On the Connes–Moscovici Hopf algebra associated to the diffeomorphisms of a manifold

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Abstract

For our own education, we reconstruct the Hopf algebra of Connes and Moscovici obtained by the action of vector fields on a crossed product of functions by diffeomorphisms. We extend the realization of that Hopf algebra in terms of rooted trees as given by Connes and Kreimer from dimension one to arbitrary dimension of the manifold. In principle there is no modification, but in higher dimension one has to be careful with the order of cuts. The order problem leads us to speculate that in quantum field theory the sum of Feynman graphs which corresponds to an element of the Connes–Moscovici Hopf algebra could have a larger symmetry than the individual graphs.

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1 Introduction

Recently two useful Hopf algebras were discovered – by Alain Connes and Henri Moscovici in noncommutative geometry [1] and by Dirk Kreimer in quantum field theory [2]. Connes and Kreimer showed that both Hopf algebras are intimately related [3], via the language of rooted trees. Recently it was pointed out [4] that the same algebra of rooted trees appears in numerical analysis. We refer to [5] for a review of all these ideas.

For a physicist, the Hopf algebra $\mathcal{H}_K$ of Kreimer [2] is not difficult to understand. The idea is to look at the divergence structure of Feynman graphs. There is a natural splitting of a Feynman graph γ with non-overlapping subdivergences into two, given by a product of selected subdivergences $\gamma_1 \cdots \gamma_n$ and the graph $\gamma \setminus (\gamma_1 \cdots \gamma_n)$ left over from γ by shrinking all γ_i, $i = 1, \dots, n$, to a point. This is a standard operation in renormalization theory. Summing over all possibilities gives a coproduct operation on the algebra of polynomials in Feynman graphs. The unique antipode reproduces precisely the combinatorics of renormalization, i.e. it produces the local counterterms to make the integral corresponding to the divergent Feynman graph finite.

The aim of this note is to review in some detail the construction of the Hopf algebra found by Connes and Moscovici, in order to ease its access for physicists interested in the Hopf algebra $\mathcal{H}_K$ of renormalization. The construction requires only some basic knowledge of classical differential geometry, which can be found in many books on this topic, for instance in [6]. More precisely, one needs the vertical and horizontal vector fields Y and X on the frame bundle over an oriented manifold and their transformation behavior under diffeomorphisms, as well as some familiarity with the push-forward and pull-back operations. These requisites are derived in section 2. New is the application of these vector fields to the crossed product, see section 3, which defines the coproduct of X, Y and leads to an additional operator δ on the crossed product. The operators X, Y, δ generate a Lie algebra. Its enveloping algebra $\mathcal{H}$ is a bialgebra with the coproduct obtained before, and there exists an antipode making it to a Hopf algebra, see section 4.

The final section is devoted to the transformation of the commutative Hopf subalgebra $\mathcal{H}_{CM}$ of $\mathcal{H}$ into the language of rooted trees so that we can compare it with $\mathcal{H}_K$. We generalize the rooted trees given in [3] from dimension 1 to arbitrary dimension of the manifold. This generalization is quite obvious, but it has several consequences which are not visible in dimension 1. An element of $\mathcal{H}_{CM}$ is a sum of decorated planar rooted trees. The root is decorated by three spacetime indices necessary to describe parallel transport whereas the other vertices are decorated by a single spacetime index. This is closer to quantum field theory, where the decoration is given by primitive Feynman graphs without subdivergences. Interestingly, elements of $\mathcal{H}_{CM}$ are invariant under permutations of the decorations, whereas the individual trees representing Feynman graphs are not. This raises the question whether the sum of Feynman graphs which corresponds to an element of $\mathcal{H}_{CM}$ has a meaning in QFT.

2 The geometry of the frame bundle

In this section we are going to derive in some detail the following well-known results on the principal fibre bundle F^+ of oriented frames on an n-dimensional manifold M:

Proposition 1 *Let $\{x^\mu\}_{\mu=1\dots,n}$ be the coordinates of $x \in M$ within a local chart of M and $\{y_i^\mu\}_{\mu,i=1,\dots n}$ be the coordinates of n linearly independent vectors of the tangent space T_xM with respect to the basis ∂_μ. On F^+ there exist the following geometrical objects, written in terms of the local coordinates (x^μ, y_i^μ) of $p \in F^+$:*

(1) an $\mathbb{R}^n$-valued (soldering) 1-form α with $\alpha^i = (y^{-1})^i_\mu dx^\mu$,

(2) a $gl(n)$-valued (connection) 1-form ω with $\omega^i_j = (y^{-1})^i_\mu(dy^\mu_j + \Gamma^\mu_{\alpha\beta}\, y^\alpha_j dx^\beta)$, where $\Gamma^\mu_{\alpha\beta}$ depends only on x^ν ,

(3) n^2 vertical vector fields $Y^i_j = y^\mu_j \partial^i_\mu$,

(4) n horizontal (with respect to ω) vector fields $X_i = y^\mu_i(\partial_\mu - \Gamma^\nu_{\alpha\mu} y^\alpha_j \partial^j_\nu)$.

A local diffeomorphism ψ of M has a lift $\tilde{\psi} : (x^\mu, y^\mu_i) \mapsto (\psi(x)^\mu, \partial_\nu\psi(x)^\mu y^\nu_i)$ to the frame bundle and induces the following transformations of the previous geometrical objects:

(1) $(\tilde{\psi}^*\alpha)\big|_p = \alpha\big|_p$.

(2) $(\tilde{\psi}^*\omega)\big|_p = (y^{-1})^i_\mu(dy^\mu_j + \tilde{\Gamma}^\mu_{\alpha\beta}\, y^\alpha_j dx^\beta)$ *is again a connection form, with* $\tilde{\Gamma}^\mu_{\alpha\beta}\big|_x = ((\partial\psi(x))^{-1})^\mu_\gamma\, \Gamma^\gamma_{\delta\epsilon}\big|_{\psi(x)}\, \partial_\alpha\psi(x)^\delta \partial_\beta\psi(x)^\epsilon + ((\partial\psi(x))^{-1})^\mu_\gamma\, \partial_\beta\partial_\alpha\psi(x)^\gamma$,

(3) $(\tilde{\psi}_* Y^j_i)\big|_p = Y^j_i\big|_p$,

(4) $(\tilde{\psi}_*^{-1} X_i)\big|_p = y^\mu_i(\partial_\mu - \tilde{\Gamma}^\nu_{\alpha\mu} y^\alpha_j \partial^j_\nu)$ *is horizontal to* $\tilde{\psi}^*\omega$.

The reader familiar with these notations can pass immediately to section 3 on page 8.

2.1 Frame bundle

Let M be an n-dimensional smooth and oriented manifold. We are going to consider the frame bundle F^+ over M defined as follows. Let T_xM be the tangent space at a given point $x \in M$. It is an n-dimensional vector space containing the tangent vectors at x of curves in M through x. A base in T_xM is given by the n tangent vectors $\partial_\mu := \frac{\partial}{\partial x^\mu}$ of the coordinate lines in M. If x has (in a given chart of its neighbourhood) the coordinates $\{x^\mu\} \equiv (x^1, \ldots, x^n)$, we compute the tangent vector to a curve $C(t) = \{x^\mu(t)\}$:

$$\frac{d\phi(C(t))}{dt}\Big|_{t=0} = \frac{dx^\mu}{dt}\Big|_{t=0} \frac{\partial}{\partial x^\mu}\phi\Big|_x , \tag{2.1}$$

where $\phi : M \to \mathbb{R}$ (or $\mathbb{C}$) is an arbitrary function on M. According to Einstein's sum convention summation over pairs of identical upper and lower indices is self-understood.

An arbitrary vector $Y_j \in T_xM$ can be decomposed with respect to that basis, $Y_j = y^\mu_j \partial_\mu$. A frame at x is now a set of n linearly independent vectors $Y_j \in T_xM$, $j = 1, \ldots, n$, parameterized by their coordinates y^μ_j, where both μ and j run from 1 to n. Linear independence is equivalent to $\det y \neq 0$, and oriented frames have the same sign of $\det y$.

The (oriented) frame bundle F^+ is now given by the base space M with the set of smooth (positively oriented) frames attached to each point $x \in M$. A point in F^+ is thus (locally) given by the collection

$$(x, \{Y_j\}) = (x^\mu, y^\mu_j)_{\mu,j=1,\ldots,n} , \qquad \det y > 0 ,$$

where x^μ are the coordinates of x and the $y^\mu_j \in Gl^+(n)$ parameterize an oriented frame $\{Y_j\}_{j=1,\ldots,n}$ at x. Here, $Gl^+(n)$ is the group of $n \times n$ matrices with positive determinant.

In the overlap of two charts U_1, U_2, a point $x \in U_1 \cap U_2 \subset M$ will have coordinates x^μ in U_1 and $x^{\nu\prime}$ in U_2. The tangent vector Y_i at a curve in $U_1 \cap U_2$ through x is given

$$\begin{aligned} \text{in } U_1 \text{ by} \quad Y_i &= \frac{df(x^\mu(t))}{dt}\Big|_{t=0} = \frac{dx^\mu(t)}{dt}\Big|_{t=0} \partial_\mu f , \\ \text{in } U_2 \text{ by} \quad Y_i &= \frac{df(x^{\nu\prime}(x^\mu(t)))}{dt}\Big|_{t=0} = \frac{dx^\mu(t)}{dt}\Big|_{t=0} \frac{\partial x^{\nu\prime}}{\partial x^\mu}\, \partial^\prime_\nu f , \end{aligned}$$

where f is an arbitrary function on F^+. Hence, the coordinates $(x^\mu, y_j^\mu) \in U_1 \times Gl^+(n)$ and $(x^{\nu\prime}, y_j^{\nu\prime}) \in U_2 \times Gl^+(n)$ label the same point in F^+ iff $x^\mu, x^{\nu\prime}$ are the coordinates in U_1, U_2 of $x \in U_1 \cap U_2$ and $y_j^{\nu\prime} = (\partial x^{\nu\prime}/\partial x^\mu) y_j^\mu$.

There is a natural action of $Gl^+(n)$ on a frame $\{Y_j\}$ at x: The matrix $g_j^i \in Gl^+(n)$ maps the vector $Y_i \in T_x M$ into the new vector $Y_i g_j^i =: Y_j' \in T_x M$, or – in coordinates – y_i^μ into $y_i^\mu g_j^i$. This $Gl^+(n)$-action extends naturally to the frame bundle, making F^+ to a $Gl^+(n)$-principal fibre bundle:

$$g: \; (x^\mu, y_i^\mu) \mapsto (x^\mu, y_i^\mu g_j^i) \; . \tag{2.2}$$

The above action can be regarded as generated by a vector field according to the following construction. Let $gl(n)$ be the Lie algebra of $Gl^+(n)$. The exponential mapping assigns to $A \in gl(n)$ a curve $\exp(tA)$ in $Gl^+(n)$, which by (2.2) induces a field of curves through every point of F^+. This field of curves provides us with a field of tangent vectors

$$\frac{df(x^\mu, y_i^\mu[\exp(tA)]_j^i}{dt}\Big|_{t=0} = \frac{\partial f}{\partial(y_k^\mu \delta_j^k)} \frac{d(y_i^\mu[\exp(tA)]_j^i)}{dt}\Big|_{t=0} = A_j^i y_i^\mu \frac{\partial}{\partial y_j^\mu} f \; ,$$

where f is a function on F^+. Hence, each such vector field associated to $A \in gl(n)$ is generated by the following (vertical) vector fields

$$Y_i^j = y_i^\mu \frac{\partial}{\partial y_j^\mu} \equiv y_i^\mu \partial_\mu^j \; . \tag{2.3}$$

The vector field $A^\# = A_j^i Y_i^j$ associated to $A \in gl(n)$ is called the fundamental vector field corresponding to A.

A somewhat tricky construction provides us with an $\mathbb{R}^n$-valued 1-form α on F^+, sometimes called soldering form or canonical 1-form. A point $p = (x, \{y_j^\mu\}) \in F^+$ assigns to a vector $\tilde{V} \in T_x M$ a vector $\Phi_p(\tilde{V}) \in \mathbb{R}^n$ by decomposing $\tilde{V}$ with respect to the basis $Y_j = y_j^\mu \partial_\mu$. Thus, $[\Phi_p(\tilde{V})]^j Y_j = \tilde{V}$. Now, the $\mathbb{R}^n$-valued 1-form α is defined by

$$\alpha(V)\big|_p = \Phi_p(\pi_* V) \; , \qquad V \in T_p F^+ \; . \tag{2.4}$$

By $\big|_p$ we denote the value of a differential form or a vector field at the point $p \in F^+$. In (2.4), π_* is the differential of the vertical projection $\pi(x, \{y_j^\mu\}) = x$ which projects the vector $V = V^\mu \partial_\mu + V_j^\mu \partial_\mu^j \in T_p F^+$ into the vector $\pi_* V = V^\mu \partial_\mu \in T_{\pi(p)} M$. In this notation we have $\pi_* V = V^\nu (y^{-1})_\nu^j Y_j$, using the obvious definition $y_i^\mu (y^{-1})_\nu^i = \delta_\nu^\mu$. This gives $[\Phi_p(\pi_* V)]^j = V^\nu (y^{-1})_\nu^j$. On the other hand, decomposing $\alpha^i = \alpha_\mu^i dx^\mu + \alpha_\mu^{ik} dy_k^\mu$ and using the definition

$$dy_i^\mu(\partial_\nu^j) = \delta_i^j \delta_\nu^\mu \; , \quad dx^\mu(\partial_\nu) = \delta_\nu^\mu \; , \quad dy_i^\mu(\partial_\nu) = 0 \; , \quad dx^\mu(\partial_\nu^j) = 0$$

of the pairing between covectors and vectors, we have $\alpha^j(V) = \alpha_\mu^j V^\mu + \alpha_\mu^{ji} V_i^\mu$, giving

$$\alpha^j = (y^{-1})_\mu^j \, dx^\mu \; . \tag{2.5}$$

The definition (2.4), although involving local coordinates in the construction, is independent of the choice of charts. Indeed, if $p \in F^+$ has the coordinates (x^μ, y^μ_j) and $(x^{\nu'}, y^{\nu'}_j)$ in two charts $U_1 \times Gl^+(n)$ and $U_2 \times Gl^+(n)$ of F^+, with $y^{\nu'}_j = (\partial x^{\nu'}/\partial x^\mu) y^\mu_j$, then $\pi_* V = V^\mu \partial_\mu \in T_{\pi(p)} U_1$ and $\pi_* V = V^{\nu'} \partial_{\nu'} \in T_{\pi(p)} U_2$, with $V^{\nu'} = (\partial x^{\nu'}/\partial x^\mu) V^\mu$. This means that $V^\mu (y^{-1})^j_\mu = V^{\nu'} (y^{-1})^j_{\nu'} \in \mathbb{R}$ give the same value for $[\Phi_p(\pi_* V)]^j$.

2.2 Connection

A connection is the splitting of the tangent space $T_p F^+$ at $p \in F^+$ into a direct sum of a vertical space $V_p F^+$ (generated by $Y^i_j = y^\mu_j \partial^i_\mu$) and a horizontal space $H_p F^+$ such that $H_{pg} F^+ = R_{g*} H_p F^+$. In the last equation, $pg \in F^+$ is the point obtained by the action (2.2) of $g \in Gl^+(n)$ and R_{g*} is the induced push-forward of a vector in $T_p F^+$ to a vector in $T_{pg} F^+$. If $V \in T_p F^+$ is the tangent vector of a curve $p(t)$ in F^+ through p, then the push-forward $R_{g*} V$ is the tangent vector of the curve $p(t)g$ through pg. Explicitly, let f be a function on F^+ and $V = V^\mu_j \partial^j_\mu + V^\mu \partial_\mu \in T_p F^+$ be tangent to the curve $C = (x^\mu(t), y^\mu_j(t))$ at p, i.e.

$$V f = (df(p)/dt) = ((dx^\mu/dt)\partial_\mu + (dy^\mu_j/dt)\partial^j_\mu) f \ . \tag{2.6}$$

Then, the push-forward is given by

$$(R_{g*} V) f = \frac{df(p(t)\,g)}{dt} = \frac{dx^\mu(t)}{dt} \partial_\mu f + \frac{d(y^\mu_j(t)\, g^j_i)}{dt} \frac{\partial f}{\partial \hat{y}^\mu_i} \ , \qquad \hat{y}^\mu_i := y^\mu_k g^k_i \ . \tag{2.7}$$

Thus, we obtain

$$R_{g*} V = V^\mu_j g^j_i \,\widehat{\partial^i_\mu} + V^\mu \partial_\mu \ , \qquad \widehat{\partial^i_\mu} := \partial / \partial \hat{y}^\mu_i \ . \tag{2.8}$$

In practice the connection is most conveniently characterized by the connection form ω, a $gl(n)$-valued differential 1-form with the following properties: For a given matrix $A \in gl(n)$ let $A^\# = A^i_j Y^j_i$ be the corresponding fundamental vector field. Then ω is defined by

$$\omega(A^\#) = A \ , \qquad \omega|_{pg}(R_{g*} V) = g^{-1}\big(\omega|_p(V)\big) g \ , \tag{2.9}$$

for $V \in T_p F^+$ and $g \in Gl^+(n)$. At the point $p = (x^\mu, y^\mu_i)$, the components ω^i_j of the connection form will have the decomposition

$$\omega^i_j = W^{ik}_{j\mu} dy^\mu_k + W^i_{j\mu} dx^\mu \ ,$$

for certain functions W. From (2.3) we get immediately

$$A^i_j \equiv \omega^i_j(A^\#) = W^{ik}_{j\mu} y^\mu_l A^l_k \ ,$$

which gives $W^{ik}_{j\mu} = \delta^k_j (y^{-1})^i_\mu$. This suggests the following ansatz

$$\omega^i_j = (y^{-1})^i_\mu (dy^\mu_j + \Gamma^\mu_{\nu\alpha} y^\nu_j dx^\alpha) \ , \tag{2.10}$$

where $\Gamma^\mu_{\nu\alpha}$ is a so far undetermined function of x and y. Using (2.8) we write down

$$\begin{aligned}\omega^i_j|_{pg}(R_{g*}V) &= (g^{-1})^i_k(y^{-1})^k_\mu(d\hat{y}^\mu_j + \Gamma^\mu_{\nu\alpha}|_{pg}\, y^\nu_k g^k_j dx^\alpha)(V^\nu_m g^m_l \widehat{\partial^l_\nu} + V^\mu\partial_\mu) \\ &= (g^{-1})^i_k(y^{-1})^k_\mu V^\mu_m g^m_j + (g^{-1})^i_k(y^{-1})^k_\mu \Gamma^\mu_{\nu\alpha}|_{pg}\, V^\alpha y^\nu_k g^k_j \ .\end{aligned}$$

On the other hand,

$$[g^{-1}(\omega|_p(V))g]^i_j = (g^{-1})^i_k(y^{-1})^k_\mu V^\mu_m g^m_j + (g^{-1})^i_k(y^{-1})^k_\mu \Gamma^\mu_{\nu\alpha}|_p\, V^\alpha y^\nu_k g^k_j \ .$$

Thus, the condition (2.9) tells us that $\Gamma^\mu_{\nu\alpha}|_p = \Gamma^\mu_{\nu\alpha}|_{pg}$, *which means that* $\Gamma^\mu_{\nu\alpha}$ *depends only on the base point* x.

Now, the horizontal vector fields X_i associated to the connection are defined as the kernel of ω and the dual of α,

$$\omega^i_j(X_k) = 0 \ , \qquad \alpha^j(X_i) = \delta^j_i \ . \tag{2.11}$$

These equations are easy to solve:

$$X_i = y^\mu_i(\partial_\mu - \Gamma^\nu_{\alpha\mu} y^\alpha_j \partial^j_\nu) \ . \tag{2.12}$$

The torsion form Θ on F^+ is an $\mathbb{R}^n$-valued differential 2-form defined as the covariant derivative of α,

$$\Theta^i = d\alpha^i + \omega^i_j \wedge \alpha^j \ . \tag{2.13}$$

Using (2.5) and (2.10) we compute

$$\begin{aligned}\Theta^i &= -(y^{-1})^i_\nu (y^{-1})^j_\mu\, dy^\nu_j \wedge dx^\mu + (y^{-1})^i_\mu (dy^\mu_j + \Gamma^\mu_{\nu\alpha} y^\nu_j dx^\alpha) \wedge (y^{-1})^j_\beta dx^\beta \\ &= (y^{-1})^i_\mu \Gamma^\mu_{\nu\alpha} dx^\alpha \wedge dx^\nu \ .\end{aligned}$$

The torsion vanishes iff the connection coefficients are symmetric, $\Gamma^\mu_{\nu\alpha} = \Gamma^\mu_{\alpha\nu}$.

2.3 Diffeomorphisms

Let ψ be a local (orientation preserving) diffeomorphism of M. By push-forward it maps a frame $\{Y_j\}$ at $x \in M$ into the frame $\{\psi_* Y_j\}$ at $\psi(x) \in M$. If Y is the tangent vector at x of a curve $C = \{x^\mu(t)\}$ through x, then $\psi_* Y$ is the tangent vector at $\psi(x)$ of the curve $\psi(C) = \{\psi(x(t))^\nu\}$. We evaluate both vectors on a function ϕ on M:

$$\begin{aligned}Y\phi &= \frac{d}{dt}\phi(x^\mu(t))\Big|_{t=0} = \frac{\partial\phi(x^\mu)}{\partial x^\mu}\,\frac{dx^\mu(t)}{dt}\Big|_{t=0} = (dx^\mu(t)/dt)|_{t=0}\,\partial_\mu\phi \ , \\ (\psi_* Y)\phi &= \frac{d}{dt}\phi(\psi(x(t))^\nu)\Big|_{t=0} = \frac{\partial\phi(\tilde{x}^\nu)}{\partial\tilde{x}^\mu}\,\frac{\partial\psi(x)^\mu}{\partial x^\nu}\,\frac{dx^\nu(t)}{dt}\Big|_{t=0} \\ &= \partial_\nu\psi(x)^\mu\,(dx^\nu(t)/dt)|_{t=0}\,\widetilde{\partial_\mu}\phi \ ,\end{aligned}$$

with $\tilde{x}^\mu = \psi(x)^\mu$ and $\widetilde{\partial_\mu} = \partial/\partial\tilde{x}^\mu$. Recall that ∂_μ and $\widetilde{\partial_\mu}$ are the bases of vector fields in T_xM and $T_{\psi(x)}M$, respectively. Hence, if y^μ_j are the coordinates of $Y_j \in T_xM$, then $\psi_* Y_j \in T_{\psi(x)}M$ has the coordinates $\partial_\nu\psi(x)^\mu\, y^\nu_j$, with respect to these bases. To summarize, the action $\tilde{\psi}$ on F^+ of a diffeomorphism ψ of M is given by

$$\tilde{\psi} : (\{x^\mu\}, \{y_i^\mu\}) \mapsto (\{\tilde{x}^\mu := \psi(x)^\mu\}, \{\tilde{y}_i^\mu := \partial_\nu \psi(x)^\mu \, y_i^\nu\}) \ . \tag{2.14}$$

Note that the (right) action (2.2) of $Gl^+(n)$ on F^+ and the (left) action (2.14) on F^+ of a diffeomorphism of M commute with each other.

We consider now the effect of a diffeomorphism ψ of M on the horizontal vector fields X_i. We use the following

Lemma 2 *The pull-back $\tilde{\psi}^*\omega$ of the connection form via the action (2.14) of the induced diffeomorphism $\tilde{\psi}$ of F^+ is again a connection form.*

Proof. We start from (2.6) and compute

$$\begin{aligned}
(\tilde{\psi}_* V) f &= \frac{d}{dt} f\Big(\tilde{x}^\mu(x^\nu(t)), \tilde{y}_i^\mu(y_j^\nu(t), x^\nu(t))\Big) \\
&= \frac{\partial f}{\partial \tilde{x}^\mu} \frac{\partial \psi(x)^\mu}{\partial x^\alpha} \frac{dx^\alpha}{dt} + \frac{\partial f}{\partial \tilde{y}_i^\mu} \Big(\frac{\partial^2 \psi(x)^\mu}{\partial x^\alpha \partial x^\beta} y_i^\beta \frac{dx^\alpha}{dt} + \partial_\alpha \psi(x)^\mu \frac{dy_i^\alpha}{dt} \Big) \\
&= \partial_\alpha \psi(x)^\mu V^\alpha \widetilde{\partial_\mu} f + (\partial_\alpha \partial_\beta \psi(x)^\mu y_i^\beta V^\alpha + \partial_\alpha \psi(x)^\mu V_i^\alpha) \widetilde{\partial_\mu^i} f \ ,
\end{aligned} \tag{2.15}$$

where $\widetilde{\partial_\mu^i} = \frac{\partial}{\partial \tilde{y}_i^\mu}$. For $V = A^\#$ we have $V^\mu = 0$ and $V_i^\mu = A_i^j y_j^\mu$, see (2.3). This means

$$(\tilde{\psi}_* A^\#)|_{\tilde{\psi}(p)} = A_i^j \partial_\alpha \psi(x)^\mu y_j^\alpha \widetilde{\partial_\mu^i} = A_i^j \widetilde{y_j^\alpha} \widetilde{\partial_\mu^i} = A^\#|_{\tilde{\psi}(p)} \ . \tag{2.16}$$

The fundamental vector field $A^\#$ is invariant under diffeomorphisms. This gives

$$(\tilde{\psi}^*\omega)(A^\#)|_p = \omega(\tilde{\psi}_* A^\#)|_{\tilde{\psi}(p)} = \omega(A^\#)|_{\tilde{\psi}(p)} = A \ .$$

The second identity to prove is

$$\begin{aligned}
&(\tilde{\psi}^*\omega)|_{pg} (R_{g*} V)|_{pg} = g^{-1} \Big((\tilde{\psi}^*\omega)(V)|_p \Big) g \\
\Rightarrow \quad &\omega|_{\tilde{\psi}(pg)} (\psi_*((R_{g*}V)|_{pg})|_{\tilde{\psi}(pg)}) = g^{-1} \Big(\omega|_{\tilde{\psi}(p)} (\tilde{\psi}_* V)|_{\tilde{\psi}(p)} \Big) g \ .
\end{aligned}$$

According to (2.7) we replace V_i^μ by $V_j^\mu g_i^j$ and y_i^μ by $y_j^\mu g_i^j$ and insert this into (2.15):

$$\tilde{\psi}_*(R_{g*}V)|_{\tilde{\psi}(pg)} = \partial_\alpha \psi(x)^\mu V^\alpha \widetilde{\partial_\mu} + (\partial_\alpha \partial_\beta \psi(x)^\mu y_j^\beta V^\alpha + \partial_\alpha \psi(x)^\mu V_j^\alpha) g_i^j \widetilde{\widetilde{\partial_\mu^i}} \ ,$$

where $\widetilde{\widetilde{\partial_\mu^i}} = \partial / \partial \widetilde{\widetilde{y_i^\mu}}$ and $\widetilde{\widetilde{y_i^\mu}} = \partial_\alpha \psi(x)^\mu y_j^\alpha g_i^j$. We must now evaluate

$$\omega_b^a|_{\tilde{\psi}(pg)} = (g^{-1})_c^a (y^{-1})_\gamma^c ((\partial\psi(x))^{-1})_\delta^\gamma \Big(d\widetilde{\widetilde{y_b^\delta}} + \Gamma_{\epsilon\zeta}^\delta|_{\psi(x)} \, \partial_\eta \psi(x)^\epsilon y_d^\eta g_b^d \, d\widetilde{x^\zeta} \Big)$$

on the above vector:

$$\begin{aligned}
&\omega_b^a|_{\tilde{\psi}(pg)} (\tilde{\psi}_*(R_{g*}V)|_{\tilde{\psi}(pg)}) \\
&= (g^{-1})_c^a \Big\{ (y^{-1})_\gamma^c ((\partial\psi(x))^{-1})_\delta^\gamma \Big((\partial_\alpha \partial_\beta \psi(x)^\delta y_d^\beta V^\alpha + \partial_\alpha \psi(x)^\delta V_d^\alpha) \\
&\qquad\qquad + \Gamma_{\epsilon\zeta}^\delta|_{\psi(x)} \, \partial_\eta \psi(x)^\epsilon y_d^\eta \, \partial_\nu \psi(x)^\zeta V^\nu \Big) \Big\} g_b^d \ .
\end{aligned} \tag{2.17}$$

Taking $g = e$ (identity matrix), it is obvious that the term in braces $\{\ \}$ equals $\omega^c_d(\tilde{\psi}_* V)|_{\tilde{\psi}(p)}$, which finishes the proof of the Lemma. $\square$

We can now rewrite the term in braces in (2.17) in a slightly different way:

$$
\begin{aligned}
&\omega^c_d(\tilde{\psi}_* V)|_{\tilde{\psi}(p)} = (\tilde{\psi}^* \omega^c_d)(V)|_p \\
&= (y^{-1})^c_\gamma V^\gamma_d + (y^{-1})^c_\gamma ((\partial\psi(x))^{-1})^\gamma_\delta \Big(\partial_\alpha\partial_\beta\psi(x)^\delta + \Gamma^\delta_{\epsilon\zeta}|_{\psi(x)}\, \partial_\beta\psi(x)^\epsilon \partial_\alpha\psi(x)^\zeta\Big) y^\beta_d V^\alpha \\
&= (y^{-1})^c_\gamma (dy^\gamma_d + \tilde{\Gamma}^\gamma_{\beta\alpha}|_x\, y^\beta_d dx^\alpha)(V^\nu_k \partial^k_\nu + V^\nu \partial_\nu) \ ,
\end{aligned}
$$

where $\tilde{\Gamma}^\gamma_{\beta\alpha}$ are the connection coefficients of the connection $\tilde{\psi}^*\omega$. This provides us with the following transformation law for the connection coefficients:

$$
\tilde{\Gamma}^\gamma_{\beta\alpha}|_x = ((\partial\psi(x))^{-1})^\gamma_\delta\, \Gamma^\delta_{\epsilon\zeta}|_{\psi(x)}\, \partial_\beta\psi(x)^\epsilon \partial_\alpha\psi(x)^\zeta + ((\partial\psi(x))^{-1})^\gamma_\delta\, \partial_\alpha\partial_\beta\psi(x)^\delta \ . \tag{2.18}
$$

Now there is an immediate question to ask: Which are the horizontal vector fields $\tilde{X}_i$ to the new connection form $\tilde{\psi}^*\omega$? We have

$$
0 = (\tilde{\psi}^*\omega)|_p(\tilde{X}_i|_p) = \omega|_{\tilde{\psi}(p)}(\tilde{\psi}_* \tilde{X}_i)|_{\psi(p)} \ ,
$$

which tells us

$$
\tilde{X}_i|_p = \tilde{\psi}_*^{-1}(X_i|_{\tilde{\psi}(p)}) = y^\mu_i (\partial_\mu - \tilde{\Gamma}^\nu_{\alpha\mu}|_x\, y^\alpha_j \partial^j_\nu) \ . \tag{2.19}
$$

The action (2.14) preserves the $\mathbb{R}^n$-valued 1-form α given in (2.4) and (2.5). Indeed, for $V = V^\mu\partial_\mu + V_i^\mu \partial^i_\mu \in T_pF^+$ we compute using (2.15)

$$
\begin{aligned}
(\tilde{\psi}^*\alpha^j)|_p(V) &= \alpha|_{\tilde{\psi}(p)}(\tilde{\psi}_* V) \\
&= (\tilde{y}^{-1})^j_\mu d\tilde{x}^\mu \Big(\partial_\alpha\psi(x)^\mu V^\alpha \widetilde{\partial_\mu} + (\partial_\alpha\partial_\beta\psi(x)^\mu y^\beta_i V^\alpha + \partial_\alpha\psi(x)^\mu V^\alpha_i)\widetilde{\partial^i_\mu}\Big) \\
&= (\tilde{y}^{-1})^j_\mu \partial_\alpha\psi(x)^\mu V^\alpha = (y^{-1})^j_\alpha V^\alpha \equiv \alpha^j|_p(V) \ .
\end{aligned}
$$

3 Crossed product

The properties listed in Proposition 1 and derived throughout section 2 are the basis for the construction of the Hopf algebra of Connes and Moscovici [1]. The idea is to apply the vertical and horizontal vector fields Y_i^j and X_i to a crossed product $\mathcal{A}$ defined below and to derive their coproduct from

$$
X_i(ab) = \Delta(X_i)\,(a \otimes b) \ , \qquad Y_i^j(ab) = \Delta(Y_i^j)\,(a\otimes b) \ , \qquad a, b \in \mathcal{A} \ . \tag{3.1}
$$

We refer to [7] for an introduction to Hopf algebras and related topics.

Let Γ be the pseudogroup of local (orientation preserving) diffeomorphisms of M. We consider the crossed product of the algebra $C^\infty_c(F^+)$ of smooth functions with compact support on the frame bundle F^+ by the action of Γ,

$$
\mathcal{A} = C^\infty_c(F^+) \rtimes \Gamma \ . \tag{3.2}
$$

As a set, $\mathcal{A}$ can be regarded as the tensor product of $C^\infty_c(F^+)$ with Γ. It is generated by the monomials

$$
fU^*_\psi \ , \qquad f \in C^\infty_c(\mathrm{Dom}(\tilde{\psi})) \ , \quad \psi \in \Gamma \ , \tag{3.3}
$$

where $\tilde{\psi}$ is the diffeomorphism of F^+ induced by $\psi \in \Gamma$ according to (2.14). As an algebra, the multiplication rule in $\mathcal{A}$ is defined by

$$f_1 U^*_{\psi_1} \; f_2 U^*_{\psi_2} := f_1 (f_2 \circ \tilde{\psi}_1) U^*_{\psi_2 \psi_1} \; . \tag{3.4}$$

In this formula, the function $f_1(f_2 \circ \tilde{\psi}_1) \in C^\infty_c(D_{\psi_1,\psi_2})$, with $D_{\psi_1,\psi_2} := \mathrm{Dom}(\tilde{\psi}_1) \cap \tilde{\psi}_1^{-1}(\mathrm{Dom}(\tilde{\psi}_2)) \subset F^+$, maps $p \in D_{\psi_1,\psi_2}$ into $f_1(p)\, f_2(\tilde{\psi}_1(p)) \in \mathbb{R}$ (or $\mathbb{C}$). The star on U^*_ψ refers to the contravariant multiplication rule $U^*_{\psi_1} U^*_{\psi_2} = U^*_{\psi_2 \psi_1}$. Associativity of $\mathcal{A}$ follows – for appropriate support of the functions – from

$$(f_1(f_2 \circ \psi_1))(f_3 \circ (\psi_2 \psi_1)) = f_1((f_2(f_3 \circ \psi_2)) \circ \psi_1) \; .$$

We consider now the action of the vertical and horizontal vector fields Y_i^j and X_i described in section 2 on the algebra $\mathcal{A}$. That action is simply defined as the action of the vector fields on the functions,

$$Y_i^j (f U^*_\psi) = Y_i^j(f) U^*_\psi \; , \qquad X_i(f U^*_\psi) = X_i(f) U^*_\psi \; . \tag{3.5}$$

The interesting effects we are looking for are obtained by application of these vector fields to the product (3.4). For any vector field V on F^+ we compute

$$\begin{aligned}
V(f_1 U^*_{\psi_1} \; f_2 U^*_{\psi_2})\big|_p &= V(f_1(f_2 \circ \tilde{\psi}_1)) U^*_{\psi_2\psi_1}|_p \\
&= \{V(f_1)\big|_p \; (f_2 \circ \tilde{\psi}_1)\big|_p + f_1\big|_p \; V(f_2 \circ \tilde{\psi}_1)\big|_p\} U^*_{\psi_2\psi_1} \\
&= V(f_1) U^*_{\psi_1}\big|_p \; f_2 U^*_{\psi_2} + f_1\big|_p \; ((\tilde{\psi}_{1*} V) f_2)\big|_{\tilde{\psi}_1(p)} U^*_{\psi_2\psi_1} \\
&= V(f_1) U^*_{\psi_1}\big|_p \; f_2 U^*_{\psi_2} + f_1\big|_p \; U^*_{\psi_1} U^*_{\psi_1^{-1}} ((\tilde{\psi}_{1*} V) f_2)\big|_{\tilde{\psi}_1(p)} U^*_{\psi_2\psi_1} \\
&= V(f_1) U^*_{\psi_1}\big|_p \; f_2 U^*_{\psi_2} + f_1 U^*_{\psi_1} \; ((\tilde{\psi}_{1*} V) f_2)\big|_{\tilde{\psi}_1^{-1} \circ \tilde{\psi}_1(p)} U^*_{\psi_2} \\
&= V(f_1 U^*_{\psi_1})\big|_p \; f_2 U^*_{\psi_2} + f_1 U^*_{\psi_1} \; (\tilde{\psi}_{1*}(V|_{\tilde{\psi}_1^{-1}(p)})) f_2 U^*_{\psi_2}\big|_p \; .
\end{aligned} \tag{3.6}$$

In the third line we have used the definition of the push-forward. In the fifth line we have commuted $U^*_{\psi_1^{-1}}$ with the function $(\tilde{\psi}_{1*}V) f_2$, evaluated at $\tilde{\psi}_1(p)$. According to (3.4), after taking $U^*_{\psi_1^{-1}}$ to the right we must evaluate the function $(\tilde{\psi}_{1*}V) f_2$ at $\tilde{\psi}_1^{-1}(\tilde{\psi}_1(p)) = p$. This means that the original field V to push forward must be taken at $\tilde{\psi}_1^{-1}(p)$.

Taking for V the vertical vector fields Y_i^j and recalling their invariance under diffeomorphisms (2.16), we obtain immediately

$$Y_i^j(ab) = Y_i^j(a)\, b + a\, Y_i^j(b) \; , \qquad a, b \in \mathcal{A} \; . \tag{3.7}$$

The behavior of the horizontal vector fields X_i is very different, because they do not commute with the diffeomorphisms. Eq. (2.19) tells us that if X_i is horizontal to ω, then $X_i^{(\psi_1)} := \tilde{\psi}_{1*}(X_i|_{\tilde{\psi}_1^{-1}(p)})$ is horizontal to $(\tilde{\psi}_1^{-1})^*\omega$. We denote the connection coefficients of $(\tilde{\psi}_1^{-1})^*\omega$ by $\hat{\Gamma}^\nu_{\alpha\mu}$. We observe from (2.12) and (2.3) that

$$\begin{aligned}
(X_i^{(\psi_1)} - X_i)\big|_p = (\Gamma^\nu_{\alpha\mu}|_x - \hat{\Gamma}^\nu_{\alpha\mu}|_x) y_i^\mu y_j^\alpha \partial^j_\nu \; &= \; (\Gamma^\nu_{\alpha\mu}|_x - \hat{\Gamma}^\nu_{\alpha\mu}|_x) y_i^\mu y_j^\alpha (y^{-1})^k_\nu Y_k^j|_p \\
&=: \hat{\gamma}^k_{ji}\big|_p^{(\psi_1)} Y_k^j\big|_p \; .
\end{aligned} \tag{3.8}$$

This gives from (3.6) for the horizontal fields X_i

$$\begin{aligned} X_i(f_1U^*_{\psi_1}\ f_2U^*_{\psi_2})|_p &= X_i(f_1U^*_{\psi_1})|_p\ f_2U^*_{\psi_2}|_p + f_1U^*_{\psi_1}|_p\ X_i^{(\psi_1)}(f_2U^*_{\psi_2})|_p \\ &= X_i(f_1U^*_{\psi_1})|_p\ f_2U^*_{\psi_2}|_p + f_1U^*_{\psi_1}|_p\ X_i(f_2U^*_{\psi_2})|_p \\ &\quad + f_1U^*_{\psi_1}|_p\ \hat{\gamma}^k_{ij}|_p^{(\psi_1)} Y^j_k(f_2U^*_{\psi_2})|_p \\ &= X_i(f_1U^*_{\psi_1})|_p\ f_2U^*_{\psi_2}|_p + f_1U^*_{\psi_1}|_p\ X_i(f_2U^*_{\psi_2})|_p \\ &\quad + f_1|_p \hat{\gamma}^k_{ij}|^{(\psi_1)}_{\tilde{\psi}_1(p)} U^*_{\psi_1}\ Y^j_k(f_2U^*_{\psi_2})|_p\ . \end{aligned} \tag{3.9}$$

Our goal is to express $\hat{\gamma}^k_{ij}|^{(\psi_1)}_{\tilde{\psi}_1(p)}$ in terms of some function evaluated at p. From (2.5) and (2.10) we conclude

$$\omega^k_j|_p - ((\tilde{\psi}^{-1})^*\omega^k_j)|_p = \hat{\gamma}^k_{ji}|^{(\psi)}_p\ \alpha^i|_p\ . \tag{3.10}$$

We take this identity at $\tilde{\psi}(p)$ and apply $\tilde{\psi}^*$, which gives

$$(\tilde{\psi}^*\omega^k_j)|_p - \omega^k_j|_p = \hat{\gamma}^k_{ji}|^{(\psi)}_{\tilde{\psi}(p)}\ (\tilde{\psi}^*\alpha^i)|_p = \hat{\gamma}^k_{ji}|^{(\psi)}_{\tilde{\psi}(p)}\ \alpha^i|_p\ , \tag{3.11}$$

using the invariance of α^i under diffeomorphisms in the last step. Replacing in (3.11) ψ by ψ^{-1} and comparing with (3.10) we get

$$\hat{\gamma}^k_{ji}|^{(\psi)}_{\tilde{\psi}(p)} = -\hat{\gamma}^k_{ji}|^{(\psi^{-1})}_p =: \gamma^k_{ji}|^{(\psi)}_p = (\tilde{\Gamma}^\nu_{\alpha\mu}|_x - \Gamma^\nu_{\alpha\mu}|_x) y^\alpha_j y^\mu_i (y^{-1})^k_\nu\ , \tag{3.12}$$

where $\tilde{\Gamma}^\nu_{\alpha\mu}$ and $\Gamma^\nu_{\alpha\mu}$ are the connection coefficients of the connections $\tilde{\psi}^*\omega$ and ω, respectively. Since $\tilde{\Gamma}^\nu_{\alpha\mu}$ is defined by the diffeomorphism ψ, we define an operator δ^k_{ji} on $\mathcal{A}$ by

$$\delta^k_{ji}(fU^*_\psi)|_p = \gamma^k_{ji}|^{(\psi)}_p\ fU^*_\psi|_p \tag{3.13}$$

and get from (3.9) and (3.12)

$$X_i(ab) = X_i(a)\,b + a\,X_i(b) + \delta^k_{ji}(a)\,Y^j_k(b)\ , \qquad a,b \in \mathcal{A}\ . \tag{3.14}$$

Next, we compute

$$\delta^k_{ji}(f_1U^*_{\psi_1}\ f_2U^*_{\psi_2})|_p = \delta^k_{ji}(f_1(f_2\circ\tilde{\psi}_1)U^*_{\psi_2\psi_1})|_p = \gamma^k_{ji}|^{(\psi_2\psi_1)}_p\ f_1|_p\, f_2|_{\tilde{\psi}_1(p)} U^*_{\psi_2\psi_1}\ . \tag{3.15}$$

Starting with (3.11) and (3.12) we compute

$$\begin{aligned} \gamma^k_{ji}|^{(\psi_2\psi_1)}_p\ \alpha^i|_p &= (\tilde{\psi}_2\tilde{\psi}_1)^*(\omega^k_j|_{(\tilde{\psi}_2\tilde{\psi}_1)(p)}) - \omega^k_j|_p \\ &= \tilde{\psi}_1^*\Big(\tilde{\psi}_2^*(\omega^k_j|_{(\tilde{\psi}_2\tilde{\psi}_1)(p)}) - \omega^k_j|_{\tilde{\psi}_1(p)}\Big) + \Big(\tilde{\psi}_1^*(\omega^k_j|_{\tilde{\psi}_1(p)}) - \omega^k_j|_p\Big) \\ &= \psi_1^*\Big(\gamma^k_{ji}|^{(\psi_2)}_{\tilde{\psi}_1(p)}\alpha^i|_{\tilde{\psi}_1(p)}\Big) + \gamma^k_{ji}|^{(\psi_1)}_p\alpha^i|_p \\ &= \Big(\gamma^k_{ji}|^{(\psi_1)}_p + \gamma^k_{ji}|^{(\psi_2)}_{\tilde{\psi}_1(p)}\Big)\alpha^i|_p\ . \end{aligned}$$

We used again the invariance of α^i under diffeomorphisms in the last line. We insert this result into (3.15) and get

$$\begin{aligned} \delta^k_{ji}(f_1U^*_{\psi_1}\ f_2U^*_{\psi_2})|_p &= \gamma^k_{ji}|^{(\psi_1)}_p f_1|_p\, f_2|_{\tilde{\psi}_1(p)}U^*_{\psi_2\psi_1} + f_1|_p\,\gamma^k_{ji}|^{(\psi_2)}_{\tilde{\psi}_1(p)}\, f_2|_{\tilde{\psi}_1(p)}U^*_{\psi_2\psi_1} \\ &= \gamma^k_{ji}|^{(\psi_1)}_p f_1|_p U^*_{\psi_1}\ f_2|_p U^*_{\psi_2} + f_1|_p U^*_{\psi_1}\ \gamma^k_{ji}|^{(\psi_2)}_p f_2|_p U^*_{\psi_2}\ , \end{aligned}$$

which means

$$\delta^k_{ji}(ab) = \delta^k_{ji}(a)\, b + a\, \delta^k_{ji}(b)\ . \tag{3.16}$$

The equations (3.7), (3.14) and (3.16) endow the operators X_i, Y^j_k and δ^k_{ji} with the structure of a coalgebra, with the coproduct (3.1) given by

$$\begin{aligned}
\Delta(Y^j_k) &= Y^k_j \otimes 1 + 1 \otimes Y^j_k\ ,\\
\Delta(X_i) &= X_i \otimes 1 + 1 \otimes X_i + \delta^k_{ji} \otimes Y^j_k\ ,\\
\Delta(\delta^k_{ji}) &= \delta^k_{ji} \otimes 1 + 1 \otimes \delta^k_{ji}\ ,\\
\Delta(1) &= 1 \otimes 1\ ,
\end{aligned} \tag{3.17}$$

with 1 being the identity on $\mathcal{A}$. It is easy to check that Δ is coassociative on the linear space $\mathbb{R}(1, X_i, Y^j_k, \delta^k_{ji})$,

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta\ . \tag{3.18}$$

4 From Lie algebra to Hopf algebra

Vector fields form a Lie algebra, so it is natural to investigate whether $X_i, Y^j_k, \delta^k_{ji}$ generate a Lie algebra. We compute the mutual commutators, starting with Y^i_j:

$$\begin{aligned}
[Y^i_j, Y^k_l](fU^*_\psi) &= (y^\mu_j \partial^i_\mu y^\nu_l \partial^k_\nu - y^\nu_l \partial^k_\nu y^\mu_j \partial^i_\mu) fU^*_\psi\\
&= (\delta^i_l Y^k_j - \delta^k_j Y^i_l)(fU^*_\psi)\ , && (4.1)\\
[Y^k_j, X_i](fU^*_\psi) &= (y^\mu_j \partial^k_\mu (y^\nu_i \partial_\nu - \Gamma^\beta_{\alpha\nu} y^\nu_i y^\alpha_l \partial^l_\beta) - (y^\nu_i \partial_\nu - \Gamma^\beta_{\alpha\nu} y^\nu_i y^\alpha_l \partial^l_\beta) y^\mu_j \partial^k_\mu) fU^*_\psi\\
&= \delta^k_i X_j (fU^*_\psi)\ , && (4.2)\\
[Y^i_j, \delta^k_{lm}](fU^*_\psi) &= (y^\mu_j \partial^i_\mu (\tilde\Gamma^\nu_{\beta\alpha} - \Gamma^\nu_{\beta\alpha}) y^\beta_l y^\alpha_m (y^{-1})^k_\nu - (\tilde\Gamma^\nu_{\beta\alpha} - \Gamma^\nu_{\beta\alpha}) y^\beta_l y^\alpha_m (y^{-1})^k_\nu y^\mu_j \partial^i_\mu) fU^*_\psi\\
&= (\delta^i_l \delta^k_{jm} + \delta^i_m \delta^k_{lj} - \delta^k_j \delta^i_{lm})(fU^*_\psi)\ . && (4.3)
\end{aligned}$$

So far we have considered the most general connection on M, even with torsion. But now, the commutator of horizontal vector fields

$$\begin{aligned}
[X_i, X_j] &= R^k_{lij} Y^l_k + \Theta^k_{ij} X_k\ , && (4.4)\\
R^k_{lij} &= (y^{-1})^k_\sigma y^\rho_l y^\mu_i y^\nu_j (\partial_\nu \Gamma^\sigma_{\rho\mu} - \partial_\mu \Gamma^\sigma_{\rho\nu} + \Gamma^\beta_{\rho\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\beta_{\rho\nu}\Gamma^\sigma_{\beta\mu})\ ,\\
\Theta^k_{ij} &= (y^{-1})^k_\rho y^\mu_i y^\nu_j (\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu})\ ,
\end{aligned}$$

leads to curvature R and torsion Θ, i.e. not to structure 'constants'. Torsion can be avoided by the choice of the connection, but we would be forced to include $R^k_{lij} Y^l_k$ and its repeated commutators with X_m in the list of generators of the Lie algebra we are looking for. To avoid these terms we follow [1] and restrict ourselves to a *flat* manifold. Locally this is always possible, and globally it is achieved via the *Morita equivalence*. For a locally finite cover of the manifold M by charts U_α, let $N = \coprod U_\alpha$ be the disjoint union of the charts. Moreover, let Γ' be the pseudogroup of local diffeomorphisms of N. Without giving the proof we recall from [1] that the two algebras $\mathcal{A} = C^\infty_c(F^+(M)) \rtimes \Gamma$ and $\mathcal{A}' = C^\infty_c(F^+(N)) \rtimes \Gamma'$ are Morita equivalent. There is a canonical connection on N, the flat connection given by $\Gamma^\alpha_{\beta\gamma} = 0$. This means that given M we pass to N and the corresponding crossed product $\mathcal{A}'$ and derive there the coproduct and Lie algebra structure of vector fields on $F^+(N)$ for the flat connection.

Thus, the horizontal vector fields take the simple form $X_i = y^\mu_i \partial_\mu$, and they now commute with each other:

$$[X_i, X_j](fU^*_\psi) = 0 \ . \tag{4.5}$$

Due to (2.18), (3.12) and (3.13), the action of δ^k_{ji} on $\mathcal{A}$ simplifies in the case of a flat manifold to

$$\delta^k_{ji}(fU^*_\psi) = ((\partial\psi(x))^{-1})^\nu_\beta\, \partial_\mu \partial_\alpha \psi(x)^\beta\, y^\mu_j y^\alpha_i (y^{-1})^k_\nu\, fU^*_\psi \ . \tag{4.6}$$

The (repeated) commutator with X_l leads to new operators on $\mathcal{A}$,

$$\begin{aligned} \delta^k_{ji,l_1\ldots l_n}(fU^*_\psi) &:= [X_{l_n}, \ldots, [X_{l_1}, \delta^k_{ji}] \ldots](fU^*_\psi) \\ &= \partial_{\lambda_n} \ldots \partial_{\lambda_1}\Big(((\partial\psi(x))^{-1})^\nu_\beta\, \partial_\mu\partial_\alpha\psi(x)^\beta\Big) y^\mu_j y^\alpha_i (y^{-1})^k_\nu\, y^{\lambda_1}_{l_1} \cdots y^{\lambda_n}_{l_n}\ fU^*_\psi \ . \end{aligned} \tag{4.7}$$

It is clear that all these operators δ commute with each other,

$$[\delta^k_{ji,l_1\ldots l_n}, \delta^c_{ba,d_1\ldots d_n}](fU^*_\psi) = 0 \ . \tag{4.8}$$

We see that the linear space generated by $X_i, Y^k_j, \delta^k_{ji,l_1\ldots l_n}$ forms a Lie algebra, and we let $\mathcal{H}$ be the corresponding enveloping algebra. This is the algebra of polynomials in the generators of the Lie algebra, with the commutation relations inherited from the Lie algebra. Thus a (Poincaré-Birkhoff-Witt) basis in $\mathcal{H}$ is given by

$$X_{i_1} \cdots X_{i_\alpha} Y^{k_1}_{j_1} \cdots Y^{k_\beta}_{j_\beta} \delta^{a_1}_{b_1 c_1} \cdots \delta^{a_\gamma}_{b_\gamma c_\gamma} \delta^{d_1}_{e_1 f_1, h_1} \cdots \delta^{d_\delta}_{e_\delta f_\delta, h_\delta} \cdots \ ,$$

with $i_1 \leq i_2 \leq \ldots \leq i_n$ and so on for the other indices. We extend the coproduct (3.17) recursively to $\mathcal{H}$ by the definition

$$\Delta(h^1 h^2) = \Delta(h^1)\,\Delta(h^2) := \sum h^1_1 h^1_2 \otimes h^2_1 h^2_2 \ , \qquad \Delta(h_i) = \sum h^1_i \otimes h^2_i \ , \tag{4.9}$$

for $h_1, h_2 \in \mathcal{H}$. The coproduct is automatically coassociative (3.18) and by construction (4.9) compatible with the multiplication in $\mathcal{H}$.

For notational convenience we abbreviate $\delta^A = \delta^k_{ji}$ with $A = 1, \ldots, n^2(n+1)/2$, due to symmetry in i, j. Moreover, we introduce a string $a = a_1 a_2 \ldots a_k$ for the repeated commutators with $X_{a_1}, \ldots, X_{a_k}$ and denote its length by $|a| = k$. Next, let $\mathcal{H}_n$ be the commutative algebra of polynomials in the variables 1 and δ^A_a, with $0 \leq |a| \leq n$. Let $\mathcal{H}^0_n$ be the ideal of polynomials vanishing at 0. We obtain a more explicit formula of the coproduct in

Lemma 3 $\Delta\delta^A_a = \delta^A_a \otimes 1 + 1 \otimes \delta^A_a + R^A_a \ , \qquad R^A_a \in \mathcal{H}^0_{n-1} \otimes \mathcal{H}^0_{n-1}$ *for* $|a| = n$.

Proof. The Lemma holds for $n=0$ with $R^A=0$. Assuming it holds for $|a|=n$ we compute for $b:=ai$ (appending the index i to the string a), $|b|=n+1$,

$$
\begin{aligned}
\Delta(\delta^A_b) &= \Delta([X_i,\delta^A_a]) = [\Delta(X_i),\Delta(\delta^A_a)] \\
&= [X_i\otimes 1 + 1\otimes X_i + \delta^k_{ji}\otimes Y^j_k, \delta^A_a\otimes 1 + 1\otimes \delta^A_a + R^A_a] \\
&= \delta^A_b\otimes 1 + 1\otimes\delta^A_b + R^A_b\;, \quad \text{with} \\
R^A_{ai} &:= [X_i\otimes 1 + 1\otimes X_i + \delta^k_{ji}\otimes Y^j_k, R^A_a] + \delta^k_{ji}\otimes[Y^j_k,\delta^A_a]\;. \qquad (4.10)
\end{aligned}
$$

For $n=1$ we get $R^A_i = \delta^k_{ji}\otimes[Y^j_k,\delta^A]\in\mathcal{H}^0_0\otimes\mathcal{H}^0_0$. The Lemma follows from the fact that the commutator with Y^j_k preserves $\mathcal{H}^0_m$ whereas the commutator with X_i sends elements of $\mathcal{H}^0_m$ to elements of $\mathcal{H}^0_{m+1}$. $\square$

For example, we obtain from (4.3) immediately

$$
\Delta(\delta^k_{ji,l}) = \delta^k_{ji,l}\otimes 1 + 1\otimes\delta^k_{ji,l} + \delta^a_{jl}\otimes\delta^k_{ai} + \delta^a_{il}\otimes\delta^k_{ja} - \delta^k_{al}\otimes\delta^a_{ji}\;. \qquad (4.11)
$$

The counit ϵ on $\mathcal{H}$ is defined by

$$
\varepsilon(1)=1\;, \qquad \varepsilon(h)=0 \quad \forall h\neq 1\;. \qquad (4.12)
$$

The counit axiom

$$
(\varepsilon\otimes\mathrm{id})\circ\Delta(h) = (\mathrm{id}\otimes\varepsilon)\circ\Delta(h) = h \qquad \forall\, h\in\mathcal{H}
$$

is clear for $h=X_i, Y^k_j, \delta^A$. For δ^A_a it follows from Lemma 3, using $\varepsilon(h^0)=0$ for $h^0\in\mathcal{H}^0_n$.

Therefore, $\mathcal{H}$ is a bialgebra (algebra+coalgebra+compatibility), and our next task is to show the existence of an antipode S on $\mathcal{H}$, making $\mathcal{H}$ to a Hopf algebra. The antipode has to satisfy the axioms

$$
\begin{aligned}
S(h_1h_2) &= S(h_2)S(h_1)\;, \\
m\circ(S\otimes\mathrm{id})\circ\Delta(h) &= \varepsilon(h)\;, \qquad (4.13) \\
m\circ(\mathrm{id}\otimes S)\circ\Delta(h) &= \varepsilon(h)\;,
\end{aligned}
$$

for $h,h_1,h_2\in\mathcal{H}$, and where m denotes the multiplication. Applying (4.13) to $1, Y^j_k, \delta^k_{ji}, X_i\in\mathcal{H}$, in that order, we get

$$
\begin{aligned}
S(1) &= 1\;, \\
S(Y^j_k) &= -Y^j_k\;, \\
S(\delta^k_{ji}) &= -\delta^k_{ji}\;, \qquad (4.14) \\
S(X_i) &= -X_i + \delta^k_{ji}Y^j_k\;,
\end{aligned}
$$

The antipode on δ^A_a is obtained from (4.13) by recursion in $|a|$, with the task to prove that the tree possible definitions coincide. First, employing the Sweedler notation $\Delta(R_a) = R_{a(1)} \otimes R_{a(2)}$ (and omitting the summation sign), we have with (4.10)

$$
\begin{aligned}
S(\delta^A_{ai}) &= -\delta^A_{ai} - m \circ (S \otimes \mathrm{id}) \circ \Delta(R^A_{ai}) \\
&= -\delta^A_{ai} - S([X_i, R^A_{a(1)}])\, R^A_{a(2)} - S(R^A_{a(1)})\, [X_i, R^A_{a(2)}] - S(\delta^k_{ji} R^A_{a(1)})\, [Y^j_k, R^A_{a(2)}] \\
&\qquad - S(\delta^k_{ji})[Y^j_k, \delta^A_a] \\
&= -\delta^A_{ai} - [X_i - \delta^k_{ji} Y^j_k, S(R^A_{a(1)})]\, R^A_{a(2)} - S(R^A_{a(1)})\, [X_i, R^A_{a(2)}] \\
&\qquad + \delta^k_{ji} S(R^A_{a(1)})\, [Y^j_k, R^A_{a(2)}] + \delta^k_{ji}[Y^j_k, \delta^A_a] \\
&= -\delta^A_{ai} + [-X_i + \delta^k_{ji} Y^j_k, S(R^A_{a(1)}) R^A_{a(2)}] + \delta^k_{ji}[Y^j_k, \delta^A_a] \\
&= [S(\delta^A_a), -X_i + \delta^k_{ji} Y^j_k] = [S(\delta^A_a), S(X_i)]\ .
\end{aligned}
\tag{4.15}
$$

In the same way one checks $-\delta^A_{ai} - m \circ (\mathrm{id} \otimes S) \circ \Delta(R^A_{ai}) = [S(\delta^A_a), S(X_i)]$. For example, one easily obtains

$$
S(\delta^k_{ji,l}) = -\delta^k_{ji,l} + \delta^a_{jl}\delta^k_{ai} + \delta^a_{il}\delta^k_{ja} - \delta^k_{al}\delta^a_{ji}\ . \tag{4.16}
$$

This finishes our review of the construction of the Connes–Moscovici Hopf algebra [1]. In their work, the cyclic cohomology of this Hopf algebra serves as an organizing principle for the computation of the cocycles in the local index formula [8]. We hope to be more specific on that point in the future.

5 Explicit solution: rooted trees

Following an idea by Connes and Kreimer [3] we will now describe the commutative Hopf algebra $\mathcal{H}_n$ of polynomials in δ^A_a, $|a| \leq n$, by graphical tools, generalized from the one-dimensional case in [3] to arbitrary dimension of the manifold M. In this way we obtain a Hopf algebra of rooted trees, which is intimately related to a Hopf algebra structure in perturbative quantum field theories as discovered by Kreimer [2]. The antipode of Kreimer's Hopf algebra achieves the renormalization of divergent Feynman graphs, see [2, 9].

We label the generator δ^k_{ji} by an indexed dot,

$$
\delta^k_{ji} = \ \bullet\, {}^{k}_{ji}\ . \tag{5.1}
$$

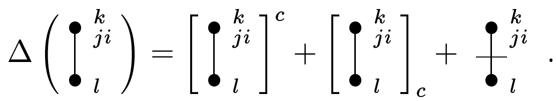
The goal is to derive the symbol for $\delta^k_{ji,l}$. This goes via the coproduct (4.11), which tells us after comparison with (4.10)

$$
\bullet\, {}^{a}_{bl} \otimes [Y^b_a,\ \bullet\, {}^{k}_{ji}] = \ \bullet\, {}^{a}_{jl} \otimes \bullet\, {}^{k}_{ai} \ + \ \bullet\, {}^{a}_{il} \otimes \bullet\, {}^{k}_{ja} \ - \ \bullet\, {}^{k}_{bl} \otimes \bullet\, {}^{b}_{ji}\ . \tag{5.2}
$$

The commutator with Y picks up one index of $\bullet\, {}^{k}_{ji}$ and moves it to the first upper or lower place in $\bullet\, {}^{a}_{bl}$, overwriting the index there. The vacant position in $\bullet\, {}^{k}_{ji}$ is filled with the remaining summation index of $\bullet\, {}^{a}_{bl}$. If the index picked up was a lower (upper) one, we count the resulting tensor product positive (negative). This leads us to think of the rhs of (5.2) as being produced by a cut of a symbol

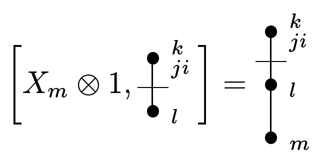
$$
\delta^k_{ji,l} = \begin{array}{l} \bullet\, {}^{k}_{ji} \\ | \\ \bullet\, {}_{l} \end{array} \quad \longrightarrow \quad \begin{array}{l} \bullet\, {}^{k}_{ji} \\ + \\ \bullet\, {}_{l} \end{array} = \begin{array}{l} \bullet\, {}^{k}_{ai} \\ \\ \bullet\, {}^{a}_{jl} \end{array} + \begin{array}{l} \bullet\, {}^{k}_{ja} \\ \\ \bullet\, {}^{a}_{il} \end{array} - \begin{array}{l} \bullet\, {}^{a}_{ij} \\ \\ \bullet\, {}^{k}_{al} \end{array}\ .
$$

We call the uppermost index which is different from the lower index the root. The graph above the cut connected with the root is called the trunk and goes to the rhs of the tensor product. A graph below the cut is called a cut branch and goes to the lhs of the tensor product. We define the action of a cut as the movement of one index of the vertex above the cut to the first position of the new root of the cut branch. The remaining position to complete the root of the cut branch is filled with a summation index and the same summation index is put into the vacant position of the trunk. In the case of cutting immediately below the root, we have to sum over the three possibilities of picking up indices of the root, adding a minus sign if we pick up the unique upper index. We thus get the following graphical interpretation of (4.11):

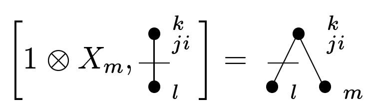

(5.3)

On the rhs, $[\delta]^c$ stands for $\delta \otimes 1$ (cutting above the entire tree) and $[\delta]_c$ for $1 \otimes \delta$ (cutting below the entire tree).

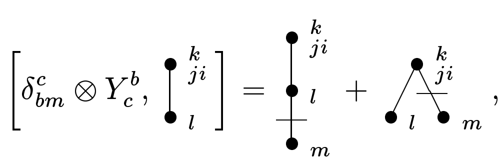
The next step is to compute $\Delta(\delta^k_{ji,lm})$ by commuting $\Delta(X_m)$ with (5.3). The term $[\delta^k_{ji,l}]^c$ has a non-vanishing commutator only with $X_m \otimes 1$. It yields $\delta^k_{ji,lm} \otimes 1$, and this trivial behavior continues to higher degrees. Next, $X_m \otimes 1$ commutes with $[\delta]_c$, whereas

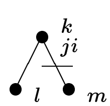
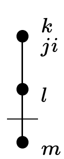

$$= \delta^a_{jl,m} \otimes \delta^k_{ai} + \delta^a_{il,m} \otimes \delta^k_{ja} - \delta^k_{al,m} \otimes \delta^a_{ji} \, . \tag{5.4}$$

Our previous definition of a cut extends without modification to that case. The term $1 \otimes X_m$ commuted with $[\delta^k_{ji,l}]_c$ gives $[\delta^k_{ji,lm}]_c$, whereas

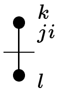
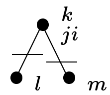

$$= \delta^a_{jl} \otimes \delta^k_{ai,m} + \delta^a_{il} \otimes \delta^k_{ja,m} - \delta^k_{al} \otimes \delta^a_{ji,m} \, . \tag{5.5}$$

The cut on the tree in the middle only sees the indices k, j, i – but not m – by the definition of a cut as affecting only the indices of the unique vertex above the cut. With this rule we get easily the corresponding expression in terms of δ's on the rhs. The commutator of $\delta^c_{bm} \otimes Y^b_c$ with $[\delta^k_{ji,l}]_c$ moves the indices k, j, i, l to their correct position in δ^c_{bm}, and this is precisely obtained as the sum of two different cuts:


(5.6)


$$= \delta^a_{jm} \otimes \delta^k_{ai,l} + \delta^a_{im} \otimes \delta^k_{ja,l} - \delta^k_{am} \otimes \delta^a_{ji,l} \, ,$$

$$= \delta^a_{lm} \otimes \delta^k_{ji,a} \, .$$

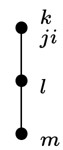
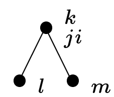
There remains one final commutator to compute, that of $\delta^c_{bm} \otimes Y^b_c$ with the graph in (5.3) already cut. For each of the tree terms corresponding to the previous cut, we have to move each of the tree indices of its root down to δ^c_{bm}. This gives the following symbolic expression of these nine tensor products:

$$\Big[\delta^c_{bm} \otimes Y^b_c \,,$$  $$\Big] =$$ 

$$= \delta^a_{jl}\delta^b_{am} \otimes \delta^k_{bi} + \delta^a_{jl}\delta^b_{im} \otimes \delta^k_{ab} - \delta^a_{jl}\delta^k_{bm} \otimes \delta^b_{ai} + \delta^a_{il}\delta^b_{jm} \otimes \delta^k_{ba} + \delta^a_{il}\delta^b_{am} \otimes \delta^k_{jb} - \delta^a_{il}\delta^k_{bm} \otimes \delta^b_{ja} - \delta^k_{al}\delta^b_{jm} \otimes \delta^a_{bi} - \delta^k_{al}\delta^b_{im} \otimes \delta^a_{jb} + \delta^k_{al}\delta^a_{bm} \otimes \delta^b_{ji} \,. \quad (5.7)$$

Note that the order of the cuts in this graph is important, we first have to cut the vertex l away and then the vertex m.

Our construction leads us to define

$$\delta^k_{ji,lm} =$$  $$+$$  $$. \quad (5.8)$$

Definition 4 *Let $\delta^A_a = \sum_{k=1}^{|a|!} t^{|a|}_k$ be recursively represented by a sum of $|a|!$ connected rooted trees, each of them having $|a|+1$ vertices. We define*

$$\delta^A_{ai} \equiv [X_i, \delta^A_a] = \sum_{k=1}^{|a|!} \sum_{j=1}^{|a|+1} t^{|a|}_{k_j} =: \sum_{\ell=1}^{|ai|!} t^{|ai|}_\ell \ , \quad (5.9)$$

where the rooted tree $t^{|a|}_{k_j}$ is obtained by attaching the new vertex i to the right of the $j^{\rm th}$ vertex of $t^{|a|}_k$.

Proposition 5 *The coproduct of $\delta^A_a = \sum_{k=1}^{|a|!} t^{|a|}_k$ is given by*

$$\Delta(\delta^A_a) = \delta^A_a \otimes 1 + 1 \otimes \delta^A_a + \sum_{k=1}^{|a|!} \sum_{\mathcal{C}} P^{\mathcal{C}}(t^{|a|}_k) \otimes R^{\mathcal{C}}(t^{|a|}_k) \ , \quad (5.10)$$

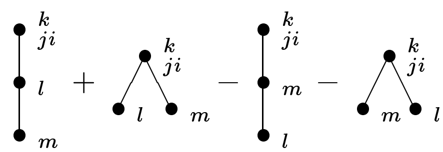
where for each $t^{|a|}_k$ the sum is over all admissible cuts $\mathcal{C}$ of $t^{|a|}_k$ (i.e. those non-empty multiple cuts for which on each path from the bottom to the root there is at most one individual cut). In eq. (5.10), $R^{\mathcal{C}}(t^{|a|}_k)$ is the trunk and $P^{\mathcal{C}}(t^{|a|}_k)$ the product of cut branches obtained by cutting $t^{|a|}_k$ via the multiple cut $\mathcal{C}$. If immediately below a vertex there are several cuts on outgoing edges, the order of the cuts is from left to right.

Proof. Commuting $\Delta(X_i)$ with $\Delta(\delta^A_a)$ to get $\Delta(\delta^A_{ai})$, the term $\delta^A_a \otimes 1$ develops into $\delta^A_{ai} \otimes 1$. Next, $X_i \otimes 1$ attaches successively a vertex i to each vertex of the cut branches $P^{\mathcal{C}}(t^{|a|}_k)$, and $1 \otimes X_i$ does the same for the trunk $R^{\mathcal{C}}(t^{|a|}_k)$ of each tree $t^{|a|}_k$ constituting δ^A_a. Finally $\delta \otimes Y$ attaches a cut-away vertex everywhere on the trunk, not on the cut branch. This excludes multiple cuts on paths from bottom to top. The result clearly reproduces our prescription of the coproduct of δ^A_{ai}, see (5.9). $\square$

We make one important observation. Although the operators δ are invariant under permutation of the indices after the comma, for instance $\delta^k_{ji,lm} = \delta^k_{ji,ml}$, see (4.7), this symmetry is lost on the level of individual trees, see for instance (5.4). However, these terms combined with the 'diagonal' terms of (5.6) are symmetric in l and m.

We recall that in Kreimer's Hopf algebra of renormalization [2, 9] a rooted tree represents the divergence structure of a Feynman graph. A divergent sector in such a graph is represented by a vertex. The root represents the overall (superficial) divergence. The construction rule for the tree is – in absence of overlapping subdivergences – to put subdivergences γ_i of a divergence γ into down-going branches of γ. Disjoint divergences are only indirectly connected via the divergence which contains them as subdivergences. Overlapping divergences have to be resolved in terms of disjoint and nested ones and give a sum of trees, see [10, 11].

The n-dimensional case treated here is closer to quantum field theory than dimension 1 because we obtain *decorated* trees – the decoration here being given by spacetime indices (three for the root) whereas in QFT it is a label for divergent Feynman graphs without subdivergences. In this sense, a (not super-) renormalizable QFT has something to do with diffeomorphisms on an infinite dimensional manifold. Our observation leads us to speculate that *the sum of Feynman graphs according to the collection of rooted trees to δ's has more symmetry than the individual Feynman graphs.* This should be checked in QFT calculations. Another interpretation would be the observation



$= 0$, (5.11)

which could possibly be regarded as a relation between Feynman graphs similar to those derived in [12][1].

Proposition 6 *The antipode S of $\delta^A_a = \sum_{k=1}^{|a|!} t^{|a|}_k$ is given by*

$$S(\delta^A_a) = -\delta^A_a - \sum_{k=1}^{|a|!} \sum_{\mathcal{C}_a} (-1)^{|\mathcal{C}_a|} \, P^{\mathcal{C}_a}(t^{|a|}_k) \, R^{\mathcal{C}_a}(t^{|a|}_k) \;, \tag{5.12}$$

where the sum is over the set of all non-empty multiple cuts $\mathcal{C}_a$ of $t^{|a|}_k$ (multiple cuts on paths from bottom to the root are allowed) consisting of $|\mathcal{C}_a|$ individual cuts. The order of cuts is from top to bottom and from left to right.

Proof. We apply the antipode axiom $m \circ (S \otimes \mathrm{id}) \circ \Delta = 0$, see (4.13), to (5.10), giving with $S(1) = 1$ the recursion

$$S(\delta^A_a) = -\delta^A_a - \sum_{k=1}^{|a|!} \sum_{\mathcal{C}} \Big(\prod_{j=1}^{|\mathcal{C}|} S(t^{|a|,\mathcal{C}}_{k,j}) \Big) R^{\mathcal{C}}(t^{|a|}_k) \;, \qquad P^{\mathcal{C}}(t^{|a|}_k) = \prod_{j=1}^{|\mathcal{C}|} t^{|a|,\mathcal{C}}_{k,j} \;,$$

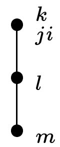
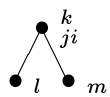
[1]Dirk Kreimer confirmed to me that (5.11) is satisfied in QFT for the leading divergences, as it can be derived from sec. V.C in [13]. For non-leading singularities there will be (probably systematic) modifications.

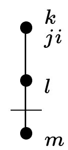
where $|\mathcal{C}|$ is the number of individual cuts in $\mathcal{C}$. For each $\{\mathcal{C}, j\}$ we have

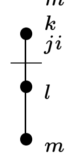
$$S(t^{|a|,\mathcal{C}}_{k,j}) = -t^{|a|,\mathcal{C}}_{k,j} - \sum_{\mathcal{C}_j} S(P^{\mathcal{C}_j}(t^{|a|,\mathcal{C}}_{k,j}))R^{\mathcal{C}_j}(t^{|a|,\mathcal{C}}_{k,j}) \;, \tag{5.13}$$

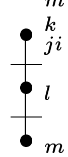
where the sum is over the set of admissible cuts $\mathcal{C}_j$ of $t^{|a|,\mathcal{C}}_{k,j}$. In the first level, the product $\prod_{j=1}^{|\mathcal{C}|}(-t^{|a|,\mathcal{C}}_{k,j})$ gives precisely $(-1)^{|\mathcal{C}|}\, P^{\mathcal{C}}(t^{|a|}_k)\, R^{\mathcal{C}}(t^{|a|}_k)$ in (5.12). In the next level, each $\mathcal{C}_j$ in (5.13) leads to a double cut on a path from some bottom vertex in $t^{|a|,\mathcal{C}}_{k,j}$ to the root of $t^{|a|}_k$, and all double cuts on paths from bottom to root of $t^{|a|}_k$ are obtained (precisely once) in this way. The second cut is below the first one so that the order of cuts is from top to bottom (and from left to right anyway). By recursion one gets all possible cuts $\mathcal{C}_a$ of $t^{|a|}_k$ contributing with the sign $(-1)^{|\mathcal{C}_a|}$ to the antipode. $\square$

For $\delta^k_{ji,lm}$, the prescription (5.12) leads to the following antipode:

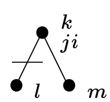
$S(\delta^k_{ji,lm}) = -$  $-$  $\longrightarrow \quad -\delta^k_{ji,lm}$

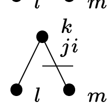
$+$  $\longrightarrow \quad +\delta^a_{lm}\delta^k_{ji,a}$

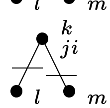
$+$  $\longrightarrow \quad +\delta^a_{jl,m}\delta^k_{ai} + \delta^a_{il,m}\delta^k_{ja} - \delta^k_{al,m}\delta^a_{ji}$

$-$  $\longrightarrow$

$$\begin{aligned} &-\delta^b_{jm}\delta^a_{bl}\delta^k_{ai} - \delta^b_{lm}\delta^a_{jb}\delta^k_{ai} + \delta^a_{bm}\delta^b_{jl}\delta^k_{ai} \\ &-\delta^b_{im}\delta^a_{bl}\delta^k_{ja} - \delta^b_{lm}\delta^a_{ib}\delta^k_{ja} + \delta^a_{bm}\delta^b_{il}\delta^k_{ja} \\ &+\delta^b_{am}\delta^k_{bl}\delta^a_{ji} + \delta^b_{lm}\delta^k_{ab}\delta^a_{ji} - \delta^k_{bm}\delta^b_{al}\delta^a_{ji} \end{aligned}$$

$+$  $\longrightarrow \quad +\delta^a_{jl}\delta^k_{ai,m} + \delta^a_{il}\delta^k_{ja,m} - \delta^k_{al}\delta^a_{ji,m}$

$+$  $\longrightarrow \quad +\delta^a_{jm}\delta^k_{ai,l} + \delta^a_{im}\delta^k_{ja,l} - \delta^k_{am}\delta^a_{ji,l}$

$-$  $\longrightarrow$

$$\begin{aligned} &-\delta^a_{jl}\delta^b_{am}\delta^k_{bi} - \delta^a_{jl}\delta^b_{im}\delta^k_{ab} + \delta^a_{jl}\delta^k_{bm}\delta^b_{ai} \\ &-\delta^a_{il}\delta^b_{jm}\delta^k_{ba} - \delta^a_{il}\delta^b_{am}\delta^k_{jb} + \delta^a_{il}\delta^k_{bm}\delta^b_{ja} \\ &+\delta^k_{al}\delta^b_{jm}\delta^a_{bi} + \delta^k_{al}\delta^b_{im}\delta^a_{jb} - \delta^k_{al}\delta^a_{bm}\delta^b_{ji} \end{aligned}$$

One checks, using (4.16) and (5.4)–(5.7), the antipode axioms (4.13).

Acknowledgements

First of all I would like to thank Alain Connes for his help on a point were I was stuck. I profited very much from seminar notes by Daniel Testard especially on the passage to the flat case. I hope to include this in a future version. It is a pleasure to thank Bruno Iochum, Thomas Krajewski, Serge Lazzarini, Thomas Schücker and Daniel Testard for encouragement and numerous discussions. I am grateful to Dirk Kreimer for a comment on eq. (5.11).

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