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Deformed QED via Seiberg-Witten Map

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Abstract. With the help of the Seiberg-Witten map for photons and fermions we define a θ-deformed QED at the classical level. Two possibilities of gauge-fixing are discussed. A possible non-Abelian extension for a pure θ-deformed Yang-Mills theory is also presented.

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1 Introduction

The Seiberg-Witten map was originally discussed in the context of string theory, where it emerged from a 2D-σ-model regularized in different ways [1]. It was argued by Seiberg and Witten that the ordinary gauge theory should be gauge-equivalent to a noncommutative counterpart. The Seiberg-Witten map may be also introduced with the help of covariant coordinates. In this way the deformed gauge theory emerges as a gauge theory of a certain noncommutative algebra [2], [4] leading to the same results as in ref.[1].

In this note we apply the idea of the Seiberg-Witten map in order to define a θ-deformed QED at the classical level, including the corresponding Seiberg-Witten map for the infinitesimal local gauge transformation of the fermions. Using the definitions of the star-product [4] and the Seiberg-Witten map for the Abelian gauge field and the fermions we define a θ-deformed QED in terms of the deformation parameter θ of a noncommuting flat space-time. This deformation parameter, which is treated as a constant external field with canonical dimension -2, leads to a field theory with infinitely many θ-dependent interaction vertices.

In order to prepare the quantization of this deformed QED (and possible non-Abelian extensions) we discuss the gauge-fixing procedure of the photon sector in the Seiberg-Witten framework. It is interesting to note that there are two possibilities of gauge-fixing. The linear one corresponds to the usual gauge-fixing of the undeformed Abelian model, whereas the second possibility is induced by a nonlinear gauge-fixing emerging from the noncommutative structure of the deformed QED. It is important to observe that the θ-deformed QED is still characterized by an Abelian gauge symmetry. The symmetry of the model is described by a linear BRST-identity.

The letter is organized as follows. Section 2 is devoted to a definition of the Seiberg-Witten map for photons and fermions. With these θ-expansions for the basic field variables of the model and the usual definition of the star-product [4] an Abelian gauge invariant deformed QED action is defined. The gauge-fixing procedure à la BRST [5] will be discussed in section 3. Higher derivative gauge invariant monomials which are allowed due to the presence of θ are analyzed in section 4. In fact such terms are required if one studies radiative corrections [6] .

In a last step we present also a non-Abelian extension of the photon sector, the pure θ-deformed noncommutative Yang-Mills (NCYM) theory, involving the discussion of the invariant action, the gauge-fixing procedure and the higher derivative gauge invariant extensions of the action.

2 Deformed QED via Seiberg-Witten Map

The starting point is the undeformed QED with its local gauge invariance

$$S_{\mathrm{inv}} = \int d^4x \left[\bar{\psi} \left(\mathrm{i}\gamma^\mu D_\mu - m \right) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \tag{1}$$

where

$$D_\mu \psi = \left(\partial_\mu - \mathrm{i} A_\mu \right) \psi, \tag{2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3}$$

$S_{\rm inv}$ is invariant with respect to infinitesimal gauge transformations:

$$\begin{aligned} \delta_\lambda A_\mu &= \partial_\mu \lambda, \\ \delta_\lambda \psi &= \mathrm{i}\lambda\psi, \\ \delta_\lambda \bar{\psi} &= -\mathrm{i}\bar{\psi}\lambda. \end{aligned} \tag{4}$$

In order to get a θ-deformed QED in the sense of Seiberg and Witten [1] one defines

$$\begin{aligned} \hat{A}_\mu &= A_\mu + A'_\mu\left(A_\nu;\theta^{\rho\sigma}\right) + O(\theta^2), \\ \hat{\psi} &= \psi + \psi'\left(\psi, A_\nu;\theta^{\rho\sigma}\right) + O(\theta^2), \\ \hat{\lambda} &= \lambda + \lambda'\left(\lambda, A_\nu;\theta^{\rho\sigma}\right) + O(\theta^2), \end{aligned} \tag{5}$$

with the corresponding Seiberg-Witten maps given by

$$\hat{A}_\mu\left(A_\nu\right) + \hat{\delta}_{\hat{\lambda}}\hat{A}_\mu(A_\nu) = \hat{A}_\mu\left(A_\nu + \delta_\lambda A_\nu\right), \tag{6}$$

$$\hat{\psi}\left(\psi, A_\nu\right) + \hat{\delta}_{\hat{\lambda}}\hat{\psi}(\psi, A_\nu) = \hat{\psi}\left(\psi + \delta_\lambda\psi, A_\nu + \delta_\lambda A_\nu\right). \tag{7}$$

The deformed infinitesimal gauge transformations are defined as usual [7]:

$$\begin{aligned} \hat{\delta}_{\hat{\lambda}}\hat{A}_\mu &= \partial_\mu\hat{\lambda} + \mathrm{i}\left[\hat{\lambda}, \hat{A}_\mu\right]_\star = \partial_\mu\hat{\lambda} + \mathrm{i}\hat{\lambda}\star\hat{A}_\mu - \mathrm{i}\hat{A}_\mu\star\hat{\lambda}, \\ \hat{\delta}_{\hat{\lambda}}\hat{\psi} &= \mathrm{i}\hat{\lambda}\star\hat{\psi}, \\ \hat{\delta}_{\hat{\lambda}}\hat{\bar{\psi}} &= -\mathrm{i}\hat{\bar{\psi}}\star\hat{\lambda}, \end{aligned} \tag{8}$$

where we have used the star-product [4]. Expanding (8) in powers of θ one gets

$$\begin{aligned} \hat{\delta}_{\hat{\lambda}}\hat{A}_\mu &= \partial_\mu\hat{\lambda} - \theta^{\rho\sigma}\,\partial_\rho\lambda\,\partial_\sigma A_\mu + O(\theta^2), \\ \hat{\delta}_{\hat{\lambda}}\hat{\psi} &= \mathrm{i}\hat{\lambda}\hat{\psi} - \frac{1}{2}\,\theta^{\rho\sigma}\,\partial_\rho\lambda\,\partial_\sigma\psi + O(\theta^2). \end{aligned} \tag{9}$$

With (6)–(8) one gets the following solutions of (5):

$$\begin{aligned} A'_\mu &= -\frac{1}{2}\,\theta^{\rho\sigma}\,A_\rho\left(\partial_\sigma A_\mu + F_{\sigma\mu}\right), \\ \psi' &= -\frac{1}{2}\,\theta^{\rho\sigma}\,A_\rho\,\partial_\sigma\psi, \\ \lambda' &= -\frac{1}{2}\,\theta^{\rho\sigma}\,A_\rho\,\partial_\sigma\lambda. \end{aligned} \tag{10}$$

In terms of the commutative fields (3) the deformed field strength becomes

$$\begin{aligned} \hat{F}_{\mu\nu} &= \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu + \theta^{\rho\sigma}\,\partial_\rho A_\mu\,\partial_\sigma A_\nu + O(\theta^2) \\ &= F_{\mu\nu} + \theta^{\rho\sigma}\left(F_{\mu\rho}\,F_{\nu\sigma} - A_\rho\,\partial_\sigma F_{\mu\nu}\right) + O(\theta^2). \end{aligned} \tag{11}$$

The infinitesimal gauge transformation of $\hat{F}_{\mu\nu}$ is calculated with $\delta_\lambda A_\mu = \partial_\mu\lambda$ (from now on we omit the symbol $O(\theta^2)$)

$$\begin{aligned} \hat{\delta}_{\hat{\lambda}}\hat{F}_{\mu\nu} &= -\theta^{\rho\sigma}\,\partial_\rho\hat{\lambda}\,\partial_\sigma\hat{F}_{\mu\nu} \\ &= -\theta^{\rho\sigma}\,\partial_\rho\lambda\,\partial_\sigma F_{\mu\nu}. \end{aligned} \tag{12}$$

Now we are able to write down the analogue of (1), leading to the following generalization

$$
\begin{aligned}
\hat{\Sigma}_{\mathrm{inv}} &= \int d^4x \, \left[\hat{\bar{\psi}} \star \left(\mathrm{i}\gamma^\mu \hat{D}_\mu - m \right) \hat{\psi} - \frac{1}{4}\, \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu} \right] \\
&= \int d^4x \, \left[\hat{\bar{\psi}} \left(\mathrm{i}\gamma^\mu \hat{D}_\mu - m \right) \hat{\psi} - \frac{1}{4}\, \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right], \qquad (13)
\end{aligned}
$$

with

$$
\hat{D}_\mu \hat{\psi} = \partial_\mu \hat{\psi} - \mathrm{i}\hat{A}_\mu \star \hat{\psi}.
$$

Using the above expansions in θ one gets

$$
\begin{aligned}
\Sigma_{\mathrm{inv}} = \int d^4x \, \Big\{ & \bar{\psi}\, \mathrm{i}\gamma^\mu \Big[D_\mu \psi - \frac{1}{2}\, \theta^{\rho\sigma} \left(\partial_\mu A_\rho \, \partial_\sigma \psi + A_\rho \, \partial_\mu \partial_\sigma \psi \right) \\
& + \frac{\mathrm{i}}{2}\, \theta^{\rho\sigma} \left(A_\rho \left(\partial_\sigma A_\mu + F_{\sigma\mu} \right) \psi + A_\mu A_\rho \, \partial_\sigma \psi - \mathrm{i}\, \partial_\rho A_\mu \partial_\sigma \psi \right) \Big] \\
& - \frac{1}{2}\, \theta^{\rho\sigma}\, A_\rho \, \partial_\sigma \bar{\psi}\, \mathrm{i}\, \gamma^\mu D_\mu \psi - m\bar{\psi}\psi + \frac{1}{2} m \theta^{\rho\sigma} \partial_\rho A_\sigma \bar{\psi}\psi \\
& - \frac{1}{4}\, F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}\, \theta^{\rho\sigma} \left(F_{\mu\rho} F_{\nu\sigma} F^{\mu\nu} - \frac{1}{4}\, F_{\rho\sigma} F_{\mu\nu} F^{\mu\nu} \right) \Big\}. \qquad (14)
\end{aligned}
$$

Equation (14) contains non-renormalizable vertices of dimension 6. The quantity $\theta^{\rho\sigma}$ will be considered as a classical unquantized external field with dimension -2. The θ-deformed action (14) is invariant with respect to (4).

3 Gauge-Fixing à la BRST

In order to quantize the system (14) we need a gauge-fixing term which allows of the calculation of the photon propagator. This can be done in a twofold manner: Corresponding to the usual BRST-quantization procedure one replaces the infinitesimal gauge parameters λ and $\hat{\lambda}$ by Faddeev-Popov ghost fields c and $\hat{c}$ leading to the following BRST-transformations for the Abelian structure

$$
sA_\mu = \partial_\mu c, \quad s\psi = \mathrm{i}c\psi. \qquad (15)
$$

Corresponding to the noncommutative structure of (9) one gets

$$
\begin{aligned}
\hat{s}\hat{A}_\mu &= \partial_\mu \hat{c} - \theta^{\rho\sigma}\, \partial_\rho c \, \partial_\sigma A_\mu, \\
\hat{s}\hat{\psi} &= \mathrm{i}\, \hat{c}\, \hat{\psi} - \frac{1}{2}\, \theta^{\rho\sigma}\, \partial_\rho c \, \partial_\sigma \psi. \qquad (16)
\end{aligned}
$$

Nilpotency of (15) and (16) implies

$$
sc = 0, \qquad (17)
$$

$$
\hat{s}\hat{c} = -\frac{1}{2}\, \theta^{\rho\sigma} \partial_\rho c \, \partial_\sigma c. \qquad (18)
$$

Clearly, due to (10) one has also

$$\hat{c} = c - \frac{1}{2}\,\theta^{\rho\sigma} A_\rho\, \partial_\sigma c. \tag{19}$$

Introducing now two BRST-doublets $\hat{\bar{c}} = \bar{c}$ and $\hat{B} = B$ with

$$\begin{aligned} \hat{s}\hat{\bar{c}} &= s\bar{c} \; = \hat{B} = B, \\ \hat{s}\hat{B} &= sB = 0, \end{aligned} \tag{20}$$

one has two possibilities to construct Faddeev-Popov actions:

$$\Sigma_{\rm gf}^{(i)} = \int d^4x\, s\left(\bar{c}\,\partial^\mu A_\mu + \frac{\alpha}{2}\bar{c}B\right) = \int d^4x\, \left(B\,\partial^\mu A_\mu - \bar{c}\,\partial^2 c + \frac{\alpha}{2}B^2\right), \tag{21}$$

and

$$\hat{\Sigma}_{\rm gf}^{(ii)} = \int d^4x\, \hat{s}\left(\hat{\bar{c}}\,\partial^\mu \hat{A}_\mu + \frac{\alpha}{2}\hat{\bar{c}}\hat{B}\right) = \int d^4x\, \left(B\,\partial^\mu \hat{A}_\mu - \bar{c}\,\partial^\mu \hat{s}\hat{A}_\mu + \frac{\alpha}{2}B^2\right). \tag{22}$$

An expansion in θ of $\hat{\Sigma}_{\rm gf}^{(ii)}$ leads to

$$\begin{aligned} \Sigma_{\rm gf}^{(ii)} = \int d^4x\, \Big\{ & B\,\partial^\mu A_\mu - \bar{c}\,\partial^2 c + \frac{\alpha}{2}B^2 \\ & -\frac{1}{2}\,\theta^{\rho\sigma}\left[B\,\partial^\mu\left(A_\rho\left(\partial_\sigma A_\mu + F_{\sigma\mu}\right)\right) + \bar{c}\,\partial^2\left(\partial_\rho c\, A_\sigma\right) - 2\,\bar{c}\,\partial^\mu\left(\partial_\rho c\,\partial_\sigma A_\mu\right)\right]\Big\}. \end{aligned} \tag{23}$$

The case described by (21) is the usual linear $\partial^\mu A_\mu = 0$ gauge, whereas the Faddeev-Popov action of (23) corresponds to a highly nonlinear gauge. However, one has to stress that both cases lead to the same photon propagator

$$\langle A_\mu A_\nu \rangle_{(0)} = \frac{1}{k^2 + {\rm i}\epsilon}\left(g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right). \tag{24}$$

In terms of (15), (17) and (20) the symmetry content of (21) and (23) may be expressed by a linearized Slavnov-Taylor identity

$$\mathcal{S}\,\Sigma_\theta^{(0)} = \int d^4x \left[\partial_\mu c\,\frac{\delta\Sigma_\theta^{(0)}}{\delta A_\mu} + B\,\frac{\delta\Sigma_\theta^{(0)}}{\delta\bar{c}}\right] = 0, \tag{25}$$

with

$$\Sigma_\theta^{(0)} = \Sigma_{\rm inv} + \left\{ \begin{array}{c} \Sigma_{\rm gf}^{(i)} \\ \Sigma_{\rm gf}^{(ii)} \end{array} \right. . \tag{26}$$

Formula (25) implies the following transversality condition for the 2-point 1PI vertex functional

$$\partial_\mu^x \frac{\delta^2\Sigma_\theta^{(0)}}{\delta A_\mu(x)\,\delta A_\nu(y)} = 0, \tag{27}$$

meaning that the "vacuum polarization" of the photon is transverse in momentum space

$$p^\mu \Pi_{\mu\nu}(p,-p) = 0. \tag{28}$$

In a companion paper [6] we have studied perturbatively this transversality condition.

However, at this stage one has to state that the total action (26) is not the whole story. Because of the deformation parameter $\theta^{\mu\nu}$ (which is treated as a constant external field) one can construct further gauge invariant quantities. Such additional terms in the BRST-invariant total action are in fact needed for the one-loop renormalization [6] procedure.

4 Higher Derivative Gauge Invariant Terms in the Action

The presence of a constant external field θ allows in principle to add "infinitely" many gauge invariant terms to the Lagrangian. Due to this fact one can construct the following bilinear BRST-invariant action in the photon sector[1]

$$\begin{aligned}\Sigma^{\rm Max}_{\rm h.d.} = \int d^4x \Big\{ & -\frac{1}{2}\, F_{\mu\nu}\, \partial^2 \tilde{F}^{\mu\nu} - \frac{1}{2}\, \partial^\mu F_{\mu\nu}\, \partial^\rho \tilde{F}_\rho{}^\nu - \frac{1}{4}\, F_{\mu\nu}\, \partial^2 \tilde{\partial}^2 F^{\mu\nu} - \frac{1}{4}\, \theta^2\, F_{\mu\nu}\, (\partial^2)^2 F^{\mu\nu} \\ & + \frac{1}{2} \left(\tilde{\partial}^\mu \partial^\nu F_{\mu\nu} \right)^2 + \frac{1}{2}\, \tilde{F}_{\mu\nu}\, (\partial^2)^2 \tilde{F}^{\mu\nu} + \text{higher derivative terms} \Big\}, \end{aligned} \tag{29}$$

with

$$\begin{aligned} \tilde{\partial}^\mu &= \theta^\mu{}_\nu \partial^\nu, \\ \tilde{\tilde{\partial}}^\mu &= \theta^\mu{}_\rho \theta^\rho{}_\sigma \partial^\sigma, \\ \theta^2 &= \theta_\mu{}^\nu \theta_\nu{}^\mu, \\ \tilde{F}^{\mu\nu} &= \theta^{\mu\rho} F_\rho{}^\nu, \\ \tilde{F} &= \tilde{F}_\mu{}^\mu. \end{aligned} \tag{30}$$

Alternatively, (29) may be rewritten as

$$\begin{aligned}\Sigma^{\rm Max}_{\rm h.d.} = \int d^4x \Big\{ & -\frac{1}{2}\, F_{\mu\nu}\, \partial^2 \tilde{F}^{\mu\nu} - \frac{1}{2}\, \partial^\mu F_{\mu\nu}\, \partial^\rho \tilde{F}_\rho{}^\nu - \frac{1}{4}\, F_{\mu\nu}\, \partial^2 \tilde{\partial}^2 F^{\mu\nu} \\ & - \frac{1}{2}\, A^\mu \left[\tilde{\partial}^2 (\partial^2)^2 g_{\mu\nu} + (\partial_\mu \tilde{\tilde{\partial}}_\nu + \partial_\nu \tilde{\tilde{\partial}}_\mu)(\partial^2)^2 + (\partial^2)^3\, \theta^\rho{}_\mu\, \theta_{\rho\nu} \right] A^\nu \\ & - \frac{1}{4}\, \theta^2\, F_{\mu\nu}\, (\partial^2)^2 F^{\mu\nu} + \frac{1}{2}\, \tilde{\partial} A\, (\partial^2)^2 \tilde{\partial} A + \text{higher derivative terms} \Big\}. \end{aligned} \tag{31}$$

[1] Note that $\int d^4x (-\frac{1}{2}\, F_{\mu\nu}\, \partial^2 \tilde{F}^{\mu\nu})$ and $\int d^4x(-\frac{1}{2}\, \partial^\mu F_{\mu\nu}\, \partial^\rho \tilde{F}_\rho{}^\nu)$ are of topological nature. Observe also that $\tilde{\partial} A$ is gauge invariant.

In a similar way one can construct with $\tilde{D}_\mu = \theta_\mu{}^\nu D_\nu$ ($D_\mu = \partial_\mu - \mathrm{i}A_\mu$) also higher derivative terms for the Dirac theory

$$\begin{aligned}\Sigma^{\mathrm{Dir}}_{\mathrm{h.d.}} = \int d^4x \Big\{ & m\,\bar{\psi}\left(D\tilde{D} + (D\tilde{D})^2 + D^2\tilde{D}^2 + \ldots\right)\psi \\ & + \bar{\psi}\gamma^\mu\,\mathrm{i}D_\mu\left(D\tilde{D} + (D\tilde{D})^2 + D^2\tilde{D}^2 + \ldots\right)\psi \\ & + \bar{\psi}\gamma^\mu\,\mathrm{i}\tilde{D}_\mu\,D^2\left(1 + D\tilde{D} + (D\tilde{D})^2 + D^2\tilde{D}^2 + \ldots\right)\psi + \mathrm{h.d.}\Big\}. \end{aligned} \tag{32}$$

It is interesting to notice that one can construct also four-fermion vertices. For example one has

$$\Sigma^{\mathrm{Dir}}_{(4),\mathrm{h.d.}} = \int d^4x \left\{\bar{\psi}\tilde{D}_\mu\psi\bar{\psi}\tilde{D}^\mu\psi + \bar{\psi}\tilde{D}_\mu\left(D\tilde{D} + (D\tilde{D})^2 + D^2\tilde{D}^2 + \ldots\right)\psi\bar{\psi}\tilde{D}^\mu\psi + \mathrm{h.d.}\right\}. \tag{33}$$

However, these terms cannot be obtained from a truly noncommutative action by the Seiberg-Witten map because the corresponding expression would be non-local. Therefore, we expect that contributions of this type, which are possible for individual Feynman graphs, cancel on the level of Green's functions. Equations (29)–(33) imply that even at the classical level one has θ-dependent terms. This entails the following form of the "vacuum polarization" of the photon in momentum space representation

$$\begin{aligned}\Pi^{(0)}_{\mu\nu}(p,-p;\theta) \approx\; & (p^2)^2\theta^2\left(p_\mu p_\nu - g_{\mu\nu}p^2\right) + \tilde{p}^2p^2\left(p_\mu p_\nu - g_{\mu\nu}p^2\right) + (p^2)^2\tilde{p}_\mu\tilde{p}_\nu \\ & + (p^2)^2\left(g_{\mu\nu}\tilde{p}^2 + p_\mu\tilde{\tilde{p}}_\nu + p_\nu\tilde{\tilde{p}}_\mu + p^2\theta^\rho{}_\mu\theta_{\rho\nu}\right), \end{aligned} \tag{34}$$

in observing that no terms linear in $\theta^{\mu\nu}$ occur. Due to

$$p\tilde{p} = 0, \qquad p\tilde{\tilde{p}} = -\tilde{p}^2, \tag{35}$$

it is obvious that (34) fulfills the transversality condition (28), where we have used a notation analogous to (30). In order to be complete the total action is given by

$$\Sigma^{(0)}_\theta = \Sigma_{\mathrm{inv}} + \left\{\begin{array}{c}\Sigma^{(i)}_{\mathrm{gf}} \\ \Sigma^{(ii)}_{\mathrm{gf}}\end{array}\right\} + \Sigma^{\mathrm{Max}}_{\mathrm{h.d.}} + \Sigma^{\mathrm{Dir}}_{\mathrm{h.d.}} + \Sigma^{\mathrm{Dir}}_{(4),\mathrm{h.d.}}. \tag{36}$$

In the limit $\theta \to 0$ one recovers the ordinary QED with a free Faddeev-Popov sector. One has to stress also that equation (29) is required for the renormalization procedure at the one-loop level for the calculation of $\Pi^{(1)}_{\mu\nu}(p,-p;\theta)$ [6].

5 Non-Abelian Extension: Pure Yang-Mills Case (NCYM)

We start with the non-Abelian extension of (5) in considering only the gluon sector. The corresponding gluon field is now Lie-algebra valued:

$$A_\mu(x) = A^a_\mu(x)T^a, \tag{37}$$

where the T^a are the generators of a $U(N)$ gauge group with

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c, \quad \{T^a, T^b\} = d^{abc} T^c,$$
$$\mathrm{Tr}\left(T^a T^b\right) = \delta^{ab}, \quad T^0 = \frac{1}{\sqrt{N}} \mathbb{1}. \tag{38}$$

The corresponding non-Abelian field strength is defined as usual

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}\left[A_\mu, A_\nu\right]. \tag{39}$$

The infinitesimal gauge transformations are

$$\delta_\lambda A_\mu = \partial_\mu \lambda + \mathrm{i}\left[\lambda, A_\mu\right] \equiv D_\mu \lambda,$$
$$\delta_\lambda F_{\mu\nu} = \mathrm{i}\left[\lambda, F_{\mu\nu}\right]. \tag{40}$$

As shown in [1] the Seiberg-Witten map for the non-Abelian extension is

$$\hat{A}_\mu = A_\mu + A'_\mu = A_\mu - \frac{1}{4}\,\theta^{\rho\sigma}\left\{A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\right\} + O(\theta^2), \tag{41}$$

$$\hat{\lambda} = \lambda + \lambda' = \lambda + \frac{1}{4}\,\theta^{\rho\sigma}\left\{\partial_\rho \lambda, A_\sigma\right\} + O(\theta^2). \tag{42}$$

From

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - \mathrm{i}\left[\hat{A}_\mu, \hat{A}_\nu\right]_\star \tag{43}$$

follows with the help of (41)

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{4}\,\theta^{\rho\sigma}\left(2\left\{F_{\mu\rho}, F_{\nu\sigma}\right\} - \left\{A_\rho, D_\sigma F_{\mu\nu} + \partial_\sigma F_{\mu\nu}\right\}\right) + O(\theta^2). \tag{44}$$

Corresponding to (40) one has

$$\hat{\delta}_{\hat{\lambda}} \hat{A}_\mu = \partial_\mu \hat{\lambda} + \mathrm{i}\left[\hat{\lambda}, \hat{A}_\mu\right]_\star,$$
$$\hat{\delta}_{\hat{\lambda}} \hat{F}_{\mu\nu} = \mathrm{i}\left[\hat{\lambda}, \hat{F}_{\mu\nu}\right]_\star. \tag{45}$$

In the sense of Seiberg and Witten (45) implies

$$\hat{\delta}_{\hat{\lambda}} \hat{A}_\mu = \partial_\mu \hat{\lambda} + \mathrm{i}\left[\hat{\lambda}, \hat{A}_\mu\right] - \frac{1}{2}\,\theta^{\rho\sigma}\left(\partial_\rho \lambda\, \partial_\sigma A_\mu - \partial_\rho A_\mu\, \partial_\sigma \lambda\right) + O(\theta^2). \tag{46}$$

Inserting (41) and (42) one arrives finally at

$$\hat{\delta}_{\hat{\lambda}} \hat{A}_\mu = D_\mu \lambda + \frac{1}{4}\,\theta^{\rho\sigma}\left(\left\{\partial_\mu \partial_\rho \lambda, A_\sigma\right\} + \left\{\partial_\rho \lambda, \partial_\mu A_\sigma\right\}\right)$$
$$+ \frac{\mathrm{i}}{4}\,\theta^{\rho\sigma}\left(\left[\left\{\partial_\rho \lambda, A_\sigma\right\}, A_\mu\right] - \left[\lambda, \left\{A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\right\}\right]\right) + O(\theta^2). \tag{47}$$

Now we are ready to construct the gauge invariant non-Abelian action. Following [3] one has at the classical level

$$\hat{\Sigma}_{\rm inv} = -\frac{1}{4g^2}\int d^4x \,{\rm Tr}\left(\hat{F}_{\mu\nu}\star\hat{F}^{\mu\nu}\right) = -\frac{1}{4g^2}\int d^4x \,{\rm Tr}\left(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right). \tag{48}$$

Using (44) one gets for (48)

$$\begin{aligned}\Sigma_{\rm inv} &= -\frac{1}{4g^2}\int d^4x \,{\rm Tr}\left(F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\,\theta^{\rho\sigma}\left(2\left\{F_{\mu\rho},F_{\nu\sigma}\right\}F^{\mu\nu} - \left\{A_\rho, D_\sigma F_{\mu\nu} + \partial_\sigma F_{\mu\nu}\right\}F^{\mu\nu}\right)\right)\\ &= -\frac{1}{4g^2}\int d^4x\,\left(F^a_{\mu\nu}F^{a\,\mu\nu}\right.\\ &\qquad\left. + \,\theta^{\rho\sigma}\left(F^a_{\mu\rho}F^b_{\nu\sigma}F^{c\,\mu\nu} - \frac{1}{2}\,A^a_\rho (D_\sigma F_{\mu\nu})^b F^{c\,\mu\nu} - \frac{1}{2}\,A^a_\rho\,\partial_\sigma F^b_{\mu\nu}F^{c\,\mu\nu}\right)d^{abc}\right).\end{aligned} \tag{49}$$

6 Gauge-Fixing for Pure Yang-Mills Theory

Similar to the QED case there are again two possibilities of gauge-fixing for the non-Abelian generalization. One replaces the infinitesimal gauge parameters by the corresponding ghost fields:

$$\lambda \to c, \quad \hat{\lambda} \to \hat{c}. \tag{50}$$

The first choice leads to

$$\begin{aligned} sA_\mu &= \partial_\mu c + {\rm i}\left[c, A_\mu\right] = D_\mu c,\\ sc &= {\rm i}\,c^2 = \frac{\rm i}{2}\left\{c,c\right\}.\end{aligned} \tag{51}$$

With the second possibility of (50) one gets from (42)

$$\hat{c} = c + \frac{1}{4}\,\theta^{\rho\sigma}\left\{\partial_\rho c, A_\sigma\right\} + O(\theta^2). \tag{52}$$

Equations (52) and (41) together with (51) allow now to calculate the BRST-variations for $\hat{A}_\mu$ and $\hat{c}$:

$$\begin{aligned} \hat{s}\hat{A}_\mu &= sA_\mu - \frac{1}{4}\,\theta^{\rho\sigma}\left(\left\{sA_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\right\} + \left\{A_\rho, \partial_\sigma sA_\mu + sF_{\sigma\mu}\right\}\right),\\ \hat{s}\hat{c} &= sc + \frac{1}{4}\,\theta^{\rho\sigma}\left(\left\{\partial_\rho sc, A_\sigma\right\} - \left[\partial_\rho c, sA_\sigma\right]\right).\end{aligned} \tag{53}$$

By explicit calculations one verifies that the equations (53) follow by θ-expansion from

$$\begin{aligned} \hat{s}\hat{A}_\mu &= \partial_\mu \hat{c} + {\rm i}\left[\hat{c}, \hat{A}_\mu\right]_\star,\\ \hat{s}\hat{c} &= {\rm i}\,\hat{c}\star\hat{c}.\end{aligned} \tag{54}$$

The above results may be summarized by the Seiberg-Witten map for the BRST-transformation

$$\hat{A}_\mu\left(A_\nu\right) + \hat{s}\hat{A}_\mu(A_\nu) = \hat{A}_\mu\left(A_\nu + sA_\nu\right). \tag{55}$$

For practical reasons we choose now the simpler possibility for the gauge-fixing procedure:

$$\Sigma_{\mathrm{gf}} = \int d^4x \, \mathrm{Tr} \left(s(\bar{c}\, \partial^\mu A_\mu + \frac{\alpha}{2} \bar{c} B) \right) = \int d^4x \mathrm{Tr} \left(B\, \partial^\mu A_\mu - \bar{c}\, \partial^\mu \left(\partial_\mu c + \mathrm{i}\, [c, A_\mu] \right) + \frac{\alpha}{2} B^2 \right), \tag{56}$$

where $\bar{c}$ and B are the corresponding non-Abelian antighost and multiplier field. To be complete one also adds further gauge invariant pieces to the total gauge-fixed action. They are obtained from (29) by replacing the derivatives by the covariant derivatives and the Abelian by the non-Abelian field strength. The total BRST-invariant action is thus

$$\begin{aligned}
\Sigma_\theta^{(0)} &= \Sigma_{\mathrm{inv}} + \Sigma_{\mathrm{gf}} + \Sigma_{\mathrm{h.d.}} \\
&= -\frac{1}{4g^2} \int d^4x \, \mathrm{Tr} \left(F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \theta^{\rho\sigma} \left(2 \left\{ F_{\mu\rho}, F_{\nu\sigma} \right\} F^{\mu\nu} - \left\{ A_\rho, D_\sigma F_{\mu\nu} + \partial_\sigma F_{\mu\nu} \right\} F^{\mu\nu} \right) \right) \\
&\quad + \int d^4x \, \mathrm{Tr} \Big(B\, \partial^\mu A_\mu - \bar{c}\, \partial^\mu \left(\partial_\mu c + \mathrm{i}\, [c, A_\mu] \right) + \frac{\alpha}{2} B^2 - \frac{1}{2} F_{\mu\nu}\, D^2 \tilde{F}^{\mu\nu} \\
&\quad - \frac{1}{2} D^\mu F_{\mu\nu}\, D^\rho \tilde{F}_\rho{}^\nu - \frac{1}{4} F_{\mu\nu}\, D\tilde{D} F^{\mu\nu} - \frac{1}{4} F_{\mu\nu}\, D^2 \tilde{D}^2 F^{\mu\nu} - \frac{1}{4} \theta^2\, F_{\mu\nu}\, (D^2)^2 F^{\mu\nu} \\
&\quad + \frac{1}{2} \left(\tilde{D}^\mu D^\nu F_{\mu\nu} \right)^2 + \frac{1}{2} \tilde{F}_{\mu\nu}\, (D^2)^2 \tilde{F}^{\mu\nu} + \text{higher derivative terms} \Big).
\end{aligned} \tag{57}$$

In order to characterize the BRST-symmetry of this model one has to introduce BRST-invariant external sources ρ^μ, σ allowing to describe the non-linear BRST-transformations consistently. This leads to

$$\Sigma_{\mathrm{ext}} = \int d^4x \, \mathrm{Tr} \left(\rho^\mu s A_\mu + \sigma s c \right) \tag{58}$$

and to the action

$$\Sigma_{\theta,\mathrm{tot}}^{(0)} = \Sigma_\theta^{(0)} + \Sigma_{\mathrm{ext}}. \tag{59}$$

The BRST-symmetry is therefore encoded at the classical level by the non-linear Slavnov-Taylor identity

$$\mathcal{S}\, \Sigma_{\theta,\mathrm{tot}}^{(0)} = \int d^4x \mathrm{Tr} \left[\frac{\delta \Sigma_{\theta,\mathrm{tot}}^{(0)}}{\delta \rho^\mu} \frac{\delta \Sigma_{\theta,\mathrm{tot}}^{(0)}}{\delta A_\mu} + \frac{\delta \Sigma_{\theta,\mathrm{tot}}^{(0)}}{\delta \sigma} \frac{\delta \Sigma_{\theta,\mathrm{tot}}^{(0)}}{\delta c} + B\, \frac{\delta \Sigma_{\theta,\mathrm{tot}}^{(0)}}{\delta \bar{c}} \right] = 0. \tag{60}$$

7 Conclusion and Outlook

We have defined a θ-deformed QED with the help of Seiberg-Witten maps for photon and fermion fields at the classical level. In the same manner we also discussed a θ-deformed Yang-Mills model.

In a next step we focus on the quantization procedure. Some preliminary work has been done in [6], where the quantization of a θ-deformed Maxwell theory is presented. There, one-loop corrections to the 1PI two-point function for the photon are investigated in the presence of non-renormalizable interaction vertices.

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