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Renormalization of noncommutative Yang-Mills theories: A simple example

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Abstract

We prove by explicit calculation that Feynman graphs in noncommutative Yang-Mills theory made of repeated insertions into itself of arbitrarily many one-loop ghost propagator corrections are renormalizable by local counterterms. This provides a strong support for the renormalizability conjecture of that model.

1 Introduction

It is now commonly admitted that our current concepts about space and time have to be changed when exploring space-time at a very small scale. Indeed, one can show that it is impossible to locate a particle with an arbitrarily small uncertainty when taking both into account the principles of quantum mechanics and general relativity [1]. Roughly speaking, one can say that measurements of coordinates on space-time are subject to uncertainty relations, thus ruining all geometrical concepts that have proved to be a guidance principle in elaborating many physical theories.

Following the previously alluded analogy with quantum mechanics, one can try to solve this puzzle by assuming that the coordinates themselves are noncommuting objects. Thus, the natural extension of geometrical ideas to this new type of coordinates has been called "noncommutative geometry". Following even closer the ideas and methods of quantum mechanics, we are led to assume that these noncommutative coordinates are represented as a subalgebra of the algebra of operators acting on a Hilbert space. This is the framework of the theory pioneered by A. Connes [2], which allows us to make use of the powerful tools of functional analysis. Within this framework, an analogue of gauge theory has been developed, even with non trivial topological properties, and it has already proved to be useful in various areas of physics, ranging from the classical description of the Higgs sector of the standard model (see [3] for a review) to recent ideas in string theory (see [4] and references therein).

This last example involves what we will call *NonCommutative Yang-Mills* (NCYM) theories in the sequel and can be thought of as a generalization of non-abelian gauge theories, whose gauge symmetry and interactions involves the noncommutative nature of the coordinates. A first example of such a theory appeared almost ten years ago, when Connes and Rieffel developed classical two dimensional Yang-Mills theory on the noncommutative torus [5]. This idea has also been generalized to higher dimensions [6].

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This naturally raises the question of the quantization of such theories, which has been tackled, at the one loop order, on the tori in [7] and on noncommutative $\mathbb{R}^D$ in [8] and [9]. Although these theories turned out to be non local, i.e. their interacting vertices involve trigonometric functions of the incoming momenta, it turns that the one-loop behavior is quite similar to the standard non-abelian case. This relies on an older work of Filk [10], who proved that the trigonometric factor of any planar graph (in a sense to be defined below) does not involve trigonometric functions of the internal momenta flowing into the loops, thus exhibiting the same divergence as the standard theory.

However, this is not true for non planar diagrams whose trigonometric factor does involve a phase depending on the internal momenta. Obviously such a phase softens the ultraviolet behavior of the corresponding diagram and it has been conjectured by several authors that such a diagram is in fact finite [7, 11, 12].

Nevertheless, it has been pointed out that this is not always the case [13]. Indeed, if the non planar diagram contains some special kind of non planar subdiagrams whose standard degree of divergence is strictly greater that zero, which is the case in a scalar field theory, the small momentum behavior of these diagrams yields a new kind of infrared divergence intimately tied up with the non locality.

This short paper is devoted to a survey of this problem in the simple case of multiloop corrections to the ghost propagator involving only nested and disjoint subdivergent one loop corrections to the ghost propagator. In the following section we shall briefly review the problem raised in [13] and then we shall present an explicit computation of the corresponding diagrams, postponing a complete probe into the renormalization of NCYM theory to a future publication.

2 Small momentum singularities induced by non planar diagrams

Before entering into the details of NCYM theory, let us recall that the noncommutative $\mathbb{R}^D$ is the algebra generated by D hermitean elements x_μ with commutator $[x_\mu, x_\nu] = -2\mathrm{i}\theta_{\mu\nu}$, where $\theta_{\mu\nu}$ denotes a real antisymmetric matrix which we will assume to be of maximal rank for convenience. Furthermore, one introduces Fourier modes $U(k) = \mathrm{e}^{\mathrm{i}k\cdot x}$, with $k\cdot x = k^\mu x_\mu$. We will always think of a smooth and at infinity rapidly decreasing function as a Fourier transform

$$f = \int d^D k\, f(k) U(k) \ ,$$

where $k \mapsto f(k)$ is itself a smooth and rapidly decreasing function on standard $\mathbb{R}^D$. The commutation relations of the coordinates endow the algebra with the star product

$$f \star_\theta g := \int d^D k\, d^D l\, f(k) g(l)\, U(k) U(l) \ , \tag{1}$$

which yields

$$(f \star_\theta g)(k) = \int d^D l\, f(k-l) g(l)\, \mathrm{e}^{\mathrm{i}\theta(k,l)}$$

with $\theta(k,l) = \theta_{\mu\nu} k^\mu l^\nu$. Finally, this algebra is equipped with the analogue of an integral defined as

$$\int f := f(0) \tag{2}$$

and partial derivatives

$$\partial_\mu f := \int d^D k \, \mathrm{i} k_\mu f(k) U(k) \ , \tag{3}$$

which satisfy most of the properties of their commutative counterparts: positivity and definiteness of the integral, Leibniz rule, commutativity of partial derivatives and integration by part, together with the tracial property of the integral

$$\int f \star_\theta g = \int g \star_\theta f \ ,$$

which proves to be fundamental in the construction of gauge invariant theories.

At that point, two additional remarks are in order. First of all, let us notice that the matrix $\theta_{\mu\nu}$ explicitly breaks Lorentz invariance (or its euclidian counterpart), which is reduced to the transformations commuting with θ, whereas translational invariance is preserved. Furthermore, as $\theta_{\mu\nu}$ is dimensionful, it also breaks scale invariance already at the classical level, for instance, in the case of four-dimensional NCYM theory or for a two-dimensional scalar field theory.

From now on, one easily constructs scalar field theories, like ϕ^4, whose euclidian action is

$$S[\phi] = \int \Big(\frac{1}{2} \partial_\mu \phi \star_\theta \partial_\mu \phi + \frac{m^2}{2} \phi \star_\theta \phi + \frac{g}{4!} \phi \star_\theta \phi \star_\theta \phi \star_\theta \phi \Big) \ , \tag{4}$$

or the NCYM action

$$\begin{aligned} S[A_\mu] &= -\frac{1}{4} \int F_{\mu\nu} \star_\theta F^{\mu\nu} \ , \qquad \text{with} \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + g(A_\mu \star_\theta A_\nu - A_\nu \star_\theta A_\mu) \ . \end{aligned} \tag{5}$$

Because of the tracial properties of the integration, the latter enjoys invariance under noncommutative gauge transformations

$$\delta_\lambda A_\mu = g(\lambda \star_\theta A_\mu - A_\mu \star_\theta \lambda) - \partial_\mu \lambda \ .$$

Perturbative quantization of these theories is easily performed within a formal functional integral point of view: The quadratic parts of the actions are equal to their commutative counterparts, whereas the interactions are non local and exhibit trigonometric functions of the incoming momenta of the interaction vertices. Thus, the total contribution of any Feynman diagram can be written as the product of a rational function by a trigonometric function. Because trigonometric functions are bounded, the standard rules of powercounting are unchanged so that Weinberg's convergence theorem remains valid.

It has been shown [10] that for planar diagrams the trigonometric function is independent of the internal momenta so that the Feynman integral reduces to the one encountered in a commutative field theory. For non planar diagrams the situation is more involved and we mainly have to distinguish two cases: whether the non planarity results from crossing of internal lines (i.e. from the non planar character of the amputed diagram), which we call type I diagrams, or whether it solely comes from crossing of internal lines whith external lines (type E). The latter case is much more tricky because the phase vanishes when the corresponding external momenta satisfy some particular relation.

The corresponding Feynman integral has been evaluated in [13] within Schwinger's regularization scheme. It turns out that a type I non planar diagram whose powercounting subdivergent non planar diagrams are all of type I will be convergent, the corresponding singularity when the Schwinger parameters goes to zero being removed. If the diagram is of type E, but does not contain any subdivergence of type E, it is non singular, except for some exceptional values of its external momenta.

Most of the trouble comes from the insertion of type E non planar subdivergences. Indeed, the latter correspond to Feynman integrals of the type

$$\int d^{nD}k\, R(k_1,\dots,k_n,p_1,\dots,p_N)\, \mathrm{e}^{\mathrm{i}(\theta(k_1,P_1)+\cdots+\theta(k_n,P_n))}, \tag{6}$$

where $k_1,\dots,k_n$ are the independent internal momenta, $p_1,\dots,p_N$ are the external momenta and $P_1,\dots,P_n$ are linear combinations of the external momenta. The rational function $R(k_1,\dots,k_n,p_1,\dots,p_N)$ is responsible for the subdivergence.

When all the P_i's which couple to internal momenta belonging to divergent integrals do not vanish, the corresponding integral can be considered as finite, being regularized by the oscillatory factor. Indeed, a computation with a cut-off introduced within Schwinger's parametric formula yields such a finiteness. However, whenever one of the P_i's coupled to a divergent loop integral vanishes, then the corresponding Feynman integral is just a usual divergent loop integral and yields a singularity. When inserted into a larger diagram, this could create some trouble when integrating over momenta approaching the subspace $P_i = 0$. The question is how fast the divergence appears compared with the smoothening property of the integration measure. A quadratic divergence seems to destroy the renormalizability [13] whereas a logarithmic divergence could be harmless. This conjecture is supported by our simple example.

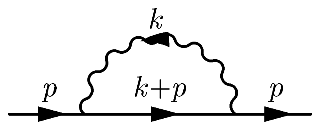
In a mathematically more satisfying manner, one can also consider such an integral as a well defined distribution which is nothing but the Fourier transform of the rational fraction R. However, we still have to face a problem when inserting the distribution into a larger Feynman diagram. In particular, the example described in [13] corresponds to a distribution which is nothing but the Feynman propagator and the infrared troubles when $p \to 0$ are quite similar to the usual small x singularities encountered in QFT.

Finally, let us point out that we did not encounter such a problem when quantizing NCYM theory on a torus [7]. The construction of the latter is similar to that of NCYM on $\mathbb{R}^4$ except for the quantization of the momenta appearing in the Fourier transform. As a consequence, there is no singularity when $p \to 0$ since such a limit cannot be taken. However, we noticed that a one loop non planar diagram with external momenta p exhibits an extra UV singularity of the type $\delta(p)/\epsilon$. Quite miraculously, all these singularities turn out to cancel, leaving us with finite one loop renormalized correlators.

In the next section, we shall evaluate the non planar contribution to some of the one loop corrections to the ghost self energy using Bessel functions, showing that they do not lead to any singularities when inserted into larger diagrams. We shall use the Feynman rules for NCYM theory without deriving them, the latter being obtained by replacing the structure constant of non-abelian gauge theory f_{abc} by $2\mathrm{i}\sin\theta(p,q)$ [8].

3 A simple example. One-loop calculation

Our goal is to compute the following 1-loop correction to the ghost propagator in NCYM theory:



(7)

Wavy lines represent gluons and straight lines ghosts. The Feynman rules derived in [8] and (for the noncommutative torus) [7] lead to the integral

$$
\begin{aligned}
I_1 &= \int d^4k\; 4g^2\hbar(-p-k)^\mu \frac{(-1)}{k^2}\Big(\delta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\Big)\frac{(-1)}{(k+p)^2}(-p)^\nu \sin\theta(k,p)\,\sin\theta(-k,k+p) \\
&= -4g^2\hbar \int d^4k\, \sin^2\theta(k,p)\Big(\frac{p^2+\alpha pk}{k^2(k+p)^2} - (1-\alpha)\frac{(pk)^2}{k^2k^2(k+p)^2}\Big)\;. \qquad (8)
\end{aligned}
$$

We work in a $D=4$ dimensional euclidian momentum space with metric $g_{\mu\nu} = \delta_{\mu\nu}$ and use obvious abbreviations such as $pk = g_{\mu\nu}p^\mu k^\nu$. Using Feynman parameters

$$
\frac{1}{A^rB^s} = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\int_0^1 \frac{x^{r-1}(1-x)^{s-1}\,dx}{(Ax+B(1-x))^{r+s}} \qquad (9)
$$

we obtain

$$
I_1 = -4g^2\hbar \int d^4k\; \sin^2\theta(k,p)\int_0^1 dx\Big(\frac{(p^2+\alpha pk)}{(k^2+2pkx+p^2x)^2} - \frac{2(1-\alpha)(1-x)\,(pk)^2}{(k^2+2pkx+p^2x)^3}\Big)\;.
$$

In the denominator we write qk instead of pk so that we reproduce the k's in the numerator by differentiation with respect to q:

$$
\begin{aligned}
I_1 = -4g^2\hbar \int d^4k\; \sin^2\theta(k,p)\Big(p^2\int_0^1 \frac{dx}{(k^2+2qkx+p^2x)^2} &- p^\mu\frac{\partial}{\partial q^\mu}\int_0^1 \frac{\alpha\,dx}{2x(k^2+2qkx+p^2x)} \\
&- p^\mu p^\nu \frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q^\nu}\int_0^1 \frac{(1-\alpha)(1-x)\,dx}{4x^2(k^2+2qkx+p^2x)}\Big)\Big|_{q=p}\;.
\end{aligned}
$$

Using $\frac{1}{A^n} = \frac{1}{\Gamma(n)}\int_0^\infty dt\, t^{n-1}\,\mathrm{e}^{-tA}$ we rewrite the integral into

$$
\begin{aligned}
I_1 &= 4g^2\hbar\int_0^1 dx\int_0^\infty dt\Big(-p^2t + \frac{\alpha}{2x}p^\mu\frac{\partial}{\partial q^\mu} + \frac{(1-\alpha)(1-x)}{4x^2}p^\mu p^\nu \frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q^\nu}\Big)K[t,p,q,x]\Big|_{q=p}\;, \\
K[t,p,q,x] &:= \int d^4k\; \mathrm{e}^{-t(k^2+2qkx+p^2x)}\sin^2\theta(k,p)\;.
\end{aligned}
$$

Developing the sine into a Fourier series we obtain for the kernel

$$
\begin{aligned}
K[t,p,q,x] = \int d^4k\; \Big(\tfrac{1}{2}\mathrm{e}^{-t(k+qx)^2-t(p^2x-q^2x^2)} &- \tfrac{1}{4}\mathrm{e}^{-t(k+qx+\mathrm{i}\theta(p)/t)^2-t(p^2x-q^2x^2)-2\mathrm{i}x\theta(p,q)-p\circ p/t} \\
&- \tfrac{1}{4}\mathrm{e}^{-t(k+qx-\mathrm{i}\theta(p)/t)^2-t(p^2x-q^2x^2)+2\mathrm{i}x\theta(p,q)-p\circ p/t}\Big)\;,
\end{aligned}
$$

where $\theta(p)^\mu := \theta^{\mu\alpha}p_\alpha$ and [13] $p\circ p := g_{\mu\nu}\theta^{\mu\alpha}p_\alpha\,\theta^{\nu\beta}p_\beta \equiv (\theta(p))^2$. Then it is easy to perform the Gaussian integration:

$$K[t,p,q,x] = \frac{\pi^2}{2t^2}\big(\mathrm{e}^{-t(p^2x-q^2x^2)} - \mathrm{e}^{-t(p^2x-q^2x^2)-p\circ p/t}\cos 2x\theta(p,q)\big)\;. \tag{10}$$

We can now perform the differentiations with the following result:

$$\begin{aligned} I_1 &= \pi^2 g^2\hbar\int_0^1 dx\int_0^\infty dt\big(\frac{1}{t}p^2(3\alpha x{-}\alpha{-}1{-}x) + 2(p^2)^2(1{-}\alpha)x^2(1{-}x)\big)\times\\ &\qquad\qquad\qquad\qquad\qquad\times\big(\mathrm{e}^{-tp^2x(1-x)} - \mathrm{e}^{-tp^2x(1-x)-p\circ p/t}\big) \\ &= p^2\pi^2g^2\hbar\int_0^1 dx\int_0^\infty dt\big(\frac{1}{t}(3\alpha x{-}\alpha{-}1{-}x) + 2(1{-}\alpha)x\big)\big(\mathrm{e}^{-t} - \mathrm{e}^{-t-x(1-x)p^2p\circ p/t}\big)\;. \end{aligned} \tag{11}$$

The integration over t diverges logarithmically. We define the projection t_p^2 onto the divergent part by

$$t_p^2(I_1) := p^2\pi^2g^2\hbar\int_0^1 dx\int_0^\infty\frac{dt}{t}(3\alpha x{-}\alpha{-}1{-}x)\mathrm{e}^{-t} = -p^2\pi^2g^2\hbar\frac{3{-}\alpha}{2}\int_0^\infty\frac{\mathrm{e}^{-t}\,dt}{t}\;. \tag{12}$$

The projection to the convergent part is

$$\begin{aligned} R_1 &:= (1-t_p^2)(I_1) \\ &= p^2\pi^2g^2\hbar\Big((1{-}\alpha) - \int_0^1 dx\int_0^\infty dt\Big(\frac{3\alpha x{-}\alpha{-}1{-}x}{t} + 2(1{-}\alpha)x\Big)\mathrm{e}^{-t-x(1-x)p^2p\circ p/t}\Big). \end{aligned} \tag{13}$$

Instead of introducing a cutoff as in [13] we prefer such a momentum subtraction before performing the divergent integral, in analogy to the BPHZ scheme, as the notation t_p^2 for the projection indicates. We believe this is advantageous for the renormalizability proof to all orders based on Zimmermann's forest formula. Please notice the difference between the power counting degree 2 entering the forest formula and the actually only logarithmic divergence in (12). This is crucial for the insertion as subdivergences and gives the reason why we will obtain local counterterms to all orders whereas there are true quadratic divergences in the scalar theory in [13]. It should be not difficult to extend such a subtraction scheme to the entire NCYM theory. It is however important to define t_p^d as the projector onto the strictly divergent part of an integral in order to produce local counterterms. This assumes one can prove that all divergent integrations give rise to such local terms, as we will do in this paper for repeated insertions of the ghost propagator.

The x-integration in (13) yields $\int_0^1 dx\; x\,\mathrm{e}^{-t-ax(1-x)/t} = \frac{1}{2}\int_0^1 dx\;\mathrm{e}^{-t-ax(1-x)/t}$ for any a so that

$$R_1 = p^2\pi^2g^2\hbar\Big((1{-}\alpha) + \int_0^1 dx\int_0^\infty dt\big(\frac{3{-}\alpha}{2t} - (1{-}\alpha)\big)\mathrm{e}^{-t-x(1-x)p^2p\circ p/t}\Big).$$

The t-integration leads to Bessel functions:

$$\begin{aligned} R_1 &= p^2\pi^2g^2\hbar\,(1{-}\alpha)\Big(1 - \int_0^1 dx\,2\sqrt{x(1{-}x)p^2p\circ p}\,K_1[2\sqrt{x(1{-}x)p^2p\circ p}]\Big) \\ &\quad + p^2\pi^2g^2\hbar\,(3{-}\alpha)\int_0^1 dx\,K_0[2\sqrt{x(1{-}x)p^2p\circ p}]\;. \end{aligned} \tag{14}$$

The Bessel function $K_0[y]$ diverges logarithmically to $+\infty$ for $y \to 0$ and converges exponentially to 0 for $y \to +\infty$. Hence, for any exponent $r > 0$ there exists a number $c_r^0 > 0$ such that

$$K_0[y] \leq c_r^0/y^r \qquad \forall\ 0 < y < \infty\ . \tag{15}$$

This will be proven algebraically in the Appendix. Graphically the situation is sketched in Figure 1. The function $yK_1[y]$ approaches 1 for $y \to 0$ and converges exponentially to 0

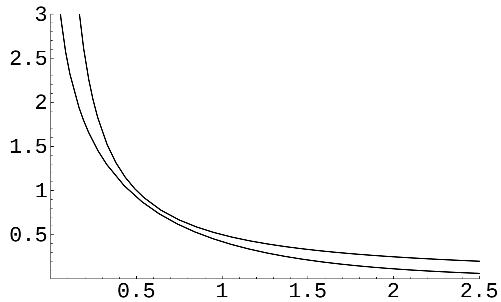


Figure 1: The Bessel function $K_0[y]$ (lower graph) is for any $0 < y < \infty$ bounded by the function c_r^0/y^r (upper graph). The situation is shown for $r = 1$ and $c_1^0 = \frac{1}{2}$.

for $y \to +\infty$. It is nevertheless convenient to regard it as $K_0[y]$ before: For any exponent $r > 0$ there exists a number $c_r^1 > 0$ such that

$$yK_1[y] \leq c_r^1/y^r \qquad \forall\ 0 < y < \infty\ . \tag{16}$$

The corresponding graphic is shown in Figure 2. Now the x-integration is easy to perform.

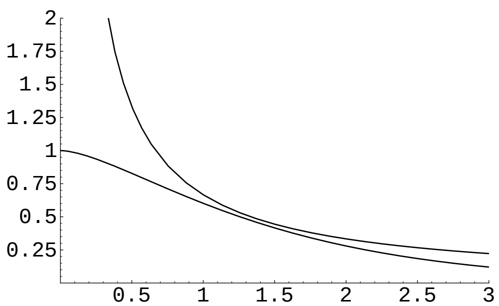


Figure 2: The function $yK_1[y]$ (lower graph) is for any $0 < y < \infty$ bounded by the function c_r^1/y^r (upper graph). The situation is shown for $r = 1$ and $c_1^1 = \frac{2}{3}$.

It converges for $0 < r < 1$, which means that infrared divergences are absent (for non exceptional momenta).

We restrict ourselves to the case where the rank of the tensor $g_{\mu\nu}\theta^{\mu\alpha}\theta^{\nu\beta}$ equals the space-time dimension (maximal noncommutativity). Then there exists some parameter m_P of dimension of a mass (the 'Planck mass') such that

$$p \circ p \geq \frac{p^2}{m_P^4} \qquad \Rightarrow \qquad \sqrt{p^2 p \circ p} \geq \frac{p^2}{m_P^2}\ . \tag{17}$$

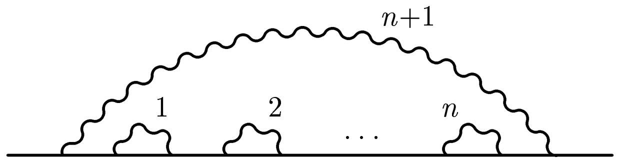
This yields the estimation

$$R_1 = p^2\pi^2 g^2 \hbar\Big((1-\alpha) + O\Big(P_r^1(\alpha)\Big(\frac{m_P^2}{p^2}\Big)^r\Big)\Big)\ , \tag{18}$$

where $P_r^n(\alpha)$ is a polynomial of homogeneous degree n in $(1-\alpha)$ and $(3-\alpha)$ with coefficients of order 1.

4 Higher loop order calculation

Now we insert n of these 1-loop propagator corrections into a propagator correction, giving an $n+1$ loop diagram:



(19)

Using (8) and the Feynman rule [7] for the ghost propagator, the corresponding integral is of order

$$
\begin{aligned}
I_{n+1} = -4g^2\hbar(-\pi^2 g^2\hbar)^n \int d^4k\ \sin^2\theta(k,p)\Big(\frac{p^2+\alpha pk}{k^2(k+p)^2} - (1-\alpha)\frac{(pk)^2}{k^2k^2(k+p)^2}\Big)\times \\
\times \sum_{j=0}^{n} P_r^j(\alpha)(1-\alpha)^{n-j}\Big(\frac{m_P^2}{(k+p)^2}\Big)^{jr}, \qquad (20)
\end{aligned}
$$

with $P_r^0(\alpha)=1$. Introduction of Feynman and Schwinger parameters as before and use of $x\Gamma(x)=\Gamma(x+1)$ leads to

$$
\begin{aligned}
I_{n+1} &= -4g^2\hbar(-\pi^2 g^2\hbar)^n \sum_{j=0}^{n}(1+jr)\,P_r^j(\alpha)\,(1-\alpha)^{n-j}m_P^{2jr}\int d^4k\ \sin^2\theta(k,p)\times \\
&\qquad \times \int_0^1 dx\Big(\frac{x^{jr}(p^2+\alpha pk)}{(k^2+2pkx+p^2x)^{2+jr}} - \frac{(2+jr)(1-\alpha)(1-x)x^{jr}(pk)^2}{(k^2+2pkx+p^2x)^{3+jr}}\Big) \\
&= 4g^2\hbar(-\pi^2 g^2\hbar)^n \sum_{j=0}^{n}\frac{1}{\Gamma(1+jr)}\,P_r^j(\alpha)\,(1-\alpha)^{n-j}m_P^{2jr}\times \\
&\quad\times \int_0^1 dx\int_0^\infty dt\Big(-x^{jr}t^{1+jr}p^2 + \tfrac{1}{2}\alpha x^{jr-1}t^{jr}p^\mu\frac{\partial}{\partial q^\mu} \\
&\qquad\qquad + \tfrac{1}{4}(1-\alpha)(1-x)x^{jr-2}t^{jr}p^\mu p^\nu\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q^\nu}\Big)K[t,p,q,x]\Big|_{q=p} \\
&= -(-\pi^2 g^2\hbar)^{n+1}\sum_{j=0}^{n}\frac{1}{\Gamma(1+jr)}\,P_r^j(\alpha)\,(1-\alpha)^{n-j}m_P^{2jr}\times \\
&\quad\times \int_0^1 dx\int_0^\infty dt\Big((3\alpha x-x-\alpha-1)x^{jr}t^{jr-1}p^2 + 2(1-\alpha)(1-x)x^{jr+2}t^{jr}(p^2)^2\Big)\times \\
&\qquad\qquad \times\big(\mathrm{e}^{-tp^2x(1-x)} - \mathrm{e}^{-tp^2x(1-x)-p\circ p/t}\big) \\
&= -(-\pi^2 g^2\hbar)^{n+1}p^2\sum_{j=0}^{n}\frac{1}{\Gamma(1+jr)}\,P_r^j(\alpha)\,(1-\alpha)^{n-j}\Big(\frac{m_P^2}{p^2}\Big)^{jr}\int_0^1\frac{dx}{(1-x)^{jr}}\times \\
&\quad\times \int_0^\infty dt\big((3\alpha x-x-\alpha-1)t^{jr-1} + 2(1-\alpha)xt^{jr}\big)\big(\mathrm{e}^{-t} - \mathrm{e}^{-t-x(1-x)p^2p\circ p/t}\big)\,. \qquad (21)
\end{aligned}
$$

The only divergent integral is for $j=0$ the projection

$$
\begin{aligned}
t_p^2(I_{n+1}) &:= -(-\pi^2 g^2\hbar)^{n+1}p^2(1-\alpha)^n\int_0^1 dx\int_0^\infty\frac{dt}{t}(3\alpha x-x-\alpha-1)\mathrm{e}^{-t} \\
&=(-\pi^2 g^2\hbar)^{n+1}p^2(1-\alpha)^n\,\frac{3-\alpha}{2}\int_0^\infty\frac{dt}{t}\mathrm{e}^{-t}\,. \qquad (22)
\end{aligned}
$$

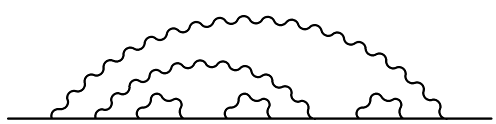
The convergent part can be evaluated to

$$
\begin{aligned}
R_{n+1} &= (1-t_p^2)(I_{n+1}) \\
&= -p^2(-\pi^2 g^2\hbar)^{n+1}(1{-}\alpha)^{n+1}\Big(1-\int_0^1 dx\, 2\sqrt{x(1{-}x)p^2 p\circ p}\, K_1[2\sqrt{x(1{-}x)p^2 p\circ p}]\Big) \\
&\quad - p^2(-\pi^2 g^2\hbar)^{n+1}(1{-}\alpha)^n(3{-}\alpha)\int_0^1 dx\, K_0[2\sqrt{x(1{-}x)p^2 p\circ p}] \\
&\quad + p^2(-\pi^2 g^2\hbar)^{n+1}\sum_{j=1}^n \frac{P_r^j(\alpha)}{jr(2{-}jr)}\,(1{-}\alpha)^{n-j}(3{-}\alpha)\Big(\frac{m_P^2}{p^2}\Big)^{jr} \\
&\quad + p^2(-\pi^2 g^2\hbar)^{n+1}\sum_{j=1}^n \frac{1}{\Gamma(1{+}jr)}\,P_r^j(\alpha)\,(1{-}\alpha)^{n-j}\Big(\frac{m_P^2}{p^2}\Big)^{jr}\times \\
&\quad \times\int_0^1 \frac{dx}{(1{-}x)^{jr}}\int_0^\infty dt\big((3\alpha x{-}x{-}\alpha{-}1)t^{jr-1}+2(1{-}\alpha)xt^{jr}\big)\mathrm{e}^{-t-x(1-x)p^2 p\circ p/t}\,.
\end{aligned}
\tag{23}
$$

The last two lines range from zero (for $p=\infty$) to minus the value of the third last line for $p=0$. Thus, in our estimation we have to neglect the last two lines. The remaining integrals over K_0 and K_1 are familiar to us, see Figures 1 and 2, and we choose the essential exponents in (15) and (16) to be $(n{+}1)r$ instead of r. Then we arrive at

$$
R_{n+1} = -p^2(-\pi^2 g^2\hbar)^{n+1}\Big((1{-}\alpha)^{n+1} + O\Big(\sum_{j=1}^{n+1} P_{r,j}^{n+1}(\alpha)\Big(\frac{m_P^2}{p^2}\Big)^{jr}\Big)\Big)\,. \tag{24}
$$

But this was precisely our starting point we inserted into (8) to obtain (20). Hence, (24) provides the structure of any renormalized $n{+}1$ loop graph made of ghost propagator corrections. The counterterm of such an n-loop graph is given by (22), and we see explicitly that the Feynman graphs made of nested 1-loop ghost propagator corrections



(25)

are renormalized by local counterterms for any order in $\hbar$. Here, locality means that the momentum dependence of the counterterm and the kinetic part of the ghost action are identical. In order to renormalize an n-loop graph (to avoid infrared divergences) the critical exponent has to be chosen $0<r<1/n$.

The essential step in this proof was the observation that the noncommutative Feynman graphs under consideration evaluate to Bessel functions, which can be estimated by a power law. It seems plausible that any Feynman graph of noncommutative Yang-Mills theory evaluates to Bessel functions, and applying the same techniques it should be possible to show that local counterterms suffice to renormalize this model.

Appendix: Proof of Eq. (15)

We prove that for each $r>0$ there is a number $c_r>0$ such that

$$
C_0(x) := \frac{c_r}{x^r} \geq K_0(x) \qquad \forall\; 0<x<\infty\,.
$$

The Bessel function $K_0(x)$ is one of the two solutions of the differential equation

$$xK_0'' + K_0' - xK_0 = 0 \ , \qquad 0 < x < \infty \ . \tag{26}$$

It is however more convenient to consider the function

$$K(x) := \sqrt{x}K_0(x) \ , \qquad K'' + \frac{1-4x^2}{4x^2}K = 0 \ ,$$

and compare it with

$$C(x) := \sqrt{x}C_0(x) \ , \qquad C'' + \frac{1-4r^2}{4x^2}C = 0 \ .$$

The derivative of the Wronskian $W(K,C) := K'C - C'K$ is

$$W' = \frac{x^2-r^2}{x^2}KC \ , \qquad \begin{array}{ll} W' > 0 & \text{for } x > r \\ W' < 0 & \text{for } x < r \end{array}$$

The asymptotic development shows that for the solution K_0 of (26) one has $W(x) < 0$ for $x \to \infty$ and $W(x) > 0$ for $x \to 0$. Therefore, there is only one zero of the Wronskian, at $x = x_r$, as illustrated in Figure 3.

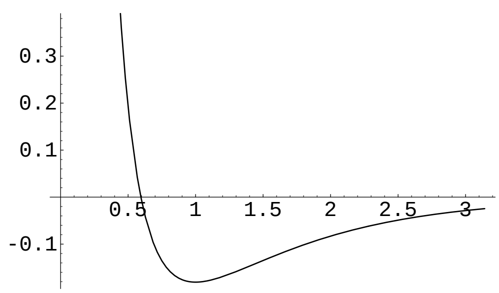


Figure 3: The Wronskian $W(K,C)$ has an extremum for $x = r$ and a zero for $x = x_r$. The situation is shown for $r = 1$ and $c_r = 1$.

We choose the normalization

$$K_0(x_r) = c_r x_r^{-r} \ , \qquad K_0'(x_r) = -rc_r x_r^{-r-1}$$

so that $W(x_r) = 0$ and $K(x_r) = C(x_r)$. Then we can integrate

$$\begin{array}{lll} x > x_r \ \Rightarrow \ W(x) < 0 : & \displaystyle\int_{x_r}^{x} \frac{K'(x)}{K(x)}dx < \int_{x_r}^{x} \frac{C'(x)}{C(x)}dx & \Rightarrow \ K(x) < C(x) \\ x < x_r \ \Rightarrow \ W(x) > 0 : & \displaystyle\int_{x}^{x_r} \frac{K'(x)}{K(x)}dx > \int_{x}^{x_r} \frac{C'(x)}{C(x)}dx & \Rightarrow \ K(x) < C(x) \end{array}$$

This finishes the proof.

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