) \big| \Big) \Big(\sum_{k, \mathcal{E}_1^s} \big| A^{(V,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{m_1 n_1; n_1 l; \sigma m\,\sigma n; k m_1}[\Lambda] \Big) \\
&\qquad \leq \frac{1}{2} C_0 \Big(\frac{\Lambda}{\mu_0} \Big)^{\delta_2 (V - \frac{N+2}{2} + 2 - 2g - (B-1))} \Big(\frac{\mu}{\Lambda} \Big)^{\delta_1 (V - V^e - \iota + 2g + (B-1) - 1 + (s+1))} \\
&\qquad\qquad \times \Big(\frac{\mu}{\Lambda} \Big)^{\delta_0 (1 + V^e + \iota - 1 - (s+1))} P^{2V - \frac{N+2}{2}} \Big[\ln \frac{\Lambda}{\Lambda_R} \Big]\;.
\end{aligned} \tag{D.26}$$

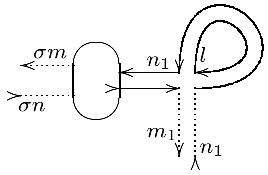
We can sum the contracting propagator over m_1 for fixed n_1, which amounts to replace one factor (4.55) by (4.56) compensated by $s = s_1$ replacing $s = s_1 - 1$.

If k cannot be a summation index in γ_1 then m_1 must be unsummed in γ. We first apply the summation over $\mathfrak{o}[l]$ for given l in γ_1. The result is independent of l so that, for given k, the l-summation can be restricted to the contracting propagator maximised over m_1, n_1. Finally, the remaining $\mathcal{E}^s$-summations are applied. We have to replace in (D.26) $(s+1)$ by s and one factor (4.55) by (4.56).

Finally, let there be no further external leg on the same boundary component as l, k. Now, the number of boundary components remains constant, $B = B_1$. Since $k - l = n_1 - m_1$ is a constant, the required summation over e.g. k has to be estimated by a volume factor (4.54). We thus replace in (D.26) $(B-1) \mapsto B$ and $(s+1) \mapsto s$ and combine one factor (4.55) and the volume factor to (4.57).

In summary, we extend after Λ-integration the scaling law (4.59) for the same degree V to a reduced number N of external lines.

Next, we study the following contraction of the second graph in the first line of (D.1) which gives rise to an inner loop:



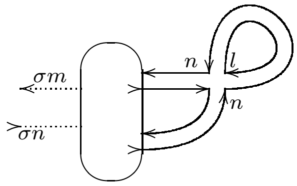
(D.27)

It describes the Λ-variation

$$\Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\sigma m\,\sigma n}[\Lambda] = -\frac{1}{2}\Big(\sum_l Q_{n_1l;ln_1}(\Lambda)\Big)\,A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{m_1n_1;n_1l;ln_1;\sigma m\,\sigma n}[\Lambda]\;. \tag{D.28}$$

The number of loops of the amputated graph is increased by 1 and the number of boundary components remains constant, giving $g = g_1$ and $B = B_1$. Note that $A^{(V,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{m_1n_1;n_1l;ln_1;\sigma m\,\sigma n}$ is independent of l so that the l-summation acts on the propagator only. We estimate the l-summed propagator by (4.57) for the product of (4.55) with a volume factor (4.54). The factor (4.57) compensates the decrease $N = N_1 - 2$, all other exponents remain unchanged when passing from γ_1 to γ. Now, the Λ-integration extends the scaling law (4.59) to a reduced N.

The third graph in the first line of (D.1) leads to the contracted graph

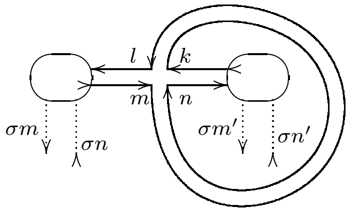


(D.29)

There is one additional loop of the amputated graph, giving $g = g_1$. We have $B = B_1$ if there are further external legs on the boundary component of n and $B = B_1 - 1$ if no further external leg exists on the contracted boundary component. Very similar to (D.27), the l-summation is restricted to the propagator maximised over n, giving a factor (4.57) which compensates $N = N_1 - 2$ in the first exponent of (4.59). For $B = B_1$ the n-summation in (D.29) is provided by the subgraph γ_1, where the additional summation $s_1 = s + 1$ compared with γ compensates the change $V_1^e = V^e + 1$ of external vertices in the second and third exponent of (4.59).

On the other hand, if $B_1 = B + 1$ we have $s = s_1$ and the summation over n has to come from a volume factor (4.54) combined with one factor (4.55) to (4.57). This verifies (4.59) for the contraction (D.29).

The last case for which contractions of two external lines at the same vertex are to investigate is the last vertex in the second line of (D.1). As before in the proof for tree-contractions, we have to distinguish whether the composed vertex under consideration appears inside a tree, in a loop but together with further composed vertices, or in a loop but as the single composed vertex. In the first case we have to analyse the graph



(D.30)

Before the contraction, the indices m,n,k,l were all located on the same loop of the amputated graph and the same boundary component. After the contraction they are split into two loops, $g=g_1$. The number of boundary components is increased by 1 if both resulting boundary components of l,m and k,n carry further external legs, $B=B_1+1$. We have $B=B_1$ if only one of the resulting boundary components of l,m or k,n carries further external legs and $B=B_1-1$ if there are no further external legs on these boundary components. We clearly have $\iota=\iota_1$ and $V^e=V_1^e-1$. Due to index conservation for segments, either k or n is an unknown summation index, and either l or m.

We first consider the case $B=B_1+1$. In both segments of γ_1 to contract there must be at least one unsummed outgoing index, which we can choose to be different from the vertex to contract. We thus take in the propagator the maximum (4.55) over all indices and restrict the required index summations over k,m to the segments of the subgraphs. This means that we have $s_1=s+2$ summations, which compensates the change of the numbers of boundary components $B_1=B-1$, external legs $N_1=N+2$ and external vertices $V_1^e=V^e+1$:

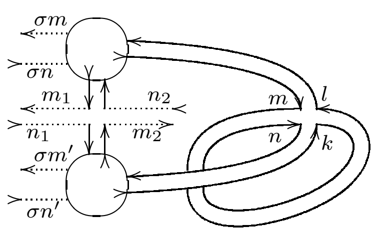
$$
\begin{aligned}
\sum_{\mathcal{E}^s,B=B_1+1} & \Big|\Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,g,\iota)\gamma}_{\sigma m\,\sigma n;\sigma m'\,\sigma n'}[\Lambda]\Big| \\
&\leq \frac{1}{2}\Big(\max_{m,n,k,l}\big|Q_{nm;lk}(\Lambda)\big|\Big)\Big(\max_{l,n}\sum_{k,m,\mathcal{E}^s}\big|A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{\sigma m\,\sigma n;mn;\sigma m'\,\sigma' n;kl}[\Lambda]\big|\Big) \\
&\leq \frac{1}{2}C_0\Big(\frac{\Lambda}{\mu}\Big)^{\delta_2(V-\frac{N+2}{2}+2-2g-(B-1))}\Big(\frac{\mu}{\Lambda}\Big)^{\delta_1(V-(V^e+1)-\iota+2g+(B-1)-1+(s+2))} \\
&\qquad\times\Big(\frac{\mu}{\Lambda}\Big)^{\delta_0(1+(V^e+1)+\iota-1-(s+2))}P^{2V-\frac{N+2}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,.
\end{aligned}
\tag{D.31}
$$

We immediately confirm (4.59). Alternatively, instead of consuming a γ_1-summation to get the k-summation we can also sum the propagator for maximised l,m and given k over n. Compared with (D.31) we have to replace $(s+2)$ by $(s+1)$ and one factor (4.55) by (4.56), ending up in the same exponents.

Next, we investigate the case $B=B_1$ where, for example, the restriction of the boundary component to the left segment does not carry another external leg than m,l. The summation over m in γ_1 is now provided by a volume factor, which means that in (D.31) we have to replace $(s+2)$ by $(s+1)$, $(B-1)$ by B and one factor (4.55) by (4.57). All exponents match again (4.59).

Finally, let us look at the possibility $B=B_1-1$ where the indices m,n,k,l to contract were the only external indices of the boundary component. We thus combine two volume factors (4.54) and two factors (4.55) to two factors (4.57), compensating $(B-1)\mapsto(B+1)$ and $(s+2)\mapsto s$. After Λ-integration we extend (4.59) to a reduced N.

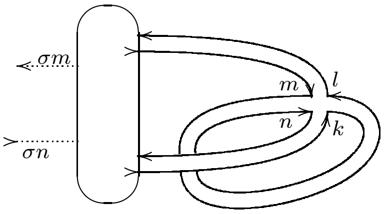
The case that the two sides of the composed vertex to contract are connected but belong to different segments, e.g.



(D.32)

is similar to treat concerning index summations, but for the interpretation of the genus there is a different situation possible. In the amputated subgraph γ_1 the indices m,n and k,l may be situated on *different* loops and thus different boundary components. The contraction joins in this case the two loops, $\tilde{L} = \tilde{L}_1 - 1$, which results due to (4.43) in $g = g_1 + 1$ and $B = B_1 - 1$. There is at least one additional external leg on each of the boundary components of m,n and k,l before the contraction, because in order to close the loop we have to pass through the vertex m_1, n_1, m_2, n_2. Now, we have to replace in (D.31) $(B-1)$ by $(B+1)$ and g by $(g-1)$, confirming (4.59) also in this case. If all indices m,n,k,l are on the same loop in γ_1, the contraction splits it into two and the entire discussion of (D.30) can be used without modification to the present example.

It remains the case



(D.33)

where the two halves of the composed vertex to contract belong to the same segment. Three of the indices m,n,k,l are now summation indices. We have $\iota = \iota_1 - 1$ and $V^e = V^e_1 - 1$. Let first the indices m,n on one hand and k,l on the other hand be situated on different loops of the amputated graph γ_1. These are joint by the contraction, yielding $g = g_1 + 1$. If there remain further external legs on the contracted loop we have $B = B_1 - 1$, otherwise $B = B_1 - 2$. We start with $B = B_1 - 1$. Due to the segmentation index present in γ_1, the induction hypothesis for γ_1 gives us the bound for two additional summations over m,k not present in γ. The third summation is provided by the propagator via (4.56). Assuming $\mathfrak{i}[k] \neq l,n$ in γ_1 we first take in the contracting propagator the maximum over m,l, then sum the m,n-boundary component over m and those indices of $\mathcal{E}^s$ which belong to the m,n-boundary component, followed by the summation of the propagator over n for given k. Finally, we sum γ_1 over the remaining indices of $\mathcal{E}^s$ and over k:

$$
\begin{aligned}
&\sum_{\mathcal{E}^s} \Big| \Lambda \frac{\partial}{\partial \Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{\sigma m\,\sigma n}[\Lambda] \Big| \\
&\qquad \leq \frac{1}{2} \Big(\max_k \sum_n \max_{l,m} \big| Q_{mn;kl}(\Lambda) \big| \Big) \sum_{k,m,\mathcal{E}^s} \big| A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{\sigma m\,\sigma n;mn;kl}[\Lambda] \big| \Big) \\
&\qquad \leq \frac{1}{2} C_1 \Big(\frac{\Lambda}{\mu}\Big)^{\delta_2(V-\frac{N+2}{2}+2-2(g-1)-(B+1))} \Big(\frac{\mu}{\Lambda}\Big)^{\delta_1(1+V-(V^e+1)-(\iota+1)+2(g-1)+(B+1)-1+(s+2))} \\
&\qquad\qquad \times \Big(\frac{\mu}{\Lambda}\Big)^{\delta_0((V^e+1)+(\iota+1)-1-(s+2))} P^{2V-\frac{N+2}{2}} \Big[\ln \frac{\Lambda}{\Lambda_R} \Big] .
\end{aligned}
\qquad \text{(D.34)}
$$

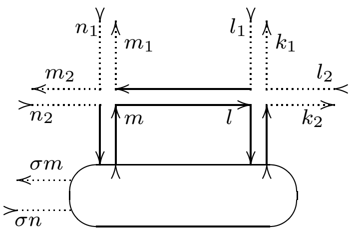
It is essential that the summation over $k - \mathfrak{i}[k]$ is independent.

If there are no external legs on the contracted loop, $B = B_1 - 2$, then we have in γ_1 either $\mathfrak{i}[m] = n$, $\mathfrak{i}[k] = l$ or $\mathfrak{i}[m] = l$, $\mathfrak{i}[k] = n$. In the first case we would first fix n,k and maximise the propagator over m,l. Now, the m-summation restricts to γ_1 with bound independent of n. Thus, the n-summation for given k restricts to the propagator and delivers a factor (4.56), independent of k. However, since $k - \mathfrak{i}[k] \equiv k - l = n - m$ is already exhausted in γ_1, the remaining k-summation has to come from a volume factor. We thus make in (D.34) the replacements $(B{+}1) \mapsto (B{+}2)$, $(s{+}2) \mapsto (s{+}1)$ and combine one factor (4.55) with a volume factor (4.54) to (4.57). The exponents match again (4.59).

Next, we investigate the situation where all indices m, n, k, l are located on the same loop of the amputated subgraph γ_1. In this case the contraction to γ splits that loop into two so that we have $g = g_1$. As before we have $B = B_1 + 1$ if both split loops contain further external legs, $B = B_1$ if only one of the split loops contains further external legs, and $B = B_1 - 1$ if the split loops do not contain further external legs. The discussion is similar as for (D.30), the difference is that three of m, n, k, l are now summations indices, which is taken into account by the replacement of ι in (D.31) by $(\iota + 1)$. We thus finish the verification of (4.59) for self-contractions of a vertex.

D.3 Loop-contractions at different vertices

It remains to check (4.59) for contractions of different vertices of the same graph. The external lines of the two vertices are arranged according to (D.1). We start with two vertices of the type shown as the second graph in (D.1). One possible contraction of their external lines is



(D.35)

assuming that the vertices to contract are located on the same segment in γ_1. One of the indices m, l is a summation index. We first consider the case that the two vertices to contract are located on the same loop of the amputated graph γ_1. The contraction to γ splits that loop into two, giving $g = g_1$. We have $B = B_1 + 1$ if the trajectory starting at l does not leave γ_1 (and γ) in m, whereas $B = B_1$ if m, l are on the same trajectory in γ_1. In case of $B = B_1 + 1$ we keep $\mathfrak{i}[m]$ in γ_1 fixed, take in the propagator the maximum over m, l and restrict the m-summation to γ_1. Due to $V_1^e = V^e$, $\iota_1 = \iota$ and $B_1 = B - 1$ we have in the case that m_1 remains unsummed

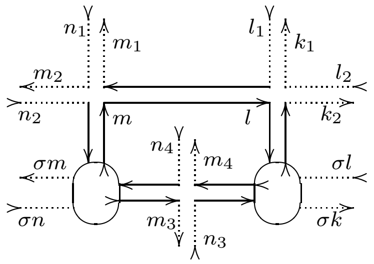
$$
\begin{aligned}
&\sum_{\mathcal{E}^s \not\ni m_1, B=B_1+1} \Big|\Lambda\frac{\partial}{\partial\Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{k_2l_2;k_1l_1;m_1n_1;m_2n_2;\sigma m\,\sigma n}[\Lambda]\Big| \\
&\qquad\leq \frac{1}{2}\Big(\max_{m,l,m_1,l_1} \big|Q_{m_1m;ll_1}(\Lambda)\big|\Big)\Big(\sum_{m,\mathcal{E}^s} \big|A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{k_2l_2;k_1l_1;l_1l;mm_1;m_1n_1;m_2n_2;\sigma m\,\sigma n}[\Lambda]\big|\Big) \\
&\qquad\leq \frac{1}{2}C_0\Big(\frac{\Lambda}{\mu}\Big)^{\delta_2(V-\frac{N+2}{2}+2-2g-(B-1))}\Big(\frac{\mu}{\Lambda}\Big)^{\delta_1(V-V^e-\iota+2g+(B-1)+1+(s+1))} \\
&\qquad\qquad\times\Big(\frac{\mu}{\Lambda}\Big)^{\delta_0(1+V^e+\iota-1-(s+1))} P^{2V-\frac{N+2}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,. \qquad (D.36)
\end{aligned}
$$

Summing additionally over m_1 we replace in (D.36) one factor (4.55) by (4.56). It is clear that this reproduces the exponents of (4.59) correctly.

If $l = \mathfrak{i}[m]$ in γ_1, we have to realise the m-summation by a volume factor. We thus replace in (D.36) $(B-1) \mapsto B$, $(s+1) \mapsto s$ and combine (4.54) with one factor (4.55) to (4.57).

Finally, the two vertices to contract in (D.35) may be located on different loops of the amputated graph γ_1. They are joint by the contraction to γ, giving $g = g_1 + 1$, and because the newly created loop obviously has external legs, we have $B = B_1 - 1$. As separated loops in γ_1, l cannot be the incoming index of the trajectory through m. Therefore, the m-summation gives the same bound as the rhs of (D.36), now with $(B-1)$ replaced by $(B+1)$ and g by $(g-1)$. We have thus extended (4.59) to a reduced N for all types of contractions (D.35).

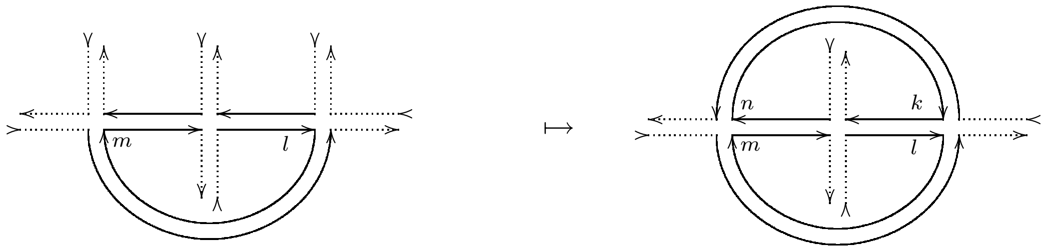
If the vertices to contract are located on different segments in γ_1, e.g.



(D.37)

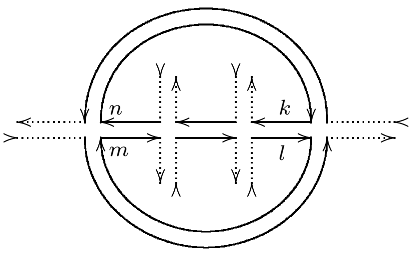
both indices m, l are determined by index conservation for segments. We can thus save an index summation compared with (D.36) and replace there and in its discussed modifications $(s+1)$ by s and $\iota_1 = \iota$ by $\iota_1 = (\iota - 1)$. Since the m-summation is not required, there is effectively an additional summation possible in agreement with (4.47). It is not possible that m and l are located on the same trajectory in γ_1 so that either $g = g_1$, $B = B_1 + 1$ or $g = g_1 + 1$, $B = B_1 - 1$.

I would like to add a few comments on the segmentation index. It is essential that the contraction joins separated segments. For instance, the contraction



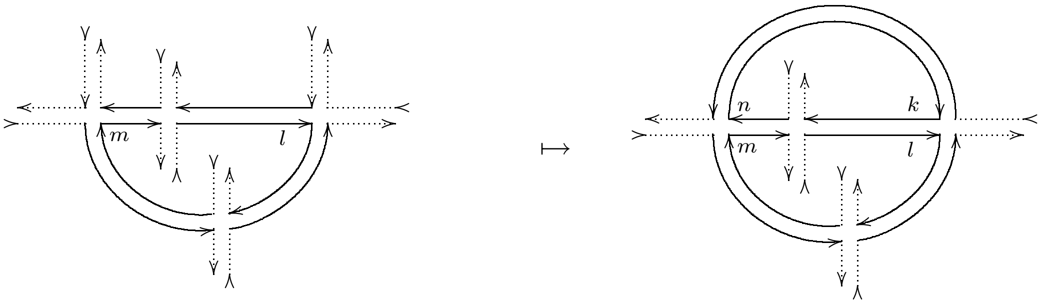
(D.38)

does not increase the segmentation index, because in agreement with Definition 8 on page 45 the number of segments remains constant. The graph on the left has $\iota = 1$, and the internal indices m, l are determined by the external ones. The graph on the right has $\iota = 1$ as well, and now one of the indices n, k becomes a summation index. Having several composed vertices in the middle link does not change the segmentation index:



(D.39)

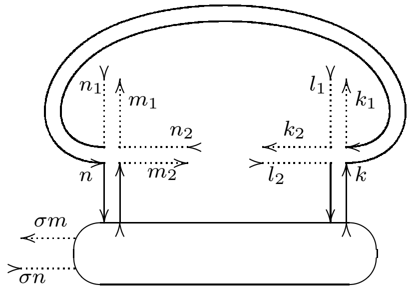
It makes, however, a difference if the two composed vertices are situated on different links:



(D.40)

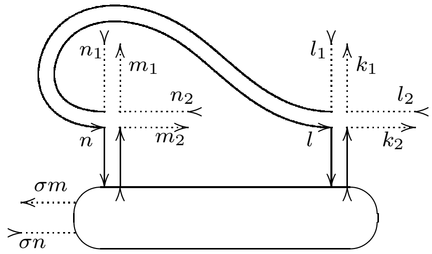
Here, the segmentation index increases from $\iota = 1$ on the left to $\iota = 2$ on the right, in agreement with Definition 8.

The case



(D.41)

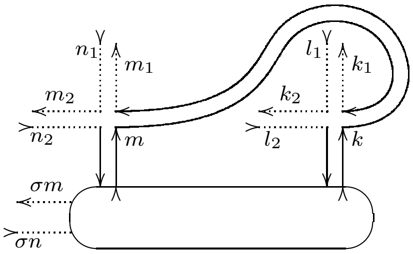
is completely identical to (D.35). In the contraction



(D.42)

the summation index n or l is provided by the propagator, replacing in (D.36) and its modifications $(s+1)$ by s and one factor (4.55) by (4.56). It is not possible that n and l are located on the same trajectory in γ_1 so that either $g = g_1$, $B = B_1 + 1$ or $g = g_1 + 1$, $B = B_1 - 1$.

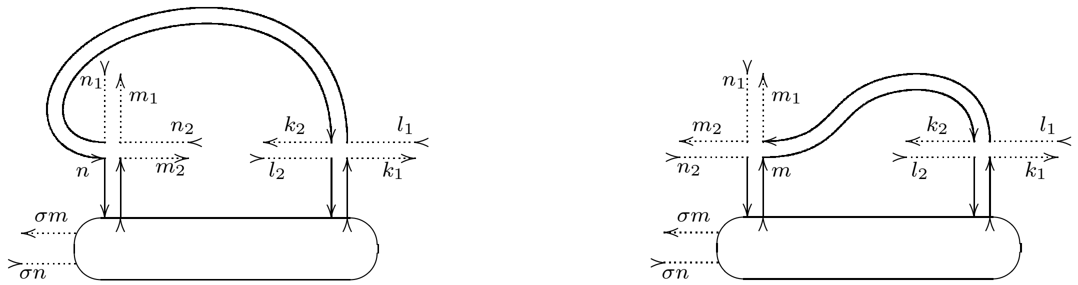
In order to treat the contraction



(D.43)

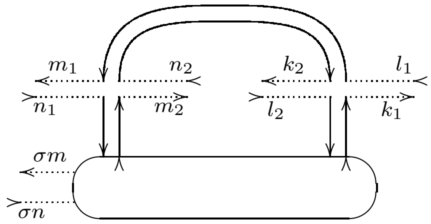
one has to use that the summation over m_1 can due to $m_1 = k + m - k_1$ be transferred as a k-summation of γ_1. The summation over the undetermined index m is applied in the last step.

Finally,



(D.44)

are similar to the ι-increased variant (D.37). The contraction

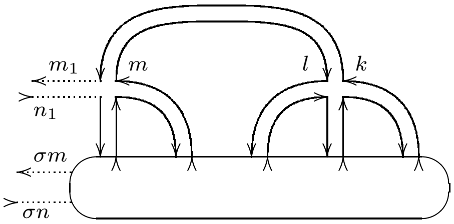


(D.45)

is an example for a realisation of (4.46) where V_c is increased by 2 and S by 1, giving again a segmentation index increased by 1.

It is obvious that the discussion of contractions involving the second and third or two of the third vertices of the first line in (D.1) is analogous.

Let us now study loop contractions which involve the first graph in the second line of (D.1), assuming first that the vertices are situated on the same segment:



(D.46)

We thus have $\iota = \iota_1$ and $V^e = V_1^e - 1$. Two of the summation indices m, k, l are undetermined. Let first the two vertices to contract be located on the same loop of the amputated subgraph γ_1. The contraction splits that loop into two, giving $g = g_1$. Next question concerns the number of boundary components. We have $B = B_1 + 1$ if there are further external legs on the loop through l, m and $B = B_1$ if $l = \mathfrak{i}[m]$ in γ_1. We start with $B = B_1 + 1$. In general, the induction hypothesis provides us with bounds for summations over m and k, because $l \neq \mathfrak{i}[m]$. If m_1 is an unsummed index we thus have

$$\begin{aligned}
\sum_{\mathcal{E}^s \not\ni m_1, B=B_1+1} & \Big| \Lambda \frac{\partial}{\partial \Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{m_1 n_1; \sigma m\, \sigma n}[\Lambda] \Big| \\
& \leq \frac{1}{2} \Big(\max_{m,l,k,m_1} \big| Q_{m_1 m; lk}(\Lambda) \big| \Big) \Big(\sum_{m,k,\mathcal{E}^s} \big| A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{kl; nm_1; m_1 n_1; \sigma m\, \sigma n}[\Lambda] \big| \Big) \\
& \leq \frac{1}{2} C_0 \Big(\frac{\Lambda}{\mu} \Big)^{\delta_2 (V - \frac{N+2}{2} + 2 - 2g - (B-1))} \Big(\frac{\mu}{\Lambda} \Big)^{\delta_1 (V - (V^e+1) - \iota + 2g + (B-1) + 1 + (s+2))} \\
& \qquad \times \Big(\frac{\mu}{\Lambda} \Big)^{\delta_0 (1 + (V^e+1) + \iota - 1 - (s+2))} P^{2V - \frac{N+2}{2}} \Big[\ln \frac{\Lambda}{\Lambda_R} \Big] . \qquad \text{(D.47)}
\end{aligned}$$

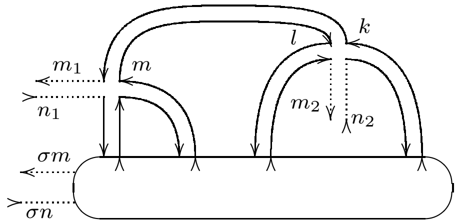
Now, an additional summation over m_1 can immediately be taken into account by replacing the maximised propagator (4.55) by the summed propagator (4.56), in agreement with $(s+2)$ replaced by $(s+1)$. The m_1-summation is applied before the k-summation is carried out.

These considerations require an unsummed outgoing index on the contracted segment of γ_1. If this is not the case, then m_1 has to be the unsummed outgoing index. Now, the l-summation for given m has to be restricted to the propagator and delivers a factor (4.56). The exponents match again (4.59).

Next, for $l = \mathfrak{i}[m]$ in γ_1 we cannot use a summation over m in γ_1 in order to account for the undetermined contraction index, because the incoming index l would change simultaneously. Instead, we have to use a volume factor (4.54) combined with one factor (4.55) to (4.57). Additionally, we have to replace in (D.47) $(s+1)$ by s and $(B-1)$ by B.

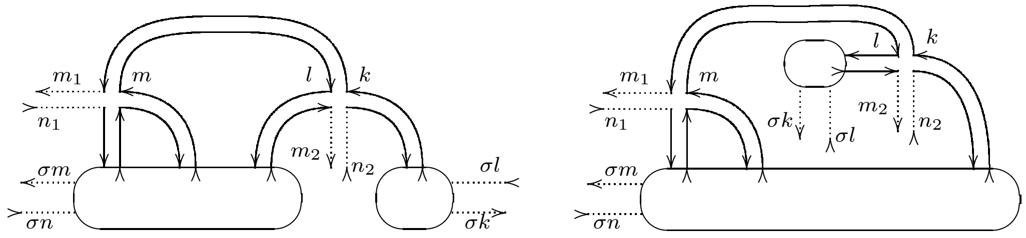
Second, the two vertices to contract may be located on different loops of the amputated graph γ_1. They are joint by the contraction, giving $g = g+1$. Because the loop carries at least the external leg m_1n_1, we necessarily have $B = B_1 - 1$. Now, $l \neq \mathfrak{i}[m]$ in γ_1 so that we use summations over m, k in γ_1, giving the same balance (D.47) for the exponents, with $(B-1) \mapsto (B+1)$ and $g \mapsto (g-1)$.

The discussion is identical for the contraction



(D.48)

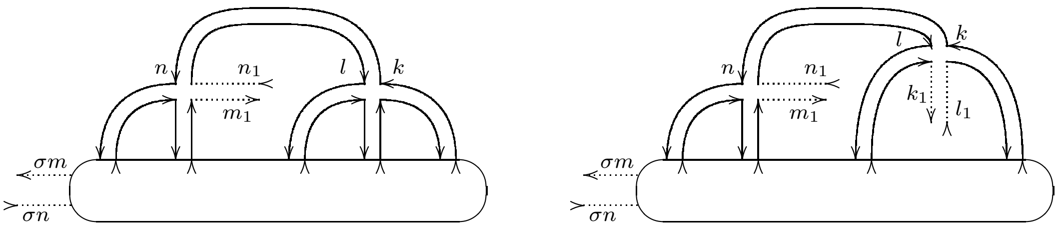
which in the case that k, l belong to the same segment in γ has two undetermined summation indices as well. We thus proceed as in (D.47) and its discussed modification and only have to replace $(V^e + 1)$ by V^e and ι by $(\iota + 1)$. If k, l are situated on different segments in γ, e.g.



(D.49)

there is only one undetermined summation index, which is reflected in the analogue of (D.47) by the fact that the segmentation index remains unchanged, $\iota = \iota_1$. Note that in the right graph of (D.49) we either have $B = B_1 - 1$, $g = g+1$ or $B = B_1 + 1$, $g = g$. Of course we get the same estimations if the segment of γ_1 with external lines $\sigma k, \sigma l$ are connected by several composed vertices to the part of γ_1 with external lines $\sigma m, \sigma n$.

The contractions

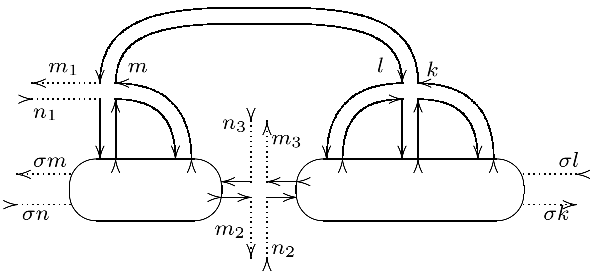


(D.50)

are a little easier because the contracting propagator does not have outgoing indices which for certain summations have to be transferred to the subgraph γ_1. If $n = \mathfrak{i}[k]$ in γ_1, the k-summation for given n, n_1 can be restricted to γ_1 after maximising the propagator over all indices. Since the result for γ_1 is independent of the starting point n, the k-summation can be regarded as a summation over all differences $k - n$. The final summation over all pairs k, n with fixed difference $k - n$ is provided by a volume factor (4.54) combined with (4.55) to (4.57). The balance of exponents is identical to (D.47) and its discussed variants.

It is clear that the analogue of (D.49) with the left vertex connected as in (D.50) is similar to treat.

Next, we discuss the variant of (D.46) where the two vertices to contract belong to different segments in the subgraph γ_1:

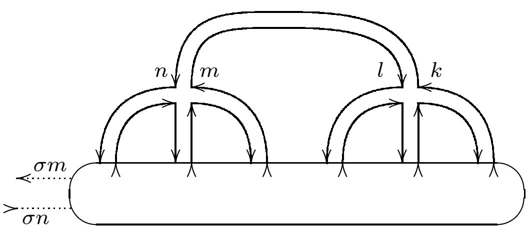


(D.51)

Now, only one of the indices m, k, l is an undetermined summation index, with k being the most natural choice. We therefore get a bound for the Λ-scaling analogous to (D.47) but with ι replaced by $\iota - 1$, reflecting the increase of the segmentation index $\iota = \iota_1 + 1$. There is now an additional index summation possible, here via (4.56) over the index m_1. Note that we have either $B = B_1 - 1$, $g = g + 1$ or $B = B_1 + 1$, $g = g$.

The discussion of the variants of (D.51) with the right vertex taken as the second one in the last line of (D.1) and/or the left vertex arranged as in (D.50) is very similar.

It remains to investigate contractions between two of the vertices in the second line of (D.1). We discuss in detail the contraction



(D.52)

All variants are similar as described between (D.46) and (D.51).

Three of the four summation indices m,n,k,l in (D.52) are undetermined. We clearly have $V_1^e = V^e+2$ and $\iota = \iota_1$. We first consider the case where the four indices m,n,k,l are located on the same loop of the amputated subgraph γ_1. The contraction will split that loop into two, giving $g = g_1$. There are three possibilities for the change of the number of boundary components after the contraction. First, if on both paths of trajectories in γ_1 from n to k and from l to m there are further external legs, we have $B = B_1 + 1$. Second, if on one of these paths there is no further external leg, we have $B = B_1$. Third, if both paths contain no further external legs, i.e. m and k are the outgoing indices of the trajectories starting at l and n, respectively, we have $B = B_1 - 1$.

We start with $B = B_1+1$. Then, $\mathfrak{i}[k]$ and $\mathfrak{i}[m]$ are fixed as external indices so that the induction hypothesis for γ_1 provides the bounds for two summations over k,m. We first apply a possible summation to the outgoing index of the trajectory starting at l. The result is maximised independently from l so that we can restrict the l-summation to the propagator, maximised over k,n with m being fixed. Finally, we apply the summations over k,m and all remaining $\mathcal{E}^s$-summations to γ_1. We thus obtain

$$
\begin{aligned}
&\sum_{\mathcal{E}^s,B=B_1+1}\Big|\Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,g,\iota)\gamma}_{\sigma m\,\sigma n}[\Lambda]\Big| \\
&\qquad\leq \frac{1}{2}\Big(\max_m\sum_l\max_{n,k}\big|Q_{nm;lk}(\Lambda)\big|\Big)\Big(\sum_{m,k,\mathcal{E}^s}\big|A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{mn;\sigma m\,\sigma n;kl}[\Lambda]\big|\Big) \\
&\qquad\leq \frac{1}{2}C_1\Big(\frac{\Lambda}{\mu}\Big)^{\delta_2(V-\frac{N+2}{2}+2-2g-(B-1))}\Big(\frac{\mu}{\Lambda}\Big)^{\delta_1(1+V-(V^e+2)+\iota+1+2g+(B-1)-1+(s+2))} \\
&\qquad\qquad\times\Big(\frac{\mu}{\Lambda}\Big)^{\delta_0((V^e+2)+\iota-1-(s+2))}P^{2V-\frac{N+2}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,. \qquad\qquad (D.53)
\end{aligned}
$$

The Λ-integration verifies (4.59) in the topological situation under consideration.

Next, we discuss the case $B = B_1$, assuming e.g. $l = \mathfrak{i}[m]$ in γ_1. We maximise the propagator over k,n for given l so that the m-summation can be restricted to γ_1. Next, we apply the $\mathcal{E}^s$-summations and the k-summation to γ_1, still for given l. The final l-summation counts the number of graphs with different l, giving the bound (4.55) of the propagator times a volume factor. In any case the required modifications of (D.53), in particular $(B-1)\mapsto B$, lead to the correct exponents of (4.59).

If $B = B_1 - 1$, i.e. $l = \mathfrak{i}[m]$ and $n = \mathfrak{i}[k]$, we take in the propagator the maximum over n,k so that for given l the m-summation can be restricted to γ_1. The result of that summation is bounded independently of l. Thus, each summand only fixes $m-l = n-k$, and the remaining freedom for the summation indices is exhausted by two volume factors and the bound (4.55) for the propagator. We thus replace in (D.53) $(s+2)\mapsto(s+1)$, $(B-1)\mapsto(B+1)$ and one factor (4.56) by (4.55). Then, two factors (4.55) are merged with two volume factors (4.54) to give two factors (4.57).

Finally, we have to consider the case where m,n are located on a different loop of the amputated subgraph γ_1 than k,l. The contraction joins these loops, giving $g = g_1+1$. If the resulting loop carries at least one external leg we have $B = B_1 - 1$, whereas we get $B = B_1-2$ if the resulting loop does not carry any external legs. We first consider the case that there is a further external leg on the n,m-loop in γ_1. We take in the propagator the maximum over k, n and sum the subgraph for given l, n over k and possibly the outgoing index of the n-trajectory. The result is independent of l, n. Next, we sum the propagator for given m over l and finally apply the remaining $\mathcal{E}^s$-summations and the summation over m to γ_1. We get the same estimates as in (D.53) with $(B-1)$ replaced by $(B+1)$ and g by $(g-1)$.

If there are no further external legs on the contracted loop we would maximise the propagator over k, n, then sum γ_1 over k for given l, next sum the propagator over l for given m. For each resulting pair k, l the remaining m-summation leaves $m-n$ constant. We thus have to use a volume factor in order to exhaust the freedom of $m-n$, combining one factor (4.55) and the volume factor (4.54) to (4.57). We thus confirm (4.59) for any contraction of the form (D.52).

It is obvious that all examples not discussed in detail are treated in the same manner. We conclude that (4.59) provides the correct bounds for the interaction coefficients of ϕ^4-matrix model with cut-off propagator described by the three exponents $\delta_0, \delta_1, \delta_2$. $\square$

The proof also shows that the genus is never decreasing in the contraction, which explains why subgraphs of planar graphs necessarily have genus $g = 0$.

E On composite propagators

E.1 Identities for differences of ribbon graphs

I continue here the discussion of Section 5.2 on composite propagators generated by differences of interaction coefficients.

After having derived (5.17) on page 56, we now have a look at (5.9). Since γ is one-particle irreducible, we get for a certain permutation π ensuring the history of integrations the following linear combination:

$$
\begin{aligned}
&\Lambda' \frac{\partial}{\partial \Lambda'} \widehat{A^{(V)\gamma}_{{}^{m^1+1}_{m^2}\,{}^{n^1+1}_{n^2};\,{}^{n^1}_{n^2}\,{}^{m^1}_{m^2}}}[\Lambda'] - \sqrt{(m^1{+}1)(n^1{+}1)}\,\Lambda' \frac{\partial}{\partial \Lambda'} \widehat{A^{(V)\gamma}_{{}^1_0{}^1_0;{}^0_0{}^0_0}}[\Lambda'] \\
&= \dots \Bigg\{ \prod_{i=1}^{p-1} Q_{{}^{n^1+1}_{n^2} k_{\pi_i}; k_{\pi_i} {}^{n^1+1}_{n^2}}(\Lambda_{\pi_i}) Q_{{}^{n^1+1}_{n^2} (k_{\pi_p}+1^+); k_{\pi_p} {}^{n^1}_{n^2}}(\Lambda_{\pi_p}) \prod_{i=p+1}^{a} Q_{{}^{n^1}_{n^2} k_{\pi_i}; k_{\pi_i} {}^{n^1}_{n^2}}(\Lambda_{\pi_i}) \\
&\qquad \times \prod_{j=1}^{q-1} Q_{{}^{m^1}_{m^2} l_{\pi_j}; l_{\pi_j} {}^{m^1}_{m^2}}(\Lambda_{\pi_j}) Q_{{}^{m^1}_{m^2} l_{\pi_q}; (l_{\pi_q}+1^+) {}^{m^1+1}_{m^2}}(\Lambda_{\pi_q}) \prod_{j=q+1}^{b} Q_{{}^{m^1+1}_{m^2} l_{\pi_j}; l_{\pi_j} {}^{m^1+1}_{m^2}}(\Lambda_{\pi_j}) \\
&\quad - \sqrt{(m^1{+}1)(n^1{+}1)} \prod_{i=1}^{p-1} Q_{{}^1_0 k_{\pi_i}; k_{\pi_i} {}^1_0}(\Lambda_{\pi_i}) Q_{{}^1_0 (k_{\pi_p}+1^+); k_{\pi_p} {}^0_0}(\Lambda_{\pi_p}) \prod_{i=p+1}^{a} Q_{{}^0_0 k_{\pi_i}; k_{\pi_i} {}^0_0}(\Lambda_{\pi_i}) \\
&\qquad \times \prod_{j=1}^{q} Q_{{}^0_0 l_{\pi_j}; l_{\pi_j} {}^0_0}(\Lambda_{\pi_j}) Q_{{}^0_0 l_{\pi_q}; (l_{\pi_q}+1^+) {}^1_0}(\Lambda_{\pi_q}) \prod_{j=q+1}^{b} Q_{{}^1_0 l_{\pi_j}; l_{\pi_j} {}^1_0}(\Lambda_{\pi_j}) \Bigg\} \\
&= \dots \Bigg\{ \Bigg(\prod_{i=1}^{p-1} Q_{{}^{n^1+1}_{n^2} k_{\pi_i}; k_{\pi_i} {}^{n^1+1}_{n^2}}(\Lambda_{\pi_i}) Q_{{}^{n^1+1}_{n^2} (k_{\pi_p}+1^+); k_{\pi_p} {}^{n^1}_{n^2}}(\Lambda_{\pi_p}) \prod_{i=p+1}^{a} Q_{{}^{n^1}_{n^2} k_{\pi_i}; k_{\pi_i} {}^{n^1}_{n^2}}(\Lambda_{\pi_i}) \\
&\qquad - \sqrt{n^1{+}1} \prod_{i=1}^{p-1} Q_{{}^1_0 k_{\pi_i}; k_{\pi_i} {}^1_0}(\Lambda_{\pi_i}) Q_{{}^1_0 (k_{\pi_p}+1^+); k_{\pi_p} {}^0_0}(\Lambda_{\pi_p}) \prod_{i=p+1}^{a} Q_{{}^0_0 k_{\pi_i}; k_{\pi_i} {}^0_0}(\Lambda_{\pi_i}) \Bigg) \\
&\quad \times \prod_{j=1}^{q-1} Q_{{}^{m^1}_{m^2} l_{\pi_j}; l_{\pi_j} {}^{m^1}_{m^2}}(\Lambda_{\pi_j}) Q_{{}^{m^1}_{m^2} l_{\pi_q}; (l_{\pi_q}+1^+) {}^{m^1+1}_{m^2}}(\Lambda_{\pi_q}) \prod_{j=q+1}^{b} Q_{{}^{m^1+1}_{m^2} l_{\pi_j}; l_{\pi_j} {}^{m^1+1}_{m^2}}(\Lambda_{\pi_j}) \qquad \text{(E.1a)} \\
&\quad + \sqrt{n^1{+}1} \prod_{i=1}^{p-1} Q_{{}^1_0 k_{\pi_i}; k_{\pi_i} {}^1_0}(\Lambda_{\pi_i}) Q_{{}^1_0 (k_{\pi_p}+1^+); k_{\pi_p} {}^0_0}(\Lambda_{\pi_p}) \prod_{i=p+1}^{a} Q_{{}^0_0 k_{\pi_i}; k_{\pi_i} {}^0_0}(\Lambda_{\pi_i}) \\
&\quad \times \Bigg(\prod_{j=1}^{q-1} Q_{{}^{m^1}_{m^2} l_{\pi_j}; l_{\pi_j} {}^{m^1}_{m^2}}(\Lambda_{\pi_j}) Q_{{}^{m^1}_{m^2} l_{\pi_q}; (l_{\pi_q}+1^+) {}^{m^1+1}_{m^2}}(\Lambda_{\pi_q}) \prod_{i=q+1}^{b} Q_{{}^{m^1+1}_{m^2} l_{\pi_j}; l_{\pi_j} {}^{m^1+1}_{m^2}}(\Lambda_{\pi_j}) \\
&\qquad - \sqrt{(m^1{+}1)} \prod_{j=1}^{q} Q_{{}^0_0 l_{\pi_j}; l_{\pi_j} {}^0_0}(\Lambda_{\pi_j}) Q_{{}^0_0 l_{\pi_q}; (l_{\pi_q}+1^+) {}^1_0}(\Lambda_{\pi_q}) \prod_{j=q+1}^{b} Q_{{}^1_0 l_{\pi_j}; l_{\pi_j} {}^1_0}(\Lambda_{\pi_j}) \Bigg) \Bigg\}\,, \qquad \text{(E.1b)}
\end{aligned}
$$

with $1^+ := {}^1_0$.

We further analyse the difference represented by the first two lines of (E.1a):

$$
\begin{aligned}
&\Big(\prod_{i=1}^{p-1} Q_{{n^1+1 \atop n^2} k_{\pi_i};k_{\pi_i} {n^1+1 \atop n^2}}(\Lambda_{\pi_i}) Q_{{n^1+1 \atop n^2}(k_{\pi_p}+1^+);k_{\pi_p} {n^1 \atop n^2}}(\Lambda_{\pi_p}) \prod_{i=p+1}^{a} Q_{{n^1 \atop n^2} k_{\pi_i};k_{\pi_i} {n^1 \atop n^2}}(\Lambda_{\pi_i}) \\
&\qquad - \sqrt{n^1{+}1} \prod_{i=1}^{p-1} Q_{{1 \atop 0} k_{\pi_i};k_{\pi_i} {1 \atop 0}}(\Lambda_{\pi_i}) Q_{{1 \atop 0}(k_{\pi_p}+1^+);k_{\pi_p} {0 \atop 0}}(\Lambda_{\pi_p}) \prod_{i=p+1}^{a} Q_{{0 \atop 0} k_{\pi_i};k_{\pi_i} {0 \atop 0}}(\Lambda_{\pi_i})\Big) \\
&= \bigg(\Big(\prod_{i=1}^{p-1} Q_{{n^1+1 \atop n^2} k_{\pi_i};k_{\pi_i} {n^1+1 \atop n^2}}(\Lambda_{\pi_i}) - \prod_{i=1}^{p-1} Q_{{1 \atop 0} k_{\pi_i};k_{\pi_i} {1 \atop 0}}(\Lambda_{\pi_i})\Big) \\
&\qquad \times Q_{{n^1+1 \atop n^2}(k_{\pi_p}+1^+);k_{\pi_p} {n^1 \atop n^2}}(\Lambda_{\pi_p}) \prod_{i=p+1}^{a} Q_{{n^1 \atop n^2} k_{\pi_i};k_{\pi_i} {n^1 \atop n^2}}(\Lambda_{\pi_i})\bigg) && \text{(E.2a)} \\
&+ \prod_{i=1}^{p-1} Q_{{1 \atop 0} k_{\pi_i};k_{\pi_i} {1 \atop 0}}(\Lambda_{\pi_i}) \Big(\mathcal{Q}^{(+\frac{1}{2})}_{{n^1+1 \atop n^2}(k_{\pi_p}+1^+);k_{\pi_p} {n^1 \atop n^2}}(\Lambda_{\pi_p})\Big) \prod_{i=p+1}^{a} Q_{{n^1 \atop n^2} k_{\pi_i};k_{\pi_i} {n^1 \atop n^2}}(\Lambda_{\pi_i}) && \text{(E.2b)} \\
&+ \sqrt{n^1{+}1}\Big(\prod_{i=1}^{p-1} Q_{{1 \atop 0} k_{\pi_i};k_{\pi_i} {1 \atop 0}}(\Lambda_{\pi_i}) Q_{{1 \atop 0}(k_{\pi_p}+1^+);k_{\pi_p} {0 \atop 0}}(\Lambda_{\pi_p}) \\
&\qquad \times \Big(\prod_{i=p+1}^{a} Q_{{n^1 \atop n^2} k_{\pi_i};k_{\pi_i} {n^1 \atop n^2}}(\Lambda_{\pi_i}) - \prod_{i=p+1}^{a} Q_{{0 \atop 0} k_{\pi_i};k_{\pi_i} {0 \atop 0}}(\Lambda_{\pi_i})\Big)\Big)\,. && \text{(E.2c)}
\end{aligned}
$$

According to (5.17), the two lines (E.2a) and (E.2c) yield graphs having one composite propagator (5.15a), whereas the line (E.2b) yields a graph having one composite propagator[38] (5.15c). In total, we get from (E.1) $a{+}b$ graphs with composite propagators (5.15a) or (5.15c). The discussion of (5.10) is similar.

Second, we treat that contribution to (5.8) which consists of graphs with constant index along the trajectories:

$$
\begin{aligned}
&\Lambda' \frac{\partial}{\partial \Lambda'} \widehat{A^{(V)\gamma}_{{m^1 \atop m^2}{n^1 \atop n^2};{n^1 \atop n^2}{m^1 \atop m^2}}}[\Lambda'] - \Lambda' \frac{\partial}{\partial \Lambda'} \widehat{A^{(V)\gamma}_{{0 \atop 0}{0 \atop 0};{0 \atop 0}{0 \atop 0}}}[\Lambda'] \\
&- m^1\Big(\Lambda' \frac{\partial}{\partial \Lambda'} \widehat{A^{(V)\gamma}_{{1 \atop 0}{0 \atop 0};{0 \atop 0}{1 \atop 0}}}[\Lambda'] - \Lambda' \frac{\partial}{\partial \Lambda'} \widehat{A^{(V)\gamma}_{{0 \atop 0}{0 \atop 0};{0 \atop 0}{0 \atop 0}}}[\Lambda']\Big) - m^2\Big(\Lambda' \frac{\partial}{\partial \Lambda'} \widehat{A^{(V)\gamma}_{{0 \atop 1}{0 \atop 0};{0 \atop 0}{0 \atop 1}}}[\Lambda'] - \Lambda' \frac{\partial}{\partial \Lambda'} \widehat{A^{(V)\gamma}_{{0 \atop 0}{0 \atop 0};{0 \atop 0}{0 \atop 0}}}[\Lambda']\Big) \\
&- n^1\Big(\Lambda' \frac{\partial}{\partial \Lambda'} \widehat{A^{(V)\gamma}_{{0 \atop 0}{1 \atop 0};{1 \atop 0}{0 \atop 0}}}[\Lambda'] - \Lambda' \frac{\partial}{\partial \Lambda'} \widehat{A^{(V)\gamma}_{{0 \atop 0}{0 \atop 0};{0 \atop 0}{0 \atop 0}}}[\Lambda']\Big) - n^2\Big(\Lambda' \frac{\partial}{\partial \Lambda'} \widehat{A^{(V)\gamma}_{{0 \atop 0}{0 \atop 1};{0 \atop 1}{0 \atop 0}}}[\Lambda'] - \Lambda' \frac{\partial}{\partial \Lambda'} \widehat{A^{(V)\gamma}_{{0 \atop 0}{0 \atop 0};{0 \atop 0}{0 \atop 0}}}[\Lambda']\Big) \\
&= \dots \bigg\{ \prod_{i=1}^{a} Q_{{n^1 \atop n^2} k_{\pi_i};k_{\pi_i} {n^1 \atop n^2}}(\Lambda_{\pi_i}) \prod_{j=1}^{b} Q_{{m^1 \atop m^2} l_{\pi_j};l_{\pi_j} {m^1 \atop m^2}}(\Lambda_{\pi_j}) - \prod_{i=1}^{a} Q_{{0 \atop 0} k_{\pi_i};k_{\pi_i} {0 \atop 0}}(\Lambda_{\pi_i}) \prod_{j=1}^{b} Q_{{0 \atop 0} l_{\pi_j};l_{\pi_j} {0 \atop 0}}(\Lambda_{\pi_j}) \\
&- \prod_{i=1}^{a} Q_{{0 \atop 0} k_{\pi_i};k_{\pi_i} {0 \atop 0}}(\Lambda_{\pi_i}) \bigg(m^1\Big(\prod_{j=1}^{b} Q_{{1 \atop 0} l_{\pi_j};l_{\pi_j} {1 \atop 0}}(\Lambda_{\pi_j}) - \prod_{j=1}^{b} Q_{{0 \atop 0} l_{\pi_j};l_{\pi_j} {0 \atop 0}}(\Lambda_{\pi_j})\Big) \\
&\qquad + m^2\Big(\prod_{j=1}^{b} Q_{{0 \atop 1} l_{\pi_j};l_{\pi_j} {0 \atop 1}}(\Lambda_{\pi_j}) - \prod_{j=1}^{b} Q_{{0 \atop 0} l_{\pi_j};l_{\pi_j} {0 \atop 0}}(\Lambda_{\pi_j})\Big)\bigg)
\end{aligned}
$$

---

[38]For the power-counting estimation one has to take into account that the estimation (5.22) yields $\sqrt{n^1{+}1}|Q_{{1 \atop 0}(k_{\pi_p}+1^+);k_{\pi_p} {0 \atop 0}}(\Lambda_{\pi_p})| \leq \frac{C}{\Omega\theta\Lambda^2}\big(\frac{n^1+1}{\theta\Lambda^2}\big)^{\frac{1}{2}}$. This means that the prefactor $\sqrt{n^1{+}1}$ in (E.2c) combines actually to the ratio $\big(\frac{n^1+1}{\theta\Lambda^2}\big)^{\frac{1}{2}}$ which is required for the (5.30)-term in Proposition 13.3.

$$
\begin{aligned}
&-\Big(n^1\Big(\prod_{i=1}^{a} Q_{{}^{1}_{0}k_{\pi_i};k_{\pi_i}{}^{1}_{0}}(\Lambda_{\pi_i}) - \prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\Big) \\
&\qquad +n^2\Big(\prod_{i=1}^{a} Q_{{}^{0}_{1}k_{\pi_i};k_{\pi_i}{}^{0}_{1}}(\Lambda_{\pi_i}) - \prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\Big)\Big) \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big\} \\
&= \ldots \Big\{\Big(\Big(\prod_{i=1}^{a} Q_{{}^{n^1}_{n^2}k_{\pi_i};k_{\pi_i}{}^{n^1}_{n^2}}(\Lambda_{\pi_i}) - \prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\Big) \\
&\qquad \times \Big(\prod_{j=1}^{b} Q_{{}^{m^1}_{m^2}l_{\pi_j};l_{\pi_j}{}^{m^1}_{m^2}}(\Lambda_{\pi_j}) - \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big)\Big) && \text{(E.3a)} \\
&+ \prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\Big(\Big(\prod_{j=1}^{b} Q_{{}^{m^1}_{m^2}l_{\pi_j};l_{\pi_j}{}^{m^1}_{m^2}}(\Lambda_{\pi_i}) - \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big) \\
&\qquad - m^1\Big(\prod_{j=1}^{b} Q_{{}^{1}_{0}l_{\pi_j};l_{\pi_j}{}^{1}_{0}}(\Lambda_{\pi_j}) - \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big) \\
&\qquad - m^2\Big(\prod_{j=1}^{b} Q_{{}^{0}_{1}l_{\pi_j};l_{\pi_j}{}^{0}_{1}}(\Lambda_{\pi_j}) - \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big)\Big) && \text{(E.3b)} \\
&+ \Big(\Big(\prod_{i=1}^{a} Q_{{}^{n^1}_{n^2}k_{\pi_i};k_{\pi_i}{}^{n^1}_{n^2}}(\Lambda_{\pi_i}) - \prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\Big) \\
&\qquad - n^1\Big(\prod_{i=1}^{a} Q_{{}^{1}_{0}k_{\pi_i};k_{\pi_i}{}^{1}_{0}}(\Lambda_{\pi_i}) - \prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\Big) \\
&\qquad -n^2\Big(\prod_{i=1}^{a} Q_{{}^{0}_{1}k_{\pi_i};k_{\pi_i}{}^{0}_{1}}(\Lambda_{\pi_i}) - \prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\Big)\Big) \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big\}\ . && \text{(E.3c)}
\end{aligned}
$$

It is clear from (5.17) that the part corresponding to the two lines (E.3a) can be written as a sum of graphs containing (at different trajectories) two composite propagators $\mathcal{Q}^{(0)}_{{}^{n^1}_{n^2}k_{\pi_i};k_{\pi_i}{}^{n^1}_{n^2}}(\Lambda_{\pi_i})$ and $\mathcal{Q}^{(0)}_{{}^{m^1}_{m^2}l_{\pi_j};l_{\pi_j}{}^{m^1}_{m^2}}$ of type (5.15a). We further analyse the expression (E.3b):

$$
\begin{aligned}
&\Big(\prod_{j=1}^{b} Q_{{}^{m^1}_{m^2}l_{\pi_j};l_{\pi_j}{}^{m^1}_{m^2}}(\Lambda_{\pi_j}) - \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big) \\
&\qquad - m^1\Big(\prod_{j=1}^{b} Q_{{}^{1}_{0}l_{\pi_j};l_{\pi_j}{}^{1}_{0}}(\Lambda_{\pi_j}) - \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big) \\
&\qquad - m^2\Big(\prod_{j=1}^{b} Q_{{}^{0}_{1}l_{\pi_j};l_{\pi_j}{}^{0}_{1}}(\Lambda_{\pi_j}) - \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big) \\
&= \mathcal{Q}^{(1)}_{{}^{m^1}_{m^2}l_{\pi_1};l_{\pi_1}{}^{m^1}_{m^2}}(\Lambda_{\pi_1}) \prod_{j=2}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j}) && \text{(E.4a)}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\mathcal{Q}^{(0)}_{{}^{m^1}_{m^2}l_{\pi_1};l_{\pi_1}{}^{m^1}_{m^2}}(\Lambda_{\pi_1})\Big(\prod_{j=2}^{b}Q_{{}^{m^1}_{m^2}l_{\pi_j};l_{\pi_j}{}^{m^1}_{m^2}}(\Lambda_{\pi_j})-\prod_{j=2}^{b}Q_{{}^0_0l_{\pi_j};l_{\pi_j}{}^0_0}(\Lambda_{\pi_j})\Big)\\
&\qquad-m^1\mathcal{Q}^{(0)}_{{}^1_0l_{\pi_1};l_{\pi_1}{}^1_0}(\Lambda_{\pi_1})\Big(\prod_{j=2}^{b}Q_{{}^1_0l_{\pi_j};l_{\pi_j}{}^1_0}(\Lambda_{\pi_j})-\prod_{j=2}^{b}Q_{{}^0_0l_{\pi_j};l_{\pi_j}{}^0_0}(\Lambda_{\pi_j})\Big)\\
&\qquad-m^2\mathcal{Q}^{(0)}_{{}^0_1l_{\pi_1};l_{\pi_1}{}^0_1}(\Lambda_{\pi_1})\Big(\prod_{j=2}^{b}Q_{{}^0_1l_{\pi_j};l_{\pi_j}{}^0_1}(\Lambda_{\pi_j})-\prod_{j=2}^{b}Q_{{}^0_0l_{\pi_j};l_{\pi_j}{}^0_0}(\Lambda_{\pi_j})\Big)\Big) \qquad \text{(E.4b)}\\
&+Q_{{}^0_0l_{\pi_1};l_{\pi_1}{}^0_0}(\Lambda_{\pi_1})\Big(\Big(\prod_{j=2}^{b}Q_{{}^{m^1}_{m^2}l_{\pi_j};l_{\pi_j}{}^{m^1}_{m^2}}(\Lambda_{\pi_j})-\prod_{j=2}^{b}Q_{{}^0_0l_{\pi_j};l_{\pi_j}{}^0_0}(\Lambda_{\pi_j})\Big)\\
&\qquad-m^1\Big(\prod_{j=2}^{b}Q_{{}^1_0l_{\pi_j};l_{\pi_j}{}^1_0}(\Lambda_{\pi_j})-\prod_{j=2}^{b}Q_{{}^0_0l_{\pi_j};l_{\pi_j}{}^0_0}(\Lambda_{\pi_j})\Big)\\
&\qquad-m^2\Big(\prod_{j=2}^{b}Q_{{}^0_1l_{\pi_j};l_{\pi_j}{}^0_1}(\Lambda_{\pi_j})-\prod_{j=2}^{b}Q_{{}^0_0l_{\pi_j};l_{\pi_j}{}^0_0}(\Lambda_{\pi_j})\Big)\Big)\;. \qquad \text{(E.4c)}
\end{aligned}
$$

The part (E.4a) gives rise to graphs with one propagator (5.15b). Due to (5.17) the part (E.4b) yields graphs with two propagators[39] (5.15a) appearing on the same trajectory. Finally, the part (E.4c) has the same structure as the lhs of the equation, now starting with $j=2$. After iteration we obtain further graphs of the type (E.4a) and (E.4b).

Finally, we look at that contribution to (5.8) which consists of graphs where one index component jumps forward and backward in the n^1-component. We can directly use the decomposition derived in (E.3) regarding, if the n^1-index jumps up,

$$
\begin{aligned}
&\prod_{i=1}^{a}Q_{{}^{n^1}_{n^2}k_{\pi_i};k_{\pi_i}{}^{n^1}_{n^2}}(\Lambda_{\pi_i})\\
&\quad\mapsto\prod_{i=1}^{p-1}Q_{{}^{n^1}_{n^2}k_{\pi_i};k_{\pi_i}{}^{n^1}_{n^2}}(\Lambda_{\pi_i})Q_{{}^{n^1}_{n^2}k_{\pi_p};(k_{\pi_p}+1){}^{n^1+1}_{n^2}}(\Lambda_{\pi_p})\prod_{i=p+1}^{q-1}Q_{{}^{n^1+1}_{n^2}k_{\pi_i};k_{\pi_i}{}^{n^1+1}_{n^2}}(\Lambda_{\pi_i})\\
&\qquad\quad\times Q_{{}^{n^1+1}_{n^2}(k_{\pi_q}+1^+);k_{\pi_q}{}^{n^1}_{n^2}}(\Lambda_{\pi_q})\prod_{i=q+1}^{a}Q_{{}^{n^1}_{n^2}k_{\pi_i};k_{\pi_i}{}^{n^1}_{n^2}}(\Lambda_{\pi_i})\;. \qquad \text{(E.5)}
\end{aligned}
$$

This requires to process (E.3) slightly differently. The two parts (E.3a) and (E.3b) need no further discussion, as they lead to graphs having a composite propagator (5.15a) on the m-trajectory. We write (E.3c) as follows:

$$
\begin{aligned}
\text{(E.3c)}=&\Big(\Big(\prod_{i=1}^{a}Q_{{}^{n^1}_{n^2}k_{\pi_i};k_{\pi_i}{}^{n^1}_{n^2}}(\Lambda_{\pi_i})-(n^1{+}1)\prod_{i=1}^{a}Q_{{}^0_0k_{\pi_i};k_{\pi_i}{}^0_0}(\Lambda_{\pi_i})\Big) \qquad \text{(E.6a)}\\
&\quad-n^1\Big(\prod_{i=1}^{a}Q_{{}^1_0k_{\pi_i};k_{\pi_i}{}^1_0}(\Lambda_{\pi_i})-2\prod_{i=1}^{a}Q_{{}^0_0k_{\pi_i};k_{\pi_i}{}^0_0}(\Lambda_{\pi_i})\Big) \qquad \text{(E.6b)}\\
&\quad-n^2\Big(\prod_{i=1}^{a}Q_{{}^0_1k_{\pi_i};k_{\pi_i}{}^0_1}(\Lambda_{\pi_i})-\prod_{i=1}^{a}Q_{{}^0_0k_{\pi_i};k_{\pi_i}{}^0_0}(\Lambda_{\pi_i})\Big)\Big)\prod_{j=1}^{b}Q_{{}^0_0l_{\pi_j};l_{\pi_j}{}^0_0}(\Lambda_{\pi_j})\;. \qquad \text{(E.6c)}
\end{aligned}
$$

---

[39]For the power-counting estimation one has to take into account that the product $m^1\mathcal{Q}^{(0)}_{{}^1_0l_{\pi_1};l_{\pi_1}{}^1_0}(\Lambda_{\pi_1})$ is according to (5.26) bounded by $\frac{C}{\Omega\theta\Lambda^2}\big(\frac{m^1}{\theta\Lambda^2}\big)$. This means that the prefactors m^1,m^2 in (E.4b) combine actually to the ratio $\frac{m^r}{\theta\Lambda^2}$ which is required for the (5.30)-term in Proposition 13.2.

The part (E.6c) leads according to (5.17) and (E.5) to graphs either with composite propagators (5.15a) or with propagators

$$Q_{\substack{1\\1}\substack{l^1+1\\l^2};\substack{l^1\\l^2}\substack{0\\1}} - Q_{\substack{1\\0}\substack{l^1+1\\l^2};\substack{l^1\\l^2}\substack{0\\0}}\;. \tag{E.7}$$

Inserting (E.5) into (E.6a) we have

$$\begin{aligned}
&\Big(\prod_{i=1}^{a} Q_{\substack{n^1\\n^2}k_{\pi_i};k_{\pi_i}\substack{n^1\\n^2}}(\Lambda_{\pi_i}) - (n^1{+}1)\prod_{i=1}^{a} Q_{\substack{0\\0}k_{\pi_i};k_{\pi_i}\substack{0\\0}}(\Lambda_{\pi_i})\Big)\\
&\xmapsto{\text{(E.5)}} \Big(\prod_{i=1}^{p-1} Q_{\substack{n^1\\n^2}k_{\pi_i};k_{\pi_i}\substack{n^1\\n^2}}(\Lambda_{\pi_i}) - \prod_{i=1}^{p-1} Q_{\substack{0\\0}k_{\pi_i};k_{\pi_i}\substack{0\\0}}(\Lambda_{\pi_i})\Big) Q_{\substack{n^1\\n^2}k_{\pi_p};(k_{\pi_p}+1)\substack{n^1+1\\n^2}}(\Lambda_{\pi_p})\\
&\qquad\times \prod_{i=p+1}^{q-1} Q_{\substack{n^1+1\\n^2}k_{\pi_i};k_{\pi_i}\substack{n^1+1\\n^2}}(\Lambda_{\pi_i}) Q_{\substack{n^1+1\\n^2}(k_{\pi_q}+1^+);k_{\pi_q}\substack{n^1\\n^2}}(\Lambda_{\pi_q}) \prod_{i=q+1}^{a} Q_{\substack{n^1\\n^2}k_{\pi_i};k_{\pi_i}\substack{n^1\\n^2}}(\Lambda_{\pi_i}) &&\text{(E.8a)}\\
&+ \prod_{i=1}^{p-1} Q_{\substack{0\\0}k_{\pi_i};k_{\pi_i}\substack{0\\0}}(\Lambda_{\pi_i}) \mathcal{Q}^{(+\frac{1}{2})}_{\substack{n^1\\n^2}k_{\pi_p};(k_{\pi_p}+1)\substack{n^1+1\\n^2}}(\Lambda_{\pi_p}) \prod_{i=p+1}^{q-1} Q_{\substack{n^1+1\\n^2}k_{\pi_i};k_{\pi_i}\substack{n^1+1\\n^2}}(\Lambda_{\pi_i})\\
&\qquad\times Q_{\substack{n^1+1\\n^2}(k_{\pi_q}+1^+);k_{\pi_q}\substack{n^1\\n^2}}(\Lambda_{\pi_q}) \prod_{i=q+1}^{a} Q_{\substack{n^1\\n^2}k_{\pi_i};k_{\pi_i}\substack{n^1\\n^2}}(\Lambda_{\pi_i}) &&\text{(E.8b)}\\
&+ \sqrt{n^1{+}1}\prod_{i=1}^{p-1} Q_{\substack{0\\0}k_{\pi_i};k_{\pi_i}\substack{0\\0}}(\Lambda_{\pi_i}) Q_{\substack{0\\0}k_{\pi_p};(k_{\pi_p}+1)\substack{1\\0}}(\Lambda_{\pi_p}) \Big(\Big(\prod_{i=p+1}^{q-1} Q_{\substack{n^1+1\\n^2}k_{\pi_i};k_{\pi_i}\substack{n^1+1\\n^2}}(\Lambda_{\pi_i})\\
&\qquad - \prod_{i=p+1}^{q-1} Q_{\substack{0\\0}k_{\pi_i};k_{\pi_i}\substack{0\\0}}(\Lambda_{\pi_i})\Big) - \Big(\prod_{i=p+1}^{q-1} Q_{\substack{1\\0}k_{\pi_i};k_{\pi_i}\substack{1\\0}}(\Lambda_{\pi_i}) - \prod_{i=p+1}^{q-1} Q_{\substack{0\\0}k_{\pi_i};k_{\pi_i}\substack{0\\0}}(\Lambda_{\pi_i})\Big)\Big)\\
&\qquad\times Q_{\substack{n^1+1\\n^2}(k_{\pi_q}+1^+);k_{\pi_q}\substack{n^1\\n^2}}(\Lambda_{\pi_q}) \prod_{i=q+1}^{a} Q_{\substack{n^1\\n^2}k_{\pi_i};k_{\pi_i}\substack{n^1\\n^2}}(\Lambda_{\pi_i}) &&\text{(E.8c)}\\
&+ \sqrt{n^1{+}1}\prod_{i=1}^{p-1} Q_{\substack{0\\0}k_{\pi_i};k_{\pi_i}\substack{0\\0}}(\Lambda_{\pi_i}) Q_{\substack{0\\0}k_{\pi_p};(k_{\pi_p}+1)\substack{1\\0}}(\Lambda_{\pi_p}) \prod_{i=p+1}^{q-1} Q_{\substack{1\\0}k_{\pi_i};k_{\pi_i}\substack{1\\0}}(\Lambda_{\pi_i})\\
&\qquad\times \mathcal{Q}^{(+\frac{1}{2})}_{\substack{n^1\\n^2}(k_{\pi_q}+1^+);k_{\pi_q}\substack{n^1\\n^2}}(\Lambda_{\pi_q}) \prod_{i=q+1}^{a} Q_{\substack{n^1\\n^2}k_{\pi_i};k_{\pi_i}\substack{n^1\\n^2}}(\Lambda_{\pi_i}) &&\text{(E.8d)}\\
&+ (n^1{+}1)\prod_{i=1}^{p-1} Q_{\substack{0\\0}k_{\pi_i};k_{\pi_i}\substack{0\\0}}(\Lambda_{\pi_i}) Q_{\substack{0\\0}k_{\pi_p};(k_{\pi_p}+1)\substack{1\\0}}(\Lambda_{\pi_p}) \prod_{i=p+1}^{q-1} Q_{\substack{1\\0}k_{\pi_i};k_{\pi_i}\substack{1\\0}}(\Lambda_{\pi_i})\\
&\qquad\times Q_{\substack{1\\0}(k_{\pi_q}+1^+);k_{\pi_q}\substack{0\\0}}(\Lambda_{\pi_q}) \Big(\prod_{i=q+1}^{a} Q_{\substack{n^1\\n^2}k_{\pi_i};k_{\pi_i}\substack{n^1\\n^2}}(\Lambda_{\pi_i}) - \prod_{i=q+1}^{a} Q_{\substack{0\\0}k_{\pi_i};k_{\pi_i}\substack{0\\0}}(\Lambda_{\pi_i})\Big)\;. &&\text{(E.8e)}
\end{aligned}$$

Thus, we obtain (recall also (5.17)) a linear combination of graphs either with composite propagator (5.15a) or with composite propagator (5.15c). In power-counting estimations, the prefactors $\sqrt{n^1{+}1}$ combine according to footnote 38 to the required ratio with the scale $\theta\Lambda^2$. The part (E.6b) is nothing but (E.6a) with $n^1 = 1$ and $n^2 = 0$.

If the index jumps down from n^1 to $n^1 - 1$, then the graph with $n^1 = 0$ does not exist.

There is no change of the discussion of (E.3a) and (E.3b), but now (E.3c) becomes

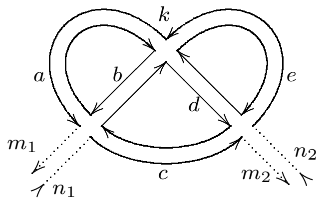
$$(\text{E.3c}) = \Big(\prod_{i=1}^{a} Q_{{}^{n^1}_{n^2}k_{\pi_i};k_{\pi_i}{}^{n^1}_{n^2}}(\Lambda_{\pi_i}) - n^1 \prod_{i=1}^{a} Q_{{}^{1}_{0}k_{\pi_i};k_{\pi_i}{}^{1}_{0}}(\Lambda_{\pi_j})\Big) \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\,. \qquad \text{(E.9)}$$

Using the same steps as in (E.8) we obtain the desired representation through graphs either with composite propagator (5.15a) or with composite propagator (5.15c).

I show in Appendix E.2 how the decomposition works in a concrete example.

E.2 Example of a difference operation for ribbon graphs

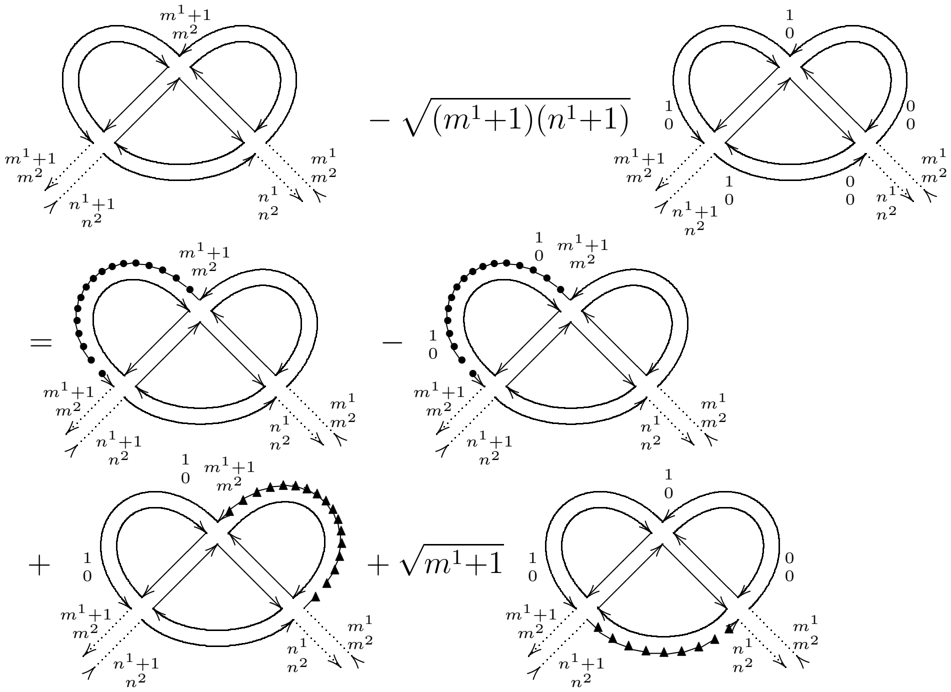
To make the considerations in Section 5.2 and Appendix E.1 about differences of graphs and composite propagators understandable, I discuss in some detail the following example of a planar two-leg graph:



(E.10)

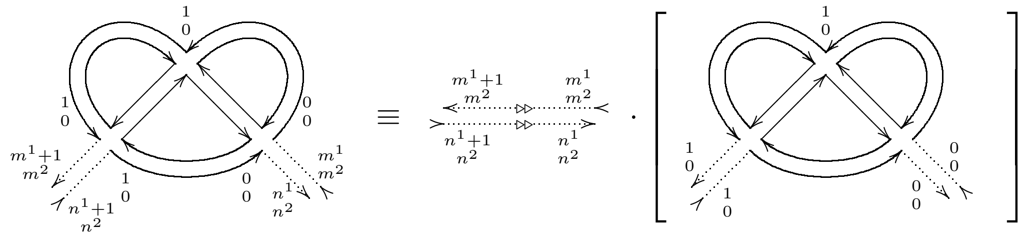
According to Proposition 13 it depends on the indices m_1, n_1, m_2, n_2, k whether this graph is irrelevant, marginal, or relevant. It depends on the history of contraction of subgraphs whether there are marginal subgraphs or not.

Let us consider $m_1 = k = {}^{m^1+1}_{m^2}$, $n_2 = {}^{m^1}_{m^2}$, $n_1 = {}^{n^1+1}_{n^2}$ and $m_2 = {}^{n^1}_{n^2}$ and the history a-c-d-e-b of contraction. Then, all resulting subgraphs are irrelevant and the total graph is marginal, which leads us to consider the following difference of graphs:



(E.11)

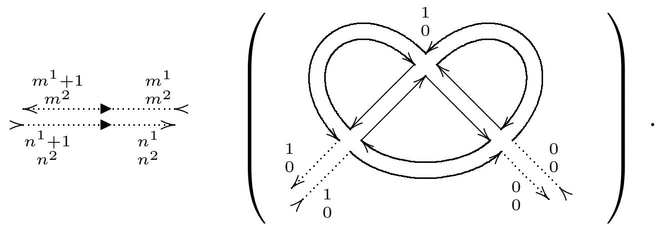
It is important to understand that according to (5.9) the indices at the external lines of the reference graph (with zero-indices) are adjusted to the external indices of the original (leftmost) graph:



(E.12)

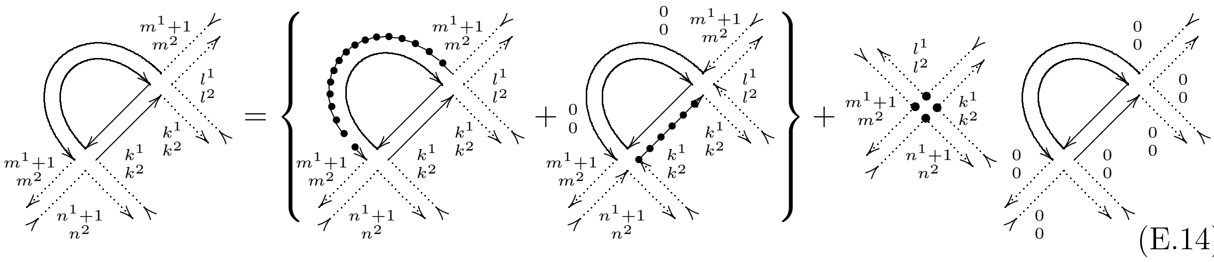
Thus, all graphs with composite propagators have the same index structure at the external legs. When further contracting these graphs, the contracting propagator matches the external indices of the original graph. The argumentation of 3 in the proof of Proposition 13 should be transparent now. In particular, it becomes understandable why the difference (E.11) is irrelevant and can be integrated from Λ_0 down to Λ. On the other hand, the reference graph to be integrated from Λ_R up to Λ becomes

$$\sqrt{(m^1+1)(n^1+1)}$$



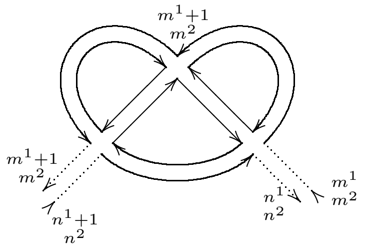
(E.13)

We cannot use the same procedure for the history *a-b-c-d-e* of contractions in (E.10), because we end up with a marginal subgraph after the *a-b* contractions. According to Definition 12.1 we have to decompose the *a-b* subgraph into an irrelevant (according to Proposition 13.1) difference and a marginal reference graph:

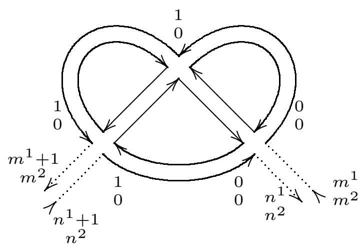


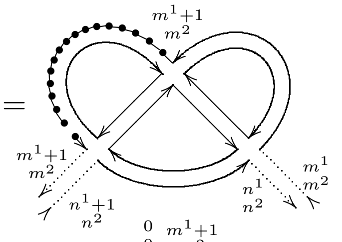
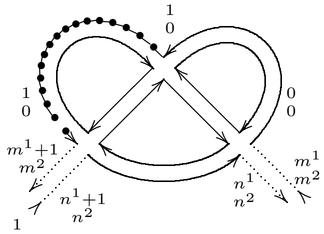
(E.14)

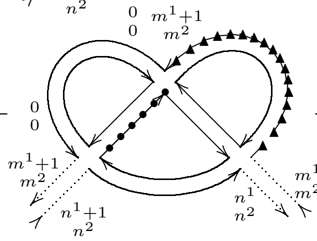
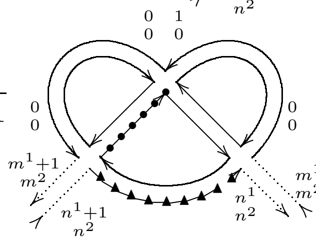
The two graphs in braces { } are irrelevant and integrated from Λ_0 down to Λ_c. The remaining piece can be written as the original ϕ^4-vertex times a graph with vanishing external indices, which is integrated from Λ_R up to Λ_c and can be bounded by $C \ln \frac{\Lambda}{\Lambda_R}$. Inserting the decomposition (E.14) into (E.10) we obtain the following decomposition valid for the history *a-b-c-d-e*:

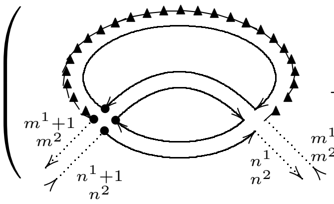
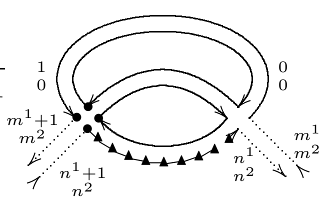
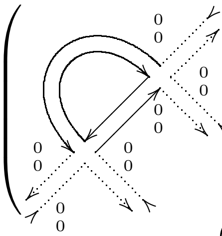


$$-\sqrt{(m^1+1)(n^1+1)}$$



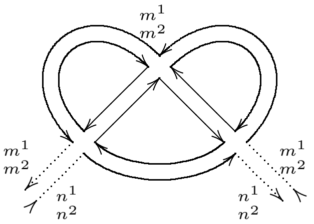
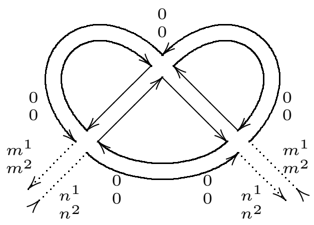
$=$  $-\sqrt{(m^1{+}1)(n^1{+}1)}$  (E.15a)

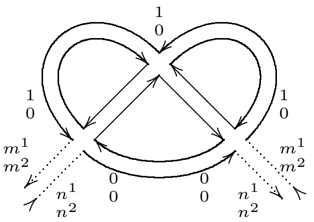
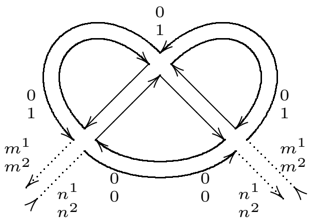
$+$  $+\sqrt{m^1{+}1}$  (E.15b)

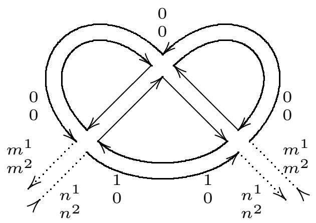
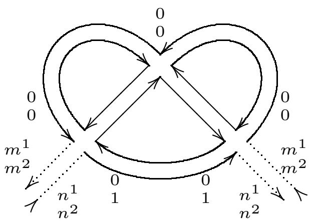
$+\Bigg($  $+\sqrt{m^1{+}1}$  $\Bigg)\Bigg($  $\Bigg)_{\Lambda_c}.$ (E.15c)

The line (E.15a) corresponds to the first graph in the braces { } of (E.14) for both graphs on the lhs of (E.15). These graphs are already irrelevant[40] so that no further decomposition is necessary. The second graph in the braces { } of (E.14), inserted into the lhs of (E.15), yields the line (E.15b). Finally, the last part of (E.14) leads to the line (E.15c).

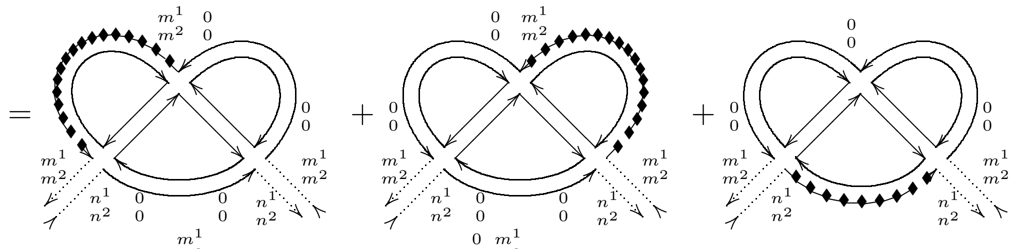
Let us also look at the relevant contribution $m_1 = k = n_2 = {}^{m^1}_{m^2}$, $n_1 = m_2 = {}^{n^1}_{n^2}$ of the graph (E.10). The history *a-c-d-e-b* contains irrelevant subgraphs only, and we get

 $+\,(m^1{+}m^2{+}n^1{+}n^2{-}1)$ 

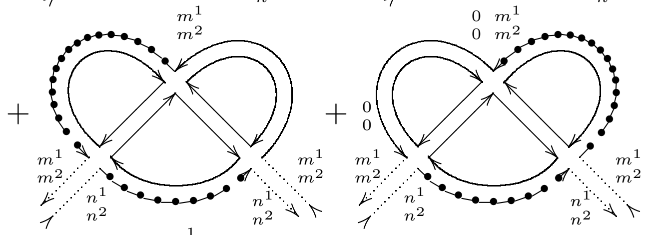
$-\,m^1$  $-\,m^2$ 

$-\,n^1$  $-\,n^2$ 

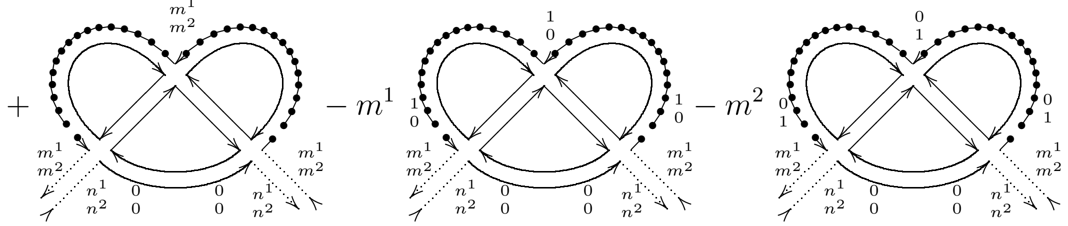
[40]In the right graph (E.15a) the composite propagator is according to (5.26) bounded by $\frac{C}{\Omega\theta\Lambda^2}\frac{1}{\theta\Lambda^2}$ so that the combination with the prefactor $\sqrt{(m^1{+}1)(n^1{+}1)}$ leads to the ratio $\sqrt{(\frac{m^1+1}{\theta\Lambda^2})(\frac{m^1+1}{\theta\Lambda^2})}$ by which (E.15a) is suppressed over the first graph on the lhs of (E.15).



(E.16a)



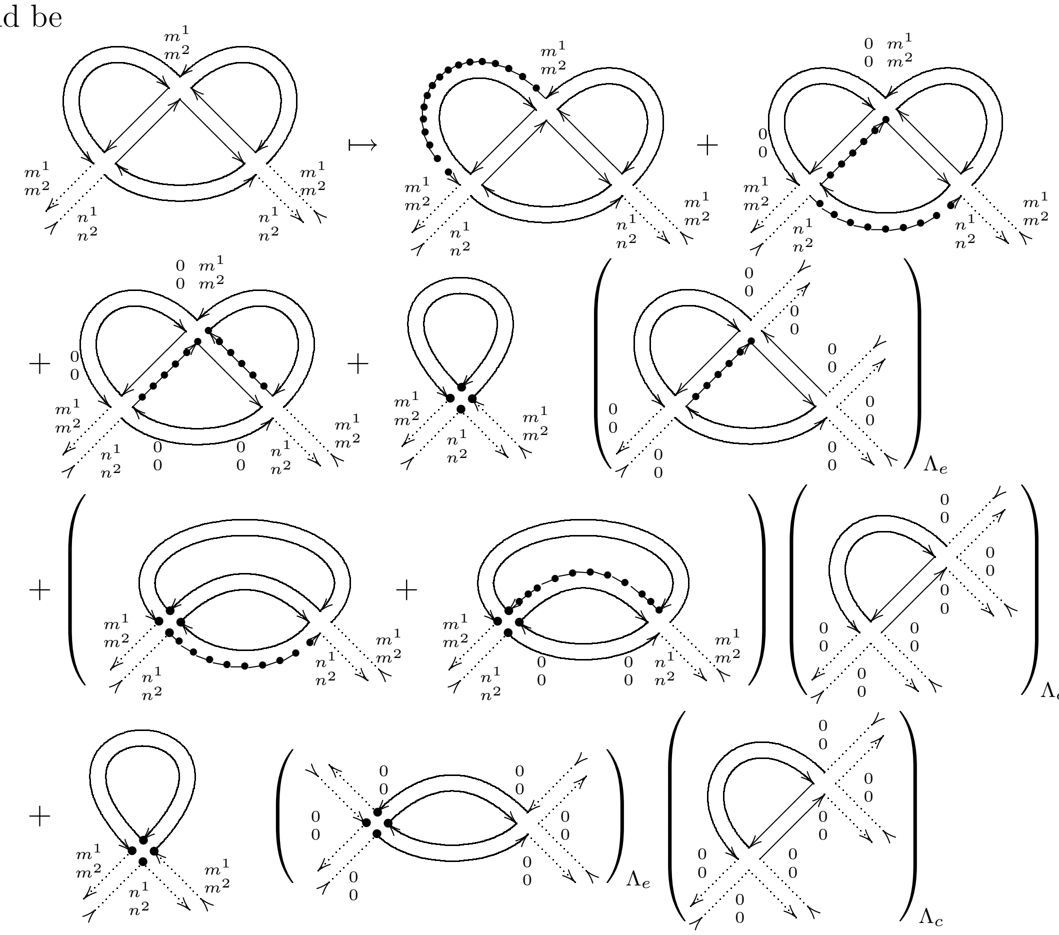
(E.16b)



(E.16c)

The line (E.16a) corresponds to (E.4a), the line (E.16b) to (E.3a) and the line (E.16c) to (E.4b).

If the history of contractions contains relevant or marginal subgraphs, we first have to decompose the subgraphs into the reference function with vanishing external indices and an irrelevant remainder. For instance, the decomposition relative to the history *a*-*b*-*c*-*d*-*e* would be



(E.17)

F Asymptotic behaviour of the propagator

For the power-counting theorem we need asymptotic formulae about the scaling behaviour of the cut-off propagator $\Delta^K_{nm;lk}$ and certain index summations. We shall restrict ourselves to the case $\theta_1 = \theta_2 = \theta$ and deduce these formulae from the numerical evaluation of the propagator for a representative class of parameters and special choices of the parameters where we can compute the propagator exactly. These formulae involve the cut-off propagator

$$\Delta^{\mathcal{C}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}} := \begin{cases} \Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}} & \text{for } \mathcal{C} \leq \max(m^1,m^2,n^1,n^2,k^1,k^2,l^1,l^2) \leq 2\mathcal{C}\;, \\ 0 & \text{otherwise}\;, \end{cases} \tag{F.1}$$

which according to (2.10) is the restriction of $\Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}$ to the support of the cut-off propagator $\Lambda\frac{\partial}{\partial\Lambda}\Delta^K_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(\Lambda)$ appearing in the Polchinski equation, with $\mathcal{C} = \theta\Lambda^2$.

Formula 1:

$$\max_{m^r,n^r,k^r,l^r} \Big|\Delta^{\mathcal{C}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}\Big|_{\mu_0=0} \approx \frac{\theta\;\delta_{m+k,n+l}}{\sqrt{\frac{1}{\pi}(16\,\mathcal{C}{+}12)} + \frac{6\Omega}{1+2\Omega^3+2\Omega^4}\mathcal{C}} \tag{F.2}$$

I demonstrate in Figure 3 for selected values of the parameters that $\theta/(\max\Delta^{\mathcal{C}}_{mn;kl})$ is asymptotically reproduced by $\sqrt{\frac{1}{\pi}(16\,\mathcal{C}{+}12)} + \frac{6\Omega}{1+2\Omega^3+2\Omega^4}\mathcal{C}$.

Formula 2:

$$\max_{m^r} \sum_{l^1,l^2\in\mathbb{N}} \max_{k^r,n^r} \Big|\Delta^{\mathcal{C}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}\Big|_{\mu_0=0} \approx \frac{\theta\,(1{+}2\Omega^3)}{7\Omega^2(\mathcal{C}+1)}\;. \tag{F.3}$$

I demonstrate in Figure 4 that $\theta/\big(\max_m \sum_l \max_{n,k}|\Delta^{\mathcal{C}}_{mn;kl}|\big)$ is asymptotically given by $7\Omega^2(\mathcal{C}+1)/(1{+}2\Omega^3)$.

Formula 3:

$$\sum_{l^1,l^2\in\mathbb{N}\;,\;\|m-l\|_1\geq 5} \max_{k^r,n^r} \Big|\Delta^{\mathcal{C}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}\Big|_{\mu_0=0} \leq \frac{\theta\,(1{-}\Omega)^4\big(15+\frac{4}{5}\|m\|_\infty+\frac{1}{25}\|m\|^2_\infty\big)}{\Omega^2(\mathcal{C}{+}1)^3}\;. \tag{F.4}$$

I verify (F.4) for several choices of the variables in Figures 5, 6 and 7.

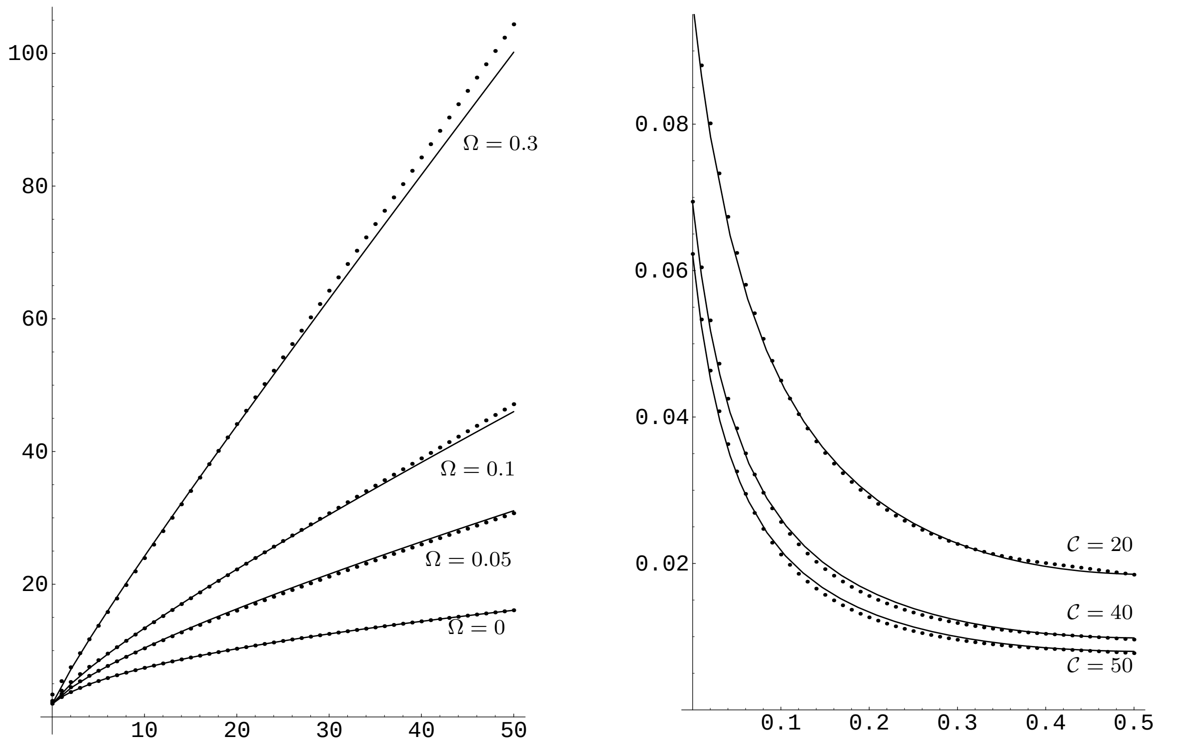


Figure 3: Comparison of $\max \Delta^{\mathcal{C}}_{mn;kl}/\theta$ (dots) with $\Big(\sqrt{\frac{1}{\pi}(16\,\mathcal{C}{+}12)}+\frac{6\Omega}{1+2\Omega^3+2\Omega^4}\mathcal{C}\Big)^{-1}$ (solid line). The left plot shows the inverses of both the propagator and its approximation over $\mathcal{C}$ for various values of Ω. The right plot shows the propagator and its approximation over Ω for various values of $\mathcal{C}$.

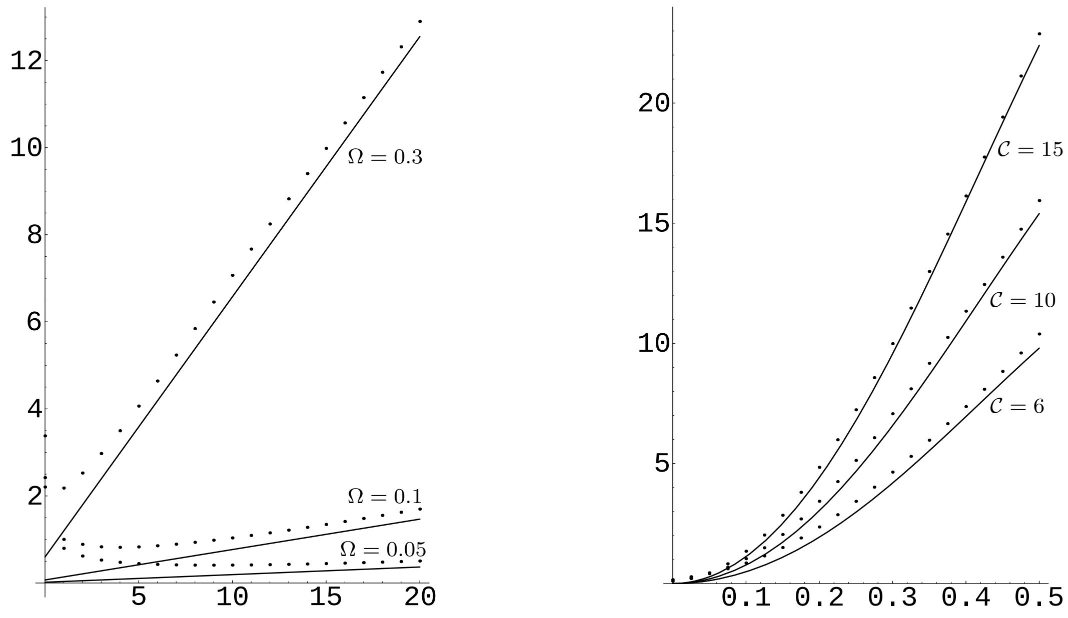


Figure 4: Comparison of $\theta/\big(\max_m \sum_l \max_{n,k} |\Delta^{\mathcal{C}}_{mn;kl}|\big)$ (dots) with $7\Omega^2(\mathcal{C}+1)/(1{+}2\Omega^2)$ (solid line). The left plot shows the inverse propagator and its approximation over $\mathcal{C}$ for three values of Ω, whereas the right plot shows the inverse propagator and its approximation over Ω for three values of $\mathcal{C}$.

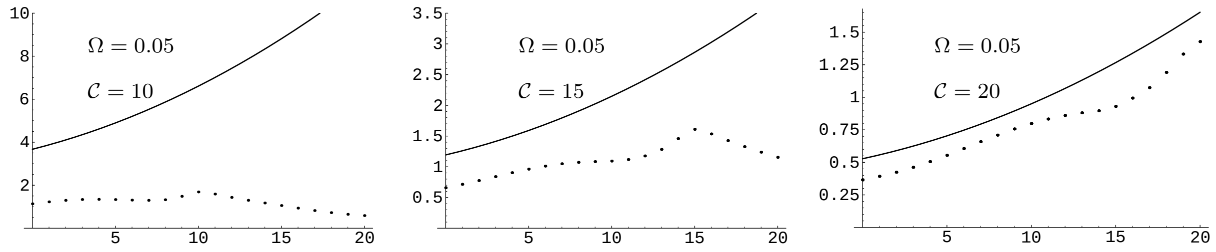


Figure 5: The index summation $\frac{1}{\theta}\Big(\sum_{l\ ,\ \|m-l\|_1\geq 5} \max_{k,r} \big|\Delta^{\mathcal{C}}_{mn;kl}\big|\Big)$ of the cut-off propagator (dots) compared with $\frac{(1-\Omega)^4\big(15+\frac{4}{5}\|m\|_\infty+\frac{1}{25}\|m\|^2_\infty\big)}{\Omega^2(\mathcal{C}+1)^3}$ (solid line), both plotted over $\|m\|_\infty$.

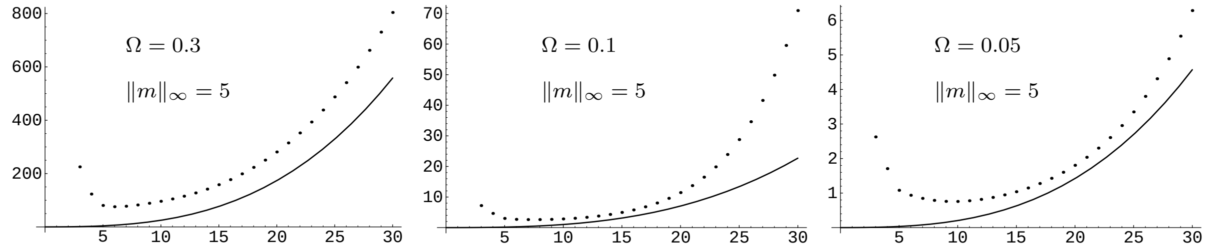


Figure 6: The inverse $\theta\Big(\sum_{l\ ,\ \|m-l\|_1\geq 5} \max_{k,r} \big|\Delta^{\mathcal{C}}_{mn;kl}\big|\Big)^{-1}$ of the summed propagator (dots) compared with $\frac{\Omega^2(\mathcal{C}+1)^3}{(1-\Omega)^4\big(15+\frac{4}{5}\|m\|_\infty+\frac{1}{25}\|m\|^2_\infty\big)}$ (solid line), both plotted over $\mathcal{C}$.

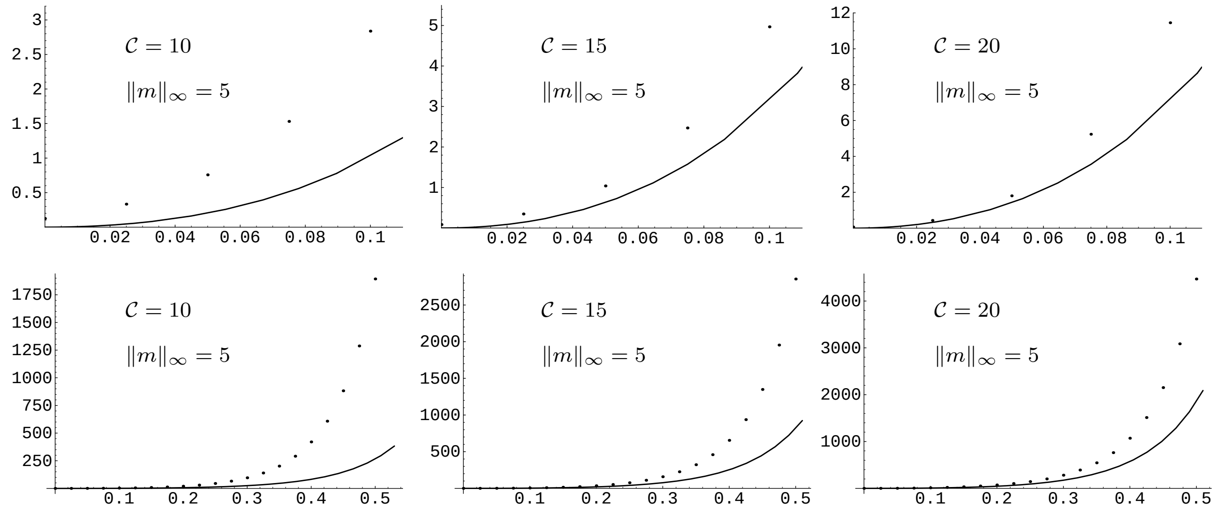


Figure 7: The inverse $\theta\Big(\sum_{l\ ,\ \|m-l\|_1\geq 5} \max_{k,r} \big|\Delta^{\mathcal{C}}_{mn;kl}\big|\Big)^{-1}$ of the summed propagator (dots) compared with $\frac{\Omega^2(\mathcal{C}+1)^3}{(1-\Omega)^4\big(15+\frac{4}{5}\|m\|_\infty+\frac{1}{25}\|m\|^2_\infty\big)}$ (solid line), both plotted over Ω.

For $\Omega = 0$ the mass μ_0 is required as a regulator. One finds

Formula 1':

$$\max_{m^r,n^r,k^r,l^r} \Big|\Delta^{\mathcal{C}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}\Big|_{\Omega=0} \approx \frac{\sqrt{\pi}\theta\ \delta_{m+k,n+l}}{(1+\sqrt{\mu_0^2\theta})\sqrt{16\,\mathcal{C}+12}}\ . \tag{F.5}$$

The corresponding asymptotics is demonstrated in Figure 8.

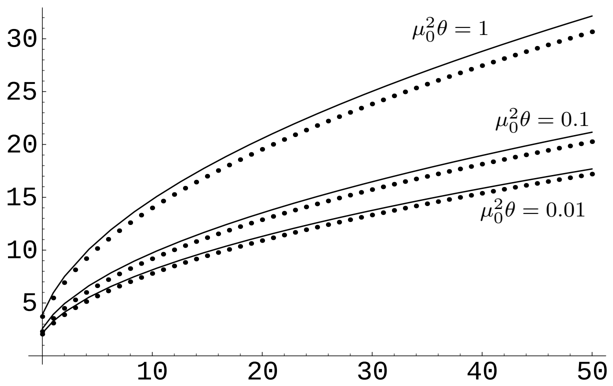


Figure 8: Comparison of $\theta/\big(\max \Delta^{\mathcal{C}}_{mn;kl}\big)$ for $\Omega = 0$ (dots) with $(1+\sqrt{\mu_0^2\theta})\sqrt{16\,\mathcal{C}+12}/\sqrt{\pi}$ (solid line), for selected values of $\mu_0^2\theta$.

Formula 2':

$$\max_{m^r} \sum_{l^1,l^2\in\mathbb{N}} \max_{k^r,n^r} \Big|\Delta^{\mathcal{C}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}\Big|_{\Omega=0} \approx \frac{1.1}{\mu_0^2}\,\frac{1+\mu_0^2\theta\mathcal{C}}{3+\mu_0^2\theta\mathcal{C}}\ . \tag{F.6}$$

The corresponding asymptotics is demonstrated in Figure 9.

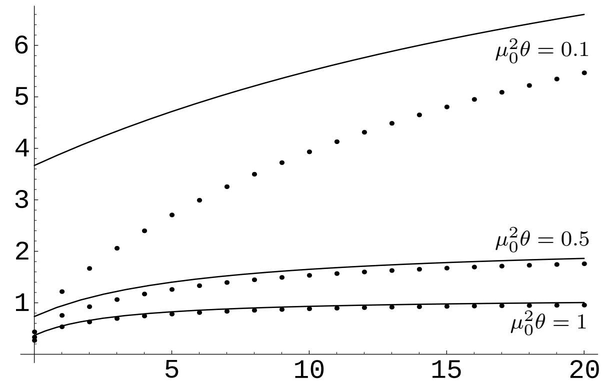


Figure 9: Comparison of $\max \Delta^{\mathcal{C}}_{mn;kl}/\theta$ for $\Omega = 0$ (dots) with $\frac{1.1}{\mu_0^2\theta}\,\frac{1+\mu_0^2\theta\mathcal{C}}{3+\mu_0^2\theta\mathcal{C}}$ (solid line), for selected values of $\mu_0^2\theta$.

G The β-function

Knowing that the duality-covariant noncommutative ϕ^4-model associated with the classical action (1.5) is renormalisable, it is interesting to compute the β_λ- and β_Ω-functions which describe the renormalisation of the coupling constant λ and of the oscillator frequency Ω. Whereas I have previously proven the renormalisability of the model in the Wilson-Polchinski approach [WK74, Pol84] adapted to non-local matrix models, I compute now the one-loop β_λ- and β_Ω-functions by standard Feynman graph calculations. Of course, these are Feynman graphs parametrised by matrix indices instead of momenta. The computation relies heavily on the power-counting behaviour given by Proposition 13, which allows us to ignore in the β-functions all non-planar graphs and the detailed index dependence of the planar two- and four-point graphs. Thus, only the lowest-order (discrete) Taylor expansion of the planar two- and four-point graphs can contribute to the β-functions. This means that I cannot refer to the usual symmetry factors of commutative ϕ^4-theory so that I have to carefully recompute the graphs.

There are interesting consequences for the limiting cases $\Omega = 1$ and $\Omega = 0$.

G.1 The renormalisation group equation

The computation of the expansion coefficients

$$\Gamma_{m_1n_1;\dots;m_Nn_N} := \frac{1}{N!}\frac{\partial^N \Gamma[\phi^{c\ell}]}{\partial\phi^{c\ell}_{m_1n_1}\dots\partial\phi^{c\ell}_{m_Nn_N}} \tag{G.1}$$

of the effective action (3.52) on page 34) involves possibly divergent sums over undetermined loop indices. Therefore, we have to introduce a (sharp) cut-off $\mathcal{N}$ for all loop indices. According to Definition 12, the expansion coefficients (G.1) can be decomposed into a relevant/marginal and an irrelevant piece. As a result of the renormalisation proof, the relevant/marginal parts have—after a rescaling of the field amplitude—the same form as in the initial action (1.5), (3.42) and (3.45), now parametrised by the "physical" mass, coupling constant and oscillator frequency:

$$\Gamma_{\mathrm{rel/marg}}[\mathcal{Z}\phi^{c\ell}] = S\big[\phi^{c\ell};\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}}\big]\;. \tag{G.2}$$

In the renormalisation process, the physical quantities μ^2_{phys}, λ_{phys} and Ω_{phys} are kept constant with respect to the cut-off $\mathcal{N}$. This is achieved by starting from a carefully adjusted initial action $S\big[\mathcal{Z}[\mathcal{N}]\phi,\mu_0[\mathcal{N}],\lambda[\mathcal{N}],\Omega[\mathcal{N}]\big]$, which gives rise to the bare effective action $\Gamma\big[\phi^{c\ell};\mu_0[\mathcal{N}\,],\lambda[\mathcal{N}\,],\Omega[\mathcal{N}\,],\mathcal{N}\big]$. Expressing the bare parameters μ_0,λ,Ω as a function of the physical quantities and the cut-off, the expansion coefficients of the renormalised effective action

$$\Gamma^R[\phi^{c\ell};\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}}] := \Gamma\big[\mathcal{Z}[\mathcal{N}]\phi^{c\ell},\mu_0[\mathcal{N}],\lambda[\mathcal{N}],\Omega[\mathcal{N}],\mathcal{N}\big]\Big|_{\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}}=\mathrm{const}} \tag{G.3}$$

are finite and convergent in the limit $\mathcal{N}\to\infty$. In other words,

$$\lim_{\mathcal{N}\to\infty}\mathcal{N}\frac{d}{d\mathcal{N}}\Big(\mathcal{Z}^N[\mathcal{N}]\Gamma_{m_1n_1;\dots;m_Nn_N}\big[\mu_0[\mathcal{N}],\lambda[\mathcal{N}],\Omega[\mathcal{N}],\mathcal{N}\big]\Big) = 0\;. \tag{G.4}$$

This implies the renormalisation group equation

$$\lim_{\mathcal{N}\to\infty}\Big(\mathcal{N}\frac{\partial}{\partial\mathcal{N}}+N\gamma+\mu_0^2\beta_{\mu_0}\frac{\partial}{\partial\mu_0^2}+\beta_\lambda\frac{\partial}{\partial\lambda}+\beta_\Omega\frac{\partial}{\partial\Omega}\Big)\Gamma_{m_1n_1;\dots;m_Nn_N}[\mu_0,\lambda,\Omega,\mathcal{N}]=0\;, \tag{G.5}$$

where

$$\beta_{\mu_0}=\frac{1}{\mu_0^2}\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\mu_0^2[\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}},\mathcal{N}]\Big)\;, \tag{G.6}$$

$$\beta_\lambda=\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\lambda[\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}},\mathcal{N}]\Big)\;, \tag{G.7}$$

$$\beta_\Omega=\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\Omega[\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}},\mathcal{N}]\Big)\;, \tag{G.8}$$

$$\gamma=\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\ln\mathcal{Z}[\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}},\mathcal{N}]\Big)\;. \tag{G.9}$$

G.2 One-loop computations

Defining $(\Delta J)_{mn}:=\sum_{p,q\in\mathbb{N}^2}\Delta_{mn;pq}J_{pq}$ we write (parts of) the generating functional (3.51), page 34, of connected Green's functions up to second order in λ:

$$\begin{aligned}
W[J]&=\ln Z[0]+4\pi^2\theta^2\sum_{m,n,k,l\in\mathbb{N}^2}\frac{1}{2}J_{mn}\Delta_{mn;kl}J_{kl}\\
&\quad-(4\pi^2\theta^2)\frac{\lambda}{4!}\sum_{m,n,k,l\in\mathbb{N}^2}\Big\{(\Delta J)_{ml}(\Delta J)_{lk}(\Delta J)_{kn}(\Delta J)_{nm}\\
&\qquad+\frac{1}{4\pi^2\theta^2}\Big(\Delta_{nm;kn}(\Delta J)_{ml}(\Delta J)_{lk}+\Delta_{kn;lk}(\Delta J)_{nm}(\Delta J)_{ml}\\
&\qquad\qquad+\Delta_{nm;ml}(\Delta J)_{lk}(\Delta J)_{kn}+\Delta_{lk;ml}(\Delta J)_{kn}(\Delta J)_{nm}\Big)\\
&\qquad+\frac{1}{4\pi^2\theta^2}\Big(\Delta_{nm;lk}(\Delta J)_{kn}(\Delta J)_{ml}+\Delta_{kn;ml}(\Delta J)_{nm}(\Delta J)_{lk}\Big)\\
&\qquad+\frac{1}{(4\pi^2\theta^2)^2}\Big(\big(\Delta_{nm;kn}\Delta_{lk;ml}+\Delta_{kn;lk}\Delta_{nm;ml}\big)+\Delta_{nm;lk}\Delta_{kn;ml}\Big)\Big\}\\
&\quad+\frac{\lambda^2}{2(4!)^2}\sum_{m,n,k,l,r,s,t,u\in\mathbb{N}^2}\Big\{\Big[\Big(\Delta_{ml;sr}\Delta_{lk;ts}(\Delta J)_{kn}(\Delta J)_{nm}+\Delta_{ml;sr}\Delta_{kn;ts}(\Delta J)_{lk}(\Delta J)_{nm}\\
&\qquad+\Delta_{ml;sr}\Delta_{nm;ts}(\Delta J)_{lk}(\Delta J)_{kn}+\Delta_{lk;sr}\Delta_{ml;ts}(\Delta J)_{kn}(\Delta J)_{nm}\\
&\qquad+\Delta_{lk;sr}\Delta_{kn;ts}(\Delta J)_{ml}(\Delta J)_{nm}+\Delta_{lk;sr}\Delta_{nm;ts}(\Delta J)_{ml}(\Delta J)_{kn}\\
&\qquad+\Delta_{kn;sr}\Delta_{ml;ts}(\Delta J)_{lk}(\Delta J)_{nm}+\Delta_{kn;sr}\Delta_{lk;ts}(\Delta J)_{ml}(\Delta J)_{nm}\\
&\qquad+\Delta_{kn;sr}\Delta_{nm;ts}(\Delta J)_{ml}(\Delta J)_{lk}+\Delta_{nm;sr}\Delta_{ml;ts}(\Delta J)_{lk}(\Delta J)_{kn}\\
&\qquad+\Delta_{nm;sr}\Delta_{lk;ts}(\Delta J)_{ml}(\Delta J)_{kn}+\Delta_{nm;sr}\Delta_{kn;ts}(\Delta J)_{ml}(\Delta J)_{lk}\Big)\\
&\qquad\qquad\times(\Delta J)_{ru}(\Delta J)_{ut}\\
&\qquad+5\text{ permutations of }{}_{ts},\,{}_{sr},\,{}_{ru},\,{}_{ut}\Big]\\
&\qquad+\text{ 1PI-contributions with }\leq 2\ J\text{'s }+\text{ 1PR-contributions}\Big\}+\mathcal{O}(\lambda^3)\;.
\end{aligned} \tag{G.10}$$

In second order in λ we get a huge number of terms so that we display only the 1PI contribution with four J's.

For the classical field (3.53) we have $\phi^{c\ell}_{mn} = \sum_{p,q\in\mathbb{N}^2} \Delta_{nm;pq} J_{pq} + \mathcal{O}(\lambda)$ so that

$$J_{pq} = \sum_{r,s\in\mathbb{N}^2} G_{qp;rs}\phi^{c\ell}_{rs} + \mathcal{O}(\lambda)\;. \tag{G.11}$$

The remaining part not displayed in (G.11) removes the 1PR-contributions when passing to $\Gamma[\phi^{c\ell}]$. We thus obtain

$$\begin{aligned}
\Gamma[\phi^{c\ell}] &= \Gamma[0] \\
&+ 4\pi^2\theta^2 \sum_{m,n,k,l\in\mathbb{N}^2} \frac{1}{2}\Big\{ G_{mn;kl} + \frac{\lambda}{6(4\pi^2\theta^2)}\Big(\delta_{ml}\sum_{p\in\mathbb{N}^2}\Delta_{pn;kp} + \delta_{kn}\sum_{p\in\mathbb{N}^2}\Delta_{mp;pl}\Big) && \text{(G.12a)}\\
&\qquad + \frac{\lambda}{6(4\pi^2\theta^2)}\Delta_{ml;kn} + \mathcal{O}(\lambda^2)\Big\}\phi^{c\ell}_{mn}\phi^{c\ell}_{kl} && \text{(G.12b)}\\
&+ 4\pi^2\theta^2 \sum_{m,n,k,l,r,s,t,u\in\mathbb{N}^2} \frac{\lambda}{4!}\Big\{\delta_{nk}\delta_{lr}\delta_{st}\delta_{um} && \text{(G.12c)}\\
&- \frac{\lambda}{2(4!)(4\pi^2\theta^2)}\Big(\sum_{p,q\in\mathbb{N}^2}\big(4\Delta_{mp;qs}\Delta_{pl;tq}\delta_{kn}\delta_{ur} + 4\Delta_{kp;qs}\Delta_{pn;tq}\delta_{ml}\delta_{ur} \\
&\qquad + 4\Delta_{pl;rq}\Delta_{mp;qu}\delta_{nk}\delta_{st} + 4\Delta_{pn;rq}\Delta_{kp;qu}\delta_{ml}\delta_{st}\big) && \text{(G.12d)}\\
&\qquad + \sum_{p\in\mathbb{N}^2}\big(4\Delta_{ml;ps}\Delta_{kn;tp}\delta_{ur} + 4\Delta_{kn;ps}\Delta_{ml;tp}\delta_{ur} + 4\Delta_{mp;ts}\Delta_{pl;ru}\delta_{nk} \\
&\qquad + 4\Delta_{pl;ts}\Delta_{mp;ru}\delta_{nk} + 4\Delta_{kp;ts}\Delta_{pn;ru}\delta_{ml} + 4\Delta_{pn;ts}\Delta_{kp;ru}\delta_{ml} \\
&\qquad + 4\Delta_{ml;rp}\Delta_{kn;pu}\delta_{st} + 4\Delta_{kn;rp}\Delta_{ml;pu}\delta_{st}\big) && \text{(G.12e)}\\
&\qquad + \sum_{p,q\in\mathbb{N}^2}\big(4\Delta_{pl;qs}\Delta_{mp;tq}\delta_{nk}\delta_{ur} + 4\Delta_{pn;qs}\Delta_{kp;tq}\delta_{ml}\delta_{ur} \\
&\qquad + 4\Delta_{kp;rq}\Delta_{pn;qu}\delta_{ml}\delta_{st} + 4\Delta_{mp;rq}\Delta_{pl;qu}\delta_{nk}\delta_{st}\big) && \text{(G.12f)}\\
&\qquad + 4\Delta_{ml;ts}\Delta_{kn;ru} + 4\Delta_{kn;ts}\Delta_{ml;ru}\Big) + \mathcal{O}(\lambda^2)\Big\}\phi^{c\ell}_{mn}\phi^{c\ell}_{kl}\phi^{c\ell}_{rs}\phi^{c\ell}_{tu} && \text{(G.12g)}\\
&+ \mathcal{O}\big((\phi^{c\ell})^6\big)\;.
\end{aligned}$$

Here, (G.12a) contains the contribution to the planar two-point function and (G.12b) the contribution to the non-planar two-point function. Next, (G.12c) and (G.12d) contribute to the planar four-point function, whereas (G.12e), (G.12f) and (G.12g) constitute three different types of non-planar four-point functions.

Introducing the cut-off $p^i, q^i \leq \mathcal{N}$ in the internal sums over $p, q \in \mathbb{N}^2$, we split the effective action according to Definition 12 (and its consistency proof, Proposition 13) as follows into a relevant/marginal and an irrelevant piece ($\Gamma[0]$ can be ignored):

$$\Gamma[\phi^{c\ell}] \equiv \Gamma_{\mathrm{rel/marg}}[\phi^{c\ell}] + \Gamma_{\mathrm{irrel}}[\phi^{c\ell}]\;, \tag{G.13}$$

$$
\begin{aligned}
\Gamma_{\mathrm{rel/marg}}[\phi^{c\ell}] = 4\pi^2\theta^2 \sum_{m,n,k,l\in\mathbb{N}^2} \frac{1}{2}\Big\{ G_{mn;kl} + \frac{\lambda}{6(4\pi^2\theta^2)}\delta_{ml}\delta_{kn}\Big(2\sum_{p^1,p^2=0}^{\mathcal{N}} \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}} \\
+ (m^1{+}n^1{+}m^2{+}n^2) \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}} - \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big)\Big) + \mathcal{O}(\lambda^2)\Big\}\phi^{c\ell}_{mn}\phi^{c\ell}_{kl} \\
+ 4\pi^2\theta^2 \sum_{m,n,k,l\in\mathbb{N}^2} \frac{\lambda}{4!}\Big\{1 - \frac{\lambda}{3(4\pi^2\theta^2)} \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big)^2 + \mathcal{O}(\lambda^2)\Big\}\phi^{c\ell}_{mn}\phi^{c\ell}_{nk}\phi^{c\ell}_{kl}\phi^{c\ell}_{lm}\,.
\end{aligned}
\tag{G.14}
$$

To the marginal four-point function and the relevant two-point function there contribute only the projections to planar graphs with vanishing external indices. The marginal two-point function is given by the next-to-leading term in the discrete Taylor expansion about vanishing external indices. At one-loop order there is no correction to the non-diagonal terms of $G_{mn;kl}$.

In a regime where $\lambda[\mathcal{N}\,]$ is so small that the perturbative expansion is valid in (G.14), the irrelevant part Γ_{irrel} can be completely ignored. Comparing (G.14) with the initial action according to (3.42) and (3.45), page 33, we have $\Gamma_{\mathrm{rel/marg}}[\mathcal{Z}\phi^{c\ell}] = S\big[\phi^{c\ell};\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}}\big]$ with

$$
\mathcal{Z} = 1 - \frac{\lambda}{192\pi^2\theta} \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}} - \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big) + \mathcal{O}(\lambda^2)\,,
\tag{G.15}
$$

$$
\begin{aligned}
\mu^2_{\mathrm{phys}} = \mu_0^2\Big(1 + \frac{\lambda}{12\pi^2\theta^2\mu_0^2} \sum_{p^1,p^2=0}^{\mathcal{N}} \big(2\Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}} - \Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}}\big) \\
- \frac{\lambda}{96\pi^2\theta} \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}} - \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big) + \mathcal{O}(\lambda^2)\Big)\,,
\end{aligned}
\tag{G.16}
$$

$$
\begin{aligned}
\lambda_{\mathrm{phys}} = \lambda\Big(1 - \frac{\lambda}{12\pi^2\theta^2} \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big)^2 \\
- \frac{\lambda}{48\pi^2\theta} \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}} - \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big) + \mathcal{O}(\lambda^2)\Big)\,,
\end{aligned}
\tag{G.17}
$$

$$
\Omega_{\mathrm{phys}} = \Omega\Big(1 + \frac{\lambda(1{-}\Omega^2)}{192\pi^2\theta\Omega^2} \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}} - \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big) + \mathcal{O}(\lambda^2)\Big)\,.
\tag{G.18}
$$

Solving (G.16), (G.17) and (G.18) for the bare quantities, we obtain to one-loop order

$$
\begin{aligned}
&\mu_0^2[\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}},\mathcal{N}\,] \\
&= \mu^2_{\mathrm{phys}}\Big(1 - \frac{\lambda_{\mathrm{phys}}}{12\pi^2\theta^2\mu^2_{\mathrm{phys}}} \sum_{p^1,p^2=0}^{\mathcal{N}} \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}} \\
&\qquad + \frac{\lambda_{\mathrm{phys}}}{96\pi^2\theta}\Big(1 + \frac{8}{\theta\mu^2_{\mathrm{phys}}}\Big) \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}} - \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big) + \mathcal{O}(\lambda^2_{\mathrm{phys}})\Big)\,,
\end{aligned}
\tag{G.19}
$$

$$
\begin{aligned}
&\lambda[\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}},\mathcal{N}]\\
&\quad=\lambda_{\mathrm{phys}}\Big(1+\frac{\lambda_{\mathrm{phys}}}{12\pi^2\theta^2}\sum_{p^1,p^2=0}^{\mathcal{N}}\big(\Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big)^2+\frac{\lambda_{\mathrm{phys}}}{48\pi^2\theta}\sum_{p^1,p^2=0}^{\mathcal{N}}\big(\Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}}-\Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big)\\
&\quad+\mathcal{O}(\lambda^2_{\mathrm{phys}})\Big)\;, \qquad\qquad (\mathrm{G.20})\\
&\Omega[\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}},\mathcal{N}]\\
&\quad=\Omega_{\mathrm{phys}}\Big(1-\frac{\lambda_{\mathrm{phys}}(1-\Omega^2_{\mathrm{phys}})}{192\pi^2\theta\Omega^2_{\mathrm{phys}}}\sum_{p^1,p^2=0}^{\mathcal{N}}\big(\Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}}-\Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big)+\mathcal{O}(\lambda^2_{\mathrm{phys}})\Big)\;. \qquad (\mathrm{G.21})
\end{aligned}
$$

Inserting (3.49), page 33, into (G.20) we can now compute the β_λ-function (G.7) up to one-loop order, omitting the index $_{\mathrm{phys}}$ on μ^2 and Ω for simplicity:

$$
\begin{aligned}
\beta_\lambda=\frac{\lambda^2_{\mathrm{phys}}}{48\pi^2}\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\sum_{p^1,p^2=0}^{\mathcal{N}}\Bigg\{&\left(\frac{{}_2F_1\Big(\genfrac{}{}{0pt}{}{1,\;\frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(p^1+p^2)}{2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)}\Big|\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)}{(1+\Omega)^2(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2))}\right)^2\\
&+\frac{p^1(1-\Omega)^2\;{}_2F_1\Big(\genfrac{}{}{0pt}{}{3,\;\frac{1+\mu_0^2\theta}{8\Omega}-\frac{1}{2}(p^1+p^2+1)}{3+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2+1)}\Big|\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)}{(1+\Omega)^4\big(\frac{1}{2}+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)\big)\big(\frac{3}{2}+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)\big)\big(\frac{5}{2}+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)\big)}\\
&+\frac{{}_2F_1\Big(\genfrac{}{}{0pt}{}{1,\;\frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(p^1+p^2+1)}{2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2+1)}\Big|\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)}{2(1+\Omega)^2(\frac{3}{2}+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2))}-\frac{{}_2F_1\Big(\genfrac{}{}{0pt}{}{1,\;\frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(p^1+p^2)}{2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)}\Big|\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)}{2(1+\Omega)^2(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2))}+\mathcal{O}(\lambda_{\mathrm{phys}})\Bigg\}\;.
\end{aligned}
\tag{G.22}
$$

Symmetrising the numerator in the second line $p^1\mapsto\frac{1}{2}(p^1+p^2)$ and using the expansions

$$
\begin{aligned}
{}_2F_1\Big(\genfrac{}{}{0pt}{}{1,\;a-p}{b+p}\Big|z\Big)&=\frac{1}{1+z}+\frac{z(a+b)+z^2(a+b-2)}{p(1+z)^3}+\mathcal{O}(p^{-2})\;,\\
{}_2F_1\Big(\genfrac{}{}{0pt}{}{3,\;a-p}{b+p}\Big|z\Big)&=\frac{1}{(1+z)^3}+\mathcal{O}(p^{-1})\;,
\end{aligned}
\tag{G.23}
$$

which are valid for large p, we obtain up to irrelevant contributions vanishing in the limit $\mathcal{N}\to\infty$

$$
\begin{aligned}
\beta_\lambda&=\frac{\lambda^2_{\mathrm{phys}}}{48\pi^2}\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\sum_{p^1,p^2=0}^{\mathcal{N}}\frac{1}{(1+\Omega^2_{\mathrm{phys}})^2}\frac{1}{\big(1+p^1+p^2\big)^2}\Big\{1+\frac{(1-\Omega^2_{\mathrm{phys}})^2}{2(1+\Omega^2_{\mathrm{phys}})}-\frac{(1+\Omega^2_{\mathrm{phys}})}{2}\Big\}\\
&\quad+\mathcal{O}(\lambda^3_{\mathrm{phys}})+\mathcal{O}(\mathcal{N}^{-1})\\
&=\frac{\lambda^2_{\mathrm{phys}}}{48\pi^2}\frac{(1-\Omega^2_{\mathrm{phys}})}{(1+\Omega^2_{\mathrm{phys}})^3}+\mathcal{O}(\lambda^3_{\mathrm{phys}})+\mathcal{O}(\mathcal{N}^{-1})\;.
\end{aligned}
\tag{G.24}
$$

Similarly, one obtains

$$
\beta_\Omega=\frac{\lambda_{\mathrm{phys}}\Omega_{\mathrm{phys}}}{96\pi^2}\frac{(1-\Omega^2_{\mathrm{phys}})}{(1+\Omega^2_{\mathrm{phys}})^3}+\mathcal{O}(\lambda^2_{\mathrm{phys}})+\mathcal{O}(\mathcal{N}^{-1})\;,
\tag{G.25}
$$

$$
\begin{aligned}
\beta_{\mu_0} &= -\frac{\lambda_{\text{phys}}}{48\pi^2\theta\mu^2_{\text{phys}}(1+\Omega^2_{\text{phys}})}\Big(4\mathcal{N}\ln(2)+\frac{(8+\theta\mu^2_{\text{phys}})\Omega^2_{\text{phys}}}{(1+\Omega^2_{\text{phys}})^2}\Big)\\
&\quad+\mathcal{O}(\lambda^2_{\text{phys}})+\mathcal{O}(\mathcal{N}^{-1})\;, && \text{(G.26)}\\
\gamma &= \frac{\lambda_{\text{phys}}}{96\pi^2}\frac{\Omega^2_{\text{phys}}}{(1+\Omega^2_{\text{phys}})^3}+\mathcal{O}(\lambda^2_{\text{phys}})+\mathcal{O}(\mathcal{N}^{-1})\;. && \text{(G.27)}
\end{aligned}
$$

G.3 Discussion

I have computed the one-loop β- and γ-functions in real four-dimensional duality-covariant noncommutative ϕ^4-theory. Remarkably, this model has a one-loop contribution to the wavefunction renormalisation which compensates partly the contribution from the planar one-loop four-point function to the β_λ-function. The one-loop β_λ-function is non-negative and vanishes in the distinguished case $\Omega = 1$ of the duality-invariant model, see (3.31). At $\Omega = 1$ also the β_Ω-function vanishes. This is of course expected (to all orders), because for $\Omega = 1$ the propagator (3.49) is diagonal, $\Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}\big|_{\Omega=1} = \frac{\delta_{m^1l^1}\delta_{k^1n^1}\delta_{m^2l^2}\delta_{k^2n^2}}{\mu_0^2+(4/\theta)(m^1+m^2+n^1+n^2+2)}$, so that the Feynman graphs never generate terms with $|m^i - l^i| = |n^i - k^i| = 1$ in (3.45).

The similarity of the duality-invariant theory with the exactly solvable models discussed in [LSZ04] suggests that also the β_λ-function vanishes to all orders for $\Omega = 1$. Whereas the model discussed here deals with real fields, the construction in [LSZ04] requires complex fields. Therefore, one has to be careful with a direct comparison of both models. However, the planar graphs of a real and a complex ϕ^4-model are very similar so that one can expect identical β_λ-functions (possibly up to a global factor) for the complex and the real model. Since a main feature of [LSZ04] was the independence on the dimension of the space, the model with $\Omega = 1$ and matrix cut-off $\mathcal{N}$ should be (more or less) equivalent to a two-dimensional model, which according to Appendix H has a mass renormalisation only. Therefore, I conjecture a vanishing β_λ-function in four-dimensional duality-invariant noncommutative ϕ^4-theory to all orders.

The most surprising result is that the one-loop β_Ω-function also vanishes for $\Omega \to 0$. We cannot directly set $\Omega = 0$, because the hypergeometric functions in (G.22) become singular and the expansions (G.23) are not valid. Moreover, the proof of Proposition 13 used to project to the relevant/marginal part of the effective action (G.14) requires $\Omega > 0$, too. However, in the same way as in the renormalisation of two-dimensional noncommutative ϕ^4-theory performed in Appendix H, it should be possible to switch off Ω very weakly with the cut-off $\mathcal{N}$, e.g. with

$$\Omega = \mathrm{e}^{-\big(\ln(1+\ln(1+\mathcal{N}))\big)^2}\;. \tag{G.28}$$

The decay (G.28) for large $\mathcal{N}$ over-compensates the growth of any polynomial in $\ln\mathcal{N}$, which according to Proposition 13 is the bound for the graphs contributing to a renormalisation of Ω. On the other hand, (G.28) does not modify the expansions (G.23). Thus, in the limit $\mathcal{N} \to \infty$, we have constructed the usual noncommutative ϕ^4-theory given by $\Omega = 0$ in (1.5) at the one-loop level. This is not so much surprising, because the UV/IR-mixing becomes a problem only at higher loop order [MVRS00]. Nevertheless, it would be very interesting to know whether this construction of the noncommutative ϕ^4-theory as the limit of a sequence (G.28) of duality-covariant ϕ^4-models can be extended to higher loop order.

We also notice that the one-loop β_λ- and β_Ω-functions are independent of the noncommutativity scale θ. There is, however a contribution to the one-loop mass renormalisation via the dimensionless quantity $\mu^2_{\text{phys}}\theta$, see (G.26).

H Renormalisation of noncommutative ϕ^4-theory in two dimensions

The traditional UV/IR-mixing problem [MVRS00] is less severe in models with only logarithmic divergences, such as the noncommutative ϕ^4-model in two dimensions. Applying the power-counting analysis of [CR00, CR01] to the real ϕ^4-model on noncommutative $\mathbb{R}^2$, one finds "that the divergences from all connected Green's functions at non-exceptional external momenta can be removed in the counter-term approach" (literally quoted from [CR01, §4.3]). However, non-exceptional momenta can become arbitrarily close to exceptional momenta so that the renormalised Green's functions are *unbounded*. Although one can probably live with that, it is not a desired feature of a quantum field theory.

I will apply here the techniques developed in the main part of the Habilitation thesis to the two-dimensional case. Again, the introduction of the harmonic oscillator potential is necessary in order to prove renormalisability. However, in contrast to the four-dimensional case, the oscillator potential can now be removed in a consistent manner with the limit $\Lambda_0 \to \infty$. I prove that there exists a Λ_0-dependence of the oscillator frequency Ω with $\lim_{\Lambda_0\to\infty} \Omega = 0$ such that the effective action at Λ_R is convergent (and thus bounded) order by order in the coupling constant in the limit $\Lambda_0 \to \infty$. This means that the partition function of the original (translation-invariant) ϕ^4-model is solved by Feynman graphs with propagators cut-off at Λ_R and vertices given by the bounded expansion coefficients of the effective action at Λ_R. Hence, this model is renormalisable (in fact super-renormalisable), and there is no problem with exceptional configurations.

H.1 The power-counting behaviour

Our starting point is the two-dimensional version of the action (1.5) on page 5, which in the matrix base developed in Section 3.3 takes the form (3.42), page 32, now with $D = 2$ and $\sqrt{\det\theta} = \theta_1 \equiv \theta$. The two-dimensional kinetic matrix is given in (3.44). Its inverse, the propagator, takes in two dimensions the form

$$\Delta_{mn;kl} = \delta_{m+k,n+l}\, \frac{\theta}{2(1+\Omega)^2} \sum_{v=\frac{|m-l|}{2}}^{\frac{\min(m+l,n+k)}{2}} \Big(\frac{1-\Omega}{1+\Omega}\Big)^{2v} B\big(\tfrac{1}{2}+\tfrac{\mu_0^2\theta}{8\Omega}+\tfrac{1}{2}(m+k)-v, 1+2v\big)$$
$$\times\, {}_2F_1\Bigg(\begin{matrix}1+2v\,,\ \frac{1}{2}+\frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(m+k)+v \\ \frac{3}{2}+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(m+k)+v\end{matrix}\Bigg|\frac{(1-\Omega)^2}{(1+\Omega)^2}\Bigg) \sqrt{\binom{n}{v+\frac{n-k}{2}}\binom{k}{v+\frac{k-n}{2}}\binom{m}{v+\frac{m-l}{2}}\binom{l}{v+\frac{l-m}{2}}}\,. \tag{H.1}$$

The limit $\Omega \to 0$ is given by

$$\Delta^{(\Omega=0)}_{mn;kl} = \frac{\theta}{2}\delta_{m+k,n+l} \sum_{v=\frac{|m-l|}{2}}^{\frac{\min(m+l,n+k)}{2}} \sqrt{\binom{n}{v+\frac{n-k}{2}}\binom{k}{v+\frac{k-n}{2}}\binom{m}{v+\frac{m-l}{2}}\binom{l}{v+\frac{l-m}{2}}}$$
$$\times\,(2v)!\,\Psi\Big(2v+1, 2v-m-k-1, \frac{\mu_0^2\theta}{2}\Big)\,. \tag{H.2}$$

We proceed with the renormalisation scheme of flow equations for non-local matrix models developed in Section 4. In particular, the effective action is expanded according to (4.51), for $D=2$, the coefficients $A^{(V)}_{m_1n_1;\dots;m_Nn_N}$ of which are interpreted by ribbon graphs γ. To renormalise the model we first have to integrate the matrix Polchinski equation (4.52) starting from mixed boundary conditions—the magic of renormalisation. The right choice, related to the initial interaction at $\Lambda=\Lambda_0$,

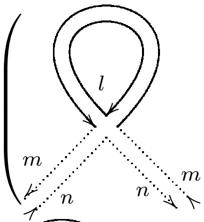
$$L[\phi,\Lambda_0,\Lambda_0,\rho^0] = \sum_{m,n\in\mathbb{N}} \frac{1}{2\pi\theta}\Big(\frac{1}{2}\rho^0\,\phi_{mn}\phi_{nm}\Big) + \sum_{m,n,k,l\in\mathbb{N}} \frac{\lambda}{4!}\,\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\;, \tag{H.3}$$

is the following:

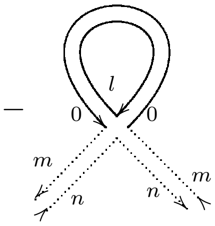
Definition 17 *We consider ribbon graphs γ which result from a history of contractions of subgraphs which at each contraction step have already been integrated according to the rules given below.*

1. *The one-vertex (V=1) planar (B=1, g=0, ι=0) (N=2)-point function is integrated as follows:*

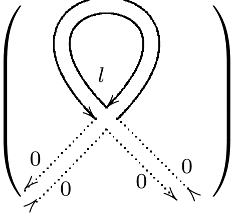
$$A^{(1,1,1,0,0)}_{mn;nm}[\Lambda,\Lambda_0,\Omega,\rho^0] := \Bigg\{ -\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\sum_{l=0}^{\infty}\Bigg(\quad$$



$$-$$



$$\Bigg)[\Lambda'] + \int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\sum_{l=0}^{\infty}\Bigg(\quad$$



$$\Bigg)[\Lambda']\Bigg\} + \Big\{m\leftrightarrow n\Big\} + A^{(1,1,1,0,0)}_{00;00}[\Lambda_R,\Lambda_0,\Omega,\rho^0]\;. \tag{H.4}$$

The propagator in the graphs of (H.4) is $Q_{ml;lm}(\Lambda')$ and $Q_{0l;l0}(\Lambda')$, respectively.

2. *Any other function (with $V+B+N>4$) is integrated according to*

$$A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda] := -\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\,\Big(\widehat{\Lambda'\frac{\partial}{\partial\Lambda'}A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda']}\Big)\;. \tag{H.5}$$

As before, the wide hat over the Λ'-derivative on the rhs of (H.5) indicates that the rhs of the matrix Polchinski equation (4.52) has to be inserted.

We identify $\rho^0\equiv\rho[\Lambda_0,\Lambda_0,\Omega,\rho^0]$, see (H.3), and put

$$\rho[\Lambda,\Lambda_0,\Omega,\rho^0] := A^{(1,1,1,0,0)}_{00;00}[\Lambda,\Lambda_0,\Omega,\rho^0]\;, \qquad \rho^R := \rho[\Lambda_R,\Lambda_0,\Omega,\rho^0]\;. \tag{H.6}$$

We are free to choose the initial condition $\rho^R=0$, which identifies the parameter μ_0 in the propagator (H.1) as the renormalised mass at the renormalisation scale Λ_R.

Since the propagator in (H.4) is the differentiated cut-off propagator (4.53), page 47, with $\mu^2 = \frac{1}{2\pi\theta}$, we can immediately perform the integration in (H.4):

$$A^{(1,1,1,0,0)}_{00;00}[\Lambda,\Lambda_0,\Omega,\rho^0] = \frac{1}{2\pi\theta}\sum_{l=0}^{\infty} 2\Big(\Delta^K_{0l;l0}(\Lambda) - \Delta^K_{0l;l0}(\Lambda_R)\Big) + \rho^R\,. \tag{H.7}$$

Using (4.10) and the cut-off function of (2.10) as well as the first equation (G.23) we thus obtain

$$\begin{aligned}\big|A^{(1,1,1,0,0)}_{00;00}[\Lambda,\Lambda_0,\Omega,\rho^0]\big| &\leq |\rho^R| + \frac{1}{\pi\theta}\sum_{l=\theta\Lambda_R^2}^{2\theta\Lambda^2} \Delta_{0l;l0}(\Lambda) \\ &= |\rho^R| + \sum_{l=\theta\Lambda_R^2}^{2\theta\Lambda^2} \frac{{}_2F_1\bigg(\begin{matrix}1\,,\;\frac{1}{2}+\frac{\mu_0^2\theta}{8\Omega}-\frac{l}{2}\\ \frac{3}{2}+\frac{\mu_0^2\theta}{8\Omega}+\frac{l}{2}\end{matrix}\bigg|\frac{(1-\Omega)^2}{(1+\Omega)^2}\bigg)}{\pi(1+\Omega)^2\big(l+1+\frac{\mu_0^2\theta}{4\Omega}\big)} \\ &= |\rho^R| + \sum_{l=\theta\Lambda_R^2}^{2\theta\Lambda^2}\Big(\frac{1}{2\pi(1+\Omega^2)\big(l+1+\frac{\mu_0^2\theta}{4\Omega}\big)} + \mathcal{O}(l^{-2})\Big) \\ &\leq P^1\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,. \end{aligned} \tag{H.8}$$

This estimation requires $\Lambda^2\Omega \gg \mu_0^2$. On the other hand, it is also valid for $\Omega = 0$ as the insertion of (H.2) into the first line of (H.8) shows.

Next, we compute the difference

$$\begin{aligned}&\big(A^{(1,1,1,0,0)}_{mn;nm} - A^{(1,1,1,0,0)}_{00;00}\big)[\Lambda,\Lambda_0,\Omega,\rho^0] \\ &= \frac{1}{2\pi\theta}\sum_{l=0}^{\infty}\Big(\big(\Delta^K_{ml;lm}(\Lambda) - \Delta^K_{0l;l0}(\Lambda)\big) - \big(\Delta^K_{ml;lm}(\Lambda_0) - \Delta^K_{0l;l0}(\Lambda_0)\big)\Big) + \{m\mapsto n\}\,. \end{aligned}\tag{H.9}$$

We have

$$\begin{aligned}&\big(\Delta_{ml;lm}(\Lambda) - \Delta_{0l;l0}(\Lambda)\big)\Big|_{l\geq m} \\ &= \frac{\theta}{2(1+\Omega)^2}\sum_{v=1}^{m}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{2v} B\big(\tfrac{1}{2}+\tfrac{\mu_0^2\theta}{8\Omega}+\tfrac{1}{2}(m+l)-v, 1+2v\big) \\ &\quad\times {}_2F_1\bigg(\begin{matrix}1+2v\,,\;\frac{1}{2}+\frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(m+l)+v\\ \frac{3}{2}+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(m+l)+v\end{matrix}\bigg|\frac{(1-\Omega)^2}{(1+\Omega)^2}\bigg)\binom{m}{v}\binom{l}{v} \qquad \text{(H.10a)} \\ &+\frac{\theta}{(1+\Omega)^2}\left(\frac{{}_2F_1\bigg(\begin{matrix}1\,,\;\frac{1}{2}+\frac{\mu_0^2\theta}{8\Omega}-\frac{m+l}{2}\\ \frac{3}{2}+\frac{\mu_0^2\theta}{8\Omega}+\frac{m+l}{2}\end{matrix}\bigg|\frac{(1-\Omega)^2}{(1+\Omega)^2}\bigg)}{\big(m+l+1+\frac{\mu_0^2\theta}{4\Omega}\big)} - \frac{{}_2F_1\bigg(\begin{matrix}1\,,\;\frac{1}{2}+\frac{\mu_0^2\theta}{8\Omega}-\frac{l}{2}\\ \frac{3}{2}+\frac{\mu_0^2\theta}{8\Omega}+\frac{l}{2}\end{matrix}\bigg|\frac{(1-\Omega)^2}{(1+\Omega)^2}\bigg)}{\big(l+1+\frac{\mu_0^2\theta}{4\Omega}\big)}\right) \qquad \text{(H.10b)}\end{aligned}$$

We insert (G.23) into the part (H.10b) and obtain

$$\text{(H.10b)} = -\frac{\theta}{2(1+\Omega^2)(m+l+1+\frac{\mu_0^2\theta}{4\Omega})}\Big(\frac{m}{l+1} + \mathcal{O}\big((\tfrac{m}{l})^2\big)\Big)\,. \tag{H.11}$$

Using (G.23), the leading contribution to the part (H.10a) comes from $v=1$. The other terms $v>1$ are suppressed with $\big(\frac{4ml}{(m+l)^2}\big)^v$:

$$(\text{H.10a}) = \frac{\theta(1-\Omega^2)}{(1+\Omega^2)^3(m+l+1+\frac{\mu_0^2\theta}{4\Omega})}\bigg(\frac{ml}{(m+l+3+\frac{\mu_0^2\theta}{4\Omega})(m+l-1+\frac{\mu_0^2\theta}{4\Omega})}+\mathcal{O}\big((\tfrac{m}{l})^2\big)\bigg)\,. \tag{H.12}$$

Taking the cut-off function into account, we thus conclude from (H.9)

$$\Big|\big(A^{(1,1,1,0,0)}_{mn;nm} - A^{(1,1,1,0,0)}_{00;00}\big)[\Lambda,\Lambda_0,\Omega,\rho^0]\Big|_{m,n\leq\theta\Lambda^2} \leq C\frac{\max(m,n)}{\theta\Lambda^2}\,. \tag{H.13}$$

We have used $\sum_{l=\theta\Lambda^2}^{2\theta\Lambda_0^2-1}\frac{1}{l^2} = \psi'(\theta\Lambda^2)-\psi'(2\theta\Lambda_0^2) = \frac{1}{\theta\Lambda^2}+\mathcal{O}\big((\theta\Lambda^2)^{-2}\big)$, where $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$.

As in the four-dimensional case, we compute numerically the asymptotic behaviour of the propagator (H.1). Up to a numerical factor, the result agrees with the four-dimensional one:

$$\big|Q_{mn;kl}(\Lambda)\big| \leq \frac{C_0}{\Omega\theta\Lambda^2}\delta_{m+k,n+l} \tag{H.14}$$

$$\max_m\sum_l\max_{n,k}\big|Q_{mn;kl}(\Lambda)\big| \leq \frac{C_1}{\Omega^2\theta\Lambda^2}\,, \tag{H.15}$$

for $\Omega>0$. I refer to [GW03b] for plots of the asymptotic behaviour. These plots involve the variable $\omega=\big(\frac{1-\Omega^2}{1+\Omega^2}\big)^2$ and were obtained without the knowledge of the explicit solution (H.1) of the propagator, but the conclusion is unchanged.

We thus insert the scaling exponents

$$\Big(\frac{\mu}{\Lambda}\Big)^{\delta_0} = \frac{1}{\Omega\theta\Lambda^2}\,, \qquad \Big(\frac{\mu}{\Lambda}\Big)^{\delta_1} = \frac{1}{\Omega^2\theta\Lambda^2}\,, \qquad \Big(\frac{\Lambda}{\mu}\Big)^{\delta_2} = \frac{1}{\Omega} \tag{H.16}$$

obtained by comparing (4.55)–(4.57) on page 47 with (H.15) into the general power-counting estimation of Theorem 10 on page 48 and conclude:

Proposition 18 *The homogeneous parts $A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}$ of the coefficients of the effective action of the duality-covariant noncommutative ϕ^4-model in two dimensions are for $2\leq N\leq 2V+2$ and $\sum_{i=1}^N(m_i-n_i)=0$ bounded by*

$$\sum_{\mathcal{E}^s}\big|A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\Omega,\rho^0]\big| \leq \Big(\frac{1}{\theta\Lambda^2}\Big)^{(V-1)+(B+2g-1)}\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}-1+B+2g-V^e-\iota+s} \times P^{V-\frac{N}{2}-B-2g+2}_{\delta_{N,2}\delta_{V+B,2}}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big]P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,. \tag{H.17}$$

We have $A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}\equiv 0$ for $N>2V+2$ or $\sum_{i=1}^N(m_i-n_i)\neq 0$.

Proof. Only the appearance of the factor (5.30), page 59, is questionable. The only source of this factor is the planar one-loop two-point function (H.13). According to (5.38), this factor survives the iteration process to more complicated graphs which contain (H.4) as a subgraph. There can be as many factors $\frac{m_i}{\theta\Lambda^2}$, for large m_i, as there are subgraphs of this kind, generated by a history of contractions where these subgraphs are independent one-loop graphs. This number is bounded by the total number of inner loops, which is $V - \frac{N}{2} - B - 2g + 2$. □

As the estimation (H.17) is independent of Λ_0 and only one initial condition in (H.4) for the mass renormalisation is necessary, we see immediately that the duality-covariant noncommutative ϕ^4_2-model is renormalisable, in fact super-renormalisable. I could easily prove a convergence theorem as in Section 6. However, I can even prove more, namely, that the limit $\Omega \to 0$ of the model exists. This is the subject of the next Section.

H.2 The limit $\Omega \to 0$

The idea is to couple the oscillator frequency Ω to the initial scale Λ_0 in such a way that

- for finite Λ_0 we have $\Omega[\Lambda_0] > 0$,
- $\lim_{\Lambda_0\to\infty} \Omega[\Lambda_0] = 0$.

Instead of (6.1) on page 68, we now consider the identity

$$\begin{aligned}&L[\Lambda_R, \Lambda_0', \Omega[\Lambda_0'], \rho^0[\Lambda_0']] - L[\Lambda_R, \Lambda_0'', \Omega[\Lambda_0''], \rho^0[\Lambda_0'']]\\ &\equiv \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0} \Big(\Lambda_0 \frac{d}{d\Lambda_0} L[\Lambda_R, \Lambda_0, \Omega[\Lambda_0], \rho^0[\Lambda_0]]\Big)\\ &= \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0} \Big(\Lambda_0 \frac{\partial L[\Lambda_R, \Lambda_0, \Omega, \rho^0]}{\partial \Lambda_0} + \Lambda_0 \frac{d\Omega}{d\Lambda_0} \frac{\partial L[\Lambda_R, \Lambda_0, \Omega, \rho^0]}{\partial \Omega} + \Lambda_0 \frac{d\rho^0}{d\Lambda_0} \frac{\partial L[\Lambda_R, \Lambda_0, \Omega, \rho^0]}{\partial \rho^0}\Big) .\end{aligned} \tag{H.18}$$

We have omitted for simplicity the dependence of L on ϕ. The model is defined by fixing the initial condition for ρ^R in (H.6) independently of Λ_0,

$$\begin{aligned}0 &= d\rho\big[\Lambda_R, \Lambda_0, \Omega[\Lambda_0], \rho^0[\Lambda_0]\big]\\ &= \frac{\partial \rho[\Lambda_R, \Lambda_0, \Omega, \rho^0]}{\partial \Lambda_0} d\Lambda_0 + \frac{\partial \rho[\Lambda_R, \Lambda_0, \Omega, \rho^0]}{\partial \Omega} d\Omega + \frac{\partial \rho[\Lambda_R, \Lambda_0, \Omega, \rho^0]}{\partial \rho^0} d\rho^0 .\end{aligned} \tag{H.19}$$

Assuming that the inverse function $\rho^0[\Lambda_R, \Lambda_0, \Omega, \rho^R]$ exists, which is the case in perturbation theory, we get

$$\frac{d\rho^0}{d\Lambda_0} = -\frac{\partial \rho^0}{\partial \rho[\Lambda_R, \Lambda_0, \Omega, \rho^0]} \frac{\partial \rho[\Lambda_R, \Lambda_0, \Omega, \rho^0]}{\partial \Lambda_0} - \frac{\partial \rho^0}{\partial \rho[\Lambda_R, \Lambda_0, \Omega, \rho^0]} \frac{\partial \rho[\Lambda_R, \Lambda_0, \Omega, \rho^0]}{\partial \Omega} \frac{d\Omega}{d\Lambda_0} . \tag{H.20}$$

Inserting (H.20) into (H.18) we obtain

$$\begin{aligned}&L[\phi, \Lambda_R, \Lambda_0', \Omega[\Lambda_0'], \rho^0[\Lambda_0']] - L[\phi, \Lambda_R, \Lambda_0'', \Omega[\Lambda_0''], \rho^0[\Lambda_0'']]\\ &\qquad = \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0} \, R[\phi, \Lambda_R, \Lambda_0, \Omega[\Lambda_0], \rho^0[\Lambda_0]] ,\end{aligned} \tag{H.21}$$

where

$$
\begin{aligned}
R[\phi,\Lambda,\Lambda_0,\Omega,\rho^0] := {}& \Lambda_0 \frac{\partial L[\phi,\Lambda,\Lambda_0,\Omega,\rho^0]}{\partial \Lambda_0} + \frac{\partial L[\phi,\Lambda,\Lambda_0,\Omega,\rho^0]}{\partial \Omega}\Lambda_0 \frac{d\Omega}{d\Lambda_0} \\
& - \frac{\partial L[\phi,\Lambda,\Lambda_0,\Omega,\rho^0]}{\partial \rho^0}\frac{\partial \rho^0}{\partial \rho[\Lambda,\Lambda_0,\Omega,\rho^0]}\Lambda_0 \frac{\partial \rho[\Lambda,\Lambda_0,\Omega,\rho^0]}{\partial \Lambda_0} \\
& - \frac{\partial L[\phi,\Lambda,\Lambda_0,\Omega,\rho^0]}{\partial \rho^0}\frac{\partial \rho^0}{\partial \rho[\Lambda,\Lambda_0,\Omega,\rho^0]}\frac{\partial \rho[\Lambda,\Lambda_0,\Omega,\rho^0]}{\partial \Omega}\Lambda_0 \frac{d\Omega}{d\Lambda_0} \,.
\end{aligned} \tag{H.22}
$$

In the same way as in (6.45) on page 80, R projects to the complement of the distinguished function (H.6). It is remarkable that the additional Ω-discussion does not modify the differential equations (6.11) and (6.13) for the Λ-derivatives. We write the first one directly in component form analogous to (6.17) on page 71:

$$
\begin{aligned}
&\Lambda \frac{\partial}{\partial \Lambda} R^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho^0] \\
&= \Big\{ \sum_{N_1=2}^{N}\sum_{V_1=1}^{V-1}\sum_{m,n,k,l\in\mathbb{N}} Q_{nm;lk}(\Lambda) A^{(V_1)}_{m_1n_1;\dots;m_{N_1-1}n_{N_1-1};mn}[\Lambda] R^{(V-V_1)}_{m_{N_1}n_{N_1};\dots;m_Nn_N;kl}[\Lambda] \\
&\quad + \Big(\binom{N}{N_1-1}-1\Big) \text{ permutations } \Big\} - \sum_{m,n,k,l\in\mathbb{N}} \frac{1}{2} Q_{nm;lk}(\Lambda) R^{(V)}_{m_1n_1;\dots;m_Nn_N;mn;kl}[\Lambda] \\
&\quad - H^{(V-1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda] \Big\{ -\frac{1}{2}\sum_{m,n,k,l\in\mathbb{N}} Q_{nm;lk}(\Lambda) R^{(1)}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 17.1}]} \,.
\end{aligned} \tag{H.23}
$$

According to Definition 17.1, only the restriction to one-vertex two-point functions can contribute. Now, in complete analogy to (6.49) on page 82, we have $R^{(1)}_{m_1n_1;\dots;m_4n_4} \equiv 0$ for the scaling of the initial interaction. Therefore, (H.23) reduces in fact to

$$
\begin{aligned}
&\Lambda \frac{\partial}{\partial \Lambda} R^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho^0] \\
&= \Big\{ \sum_{N_1=2}^{N}\sum_{V_1=1}^{V-1}\sum_{m,n,k,l\in\mathbb{N}} Q_{nm;lk}(\Lambda) A^{(V_1)}_{m_1n_1;\dots;m_{N_1-1}n_{N_1-1};mn}[\Lambda] R^{(V-V_1)}_{m_{N_1}n_{N_1};\dots;m_Nn_N;kl}[\Lambda] \\
&\quad + \Big(\binom{N}{N_1-1}-1\Big) \text{ permutations } \Big\} - \sum_{m,n,k,l\in\mathbb{N}} \frac{1}{2} Q_{nm;lk}(\Lambda) R^{(V)}_{m_1n_1;\dots;m_Nn_N;mn;kl}[\Lambda] \,,
\end{aligned} \tag{H.24}
$$

so that there is no need to discuss the H-functions. In analogy to (6.48) on page 81 we derive the initial condition

$$R^{(1,1,1,0,0)}_{m_1n_1;\dots;m_4n_4}[\Lambda_0,\Lambda_0,\Omega,\rho^0] = 0 \,, \tag{H.25}$$

$$R^{(1,1,1,0,0)}_{m_1n_1;m_2n_2}[\Lambda_0,\Lambda_0,\Omega,\rho^0] = -\Big(\Lambda\frac{\partial}{\partial\Lambda}\big(A^{(1,1,1,0,0)}_{m_1n_1;m_2n_2} - A^{(1,1,1,0,0)}_{00;00}\big)[\Lambda,\Lambda_0,\Omega,\rho^0]\Big)_{\Lambda=\Lambda_0} \,, \tag{H.26}$$

$$R^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\Omega,\rho^0]\Big|_{V+B>2} = -\Big(\Lambda\frac{\partial}{\partial\Lambda} A^{(V,V^e,B,g,\iota)}_{m_1n_1;m_2n_2}[\Lambda,\Lambda_0,\Omega,\rho^0]\Big)_{\Lambda=\Lambda_0} \,. \tag{H.27}$$

We thus obtain from Proposition 18 and the considerations in the proof of Proposition 15

Lemma 19 *The expansion coefficients $R^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}$ of the Λ_0-varied effective action describing the duality-covariant noncommutative ϕ^4-model in two dimensions are for $2 \leq N \leq 2V{+}2$ and $\sum_{i=1}^N (m_i{-}n_i) = 0$ bounded by*

$$R^{(1,1,1,0,0)}_{m_1n_1;\dots;m_4n_4}[\Lambda,\Lambda_0,\Omega,\rho^0] = 0\;, \tag{H.28}$$

$$R^{(1,1,1,0,0)}_{m_1n_1;m_2n_2}[\Lambda,\Lambda_0,\Omega,\rho^0] \leq \Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)\Big(\frac{1}{\Omega}\Big) P^1_1\Big[\frac{m_1n_1;m_2n_2}{\theta\Lambda^2}\Big]\delta_{m_1,n_2}\delta_{m_2,n_1}\;, \tag{H.29}$$

$$\begin{aligned}\sum_{\mathcal{E}^s} \big|R^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\Omega,\rho^0]\big|_{V+B>2} &\\ \leq \Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)\Big(\frac{1}{\theta\Lambda^2}\Big)^{(V-1)+(B+2g-1)}&\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}-1+B+2g-V^e-\iota+s} \\ \times P_0^{V-\frac{N}{2}-B-2g+2}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] &P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\;.\end{aligned} \tag{H.30}$$

We have $R^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N} \equiv 0$ for $N > 2V{+}2$ or $\sum_{i=1}^N (m_i{-}n_i) \neq 0$. □

The estimations in Lemma 19 hold for any function $\Omega[\Lambda_0]$, because the integration only involved the Λ-dependence. We can now choose $\Omega[\Lambda_0]$ in such a way that the limit $\Lambda_0 \to \infty$ in (H.21) exists. Such a choice is

$$\Omega[\Lambda_0] = \Big(1 + \ln\frac{\Lambda_0}{\Lambda_R}\Big)^{-1}\;. \tag{H.31}$$

This yields

Theorem 20 *The usual ϕ^4-model on the two-dimensional Moyal plane is (order by order in the coupling constant) renormalisable in the matrix base by adjusting the coefficient $\rho^0[\Lambda_0]$ of the initial interaction to give $A^{(1,1,1,0,0)}_{00;00}\big[\Lambda_R,\Lambda_0,\Omega[\Lambda_0],\rho^0[\Lambda_0]\big] = 0$ and by performing the limit $\Lambda_0 \to \infty$ along the path of duality-covariant models characterised by the frequency $\Omega[\Lambda_0] = (1 + \ln\frac{\Lambda_0}{\Lambda_R})^{-1}$. The limit $A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty] := \lim_{\Lambda_0\to\infty} A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}\big[\Lambda_R,\Lambda_0,\Omega[\Lambda_0],\rho^0[\Lambda_0]\big]$ of the expansion coefficients of the effective action $L[\phi,\Lambda_R,\Lambda_0,\Omega[\Lambda_0],\rho^0[\Lambda_0]]$, see (4.51), exists and satisfies*

$$\begin{aligned}&\lambda\big(2\pi\theta\lambda\big)^{V-1}\Big|A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty] - A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}\big[\Lambda_R,\Lambda_0,(1{+}\ln\tfrac{\Lambda_0}{\Lambda_R})^{-1},\rho^0[\Lambda_0]\big]\Big| \\ &\leq \frac{\Lambda_R^4}{\Lambda_0^2}\Big(\frac{\lambda}{\Lambda_R^2}\Big)^V\Big(\frac{1{+}\ln\frac{\Lambda_0}{\Lambda_R}}{\theta\Lambda_R^2}\Big)^{B+2g-1} P^{V-\frac{N}{2}-B-2g+2}_{\delta_{N,2}\delta_{V+B,2}}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda_R^2}\Big] P^{5V-N-V^e-\iota}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\;.\end{aligned} \tag{H.32}$$

Proof. The existence of the limit and its property (H.32) follow from inserting (H.28)–(H.30) and (H.31) into (H.21) and Cauchy's criterion. I also recall that $\int \frac{dx}{x^3} P^q[\ln x] = \frac{1}{x^2}P'^q[\ln x]$. □

Note that Proposition 18 and Theorem 20 combine to

$$\begin{aligned}
&\big(2\pi\theta\lambda\big)^{V-1}\Big|A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty]\Big| \\
&\leq \Big(\frac{\lambda}{\Lambda_R^2}\Big)^{V-1}\Big(\frac{1+\ln\frac{\Lambda_0}{\Lambda_R}}{\theta\Lambda_R^2}\Big)^{B+2g-1} P^{V-\frac{N}{2}-B-2g+2}_{\delta_{N,2}\delta_{V+B,2}}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda_R^2}\Big]\, P^{3V-\frac{N}{2}-V^e-\iota}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big] \\
&+\frac{\Lambda_R^2}{\Lambda_0^2}\Big(\frac{\lambda}{\Lambda_R^2}\Big)^{V-1}\Big(\frac{1+\ln\frac{\Lambda_0}{\Lambda_R}}{\theta\Lambda_R^2}\Big)^{B+2g-1} P^{V-\frac{N}{2}-B-2g+2}_{\delta_{N,2}\delta_{V+B,2}}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda_R^2}\Big]\, P^{5V-N-V^e-\iota}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big].
\end{aligned} \tag{H.33}$$

Regarded as a function of Λ_0, the rhs of (H.33) has a minimum at intermediate values of Λ_0, whereas for very large Λ_0 the estimation gets worse. Similarly, one has to avoid in Proposition 18 huge values of Λ_0 in connection with $\Omega[\Lambda_0]$ from (H.31). Of course, this is an artifact of the estimation procedure, because we know form Theorem 20 that at very large Λ_0 the function $\big(2\pi\theta\lambda\big)^{V-1}A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}\big[\Lambda_R,\Lambda_0,(1+\ln\frac{\Lambda_0}{\Lambda_R})^{-1},\rho^0[\Lambda_0]\big]$ converges to $\big(2\pi\theta\lambda\big)^{V-1}A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty]$, which is bounded by the optimum in (H.33) with respect to Λ_0.

To summarise, I have proven that the standard noncommutative ϕ^4-model in two dimensions is renormalisable to all orders. The model is actually super-renormalisable because mixed boundary conditions are necessary for only a finite number of graphs. It was crucial to work in the matrix base and to define the model at the initial scale Λ_0 by the ϕ^4-action (H.3) supplemented by a harmonic oscillator potential entering in (H.1). The renormalisation is achieved by a suitable Λ_0-dependence of the bare mass and the oscillator frequency. The model constructed in this way differs from the naïve Feynman graph approach in momentum space. In contrast to the latter, our renormalised Green's functions are bounded and convergent for any configuration of the external parameters (matrix indices versus momenta).

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