hep-th/0203141
TUW 02-05
UWThPh-2002-09
CERN-TH/2002-051

Non-commutative $U(1)$ Super-Yang–Mills Theory: Perturbative Self-Energy Corrections

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Abstract. The quantization of the non-commutative $\mathcal{N}=1$, $U(1)$ super-Yang–Mills action is performed. We calculate the one-loop corrections to the self-energy of the super vector field. Although the power-counting theorem predicts quadratic ultraviolet and infrared divergences, there are actually only logarithmic UV and IR divergences unless one chooses the Wess–Zumino gauge, for which the divergences are indeed quadratic. This could indicate that UV/IR mixing might be unphysical.

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Dedicated to Olivier Piguet on the occasion of his 60^{th} birthday

1 Introduction

We know that the concept of space-time as a differentiable manifold cannot be reasonably applied to extremely short distances [1]. Simple heuristic arguments show that it is impossible to locate a particle with arbitrarily small uncertainty [2]. An interesting concept in order to replace standard differential geometry is *non-commutative geometry* pioneered by Connes [3, 4]. Non-commutative geometry can be regarded as an extension of the principles of quantum mechanics to geometry itself: space-time coordinates become non-commutative operators.

The general strategy in non-commutative geometry is to generalize the mathematical structures encountered in ordinary physics. Standard quantum field theories deal with problems of interactions at short distances. Quantum field theory (QFT) on spaces with different short-distance structure may therefore show interesting features. Since singularities in standard QFT are a consequence of point-like interactions, there has been hope that 'smearing out the points' [5] avoids these UV divergences. However, it was first noticed by Filk [6] that divergences are not avoided on non-commutative $\mathbb{R}^4$. This raised the question of whether the QFT is renormalizable, or not. On the one-loop level this was affirmed for Yang–Mills theory on non-commutative $\mathbb{R}^4$ [7] and the non-commutative 4-torus [8] as well as for supersymmetric Yang–Mills theory in (2+1) dimensions, with space being the non-commutative 2-torus [9]. QED on non-commutative $\mathbb{R}^4$ was treated in [10] and BF–Yang–Mills theory in [11]. The Chern–Simons model on non-commutative space was treated in [12], see also [13], and the Wess–Zumino model in [14].

Concerning supersymmetry, also a deformation of the anticommutator of the fermionic superspace coordinates was considered [15], but this deformation is not compatible with supertranslations and chiral fields. A superspace formulation (at the classical level) of the Wess–Zumino model and of super-Yang–Mills theory was given in [16]. Non-commutative $\mathcal{N} = 1, 2$ super-Yang–Mills theories were studied by Zanon in [17], using the background field method, with the result that at one loop there are only logarithmic divergences in the self-energy. This is remarkable because the power-counting theorem predicts *quadratic divergences* for $\mathcal{N} = 1$ super-Yang–Mills theory, which would lead, according to the power-counting analysis of non-commutative field theories by Chepelev and Roiban [18], to non-renormalizability on non-commutative space-time. The lowering of the degree of divergence from quadratic to logarithmic seems to be governed by non-renormalization theorems, see [19].

In this paper we reinvestigate the question of UV/IR mixing in non-commutative $\mathcal{N} = 1$ super-Yang–Mills theory. We work in the non-commutative superfield formalism [14], which allows us to easily switch between a general superfield and one in the Wess–Zumino gauge. It turns out that the one-loop self-energy of the superfield is *quadratically* IR-divergent in the Wess–Zumino gauge and *logarithmically* IR-divergent for a general superfield. UV divergences are multiplicatively renormalizable in both cases. Assuming that this behaviour continues to all orders, non-commutative $\mathcal{N} = 1$ super-Yang–Mills theory would be renormalizable, according to [18], when using general superfields (provided that commutants-type divergences are absent), and non-renormalizable when choosing the Wess–Zumino gauge. Since such a conclusion seems to be very unnatural, we rather interpret our result as an indication that UV/IR mixing is not physical after all: renormalization of non-commutative field theories should be possible also in the presence of quadratic IR divergences. Possible ways out could be hard non-commutative loops resummation [20] or the use of field redefinitions [21].

The paper is organized as follows: Section 2 presents the Moyal product applied to superfields, while section 3 treats the action of our model. In section 4 the Legendre transformation and the perturbative expansion are performed and, after a short power-counting argument given in section 5, the self-energy of the super-vector field is calculated at the one-loop level (section 6). Appendices contain some calculations and conventions.

2 Moyal Product for Superfields

We consider a non-commutative ($\mathcal{N}=1$) superspace characterized by the algebra

$$[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}, \tag{1}$$

where $\Theta^{\mu\nu}$ is an antisymmetric, constant and real matrix. We do not deform the anti-commuting coordinates θ_α and $\bar{\theta}^{\dot{\alpha}}$, i.e. we assume

$$\{\theta_\alpha, \theta_\beta\} = \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \{\theta_\alpha, \bar{\theta}^{\dot{\alpha}}\} = [\hat{x}^\mu, \theta_\alpha] = [\hat{x}^\mu, \bar{\theta}^{\dot{\alpha}}] = 0. \tag{2}$$

The non-commutative algebra is represented on an ordinary manifold by the Moyal product [6]. The Moyal product of two vector superfields can be written as [14]

$$\begin{aligned}(\phi \star \phi')(x, \theta_1, \bar{\theta}_1) = \int dP_{V2}\, dP_{V3}\, \tilde{\delta}_V(1,2)\tilde{\delta}_V(1,3) \\ \times\ \tilde{\phi}(p_2, \theta_2, \bar{\theta}_2)\tilde{\phi}'(p_3, \theta_3, \bar{\theta}_3) e^{-i(p_2+p_3)x} e^{-ip_2\wedge p_3}.\end{aligned} \tag{3}$$

The Moyal product has the important property

$$\int dV_1\, (\phi \star \phi')(1) = \int dV_1\, (\phi' \star \phi)(1) = \int dV_1\, \phi(1)\phi'(1). \tag{4}$$

This implies in particular that one can perform cyclic rotations of the fields under the integral.

For definiteness we have used vector superfields in (3) and (4). Of course, one can easily write down analogous formulae for (anti-)chiral superfields.

3 The Action

For simplicity we choose the gauge group $U(1)$. We introduce a vector superfield ϕ whose gauge transformation is given by [16]:

$$(e^{\phi'})_\star = (e^{-i\bar{\Lambda}})_\star \star (e^{\phi})_\star \star (e^{i\Lambda})_\star, \tag{5}$$

with a chiral superfield Λ (gauge parameter). With the help of the Baker–Campbell–Hausdorff formula we obtain the infinitesimal gauge transformation of ϕ itself:

$$\phi' = \phi + i(\Lambda - \bar{\Lambda}) + \frac{i}{2}[\phi, \Lambda + \bar{\Lambda}]_\star + \frac{i}{12}[\phi, [\phi, \Lambda - \bar{\Lambda}]_\star]_\star + \ldots, \tag{6}$$

where the dots denote terms that contain three or more powers of ϕ. The gauge-invariant NCSYM action is given by

$$S_{inv} = -\frac{1}{128g^2}\int dS\, W^\alpha W_\alpha, \tag{7}$$

with

$$W_\alpha := \bar{D}^2\left((e^{-\phi})_\star \star D_\alpha (e^\phi)_\star\right). \tag{8}$$

We perform a Taylor expansion of the integrand,

$$S_{inv} = -\frac{1}{128g^2}\int dV\Big[-\phi D^\alpha \bar{D}^2 D_\alpha \phi - (\bar{D}^2 D^\alpha \phi)[\phi, D_\alpha \phi]_\star \\ -\frac{1}{3}[\phi, \bar{D}^2 D^\alpha \phi]_\star [\phi, D_\alpha \phi]_\star + \frac{1}{4}[\phi, D^\alpha \phi]_\star \bar{D}^2 [\phi, D_\alpha \phi]_\star + \mathcal{O}(\phi^5)\Big]. \tag{9}$$

In order to prepare the quantization, we introduce a chiral superfield B (multiplier field) and two chiral anticommuting superfields c_+ (ghost) and c_- (antighost). The BRS transformations are given by:

$$\begin{aligned} s\phi &= c_+ - \bar{c}_+ + \frac{1}{2}[\phi, c_+ + \bar{c}_+]_\star + \frac{1}{12}[\phi,[\phi, c_+ - \bar{c}_+]_\star]_\star + \dots \\ &=: Q_s(\phi, c_+), \\ sc_+ &= -c_+ \star c_+, & s\bar{c}_+ &= -\bar{c}_+ \star \bar{c}_+, \\ sc_- &= B, & s\bar{c}_- &= \bar{B}, \\ sB &= 0, & s\bar{B} &= 0. \end{aligned} \tag{10}$$

Now we can write down the BRS-invariant total action:

$$S_{tot} = S_{inv} + S_{gf} + S_{\phi\pi}, \tag{11}$$

where S_{inv} is given by (9) and the gauge fixing and the Faddeev–Popov terms are given by [22]:

$$S_{gf} = -\frac{1}{128}\int dV (B + \bar{B})\phi, \tag{12}$$

$$S_{\phi\pi} = \frac{1}{128}\int dV (c_- + \bar{c}_-) Q_s(\phi, c_+). \tag{13}$$

Using (10), the Faddeev–Popov term can be rewritten as

$$S_{\phi\pi} = \frac{1}{128}\int dV \Bigg(\bar{c}_- c_+ - c_- \bar{c}_+ \\ + \Big(c_- + \bar{c}_-\Big)\Big(\frac{1}{2}[\phi, c_+ + \bar{c}_+]_\star + \frac{1}{12}[\phi,[\phi, c_+ - \bar{c}_+]_\star]_\star + \dots\Big)\Bigg). \tag{14}$$

Again the dots denote terms with three or more powers of ϕ.

In the Wess–Zumino gauge [23] we have $\phi^3 = 0$, and the infinitesimal gauge transformation is given by

$$\phi' = \phi + i(\Lambda - \bar{\Lambda}) + \frac{i}{2}[\phi, \Lambda + \bar{\Lambda}]_\star. \tag{15}$$

It is convenient to introduce a switch β for the Wess–Zumino gauge: $\beta = 1$ is the general case, $\beta = 0$ corresponds to the Wess–Zumino gauge. This leads to

$$\begin{aligned} S_{inv} &= -\frac{1}{128g^2}\int dV \Big[-\phi D^\alpha \bar{D}^2 D_\alpha \phi - (\bar{D}^2 D^\alpha \phi)[\phi, D_\alpha \phi]_\star \\ &\quad - \frac{\beta}{3}[\phi, \bar{D}^2 D^\alpha \phi]_\star [\phi, D_\alpha \phi]_\star + \frac{1}{4}[\phi, D^\alpha \phi]_\star \bar{D}^2 [\phi, D_\alpha \phi]_\star + \beta\ \mathcal{O}(\phi^5)\Big], \end{aligned} \tag{16}$$

$$S_{gf} = -\frac{1}{128}\int dV (B + \bar{B})\phi, \tag{17}$$

$$\begin{aligned} S_{\phi\pi} &= \frac{1}{128}\int dV \Bigg(\bar{c}_- c_+ - c_- \bar{c}_+ \\ &\quad + \Big(c_- + \bar{c}_-\Big)\Big(\frac{1}{2}[\phi, c_+ + \bar{c}_+]_\star + \frac{\beta}{12}[\phi, [\phi, c_+ - \bar{c}_+]_\star]_\star + \beta\ \mathcal{O}(\phi^3)\Big)\Bigg). \end{aligned} \tag{18}$$

In the following we will also include a mass term in the total action,

$$S_{mass} = \frac{1}{16g^2}\int dV M^2 \phi^2, \tag{19}$$

in order to avoid an IR divergence in the propagator of the vector superfield.

4 Generating Functionals

The generating functional of connected Green's functions for the free theory can be obtained from the bilinear part S_{bil} of $S_{tot} + S_{mass}$ via a Legendre transformation:

$$\begin{aligned} Z^c_{bil} &= S_{bil} + \int dV J\phi + \int dS \left(J_B B + \eta_- c_+ + \eta_+ c_-\right) \\ &\quad + \int d\bar{S}\left(J_{\bar{B}}\bar{B} + \bar{\eta}_- \bar{c}_+ + \bar{\eta}_+ \bar{c}_-\right) \\ &= S_{bil} + \int dP_V \tilde{J}_{-p}\tilde{\phi}_p + \int dP_S \left(\tilde{J}_{B,-p}\tilde{B}_p + \tilde{\eta}_{-,-p}\tilde{c}_{+,p} + \tilde{\eta}_{+,-p}\tilde{c}_{-,p}\right) \\ &\quad + \int dP_{\bar{S}}\left(\tilde{J}_{\bar{B},-p}\tilde{\bar{B}}_p + \tilde{\bar{\eta}}_{-,-p}\tilde{\bar{c}}_{+,p} + \tilde{\bar{\eta}}_{+,-p}\tilde{\bar{c}}_{-,p}\right), \end{aligned} \tag{20}$$

where ϕ, B, $\bar{B}$, $c_\pm$ and $\bar{c}_\pm$ are replaced by the inverse solutions of

$$
\begin{aligned}
&\frac{\delta_V S_{bil}}{\delta_V \tilde{\phi}_{-p}} = -\tilde{J}_p, && \\
&\frac{\delta_S S_{bil}}{\delta_S \tilde{B}_{-p}} = -\tilde{J}_{B,p}, && \frac{\delta_{\bar{S}} S_{bil}}{\delta_{\bar{S}} \tilde{\bar{B}}_{-p}} = -\tilde{J}_{\bar{B},p}, \\
&\frac{\delta_S S_{bil}}{\delta_S \tilde{c}_{-,-p}} = \tilde{\eta}_{+,p}, && \frac{\delta_S S_{bil}}{\delta_S \tilde{c}_{+,-p}} = \tilde{\eta}_{-,p}, \\
&\frac{\delta_{\bar{S}} S_{bil}}{\delta_{\bar{S}} \tilde{\bar{c}}_{-,-p}} = \tilde{\bar{\eta}}_{+,p}, && \frac{\delta_{\bar{S}} S_{bil}}{\delta_{\bar{S}} \tilde{\bar{c}}_{+,-p}} = \tilde{\bar{\eta}}_{-,p}.
\end{aligned} \tag{21}
$$

This leads to

$$
\begin{aligned}
Z^c_{bil} = &\int dP_{V1}\, dP_{V2}\, \frac{1}{2} \tilde{J}_{-p_1} \Delta_{\phi\phi}(1,2) \tilde{J}_{-p_2} + \int dP_{V1}\, dP_{S2}\, \tilde{J}_{-p_1} \Delta_{\phi B}(1,2) \tilde{J}_{B,-p_2} \\
&+ \int dP_{V1}\, dP_{\bar{S}2}\, \tilde{J}_{-p_1} \Delta_{\phi\bar{B}}(1,2) \tilde{J}_{\bar{B},-p_2} + \int dP_{S1}\, dP_{\bar{S}2}\, \tilde{J}_{B,-p_1} \Delta_{B\bar{B}}(1,2) \tilde{J}_{\bar{B},-p_2} \\
&- \int dP_{\bar{S}1}\, dP_{S2}\, \tilde{\bar{\eta}}_{-,-p_1} \Delta_{\bar{c}_+ c_-}(1,2) \tilde{\eta}_{+,-p_2} - \int dP_{\bar{S}1}\, dP_{S2}\, \tilde{\bar{\eta}}_{+,-p_1} \Delta_{\bar{c}_- c_+}(1,2) \tilde{\eta}_{-,-p_2},
\end{aligned} \tag{22}
$$

where the propagators are given by

$$
\begin{aligned}
\Delta_{\phi\phi}(1,2) &= -g^2 (2\pi)^4 \delta^4(p_1+p_2) e^{p_{2,\mu}(\theta_2\sigma^\mu\bar{\theta}_1 - \theta_1\sigma^\mu\bar{\theta}_2)} \frac{1 - \frac{1}{4}\theta_{21}^2 \bar{\theta}_{21}^2 p_2^2}{p_2^2(p_2^2 - M^2)}, \\
\Delta_{\phi B}(1,2) &= 8(2\pi)^4 \delta^4(p_1+p_2) e^{p_{2,\mu}(\theta_2\sigma^\mu\bar{\theta}_1 - \theta_1\sigma^\mu\bar{\theta}_2)} \frac{1 - (\theta_{21}\sigma^\rho\bar{\theta}_{21}) p_{2,\rho} + \frac{1}{4}\theta_{21}^2\bar{\theta}_{21}^2 p_2^2}{p_2^2}, \\
\Delta_{\phi\bar{B}}(1,2) &= 8(2\pi)^4 \delta^4(p_1+p_2) e^{p_{2,\mu}(\theta_2\sigma^\mu\bar{\theta}_1 - \theta_1\sigma^\mu\bar{\theta}_2)} \frac{1 + (\theta_{21}\sigma^\rho\bar{\theta}_{21}) p_{2,\rho} + \frac{1}{4}\theta_{21}^2\bar{\theta}_{21}^2 p_2^2}{p_2^2}, \\
\Delta_{\bar{c}_+ c_-}(1,2) &= \Delta_{\bar{c}_- c_+}(1,2) = \Delta_{\phi B}(1,2), \\
\Delta_{B\bar{B}}(1,2) &= \frac{16M^2}{g^2} \Delta_{\phi\bar{B}}.
\end{aligned} \tag{23}
$$

The generating functional of general Green's functions is given by

$$
Z = \mathcal{N}^{-1} e^{\frac{i}{\hbar}\Gamma_{int}} e^{\frac{i}{\hbar} Z^c_{bil}}, \tag{24}
$$

where $\mathcal{N}$ is a normalization factor such that $Z[0] = 1$ and

$$
\begin{aligned}
\Gamma_{int} = &\frac{1}{4!}\left(\frac{\hbar}{i}\right)^4 \int dP_{V1}\, dP_{V2}\, dP_{V3}\, dP_{V4}\, \Gamma_{\phi^4}(1,2,3,4) \frac{\delta_V^4}{\delta_V \tilde{J}_{-p_1} \delta_V \tilde{J}_{-p_2} \delta_V \tilde{J}_{-p_3} \delta_V \tilde{J}_{-p_4}} \\
&+ \frac{1}{3!}\left(\frac{\hbar}{i}\right)^3 \int dP_{V1}\, dP_{V2}\, dP_{V3}\, \Gamma_{\phi^3}(1,2,3) \frac{\delta_V^3}{\delta_V \tilde{J}_{-p_1} \delta_V \tilde{J}_{-p_2} \delta_V \tilde{J}_{-p_3}} \\
&+ \left(\frac{\hbar}{i}\right)^3 \int dP_{V1}\, dP_{V2}\, dP_{V3}\, \Gamma_{c_- c_+ \phi}(1,2,3) \frac{\delta_S^2 \delta_V}{\delta_S \tilde{\eta}_{-,-p_2} \delta_S \tilde{\eta}_{+,-p_1} \delta_V \tilde{J}_{-p_3}}
\end{aligned}
$$

$$+\left(\frac{\hbar}{i}\right)^3\int dP_{V1}\,dP_{V2}\,dP_{V3}\,\Gamma_{\bar{c}_-c_+\phi}(1,2,3)\frac{\delta_S\delta_{\bar{S}}\delta_V}{\delta_S\tilde{\eta}_{-,-p_2}\delta_{\bar{S}}\tilde{\bar{\eta}}_{+,-p_1}\delta_V\tilde{J}_{-p_3}}$$
$$+\left(\frac{\hbar}{i}\right)^3\int dP_{V1}\,dP_{V2}\,dP_{V3}\,\Gamma_{c_-\bar{c}_+\phi}(1,2,3)\frac{\delta_{\bar{S}}\delta_S\delta_V}{\delta_{\bar{S}}\tilde{\bar{\eta}}_{-,-p_2}\delta_S\tilde{\eta}_{+,-p_1}\delta_V\tilde{J}_{-p_3}}$$
$$+\left(\frac{\hbar}{i}\right)^3\int dP_{V1}\,dP_{V2}\,dP_{V3}\,\Gamma_{\bar{c}_-\bar{c}_+\phi}(1,2,3)\frac{\delta_{\bar{S}}^2\delta_V}{\delta_{\bar{S}}\tilde{\bar{\eta}}_{-,-p_2}\delta_{\bar{S}}\tilde{\bar{\eta}}_{+,-p_1}\delta_V\tilde{J}_{-p_3}}$$
$$+\frac{1}{2!}\left(\frac{\hbar}{i}\right)^4\int dP_{V1}\,dP_{V2}\,dP_{V3}\,dP_{V4}\,\Gamma_{c_-c_+\phi^2}(1,2,3,4)\frac{\delta_S^2\delta_V^2}{\delta_S\tilde{\eta}_{-,-p_2}\delta_S\tilde{\eta}_{+,-p_1}\delta_V\tilde{J}_{-p_3}\delta_V\tilde{J}_{-p_4}}$$
$$+\frac{1}{2!}\left(\frac{\hbar}{i}\right)^4\int dP_{V1}\,dP_{V2}\,dP_{V3}\,dP_{V4}\,\Gamma_{\bar{c}_-c_+\phi^2}(1,2,3,4)\frac{\delta_S\delta_{\bar{S}}\delta_V^2}{\delta_S\tilde{\eta}_{-,-p_2}\delta_{\bar{S}}\tilde{\bar{\eta}}_{+,-p_1}\delta_V\tilde{J}_{-p_3}\delta_V\tilde{J}_{-p_4}}$$
$$+\frac{1}{2!}\left(\frac{\hbar}{i}\right)^4\int dP_{V1}\,dP_{V2}\,dP_{V3}\,dP_{V4}\,\Gamma_{c_-\bar{c}_+\phi^2}(1,2,3,4)\frac{\delta_{\bar{S}}\delta_S\delta_V^2}{\delta_{\bar{S}}\tilde{\bar{\eta}}_{-,-p_2}\delta_S\tilde{\eta}_{+,-p_1}\delta_V\tilde{J}_{-p_3}\delta_V\tilde{J}_{-p_4}}$$
$$+\frac{1}{2!}\left(\frac{\hbar}{i}\right)^4\int dP_{V1}\,dP_{V2}\,dP_{V3}\,dP_{V4}\,\Gamma_{\bar{c}_-\bar{c}_+\phi^2}(1,2,3,4)\frac{\delta_{\bar{S}}^2\delta_V^2}{\delta_{\bar{S}}\tilde{\bar{\eta}}_{-,-p_2}\delta_{\bar{S}}\tilde{\bar{\eta}}_{+,-p_1}\delta_V\tilde{J}_{-p_3}\delta_V\tilde{J}_{-p_4}}, \tag{25}$$

where we have defined

$$\Gamma_{\phi^3}(1,2,3)=\frac{\delta_V^3 S_{int}}{\delta_V\tilde{\phi}_{p_1}\delta_V\tilde{\phi}_{p_2}\delta_V\tilde{\phi}_{p_3}}\bigg|_0,$$
$$\Gamma_{\phi^4}(1,2,3,4)=\frac{\delta_V^4 S_{int}}{\delta_V\tilde{\phi}_{p_1}\delta_V\tilde{\phi}_{p_2}\delta_V\tilde{\phi}_{p_3}\delta_V\tilde{\phi}_{p_4}}\bigg|_0,$$
$$\Gamma_{c_-c_+\phi}(1,2,3)=\frac{\delta_V^3 S_{int}}{\delta_V\tilde{c}_{-,p_1}\delta_V\tilde{c}_{+,p_2}\delta_V\tilde{\phi}_{p_3}}\bigg|_0,$$
$$\Gamma_{c_-c_+\phi^2}(1,2,3,4)=\frac{\delta_V^4 S_{int}}{\delta_V\tilde{c}_{-,p_1}\delta_V\tilde{c}_{+,p_2}\delta_V\tilde{\phi}_{p_3}\delta_V\tilde{\phi}_{p_4}}\bigg|_0, \tag{26}$$

and similarly for $\Gamma_{\bar{c}_-c_+\phi}(1,2,3)$, $\Gamma_{c_-\bar{c}_+\phi}(1,2,3)$, $\Gamma_{\bar{c}_-\bar{c}_+\phi}(1,2,3)$, $\Gamma_{\bar{c}_-c_+\phi^2}(1,2,3,4)$, $\Gamma_{c_-\bar{c}_+\phi^2}(1,2,3,4)$ and $\Gamma_{\bar{c}_-\bar{c}_+\phi^2}(1,2,3,4)$. Here, S_{int} is the interaction part of S_{tot}, and the subscript 0 means that all the fields have to be set to zero after the functional derivatives have been performed. The mixture of (anti-)chiral and vectorial field derivatives in the ghost sector is due to our convention that source terms for the ghosts involve the (anti-)chiral measure (20), whereas interactions between ghosts and vector superfields are defined in terms of the vectorial measure (18). One should notice here that these are all necessary vertices for the calculation of the one-loop self-energy part of the vector superfield. The final results for these vertex functions (26) are rather complicated and can be looked up in appendix A.

Furthermore, we mention the generating functional of connected Green's functions, which is given by

$$Z^c=\frac{\hbar}{i}\ln Z. \tag{27}$$

5 Power Counting

We note that, apart from the exponentials and the θ-factors in the numerator, the vector field propagators are of order $\frac{1}{(p^2)^2}$ and the ghost propagators of order $\frac{1}{p^2}$. We consider the exponentials and the θ-factors. From the invariance of Green's functions with respect to translations and supersymmetry transformations one finds that a one-particle irreducible Green's function in momentum space can always be written as [22]:

$$\begin{aligned}\tilde{\Gamma}(1,\ldots,n) &= \frac{\delta^n\Gamma[\phi_i]}{\delta\tilde{\phi}(p_1)\ldots\delta\tilde{\phi}(p_n)} \\ &= (2\pi)^4\delta^4(\sum_{j=1}^{n}p_j)e^{-\sum_{i=2}^{n}p_{i,\mu}(\theta_i\sigma^\mu\bar{\theta}_1-\theta_1\sigma^\mu\bar{\theta}_i)}\tilde{f}(-p_2,...,-p_n,\theta_{i1},\bar{\theta}_{i1}). \qquad (28)\end{aligned}$$

This general structure is true in particular for propagators and vertices. Thus, the general structure of the integrand of the superspace integral corresponding to an arbitrary Feynman graph is

$$I = \exp\underbrace{\Big(-\sum_{i\leq j}\sum_{\tau} l_{ij,\tau,\mu}(\theta_i\sigma^\mu\bar{\theta}_j - \theta_j\sigma^\mu\bar{\theta}_i)\Big)}_{E(p,k,\theta_l,\bar{\theta}_l)}\prod_{i\leq j}\prod_{\tau}\tilde{f}_{ij,\tau}(-l_{ij,\tau},\theta_{ij},\bar{\theta}_{ij}). \qquad (29)$$

Here, $l_{ij,\tau}$ is the momentum running from point i to point j, and τ counts momenta running between the same pair of points. We have chosen a basis for $(l_{ij,\tau}) = (p,k)$, where p and k are the external and internal momenta, respectively. With momentum conservation,

$$\sum_{j,\tau} l_{ij,\tau} = p_i, \qquad (30)$$

we find

$$E(p,k,\theta_l,\bar{\theta}_l) = E(p,k,\theta_{l1},\bar{\theta}_{l1}) - \sum_i(\theta_i\sigma^\mu\bar{\theta}_1 - \theta_1\sigma^\mu\bar{\theta}_i)p_{i,\mu}. \qquad (31)$$

Therefore, the exponentials appearing in the formulae for the propagators and vertices can be rewritten as

$$e^{k_\mu(\theta_i\sigma^\mu\bar{\theta}_j-\theta_j\sigma^\mu\bar{\theta}_i)} \Rightarrow e^{k_\mu(\theta_{i1}\sigma^\mu\bar{\theta}_{j1}-\theta_{j1}\sigma^\mu\bar{\theta}_{i1})}, \qquad (32)$$

if and only if k is an internal momentum. A Taylor expansion of the exponentials shows that from the θ-factors and the exponential we will at most get terms like $\theta_{i1}^2\bar{\theta}_{i1}^2k^2$. The highest power of k^2 that can appear is just the number of independent differences θ_{ij} (with $j=1$ in the calculation above) that can be constructed, which is exactly $n-1$, n being the number of vertices. So we find for the superficial divergence degree of an 1PI-graph

$$d(\Gamma) = 4L - 2G - 4V + 2(n_G + n_V - 1) + 2n_V. \qquad (33)$$

Here, L is the number of loop integrations, G and V are the numbers of ghost and vector superfield propagators, respectively, n_G and n_V count the ghost-vector superfield and the pure-vector superfield vertices. The last term, $2n_V$, has to be included in (33) because of the four covariant derivatives that appear in the parts of the Lagrangian corresponding to the three and four vertices of the vector superfield.

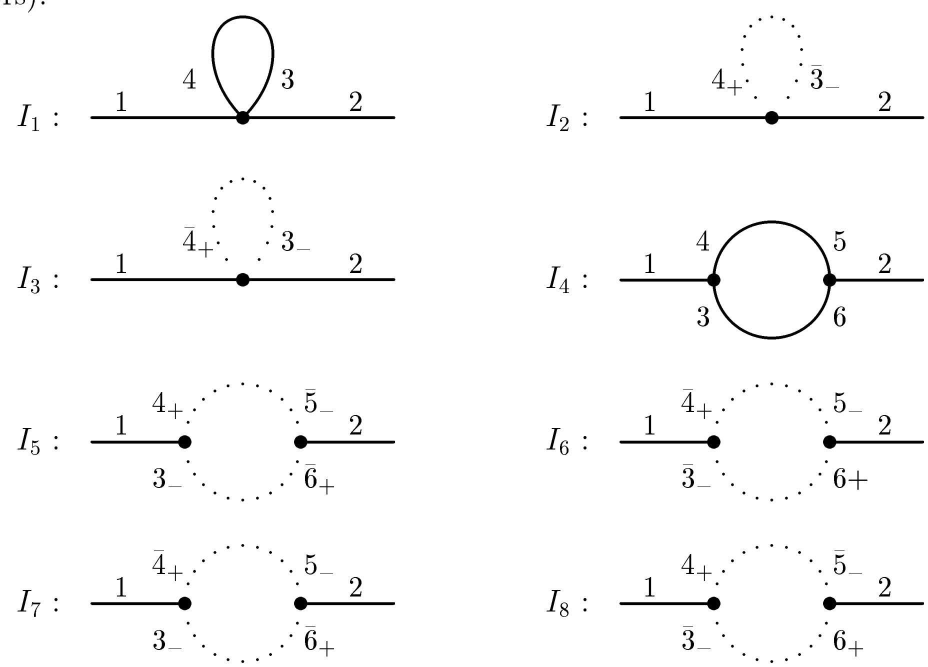
Using the topological relation $L = G + V - n_G - n_V + 1$ and charge conservation for the ghost fields, $2n_G = 2G + N_G$ (N_G being the number of external ghost fields), we find

$$d(\Gamma) = 2 - N_G. \tag{34}$$

For the vector superfield self-energy ($N_G = 0$), this means a superficial degree of divergence of 2.

6 Self-Energy of the Vector Superfield

The following Feynman graphs contribute to the self-energy of the vector superfield at the one-loop level (continuous lines denote vector superfield propagators, dotted lines ghost propagators):



(35)

From the generating functional (27) we obtain the following integrals corresponding to these graphs (after amputation of the external lines):

$$I_1 = \frac{\hbar}{2i}\int dP_{V3}dP_{V4}\,\Gamma_{\phi^4}(3,4,1,2)\Delta_{\phi\phi}(3,4),$$
$$I_2 = -\frac{\hbar}{i}\int dP_{V3}dP_{V4}\,\Gamma_{\bar{c}_-c_+\phi^2}(3,4,1,2)\Delta_{\bar{c}_-c_+}(3,4),$$
$$I_3 = -\frac{\hbar}{i}\int dP_{V3}dP_{V4}\,\Gamma_{c_-\bar{c}_+\phi^2}(3,4,1,2)\Delta_{c_-\bar{c}_+}(3,4),$$
$$I_4 = \frac{\hbar}{2i}\int dP_{V3}dP_{V4}dP_{V5}dP_{V6}\,\Gamma_{\phi^3}(3,4,1)\Delta_{\phi\phi}(3,6)\Gamma_{\phi^3}(5,6,2)\Delta_{\phi\phi}(5,4),$$
$$I_5 = -\frac{\hbar}{i}\int dP_{V3}dP_{V4}dP_{V5}dP_{V6}\,\Gamma_{c_-c_+\phi}(3,4,1)\Delta_{c_-\bar{c}_+}(3,6)\Gamma_{\bar{c}_-\bar{c}_+\phi}(5,6,2)\Delta_{\bar{c}_-c_+}(5,4),$$

$$I_6 = -\frac{\hbar}{i}\int dP_{V3}dP_{V4}dP_{V5}dP_{V6}\,\Gamma_{\bar{c}_-\bar{c}_+\phi}(3,4,1)\Delta_{\bar{c}_-c_+}(3,6)\Gamma_{c_-c_+\phi}(5,6,2)\Delta_{c_-\bar{c}_+}(5,4),$$
$$I_7 = -\frac{\hbar}{i}\int dP_{V3}dP_{V4}dP_{V5}dP_{V6}\,\Gamma_{c_-\bar{c}_+\phi}(3,4,1)\Delta_{c_-\bar{c}_+}(3,6)\Gamma_{c_-\bar{c}_+\phi}(5,6,2)\Delta_{c_-\bar{c}_+}(5,4),$$
$$I_8 = -\frac{\hbar}{i}\int dP_{V3}dP_{V4}dP_{V5}dP_{V6}\,\Gamma_{\bar{c}_-c_+\phi}(3,4,1)\Delta_{\bar{c}_-c_+}(3,6)\Gamma_{\bar{c}_-c_+\phi}(5,6,2)\Delta_{\bar{c}_-c_+}(5,4). \quad (36)$$

(Note that $\Delta_{c_-\bar{c}_+}(3,6) = -\Delta_{\bar{c}_+c_-}(6,3)$.) We now insert the explicit expressions for the propagators and vertices into the eight integrals in (36). After some lengthy simplifications of the integrands (see appendix A) we arrive at

$$I_1 = \frac{\hbar}{128i}\delta^4(p_1+p_2)e^{-p_{1,\mu}(\theta_1\sigma^\mu\bar{\theta}_2-\theta_2\sigma^\mu\bar{\theta}_1)}\int d^4k\,\sin^2(p_1\wedge k)\frac{1}{k^2(k^2-M^2)}$$
$$\times\left(\left(2-\frac{8\beta}{3}\right)-\frac{1}{2}\theta_{12}^2\bar{\theta}_{12}^2\left(1-\frac{4\beta}{3}\right)(2k^2+p_1^2)\right), \quad (37)$$
$$I_2 = \frac{\hbar\beta}{384i}\delta^4(p_1+p_2)e^{-p_{1,\mu}(\theta_1\sigma^\mu\bar{\theta}_2-\theta_2\sigma^\mu\bar{\theta}_1)}\int d^4k\,\sin^2(p_1\wedge k)\frac{1}{k^2}\theta_{12}^2\bar{\theta}_{12}^2, \quad (38)$$
$$I_3 = \frac{\hbar\beta}{384i}\delta^4(p_1+p_2)e^{-p_{1,\mu}(\theta_1\sigma^\mu\bar{\theta}_2-\theta_2\sigma^\mu\bar{\theta}_1)}\int d^4k\,\sin^2(p_1\wedge k)\frac{1}{k^2}\theta_{12}^2\bar{\theta}_{12}^2, \quad (39)$$
$$I_4 = \frac{\hbar}{128i}\delta^4(p_1+p_2)e^{-p_{1,\mu}(\theta_1\sigma^\mu\bar{\theta}_2-\theta_2\sigma^\mu\bar{\theta}_1)}\int d^4k\,\sin^2(p_1\wedge k)$$
$$\times\frac{1}{k^2(k^2-M^2)(k+p_1)^2((k+p_1)^2-M^2)}$$
$$\times\Big(-4(k^2)^2-8k^2(kp_1)-5k^2p_1^2-p_1^2(kp_1)-\frac{1}{4}\theta_{12}^2\bar{\theta}_{12}^2\Big(-4(k^2)^2p_1^2$$
$$+8k^2(kp_1)^2+8(kp_1)^3+8(kp_1)^2p_1^2+(kp_1)(p_1^2)^2-3(p_1^2)^2k^2\Big)\Big), \quad (40)$$
$$I_5 = \frac{\hbar}{256i}\delta^4(p_1+p_2)e^{-p_{1,\mu}(\theta_1\sigma^\mu\bar{\theta}_2-\theta_2\sigma^\mu\bar{\theta}_1)}\left(1-(\theta_{12}\sigma^\rho\bar{\theta}_{12})p_{1,\rho}+\frac{1}{4}\theta_{12}^2\bar{\theta}_{12}^2p_1^2\right)$$
$$\times\int d^4k\,\sin^2(p_1\wedge k)\frac{1}{k^2(p_1+k)^2}, \quad (41)$$
$$I_6 = \frac{\hbar}{256i}\delta^4(p_1+p_2)e^{-p_{1,\mu}(\theta_1\sigma^\mu\bar{\theta}_2-\theta_2\sigma^\mu\bar{\theta}_1)}\left(1+(\theta_{12}\sigma^\rho\bar{\theta}_{12})p_{1,\rho}+\frac{1}{4}\theta_{12}^2\bar{\theta}_{12}^2p_1^2\right)$$
$$\times\int d^4k\,\sin^2(p_1\wedge k)\frac{1}{k^2(p_1+k)^2}, \quad (42)$$
$$I_7 = -\frac{\hbar}{256i}\delta^4(p_1+p_2)e^{-p_{1,\mu}(\theta_1\sigma^\mu\bar{\theta}_2-\theta_2\sigma^\mu\bar{\theta}_1)}\int d^4k\,\sin^2(p_1\wedge k)\frac{1}{k^2(p_1+k)^2}$$
$$\times\left(1-(\theta_{12}\sigma^\rho\bar{\theta}_{12})(p_1+2k)_\rho+\theta_{12}^2\bar{\theta}_{12}^2\left(\frac{1}{4}p_1^2+p_1k+k^2\right)\right), \quad (43)$$
$$I_8 = -\frac{\hbar}{256i}\delta^4(p_1+p_2)e^{-p_{1,\mu}(\theta_1\sigma^\mu\bar{\theta}_2-\theta_2\sigma^\mu\bar{\theta}_1)}\int d^4k\,\sin^2(p_1\wedge k)\frac{1}{k^2(p_1+k)^2}$$
$$\times\left(1+(\theta_{12}\sigma^\rho\bar{\theta}_{12})(p_1+2k)_\rho+\theta_{12}^2\bar{\theta}_{12}^2\left(\frac{1}{4}p_1^2+p_1k+k^2\right)\right). \quad (44)$$

This gives up to terms in the integrand, which evaluate for $M \neq 0$ to finite quantities

$$\sum_{i=1}^{8} I_i = \frac{\hbar}{128i}\delta^4(p_1+p_2)e^{-p_{1,\mu}(\theta_1\sigma^\mu\bar\theta_2-\theta_2\sigma^\mu\bar\theta_1)}\int d^4k\,\frac{\sin^2(p_1\wedge k)}{k^2(k+p_1)^2}\Big((-2-\frac{8\beta}{3}) - \frac{1}{4}\theta_{12}^2\bar\theta_{12}^2\Big(8(1-\beta)k^2+(12-16\beta)kp_1+(2-\frac{32\beta}{3})p_1^2+\frac{8(kp_1)^2}{k^2-M^2}\Big)\Big). \qquad (45)$$

As usual we write $\sin^2(p_1\wedge k) = \frac{1}{2}-\frac{1}{2}\cos(2p_1\wedge k)$ and refer to the part corresponding to $\frac{1}{2}$ as 'planar' and the part corresponding to $\frac{1}{2}\cos(2p_1\wedge k)$ as 'non-planar'. The planar part of (45) is UV-divergent and evaluated in dimensional or analytic regularization to

$$\Big(\sum_{i=1}^{8} I_i\Big)^{reg}_{planar} = -\Big(1+\frac{4\beta}{3}\Big)\frac{\hbar\pi^2}{128\varepsilon}\,\delta^4(p_1+p_2)e^{-p_{1,\mu}(\theta_1\sigma^\mu\bar\theta_2-\theta_2\sigma^\mu\bar\theta_1)}\Big(1-\frac{1}{4}\theta_{12}^2\bar\theta_{12}^2p_1^2\Big)+\mathcal{O}(1)\,. \qquad (46)$$

This means that the divergence in the planar part of the self-energy is transversal, so that it can be removed by multiplicative renormalization. Because of the oscillating integrand, the non-planar part of (45) turns out to be finite for $p_1 \neq 0$ and is evaluated to (with $\tilde{p}_1^\mu := \Theta^{\mu\nu}p_{1,\nu}$):

$$\Big(\sum_{i=1}^{8} I_i\Big)_{non\text{-}planar} = \frac{\hbar\pi^2}{64}\delta^4(p_1+p_2)\,e^{-p_{1,\mu}(\theta_1\sigma^\mu\bar\theta_2-\theta_2\sigma^\mu\bar\theta_1)}\Big(\frac{2(1-\beta)}{\tilde{p_1}^2}\theta_{12}^2\bar\theta_{12}^2 + \Big(1+\frac{4\beta}{3}\Big)\Big(1-\frac{1}{4}\theta_{12}^2\bar\theta_{12}^2p_1^2\Big)\int_0^1 dx\,K_0\Big(\sqrt{x(1-x)p_1^2\tilde{p_1}^2}\Big)\Big)+\mathcal{O}(1)\,, \quad (47)$$

where $\mathcal{O}(1)$ in (47) collects terms that are regular for $p_1 \to 0$, and $K_0(y) = \log 2 - \log y - \gamma + \mathcal{O}(y^2)$ is the modified Bessel function of second kind (with Euler's $\gamma = 0.577...$). Thus, the non-planar part of the self-energy is quadratically divergent for $p_1 \to 0$ when choosing the Wess–Zumino gauge $\beta = 0$, but only logarithmically divergent for a general superfield, which corresponds to $\beta = 1$.

7 Conclusions

We have computed the one-loop self-energy of the superfield in non-commutative $\mathcal{N} = 1$, $U(1)$ super-Yang–Mills theory, for a general superfield and one in the Wess–Zumino gauge. Our results can be summarized as follows:

- UV divergences can be renormalized independently of the choice of Wess–Zumino gauge or not.
- If we do not use the Wess–Zumino gauge, the one-loop self-energy is only logarithmically IR-divergent (confirming the result of [17]).
- In the Wess–Zumino gauge, the one-loop self-energy is quadratically IR-divergent.

Quadratic IR divergences at one loop make a non-commutative field theory non-renormalizable at higher loop order. If there are only logarithmic divergences, a non-commutative field theory is, according to [18], power-counting renormalizable (assuming there is no problem with commutants). This would imply that non-commutative $\mathcal{N}=1$ super-Yang–Mills theory is renormalizable, unless one chooses the Wess–Zumino gauge. As renormalizability should be gauge-independent, our results seem to indicate that models showing UV/IR mixing could nevertheless be renormalizable.

A Calculations

A.1 ϕ^4-vertex

Let us compute the ϕ^4-vertex given in (26), $\Gamma_{\phi^4}(1,2,3,4) = \dfrac{\delta_V^4 S_{\phi^4}}{\delta_V\tilde{\phi}_{p_1}\delta_V\tilde{\phi}_{p_2}\delta_V\tilde{\phi}_{p_3}\delta_V\tilde{\phi}_{p_4}}$, where S_{ϕ^4} denotes the part of the total action which is quartic in ϕ:

$$S_{\phi^4} = -\frac{1}{128g^2}\int dV\left(\frac{1}{4}[\bar{D}^2\phi, D^\alpha\phi]_\star + \left(\frac{1}{4}-\frac{\beta}{3}\right)[\phi, \bar{D}^2 D^\alpha\phi]_\star \right.$$
$$\left. + \frac{1}{2}[\bar{D}_{\dot\alpha}\phi, \bar{D}^{\dot\alpha}D^\alpha\phi]_\star\right)[\phi, D_\alpha\phi]_\star. \qquad (A.1)$$

Using the definition of the Moyal product (3) and integrating by parts several times, this expression can be written as

$$S_{\phi^4} = -\frac{1}{256g^2}\int dP_{V5}\,dP_{V6}\,dP_{V7}\,dP_{V8}\,(2\pi)^4\delta^4(p_5+p_6+p_7+p_8)e^{-ip_5\wedge p_6 - ip_7\wedge p_8}$$
$$\times\,\tilde{\phi}_{p_5}\tilde{\phi}_{p_6}\tilde{\phi}_{p_7}\tilde{\phi}_{p_8}\mathcal{V}(5,6,7,8)\tilde{\delta}_V(5,6)\tilde{\delta}_V(5,7)\tilde{\delta}_V(5,8), \qquad (A.2)$$

where the differential operator $\mathcal{V}$ is given by

$$\begin{aligned}\mathcal{V}(5,6,7,8) &\\ = \left(\tilde{D}^\alpha_{8,-p_8} - \tilde{D}^\alpha_{7,-p_7}\right)\tilde{D}_{\alpha,6,-p_6}&\left[\frac{1}{2}\tilde{\bar{D}}^2_{5,-p_5} + \left(\frac{1}{2}-\frac{2\beta}{3}\right)\tilde{\bar{D}}^2_{6,-p_6} + \tilde{\bar{D}}_{\dot\alpha,6,-p_6}\tilde{\bar{D}}^{\dot\alpha}_{5,-p_5}\right]\\ + \left(\tilde{D}^\alpha_{7,-p_7} - \tilde{D}^\alpha_{8,-p_8}\right)\tilde{D}_{\alpha,5,-p_5}&\left[\frac{1}{2}\tilde{\bar{D}}^2_{6,-p_6} + \left(\frac{1}{2}-\frac{2\beta}{3}\right)\tilde{\bar{D}}^2_{5,-p_5} + \tilde{\bar{D}}_{\dot\alpha,5,-p_5}\tilde{\bar{D}}^{\dot\alpha}_{6,-p_6}\right].\end{aligned} \qquad (A.3)$$

We notice that $\mathcal{V}$ has the following symmetry properties:

$$\mathcal{V}(5,6,7,8) = -\mathcal{V}(6,5,7,8) = -\mathcal{V}(5,6,8,7) = \mathcal{V}(6,5,8,7). \qquad (A.4)$$

Using these symmetries we can write Γ_{ϕ^4} as

$$\begin{aligned}\Gamma_{\phi^4}(1,2,3,4) = & \frac{1}{64g^2}(2\pi)^4\delta^4(p_1+p_2+p_3+p_4) \\ & \times \Big[\sin(p_1\wedge p_2)\sin(p_3\wedge p_4)\left(\mathcal{V}(1,2,3,4)+\mathcal{V}(3,4,1,2)\right) \\ & \quad + \sin(p_1\wedge p_3)\sin(p_4\wedge p_2)\left(\mathcal{V}(1,3,4,2)+\mathcal{V}(4,2,1,3)\right) \\ & \quad + \sin(p_1\wedge p_4)\sin(p_2\wedge p_3)\left(\mathcal{V}(1,4,2,3)+\mathcal{V}(2,3,1,4)\right)\Big] \\ & \times \tilde{\delta}_V(1,2)\tilde{\delta}_V(1,3)\tilde{\delta}_V(1,4). \end{aligned} \tag{A.5}$$

In order to simplify this expression we have to evaluate terms like

$$A(1,2,3,4) := \tilde{D}^{\alpha}_{4,-p_4}\tilde{D}_{\alpha,2,-p_2}\tilde{\bar{D}}_{\dot{\alpha},2,-p_2}\tilde{\bar{D}}^{\dot{\alpha}}_{1,-p_1}\tilde{\delta}_V(1,2)\tilde{\delta}_V(1,3)\tilde{\delta}_V(1,4). \tag{A.6}$$

This can be done most easily if we first insert exponentials in the following way:

$$\begin{aligned} A(1,2,3,4) = & \left(\tilde{D}^{\alpha}_{4,-p_4} e^{p_{4,\mu}(\theta_3\sigma^\mu\bar{\theta}_4-\theta_4\sigma^\mu\bar{\theta}_3)}\tilde{\delta}_V(4,3)\right) \\ \times & \left(\tilde{D}_{\alpha,2,-p_2}\tilde{\bar{D}}_{\dot{\alpha},2,-p_2} e^{p_{2,\nu}(\theta_3\sigma^\nu\bar{\theta}_2-\theta_2\sigma^\nu\bar{\theta}_3)}\tilde{\delta}_V(2,3)\right)\left(\tilde{\bar{D}}^{\dot{\alpha}}_{1,-p_1} e^{p_{1,\rho}(\theta_3\sigma^\rho\bar{\theta}_1-\theta_1\sigma^\rho\bar{\theta}_3)}\tilde{\delta}_V(1,3)\right), \end{aligned} \tag{A.7}$$

which is allowed because of $\theta^3 = \bar{\theta}^3 = 0$. We now use the identities

$$\tilde{D}_{\alpha,i,p} e^{p_\mu(\theta_i\sigma^\mu\bar{\theta}_j-\theta_j\sigma^\mu\bar{\theta}_i)} = e^{p_\mu(\theta_i\sigma^\mu\bar{\theta}_j-\theta_j\sigma^\mu\bar{\theta}_i)}\tilde{D}_{\alpha,ij,p}, \tag{A.8}$$

$$\tilde{\bar{D}}_{\dot{\alpha},i,p} e^{p_\mu(\theta_i\sigma^\mu\bar{\theta}_j-\theta_j\sigma^\mu\bar{\theta}_i)} = e^{p_\mu(\theta_i\sigma^\mu\bar{\theta}_j-\theta_j\sigma^\mu\bar{\theta}_i)}\tilde{\bar{D}}_{\dot{\alpha},ij,p}, \tag{A.9}$$

which can easily be verified, leading to

$$A(1,2,3,4) = e^{E_3^{(4)}}\left(\tilde{D}^{\alpha}_{43,-p_4}\tilde{\delta}_V(4,3)\right)\left(\tilde{D}_{\alpha,23,-p_2}\tilde{\bar{D}}_{\dot{\alpha},23,-p_2}\tilde{\delta}_V(2,3)\right)\left(\tilde{\bar{D}}^{\dot{\alpha}}_{13,-p_1}\tilde{\delta}_V(1,3)\right), \tag{A.10}$$

where we have used the notation (B.11). We can readily evaluate the covariant derivatives of the delta functions, which give

$$A(1,2,3,4) = \frac{1}{256}e^{E_3^{(4)}}\theta^{\alpha}_{43}\bar{\theta}^2_{43}\left(\theta_{23,\alpha}\bar{\theta}_{23,\dot{\alpha}} + \frac{1}{2}(\sigma^\mu\bar{\theta}_{23})_\alpha p_{2,\mu}\bar{\theta}_{23,\dot{\alpha}}\theta^2_{23}\right)\bar{\theta}^{\dot{\alpha}}_{13}\theta^2_{13}. \tag{A.11}$$

With the help of the Fierz identity (C.2), this expression can be rewritten as

$$A(1,2,3,4) = \frac{1}{256}e^{E_3^{(4)}-p_{2,\mu}(\theta_{23}\sigma^\mu\bar{\theta}_{23})}(\theta_{43}\theta_{23})\bar{\theta}^2_{43}(\bar{\theta}_{23}\bar{\theta}_{13})\theta^2_{13}. \tag{A.12}$$

By applying this procedure to all the terms in (A.5), we arrive at

$$\Gamma_{\phi^4}(1,2,3,4) = \frac{1}{64g^2}(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)$$
$$\times\Big[\sin(p_1\wedge p_2)\sin(p_3\wedge p_4)\Big(F(1,2,3,4)-F(2,1,3,4)-F(1,2,4,3)$$
$$+F(2,1,4,3)+F(3,4,1,2)-F(4,3,1,2)-F(3,4,2,1)+F(4,3,2,1)\Big)$$
$$+\sin(p_1\wedge p_3)\sin(p_4\wedge p_2)\Big(F(1,3,4,2)-F(3,1,4,2)-F(1,3,2,4)$$
$$+F(3,1,2,4)+F(4,2,1,3)-F(2,4,1,3)-F(4,2,3,1)+F(2,4,3,1)\Big)$$
$$+\sin(p_1\wedge p_4)\sin(p_2\wedge p_3)\Big(F(1,4,2,3)-F(4,1,2,3)-F(1,4,3,2)$$
$$+F(4,1,3,2)+F(2,3,1,4)-F(3,2,1,4)-F(2,3,4,1)+F(3,2,4,1)\Big)\Big], \tag{A.13}$$

where F is given by

$$F(i,j,k,l) = \frac{1}{256}e^{E_k^{(4)}-p_{j,\mu}(\theta_{jk}\sigma^\mu\bar\theta_{jk})}(\theta_{lk}\theta_{jk})\bar\theta_{lk}^2\theta_{ik}^2\left(-\frac{1}{2}\bar\theta_{ij}^2+\frac{2\beta}{3}\bar\theta_{ik}^2\right). \tag{A.14}$$

A.2 The other vertices

The other vertices can be evaluated in a similar way as the ϕ^4-vertex. Since the calculations are much easier than in the case of the ϕ^4-vertex, we only give the results:

$$\Gamma_{\phi^3}(1,2,3) = -\frac{i}{64g^2}(2\pi)^4\delta^4(p_1+p_2+p_3)\sin(p_2\wedge p_3)$$
$$\times\, e^{p_{2,\mu}(\theta_1\sigma^\mu\bar\theta_2-\theta_2\sigma^\mu\bar\theta_1)+p_{3,\mu}(\theta_1\sigma^\mu\bar\theta_3-\theta_3\sigma^\mu\bar\theta_1)}$$
$$\times\Big(\frac{1}{8}(\theta_{21}^2-\theta_{31}^2)(\bar\theta_{21}\bar\theta_{31})-\frac{1}{8}(\theta_{21}\theta_{31})(\bar\theta_{21}^2-\bar\theta_{31}^2)-\frac{1}{16}\theta_{21}^2\bar\theta_{31}^2$$
$$+\frac{1}{16}\theta_{31}^2\bar\theta_{21}^2-\frac{1}{4}(p_{2,\rho}\theta_{21}\sigma^\rho\bar\theta_{21}-p_{3,\rho}\theta_{31}\sigma^\rho\bar\theta_{31})(\theta_{21}\theta_{31})(\bar\theta_{21}\bar\theta_{31})\Big), \tag{A.15}$$

$$\Gamma_{c_-c_+\phi^2}(1,2,3,4) = -\frac{\beta}{384}(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\tilde\delta_V(1,2)\tilde\delta_V(1,3)\tilde\delta_V(1,4)$$
$$\times\Big(\sin(p_1\wedge p_4)\sin(p_2\wedge p_3)+\sin(p_1\wedge p_3)\sin(p_2\wedge p_4)\Big), \tag{A.16}$$

$$\Gamma_{\bar c_-\bar c_+\phi^2}(1,2,3,4) = -\Gamma_{\bar c_-c_+\phi^2}(1,2,3,4) = \Gamma_{c_-\bar c_+\phi^2}(1,2,3,4) = -\Gamma_{c_-c_+\phi^2}(1,2,3,4), \tag{A.17}$$

$$\Gamma_{c_-c_+\phi}(1,2,3) = -\frac{i}{128}\tilde\delta_V(1,2)\tilde\delta_V(1,3)(2\pi)^4\delta^4(p_1+p_2+p_3)\sin(p_2\wedge p_3), \tag{A.18}$$

$$\Gamma_{\bar c_-\bar c_+\phi}(1,2,3) = \Gamma_{\bar c_-c_+\phi}(1,2,3) = \Gamma_{c_-\bar c_+\phi}(1,2,3) = \Gamma_{c_-c_+\phi}(1,2,3). \tag{A.19}$$

A.3 The integrals (37)–(44)

As an example, we show how to compute I_7 in (36). Inserting the formulae for the propagators and vertices we obtain

$$\begin{aligned}
I_7 = &-\frac{\hbar}{i}\int dP_{V3}dP_{V4}dP_{V5}dP_{V6}\left(-\frac{i}{128}\right)\tilde{\delta}_V(3,1)\tilde{\delta}_V(4,1)(2\pi)^4\delta^4(p_3+p_4+p_1)\\
&\times\sin(p_4\wedge p_1)(-8)(2\pi)^4\delta^4(p_6+p_3)e^{p_{3,\mu}(\theta_3\sigma^\mu\bar{\theta}_6-\theta_6\sigma^\mu\bar{\theta}_3)}\\
&\times\frac{1-(\theta_{36}\sigma^\rho\bar{\theta}_{36})p_{3,\rho}+\frac{1}{4}p_3^2\theta_{36}^2\bar{\theta}_{36}^2}{p_3^2}\left(-\frac{i}{128}\right)\tilde{\delta}_V(5,2)\tilde{\delta}_V(6,2)(2\pi)^4\delta^4(p_5+p_6+p_2)\\
&\times\sin(p_6\wedge p_2)(-8)(2\pi)^4\delta^4(p_4+p_5)\,e^{p_{5,\nu}(\theta_5\sigma^\nu\bar{\theta}_4-\theta_4\sigma^\nu\bar{\theta}_5)}\frac{1-(\theta_{54}\sigma^\lambda\bar{\theta}_{54})p_{5,\lambda}+\frac{1}{4}p_5^2\theta_{54}^2\bar{\theta}_{54}^2}{p_5^2}\\
&=-\frac{\hbar}{256i}\delta^4(p_1+p_2)\int d^4k\,\partial_3^2\bar{\partial}_3^2\ldots\partial_6^2\bar{\partial}_6^2\Big[\sin^2(p_1\wedge k)\tilde{\delta}_V(3,1)\tilde{\delta}_V(4,1)\\
&\times\tilde{\delta}_V(5,2)\tilde{\delta}_V(6,2)\,e^{k_\mu(\theta_3\sigma^\mu\bar{\theta}_6-\theta_6\sigma^\mu\bar{\theta}_3)}e^{(p_1+k)_\nu(\theta_5\sigma^\nu\bar{\theta}_4-\theta_4\sigma^\nu\bar{\theta}_5)}\frac{1}{k^2(p_1+k)^2}\\
&\times\left(1-(\theta_{36}\sigma^\rho\bar{\theta}_{36})k_\rho+\frac{1}{4}k^2\theta_{36}^2\bar{\theta}_{36}^2\right)\left(1-(\theta_{54}\sigma^\lambda\bar{\theta}_{54})(p_1+k)_\lambda+\frac{1}{4}(p_1+k)^2\theta_{54}^2\bar{\theta}_{54}^2\right)\Big],
\end{aligned}\tag{A.20}$$

where we have carried out the p_4-, p_5- and p_6-integrations with the help of the delta functions and have renamed the remaining loop momentum $p_3 =: k$. As usual ∂_i^2 is a short-hand notation for $\frac{\partial}{\partial\theta_{i,\alpha}}\frac{\partial}{\partial\theta_i^\alpha}$. In the last three lines of (A.20) we can replace θ_3 and θ_4 with θ_1, and θ_5 and θ_6 with θ_2, because of the superspace delta functions that are contained in the integrand. Thus, we obtain

$$\begin{aligned}
I_7 = &-\frac{\hbar}{256i}\delta^4(p_1+p_2)e^{-p_{1,\mu}(\theta_1\sigma^\mu\bar{\theta}_2-\theta_2\sigma^\mu\bar{\theta}_1)}\int d^4k\,\sin^2(p_1\wedge k)\\
&\times\left[\partial_3^2\bar{\partial}_3^2\ldots\partial_6^2\bar{\partial}_6^2\tilde{\delta}_V(3,1)\tilde{\delta}_V(4,1)\tilde{\delta}_V(5,2)\tilde{\delta}_V(6,2)\right]\frac{1}{k^2(p_1+k)^2}\\
&\times\left(1-(\theta_{12}\sigma^\rho\bar{\theta}_{12})k_\rho+\frac{1}{4}k^2\theta_{12}^2\bar{\theta}_{12}^2\right)\left(1-(\theta_{12}\sigma^\lambda\bar{\theta}_{12})(p_1+k)_\lambda+\frac{1}{4}(p_1+k)^2\theta_{12}^2\bar{\theta}_{12}^2\right).
\end{aligned}\tag{A.21}$$

The term between the square brackets of this expression simply gives a factor 1. Applying the identity (C.1) in the last two lines of (A.21) we arrive at

$$\begin{aligned}
I_7 = &-\frac{\hbar}{256i}\delta^4(p_1+p_2)e^{-p_{1,\mu}(\theta_1\sigma^\mu\bar{\theta}_2-\theta_2\sigma^\mu\bar{\theta}_1)}\int d^4k\,\sin^2(p_1\wedge k)\frac{1}{k^2(p_1+k)^2}\\
&\times\left(1-(\theta_{12}\sigma^\rho\bar{\theta}_{12})(p_1+2k)_\rho+\theta_{12}^2\bar{\theta}_{12}^2\left(\frac{1}{4}p_1^2+p_1k+k^2\right)\right).
\end{aligned}\tag{A.22}$$

B Notations and Conventions

We frequently use the definition

$$\frac{1}{2}k_ip_j\Theta^{ij} =: k\wedge p.\tag{B.1}$$

The Fourier transform of a field is

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} \tilde{\phi}(p) e^{-ipx}. \tag{B.2}$$

The covariant derivatives are defined as

$$D_\alpha = \partial_\alpha - i(\sigma^\mu \bar{\theta})_\alpha \partial_\mu, \qquad \bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + i(\theta\sigma^\mu)_{\dot{\alpha}} \partial_\mu. \tag{B.3}$$

In momentum space the covariant derivatives read

$$\tilde{D}_{\alpha,p} = \partial_\alpha - p_\mu \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}, \qquad \tilde{\bar{D}}_{\dot{\alpha},p} = -\bar{\partial}_{\dot{\alpha}} + p_\mu \sigma^\mu_{\alpha\dot{\alpha}} \theta^\alpha. \tag{B.4}$$

We use the following definitions concerning Grassmann-valued objects,

$$\chi\eta := \chi^\alpha \eta_\alpha = -\eta_\alpha \chi^\alpha = \eta^\alpha \chi_\alpha = \eta\chi, \qquad \bar{\chi}\bar{\eta} := \bar{\chi}_{\dot{\alpha}} \bar{\eta}^{\dot{\alpha}} = \bar{\eta}\bar{\chi}. \tag{B.5}$$

We have the integration measures,

$$\int dV := \int d^4x D^2 \bar{D}^2, \qquad \int dS := \int d^4x D^2, \qquad \int d\bar{S} := \int d^4x \bar{D}^2. \tag{B.6}$$

The integration over x cancels the total divergence parts of the covariant derivatives. Therefore, in momentum space we have to define

$$\begin{aligned} &\int dP_V := \int \frac{d^4p}{(2\pi)^4} \tilde{D}^2_{p\mapsto 0} \tilde{\bar{D}}^2_{p\mapsto 0}, \\ &\int dP_S := \int \frac{d^4p}{(2\pi)^4} \tilde{D}^2_{p\mapsto 0}, \qquad \int dP_{\bar{S}} := \int \frac{d^4p}{(2\pi)^4} \tilde{\bar{D}}^2_{p\mapsto 0}. \end{aligned} \tag{B.7}$$

The delta functions and their Fourier transforms are given by

$$\begin{aligned} \delta_V(1,2) &= \frac{1}{16} \theta_{12}^2 \bar{\theta}_{12}^2 \delta^4(x_1 - x_2), & \tilde{\delta}_V(1,2) &= \frac{1}{16} \theta_{12}^2 \bar{\theta}_{12}^2, \\ \delta_S(1,2) &= -\frac{1}{4} \theta_{12}^2 \delta^4(x_1 - x_2), & \tilde{\delta}_S(1,2) &= -\frac{1}{4} \theta_{12}^2, \\ \delta_{\bar{S}}(1,2) &= -\frac{1}{4} \bar{\theta}_{12}^2 \delta^4(x_1 - x_2), & \tilde{\delta}_{\bar{S}}(1,2) &= -\frac{1}{4} \bar{\theta}_{12}^2. \end{aligned} \tag{B.8}$$

Here $\theta_{ij}^\alpha := \theta_i^\alpha - \theta_j^\alpha$. We use functional derivation in superspace:

$$\frac{\delta_V \phi_i}{\delta_V \phi_j} = \delta^4(x_i - x_j) \delta_V(i,j). \tag{B.9}$$

In momentum space we get an extra factor $(2\pi)^4$:

$$\frac{\delta_V \tilde{\phi}_{p_i}}{\delta_V \tilde{\phi}_{p_j}} = (2\pi)^4 \delta^4(p_i - p_j) \tilde{\delta}_V(i,j). \tag{B.10}$$

Finally, we use the definition

$$E_i^{(n)} := \sum_{j=1,\ldots,n,\ j\neq i} p_{j,\mu} \left(\theta_i \sigma^\mu \bar{\theta}_j - \theta_j \sigma^\mu \bar{\theta}_i \right). \tag{B.11}$$

C Useful Formulae

These helpful formulae are used throughout our calculations:

$$\theta_{ji}\theta_{ki} = \frac{1}{2}(\theta_{ji}^2 + \theta_{ki}^2 - \theta_{jk}^2), \qquad (\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) = \frac{1}{2}\eta^{\mu\nu}\theta^2\bar{\theta}^2, \tag{C.1}$$

$$\theta_\alpha(\theta\psi) = -\frac{1}{2}\theta^2\psi_\alpha, \qquad \bar{\theta}_{\dot{\alpha}}(\bar{\theta}\bar{\psi}) = -\frac{1}{2}\bar{\theta}^2\bar{\psi}_{\dot{\alpha}}. \tag{C.2}$$

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