DISCRETE KALUZA-KLEIN FROM SCALAR FLUCTUATIONS IN NONCOMMUTATIVE GEOMETRY

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Abstract

We compute the metric associated to noncommutative spaces described by a tensor product of spectral triples. Well-known results of the two-sheets model (distance on a sheet, distance between the sheets) are extended to any product of two spectral triples. The distance between different points on different fibres is investigated. When one of the triples describes a manifold, one finds a Pythagorean theorem as soon as the direct sum of the internal states (viewed as projections) commutes with the internal Dirac operator. Scalar fluctuations yield a discrete Kaluza-Klein model in which the extra component of the metric is given by the internal part of the geometry. In the standard model, this extra component comes from the Higgs field.

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I Introduction.

In the noncommutative approach to the standard model of elementary particles[10], space-time appears as the product (in the sense of fibre bundles) of a continuous manifold by a discrete space. In precedent papers, we have studied the metric aspect of several classes of discrete spaces[17], and the metric of the continuum has been approached from a Lie-algebraic approach[34]. Here, within the framework of noncommutative geometry, we investigate how the distance in the continuum evolves when the space-time of euclidean general relativity is tensorised by an internal space. We find that in many cases the relevant picture is the two-sheets model[8,9]. Indeed, under precise conditions, the metric aspect of "continuum $\times$ discrete" spaces reduces to the simple picture of two copies of the manifold. It was known[11,5] that the distance on each copy is the geodesic distance while the distance between the copies – the distance on the fibre – is a constant. But this does not give a complete description of the geometry, in particular the distance between different points on different copies. In this paper we show that this distance coincides with the geodesic distance within a (4+1)-dimensional manifold whose fifth component comes from the internal part of the geometry. This component is a constant in the simpliest cases and becomes a function of the manifold when the metric fluctuates. Restricting ourselves to scalar fluctuations of the metric, which correspond to the Higgs sector in the standard model, it appears that the Higgs field describes the internal part of the metric in terms of a discrete Kaluza-Klein model.

The aim of this paper is to investigate the metric aspect of the standard model geometry. This goal is only partially achieved because we focus on scalar fluctuations and we mention only very briefly mathematical aspects such as the Gromov distance. For a comprehensive approach of these questions, the reader is invited to consult ref.[30]. Other works on distance in noncommutative geometry mainly concern lattices[1,2,14,28] and finite spaces. A larger bibliography can be found in ref.[17]. Naturally, using a Kaluza-Klein picture in noncommutative geometry is not a new idea and one can refer to refs.[24,12] for instance as well as the textbook[25]. Particularly, that the distance between the sheets depends on the manifold has been shown in refs.[4,5]. Last but not least, for a comprehensive approach of the subject, the most recent and complete reference is the book[16].

The paper is written for a 4-dimensional manifold but generalisation to higher dimension should be straightforward. The next two sections introduce classical notions of distance in noncommutative geometry and a simple proof that, on a manifold, this distance coincides with the geodesic distance. Section IV extends known results of the two-sheets model – distance on each copy, distance between the copies – to the product of any two spaces (not necessarily a manifold $\times$ a discrete space). In section V we show that, under conditions on the internal part of the Dirac operator, a large number of examples actually reduce to a two points fibre space. In the simplest case the internal space is orthogonal to the continuum in the sense of Pythagorean theorem (in finite spaces, the Pythagorean theorem has already been mentioned by ref.[13]). Section VI studies the scalar fluctuations (terminology is precised there) of this metric. The last part presents examples, among them the standard model, and precises the link between the Higgs field and the metric.

II The distance formula.

Let $\mathcal{A}$ be a unital C^*-algebra represented over a complex Hilbert space $\mathcal{H}$ equipped with a scalar product $\langle .,. \rangle$ defining the norm $\|\psi\|^2_{\mathcal{H}} \doteq |\langle \psi, \psi \rangle|$ for $\psi \in \mathcal{H}$. The C^*-norm of $\mathcal{A}$ is the operator norm in $\mathcal{H}$

$$\|a\|_{\mathcal{A}} \doteq \sup_{\psi \in \mathcal{H}} \frac{\|\pi(a)(\psi)\|_{\mathcal{H}}}{\|\psi\|_{\mathcal{H}}}$$

where π is the representation. The so called Dirac operator D is a selfadjoint operator in $\mathcal{H}$, possibly unbounded. When the spectral dimension is even[8], the chirality χ is a hermitean operator which anticommutes with D and commutes with $\pi(\mathcal{A})$. The set $(\mathcal{A}, \mathcal{H}, D, \pi, \chi)$ is called a spectral triple. The terminology is justified because π is usually infered in the notation $\mathcal{H}$, and once given $(\mathcal{A}, \mathcal{H}, D)$, χ – if it exists – is uniquely determined by the axioms of noncommutative geometry[10]. Since the algebra appears through its representation, we can, without loss of generality, replace $\mathcal{A}$ by $\mathcal{A}/\ker(\pi)$ and assume that π is faithful. To improve the readability we omit the symbol π unless necessary.

We denote by $\mathcal{P}(\mathcal{A})$ the set of pure states of $\mathcal{A}$. The distance d between two of its elements ω_1, ω_2 is

$$d(\omega_1, \omega_2) \doteq \sup_{a \in \mathcal{A}} \{\, |\omega_1(a) - \omega_2(a)| \,/\, \|[D,a]\| \leq 1 \} \,,$$

where $\|.\|$ is the operator norm in $\mathcal{H}$ (we do not write $\|[D,a]\|_{\mathcal{A}}$ because $[D,a]$ may not be the representation of an element of $\mathcal{A}$). This supremum is reached[17] by a positive element such that $\|[D, \pi(a)]\| = 1$:

$$d(\omega_1, \omega_2) = \sup_{a \in \mathcal{A}_+} \{\, |\omega_1(a) - \omega_2(a)| \,/\, \|[D,a]\| = 1 \} \,. \tag{1}$$

This formula is invariant under several transformations, including unitary transformation and projection. First, a unitary element u of $\mathcal{A}$ defines both an automorphism of the algebra $\alpha_u(a) \doteq uau^*$ and a *unitary equivalent* triple $(\mathcal{A}, \mathcal{H}, uDu^*, \pi \circ \alpha_u)$. Obviously distances are not changed under such a transformation because $\|[D,a]\| = \|[uDu^*, \alpha_u(a)]\|$. More interesting is the action of a projection $e \in \mathcal{A}$ ($e^2 = e^* = e$) through the endomorphism of $\mathcal{A}$

$$\alpha_e(a) \doteq eae,$$

which defines the *restricted* spectral triple

$$(\mathcal{A}_e \doteq \alpha_e(\mathcal{A}), \quad \mathcal{H}_e \doteq e\mathcal{H} \doteq \operatorname{ran} e, \quad D_e \doteq eDe\big|_{\mathcal{H}_e}, \quad \pi_e \doteq \pi\big|_{\mathcal{H}_e})$$

whose corresponding distance is denoted by d_e. α_e being not injective, for a pure state $\omega \in \mathcal{P}(\mathcal{A})$ the linear form $\omega \circ \alpha_e$ is not necessarily a state of $\mathcal{A}$ (for instance if e is in the kernel of ω). However it is a pure state of the subalgebra $\mathcal{A}_e$. Conversely, any pure state ω_e of $\mathcal{A}_e$ is made a pure state of $\mathcal{A}$ by writing $\omega_e \circ \alpha_e$. In other words, $\mathcal{P}(\mathcal{A}_e) = \mathcal{P}(\mathcal{A}) \circ \alpha_e \subset \mathcal{P}(\mathcal{A})$.

Lemma 1. *If a projection e is such that $[D, e] = 0$, the distance between two pure states ω_1, ω_2 of $\mathcal{A}_e$ is invariant by projection: $d_e(\omega_1, \omega_2) = d(\omega_1 \circ \alpha_e, \omega_2 \circ \alpha_e)$.*

Proof. For $a_e \in \mathcal{A}_e$, $\|[D_e, \pi_e(a_e)]\| = \|[\pi(e)D\pi(e), \pi(a_e)]\| = \|[D, \pi(a_e)]\|$ therefore

$$\begin{aligned} d_e(\omega_1, \omega_2) &= \sup_{a_e \in \mathcal{A}_e} \{ |(\omega_1 - \omega_2)(a_e)| \,/\, \|[D, \pi(a_e)]\| \leq 1 \} \,, \\ &\leq \sup_{a \in \mathcal{A}} \{ |(\omega_1 \circ \alpha_e - \omega_2 \circ \alpha_e)(a)| \,/\, \|[D, \pi(a)]\| \leq 1 \} = d(\omega_1 \circ \alpha_e, \omega_2 \circ \alpha_e). \end{aligned}$$

This upper bound is reached by $\alpha_e(a)$ where $a \in \mathcal{A}$ reaches the supremum for the distance d, namely $\|[D, \pi(a)]\| = 1$ and $d(\omega_1 \circ \alpha_e, \omega_2 \circ \alpha_e) = \omega_1 \circ \alpha_e(a) - \omega_2 \circ \alpha_e(a)$. ■

III Distance in a manifold.

The spectral triple of a Riemannian spin manifold $\mathcal{M}$ of dimension 4 with a metric g is

$$\mathcal{A} = C^\infty(\mathcal{M}), \qquad \mathcal{H} = L_2(\mathcal{M}, S), \qquad D = i\gamma^\mu \partial_\mu = i\partial\!\!\!/\,, \tag{2}$$

where $L_2(\mathcal{M}, S)$ is the set of square integrable spinors on $\mathcal{M}$. The Riemannian gamma matrices $\gamma^\mu = \gamma^{\mu *} = e^\mu_a \gamma^a$ are obtained via the vierbein field e^μ_a from the Euclidean gamma matrices γ^a of the associated Clifford algebra. Using $\delta^{ab} e^\mu_a e^\nu_b = g^{\mu\nu}$ and $\gamma^a\gamma^b + \gamma^b\gamma^a = 2\delta^{ab}\mathbb{I}$ one has $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{I}$. The spectral dimension is the dimension of the manifold, so there is a chirality $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ made of the Euclidean γ^a's. The scalar product of $\mathcal{H}$ is $\langle \psi, \phi \rangle \doteq \int_{\mathcal{M}} \bar\psi(x)\phi(x)\, dx$ and an element $f \in \mathcal{A}$ is represented over $\mathcal{H}$ by the pointwise multiplication, $\pi(f) \doteq f\mathbb{I}$, so that

$$\|f\|_{\mathcal{A}} = \sup_{\psi \in \mathcal{H}} \left(\frac{\int_{\mathcal{M}} (\bar f \bar\psi)(x)(f\psi)(x) dx}{\int_{\mathcal{M}} \bar\psi(x)\psi(x) dx} \right)^{\frac{1}{2}} = \sup_{x \in \mathcal{M}} |f(x)|\,.$$

By Gelfand transform, $\mathcal{P}(\mathcal{A}) \simeq \mathcal{M}$. The isomorphism $x \in \mathcal{M} \leftrightarrow \omega_x \in \mathcal{P}(\mathcal{A})$ is defined by $\omega_x(f) \doteq f(x)$. The noncommutative distance (1)

$$d(x, y) = \sup_{f \in C^\infty(\mathcal{M})} \left\{ |f(x) - f(y)| \,/\, \|[i\partial\!\!\!/, f\mathbb{I}]\| \leq 1 \right\}\,,$$

coincides with the geodesic distance $L(x, y)$ between points x, y of $\mathcal{M}$. This is a classical result[8] but the proof introduces ideas and notations important for further presentation so that we shall give it in detail (this version of the proof comes from ref.[22]).

The supremum is reached on $\mathcal{A}_+$, so f is real. For $\psi \in \mathcal{H}$, $[i\partial\!\!\!/, f\mathbb{I}]\psi = i(\partial\!\!\!/ f)\psi$, so

$$\|[i\partial\!\!\!/, f\mathbb{I}]\|^2 = \|(i\partial\!\!\!/ f)^* i\partial\!\!\!/ f\| = \|\gamma^\mu \partial_\mu f \gamma^\nu \partial_\nu f\| = \|g^{\mu\nu} \partial_\mu f \partial_\nu f \mathbb{I}\| = \sup_{q \in \mathcal{M}} \{ g^{\mu\nu}(q) \partial_\mu f(q) \partial_\nu f(q) \}\,.$$

The gradient $\overrightarrow{\nabla}$ in the usual sense is the exterior derivative d (not to be confused with the distance) which maps 0-forms (i.e. smooth functions over $\mathcal{M}$) onto 1-forms:

$$\overrightarrow{\nabla} f \doteq (\partial_\mu f) dx^\mu \in T^*\mathcal{M}\,.$$

By definition[27] g defines an inner product (thus, a norm) in each cotangent space $T^*_q\mathcal{M}$ in such a manner that

$$\left\| \overrightarrow{\nabla} f(q) \right\|^2_{T^*_q\mathcal{M}} = g^{\mu\nu}(q) \partial_\mu f(q) \partial_\nu f(q)\,.$$

Omitting the index $T_q{}^*\mathcal{M}$, one writes

$$\|[i\partial\!\!\!/, f\mathbb{I}]\| = \sup_{q \in \mathcal{M}} \left\| \overrightarrow{\nabla} f(q) \right\|.$$

Now, let $c\colon t \in [0, 1] \rightarrow \mathcal{M}$ be the minimal geodesic between x and y and let $\dot{}$ denote the total derivative with respect to t. For any $f \in C^\infty(\mathcal{M})$

$$f(x) - f(y) = \int_0^1 \dot f(c(t))\, dt = \int_0^1 \partial_\mu f(p)\, \dot{c}^\mu(t) dt$$

where $p \doteq c(t)$. The metric defines an isomorphism $T_p\mathcal{M} \simeq T_p^*\mathcal{M}$ such that

$$\partial_\mu f(p)\, \dot{c}^\mu(t) = g^{\mu\nu}(p)\, \partial_\mu f(p)\, \dot{c}_\nu(t) = \langle \overrightarrow{\nabla} f(p), \dot{c}_\nu(t) dx^\nu \rangle \,,$$

thus, by Cauchy-Schwarz, $|\partial_\mu f(p)\, \dot{c}^\mu(t)| \leq \left\| \overrightarrow{\nabla} f(p) \right\| \|\dot{c}_\nu(t) dx^\nu\|$. Assuming that f reaches the supremum, one has $\left\| \overrightarrow{\nabla} f \right\| \leq 1$, so

$$d(x,y) = |f(x) - f(y)| \leq \int_0^1 \|\dot{c}_\nu(t) dx^\nu\| \; dt = L(x,y)\,.$$

This upper bound is reached by the function

$$L : q \mapsto L(q,y). \tag{3}$$

Indeed, $L(x) - L(y) = L(x,y)$ and

$$\sup_{q\in\mathcal{M}} \left\| \overrightarrow{\nabla} L(q) \right\| \leq 1\,. \tag{4}$$

To prove (4), take $q, q' \in \mathcal{M}$ with coordinates q^μ, q'^μ in a given chart such that q' comes from q by the infinitesimal transformation $\sigma(\epsilon)$, $\epsilon \ll 1$, where σ is the flow generated by the vector field $g^{\mu\nu}(\partial_\nu L)\partial_\mu$ with initial condition $\sigma(0) = q$. Then, writing $dq^\mu \doteq q'^\mu - q^\mu$,

$$q^\mu + dq^\mu = q'^\mu = \sigma^\mu(\epsilon) = \sigma^\mu(0) + \epsilon\, \frac{d\sigma^\mu}{dt}(0) + \mathcal{O}(\epsilon^2) = q^\mu + \epsilon\, g^{\mu\nu}(q)\partial_\nu L(q) + \mathcal{O}(\epsilon^2)\,,$$

which means that

$$dq^\mu = \epsilon\, g^{\mu\nu}(q)\partial_\nu L(q) + \mathcal{O}(\epsilon^2)\,. \tag{5}$$

As $L(q',y)$ is the shortest length from q' to y, $L(q',y) \leq L(q',q) + L(q,y)$, and one has

$$L(q+dq) \leq L(q',q) + L(q)\,. \tag{6}$$

Using (5),

$$L(q',q) \doteq \sqrt{g_{\lambda\rho}(q) dq^\lambda dq^\rho} = \sqrt{\epsilon^2 g_{\lambda\rho}(q) g^{\lambda\mu}(q)\partial_\mu L(q)\, g^{\rho\nu}(q)\partial_\nu L(q)} = \epsilon\sqrt{g^{\mu\nu}\partial_\mu L(q)\, \partial_\nu L(q)}\,.$$

Inserting into the r.h.s. of (6) whose l.h.s. is developed with respect to ϵ yields

$$L(q) + \partial_\mu L(q)\, dq^\mu = L(q) + \epsilon\, g^{\mu\nu}(q)\partial_\mu L(q)\partial_\nu L(q) + \mathcal{O}(\epsilon^2) \leq \epsilon\sqrt{g^{\mu\nu}\partial_\mu L(q)\, \partial_\nu L(q)} + L(q) + \mathcal{O}(\epsilon^2),$$

which is true for all q, hence (4) and finally $d(x,y) = L(x,y)$.

IV Tensor product of spectral triples.

The tensor product of an even spectral triple $T_I = (\mathcal{A}_I, \mathcal{H}_I, D_I, \pi_I)$ with chirality χ_I by the spectral triple $T_E = (\mathcal{A}_E, \mathcal{H}_E, D_E, \pi_E)$ is the spectral triple $T_I \otimes T_E \doteq (\mathcal{A}', \mathcal{H}', D')$ defined by

$$\mathcal{A}' \doteq \mathcal{A}_I \otimes \mathcal{A}_E, \quad \mathcal{H}' \doteq \mathcal{H}_I \otimes \mathcal{H}_E, \quad D' \doteq D_I \otimes \mathbb{I}_E + \chi_I \otimes D_E\,,$$

where the representation of $\mathcal{A}'$ is $\pi' \doteq \pi_I \otimes \pi_E$. The notation $T_I \otimes T_E$ is a matter of convention for spectral triples do not form a vector space. The product of spectral triples is commutative in the sense that when T_E is even with chirality χ_E, then $T_E \otimes T_I \doteq (\mathcal{A}, \mathcal{H}, D)$ is well defined by permutation of factors,

$$\mathcal{A} \doteq \mathcal{A}_E \otimes \mathcal{A}_I, \quad \mathcal{H} \doteq \mathcal{H}_E \otimes \mathcal{H}_I, \quad D \doteq D_E \otimes \mathbb{I}_I + \chi_E \otimes D_I\,, \tag{7}$$

$\pi = \pi_E \otimes \pi_I$, and is equivalent to $T_I \otimes T_E$ up to the unitary operator

$$U \doteq (\frac{\mathbb{I}_I + \chi_I}{2} \otimes \mathbb{I}_E + \frac{\mathbb{I}_I - \chi_I}{2} \otimes \chi_E)\,.$$

For physics it is interesting to take for this tensor product the product of the continuum by the discrete, namely to study the geometry of the four-dimensional space-time of Euclidean general relativity together with an internal discrete space. In the standard model, the internal space describes the electroweak and strong interactions and is defined by a spectral triple T_I in which the algebra $\mathcal{A}_I$ is chosen such that its unitarities are related to the gauge group of interactions while $\mathcal{H}_I$ is the space of fermions. Both $\mathcal{A}_I$ and $\mathcal{H}_I$ are finite dimensional, so T_I is a finite spectral triple[21] and T_E is the usual spectral triple (2) of a manifold. The spectral dimension of a finite spectral triple is 0 and $\dim(T_E) = \dim(\mathcal{M}) = 4$: both T_E and T_I are even therefore both $T_E \otimes T_I$ and $T_I \otimes T_E$ are defined.

In this section, we give general results that do not require neither T_E to be the spectral triple of a manifold nor T_I to be finite. To fix notations we simply assume that T_E is even so that we work with $T_E \otimes T_I$. To study the metric of a noncommutative space, the first goal is to make explicit the set of pure states of the associated algebra. For ω_E and ω_I being pure states of $\mathcal{A}_E$ and $\mathcal{A}_I$, the pair (ω_E, ω_I) is a state of $\mathcal{A}$ which acts as $\omega_E \otimes \omega_I$ (that $\mathbb{I}$ maps to 1 is obvious, the positivity can be seen in ref.[26] for instance) but this is not necessarily a pure state. Moreover there can be pure states of $\mathcal{A}$ that cannot be written as tensor products. However, as soon as one of the algebras is abelian, one obtains[19] that $\mathcal{P}(\mathcal{A}_E \otimes \mathcal{A}_I) \simeq \mathcal{P}(\mathcal{A}_E) \times \mathcal{P}(\mathcal{A}_I)$ and any pure state ω of $\mathcal{A}$ writes $\omega = \omega_E \otimes \omega_I$.

In the two sheets-model $\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$, therefore any pure state is $\omega_x \otimes \omega_i$ where ω_i, $i = 1, 2$, is a pure state of $\mathbb{C}^2$ and labels the sheets. It is known[8] that $d(\omega_x \otimes \omega_i,\, \omega_y \otimes \omega_i)$ is the geodesic distance $L(x, y)$ while $d(\omega_x \otimes \omega_i,\, \omega_x \otimes \omega_j)$ is a constant. This extends to any product of spectral triples. Once fixed a pure state ω_E, $d(\omega_E \otimes \omega_I, \omega_E \otimes \omega'_I)$ depends only on the spectral triple T_I and, similarly, $d(\omega_E \otimes \omega_I, \omega'_E \otimes \omega_I)$ depends only on T_E. This is true even when none of the algebra is commutative: the distance is then defined between states that may be not pure.

Theorem 2. *Let d_E, d_I, d be the distance in T_E, T_I, $T_E \otimes T_I$ respectively. For ω_E, ω'_E in $\mathcal{P}(\mathcal{A}_E)$ and ω_I, ω'_I in $\mathcal{P}(\mathcal{A}_I)$,*

$$d\left(\omega_E \otimes \omega_I, \omega_E \otimes \omega'_I\right) = d_I\left(\omega_I, \omega'_I\right),$$
$$d\left(\omega_E \otimes \omega_I, \omega'_E \otimes \omega_I\right) = d_E\left(\omega_E, \omega'_E\right).$$

Proof. Let f_j denote the elements of $\mathcal{A}_E$ and m_i those of $\mathcal{A}_I$. A generic element of $\mathcal{A}$ is $a = f^i \otimes m_i$, where the summation index i runs over a finite subset of $\mathbb{N}$. Definition (7) yields

$$[D, a] = [D_E, f^i] \otimes m_i + f^i \chi_E \otimes [D_I, m_i]\,.$$

Multiplying on left and right by the unitary operator $\chi_E \otimes \mathbb{I}_I$ allows to write

$$\left\|[D_E, f^i] \otimes m_i + f^i \chi_E \otimes [D_I, m_i]\right\| = \left\|-[D_E, f^i] \otimes m_i + f^i \chi_E \otimes [D_I, m_i]\right\| ,$$

where we use that $\chi_E = \chi_E^*$ commutes with f^i and anticommutes with D_E. For u, v in a normed space, $2\,\|u\| \leq \|u+v\| + \|u-v\|$, thus

$$\left\|[D_E, f^i] \otimes m_i\right\| \leq \|[D, a]\| \ , \tag{8}$$

and $\|f^i \chi_E \otimes [D_I, m_i]\| \leq \|[D, a]\|$. One can factorise the left-hand side of this last equation by $\chi_E \otimes \mathbb{I}_I$ in order to have

$$\left\|f^i \otimes [D_I, m_i]\right\| \leq \|[D, a]\| \ . \tag{9}$$

For any $\omega_E \in \mathcal{P}(\mathcal{A}_E)$ and $a \in \mathcal{A}_+$, let us define $a_E \in \mathcal{A}_I$ by

$$a_E \doteq \omega_E(f^i) m_i.$$

a_E is selfadjoint. Indeed, positivity of a, i.e. $a = (f^{p*} \otimes m_p{}^*)(f^q \otimes m_q) = \frac{1}{2}(f^{pq} \otimes m_{pq} + f^{pq*} \otimes m_{pq}{}^*)$ where $f^{pq} \doteq f^{p*} f^q$ and $m_{pq} = m_p^* m_q$, yields

$$a_E = \frac{1}{2}(\omega_E(f^{pq}) m_{pq} + \omega_E(f^{pq*}) m_{pq}{}^*) = a_E^*.$$

Thus

$$i[D_I, a_E] = i\,(\omega_E \otimes \mathbb{I}_I)\left(f^i \otimes [D_I, m_I]\right)$$

in $\mathcal{B}(\mathcal{H}_I)$ is normal. One knows[19] that for any normal element a of a C^*-algebra, $\|a\| = \sup_{\tau \in \mathcal{S}} |\tau(a)|$, where $\mathcal{S}$ is the set of states. Thus, with $\mathcal{S}_I$ the set of states of $\mathcal{B}(\mathcal{H}_I)$,

$$\begin{aligned}
\|[D_I, a_E]\| &= \sup_{\tau_I \in \mathcal{S}_I} |\tau_I([D_I, a_E])|, \\
&\leq \sup_{(\tilde{\omega}_E, \tau_I) \in \mathcal{P}(\mathcal{A}_E) \times \mathcal{S}_I} |(\tilde{\omega}_E \otimes \tau_I)(f^i \otimes [D_I, m_i])|, \\
&\leq \sup_{(\tau_E, \tau_I) \in \mathcal{S}_E \times \mathcal{S}_I} |(\tau_E \otimes \tau_I)\left(f^i \otimes [D_I, m_i]\right)| = \left\|f^i \otimes [D_I, m_i]\right\|,
\end{aligned}$$

where we use that $if^i \otimes [D_I, m_i] \in \mathcal{B}(\mathcal{H})$ is also normal. Together with (9),

$$\|[D_I, a_E]\| \leq \|[D, a]\| .$$

Since $(\omega_E \otimes \omega_I)(a) - (\omega_E \otimes \omega_I')(a) = \omega_I(a_E) - \omega_I'(a_E)$,

$$d(\omega_E \otimes \omega_I, \omega_E \otimes \omega_I') \leq d_I(\omega_I, \omega_I').$$

This upper bound is reached by $\mathbb{I}_E \otimes a_I$ where $a_I \in \mathcal{A}_I$ reaches the supremum for T_I alone, namely $d_I(\omega_I, \omega_I') \doteq |(\omega_I - \omega_I')(a_I)|$ and $1 = \|[D_I, \pi_I(a_I)]\|$.

The proof for $d(\omega_E \otimes \omega_I, \omega_E' \otimes \omega_I)$ is similar, using (8) instead of (9). ∎

V Metric in the continuum $\times$ discrete.

The key points of Theorem 2 are equations (8) and (9). The first one allows to forget about the internal part of the commutator and makes sense for states of $\mathcal{A}$ defined by different pure states on $\mathcal{A}_E$ but the same pure state on $\mathcal{A}_I$. When T_E is the spectral triple of a manifold and T_I a finite spectral triple, the noncommutative space described by $T_E \times T_I$ is a fibre bundle over the manifold with a discrete fibre. This can also be seen as the union of several copies of the manifold, indexed by the element of the fibre. Theorem 2 simply says that each of the copies is endowed with the metric of the base. Note that the discussion about the Gromov distance between manifolds with distinct metrics in ref.[8] may not be transposed here because such manifolds are not described by a tensor product of spectral triples.

In contrast, (9) does not take into account the external part of the commutator and is sufficient to determine the distance between states defined by the same pure state on $\mathcal{A}_I$ (i.e. points on the same fibre within the picture of a continuum $\times$ discrete space). Of course the mixed case $d\left(\omega_E \otimes \omega_I, \, {\omega_E}' \otimes {\omega_I}'\right)$ – the distance between different points on different copies of the manifold – requires to take into account both the internal and the external part of the commutator. This makes the computation more difficult. However, for continuum $\times$ discrete spaces, some of these distances have a nice interpretation in terms of a discrete Kaluza-Klein model: although the internal space is discrete, the distance appears as the geodesic distance in a "virtual" (4+1)-dimensional manifold ("virtual" means that the points between the sheets are not part of the geometry, the embedding into a higher dimensional continuum space is a practical intermediate).

Let us first give a semi-general result which does not require T_E to be the spectral triple of a manifold (T_E is just supposed to be even to fix notations) but which assumes

$$\mathcal{A}_I \doteq \bigoplus_k \mathcal{A}_k \,,$$

where k runs over a finite subset of $\mathbb{N}$ and the $\mathcal{A}_k$'s are von Neumann algebras – i.e. their universal representation $\{\pi_u, \mathcal{H}_u\}$ is a von Neumann algebra – on $\mathbb{C}$. Note that a pure state of a direct sum of algebras is a pure state of one of the algebras, that is

$$\mathcal{P}(A_I) = \bigcup_k \mathcal{P}(\mathcal{A}_k).$$

The reason why we restrict to von Neumann algebras is that to any pure states ω of $\mathcal{A}_k$ corresponds a projection $\rho \in \mathcal{A}_k$ such that

$$\alpha_\rho(a) \doteq \rho a \rho = \omega(a)\rho. \tag{10}$$

This result comes from the proof of proposition 2.16 of ref.[33] in which is assumed, by hypothesis, that the universal enveloping von Neumann algebra $\tilde{\mathcal{A}}_k$ equals $\pi_u(\mathcal{A}_k)$. Strictly speaking this proof is written for complex algebras. However in the standard model, we shall explicitly exhibit such a projection for the real internal algebra so that, in the following, we deal with algebra over $\mathbb{K}$ where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Typically, in physical examples, the $\mathcal{A}_k$ are matrix algebras and ρ is a density matrix. When pure states of different components $\mathcal{A}_k$ are involved and D_I commutes with the direct sum of the corresponding projectors, one obtains as an immediate consequence of Lemma 1 that $\mathcal{A}_I$ reduces to $\mathbb{K}^2$.

Proposition 3. *For $\omega_k \in \mathcal{P}(\mathcal{A}_k)$, let ρ be the corresponding projection in $\mathcal{A}_k$. Define similarly ρ' for $k' \neq k$ and let $p \doteq \rho \oplus \rho'$. If $[D_I, p] = 0$ then, for any $\omega_E, \omega'_E \in \mathcal{P}(\mathcal{A}_E)$,*

$$d\left(\omega_E \otimes \omega_k,\, \omega'_E \otimes \omega_{k'}\right) = d_e\left(\omega_E \otimes \omega_1,\, \omega'_E \otimes \omega_2\right)$$

where ω_1, ω_2 are the pure states of $\mathbb{K}^2$ and d_e is the distance associated to $T_e \doteq T_E \otimes T_r$ with

$$\mathcal{A}_r \doteq \mathbb{K}^2, \quad \mathcal{H}_r \doteq p\mathcal{H}_I, \quad D_r \doteq pD_Ip\big|_{\mathcal{H}_r}.$$

Proof. The projection $e \doteq \mathbb{I}_E \otimes p \in \mathcal{A}$ defines the restricted triple $T_e \doteq (\mathcal{A}_e, \mathcal{H}_e, D_e)$ in which

$$\mathcal{A}_e \doteq \alpha_e(\mathcal{A}) = \mathcal{A}_E \otimes \alpha_p\left(\mathcal{A}_I\right).$$

Since ρ and ρ' correspond to different components of $\mathcal{A}_I$ they are orthogonal, therefore

$$\alpha_p(\mathcal{A}_I) = \alpha_\rho\left(\mathcal{A}_k\right) \oplus \alpha_{\rho'}\left(\mathcal{A}_{k'}\right) = \omega_k(\mathcal{A}_k)\rho \oplus \omega_{k'}(\mathcal{A}_{k'})\rho'$$

by (10). $\omega_k, \omega_{k'}$ being surjective on $\mathbb{K}$, $\omega_k(\mathcal{A}_k)\rho$ and $\omega_{k'}(\mathcal{A}_{k'})\rho'$ are isomorphic to $\mathbb{K}$. Hence

$$\mathcal{A}_e = \mathcal{A}_E \otimes \mathbb{K}^2.$$

The state $\omega_i \in \mathcal{P}(\mathbb{K}^2)$ extracts the i^{th} component of a pair of elements of $\mathbb{K}$. In detail, for $a_I \in \mathcal{A}_I$,

$$\alpha_p(a_I) = \omega_k(a_I)\rho \,\oplus\, \omega_{k'}(a_I)\rho'\,, \tag{11}$$

so that $\omega_1 \circ \alpha_p(a_I) = \omega_k(a_I)$. Since e acts like the identity on $\mathcal{A}_E$,

$$\left(\omega_E \otimes \omega_1\right) \circ \alpha_e = \omega_E \otimes \left(\omega_1 \circ \alpha_p\right) = \omega_E \otimes \omega_k,$$

and $(\omega'_E \otimes \omega_2) \circ \alpha_e = \omega'_E \otimes \omega_{k'}$. By hypothesis $[D, e] = \chi_E \otimes [D_I, p] = 0$ so Lemma 1 yields

$$d\left(\omega_E \otimes \omega_k,\, \omega'_E \otimes \omega_{k'}\right) = d_e\left(\omega_E \otimes \omega_1,\, \omega'_E \otimes \omega_2\right).$$

$\mathcal{H}_r$ and D_r are given by Lemma 1. ■

To explicitly compute d_e, we now focus on the case of a continuum $\times$ discrete space and we take for T_E the spectral triple of a manifold (2). To simplify the notations, the pure state $\omega_x \otimes \omega_k$ is denoted by x_k. The main result of this section is that the internal space is orthogonal to the manifold, in the sense of Pythagorean theorem, as soon as the Dirac operator commutes with the sum of the density matrices.

Theorem 4. *Let $\omega_k, \omega_{k'} \in \mathcal{P}(\mathcal{A}_k), \mathcal{P}(\mathcal{A}_{k'})$, $k \neq k'$. Let ρ, ρ' be the associated projections and $p \doteq \rho \oplus \rho'$. If $[D_I, p] = 0$, then for any points x, y in $\mathcal{M}$*

$$d(x_k, y_{k'})^2 = d(x_k, y_k)^2 + d(y_k, y_{k'})^2.$$

Proof. The proof consists of three steps. First the problem is reduced to a two-sheets model. Then the distance is shown to be the geodesic distance within a (4+1)-dimensional Riemannian manifold which, third, satisfies Pythagorean theorem.

1) With notations of Proposition 3,

$$d(x_k, y_{k'}) = d_e(x_1, y_2). \tag{12}$$

Let us be more explicit on $\mathcal{H}_r, \pi_r$ and D_r.

$$\mathcal{H}_r \doteq p\mathcal{H}_I = \mathcal{H}_k \oplus \mathcal{H}_{k'}, \tag{13}$$

where $\mathcal{H}_k \doteq \rho\mathcal{H}_I$ and $\mathcal{H}_{k'} \doteq \rho'\mathcal{H}_I$. Following (11), one lets $a_r = \omega_k(a_I)\rho \oplus \omega_{k'}(a_I)\rho'$ denote a generic element of $\mathcal{A}_r$. Clearly $\pi_r(\rho) = \mathbb{I}_k$ so

$$\pi_r(a_r) = \omega_k(a_I)\mathbb{I}_k \oplus \omega_{k'}(a_I)\mathbb{I}_{k'}. \tag{14}$$

D_r is the restriction to $\mathcal{H}_r$ of the projection of D_I on $\mathcal{H}_r$, namely

$$D_r \doteq \begin{pmatrix} V & M \\ M^* & W \end{pmatrix} \tag{15}$$

where M is a linear application from $\mathcal{H}_k$ to $\mathcal{H}_{k'}$, and V, W are endomorphisms of $\mathcal{H}_k$, $\mathcal{H}_{k'}$ respectively. M is supposed to be non zero for the contrary makes D_r commuting with π_r, that is all states of $\mathcal{A}$ defined by ω_k are at infinite distance from any states defined by $\omega_{k'}$.

Equations (13, 14, 15) associated to (11) fully determine the triple T_r, and thus T_e. Omitting ρ and ρ' appearing in (11), a generic element of $\mathcal{A}_e$ writes

$$a = f^i \otimes \omega_k(m_i) \;\oplus\; f^i \otimes \omega_{k'}(m_i) = f \oplus g,$$

where $m_i \in \mathcal{A}_I$ and $f^i, f \doteq f^i\omega_k(m_i), g \doteq f^i\omega_{k'}(m_i) \in C^\infty(\mathcal{M})$. In accordance with (1), we assume that $f \oplus g$ is positive, i.e. f and g are real functions. x_1 and y_2 act as

$$x_1(a) = f(x), \qquad y_2(a) = g(y).$$

a is represented by

$$f\mathbb{I}_E \otimes \mathbb{I}_k \;\oplus\; g\mathbb{I}_E \otimes \mathbb{I}_{k'}$$

and the Dirac operator $D_e = i\partial\!\!\!/ \otimes \mathbb{I}_I + \gamma^5 \otimes D_r$ is such that

$$[D_e, a] = \begin{pmatrix} i\partial\!\!\!/ f \otimes \mathbb{I}_k & (g-f)\gamma^5 \otimes M \\ \overline{(f-g)\gamma^5 \otimes M^*} & i\partial\!\!\!/ \otimes \mathbb{I}_{k'} \end{pmatrix}. \tag{16}$$

2) Let us show that d_e coincides with the geodesic distance on the compact manifold

$$\mathcal{M}' \doteq [0,1] \times \mathcal{M}\,,$$

with coordinates $x'^a = (t, x^\mu)$, equipped with the metric

$$\{g^{ab}(x')\} \doteq \begin{pmatrix} \|M\|^2 & 0 \\ 0 & g^{\mu\nu}(x) \end{pmatrix},$$

and made a spin manifold by adding to the previous γ-matrices

$$\gamma^t = \|M\|\,\gamma^5.$$

Thanks to section III, it is enough to show that d_e coincides with the distance L' of the triple

$$\mathcal{A}' = C^\infty(\mathcal{M}'), \qquad \mathcal{H}' = L_2(\mathcal{M}', S), \qquad D' = i\gamma^a\partial_a = i\gamma^t\partial_t + i\partial\!\!\!/\,.$$

To proceed, let $\mathcal{A}''$ be the subset of $\mathcal{A}'_+$ consisting of all functions

$$\phi(t,x) \doteq (1-t)f(x) + tg(x),$$

where f and g are any real functions on $\mathcal{M}$. Then

$$\begin{aligned} \|[D',\phi]\|^2 = \|\gamma^a \partial_a \phi\|^2 &= \sup_{(t,x)\in\mathcal{M}'} \left\{ g^{ab}(t,x)\, \partial_a\phi(t,x)\, \partial_b\phi(t,x) \right\} \\ &\leq \sup_{x\in\mathcal{M}} \left\{ |(f-g)(x)|^2 \|M\|^2 + \sup_{t\in[0,1]} P(t,x) \right\}, \end{aligned}$$

where

$$P(t,x) \doteq t^2 \left\| \overrightarrow{\nabla}(f-g)(x) \right\|^2 + 2tg^{\mu\nu}(x)\, \partial_\mu (f-g)(x)\, \partial_\nu g(x) + \left\| \overrightarrow{\nabla} g(x) \right\|^2$$

is a parabola in t of positive leading coefficient, i.e. which reaches its maximum for $t=0$ or 1. Note that

$$P(0,x) = \left\| \overrightarrow{\nabla} g(x) \right\|^2, \qquad P(1,x) = \left\| \overrightarrow{\nabla} f(x) \right\|^2$$

and, thanks to (16),

$$\begin{aligned} &\left\| \begin{pmatrix} \mathbb{I}_E \otimes \mathbb{I}_k & 0 \\ 0 & 0 \end{pmatrix} [D_e, a] \begin{pmatrix} \mathbb{I}_E \otimes \mathbb{I}_k & 0 \\ 0 & \gamma^5 \otimes \mathbb{I}_{k'} \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} i\partial\!\!\!/ f \otimes \mathbb{I}_k & (g-f)\mathbb{I}_E \otimes M \\ 0 & 0 \end{pmatrix} \right\|^2 \\ &= \sup_{x\in\mathcal{M}} \left\{ \left\| \overrightarrow{\nabla} f(x) \right\|^2 + |f(x)-g(x)|^2 \|M\|^2 \right\} \leq \|[D_e,a]\|^2 . \end{aligned}$$

Similarly, one has $\sup_{x\in\mathcal{M}} \left\{ \left\| \overrightarrow{\nabla} g(x) \right\|^2 + |f(x)-g(x)|^2 \|M\|^2 \right\} \leq \|[D_e,a]\|^2$, hence

$$\|[D',\phi]\| \leq \|[D_e,a]\| .$$

Consequently, since $x_1(a) - y_2(a) = \phi(0,x) - \phi(1,y)$,

$$d_e(x_1,y_2) \leq \sup_{\phi\in\mathcal{A}''} \left\{ |\phi(0,x)-\phi(1,y)| \,/\, \|[D',\phi] \leq 1\| \right\} \leq L'\left((0,x),(1,y)\right) . \tag{17}$$

Proving the converse inequality calls for more precisions on the geometry of $\mathcal{M}'$. Because $\{g^{ab}(x')\}$ is block diagonal and does not depend on t, the coefficients of the Levi-Civita connexion are

$$\begin{gathered} \Gamma^t_{t\mu} = \Gamma^t_{\mu t} = \tfrac{1}{2} g^{tt} \partial_\mu g_{tt} , \qquad \Gamma^\mu_{tt} = -\tfrac{1}{2} g^{\mu\nu} \partial_\nu g_{tt} , \\ \Gamma^\mu_{t\nu} = \Gamma^\mu_{\nu t} = \Gamma^t_{tt} = \Gamma^t_{\mu\nu} = 0 , \end{gathered}$$

where $g_{tt} = (g^{tt})^{-1} = \|M\|^{-2}$. The geodesic equations read

$$\frac{d^2 t}{d\tau^2} + g^{tt}(\partial_\mu g_{tt}) \frac{dt}{d\tau}\frac{dx^\mu}{d\tau} = 0 , \tag{18}$$

$$\frac{d^2 x^\mu}{d\tau^2} - \frac{1}{2} g^{\mu\nu}(\partial_\nu g_{tt}) \frac{dt}{d\tau}\frac{dt}{d\tau} + \Gamma^\mu_{\lambda\rho} \frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} = 0 , \tag{19}$$

and, because g_{tt} does not depend on x^μ, reduce to

$$\frac{dt}{d\tau} = \text{constant} \doteq g^{tt} K \quad \text{and} \quad \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\lambda\rho} \frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} = 0 , \tag{20}$$

where K is a real constant. In other terms, the projection to $\mathcal{M}$ of a geodesic $\mathcal{G}'$ of $\mathcal{M}'$ is a geodesic $\mathcal{G}$ of $\mathcal{M}$, and the projection of $\mathcal{G}'$ to the submanifold $[0,1]\times\mathcal{G}$ is a straight line (i.e. a geodesic of the submanifold). Let $\{x^a(\tau)\}$ be a geodesic in $\mathcal{M}'$ parametrised by its length element $d\tau$. Note that, using (20),

$$1 = \frac{d\tau^2}{d\tau^2} = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + g^{tt}K^2. \tag{21}$$

Let ds be the line element of $\mathcal{M}$. Assuming that $g^{tt}K^2 \neq 1$ (this will be discussed later),

$$d\tau^2 = \frac{ds^2}{1-g^{tt}K^2}, \quad dt = \frac{dt}{d\tau}d\tau = \frac{g^{tt}K ds}{\sqrt{1-g^{tt}K^2}}. \tag{22}$$

For q in $\mathcal{M}$, let $\mathcal{G}'_q$ be the minimum geodesic of $\mathcal{M}'$ between $(0,q)$ and $(1,y)$, and $\mathcal{G}_q$ its projection on $\mathcal{M}$. Let us define $f_0 \in C^\infty(\mathcal{M})$ by

$$f_0(q) = \sqrt{1-g^{tt}K^2}L(q) = \sqrt{1-g^{tt}K^2}\int_{\mathcal{G}_q} ds,$$

where L has been defined in (3). Take $a_0 = (f_0, g_0) \in \mathcal{A}_e$, where $g_0 = f_0 - K$. Then

$$x_1(a_0) - y_2(a_0) = f_0(x) - g_0(y) = f_0(x) + K. \tag{23}$$

But the second equation (22) gives

$$1 = \int_{\mathcal{G}'_x} dt = \frac{g^{tt}K}{\sqrt{1-g^{tt}K^2}}\int_{\mathcal{G}_x} ds,$$

inserted in (23) as $K1$,

$$x_1(a_0) - y_2(a_0) = \sqrt{1-g^{tt}K^2}\int_{\mathcal{G}_x} ds + \frac{g^{tt}K^2}{\sqrt{1-g^{tt}K^2}}\int_{\mathcal{G}_x} ds = \frac{1}{\sqrt{1-g^{tt}K^2}}\int_{\mathcal{G}_x} ds.$$

Using the first equation (22) one obtains

$$x_1(a_0) - y_2(a_0) = \int_{\mathcal{G}'_x} d\tau = L'\left((0,x),(1,y)\right). \tag{24}$$

Moreover, $\partial\!\!\!/ f_0 = \partial\!\!\!/ g_0$ and $\partial_\mu f_0 = \sqrt{1-g^{tt}K^2}\partial_\mu L$, so (16) yields

$$\begin{aligned}\left\|[D_e, a_0]\right\|^2 &= \sup_{q\in\mathcal{M}}\left\{g^{\mu\nu}(q)\partial_\mu f_0(q)\partial_\nu f_0(q) + g^{tt}K^2\right\} \\ &= \sup_{q\in\mathcal{M}}\left\{(1-g^{tt}K^2)\left\|\overrightarrow{\nabla} L(q)\right\|^2 + g^{tt}K^2\right\}.\end{aligned}$$

Recalling (4), this gives $\|[D_e, a_0]\| \leq 1$ so, with (24),

$$d_e(x_1, y_2) \geq L'((0,x),(1,y)).$$

Together with (17) and (12),

$$d(x_k, y_{k'}) = L'((0,x),(1,y)). \tag{25}$$

This result holds as long as $g^{tt}K^2 \neq 1$. If this is not true, then $U \doteq \frac{dx^\mu}{d\tau}\partial_\mu \in TM$ is zero for (21) indicates that $g(U,U) = 0$ and $\mathcal{M}$ is Riemannian. In other words, $x^\mu(\tau)$ is a constant. This cannot be the equation of $\mathcal{G}'_x$ unless $x = y$. As a conclusion, (25) holds as soon as $x \neq y$.

When $x = y$, (12) gives $d(y_k, y_{k'}) = d_e(y_1, y_2)$. With d_r denoting the distance associated to the triple T_r alone, Proposition (2) yields $d_e(y_1, y_2) = d_r(\omega_1, \omega_2)$, which is nothing but the distance of the simplest two-points space and equals[8] $\frac{1}{\|M\|}$. Thus

$$d(y_k, y_{k'}) = \frac{1}{\|M\|}. \tag{26}$$

The projection $\mathcal{G}_y$ of the geodesic $\mathcal{G}'_x = \mathcal{G}'_y$ is, by (19), a geodesic between y and y, that is to say a point. $\mathcal{G}'_y$ reduces to a straight line in the hyperplane. Thus $d\tau^2 = g_{tt}dt^2$ and

$$L'\left((0,y),(1,y)\right) = \sqrt{g_{tt}}\int_{\mathcal{G}'_y} dt = \sqrt{g_{tt}} = \frac{1}{\|M\|}.$$

Consequently $d(y_k, y_{k'}) = L'\left((0,y),(1,y)\right)$ and (25) holds even if $x = y$.

3) The last step is to show that (25) satisfies Pythagorean equality. g^{tt} being a constant, equation (22) indicates that $d\tau$ and ds are equal up to a constant factor. In this way, one may parametrise a geodesic of $\mathcal{M}'$ by ds rather than $d\tau$ and obtains, thanks to the geodesic equations,

$$dt = g^{tt}K'ds$$

where K' is a real constant. Then

$$d\tau^2 = g_{tt}dt^2 + ds^2 = ds^2(1 + g^{tt}K'^2).$$

Thus

$$\begin{aligned} L'\left((0,x),(1,y)\right) &= \sqrt{1 + g^{tt}K'^2}\int_{\mathcal{G}'_x} ds = \sqrt{1 + g^{tt}K'^2}\,L(x,y) \\ &= \sqrt{L(x,y)^2 + g^{tt}K'^2 L(x,y)^2}. \end{aligned} \tag{27}$$

On one side, Theorem 2 gives $L(x,y) = d(x_k, y_k)$. On the other side,

$$g^{tt}K'^2 L(x,y)^2 = g_{tt}\left(\int_{\mathcal{G}'_x} g^{tt}K'ds\right)^2 = g_{tt}\left(\int_{\mathcal{G}'_x} dt\right)^2 = g_{tt} = \frac{1}{\|M\|^2} = d^2(y_k, y_{k'})$$

by (26). Together with (25) and (27),

$$d(x_k, y_{k'})^2 = d(x_k, y_k)^2 + d^2(y_k, y_{k'}) \,. \qquad \blacksquare$$

VI Fluctuations of the metric.

For a complete presentation of the material of this section and a justification of the terminology, see refs.[8,10]. To a triple $(\mathcal{A}, \mathcal{H}, D)$, the axiom of reality adds an operator J, called the real structure, such that $[JaJ^{-1}, b] = 0$ for any $a, b \in \mathcal{A}$. This allows to define a right action of $\mathcal{A}$ over $\mathcal{H}$ which makes sense because of the noncommutativity of the algebra. To define a notion of unitarily equivalent spectral triples preserving the operator J, a unitary element u of $\mathcal{A}$ is implemented by the operator $U \doteq uJuJ^{-1}$ rather than the operator u. Then the action of u defines the *gauge transformed* triple $(\mathcal{A}, \mathcal{H}, D_A)$ where

$$D_A \doteq UDU^* = D + A + JAJ^{-1} \tag{28}$$

with

$$A \doteq u[D, u^{-1}] .$$

The selfadjoint operator A governs the failure of invariance of D under a gauge transformation[8]. Under a gauge transformation, A transforms like a usual vector potential. Since in electrodynamics the vector potential is a 1-form, one defines the space Ω^1 of 1-form of the noncommutative space $(\mathcal{A}, \mathcal{H}, D)$ as the set of elements

$$a^i[D, b_i]$$

where $a^i, b_i \in \mathcal{A}$. Note that we use the simplifying notation Ω^1 rather than Ω^1_D, more common in the literature, because we only deal with 0-forms and 1-forms (Ω^n_D differs from Ω^n for $n \geq 2$). Since A is selfadjoint, the set of vector potentials is simply the subset of selfadjoint elements of Ω^1. For any vector potential A, D_A defined by (28) is called the covariant Dirac operator.

The distance is not invariant under a gauge transformation and the metric is said to fluctuate. To study such fluctuations, one has to replace D by D_A everywhere in the preceding sections. A well known result makes this replacement less studious than it seems.

Lemma 5. $[a, J\omega J^{-1}] = 0,\ \forall \omega \in \Omega^1, a \in \mathcal{A}$.

Proof. $[J^{-1}aJ, [D, b_i]] = 0$ (first order axiom) and $[a, Ja^iJ^{-1}] = 0$ (axiom of reality) yield

$$\begin{aligned}
[a, J\omega J^{-1}] &= [a, Ja^i[D, b_i]J^{-1}] \\
&= aJa^iJ^{-1}J[D, b_i]J^{-1} - Ja^i[D, b_i]J^{-1}a \\
&= Ja^i[D, b_i]J^{-1}a - Ja^i[D, b_i]J^{-1}a = 0 .
\end{aligned}$$

■

As an immediate consequence,

$$[D_A, a] = [D + A, a] . \tag{29}$$

Let us now work out the 1-forms of a tensor product triple $T_E \otimes T_I$. In refs.[20,32] it is shown that

$$\Omega^1 = \Omega^1_E \otimes \Omega^0_I + \chi_E \Omega^0_E \otimes \Omega^1_I ,$$

where $\Omega^0_E = \mathcal{A}_E$ is the set of 0-forms of $\mathcal{A}_E$, and similar definitions for the other terms. When T_E is the spectral triple of a manifold,

$$\Omega^1_E \ni f^j[i\partial\!\!\!/, g_j \mathbb{I}_E] = if^j(\gamma^\mu \partial_\mu g_j) = i\gamma^\mu f_\mu ,$$

where $f^j, g_j, f_\mu \doteq f^j \partial_\mu g_j \in C^\infty(\mathcal{M})$. A 1-form of the total spectral triple is

$$\Omega^1 \ni i\gamma^\mu f^i_\mu \otimes a_i + \gamma^5 h^j \otimes m_j$$

where $a_i \in \mathcal{A}_I$, $h^j \in C^\infty(\mathcal{M})$, $m_j \in \Omega^1_I$. A vector potential is

$$A = i\gamma^\mu \otimes A_\mu + \gamma^5 \otimes H \tag{30}$$

with $A_\mu \doteq f^i{}_\mu a_i$ an $\mathcal{A}_I$-valued skew-adjoint vector field (over $\mathcal{M}$) and $H \doteq h^j m_j$ an Ω^1_I-valued selfadjoint scalar field. For a matrix algebra (or a direct sum of matrix algebras), the skew-adjoint elements form the Lie algebra of the Lie group of unitarities. This Lie group represents the gauge group of the theory, thus A_μ is a gauge potential. In ref.[10] a formula is given for the fluctuations of the metrics due to A_μ. Here, we focus on the fluctuations coming from the scalar field H only, and we assume that $A_\mu = 0$. Then (29) becomes

$$[D_A, a] = [D + \gamma^5 \otimes H, a]. \tag{31}$$

From now on, we write $D_A \doteq D + \gamma^5 \otimes H$. For simplicity, d still denotes the distance associated to the triple $(\mathcal{A}, \mathcal{H}, D_A)$. Remembering definition (7), a scalar fluctuation substitutes

$$D_H \doteq D_I + H$$

for D_I. The main difference is that the internal Dirac operator D_H now depends on x so that each point x of the manifold defines an internal triple

$$T_I^x \doteq (\mathcal{A}_I, \mathcal{H}_I, D_H(x))\,.$$

This interpretation of scalar fluctuations perfectly fits to the adaptation of Theorem 2.

Theorem 2'. *Let L be the geodesic distance in $\mathcal{M}$ and d_x the distance of the spectral triple T_I^x alone. For $x, y \in \mathcal{M}$ and $\omega_k, \omega_{k'} \in \mathcal{P}(\mathcal{A}_I)$,*

$$\begin{aligned} d(x_k, x_{k'}) &= d_x(\omega_k, \omega_{k'}), \\ d(x_k, y_k) &= L(x, y). \end{aligned}$$

Proof. The adaptation of the proof of Theorem 2 is straightforward. Notations are similar except that ω_E is now ω_x so that a_E is replaced by a_x. With $H = h^j m_j$,

$$\begin{aligned} [D_H(x), a_x] &= [D_I + \omega_x(h^j) m_j, \omega_x(f^i) m_i] \\ &= \omega_x(f^i)[D_I, m_i] + \omega_x(h^j)\omega_x(f^i)[m_j, m_i] \\ &= (\omega_x \otimes \mathbb{I}_I)\left(f^i \otimes [D_I, m_i] + h^j f^i \otimes [m_j, m_i]\right) \\ &= (\omega_x \otimes \mathbb{I}_I)\left(f^i \otimes [D_H, m_i]\right). \end{aligned} \tag{32}$$

Then, $i[D_H(x), a_x]$ being normal,

$$\begin{aligned} \|[D_H(x), a_x]\| &= \sup_{\tau_I \in \mathcal{S}_I} |\tau_I([D_H(x), a_x])| \\ &= \sup_{\tau_I \in \mathcal{S}_I} \left|(\omega_x \otimes \tau_I)\left(f^i \otimes [D_H, m_i]\right)\right| \\ &\leq \sup_{\tilde\omega_E \otimes \tau_I \in \mathcal{P}(\mathcal{A}_E)\otimes \mathcal{S}_I} \left|(\tilde\omega_E \otimes \omega_I)\left(f^i \otimes [D_H, m_i]\right)\right| \\ &\leq \left\| f^i \otimes [D_H, m_i] \right\|. \end{aligned}$$

Equation (9) being replaced by $\|f^i \otimes [D_H, m_i]\| \leq \|[D_A, a]\|$, one obtains

$$\|[D_H(x), a_x]\| \leq \|[D_A, a]\|\,.$$

The rest of the proof is then similar as in Theorem 2. ■

Note that in (32) we use that ω_x is a character, i.e. that $\mathcal{A}_E$ is Abelian.

Applied to the two sheets-model, Theorem 2 simply says that the distance between the sheets is encoded by a scalar field, as it has already been shown in ref.[4] (see also ref.[5] for a $M_2(\mathbb{C}) \oplus \mathbb{C}$ model). Theorem 4 is modified in a more serious way for the fluctuation introduces an x-dependence for the coefficients of the Kaluza-Klein metric.

Theorem 4'. *Let $\omega_k, \omega_{k'} \in \mathcal{P}(\mathcal{A}_k), \mathcal{P}(\mathcal{A}_{k'})$, $k \neq k'$. Let ρ, ρ' be the associated projections and $p \doteq \rho \oplus \rho'$. If $[D_H, p] = 0$ for any points of $\mathcal{M}$, then for any points $x, y \in \mathcal{M}$,*

$$d(x_k, y_{k'}) = L'((0,x),(1,y))$$

where L' is the geodesic distance of the spin manifold $\mathcal{M}' \doteq [0,1] \times \mathcal{M}$ equipped with the metric

$$\begin{pmatrix} \|M(x)\|^2 & 0 \\ 0 & g^{\mu\nu}(x) \end{pmatrix}$$

in which $g^{\mu\nu}$ is the metric of $\mathcal{M}$ and M is the restriction to the representation of $\mathcal{A}_{k'}$ of the projection of D_H on the representation of $\mathcal{A}_k$.

Proof. Unless otherwise made precise, notations are similar to Theorem 4. The first part of the proof is hardly modified. Let $\psi^r \otimes \xi_r \in \mathcal{H}$. Recalling (31) and the definition (30) of H,

$$[D_A, a]\psi^r \otimes \xi_r = \gamma^5 \psi_r \otimes [D_I, p]\xi_r + \gamma^5 h^j \psi_r \otimes [m_j, p]\xi_r \in \mathcal{H}.$$

Evaluated at $x \in \mathcal{M}$, the above expression yields

$$[D_A, a]\psi^r(x) \otimes \xi_r = \gamma^5 \psi_r(x) \otimes [D_I + H(x), p]\xi_r = 0$$

by hypothesis, which means that $[D_A, a]$ is the zero endomorphism of $\mathcal{H}$ so that Lemma 3 applies and

$$d(x_k, y_{k'}) = d_e(x_1, y_2).$$

The only difference with Theorem 4 is that D_r now depends on x. For instance when $\mathcal{A}_I$ is finite dimensional then M is a matrix whose entries are scalar fields on $\mathcal{M}$.

Now $g^{tt}(x) \doteq \|M(x)\|^2$ depends on x but is still independent with respect to t. The geodesic equations (18, 19) no longer reduce to (20) but

$$\begin{aligned} \frac{d}{d\tau}(g_{tt}\frac{dt}{d\tau}) &= (\frac{d}{d\tau}g_{tt})\frac{dt}{d\tau} + g_{tt}\frac{d}{d\tau}(\frac{dt}{d\tau}) \\ &= (\partial_\mu g_{tt})\frac{dt}{d\tau}\frac{dx^\mu}{d\tau} + g_{tt}\frac{d^2t}{d\tau^2} \\ &= g_{tt}\left(g^{tt}(\partial_\mu g_{tt})\frac{dt}{d\tau}\frac{dx^\mu}{d\tau} + \frac{d^2t}{d\tau^2}\right) = 0 \end{aligned}$$

by (18). Thus $g_{tt}\frac{dt}{d\tau} = K$ is a constant. This is almost the first equation (20), except that

$$\frac{dt}{d\tau} = Kg^{tt}(x) \tag{33}$$

now depends on x. $a_0 = (f_0, g_0)$ is defined by

$$f_0(q) \doteq \int_{\mathcal{G}_q} \sqrt{1 - K^2 g^{tt}} ds\,, \qquad g_0 \doteq f_0 - K\,, \tag{34}$$

where $\mathcal{G}'_q$ is the minimal geodesics from $(0, q)$ to the fixed point $(1, y)$ and $\mathcal{G}_q$ its projection to $\mathcal{M}$ (note that $\mathcal{G}_q$ is no longer a geodesic of $\mathcal{M}$). Assuming that

$$K^2 g^{tt}(p) \neq 1 \tag{35}$$

for any $p \in \mathcal{G}_q$ allows to write $d\tau = \frac{ds}{\sqrt{1-K^2 g^{tt}}}$ and then

$$1 = \int_{\mathcal{G}'_q} dt = \int_{\mathcal{G}'_q} \frac{dt}{d\tau} d\tau = \int_{\mathcal{G}_q} \frac{K g^{tt}}{\sqrt{1 - K^2 g^{tt}}} ds\,. \tag{36}$$

If (35) does not hold, we call G the set of points p of $\mathcal{G}_q$ for which $1 - K^2 g^{tt}(p) = 0$. G' is the corresponding subset of $\mathcal{G}'_q$. For any $p' \in \mathcal{G}'{}_q$, (33) yields

$$\frac{dt}{d\tau} d\tau = K^{-1} d\tau\,,$$

and (36) is replaced by

$$1 = \int_{\mathcal{G}_q/G} \frac{K g^{tt}}{\sqrt{1 - K^2 g^{tt}}} ds + \int_{G'} K^{-1} d\tau\,.$$

Inserted as $K1$ in $x_1(a_0) - y_2(a_0) = f_0(x) + K$, this gives

$$\begin{aligned} x_1(a_0) - y_2(a_0) &= \int_{\mathcal{G}_x} \sqrt{1 - K^2 g^{tt}} ds + \int_{\mathcal{G}_x/G} \frac{K^2 g^{tt}}{\sqrt{1 - K^2 g^{tt}(x)}} ds + \int_{G'} d\tau \\ &= \int_G \sqrt{1 - K^2 g^{tt}} ds + \int_{\mathcal{G}_x/G} \frac{ds}{\sqrt{1 - K^2 g^{tt}(x)}} + \int_{G'} d\tau \\ &= \int_{\mathcal{G}'_x/G'} d\tau + \int_{G'} d\tau = L'\left((0, x), (1, y)\right). \end{aligned}$$

The function $f_0(q)$ is in the vicinity of q by definition (34) constant on a codimension 1 hypersurface through q. Choosing an adapted reference frame with $\{x^1, x^2, x^3\}$ being the coordinates in the hypersurface and x^0 the normal coordinate, one has $ds(q) = \sqrt{g_{00}(q)} dx^0$ and $\partial_\mu f_0(q) = \delta^0_\mu \partial_0 f_0(q)$, giving

$$\begin{aligned} \partial_\mu f_0(q) &= \delta^0_\mu \sqrt{1 - K^2 g^{tt}(q)} \sqrt{g_{00}(q)}\,, \\ g^{\mu\nu}(q) \partial_\mu f_0(q) \partial_\nu f_0(q) &= g^{00}(1 - g^{tt} K^2) g_{00} = 1 - g^{tt} K^2\,, \end{aligned}$$

which leads to $\|[D_e, a_0]\| = 1$. Hence the result. ■

Few comments about this theorem. First, since all the coefficients of the metric depend on x, there is no way that the geodesic distance satisfies Pythagorean theorem. Second, a metric is non-degenerate by definition, and we implicitly assume that $M(x)$ never cancels. This was assumed in Theorem 4 to make the distance finite. Here the point is more subtle for the field M

may be zero for some points x. Let $\ker(M) \subset \mathcal{M}$ be the set of such points. For any $q \in \ker(M)$, $d\left((0,q),(1,q)\right) = +\infty$ by Proposition 2'. Moreover,

$$\begin{aligned} d\left((0,q),(1,q)\right) &\leq d\left((0,q),(0,x)\right) + d\left((0,x),(1,y)\right) + d\left((1,y),(1,q)\right) \\ &\leq L(p,x) + d\left((0,x),(1,y)\right) + L(y,q)\,, \end{aligned}$$

so $d\left((0,x),(1,y)\right) = +\infty$ for any $x, y \in \mathcal{M}$, which contradicts Theorem 4' if $x = y \notin \ker(M)$. One solution is to assume that any point (t,q) with $q \in \ker(M)$ is at infinite distance from any other point, and define $\mathcal{M}'$ as $[0,1] \times \mathcal{M}/\ker(M)$. If any path between x and y crosses $\ker(M)$, this operation splits $\mathcal{M}'$ into disconnected parts. A better solution is to take into account the non-scalar part A_μ of the fluctuation[a]. This goes beyond the aim of this paper and the reader should refer to ref.[10].

VII The standard model and other examples.

We shall investigate the metric of spaces whose internal part is one of those described in ref.[17]. We also give some indications on the distance in the standard model.

Commutative spaces.

We call commutative space a spectral triple whose internal algebra is $\mathbb{C}^k$, $k \in \mathbb{N}$. Any k-tuple of complex numbers $a = (a^1, ..., a^k)$ is represented by a diagonal matrix. For two pure states ω_u, ω_v $(u, v \in [1,k])$, $\rho_u \oplus \rho_v$ is the matrix with null coefficients except 1 on the u^{th} and v^{th} elements of the diagonal. Within the graphical framework of ref.[17], one shows that the internal distance only depends on points that are on some path between u and v. In other terms

$$d_I(u,v) = \tilde{d}_I(u,v)$$

where $\tilde{d}_I$ denotes the distance computed with the Dirac operator $\tilde{D}_I = \rho D_I \rho$ in which

$$\rho \doteq \bigoplus_{i \in P \cup Q} \rho_i,$$

with P the set of points that are not connected to u nor v, and Q the set of points that are connected to u or - this is an exclusive "or"- v by one and only one path. Note that, for any internal 1-form,

$$\rho\, a_i\, [D_I, b^i] \rho = a_i [\tilde{D}, b^i]$$

so that the ˜ notation is coherent with the scalar fluctuation. At any point x of the manifold

$$d_x(\omega_u, \omega_v) = \tilde{d}_x(\omega_u, \omega_v)$$

therefore, to apply Theorem 2, it is enough to check that $[\tilde{D}_H, \rho] = 0$. One verifies that whenever a component of the internal Dirac operator is zero, the corresponding component of any internal 1-form is also zero, so that $[\tilde{D}_H, \rho_I] = 0$ as soon as $[\tilde{D}_I, \rho_I] = 0$.

[a]In physical models, $M(x)$ is the representation of the Higgs field in the unbroken phase. Then, at $M = 0$ the Higgs potential reaches its local maximum. Neglecting the gauge potential A_μ, the Faddeev-Popov determinant of the t'Hooft gauge-fixing condition is zero at the maximum of the Higgs potential. This leads to a Gribov problem and questions a quantum treatment of $M(x)$ without gauge field. (observation by Helmuth Hüffel)

This means that the only path between u and v is the link $u - v$ itself. The simplest case, $k = 2$, endows the two-sheets model with a cylindrical metric. The other examples of commutative spaces given in ref.[17] do not fit the required condition and our next examples will be noncommutative.

Two-points space.

Let $\mathcal{A}_I = M_n(\mathbb{C}) \oplus \mathbb{C}$ be represented over $\mathbb{C}^{n+1}$ by

$$\begin{pmatrix} m & 0 \\ 0 & c \end{pmatrix} \tag{37}$$

where $m \in M_n(\mathbb{C})$ and $c \in \mathbb{C}$. Possible chirality K and Dirac operator Δ are

$$K = \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & -1 \end{pmatrix}, \quad \Delta = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix},$$

where $M \in \mathbb{C}^n$. But there is no operator J to fluctuate the metric. A solution is to make (37) acting over $\mathcal{H}_I = M_{n+1}(\mathbb{C})$ and define

$$D\psi \doteq \Delta\psi + \psi\Delta, \qquad \chi_I\psi \doteq K\psi + \psi K, \qquad J\psi \doteq \psi^*$$

for any $\psi \in \mathcal{H}_I$. Since $J\Delta J^{-1}\psi = J\Delta\psi^* = (\Delta\psi^*)^* = \psi\Delta$, one has $D\psi = \Delta\psi + J\Delta J^{-1}\psi$. Moreover, for any $a \in \mathcal{A}_I$, $[J\Delta J^{-1}, a]\psi = a\psi\Delta - a\psi\Delta = 0$, so $[D_I, a] = [\Delta, a]$. Note that this result comes directly from Lemma 5 as soon as one knows that Δ is a 1-form[21]. Since the operator norm over $\mathbb{C}^n$ is equal to the operator norm over $M_n(\mathbb{C})$,

$$\|[D, a]\| = \|[\Delta, a]\|$$

and the distance is in fact the same as the one computed with the spectral triple $(\mathcal{A}_I, \mathbb{C}^{n+1}, \Delta)$. Note that this point is assumed in ref.[31].

Let ρ_1 be the density matrix associated to a pure state ω_1 of $M_n(\mathbb{C})$ and ρ_0 the one corresponding to the pure state ω_0 of $\mathbb{C}$. Then

$$\rho_1 \oplus \rho_0 = \begin{pmatrix} \rho_1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that $[D_I, \rho_1 \oplus \rho_0] = 0$ is equivalent to $\rho_1 M = M$. In other terms, M is colinear to the range of ρ_1. An happy coincidence makes that this is precisely the condition under which the internal distance $d_I(\omega_1, \omega_0) = \frac{1}{\|M\|}$ is finite[17]. Theorem 4 is true for any Dirac operator – $d_I(\omega_0, \omega_1) = +\infty$ makes $d(x_0, y_1) = +\infty$ for any x, y in $\mathcal{M}$ – so

$$d(x_0, y_1) = \sqrt{L(x,y)^2 + \frac{1}{\|M\|^2}}$$

when M is in the range of ρ_1, is infinite otherwise.

The standard model.

The spectral triple of the standard model (see refs.[8,10,7] and ref.[3,6] for a physical expectation of the Higgs mass) is the tensor product of the usual spectral triple of a manifold T_E by an internal triple in which

$$\mathcal{A}_I = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$$

($\mathbb{H}$ is the real algebra of quaternions) is represented over

$$\mathcal{H}_I = \mathbb{C}^{90} = \mathcal{H}^P \oplus \mathcal{H}^A = \mathcal{H}_L^P \oplus \mathcal{H}_R^P \oplus \mathcal{H}_L^A \oplus \mathcal{H}_R^A .$$

The basis of $\mathcal{H}_L^P = \mathbb{C}^{24}$ consists of the left-handed fermions

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \mu_\tau \\ \tau \end{pmatrix}_L,$$

and the basis of $\mathcal{H}_R^P = \mathbb{C}^{21}$ is labelled by the right-handed fermions u_R, d_r, c_R, s_R, t_R, b_R and e_R, μ_R, τ_R (the model assumes massless neutrinos). The colour index for the quarks has been omitted. $\mathcal{H}_R^A$ and $\mathcal{H}_L^A$ correspond to the antiparticles. ($a \in \mathbb{H}$, $b \in \mathbb{C}, c \in M_3(\mathbb{C})$) is represented by

$$\pi_I(a,b,c) \doteq \pi^P(a,b) \oplus \pi^A(b,c) \doteq \pi_L^P(a) \oplus \pi_R^P(b) \oplus \pi_L^A(b,c) \oplus \pi_R^A(b,c)$$

where, writing $B \doteq \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix} \in \mathbb{H}$ and $N = 3$ for the number of fermion generations,

$$\pi_L^P(a) \doteq a \otimes \mathbb{I}_N \otimes \mathbb{I}_3 \,\oplus\, a \otimes \mathbb{I}_N , \qquad \pi_R^P(b) \doteq B \otimes \mathbb{I}_N \otimes \mathbb{I}_3 \,\oplus\, \bar{b} \otimes \mathbb{I}_N ,$$
$$\pi_L^A(b,c) \doteq \mathbb{I}_2 \otimes \mathbb{I}_N \otimes c \,\oplus\, \bar{b}\mathbb{I}_2 \otimes \mathbb{I}_N , \qquad \pi_R^A(b,c) \doteq \mathbb{I}_2 \otimes \mathbb{I}_N \otimes c \,\oplus\, \bar{b}\mathbb{I}_n .$$

One defines a real structure

$$J_I = \begin{pmatrix} 0 & \mathbb{I}_{15N} \\ \mathbb{I}_{15N} & 0 \end{pmatrix} \circ \,\bar{}$$

where $\bar{}$ denotes the complex conjugation, and an internal Dirac operator

$$D_I \doteq \begin{pmatrix} D_P & 0 \\ 0 & \bar{D}_P \end{pmatrix} = \begin{pmatrix} D_P & 0 \\ 0 & 0 \end{pmatrix} + J_I \begin{pmatrix} D_P & 0 \\ 0 & 0 \end{pmatrix} J_I^{-1}$$

whose diagonal blocks are $15N \times 15N$ matrices

$$D_P \doteq \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix},$$

with M a $8N \times 7N$ matrix

$$M \doteq \begin{pmatrix} (e_{11} \otimes M_u + e_{22} \otimes M_d) \otimes \mathbb{I}_3 & 0 \\ 0 & e_2 \otimes M_e \end{pmatrix}. \qquad (38)$$

Here, $\{e_{ij}\}$ and $\{e_i\}$ denote the canonical basis of $M_2(\mathbb{C})$ and $\mathbb{C}^2$ respectively. M_u, M_d, M_e are the mass matrices

$$M_u = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \quad M_d = C_{KM} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \quad M_e = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}$$

whose coefficients are the masses of the elementary fermions, pondered by the unitary Cabibbo-Kobayashi-Maskawa matrix. The chirality, last element of the spectral triple, is

$$\chi_I = (-\mathbb{I}_{8N}) \oplus \mathbb{I}_{7N} \oplus (-\mathbb{I}_{8N}) \oplus \mathbb{I}_{7N}.$$

The presence of the conjugate representation $\bar{b}$ in π_I requires to view $\mathbb{C}$ as a real algebra. Therefore, the pure state ω_0 of $\mathbb{C}$ is no longer the identity but an $\mathbb{R}$-linear function with value in $\mathbb{R}$ which maps 1 to 1. In other words, ω_0 is the real part: $\omega_0(b) = \mathrm{Re}(b)$. As a quaternionic algebra, $\mathbb{H}$ has a single pure state and this remains true for $\mathbb{H}$ seen as a real algebra.

Lemma 6. *The single pure state ω_1 of $\mathbb{H}$ is $\omega_1(a) = \frac{1}{2} Tr(\mathbb{I}_{\mathbb{H}}\, a)$.*

Proof. The representation of $\mathbb{H}$ over the four-dimensional real vector space with basis $\{1, i, j, k\}$ such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$, is

$$a = \alpha + \beta i + \gamma j + \delta k$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Since $\bar{a} \doteq \alpha - \beta i - \gamma j - \delta k$, $a\bar{a} \in \mathbb{R}^+$ so any $\mathbb{R}$-linear form is positive. Therefore a state is any $\mathbb{R}$-linear form that maps $\mathbb{I}_{\mathbb{H}} = 1$ to 1. Let ω be such a state. By linearity,

$$\omega(a) = \alpha + \beta\omega(i) + \gamma\omega(j) + \delta\omega(k),$$

so ω is uniquely determined by its values on i, j, k. Let $\omega_{\omega(i)}$ be the linear form defined by $\omega_{\omega(i)}(i) = \omega(i)$, $\omega_{\omega(i)}(1) = \omega_{\omega(i)}(j) = \omega_{\omega(i)}(k) = 0$. Define similarly $\omega_{\omega(1)}, \omega_{\omega(j)}, \omega_{\omega(k)}$. Then

$$\begin{aligned} \omega &= \omega_{\omega(1)} + \omega_{\omega(i)} + \omega_{\omega(j)} + \omega_{\omega(k)} \\ &= \lambda(\omega_{\omega(1)} + \omega_{\kappa\omega(i)} + \omega_{\kappa\omega(j)} + \omega_{\kappa\omega(k)}) + (1-\lambda)(\omega_{\omega(1)} + \omega_{\kappa'\omega(i)} + \omega_{\kappa'\omega(j)} + \omega_{\kappa'\omega(k)}), \end{aligned} \tag{39}$$

where $\lambda, \kappa \in \mathbb{R}/\{1\}$ and $\kappa' \doteq \frac{1-\lambda\kappa}{1-\lambda}$. Both factors of the right hand side of (39) map 1 to 1, so they are states and ω is not pure unless $\omega(i) = \omega(j) = \omega(k) = 0$. Hence the only pure state of $\mathbb{H}$ is $\omega_1 \doteq \omega_{\omega(1)}$.

The quaternion a can also be represented over $\mathbb{C}^2$ by $\begin{pmatrix} \theta & \rho \\ -\bar{\rho} & \bar{\theta} \end{pmatrix}$ where $\theta \doteq \alpha + i\beta$. Then $\mathrm{Tr}(a) \doteq 2\mathrm{Re}(\theta) = 2\alpha = 2\omega_1(a)$, that is $\omega_1(a) = \mathrm{Tr}(\frac{1}{2}\mathbb{I}_H\, a)$. ■

With regard to $\mathcal{P}(M_3(\mathbb{C}))$, we shall only need the following well-known lemma:

Lemma 7. *Let $\omega, \omega' \in \mathcal{P}(\mathcal{A}_I)$. Then $\omega = \omega'$ if and only if $\ker(\omega) = \ker(\omega')$.*

Proof. Pure states are linear form, so if they have the same kernel they are proportional. Since they coincide on the identity, they are equal. ■

Noncommutative geometry gives an interpretation of the Higgs field as a 1-form of the internal space. By scalar fluctuation, 1-forms closely interfere with the metric. Thus the Higgs field has an interpretation in term of an internal metric. The conclusive result of this paper is a precision of this link between Higgs and metric when the gauge field A_μ is neglected.

Proposition 8. *The finite part of the geometry of the standard model with scalar fluctuations of the metric consists of a two-sheets model labelled by the single states of $\mathbb{C}$ and $\mathbb{H}$. Each of* *the sheets is a copy of the Riemannian four-dimensional space-time endowed with its metric. The fifth component of the metric, corresponding to the discrete dimension, is*

$$g^{tt}(x) = \left(|1+h_1(x)|^2 + |h_2(x)|^2\right) m_t^2$$

where $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ *is the Higgs doublet and* m_t *the mass of the quark top.*

Proof. π_I stands for $\pi_I(a,b,c)$ and $\Delta \doteq \begin{pmatrix} D_P & 0 \\ 0 & 0 \end{pmatrix}$ so that $D_I = \Delta + J\Delta J^{-1}$. Since Δ is a 1-form[21], Lemma 5 yields $[J_I \Delta J_I^{-1}, \pi_I] = 0$, so that we can take $D_H = \Delta + H$. By explicit calculation[18],

$$H = \begin{pmatrix} 0 & \pi_L^P(h)M & 0 & 0 \\ M^*\pi_L^P(h^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where h is a quaternion-valued scalar field. Thus

$$D_H = \begin{pmatrix} 0 & \Phi M & 0 & 0 \\ M^*\Phi^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{40}$$

where

$$\Phi \doteq (h + \mathbb{I}_{\mathbb{H}}) \otimes \mathbb{I}_{4N} = \begin{pmatrix} 1+h_1 & h_2 \\ -\bar{h}_2 & 1+\bar{h}_1 \end{pmatrix} \otimes \mathbb{I}_{4N},$$

with h_1 and h_2 being two complex scalar fields.

By (1), the metric of the standard model is identical to the metric associated to the triple $(\mathcal{A}_s,\, \mathcal{H},\, D)$, where $\mathcal{A}_s = C^\infty(\mathcal{M})_s \otimes \mathcal{A}_{Is}$ is the subalgebra of selfadjoint elements of $\mathcal{A}$, with

$$\mathcal{A}_{Is} = \mathbb{C}_s \oplus \mathbb{H}_s \oplus M_3(\mathbb{C})_s = \mathbb{R} \oplus \mathbb{R} \oplus M_3(\mathbb{C})_s.$$

The representation π_s associated to this triple coincides with the restriction of π to $\mathcal{A}_s$. Concerning the quaternion, π_s substitutes

$$\begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} \theta & \bar{\rho} \\ -\bar{\rho} & \bar{\theta} \end{pmatrix}.$$

In other words, to each representation of $\mathbb{H}$ there corresponds the direct sum of twice the fundamental representation of $\mathbb{R} = \mathbb{H}_s$. Now ω_1 seen as a pure state of $\mathbb{H}_s$ is nothing but the identity. The associated projection $\rho_1 \in \mathbb{H}_s$ is nothing but the real number 1 which obviously satisfies (10). The same is true for ω_0 seen as a pure state of $\mathbb{R} = \mathbb{C}_s$. Hence

$$\pi_s(\rho_0 \oplus \rho_1) = \begin{pmatrix} \mathbb{I}_{15N} & 0 \\ 0 & \begin{pmatrix} 0_{6N} & & & \\ & \mathbb{I}_{2N} & & \\ & & 0_{6N} & \\ & & & \mathbb{I}_N \end{pmatrix} \end{pmatrix}$$

commutes with D_H defined in (40). Proposition 4' applies to the distance between pure states of $\mathcal{A}$ defined by ω_0 and ω_1. Here

$$\pi_s(\text{ran }\rho_1) = \mathcal{H}_L^P \;\text{ and }\; \pi_s(\text{ran }\rho_0) = \mathcal{H}_R^P \oplus \mathcal{H}_{lep}^A,$$

where $\mathcal{H}^A_{lep} = \mathbb{C}^{3N}$ is the subset of $\mathcal{H}^A$ generated by the anti-leptons. Thus the extra metric component is

$$g^{tt}(x) = \|\Phi(x)M\|^2 .$$

Note that, as desired, ΦM is a $2\alpha^{\mathbb{H}} \times \left(\alpha^{\mathbb{C}} + \alpha^{\bar{\mathbb{C}}}\right)$ matrix, where $\alpha^{\mathbb{H}} = 4N$ is the degeneracy of the representation of $\mathbb{H}_s$ in π^P_L, and $\alpha^{\mathbb{C}} = 3N$, $\alpha^{\bar{\mathbb{C}}} = 4N$ are defined as well. Using the explicit form (38),

$$\begin{aligned}
\|\Phi(x)M\|^2 &= \max\left\{ \|(\Phi(x)\otimes \mathbb{I}_3)(e_{11}\otimes M_u + e_{22}\otimes M_d)\|^2 , \|(\Phi(x)\otimes\mathbb{I}_3)(e_2\otimes M_e)\|^2 \right\}\\
&= \left(|1+h_1(x)|^2 + |h_2(x)|^2\right)\max\left\{ {m_t}^2, {m_\tau}^2\right\}\\
&= \left(|1+h_1(x)|^2 + |h_2(x)|^2\right) {m_t}^2.
\end{aligned}$$

The other distances, involving the pure states of $M_3(\mathbb{C})$, are not finite. Indeed,

$$\|[D_H, \pi_I(a,b,c)]\| = \left\| \left[\begin{pmatrix} 0 & \Phi M \\ M^*\Phi^* & 0\end{pmatrix}, \pi^P(a,b)\right]\right\|$$

does not put any constraint on c, thus for $\omega_2 \in \mathcal{P}\left(M_3(\mathbb{C})\right)$ and $\omega \in \mathcal{P}(\mathcal{A}_I)$,

$$d_I(\omega_2, \omega) \geq \sup_{c\in M_3(\mathbb{C})} |\omega_2(c) - \omega_(c)|.$$

For $\omega = \omega_0$, $c = \lambda\mathbb{I}_3$ with $\lambda \to \infty$ makes the distance $d_I(\omega_2, \omega_0)$ infinite. Then

$$d_I(\omega_2,\omega_0) = d(x_2, x_0) \leq d(x_2, y_0) + d(y_0, x_0) \leq d(x_2, y_0) + L(x,y)$$

by Theorem 4', so that $d(x_2, y_0) = +\infty$. The same is true for $\omega = \omega_1$. The same is also true when $\omega \in \mathcal{P}(M_3(\mathbb{C}))$ because, by Lemma 7, there exists $c' \in ker(\omega_2)$, $c' \notin ker(\omega)$ which makes $d_I(\omega_2, \omega)$ infinite. ■

VIII Conclusion.

Noncommutative geometry intrinsically links the Higgs field with the metric structure of space-time. We have not considered the gauge field A_μ so it is not clear whether or not the interpretation of the Higgs as an extra metric component has a direct physical meaning. It is important to study the influence of the gauge fluctuation and, particularly, how it probably makes the metric of the strong interaction part finite.

Since $\mathbb{H}$ has only one pure state, the problem of the distance between states defined by distinct pure states of the same component of the internal algebra is not questioned here. One may be tempted to consider states τ of $\mathbb{H}$ that are not pure. But asking $\tau(\bar{q}) = \bar{\tau}(q)$ - which is part of the definition of a real state[15] and does not come as a consequence like in the complex case- precisely means that $\tau = \omega_1$. To extend the field of investigation, one can consider states that do not preserve the conjugation – then the supremum is no longer reached by a positive element– but this contradicts the spirit of density matrices in quantum mechanics. More interesting is probably to take into account complexified states, that is real linear functions with value in $\mathbb{C}$.

The reduction of $\mathcal{A}_I$ to $\mathbb{K}^2$ (Proposition 3) is made possible by the orthogonality of the projections. When the two internal pure states are no longer orthogonal, there is no reason why the relevant picture should remain the two-sheets model. The same is true for two orthogonal states whose sum of the projections does not commute with the Dirac operator. In this sense, if these cases do not support a simple "classical" picture (such as being the geodesic distance of a discrete Kaluza-Klein manifold), they reflect a purely noncommutative aspect of space-time.

Note that the result – before fluctuation – concerning states defined by the same pure state of one of the algebras (Theorem 2), as well as the reduction from $\mathcal{A}_I$ to $\mathbb{K}^2$, do not assume that $\mathcal{A}_E$ is Abelian. It is only later, to establish the orthogonality between the internal and the external spaces, that T_E is taken as the spectral triple of a manifold. It would be interesting to clarify the importance, or the unimportance, of the commutativity regarding Pythagorean theorem.

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