Power-counting theorem for non-local matrix models and renormalisation

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Abstract: Solving the exact renormalisation group equation à la Wilson-Polchinski perturbatively, we derive a power-counting theorem for general matrix models with arbitrarily non-local propagators. The power-counting degree is determined by two scaling dimensions of the cut-off propagator and various topological data of ribbon graphs. As a necessary condition for the renormalisability of a model, the two scaling dimensions have to be large enough relative to the dimension of the underlying space. In order to have a renormalisable model one needs additional locality properties—typically arising from orthogonal polynomials—which relate the relevant and marginal interaction coefficients to a finite number of base couplings. The main application of our power-counting theorem is the renormalisation of field theories on noncommutative $\mathbb{R}^D$ in matrix formulation.

1. Introduction

Noncommutative quantum field theories show in most cases a phenomenon called *UV/IR-mixing* [1] which seems to prevent the perturbative renormalisation. There is an enormous number of articles on this problem, most of them performing one-loop calculations extrapolated to higher order. A systematic analysis of noncommutative (massive) field theories at *any loop order* was performed by Chepelev and Roiban [2,3]. They calculated the integral of an arbitrary Feynman graph using the parametric integral representation and expressed the result in terms of determinants involving the incidence matrix and the intersection matrix. They succeeded to evaluate the leading contribution to the determinants in terms of topological properties of *ribbon graphs* wrapped around Riemann surfaces. In this way a power-counting theorem was established which led to the identification of two power-counting non-renormalisable classes of ribbon graphs. The *Rings*-type class consists of graphs with classically divergent subgraphs wrapped around the same handle of the Riemann surface. The *Com*-type class consists of planar graphs with external legs ending at several disconnected boundary components with the momentum flow into a boundary component being identically zero[1].

[1] These graphs were called "Swiss Cheese" in [4]

Except for models with enough symmetry, noncommutative field theories are not renormalisable by standard techniques. One may speculate that the reason is the too naïve way of performing the various limits. Namely, a field theory is a dynamical system with infinitely many degrees of freedom defined by a certain limiting procedure of a system with finitely many degrees of freedom. One may perform the limits formally to the path integral and evaluate it by Feynman graphs which are often meaningless. It is the art of renormalisation to give a meaning to these graphs. This approach works well in the commutative case, but in the noncommutative situation it seems to be not successful.

A procedure which deals more carefully with the limits is the renormalisation group approach due to Wilson [5], which was adapted by Polchinski to a very efficient renormalisability proof of commutative ϕ^4-theory in four dimensions [6]. There are already some attempts [7] to use Polchinski's method to renormalise noncommutative field theories. We are, however, not convinced that the claimed results (UV-renormalisability) are so easy to obtain. The main argument in [7] is that the Polchinski equation is a one-loop equation so that the authors simply compute an integral having exactly one loop. It is, however, not true that nothing new happens at higher loop order. For instance, all one-loop graphs can be drawn on a genus-zero Riemann surface. The entire complexity of Riemann surfaces of higher genus as discussed by Chepelev and Roiban [2,3] shows up at higher loop order and is completely ignored by the authors of [7]. As we show in this paper, the same discussion of Riemann surfaces is necessary in the renormalisation group approach, too.

The mentioned complexity is due to the phase factors described by the intersection matrix which result in convergent but not absolutely convergent momentum integrals. As such it is very difficult to access this complexity by Polchinski's procedure [6] which is based on taking *norms* of the contributions. Moreover, Chepelev and Roiban established the link between power-counting and the topology of the Riemann surface via the parametric integral representation, which is based on Gaußian integrations. These are not available for Polchinski's method where we deal with cut-off integrals. In conclusion, we believe it is extremely difficult (if not impossible) to use the exact renormalisation group equation for noncommutative field theories in momentum space. The best one can hope is to restrict oneself to limiting cases where e.g. the non-planar graphs are suppressed [4,8]. Even this restricted model has rich topological features.

Fortunately, there exists a base f_{mn} for the algebra under consideration[2] where the $\star$-product is reduced to an ordinary product of (infinite) matrices, $f_{mn} \star f_{kl} = \delta_{nk} f_{ml}$, see [10]. The interaction $\int d^D x(\phi \star \phi \star \phi \star \phi)$ can then be written as $\mathrm{tr}(\phi^4)$ where ϕ is now an infinite matrix (with entries of rapid decay). The price for the simplification of the interaction is that the kinetic matrix, or rather its inverse, the propagator, becomes very complicated. However, in Polchinski's approach the propagator is anyway made complicated when multiplying it with the smooth cut-off function $K[\Lambda]$. The parameter Λ is an energy scale which varies between the renormalisation scale Λ_R and the initial scale $\Lambda_0 \gg \Lambda_R$. Introducing in the bilinear (kinetic) part of the action the cut-off function and replacing the ϕ^4-interaction by a Λ-dependent effective action $L[\phi, \Lambda]$, the philosophy is to determine $L[\phi, \Lambda]$ such that the generating functional $Z[J, \Lambda]$ is actually independent of Λ.

In this paper we provide the prerequisites to investigate the renormalisation of general non-local matrix models. We prove a power-counting theorem for the effective action $L[\phi, \Lambda]$ by solving (better: estimating) the Polchinski equation perturbatively. Our derivation and solution of the matrix Polchinski equation combines

[2] For another matrix realisation of the noncommutative $\mathbb{R}^D$ and its treatment by renormalisation group methods, see [9].

the original ideas of [6] with some of the improvements made in [11]. In particular, we follow [11] to obtain Λ_0-independent estimations for the interaction coefficients. The Polchinski equation for matrix models can be visualised by ribbon graphs. The power-counting degree of divergence of a ribbon graph depends on the topological data of the graph and on two scaling dimensions of the cut-off propagator. In this way, suitable scaling dimensions provide a simple criterion to decide whether a non-local matrix model has the chance of resulting in a renormalisable or not. However, having the right scaling dimensions is not sufficient for the renormalisability of a model, because a divergent interaction is parametrised by an infinite number of matrix indices. Thus, a renormalisable model needs further structures[3] which relate these infinitely many interaction coefficients to a finite number of base couplings.

Nevertheless, the dimension criterion proven in this paper is of great value. For instance, it discards immediately the standard ϕ^4-models on noncommutative $\mathbb{R}^D$, $D = 2, 4$, in the matrix base. These models have the wrong scaling dimension, which is nothing but the manifestation of the old UV/IR-mixing problem [1]. Looking closer at the origin of the wrong scaling dimensions it is not difficult to find a deformation of the free action which has the chance to be a renormalisable model. In the matrix base of the noncommutative $\mathbb{R}^D$, the Laplace operator becomes a tri-diagonal band matrix. The main diagonal behaves nicely, but the two adjacent diagonals are "too big" and compensate the desired behaviour of the main diagonal. Making the adjacent diagonals "smaller" one preserves the properties of the main diagonal and obtains the good scaling dimensions required for a renormalisable model. The deformation of the adjacent diagonals corresponds to the inclusion of a harmonic oscillator potential in the free field action.

We treat in [13] the ϕ^4-model on noncommutative $\mathbb{R}^2$ within the Wilson-Polchinski approach in more detail. We prove that this model is renormalisable when adding the harmonic oscillator potential. Remarkably, the model remains renormalisable when scaling the oscillator potential in a certain way to zero with the removal $\Lambda_0 \to \infty$ of the cut-off.

We prove in [14] that the ϕ^4-model on noncommutative $\mathbb{R}^4$ is renormalisable to all orders by imposing normalisation conditions for the physical mass, the field amplitude, the frequency of the harmonic oscillator potential and the coupling constant. In particular, the harmonic oscillator potential cannot be removed from the model. It gives the explicit solution of the UV/IR-duality which suggests that noncommutativity relevant at short distances goes hand in hand with a different physics at very large distances. The oscillator potential makes the ϕ^4-action covariant with respect to a duality transformation [15] between positions and momenta.

We stress that the power-counting theorem proven in this paper was indispensable to have from the start the right ϕ^4-model for the renormalisation proof [13, 14]. Many noncommutative field theories have a matrix formulation, too. We think of fuzzy spaces and q-deformed models. Our general power-counting theorem can play an important rôle in the renormalisation proof of these examples.

2. The exact renormalisation group equation

We consider a ϕ^4-matrix model with a general (non-diagonal) kinetic term,

$$S[\phi] = \mathcal{V}_D\Big(\sum_{m,n,k,l} \frac{1}{2} G_{mn;kl} \phi_{mn} \phi_{kl} + \sum_{m,n,k,l} \frac{\lambda}{4!} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} \Big) , \tag{2.1}$$

[3] In the first version of this paper we had proposed a "reduction-of-couplings" mechanism [12] to get a finite number of relevant/marginal base couplings. Meanwhile it turned out [13,14] that the models of interest provide automatically such structures in form of orthogonal polynomials.

where $m,n,k,l \in \mathbb{N}^q$. For the noncommutative $\mathbb{R}^D$, D even, we have $q = \frac{D}{2}$. The factor $\mathcal{V}_D$ is the volume of an elementary cell. The choice of ϕ^4 is no restriction but for us the most natural one because we are interested in four-dimensional models. Standard matrix models are given by

$$q = 1\;, \qquad G_{mn;kl} = \frac{1}{\mu_0^2}\delta_{ml}\delta_{nk}\;. \tag{2.2}$$

For reviews on matrix models and their applications we refer to [16,17]. The idea to apply renormalisation group techniques to matrix models is also not new [18]. The difference of our approach is that we will not demand that the action can be written as the trace of a polynomial in the field, that is, we allow for matrix-valued kinetic terms. The only restriction we are imposing is

$$G_{mn;kl} = 0 \qquad \text{unless } m+k = n+l\;. \tag{2.3}$$

The restriction (2.3) is due to the fact that the action comes from a trace. It is verified for the noncommutative $\mathbb{R}^D$ due to the $\big(O(2)\big)^{\frac{D}{2}}$-symmetry of both the interaction and the kinetic term. The kinetic matrix $G_{mn;kl}$ contains the entire information about the differential calculus, including the underlying (Riemannian) geometry, and the masses of the model. More important than the kinetic matrix G will be its inverse, the propagator Δ defined by

$$\sum_{k,l} G_{mn;kl}\Delta_{lk;sr} = \sum_{k,l} \Delta_{nm;lk}G_{kl;rs} = \delta_{mr}\delta_{ns}\;. \tag{2.4}$$

Due to (2.3) we have the same index restrictions for the propagator:

$$\Delta_{nm;lk} = 0 \qquad \text{unless } m+k = n+l\;. \tag{2.5}$$

Let us introduce a notion of locality:

Definition 1 *A matrix model is called* <u>*local*</u> *if* $\Delta_{nm;lk} = \Delta(m,n)\delta_{ml}\delta_{nk}$ *for some function* $\Delta(m,n)$*, otherwise* <u>*non-local*</u>*.*

We add sources J to the action (2.1) and define a (Euclidean) quantum field theory by the generating functional (partition function)

$$Z[J] = \int \Big(\prod_{a,b} d\phi_{ab}\Big)\,\exp\big(-S[\phi] - \mathcal{V}_D\sum_{m,n}\phi_{mn}J_{nm}\big)\;. \tag{2.6}$$

According to Polchinski's derivation of the exact renormalisation group equation we now consider a (at first sight) different problem than (2.6). Via a cut-off function $K[m,\Lambda]$, which is smooth in Λ and satisfies $K[m,\infty]=1$, we modify the weight of a matrix index m as a function of a certain *scale* Λ:

$$Z[J,\Lambda] = \int \Big(\prod_{a,b} d\phi_{ab}\Big)\,\exp\big(-S[\phi,J,\Lambda]\big)\;, \tag{2.7}$$

$$\begin{aligned} S[\phi,J,\Lambda] = \mathcal{V}_D\Big(&\sum_{m,n,k,l}\frac{1}{2}\phi_{mn}G^K_{mn;kl}(\Lambda)\,\phi_{kl} + L[\phi,\Lambda] \\ &+ \sum_{m,n,k,l}\phi_{mn}F_{mn;kl}[\Lambda]J_{kl} + \sum_{m,n,k,l}\frac{1}{2}J_{mn}E_{mn;kl}[\Lambda]J_{kl} + C[\Lambda]\Big)\;, \end{aligned} \tag{2.8}$$

$$G^K_{mn;kl}(\Lambda) := \Big(\prod_{i\in m,n,k,l} K[i,\Lambda]^{-1}\Big)G_{mn;kl}\;, \tag{2.9}$$

with $L[0,\Lambda]=0$. Accordingly, we define

$$\Delta^K_{nm;lk}(\Lambda)=\Big(\prod_{i\in m,n,k,l}K[i,\Lambda]\Big)\Delta_{nm;lk}\;. \tag{2.10}$$

For indices $m=(m^1,\dots,m^{\frac{D}{2}})\in\mathbb{N}^{\frac{D}{2}}$ we would write the cut-off function as a product $K[m,\Lambda]=\prod_{i=1}^{\frac{D}{2}}K\Big(\frac{m^i}{(\mathcal{V}_D)^{\frac{2}{D}}\Lambda^2}\Big)$ where $K(x)$ is a smooth function on $\mathbb{R}^+$ with $K(x)=1$ for $0\leq x\leq 1$ and $K(x)=\epsilon$ for $x\geq 2$. In the limit $\epsilon\to 0$, the partition function (2.7) vanishes unless $\phi_{mn}=0$ for $\max_i(m^i,n^i)\geq 2(\mathcal{V}_D)^{\frac{2}{D}}\Lambda^2$, thus implementing a cut-off of the measure $\prod_{a,b}d\phi_{ab}$ in (2.7). All other formulae involve positive powers of $K\Big(\frac{m^i}{(\mathcal{V}_D)^{\frac{2}{D}}\Lambda^2}\Big)$ which multiply through the cut-off propagator (2.10) the appearing matrix indices. In the limit $\epsilon\to 0$, $K[m,\Lambda]$ has finite support in m so that all infinite-sized matrices are reduced to finite ones.

The function $C[\Lambda]$ is the vacuum energy and the matrices E and F, which are not necessary in the commutative case, must be introduced because the propagator Δ is non-local. It is, in general, not possible to separate the support of the sources J from the support of the Λ-variation of K. Due to $K[m,\infty]=1$ we formally obtain (2.6) for $\Lambda\to\infty$ in (2.9) if we set

$$\begin{aligned}L[\phi,\infty]&=\sum_{m,n,k,l}\frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\;,\\ C[\infty]=0\;,\qquad & E_{mn;kl}[\infty]=0\;,\qquad F_{mn;kl}[\infty]=\delta_{ml}\delta_{nk}\;.\end{aligned} \tag{2.11}$$

However, we shall expect divergences in the partition function which require a renormalisation, i.e. additional (divergent) counterterms in $L[\phi,\infty]$. In the Feynman graph solution of the partition function one carefully adapts these counterterms so that all divergences disappear. If such an adaptation is possible with a *finite number* of *local* counterterms, the model is considered as perturbatively renormalisable.

Following Polchinski [6] we proceed differently to prove renormalisability. We first ask ourselves how to choose L,C,E,F in order to make $Z[J,\Lambda]$ *independent* of Λ. After straightforward calculation one finds the answer

$$\frac{\partial}{\partial\Lambda}Z[J,\Lambda]=0\qquad\text{iff} \tag{2.12}$$

$$\Lambda\frac{\partial L[\phi,\Lambda]}{\partial\Lambda}=\sum_{m,n,k,l}\frac{1}{2}\Lambda\frac{\partial\Delta^K_{nm;lk}(\Lambda)}{\partial\Lambda}\Big(\frac{\partial L[\phi,\Lambda]}{\partial\phi_{mn}}\frac{\partial L[\phi,\Lambda]}{\partial\phi_{kl}}-\frac{1}{\mathcal{V}_D}\Big[\frac{\partial^2 L[\phi,\Lambda]}{\partial\phi_{mn}\,\partial\phi_{kl}}\Big]_\phi\Big)\;, \tag{2.13}$$

$$\Lambda\frac{\partial F_{mn;kl}[\Lambda]}{\partial\Lambda}=-\sum_{m',n',k',l'}G^K_{mn;m'n'}(\Lambda)\,\Lambda\frac{\partial\Delta^K_{n'm';l'k'}(\Lambda)}{\partial\Lambda}F_{k'l';kl}[\Lambda]\;, \tag{2.14}$$

$$\Lambda\frac{\partial E_{mn;kl}[\Lambda]}{\partial\Lambda}=-\sum_{m',n',k',l'}F^T_{mn;m'n'}[\Lambda]\,\Lambda\frac{\partial\Delta^K_{n'm';l'k'}(\Lambda)}{\partial\Lambda}F_{k'l';kl}[\Lambda]\;, \tag{2.15}$$

$$\begin{aligned}\Lambda\frac{\partial C[\Lambda]}{\partial\Lambda}&=\frac{1}{\mathcal{V}_D}\sum_{m,n}\Lambda\frac{\partial}{\partial\Lambda}\ln\big(K[m,\Lambda]K[n,\Lambda]\big)\\ &\qquad-\frac{1}{2\mathcal{V}_D}\sum_{m,n,k,l}\Lambda\frac{\partial\Delta^K_{nm;lk}(\Lambda)}{\partial\Lambda}\frac{\partial^2 L[\phi,\Lambda]}{\partial\phi_{mn}\,\partial\phi_{kl}}\Big|_{\phi=0}\;,\end{aligned} \tag{2.16}$$

where $\big[f[\phi]\big]_\phi := f[\phi]-f[0]$. Naïvely we would integrate (2.13)–(2.16) for the initial conditions (2.11). Technically, this would be achieved by imposing the conditions (2.11) not at $\Lambda=\infty$ but at some finite scale $\Lambda=\Lambda_0$, followed by taking the limit $\Lambda_0\to\infty$. This is easily done for (2.14)–(2.16):

$$F_{mn;kl}[\Lambda] = \sum_{m',n'} G^K_{mn;m'n'}(\Lambda)\Delta^K_{n'm';kl}(\Lambda_0)\;, \tag{2.17}$$

$$E_{mn;kl}[\Lambda] = \sum_{m',n',k',l'} \Delta^K_{mn;m'n'}(\Lambda_0)\Big(G^K_{n'm';l'k'}(\Lambda) - G^K_{n'm';l'k'}(\Lambda_0)\Big)\Delta^K_{k'l';kl}(\Lambda_0)\;, \tag{2.18}$$

$$\begin{aligned} C[\Lambda] &= \frac{2}{\mathcal{V}_D}\ln\Big(\prod_m K[m,\Lambda]K^{-1}[m,\Lambda_0]\Big) \\ &\quad + \frac{1}{2\mathcal{V}_D}\int_\Lambda^{\Lambda_0} d\Lambda' \sum_{m,n,k,l} \frac{\partial\Delta^K_{nm;lk}(\Lambda')}{\partial\Lambda'}\frac{\partial^2 L[\phi,\Lambda']}{\partial\phi_{mn}\,\partial\phi_{kl}}\Big|_{\phi=0}\;. \end{aligned} \tag{2.19}$$

At $\Lambda=\Lambda_0$ the functions F,E,C become independent of Λ_0 and satisfy, in particular, (2.11) in the limit $\Lambda_0\to\infty$.

The partition function $Z[J,\Lambda]$ is evaluated by Feynman graphs with vertices given by the Taylor expansion coefficients

$$L_{m_1n_1;\dots;m_Nn_N}[\Lambda] := \frac{1}{N!}\Big(\frac{\partial^N L[\phi,\Lambda]}{\partial\phi_{m_1n_1}\,\partial\phi_{m_2n_2}\,\dots\partial\phi_{m_Nn_N}}\Big)_{\phi=0} \tag{2.20}$$

connected with each other by internal lines $\Delta^K(\Lambda)$ and to sources J by external lines $\Delta^K(\Lambda_0)$. As $K[m,\Lambda]$ has finite support in m for finite Λ, the summation variables in the above Feynman graphs are via the propagator $\Delta^K(\Lambda)$ restricted to a finite set. Thus, loop summations are finite, provided that the interaction coefficients $L_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ are bounded. In other words, for the renormalisation of a non-local matrix model it is necessary to prove that the differential equation (2.13) admits a regular solution. As pointed out in the Introduction, to obtain a physically reasonable quantum field theory one has additionally to prove that there is a regular solution of (2.13) which depends on a *finite number of initial conditions* only. This requirement is difficult to fulfil because there is, a priori, an infinite number of degrees of freedom given by the Taylor expansion coefficients (2.20). This is the reason for the fact that renormalisable (four-dimensional) quantum field theories are rare.

We are going to integrate (2.13) between a certain renormalisation scale Λ_R and the initial scale Λ_0. We assume that $L_{m_1n_1;\dots;m_Nn_N}$ can be decomposed into parts $L^{(i)}_{m_1n_1;\dots;m_Nn_N}$ which for $\Lambda_R\leq\Lambda\leq\Lambda_0$ scale homogeneously:

$$\Big|\Lambda\frac{\partial L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]}{\partial\Lambda}\Big| \leq \Lambda^{r_i}\;P^{q_i}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;. \tag{2.21}$$

Here, $P^q[X]\geq 0$ stands for some polynomial of degree q in $X\geq 0$. Clearly, $P^q[X]$, for $X\geq 0$, can be further bound by a polynomial with non-negative coefficients. As usual we define

Definition 2 *Homogeneous parts $L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ in (2.21) with $r_i>0$ are called relevant, with $r_i<0$ irrelevant and with $r_i=0$ marginal.*

There are two possibilities for the integration, either from Λ_0 down to Λ or from Λ_R up to Λ, corresponding to the identities

$$\begin{aligned} &L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda] \\ &= L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0] - \int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'}\Big(\Lambda' \frac{\partial}{\partial \Lambda'} L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda']\Big) \qquad (2.22a) \\ &= L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R] + \int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'}\Big(\Lambda' \frac{\partial}{\partial \Lambda'} L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda']\Big) . \qquad (2.22b) \end{aligned}$$

One has

$$\int dx\, x^{r-1}\Big(\ln\frac{x}{x_R}\Big)^q = \begin{cases} \dfrac{(-1)^q q!}{r^{q+1}} x^r \displaystyle\sum_{j=0}^{q} \dfrac{\big(-r\ln\frac{x}{x_R}\big)^j}{j!} + \text{const} & \text{for } r \neq 0\,, \\ \dfrac{1}{q+1}\Big(\ln\dfrac{x}{x_R}\Big)^{q+1} + \text{const} & \text{for } r = 0\,. \end{cases} \qquad (2.23)$$

At the end we are interested in the limit $\Lambda_0 \to \infty$. This requires that positive powers of Λ_0 must be prevented in the estimations. For $r_i < 0$ we we can safely take the direction (2.22a) of integration and then, because all coefficients are positive, the limit $\Lambda_0 \to \infty$ in the integral of (2.22a). Thus,

$$\begin{aligned} \big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| &\leq \big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0]\big| + \int_{\Lambda}^{\infty} \frac{d\Lambda'}{\Lambda'}\Big|\Lambda' \frac{\partial}{\partial \Lambda'} L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda']\Big| \\ &\leq \big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0]\big| + \Lambda^{-|r_i|} P^{q_i}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,, \qquad \text{for } r_i < 0\,. \end{aligned} \qquad (2.24)$$

Here, P^{q_i} is a new polynomial of degree q_i with non-negative coefficients. Now, the limit $\Lambda_0 \to \infty$ carried out later requires that $\big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0]\big|$ in (2.24) is bounded, i.e. $\big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0]\big| < \frac{C}{\Lambda_0^{s_i}}$, with $s_i > 0$. As the resulting estimation (2.24) is further iterated, s_i must be sufficiently large. We do not investigate this question in detail and simply note that it is safe to require

$$\big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0]\big| < \Lambda_0^{-|r_i|} P^{q_i}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,, \qquad \text{for } r_i < 0\,, \qquad (2.25)$$

for the boundary condition.

In the other case $r_i \geq 0$, the integration direction (2.22a) will produce divergences in $\Lambda_0 \to \infty$. Thus, we have to choose the other direction (2.22b). The integration (2.23) produces alternating signs, but these can be ignored in the maximisation. The only contribution from the lower bound Λ_R in the integral of (2.22b) is the term with $j = 0$ in (2.23). There, we can obviously ignore it in the difference $\Lambda^r - \Lambda_R^r$. We thus obtain from (2.23) the estimation

$$\big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \leq \big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R]\big| + \begin{cases} \Lambda^{r_i} P^{q_i}\big[\ln\frac{\Lambda}{\Lambda_R}\big] & \text{for } r_i > 0 \\ P^{q_i+1}\big[\ln\frac{\Lambda}{\Lambda_R}\big] & \text{for } r_i = 0\,. \end{cases} \qquad (2.26)$$

The reduction from $P\big[\ln\frac{\Lambda_0}{\Lambda_R}\big]$ in Polchinski's original work [6] to $P\big[\ln\frac{\Lambda}{\Lambda_R}\big]$ is due to [11]. We can summarise these considerations as follows:

Definition/Lemma 3 *Let* $\big|\Lambda\frac{\partial}{\partial\Lambda}L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big|$ *be bounded by (2.21),*

$$\Big|\Lambda\frac{\partial L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]}{\partial\Lambda}\Big| \leq \Lambda^{r_i}P^{q_i}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;. \tag{2.27}$$

The integration of (2.27) is for irrelevant interactions performed from Λ_0 *down to* Λ *starting from an initial condition bounded by* $\big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0]\big| < \Lambda_0^{-|r_i|}P^{q_i}\big[\ln\frac{\Lambda_0}{\Lambda_R}\big]$. *For relevant and marginal interactions we have to integrate (2.27) from* Λ_R *up to* Λ, *starting from an initial condition* $L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R] < \infty$. *Under these conventions we have*

$$\big|L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \leq \Lambda^{r_i}P^{q_i+1}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;. \tag{2.28}$$

Let us give a few comments:

- The stability (2.27) versus (2.28) of the estimation will be very useful in the iteration process.
- Integrations according to the direction (2.22b), which entail an initial condition $L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R]$, are expensive for renormalisation, because each such condition (even the choice $L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R] = 0$) corresponds to a normalisation experiment. In order to have a meaningful theory, there has to be only a finite number of required normalisation experiments. Initial data at Λ_0 do not correspond to normalisation conditions, because the interaction at $\Lambda_0 \to \infty$ is experimentally not accessible. Moreover, unless artificially kept alive[4], an irrelevant coupling scales away for $\Lambda_0 \to \infty$ via its own dynamics. The property $\lim_{\Lambda_0\to\infty} L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0] = 0$ for an irrelevant coupling is, therefore, a result and no condition.
- There might be cases where the direction (2.22b) for $r_i < 0$ gives convergence for $\Lambda_0 \to \infty$ nevertheless. This corresponds to the over-subtractions in the BPHZ renormalisation scheme. We shall not exploit this possibility.

Unless there are further correlations between functions with different indices, specifying $L^{(i)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R]$ means to impose an *infinite number* of normalisation conditions (because of $m_i, n_i \in \mathbb{N}^{D/2}$). Hence, a non-local matrix model with relevant and/or marginal interactions can only be renormalisable if some additional structures exist which relate all divergent functions to a finite number of relevant/marginal base interactions. Such a distinguished property depends crucially on the model. Presumably, the class of models where such a reduction is possible is rather small. It cannot be the purpose of this paper to analyse these reductions. Instead, our strategy is to find the general power-counting behaviour of a non-local matrix model which limits the class of divergent functions among which the reduction has to be studied in detail. For example, we will find that under very general conditions on the propagator all non-planar graphs (as defined below) are irrelevant. Such a result is already an enormous gain[5] for the detailed investigation of a model.

Thus, our strategy is to integrate the Polchinski equation (2.13) perturbatively between two scales Λ_R and Λ_0 for a self-determined choice of the boundary condition according to Definition/Lemma 3. The resulting normalisation condition for relevant and marginal interactions will not be the correct choice for a renormalisable model. Nevertheless, the resulting estimation (2.28) is compatible with a more

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[4] An example of an irrelevant coupling which remains present for $\Lambda_0 \to \infty$ is the initial ϕ^4-interaction in two-dimensional models [13].

[5] We recall [1] that non-planar graphs produce the trouble in noncommutative quantum field theories in momentum space.

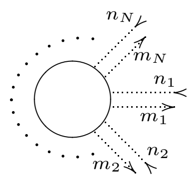
careful treatment. Taking the example [14] of the ϕ^4-model on noncommutative $\mathbb{R}^4$, we would replace

- almost all of the relevant functions with bound $\frac{\Lambda^2}{\mu^2} P^q[\ln \frac{\Lambda}{\Lambda_R}]$ in (2.28) by irrelevant functions with bound $\big(\max(m_1, n_1, \dots, m_N, n_N)\big)^2 \frac{\mu^2}{\Lambda^2} P^q[\ln \frac{\Lambda}{\Lambda_R}]$, and
- almost all marginal functions with bound $P^q[\ln \frac{\Lambda}{\Lambda_R}]$ in (2.28) by irrelevant functions with bound $\max(m_1, n_1, \dots, m_N, n_N) \frac{\mu^2}{\Lambda^2} P^q[\ln \frac{\Lambda}{\Lambda_R}]$,

for some reference scale μ.

3. Ribbon graphs and their topologies

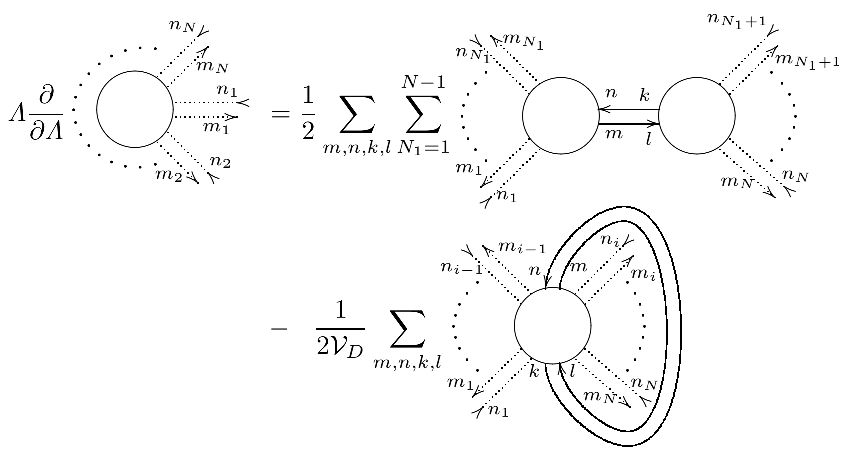
We can symbolise the expansion coefficients $L_{m_1 n_1;\dots;m_N n_N}$ as



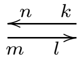
$+ (N!-1)$ permutations of $\{m_i n_i\}$. (3.1)

The big circle stands for a possibly very complex interior and the outer (dotted) double lines stand for the valences produced by differentiation (2.20) with respect to the N fields $\phi_{m_i n_i}$. The arrows are merely added for bookkeeping purposes in the proof of the power-counting theorem. Since we work with real fields, i.e. $\phi_{mn} = \overline{\phi_{nm}}$, the expansion coefficients $L_{m_1 n_1;\dots;m_N n_N}$ have to be unoriented. The situation is different for complex fields where $\phi \neq \phi^*$ leads to an orientation of the lines. In this case we would draw both arrows at the double line either incoming or outgoing.

The graphical interpretation of the Polchinski equation (2.13) is found when differentiating it with respect to the fields $\phi_{m_i n_i}$:



(3.2)

Combinatorical factors are not shown and a symmetrisation in all indices $m_i n_i$ has to be performed. On the rhs of (3.2) the two valences mn and kl of the subgraphs are connected to the ends of a *ribbon* which symbolises the differentiated propagator  $= \Lambda \frac{\partial}{\partial \Lambda} \Delta^K_{nm;lk}$. For local matrix models in the sense of Definition 1 we can regard the ribbon as a product of single lines with interaction given by $\Delta(m,n)$. For non-local matrix models there is an exchange of indices within the entire ribbon.

We can regard (2.13) as a formal construction scheme for $L[\phi,\Lambda]$ if we introduce a grading $L[\phi,\Lambda]=\sum_{V=1}^{\infty}\lambda^V L^{(V)}[\phi,\Lambda]$ and additionally impose a cut-off in N for $V=1$, i.e

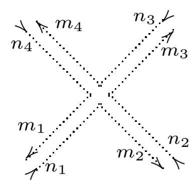
$$L^{(1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]=0 \qquad \text{for } N>N_0\,. \tag{3.3}$$

In order to obtain a ϕ^4-model we choose $N_0=4$ and the grading as the degree V in the coupling constant λ. We conclude from (2.13) that $L^{(1)}_{m_1n_1;\dots;m_4n_4}$ is independent of Λ so that it is identified with the original $(\lambda/4!)\phi^4$-interaction in (2.1):

$$\begin{aligned}
&L^{(1)}_{m_1n_1;m_2n_2;m_3n_3;m_4n_4}[\Lambda]\\
&\qquad=\frac{1}{4!\,6}\Big(\delta_{n_1m_2}\delta_{n_2m_3}\delta_{n_3m_4}\delta_{n_4m_1}+\delta_{n_1m_3}\delta_{n_3m_4}\delta_{n_4m_2}\delta_{n_2m_1}\\
&\qquad\quad+\delta_{n_1m_4}\delta_{n_4m_2}\delta_{n_2m_3}\delta_{n_3m_1}+\delta_{n_1m_4}\delta_{n_4m_3}\delta_{n_3m_2}\delta_{n_2m_1}\\
&\qquad\quad+\delta_{n_1m_3}\delta_{n_3m_2}\delta_{n_2m_4}\delta_{n_4m_1}+\delta_{n_1m_2}\delta_{n_2m_4}\delta_{n_4m_3}\delta_{n_3m_1}\Big)\,.
\end{aligned}\tag{3.4}$$

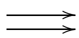
To the first term on the rhs of (3.4) we associate the graph

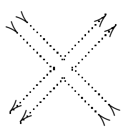
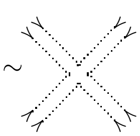
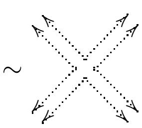
$$\delta_{n_1m_2}\delta_{n_2m_3}\delta_{n_3m_4}\delta_{n_4m_1} =$$



(3.5)

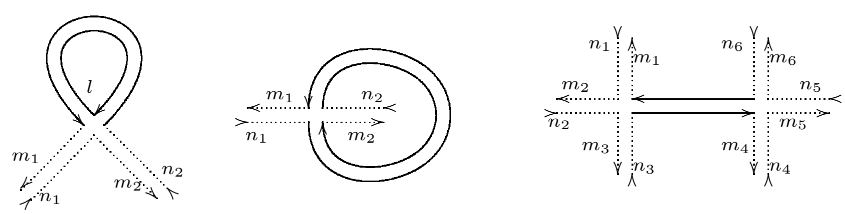
The graphs for the other five terms are obtained by permutation of indices.

As mentioned before, a complex ϕ^4-model would be given by oriented propagators  and examples for vertices are

$\phi\phi^*\phi\phi^* \sim$  $\phi\phi\phi\phi \sim$  $\phi^*\phi^*\phi^*\phi^* \sim$  (3.6)

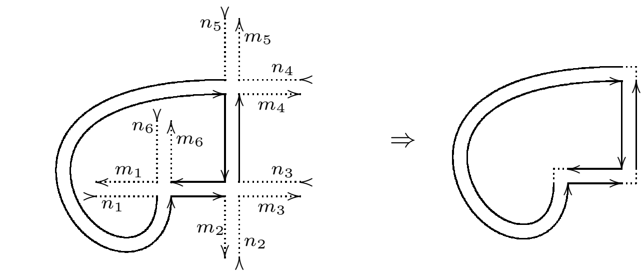
The consequence is that many graphs of the real ϕ^4-model are now excluded. We can thus obtain the complex ϕ^4-model from the real one by deleting the impossible graphs.

The iteration of (3.2) with starting point (3.5) leads to *ribbon graphs*. The first examples of the iteration are

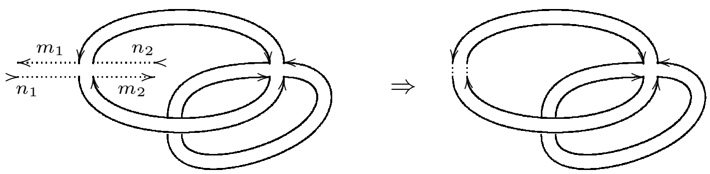


(3.7)

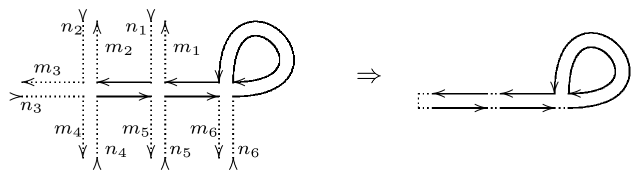
We can obviously build very complicated ribbon graphs with crossings of lines which cannot be drawn any more in a plane. A general ribbon graph can, however, be drawn on a *Riemann surface* of some *genus* g. In fact, a ribbon graph *defines* the Riemann surfaces topologically through the *Euler characteristic* χ. We have to regard here the external lines of the ribbon graph as amputated (or closed), which means to directly connect the single lines m_i with n_i for each external leg $m_i n_i$. A few examples may help to understand this procedure:



$$\tilde{L}=2 \quad B=2 \qquad I=3 \quad N=6 \qquad V=3 \quad V^e=3 \qquad g=0 \quad \iota=0 \tag{3.8}$$



$$\tilde{L}=1 \quad B=1 \qquad I=3 \quad N=2 \qquad V=2 \quad V^e=1 \qquad g=1 \quad \iota=1 \tag{3.9}$$



$$\tilde{L}=2 \quad B=1 \qquad I=3 \quad N=6 \qquad V=3 \quad V^e=3 \qquad g=0 \quad \iota=0 \tag{3.10}$$

The genus is computed from the number $\tilde{L}$ of single-line loops of the closed graph, the number I of internal (double) lines and the number V of vertices of the graph according to

$$\chi = 2 - 2g = \tilde{L} - I + V\ . \tag{3.11}$$

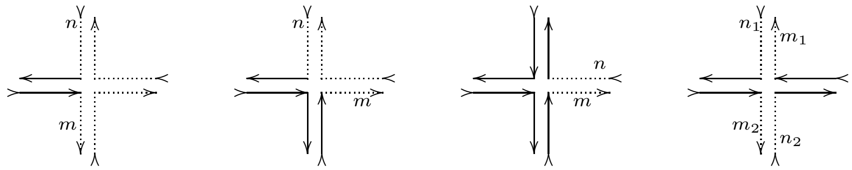
There can be several possibilities to draw the graph and its Riemann surface, but $\tilde{L}, I, V$ and thus g remain unchanged. Indeed, the Polchinski equation (2.13) interpreted as in (3.2) tells us which external legs of the vertices are connected. It is completely irrelevant how the ribbons are drawn between these legs. In particular, there is no distinction between overcrossings and undercrossings.

There are two types of loops in (amputated) ribbon graphs:

– Some of them carry at least one external leg. They are called *boundary components* (or holes of the Riemann surface). Their number is B.
– Some of them do not carry any external leg. They are called *inner loops*. Their number is $\tilde{L}_0 = \tilde{L} - B$.

Boundary components consist of a concatenation of *trajectories* from an incoming index n_i to an outgoing index m_j. In the example (3.8) the inner boundary component consists of the single trajectory $\overrightarrow{n_1 m_6}$ whereas the outer boundary component is made of two trajectories $\overrightarrow{n_3 m_4}$ and $\overrightarrow{n_5 m_2}$. We let $\mathfrak{o}[n_j]$ be the outgoing index to n_j and $\mathfrak{i}[m_j]$ be the incoming index to m_j.

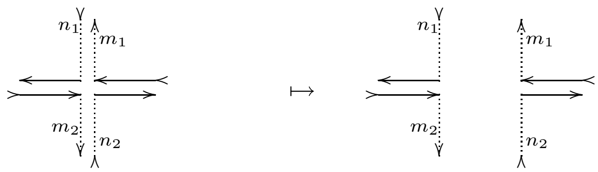
We have to introduce a few additional notations for ribbon graphs. An *external vertex* is a vertex which has at least one external leg. We denote by V^e the total number of external vertices. For the arrangement of external legs at an external vertex there are the following possibilities:



(3.12)

We call the first three types of external vertices *simple vertices*. They provide one starting point and one end point of trajectories through a ribbon graph. The fourth vertex in (3.12) is called *composed vertex*. It has two starting points and two end points of trajectories.

A composed vertex can be decomposed by pulling the two propagators with attached external lines apart:



(3.13)

In this way a given graph with composed vertices is decomposed into S *segments*. The external vertices of the segments are either true external vertices or the halves of a composed vertex. If composed vertices occur in loops, their decomposition does not always increase the number of segments. We need the following

Definition 4 *The segmentation index ι of a graph is the maximal number of decompositions of composed vertices which keep the graph connected.*

It follows immediately that if V^c is the number of composed vertices of a graph and S the number of segments obtained by decomposing all composed vertices we have

$$\iota = V^c - S + 1 \,. \tag{3.14}$$

In order to evaluate $L_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ by connection and contraction of subgraphs according to (3.2) we need estimations for *index summations* of ribbon graphs. Namely, our strategy is to apply the summations in (3.2) either to the propagator or the subgraph only and to maximise the other object over the summation indices. We agree to fix all starting points of trajectories and sum over the end points of trajectories. However, due to (2.5) and (3.4) not all summations are independent: The sum of outgoing indices equals for each segment the sum of incoming indices. Since there are $V^e + V^c$ (end points of) trajectories in a ribbon graph, there are

$$s \leq V^e + V^c - S = V^e + \iota - 1 \tag{3.15}$$

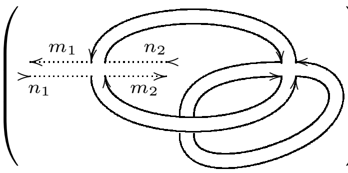
independent index summations. The inequality (3.15) also holds for the restriction to each segment if V^e includes the number of halves of composed vertices belonging to the segment. We let $\mathcal{E}^s$ be the set of s end points of trajectories in a graph over which we are going to sum, keeping the starting points of these trajectories fixed. We define

$$\sum_{\mathcal{E}^s} \equiv \sum_{m_1} \cdots \sum_{m_s} \Big|_{\mathfrak{i}[m_j]=\text{const}} \qquad \text{if } \mathcal{E}^s = \{m_1, m_2, \dots, m_s\} \,. \tag{3.16}$$

Taking the example of the graph (3.8), we can due to $V^e + \iota = 3$ apply up to two index summations, i.e. a summation over at most two of the end points of trajectories m_2, m_4, m_6, where the corresponding incoming indices $\mathfrak{i}[m_2] = n_5$, $\mathfrak{i}[m_4] = n_3$ and

$\mathfrak{i}[m_6] = n_1$ are kept fixed. For the example of the graph (3.9) we can due to $V^e + \iota = 2$ apply at most one index summation, either over m_1 for fixed $\mathfrak{i}[m_1] = n_2$ or over $\mathfrak{i}[m_2] = n_1$. For $\mathcal{E}^1 = \{m_2\}$ we would consider

$$\sum_{m_2} \Bigg(\quad \Bigg)_{n_1=\mathrm{const}} \tag{3.17}$$



Note that for given n_2 the other outgoing index is determined to $m_1 = n_1 + n_2 - m_2$ through index conservation at propagators (2.5) and vertices (3.5). It is part of the proof to show that the index summation (3.17) is bounded independently of the incoming indices n_1, n_2.

4. Formulation of the power-counting theorem

We first have to transform the Polchinski equation (2.13) into a dimensionless form. It is important here that in the class of models we consider there is always a dimensionful parameter,

$$\mu = \big(\mathcal{V}_D\big)^{-\frac{1}{D}} , \tag{4.1}$$

which instead of Λ can be used to absorb the mass dimensions. The effective action $L[\phi, \Lambda]$ has total mass dimension D, a field ϕ has dimension $\frac{D-2}{2}$ and the dimension of the coupling constant for the $\lambda\phi^4$ interaction is $4-D$. We thus decompose $L[\phi, \Lambda]$ according to the number of fields and the order in the coupling constant:

$$L[\phi, \Lambda] = \sum_{V=1}^{\infty} \sum_{N=2}^{2V+2} \frac{1}{N!} \sum_{m_i, n_i} \Big(\frac{\lambda}{\mu^{4-D}}\Big)^V \mu^D A^{(V)}_{m_1 n_1;\dots;m_N n_N}[\Lambda] \Big(\frac{\phi_{m_1 n_1}}{\mu^{\frac{D-2}{2}}}\Big) \cdots \Big(\frac{\phi_{m_N n_N}}{\mu^{\frac{D-2}{2}}}\Big) . \tag{4.2}$$

The functions $A^{(V)}_{m_1 n_1;\dots;m_N n_N}[\Lambda]$ are assumed to be symmetric in their indices $m_i n_i$. Inserted into (2.13) we get

$$\begin{aligned} &\Lambda \frac{\partial}{\partial \Lambda} A^{(V)}_{m_1 n_1;\dots;m_N n_N}[\Lambda] \\ &= \sum_{m,n,k,l} \frac{1}{2} Q_{nm;lk}(\Lambda) \Bigg\{ \sum_{N_1=2}^{N} \sum_{V_1=1}^{V-1} A^{(V_1)}_{m_1 n_1;\dots;m_{N_1-1} n_{N_1-1};mn}[\Lambda] A^{(V-V_1)}_{m_{N_1} n_{N_1};\dots;m_N n_N;kl}[\Lambda] \\ &\qquad\qquad + \Big(\binom{N}{N_1-1} - 1\Big) \text{ permutations} \\ &\qquad - A^{(V)}_{m_1 n_1;\dots;m_N n_N;mn;kl}[\Lambda] \Bigg\} , \end{aligned} \tag{4.3}$$

where

$$Q_{nm;lk}(\Lambda) := \mu^2 \Lambda \frac{\partial}{\partial \Lambda} \Delta^K_{nm;lk}(\Lambda) . \tag{4.4}$$

The permutations refer to the possibilities to choose $N_1 - 1$ of the pairs of indices $m_1 n_1, \dots, m_N n_N$ which label the external legs of the first A-function.

The cut-off function K in (2.10) has to be chosen such that for finite Λ there is a finite number of indices m,n,k,l with $Q_{nm;lk}(\Lambda)\neq 0$. By suitable normalisation we can achieve that the volume of the support of $Q_{nm;lk}(\Lambda)$ with respect to a chosen index scales as Λ^D:

$$\sum_m \mathrm{sign}\big|K[m,\Lambda]\big| \leq C_D\Big(\frac{\Lambda}{\mu}\Big)^D\;, \tag{4.5}$$

for some constant C_D independent of Λ. For such a normalisation we define two exponents δ_0,δ_1 by

$$\max_{m,n,k,l}|Q_{nm;lk}(\Lambda)| \leq C_0\Big(\frac{\mu}{\Lambda}\Big)^{\delta_0}\delta_{m+k,n+l}\;, \tag{4.6}$$

$$\max_n\Big(\sum_k\Big(\max_{m,l}|Q_{nm;lk}(\Lambda)|\Big)\Big) \leq C_1\Big(\frac{\mu}{\Lambda}\Big)^{\delta_1}\;. \tag{4.7}$$

In (4.7) the index n is kept constant for the summation over k. It is convenient to encode the dimension D in a further exponent δ_2 which describes the product of (4.5) with (4.6):

$$\max_{m,n,k,l}|Q_{nm;lk}(\Lambda)|\;\sum_m \mathrm{sign}\big|K[m,\Lambda]\big| \leq C_2\Big(\frac{\Lambda}{\mu}\Big)^{\delta_2}\;. \tag{4.8}$$

We have obviously $C_2=C_DC_0$ and $\delta_2=D-\delta_0$.

Definition 5 *A non-local matrix model defined by the cut-off propagator $Q_{nm;kl}$ given by (2.10) and (4.4) and the normalisation (4.5) of the cut-off function is called* regular *if $\delta_0=\delta_1=2$, otherwise* anomalous*.*

The three exponents $\delta_0,\delta_1,\delta_2$ play an essential rôle in the power-counting theorem which yields the Λ-scaling of a homogeneous part $A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ of the interaction coefficients

$$\begin{aligned}&A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\\ &=\sum_{1\leq V^e\leq V}\;\sum_{1\leq B\leq N}\;\sum_{0\leq g\leq 1+\frac{V}{2}-\frac{N}{4}-\frac{B}{2}}\;\sum_{0\leq\iota\leq B-1} A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\Big|_{2\leq N\leq 2V+2}\;.\end{aligned} \tag{4.9}$$

It is important that the sums over the graphical (topological) data V^e,B,g,ι in (4.9) are finite. We are going to prove

Theorem 6 *The homogeneous parts $A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ of the coefficients of the effective action describing a ϕ^4-matrix model with initial interaction (3.4) and cut-off propagator characterised by the three exponents $\delta_0,\delta_1,\delta_2$ are for $2\leq N\leq 2V{+}2$ and $\sum_{i=1}^N(m_i{-}n_i)=0$ bounded by*

$$\begin{aligned}\sum_{\mathcal{E}^s}\big|A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| &\leq \Big(\frac{\Lambda}{\mu}\Big)^{\delta_2(V-\frac{N}{2}+2-2g-B)}\Big(\frac{\mu}{\Lambda}\Big)^{\delta_1(V-V^e-\iota+2g+B-1+s)}\\ &\qquad\times\Big(\frac{\mu}{\Lambda}\Big)^{\delta_0(V^e+\iota-1-s)}P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;,\end{aligned} \tag{4.10}$$

provided that for all $V'{<}V$, $2\leq N'\leq 2V'{+}2$ and $V'{=}V$, $N{+}2\leq N'\leq 2V{+}2$ the initial conditions for relevant / marginal (irrelevant) $A^{(V',V'^e,B',g',\iota')}_{m_1'n_1';\dots;m_{N'}'n_{N'}'}[\Lambda]$ are imposed at Λ_R (Λ_0), respectively, according to Definition/Lemma 3. The bound (4.10) is independent of the unsummed indices $m_i,n_i\notin\mathcal{E}^s$. We have $A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\equiv 0$ for $N>2V{+}2$ or $\sum_{i=1}^N(m_i{-}n_i)\neq 0$.

The proof will be given in Section 5. We remark that $\tilde{L}_0 = V - \frac{N}{2} + 2 - 2g - B$ is the number of inner loops of a graph.

The power-counting estimation (4.10) does not make any reference to the initial scale Λ_0 [11] so that we can safely take the limit $\Lambda_0 \to \infty$. In this way we have constructed a regular solution of the Polchinski equation (2.13) associated with the non-local matrix model. However, this solution remains useless unless it can be achieved by a *finite number* of integrations from Λ_R to Λ depending on a finite number of initial conditions at Λ_R. We refer to the remarks following Definition/Lemma 3. A first step would be to achieve regular scaling dimensions:

Corollary 7 *For regular matrix models according to Definition 5 we have independently of the segmentation index and the numbers of external vertices*

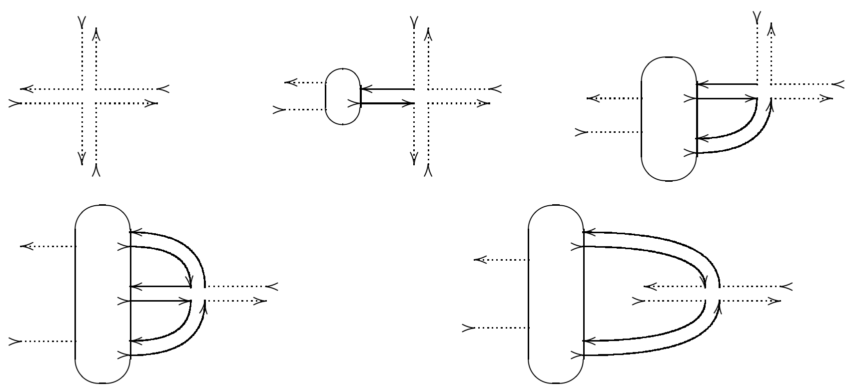
$$\sum_{\mathcal{E}^s} \big|A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \leq \Big(\frac{\Lambda}{\mu}\Big)^{\omega-D(2g+B-1)} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,, \qquad (4.11)$$

where $\omega = D+V(D-4)-N\frac{D-2}{2}$ *is the classical power-counting degree of divergence.*

We have derived the relation (4.11) with respect to the classical power-counting degree of divergence only for ϕ^4-matrix models, but it is plausible that it also holds for more general interactions.

5. Proof of the power-counting theorem

We provide here the proof of Theorem 6, which is quite long and technical. The proof amounts to study all possible connections of two external legs of either different graphs or the same graph. It will be essential how the legs to connect are situated with respect to the remaining part of the graph. There are the following arrangements of the external legs at the distinguished vertex one (or two) of which we are going to connect:



(5.1)

A big oval stands for other parts of the graph the specification of which is not necessary for the proof. Dotted lines entering and leaving the oval stand for the set of all external legs different from the external legs of the distinguished vertex to contract. If two or three internal lines are connected to the oval this does not necessarily mean that these two lines are part of an inner loop.

We are going to integrate the Polchinski equation (4.3) by induction upward in V and for constant V downward in N. Due to the grading (V,N), the differential equation (4.3) is actually constructive. We consider in Section 5.1 the connection of two smaller graphs of (V_1,N_1) and (V_2,N_2) vertices and external legs and in Sections 5.2 and 5.3 the self-contraction of a graph with $(V_1 = V, N_1 = N+2)$ vertices and external legs. These graphs are further characterised by V_i^e, B_i, g_i, ι_i external vertices, boundary components, genera and segmentation indices, respectively. Since the sums in (4.9) and the number of arrangements of legs in (5.1) are finite, it is sufficient to regard the contraction of subgraphs individually. That is, we consider individual subgraphs γ_1, γ_2 the contraction of which produces an individual graph γ. We also ignore the problem of making the graphs symmetric in the indices $m_i n_i$ of the external legs. At the very end we project the sum of graphs γ to homogeneous degree (V, V^e, B, g, ι). To these homogeneous parts there contributes according to (4.9) a finite number of contractions of γ_i. We thus get the bound (4.10) if we can prove it for any individual contraction.

The Theorem is certainly correct for the initial ϕ^4-interaction (3.4) which due to (4.2) gives $|A^{(1,1,1,0,0)}_{m_1n_1;\dots;m_4n_4}[\Lambda]| \leq 1$.

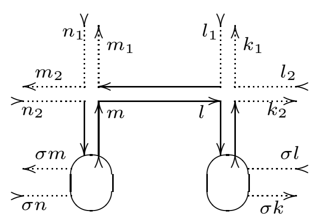
5.1. Tree-contractions of two subgraphs. We start with the first term on the rhs of (4.3) which describes the connection of two smaller subgraphs γ_1, γ_2 of V_1, V_2 vertices and N_1, N_2 external legs via a propagator. The total graph γ for a tree-contraction has

$$\begin{aligned} V &= V_1 + V_2 \text{ vertices}\,, & N &= N_1 + N_2 - 2 \text{ external legs}\,, \\ I &= I_1 + I_2 + 1 \text{ propagators}\,, & \tilde{L} &= \tilde{L}_1 + \tilde{L}_2 - 1 \text{ loops}\,, \end{aligned} \tag{5.2}$$

because two loops of the subgraphs are merged to a new loop in the total graph. It follows from (3.11) that for tree-contractions we always have additivity of the genus,

$$g = g_1 + g_2\,. \tag{5.3}$$

As an example for a contraction between graphs in the first line of (5.1) let us consider



(5.4)

where σm and σn stand for the set of all other outgoing and incoming indices via external legs at the remaining part of the left subgraph γ_1 and similarly for σk and σl for the right subgraph γ_2. The two boundary components to which the contracted vertices belong are joint in the total graph, i.e. $B = B_1 + B_2 - 1$. Moreover, we obviously have $V^e = V_1^e + V_2^e$ and $\iota = \iota_1 + \iota_2$. The graph (5.4) determines the Λ-scaling

$$\begin{aligned} &\Lambda \frac{\partial}{\partial \Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;m_2n_2;\sigma m\, \sigma n;\sigma k\, \sigma l;k_2l_2;k_1l_1}[\Lambda] \\ &= \frac{1}{2} \sum_{m,l} A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{m_1n_1;m_2n_2;\sigma m\, \sigma n;mm_1}[\Lambda]\, Q_{m_1m;ll_1}(\Lambda)\, A^{(V_2,V_2^e,B_2,g_2,\iota_2)\gamma_2}_{l_1l;\sigma k\, \sigma l;k_2l_2;k_1l_1}[\Lambda]\,. \end{aligned} \tag{5.5}$$

Due to the conservation of the total amount of indices in γ_1 and γ_2 by induction hypothesis (4.10), both

$$m = \sigma n - \sigma m + n_2 \qquad \text{and} \qquad l = \sigma k - \sigma l + k_2 \tag{5.6}$$

are completely fixed by the other external indices so that from the sum over m and l there survives a single term only. Then, because of the relation $m_1+l=m+l_1$ from the propagator $Q_{m_1m;ll_1}(\Lambda)$, see (4.6), it follows that the total amount of indices for $A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;m_2n_2;\sigma m\,\sigma n;\sigma k\,\sigma l;k_2l_2;k_1l_1}$ is conserved as well.

Let $\bar{V}^e_i$ and $\bar{\iota}_i$ be the numbers of external vertices and segmentation indices on the segments of the subgraphs γ_i on which the contracted vertices are situated. The induction hypothesis (4.10) gives us the bound if these segments carry $\bar{s}_i \leq \bar{V}^e_i+\bar{\iota}_i-1$ index summations. The new segment of the total graph γ created by connecting the boundary components of γ_i carries $\bar{V}^e_1+\bar{V}^e_2$ external vertices and $\bar{\iota}_1+\bar{\iota}_2$ segmentation indices and therefore admits up to $\bar{s}_1+\bar{s}_2+1$ index summations. In (5.4) that additional index summation will be the m_1-summation.

Due to (3.15) (for segments) there has to be an external leg on each segment the outgoing index of which is not allowed to be summed. If on the γ_2-part of the contracted segment there is an unsummed external leg, we can choose m as that particular index in γ_1. In this case we take in the propagator the maximum over m,l and sum the part γ_2 for given l over those indices which belong to $\mathcal{E}^s$. The result is bounded independently of l and all other incoming indices. Next, we sum over the indices in $\mathcal{E}^s$ which belong to γ_1, regarding m as an unsummed index. There is the possibility of an m_1-summation applied to the propagator in the last step, with l_1 kept fixed, for which the bound is given by (4.7). In this case we therefore get

$$
\begin{aligned}
&\sum_{\mathcal{E}^s,\ \bar{s}_2\leq \bar{V}^e_2+\bar{\iota}_2-1,\ m_1\in\mathcal{E}^s} \Big|\Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;m_2n_2;\sigma m\,\sigma n;\sigma k\,\sigma l;k_2l_2;k_1l_1}[\Lambda]\Big| \\
&\quad\leq \frac{1}{2}\Big(\sum_{\mathcal{E}^{s_1}_1}\big|A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{m_1n_1;m_2n_2;\sigma m\,\sigma n;mm_1}[\Lambda]\big|\Big)\Big(\max_{l_1}\sum_{m_1}\max_{m,l}\big|Q_{m_1m;ll_1}(\Lambda)\big|\Big) \\
&\qquad\times\Big(\sum_{\mathcal{E}^{s_2}_2}\big|A^{(V_2,V_2^e,B_2,g_2,\iota_2)\gamma_2}_{l_1l;\sigma k\,\sigma l;k_2l_2;k_1l_1}[\Lambda]\big|\Big) \\
&\quad\leq \frac{1}{2}C_1\Big(\frac{\Lambda}{\mu}\Big)^{\delta_2(V-\frac{N+2}{2}+4-2g-(B+1))}\Big(\frac{\mu}{\Lambda}\Big)^{\delta_1(1+V-V^e-\iota+2g+(B+1)-2+(s-1))} \\
&\qquad\times\Big(\frac{\mu}{\Lambda}\Big)^{\delta_0(V^e+\iota-2-(s-1))}P^{2V-\frac{N+2}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;.
\end{aligned}
\tag{5.7}
$$

We have used the induction hypothesis (4.10) for the subgraphs as well as (4.7) for the propagator and have inserted $N_1+N_2=N+2$, $V_1+V_2=V$, $V_1^e+V_2^e=V^e$, $\iota_1+\iota_2=\iota$, $B_1+B_2=B+1$, $g_1+g_2=g$ and $s_1+s_2=s-1$, because there is an additional summation over m_1 which belongs to $\mathcal{E}^s$ but not to $\mathcal{E}^{s_i}_i$. If $m_1\notin\mathcal{E}^s$ we take instead the unsummed propagator and replace in (5.7) one factor (4.7) by (4.6) as well as $(s-1)$ by s. The total exponents of γ remain unchanged.

Next, let there be no unsummed external leg on the contracted segment of γ_2 viewed from γ. Now, we cannot directly use the induction hypothesis. On the other hand, for a given index configuration of γ_2 and the propagator, the index k_2 is not an independent summation index:

$$
k_2 = l+\sigma l-\sigma k = m-m_1+l_1+\sigma l-\sigma k\;. \tag{5.8}
$$

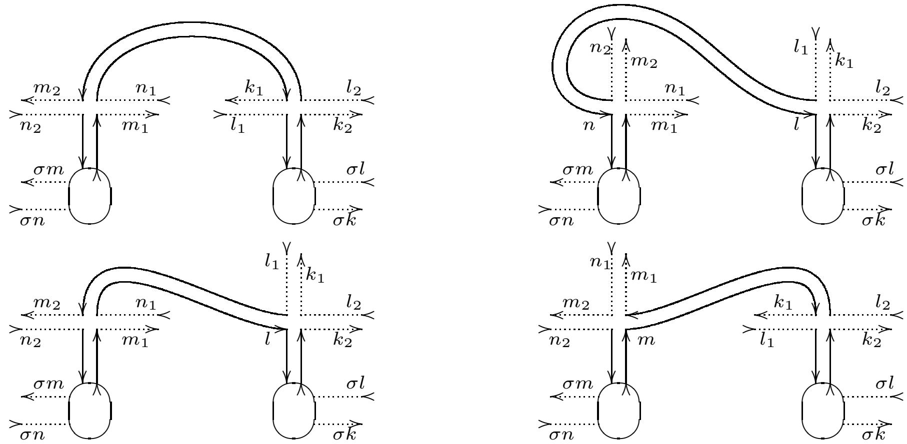
See also (5.6). If $m_1\in\mathcal{E}^s$ there must be an unsummed outgoing index on the contracted segment of γ_1. We can thus realise the k_2-summation as a summation over m in γ_1 for fixed index configuration of γ_2 and m_1,l_1. This m-summation is applied together summation over the γ_1-indices of $\mathcal{E}^s$ to γ_1 as the first step, taking again the maximum of the propagator over m,l. In the second step we sum over the restriction of $\mathcal{E}^s$ to γ_2 and the propagator. It is obvious that the estimation (5.7) remains unchanged, in particular, $s_1+s_2=(s_1+1)+(s_2-1)=s-1$. If m_1 is the only unsummed index we realise the k_2-summation as a summation of the propagator over l. Here, one has to take into account that the subgraph γ_2 is bounded independently of the incoming index l. Again we get the same exponents as in (5.7).

We can summarise (5.7) and its discussed modification to

$$\begin{aligned}&\sum_{\mathcal{E}^s}\Big|\Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;m_2n_2;\sigma m\,\sigma n;\sigma k\,\sigma l;k_2l_2;k_1l_1}[\Lambda]\Big|\\&\leq\Big(\frac{\Lambda}{\mu}\Big)^{\delta_2(V-\frac{N}{2}+2-2g-B)}\Big(\frac{\mu}{\Lambda}\Big)^{\delta_1(V-V^e-\iota+2g+B-1+s)}\Big(\frac{\mu}{\Lambda}\Big)^{\delta_0(V^e+\iota-1-s)}\\&\qquad\times P^{2V-\frac{N}{2}-1}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,.\end{aligned}\qquad(5.9)$$

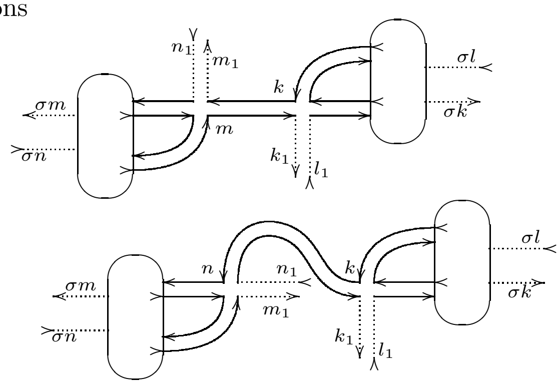
For the choice of the boundary conditions according to Definition/Lemma 3, the Λ-integration increases (again according to Definition/Lemma 3) the degree of the polynomial in $\ln\frac{\Lambda}{\Lambda_R}$ by 1. Hence, we have extended (4.10) to a bigger degree V for contractions of type (5.4). In particular, the bound is (by induction starting with (4.7), which represents the third graph in (3.7)) independent of the incoming indices n_i, l_i.

The verification of (4.10) for any contraction between graphs of the first line in (5.1) is performed in a similar manner. Taking the same subgraphs as in (5.4), but with a contraction of other legs, the discussion is in fact a little easier because there are no trajectories going through both subgraphs:



(5.10)

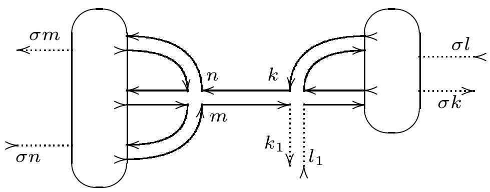
The contractions



(5.11)

are treated in the same way. The point is that the summation indices of the propagator (m,k for the upper graph and n,k for the lower graph in (5.11)) are fixed by index conservation for the subgraphs. In the same way one also discusses any contraction between the second and third graph in (5.1).

Let us now contract the left graph in the second line of (5.1) with any graph of the first line of (5.1), e.g.



(5.12)

The number of boundary components is reduced by 1, giving $B_1+B_2=B+1$. We clearly have $\iota=\iota_1+\iota_2$, but there is now one external vertex less on which we can apply an index summation, $V^e=V_1^e+V_2^e-1$. At the same time we need the index summation from the subgraph, because in the Λ-scaling

$$\Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,g,\iota)\gamma}_{k_1l_1;\sigma k\,\sigma l;\sigma m\,\sigma n}[\Lambda] = \frac{1}{2}\sum_{m,n,k} A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{mn;\sigma m\,\sigma n}[\Lambda]\,Q_{nm;k_1k}(\Lambda)\,A^{(V_2,V_2^e,B_2,g_2,\iota_2)\gamma_2}_{kk_1;k_1l_1;\sigma k\,\sigma l}[\Lambda] \tag{5.13}$$

there is now one undetermined summation index:

$$k=l_1+\sigma l-\sigma k\;,\qquad m(n)=n+\sigma n-\sigma m\;. \tag{5.14}$$

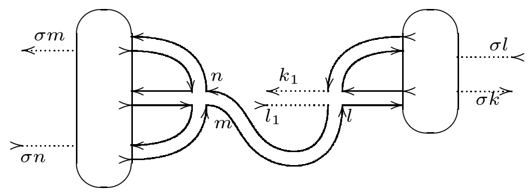
First, let there be an additional unsummed external leg on the segment of m,n in γ_1. Then, the induction hypothesis (4.10) gives the bound for a summation over m. We thus fix n,k and all indices of γ_2 in the first step and realise a possible k_1-summation due to $k_1=m+k-n$ as an m-summation, which is applied together with the summation over the γ_1-indices of $\mathcal{E}^s$, after maximising the propagator over m,k_1. The result is independent of n. We thus restrict the n-summation to the propagator, see (4.7), and apply the remaining $\mathcal{E}^s$-summations to γ_2, where k remains unsummed. We have $s_1+s_2=s$ and get the estimation

$$\begin{aligned}
&\sum_{\mathcal{E}^s\ni k_1,\;\bar{s}_1\le\bar{V}_1^e+\bar{\iota}_1-2}\Big|\Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,g,\iota)\gamma}_{k_1l_1;\sigma k\,\sigma l;\sigma m\,\sigma n}[\Lambda]\Big| \\
&\le\frac{1}{2}\Big(\sum_{m,\mathcal{E}_1^{s_1}}\big|A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{mn;\sigma m\,\sigma n}[\Lambda]\big|\Big)\Big(\max_k\sum_n\max_{m,k_1}\big|Q_{nm;k_1k}(\Lambda)\big|\Big) \\
&\qquad\times\Big(\sum_{\mathcal{E}_2^{s_2}\not\ni k_1}\big|A^{(V_2,V_2^e,B_2,g_2,\iota_2)\gamma_2}_{kk_1;k_1l_1;\sigma k\,\sigma l}[\Lambda]\big|\Big) \\
&\le\frac{1}{2}C_1\Big(\frac{\Lambda}{\mu}\Big)^{\delta_2(V-\frac{N+2}{2}+4-2g-(B+1))}\Big(\frac{\mu}{\Lambda}\Big)^{\delta_1(1+V-(V^e+1)-\iota+2g+(B+1)-2+s)} \\
&\qquad\times\Big(\frac{\mu}{\Lambda}\Big)^{\delta_0((V^e+1)+\iota-2-s)}P^{2V-\frac{N+2}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;.
\end{aligned} \tag{5.15}$$

If $k_1\notin\mathcal{E}^s$ we do not need the m-summation on γ_1. Again we have $s=s_1+s_2$ and (5.15) remains unchanged. Here, we may allow for index summations at all other external legs on the segment of m,n in γ_1.

If there is no unsummed external leg on the segment of m,n in γ_1, we must realise the k_1-summation as follows: We proceed as before up to the step where we sum the propagator for given k over n. For each term in this sum we have $k_1 = k+\sigma n-\sigma m$. We thus achieve a different k_1 if for given $\sigma m, \sigma n$ we start from a different k. Since the result of the summations over γ_1 and the propagator is independent of k, see (4.7), we realise the k_1-summation as a sum over k restricted to γ_2. We now get the same exponents as in (5.15) also for this case. According to Definition/Lemma 3, the Λ-integration extends for contractions of type (5.12) the bound (4.10) to a bigger order V.

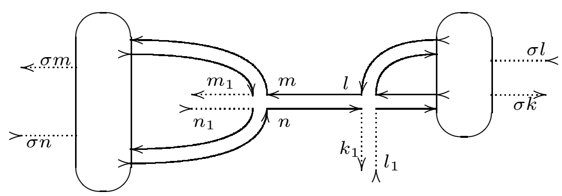
The contraction of the other leg of the right vertex



(5.16)

is easier to discuss because the k_1-summation is directly applied to γ_2. Taking the second vertex of the first line of (5.1) instead, we have two contractions which are identical to (5.12) and (5.16) and a third one with contractions as in the first and last graphs of (5.10) where γ_1 and γ_2 form different segments in γ. This case is much easier because there is no trajectory involving both subgraphs.

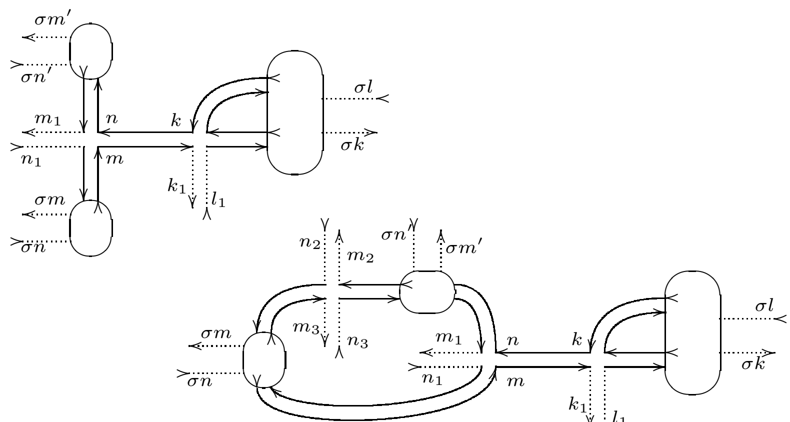
Moreover, contracting the last instead of the first vertex of the second line of (5.1) gives the same estimates if the two propagators between the vertex and the oval belong to the same segment:



(5.17)

The only modification to (5.15) and its variants is to replace (V^e+1) by V and ι by $(\iota+1)$, because the total number of external vertices is unchanged whereas the total segmentation index is reduced by 1.

If we contract the second vertex of the last line in (5.1) in such a way that the contracted indices m,n belong to different segments of γ_1, e.g.

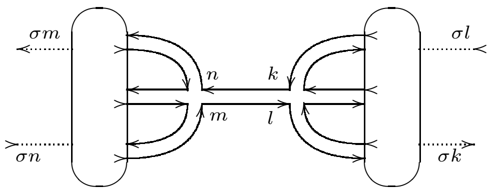


(5.18)

they are actually determined by index conservation for the segments. The entire discussion of these examples is therefore similar to the graph (5.4) with bound (5.7) and its modifications. Note that we have $V^e = V_1^e + V_2^e$ and $\iota = \iota_1 + \iota_2$ in (5.18).

Accordingly, we can replace in all previous examples a vertex of the first line of (5.1) by the composed vertex under the condition that the two contracted trajectories at the composed vertex belong to different segments.

It remains to study the contraction



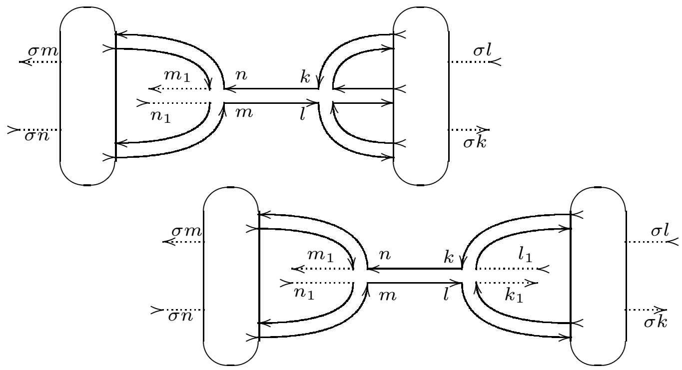
(5.19)

where two contraction indices (m or n and k or l) are undetermined. We have $V^e = V_1^e + V_2^e - 2$ and $\iota = \iota_1 + \iota_2$. We first assume that at least one of the boundary components of γ_i to contract carries more than one external vertex. In this case we have $B = B_1 + B_2 - 1$. There has to be at least one unsummed external vertex on the segment, say on γ_2. We fix the indices of γ_2 as well as n in the first step, take in the propagator the maximum over m, l and sum over the γ_1-indices of $\mathcal{E}^s$. Here, m can be regarded as an unsummed index. We take the maximum of γ_1 over n so that the n-summation restricts to the propagator only. We take in the summed propagator the maximum over k so that the remaining k-summation is applied together with the summation over the γ_2-indices of $\mathcal{E}^s$. We thus need $s_1 + s_2 = s + 1$ summations and the bound (4.7) for the propagator:

$$
\begin{aligned}
&\sum_{\mathcal{E}^s} \Big| \Lambda \frac{\partial}{\partial \Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{\sigma k\,\sigma l;\sigma m\,\sigma n}[\Lambda] \Big| \\
&\leq \frac{1}{2} \Big(\max_n \sum_{\mathcal{E}_1^{s_1}} \big| A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{mn;\sigma m\,\sigma n}[\Lambda] \big| \Big) \Big(\max_k \sum_n \max_{m,l} \big| Q_{nm;lk}(\Lambda) \big| \Big) \\
&\qquad\qquad \times \Big(\sum_{k,\mathcal{E}_2^{s_2}} \big| A^{(V_2,V_2^e,B_2,g_2,\iota_2)\gamma_2}_{kl;\sigma k\,\sigma l}[\Lambda] \big| \Big) \\
&\leq \frac{1}{2} C_1 \Big(\frac{\Lambda}{\mu} \Big)^{\delta_2 (V - \frac{N+2}{2} + 4 - 2g - (B+1))} \Big(\frac{\mu}{\Lambda} \Big)^{\delta_1 (1 + V - (V^e+2) - \iota + 2g + (B+1) - 2 + (s+1))} \\
&\qquad\qquad \times \Big(\frac{\mu}{\Lambda} \Big)^{\delta_0 ((V^e+2) + \iota - 2 - (s+1))} P^{2V - \frac{N+2}{2}} \Big[\ln \frac{\Lambda}{\Lambda_R} \Big] .
\end{aligned}
\tag{5.20}
$$

Finally, we have to consider the case where the only external vertices of both boundary components of γ_i to contract are just the contracted vertices. In this case the contraction removes these two boundary components at expense of a completely inner loop, giving $B = B_1 + B_2 - 2$. The differences $n - m$ and $k - l$ are fixed by the remaining indices of γ_i. For given m we may thus take the maximum of γ_2 over l and realise the l-summation as a summation (4.7) over the propagator. We thus exhaust all differences $m - l$. In order to exhaust all values of m we take the maximum of γ_1 over m, n and multiply the result by a volume factor (4.5). We thus replace in (5.20) $(s+1) \mapsto s$ and $(B+1) \mapsto (B+2)$, and combine one factor (4.6) and a volume factor (4.5) to (4.8). We thus get the same total exponents as in (4.10) so that the Λ-integration extends (4.10) to a bigger order V for all contractions represented by (5.19).

The contractions



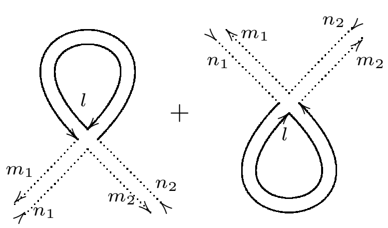
(5.21)

are treated in the same way as (5.19), now with the two unknown summation indices taken into account by a reduction of $V^e + \iota = (V^e_1 + \iota_1) + (V^e_2 + \iota_2) - 2$. In particular, there is also the situation where m, n and k, l are the only external legs of their boundary components before the contraction. In this case the number of boundary components drops by 2, which requires a volume factor in order to realise the sum over the starting point of the inner loop.

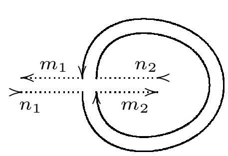
Thus, (4.10) is proven for any contractions produced by the first (bilinear) term on the rhs of (4.3).

5.2. Loop-contractions at the same vertex. It remains to verify the scaling formula (4.10) for the second term (the last line) on the rhs of the Polchinski equation (4.3), which describes self-contractions of graphs. The graphical data for the subgraph will obtain a subscript 1, such as the number of external vertices V^e_1, the segmentation index ι_1 and the set $\mathcal{E}^{s_1}_1$ of summation indices. We always have $V_1 = V$ and $N_1 = N + 2$. We first consider contractions of external lines at the same vertex, for which we have the possibilities shown in (5.1).

The very first vertex leads to two different self-contractions:



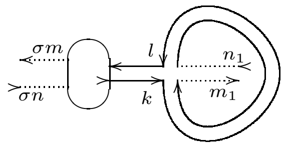
$$\Big|\Lambda\frac{\partial}{\partial\Lambda}A^{(1,1,1,0,0)}_{m_1n_1;m_2n_2}[\Lambda]\Big| = \sum_l |Q_{m_1l;ln_2}(\Lambda)|\,\delta_{n_1m_2} + \sum_l |Q_{ln_1;m_2l}(\Lambda)|\,\delta_{n_2m_1}\;, \tag{5.22}$$



$$\Big|\Lambda\frac{\partial}{\partial\Lambda}A^{(1,1,2,0,1)}_{m_1n_1;m_2n_2}[\Lambda]\Big| = |Q_{m_1n_2;m_2n_1}(\Lambda)|\;. \tag{5.23}$$

For the planar contraction (5.22) we estimate the l-summation by a volume factor so that we obtain (4.10) from (4.8). For the non-planar graph (5.23) we obtain (4.10) for $s = 0$ directly from (4.6). According to (3.15) we can apply one index summation which yields (4.10) via (4.7).

For the second graph in the first line of (5.1) we first investigate the contraction



(5.24)

The number of loops of the amputated graph is increased by 1, $\tilde{L} = \tilde{L}_1 + 1$, so that due to (3.11) and $I = I_1 + 1$ we get $g = g_1$. The graph (5.24) determines the Λ-variation

$$\Lambda \frac{\partial}{\partial \Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{m_1 n_1;\sigma m\,\sigma n}[\Lambda] = -\frac{1}{2} \sum_{k,l} Q_{m_1 k; l n_1}(\Lambda)\, A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{m_1 n_1; n_1 l; \sigma m\,\sigma n; k m_1}[\Lambda]\;, \tag{5.25}$$

with one of the indices k, l being undetermined. First, let there be at least one further external leg on the same boundary component as l, k. In this case the number of boundary components is increased by 1, $B = B_1 + 1$. If there is an unsummed index on the segment of k, l we can realise the k-summation in γ as a summation in γ_1 after taking in the propagator the maximum over k, l. We thus have $s_1 = s + 1$ and consequently

$$\begin{aligned}
\sum_{\mathcal{E}^s,\; m_1 \notin \mathcal{E}^s} & \Big| \Lambda \frac{\partial}{\partial \Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{m_1 n_1;\sigma m\,\sigma n}[\Lambda] \Big| \\
&\leq \frac{1}{2} \Big(\max_{k,l} \big| Q_{m_1 k; l n_1}(\Lambda) \big| \Big) \Big(\sum_{k, \mathcal{E}_1^s} \big| A^{(V,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{m_1 n_1; n_1 l; \sigma m\,\sigma n; k m_1}[\Lambda] \Big) \\
&\leq \frac{1}{2} C_0 \Big(\frac{\Lambda}{\mu_0}\Big)^{\delta_2 (V - \frac{N+2}{2} + 2 - 2g - (B-1))} \Big(\frac{\mu}{\Lambda}\Big)^{\delta_1 (V - V^e - \iota + 2g + (B-1) - 1 + (s+1))} \\
&\qquad \times \Big(\frac{\mu}{\Lambda}\Big)^{\delta_0 (1 + V^e + \iota - 1 - (s+1))} P^{2V - \frac{N+2}{2}} \Big[\ln \frac{\Lambda}{\Lambda_R} \Big]\;.
\end{aligned} \tag{5.26}$$

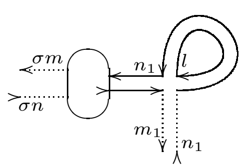
We can sum the contracting propagator over m_1 for fixed n_1, which amounts to replace one factor (4.6) by (4.7) compensated by $s = s_1$ replacing $s = s_1 - 1$.

If k cannot be a summation index in γ_1 then m_1 must be unsummed in γ. We first apply the summation over $\mathfrak{o}[l]$ for given l in γ_1. The result is independent of l so that, for given k, the l-summation can be restricted to the contracting propagator maximised over m_1, n_1. Finally, the remaining $\mathcal{E}^s$-summations are applied. We have to replace in (5.26) $(s + 1)$ by s and one factor (4.6) by (4.7).

Finally, let there be no further external leg on the same boundary component as l, k. Now the number of boundary components remains constant, $B = B_1$. Since $k - l = n_1 - m_1$ is a constant, the required summation over e.g. k has to be estimated by a volume factor (4.5). We thus replace in (5.26) $(B - 1) \mapsto B$ and $(s + 1) \mapsto s$ and combine one factor (4.6) and the volume factor to (4.8).

In summary, we extend after Λ-integration the scaling law (4.10) for the same degree V to a reduced number N of external lines.

Next, we study the following contraction of the second graph in (5.1) which gives rise to an inner loop:



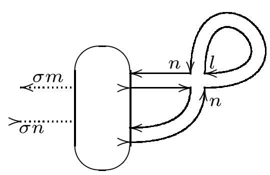
(5.27)

It describes the Λ-variation

$$\Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\sigma m\,\sigma n}[\Lambda] = -\frac{1}{2}\Big(\sum_l Q_{n_1l;ln_1}(\Lambda)\Big)\,A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{m_1n_1;n_1l;ln_1;\sigma m\,\sigma n}[\Lambda]\;. \tag{5.28}$$

The number of loops of the amputated graph is increased by 1 and the number of boundary components remains constant, giving $g = g_1$ and $B = B_1$. Note that $A^{(V,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{m_1n_1;n_1l;ln_1;\sigma m\,\sigma n}$ is independent of l so that the l-summation acts on the propagator only. We estimate the l-summed propagator by (4.8) for the product of (4.6) with a volume factor (4.5). The factor (4.8) compensates the decrease $N = N_1 - 2$, all other exponents remain unchanged when passing from γ_1 to γ. Now the Λ-integration extends the scaling law (4.10) to a reduced N.

The third graph in the first line of (5.1) leads to the contracted graph

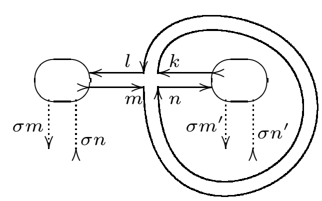


(5.29)

There is one additional loop of the amputated graph, giving $g = g_1$. We have $B = B_1$ if there are further external legs on the boundary component of n and $B = B_1 - 1$ if no further external leg exists on the contracted boundary component. Very similar to (5.27), the l-summation is restricted to the propagator maximised over n, giving a factor (4.8) which compensates $N = N_1 - 2$ in the first exponent of (4.10). For $B = B_1$ the n-summation in (5.29) is provided by the subgraph γ_1, where the additional summation $s_1 = s + 1$ compared with γ compensates the change $V_1^e = V^e + 1$ of external vertices in the second and third exponent of (4.10).

On the other hand, if $B_1 = B + 1$ we have $s = s_1$ and the summation over n has to come from a volume factor (4.5) combined with one factor (4.6) to (4.8). This verifies (4.10) for the contraction (5.29).

The last case for which contractions of two external lines at the same vertex are to investigate is the last vertex in the second line of (5.1). As before in the proof for tree-contractions, we have to distinguish whether the composed vertex under consideration appears inside a tree, in a loop but together with further composed vertices, or in a loop but as the single composed vertex. In the first case we have to analyse the graph



(5.30)

Before the contraction, the indices m, n, k, l were all located on the same loop of the amputated graph and the same boundary component. After the contraction they are split into two loops, $g = g_1$. The number of boundary components is increased by 1 if both resulting boundary components of l, m and k, n carry further external legs, $B = B_1 + 1$. We have $B = B_1$ if only one of the resulting boundary components of l, m or k, n carries further external legs and $B = B_1 - 1$ if there are no further external legs on these boundary components. We clearly have $\iota = \iota_1$ and

$V^e = V_1^e - 1$. Due to index conservation for segments, either k or n is an unknown summation index, and either l or m.

We first consider the case $B = B_1 + 1$. In both segments of γ_1 to contract there must be at least one unsummed outgoing index, which we can choose to be different from the vertex to contract. We thus take in the propagator the maximum (4.6) over all indices and restrict the required index summations over k, m to the segments of the subgraphs. This means that we have $s_1 = s+2$ summations, which compensates the change of the numbers of boundary components $B_1 = B - 1$, external legs $N_1 = N + 2$ and external vertices $V_1^e = V^e + 1$:

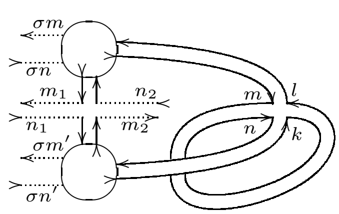
$$\begin{aligned}
&\sum_{\mathcal{E}^s, B=B_1+1} \Big| \Lambda \frac{\partial}{\partial \Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{\sigma m\, \sigma n; \sigma m'\, \sigma n'}[\Lambda] \Big| \\
&\leq \frac{1}{2} \Big(\max_{m,n,k,l} \big| Q_{nm;lk}(\Lambda) \big| \Big) \Big(\max_{l,n} \sum_{k,m,\mathcal{E}^s} \big| A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{\sigma m\, \sigma n; mn; \sigma m'\, \sigma' n; kl}[\Lambda] \big| \Big) \\
&\leq \frac{1}{2} C_0 \Big(\frac{\Lambda}{\mu} \Big)^{\delta_2 (V - \frac{N+2}{2} + 2 - 2g - (B-1))} \Big(\frac{\mu}{\Lambda} \Big)^{\delta_1 (V - (V^e+1) - \iota + 2g + (B-1) - 1 + (s+2))} \\
&\qquad \times \Big(\frac{\mu}{\Lambda} \Big)^{\delta_0 (1 + (V^e+1) + \iota - 1 - (s+2))} P^{2V - \frac{N+2}{2}} \Big[\ln \frac{\Lambda}{\Lambda_R} \Big] . \qquad (5.31)
\end{aligned}$$

We immediately confirm (4.10). Alternatively, instead of consuming a γ_1-summation to get the k-summation we can also sum the propagator for maximised l, m and given k over n. Compared with (5.31) we have to replace $(s+2)$ by $(s+1)$ and one factor (4.6) by (4.7), ending up in the same exponents.

Next, we investigate the case $B = B_1$ where, for example, the restriction of the boundary component to the left segment does not carry another external leg than m, l. The summation over m in γ_1 is now provided by a volume factor, which means that in (5.31) we have to replace $(s+2)$ by $(s+1)$, $(B-1)$ by B and one factor (4.6) by (4.8). All exponents match again (4.10).

Finally, let us look at the possibility $B = B_1 - 1$ where the indices m, n, k, l to contract were the only external indices of the boundary component. We thus combine two volume factors (4.5) and two factors (4.6) to two factors (4.8), compensating $(B-1) \mapsto (B+1)$ and $(s+2) \mapsto s$. After Λ-integration we extend (4.10) to a reduced N.

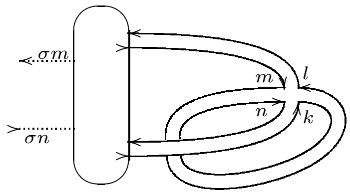
The case that the two sides of the composed vertex to contract are connected but belong to different segments, e.g.



(5.32)

is similar to treat concerning index summations, but for the interpretation of the genus there is a different situation possible. In the amputated subgraph γ_1 the indices m, n and k, l may be situated on *different* loops and thus different boundary components. The contraction joins in this case the two loops, $\tilde{L} = \tilde{L}_1 - 1$, which results due to (3.11) in $g = g_1 + 1$ and $B = B_1 - 1$. There is at least one additional external leg on each of the boundary components of m, n and k, l before the contraction, because in order to close the loop we have to pass through the vertex m_1, n_1, m_2, n_2. Now we have to replace in (5.31) $(B-1)$ by $(B+1)$ and g by $(g-1)$, confirming (4.10) also in this case. If all indices m,n,k,l are on the same loop in γ_1, the contraction splits it into two and the entire discussion of (5.30) can be used without modification to the present example.

It remains the case



(5.33)

where the two halves of the composed vertex to contract belong to the same segment. Three of the indices m,n,k,l are now summation indices. We have $\iota = \iota_1 - 1$ and $V^e = V_1^e - 1$. Let first the indices m,n on one hand and k,l on the other hand be situated on different loops of the amputated graph γ_1. These are joint by the contraction, yielding $g = g_1 + 1$. If there remain further external legs on the contracted loop we have $B = B_1 - 1$, otherwise $B = B_1 - 2$. We start with $B = B_1 - 1$. Due to the segmentation index present in γ_1, the induction hypothesis for γ_1 gives us the bound for two additional summations over m,k not present in γ. The third summation is provided by the propagator via (4.7). Assuming $\mathtt{i}[k] \neq l,n$ in γ_1 we first take in the contracting propagator the maximum over m,l, then sum the m,n-boundary component over m and those indices of $\mathcal{E}^s$ which belong to the m,n-boundary component, followed by the summation of the propagator over n for given k. Finally, we sum γ_1 over the remaining indices of $\mathcal{E}^s$ and over k:

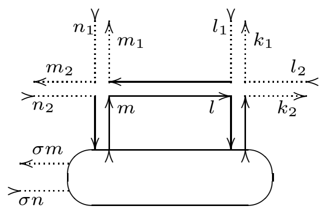
$$
\begin{aligned}
&\sum_{\mathcal{E}^s}\Big|\Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,g,\iota)\gamma}_{\sigma m\,\sigma n}[\Lambda]\Big| \\
&\quad\leq \frac{1}{2}\Big(\max_k\sum_n\max_{l,m}\big|Q_{mn;kl}(\Lambda)\big|\Big)\sum_{k,m,\mathcal{E}^s}\big|A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{\sigma m\,\sigma n;mn;kl}[\Lambda]\big|\Big) \\
&\quad\leq \frac{1}{2}C_1\Big(\frac{\Lambda}{\mu}\Big)^{\delta_2(V-\frac{N+2}{2}+2-2(g-1)-(B+1))} \\
&\qquad\qquad\times\Big(\frac{\mu}{\Lambda}\Big)^{\delta_1(1+V-(V^e+1)-(\iota+1)+2(g-1)+(B+1)-1+(s+2))} \\
&\qquad\qquad\times\Big(\frac{\mu}{\Lambda}\Big)^{\delta_0((V^e+1)+(\iota+1)-1-(s+2))}P^{2V-\frac{N+2}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;.
\end{aligned}
\qquad (5.34)
$$

It is essential that the summation over $k - \mathtt{i}[k]$ is independent.

If there are no external legs on the contracted loop, $B = B_1 - 2$, then we have in γ_1 either $\mathtt{i}[m] = n,\ \mathtt{i}[k] = l$ or $\mathtt{i}[m] = l,\ \mathtt{i}[k] = n$. In the first case we would first fix n,k and maximise the propagator over m,l. Now the m-summation restricts to γ_1 with bound independent of n. Thus, the n-summation for given k restricts to the propagator and delivers a factor (4.7), independent of k. However, since $k - \mathtt{i}[k] \equiv k - l = n - m$ is already exhausted in γ_1, the remaining k-summation has to come from a volume factor. We thus make in (5.34) the replacements $(B{+}1) \mapsto (B{+}2)$, $(s{+}2) \mapsto (s{+}1)$ and combine one factor (4.6) with a volume factor (4.5) to (4.8). The exponents match again (4.10).

Next, we investigate the situation where all indices m,n,k,l are located on the same loop of the amputated subgraph γ_1. In this case the contraction to γ splits that loop into two so that we have $g = g_1$. As before we have $B = B_1 + 1$ if both split loops contain further external legs, $B = B_1$ if only one of the split loops contains further external legs, and $B = B_1 - 1$ if the split loops do not contain further external legs. The discussion is similar as for (5.30), the difference is that three of m, n, k, l are now summations indices, which is taken into account by the replacement of ι in (5.31) by $(\iota + 1)$. We thus finish the verification of (4.10) for self-contractions of a vertex.

5.3. Loop-contractions at different vertices. It remains to check (4.10) for contractions of different vertices of the same graph. The external lines of the two vertices are arranged according to (5.1). We start with two vertices of the type shown as the second graph in (5.1). One possible contraction of their external lines is



(5.35)

assuming that the vertices to contract are located on the same segment in γ_1. One of the indices m, l is a summation index. We first consider the case that the two vertices to contract are located on the same loop of the amputated graph γ_1. The contraction to γ splits that loop into two, giving $g = g_1$. We have $B = B_1 + 1$ if the trajectory starting at l does not leave γ_1 (and γ) in m, whereas $B = B_1$ if m, l are on the same trajectory in γ_1. In case of $B = B_1 + 1$ we keep $\mathfrak{i}[m]$ in γ_1 fixed, take in the propagator the maximum over m, l and restrict the m-summation to γ_1. Due to $V_1^e = V^e$, $\iota_1 = \iota$ and $B_1 = B - 1$ we have in the case that m_1 remains unsummed

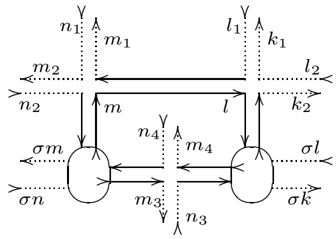
$$
\begin{aligned}
&\sum_{\mathcal{E}^s \not\ni m_1, B=B_1+1} \Big| \Lambda \frac{\partial}{\partial \Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{k_2 l_2; k_1 l_1; m_1 n_1; m_2 n_2; \sigma m\, \sigma n}[\Lambda] \Big| \\
&\leq \frac{1}{2} \Big(\max_{m,l,m_1,l_1} \big| Q_{m_1 m; l l_1}(\Lambda) \big| \Big) \Big(\sum_{m, \mathcal{E}^s} \big| A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{k_2 l_2; k_1 l_1; l_1 l; m m_1; m_1 n_1; m_2 n_2; \sigma m\, \sigma n}[\Lambda] \big| \Big) \\
&\leq \frac{1}{2} C_0 \Big(\frac{\Lambda}{\mu} \Big)^{\delta_2 (V - \frac{N+2}{2} + 2 - 2g - (B-1))} \Big(\frac{\mu}{\Lambda} \Big)^{\delta_1 (V - V^e - \iota + 2g + (B-1) + 1 + (s+1))} \\
&\qquad \times \Big(\frac{\mu}{\Lambda} \Big)^{\delta_0 (1 + V^e + \iota - 1 - (s+1))} P^{2V - \frac{N+2}{2}} \Big[\ln \frac{\Lambda}{\Lambda_R} \Big] .
\end{aligned}
\tag{5.36}
$$

Summing additionally over m_1 we replace in (5.36) one factor (4.6) by (4.7). It is clear that this reproduces the exponents of (4.10) correctly.

If $l = \mathfrak{i}[m]$ in γ_1, we have to realise the m-summation by a volume factor. We thus replace in (5.36) $(B-1) \mapsto B$, $(s+1) \mapsto s$ and combine (4.5) with one factor (4.6) to (4.8).

Finally, the two vertices to contract in (5.35) may be located on different loops of the amputated graph γ_1. They are joint by the contraction to γ, giving $g = g_1 + 1$, and because the newly created loop obviously has external legs, we have $B = B_1 - 1$. As separated loops in γ_1, l cannot be the incoming index of the trajectory through m. Therefore, the m-summation gives the same bound as the rhs of (5.36), now with $(B-1)$ replaced by $(B+1)$ and g by $(g-1)$. We have thus extended (4.10) to a reduced N for all types of contractions (5.35).

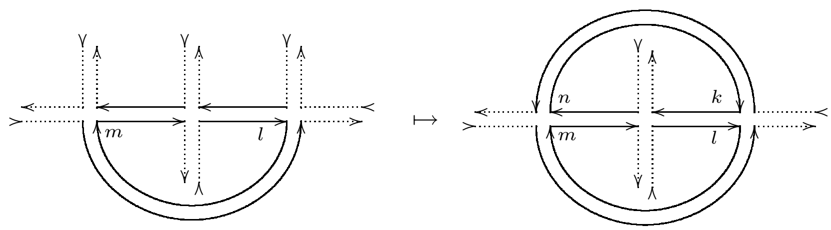
If the vertices to contract are located on different segments in γ_1, e.g.



(5.37)

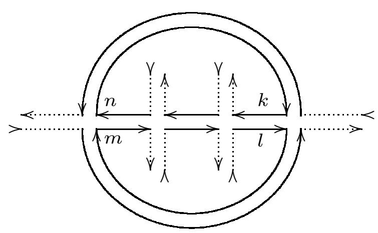
both indices m, l are determined by index conservation for segments. We can thus save an index summation compared with (5.36) and replace there and in its discussed modifications $(s+1)$ by s and $\iota_1 = \iota$ by $\iota_1 = (\iota - 1)$. Since the m-summation is not required, there is effectively an additional summation possible in agreement with (3.15). It is not possible that m and l are located on the same trajectory in γ_1 so that either $g = g_1, B = B_1 + 1$ or $g = g_1 + 1, B = B_1 - 1$.

Let us make a few more comments on the segmentation index. It is essential that the contraction joins separated segments. For instance, the contraction



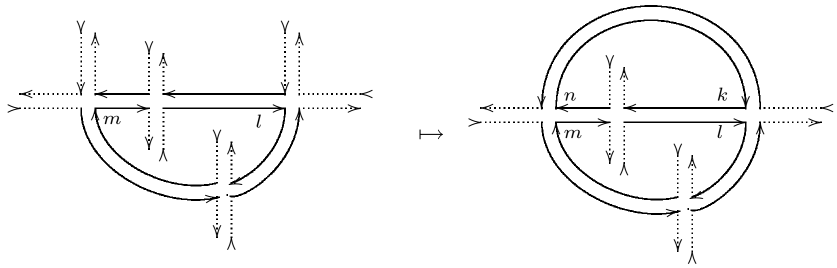
(5.38)

does not increase the segmentation index, because in agreement with Definition 4 the number of segments remains constant. The graph on the left has $\iota = 1$, and the internal indices m, l are determined by the external ones. The graph on the right has $\iota = 1$ as well, and now one of the indices n, k becomes a summation index. Having several composed vertices in the middle link does not change the segmentation index:



(5.39)

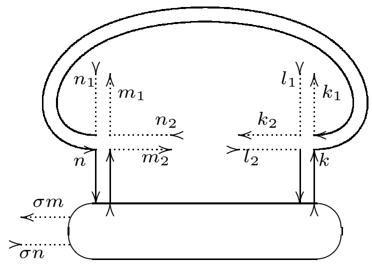
It makes, however, a difference if the two composed vertices are situated on different links:



(5.40)

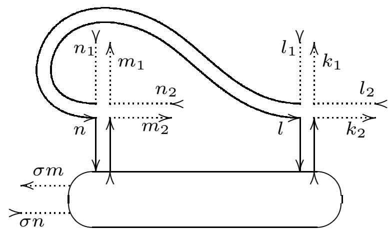
Here, the segmentation index increases from $\iota = 1$ on the left to $\iota = 2$ on the right, in agreement with Definition 4.

The case



(5.41)

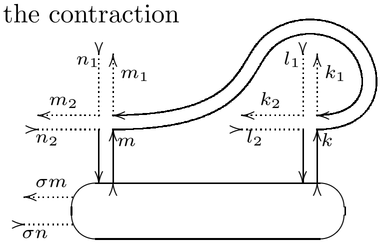
is completely identical to (5.35). In the contraction



(5.42)

the summation index n or l is provided by the propagator, replacing in (5.36) and its modifications $(s+1)$ by s and one factor (4.6) by (4.7). It is not possible that n and l are located on the same trajectory in γ_1 so that either $g = g_1, B = B_1 + 1$ or $g = g_1 + 1, B = B_1 - 1$.

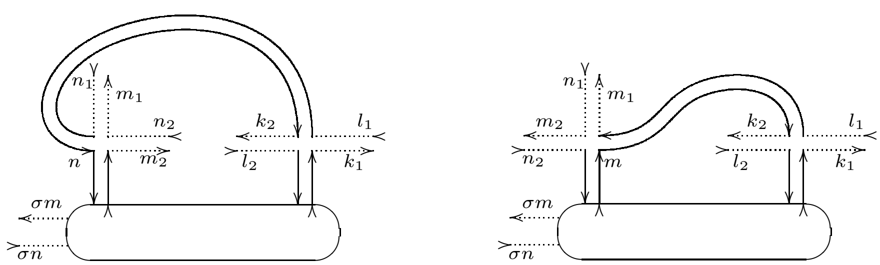
In order to treat the contraction



(5.43)

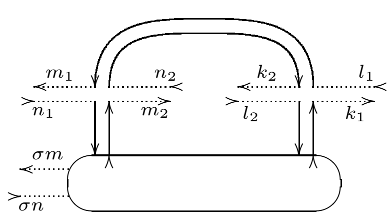
one has to use that the summation over m_1 can due to $m_1 = k+m-k_1$ be transferred as a k-summation of γ_1. The summation over the undetermined index m is applied in the last step.

Finally,



(5.44)

are similar to the ι-increased variant (5.37). The contraction

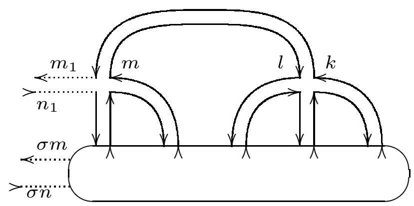


(5.45)

is an example for a realisation of (3.14) where V_c is increased by 2 and S by 1, giving again a segmentation index increased by 1.

It is obvious that the discussion of contractions involving the second and third or two of the third vertices of the first line in (5.1) is analogous.

Let us now study loop contractions which involve the first graph in the second line of (5.1), assuming first that the vertices are situated on the same segment:



(5.46)

We thus have $\iota = \iota_1$ and $V^e = V_1^e - 1$. Two of the summation indices m, k, l are undetermined. Let first the two vertices to contract be located on the same loop of the amputated subgraph γ_1. The contraction splits that loop into two, giving $g = g_1$. Next question concerns the number of boundary components. We have $B = B_1 + 1$ if there are further external legs on the loop through l, m and $B = B_1$ if $l = \mathfrak{i}[m]$ in γ_1. We start with $B = B_1 + 1$. In general, the induction hypothesis provides us with bounds for summations over m and k, because $l \neq \mathfrak{i}[m]$. If m_1 is an unsummed index we thus have

$$
\begin{aligned}
&\sum_{\mathcal{E}^s \not\ni m_1, B=B_1+1} \Big|\Lambda \frac{\partial}{\partial \Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{m_1 n_1;\sigma m\, \sigma n}[\Lambda]\Big| \\
&\leq \frac{1}{2}\Big(\max_{m,l,k,m_1} \big|Q_{m_1 m;lk}(\Lambda)\big|\Big)\Big(\sum_{m,k,\mathcal{E}^s} \big|A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{kl;nm_1;m_1n_1;\sigma m\,\sigma n}[\Lambda]\big|\Big) \\
&\leq \frac{1}{2} C_0 \Big(\frac{\Lambda}{\mu}\Big)^{\delta_2(V - \frac{N+2}{2} + 2 - 2g - (B-1))} \Big(\frac{\mu}{\Lambda}\Big)^{\delta_1(V - (V^e+1) - \iota + 2g + (B-1) + 1 + (s+2))} \\
&\qquad\qquad \times \Big(\frac{\mu}{\Lambda}\Big)^{\delta_0(1 + (V^e+1) + \iota - 1 - (s+2))} P^{2V - \frac{N+2}{2}}\Big[\ln \frac{\Lambda}{\Lambda_R}\Big]\,. \qquad (5.47)
\end{aligned}
$$

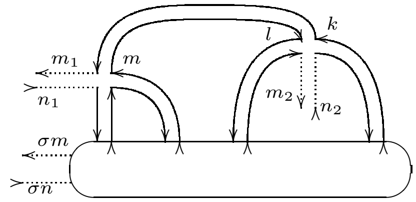
Now an additional summation over m_1 can immediately be taken into account by replacing the maximised propagator (4.6) by the summed propagator (4.7), in agreement with $(s+2)$ replaced by $(s+1)$. The m_1-summation is applied before the k-summation is carried out.

These considerations require an unsummed outgoing index on the contracted segment of γ_1. If this is not the case then m_1 has to be the unsummed outgoing index. Now the l-summation for given m has to be restricted to the propagator and delivers a factor (4.7). The exponents match again (4.10).

Next, for $l = \mathfrak{i}[m]$ in γ_1 we cannot use a summation over m in γ_1 in order to account for the undetermined contraction index, because the incoming index l would change simultaneously. Instead we have to use a volume factor (4.5) combined with one factor (4.6) to (4.8). Additionally we have to replace in (5.47) $(s+1)$ by s and $(B-1)$ by B.

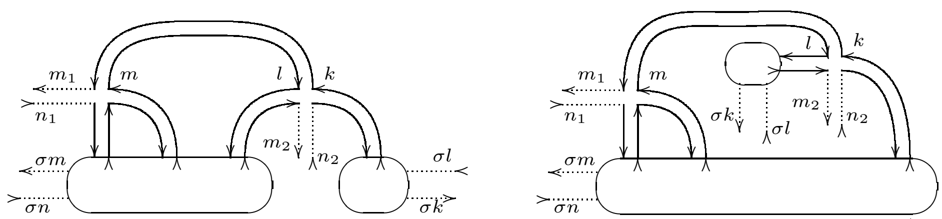
Second, the two vertices to contract may be located on different loops of the amputated graph γ_1. They are joint by the contraction, giving $g = g + 1$. Because the loop carries at least the external leg $m_1 n_1$, we necessarily have $B = B_1 - 1$. Now, $l \neq \mathfrak{i}[m]$ in γ_1 so that we use summations over m, k in γ_1, giving the same balance (5.47) for the exponents, with $(B-1) \mapsto (B+1)$ and $g \mapsto (g-1)$.

The discussion is identical for the contraction



(5.48)

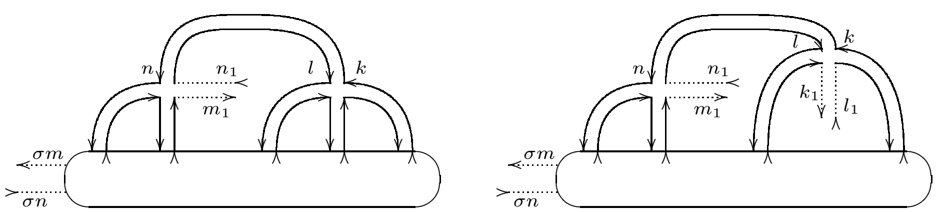
which in the case that k, l belong to the same segment in γ has two undetermined summation indices as well. We thus proceed as in (5.47) and its discussed modification and only have to replace $(V^e + 1)$ by V^e and ι by $(\iota + 1)$. If k, l are situated on different segments in γ, e.g.



(5.49)

there is only one undetermined summation index, which is reflected in the analogue of (5.47) by the fact that the segmentation index remains unchanged, $\iota = \iota_1$. Note that in the right graph of (5.49) we either have $B = B_1 - 1, g = g + 1$ or $B = B_1 + 1, g = g$. Of course we get the same estimations if the segment of γ_1 with external lines $\sigma k, \sigma l$ are connected by several composed vertices to the part of γ_1 with external lines $\sigma m, \sigma n$.

The contractions

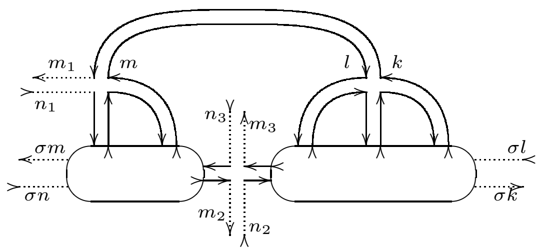


(5.50)

are a little easier because the contracting propagator does not have outgoing indices which for certain summations had to be transferred to the subgraph γ_1. If $n = \mathrm{i}[k]$ in γ_1, the k-summation for given n, n_1 can be restricted to γ_1 after maximising the propagator over all indices. Since the result for γ_1 is independent of the starting point n, the k-summation can be regarded as a summation over all differences $k-n$. The final summation over all pairs k, n with fixed difference $k - n$ is provided by a volume factor (4.5) combined with (4.6) to (4.8). The balance of exponents is identical to (5.47) and its discussed variants.

It is clear that the analogue of (5.49) with the left vertex connected as in (5.50) is similar to treat.

Next, we discuss the variant of (5.46) where the two vertices to contract belong to different segments in the subgraph γ_1:

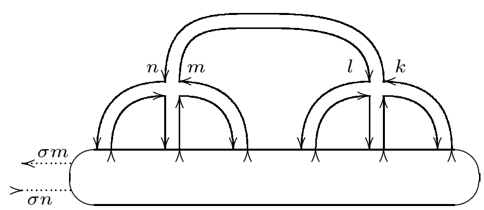


(5.51)

Now only one of the indices m, k, l is an undetermined summation index, with k being the most natural choice. We therefore get a bound for the Λ-scaling analogous to (5.47) but with ι replaced by $\iota - 1$, reflecting the increase of the segmentation index $\iota = \iota_1 + 1$. There is now an additional index summation possible, here via (4.7) over the index m_1. Note that we have either $B = B_1 - 1, g = g + 1$ or $B = B_1 + 1, g = g$.

The discussion of the variants of (5.51) with the right vertex taken as the second one in the last line of (5.1) and/or the left vertex arranged as in (5.50) is very similar.

It remains to investigate contractions between two of the vertices in the second line of (5.1). We discuss in detail the contraction



(5.52)

All variants are similar as described between (5.46) and (5.51).

Three of the four summation indices m, n, k, l in (5.52) are undetermined. We clearly have $V_1^e = V^e + 2$ and $\iota = \iota_1$. We first consider the case where the four indices m, n, k, l are located on the same loop of the amputated subgraph γ_1. The contraction will split that loop into two, giving $g = g_1$. There are three possibilities for the change of the number of boundary components after the contraction. First, if on both paths of trajectories in γ_1 from n to k and from l to m there are further external legs, we have $B = B_1 + 1$. Second, if on one of these paths there is no further external leg, we have $B = B_1$. Third, if both paths contain no further external legs, i.e. m and k are the outgoing indices of the trajectories starting at l and n, respectively, we have $B = B_1 - 1$.

We start with $B = B_1 + 1$. Then, $\mathfrak{i}[k]$ and $\mathfrak{i}[m]$ are fixed as external indices so that the induction hypothesis for γ_1 provides the bounds for two summations over k, m. We first apply a possible summation to the outgoing index of the trajectory starting at l. The result is maximised independently from l so that we can restrict the l-summation to the propagator, maximised over k, n with m being fixed. Finally, we apply the summations over k, m and all remaining $\mathcal{E}^s$-summations to γ_1. We thus obtain

$$\begin{aligned}
&\sum_{\mathcal{E}^s, B=B_1+1} \Big| \Lambda \frac{\partial}{\partial \Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{\sigma m\,\sigma n}[\Lambda] \Big| \\
&\leq \frac{1}{2} \Big(\max_m \sum_l \max_{n,k} \big| Q_{nm;lk}(\Lambda) \big| \Big) \Big(\sum_{m,k,\mathcal{E}^s} \big| A^{(V_1,V_1^e,B_1,g_1,\iota_1)\gamma_1}_{mn;\sigma m\,\sigma n;kl}[\Lambda] \big| \Big) \\
&\leq \frac{1}{2} C_1 \Big(\frac{\Lambda}{\mu} \Big)^{\delta_2(V-\frac{N+2}{2}+2-2g-(B-1))} \Big(\frac{\mu}{\Lambda} \Big)^{\delta_1(1+V-(V^e+2)+\iota+1+2g+(B-1)-1+(s+2))} \\
&\qquad \times \Big(\frac{\mu}{\Lambda} \Big)^{\delta_0((V^e+2)+\iota-1-(s+2))} P^{2V-\frac{N+2}{2}} \Big[\ln \frac{\Lambda}{\Lambda_R} \Big]\;. \qquad (5.53)
\end{aligned}$$

The Λ-integration verifies (4.10) in the topological situation under consideration.

Next, we discuss the case $B = B_1$, assuming e.g. $l = \mathfrak{i}[m]$ in γ_1. We maximise the propagator over k, n for given l so that the m-summation can be restricted to γ_1. Next, we apply the $\mathcal{E}^s$-summations and the k-summation to γ_1, still for given l. The final l-summation counts the number of graphs with different l, giving the bound (4.6) of the propagator times a volume factor. In any case the required modifications of (5.53), in particular $(B-1) \mapsto B$, lead to the correct exponents of (4.10).

If $B = B_1 - 1$, i.e. $l = \mathfrak{i}[m]$ and $n = \mathfrak{i}[k]$, we take in the propagator the maximum over n, k so that for given l the m-summation can be restricted to γ_1. The result of that summation is bounded independently of l. Thus, each summand only fixes $m - l = n - k$, and the remaining freedom for the summation indices is exhausted by two volume factors and the bound (4.6) for the propagator. We thus replace in (5.53) $(s+2) \mapsto (s+1)$, $(B-1) \mapsto (B+1)$ and one factor (4.7) by (4.6). Then two factors (4.6) are merged with two volume factors (4.5) to give two factors (4.8).

Finally, we have to consider the case where m, n are located on a different loop of the amputated subgraph γ_1 than k, l. The contraction joins these loops, giving $g = g_1 + 1$. If the resulting loop carries at least one external leg we have $B = B_1 - 1$, whereas we get $B = B_1 - 2$ if the resulting loop does not carry any external legs. We first consider the case that there is a further external leg on the n, m-loop in γ_1. We take in the propagator the maximum over k, n and sum the subgraph for given l, n over k and possibly the outgoing index of the n-trajectory. The result is independent of l, n. Next, we sum the propagator for given m over l and finally apply the remaining $\mathcal{E}^s$-summations and the summation over m to γ_1. We get the same estimates as in (5.53) with $(B-1)$ replaced by $(B+1)$ and g by $(g-1)$.

If there are no further external legs on the contracted loop we would maximise the propagator over k, n, then sum γ_1 over k for given l, next sum the propagator over l for given m. For each resulting pair k, l the remaining m-summation leaves $m-n$ constant. We thus have to use a volume factor in order to exhaust the freedom of $m - n$, combining one factor (4.6) and the volume factor (4.5) to (4.8). We thus confirm (4.10) for any contraction of the form (5.52).

It is obvious that all examples not discussed in detail are treated in the same manner. We conclude that (4.10) provides the correct bounds for the interaction coefficients of ϕ^4-matrix model with cut-off propagator described by the three exponents $\delta_0, \delta_1, \delta_2$. □

6. Discussion

By solving the Polchinski equation perturbatively we have derived a power-counting theorem for non-local matrix models with arbitrary propagator.

Our main motivation for the renormalisation group investigation of non-local matrix models was to tackle the renormalisation problem of field theories on non-commutative $\mathbb{R}^D$ from a different perspective. The momentum integrals leading to the parametric integral representation are not absolutely convergent; nevertheless one exchanges the order of integration. In momentum space one can therefore not exclude the possibility that the UV/IR-mixing is due to the mathematically questionable exchange of the order of integration.

The renormalisation group approach to noncommutative field theories in matrix formulation avoids these problems. We work with cut-off propagators leading to finite sums and take absolute values of the interaction coefficients throughout. Oscillating phases never appear; they are not required for convergence of certain graphs.

Our power-counting theorem provides a necessary condition for renormalisability: The two scaling exponents δ_0, δ_1 of the cut-off propagator have to be large enough relative to the dimension of the underlying space. In [13,14] we determine these exponents for ϕ^4-theory on noncommutative $\mathbb{R}^D$, $D = 2, 4$:

Proposition 8 *The propagator for the real scalar field on noncommutative $\mathbb{R}^D$, $D = 2, 4$, is characterised by the scaling exponents $\delta_0 = 1$ and $\delta_1 = 0$. Adding a harmonic oscillator potential to the action one achieves $\delta_0 = \delta_1 = 2$.*

We thus conclude that scalar models on noncommutative $\mathbb{R}^D$ are anomalous unless one adds the regulating harmonic oscillator potential.

The weak decay $\sim \Lambda^{-1}$ of the propagator leads to divergences in $\Lambda \sim \Lambda_0 \to \infty$ of arbitrarily high degree. The appearance of unbounded degrees of divergences in field theories on noncommutative $\mathbb{R}^4$ is often related to the so-called UV/IR-mixing [1]. We learn from the power-counting theorem (Theorem 6) that similar effects will show up in any matrix model in which the propagator decays too slowly with Λ. This means that the correlation between distant modes is too strong, i.e. the model is too non-local.

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