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Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^2$ in the matrix base

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Abstract

As a first application of our renormalisation group approach to non-local matrix models [hep-th/0305066], we prove (super-) renormalisability of Euclidean two-dimensional noncommutative ϕ^4-theory. It is widely believed that this model is renormalisable in momentum space arguing that there would be logarithmic UV/IR-divergences only. Although momentum space Feynman graphs can indeed be computed to any loop order, the logarithmic UV/IR-divergence appears in the renormalised two-point function—a hint that the renormalisation is not completed. In particular, it is impossible to define the squared mass as the value of the two-point function at vanishing momentum. In contrast, in our matrix approach the renormalised N-point functions are bounded everywhere and nevertheless rely on adjusting the mass only. We achieve this by introducing into the cut-off model a translation-invariance breaking regulator which is scaled to zero with the removal of the cut-off. The naïve treatment without regulator would not lead to a renormalised theory.

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1 Introduction

In spite of enormous efforts, the renormalisation of quantum field theories on the noncommutative $\mathbb{R}^D$ is not achieved. These models show a phenomenon called *UV/IR-mixing* [1] which was analysed to all orders by Chepelev and Roiban [2, 3]. The conclusion of the power-counting theorem is that, in general, field theories on noncommutative $\mathbb{R}^D$ are not renormalisable if their commutative counterparts are worse than logarithmically divergent. The situation is better for models with at most logarithmic divergences. Applying the power-counting analysis to the real ϕ^4-model on noncommutative $\mathbb{R}^2$, one finds "that the divergences from all connected Green's functions at non-exceptional external momenta can be removed in the counter-term approach" (literally quoted from [3, §4.3]). The problem is, however, that non-exceptional momenta can become arbitrarily close to exceptional momenta so that the renormalised Green's functions are *unbounded*. Although one can probably live with that, it is not a desired feature of a quantum field theory.

We have elaborated in [4] the Wilson-Polchinski renormalisation group approach [5, 6] for dynamical matrix models where the propagator is neither diagonal nor constant. We have derived a power-counting theorem for ribbon graphs by solving the exact renormalisation group equation perturbatively. The power-counting degree of divergence of a ribbon graph is determined by its topology and the asymptotic behaviour of the cut-off propagator. Our motivation was to provide a renormalisation scheme for very general noncommutative field theories, because the typical noncommutative geometries are matrix geometries. The noncommutative $\mathbb{R}^D$ is no exception as there exists a matrix base [7] in which the $\star$-product interaction becomes the trace of an ordinary product of matrices. The propagator becomes complicated in the matrix base but as we show in this paper, the difficulties can be overcome.

In [4] we have only completed the first (but most essential) step of Polchinski's approach [6], namely the integration of the flow equation between a finite initial scale Λ_0 and the renormalisation scale Λ_R. In order to prove renormalisability the limit $\Lambda_0 \to \infty$ has to be taken. This step is model dependent. We focus in this paper on the real ϕ^4-theory on noncommutative $\mathbb{R}^2$. The naïve idea would be to take the standard ϕ^4-action at the initial scale Λ_0, with Λ_0-dependent bare mass to be adjusted such that at Λ_R it is scaled down to the renormalised mass. Unfortunately, this does not work. In the limit $\Lambda_0 \to \infty$ one obtains an unbounded power-counting degree of divergence for the ribbon graphs. The solution is the observation that the cut-off action at Λ_0 is (due to the cut-off) not translation invariant. We are therefore free to break the translational symmetry of the action at Λ_0 even more by adding a harmonic oscillator potential for the fields ϕ. We prove that there exists a Λ_0-dependence of the oscillator frequency Ω with $\lim_{\Lambda_0\to\infty} \Omega = 0$ such that the effective action at Λ_R is convergent (and thus bounded) order by order in the coupling constant in the limit $\Lambda_0 \to \infty$. This means that the partition function of the original (translation-invariant) ϕ^4-model without cut-off and with suitable divergent bare mass is solved by Feynman graphs with propagators cut-off at Λ_R and vertices given by the bounded expansion coefficients of the effective action at Λ_R. Hence, this model is renormalisable, and there is no problem with exceptional configurations.

We are optimistic that in the same way we can renormalise the ϕ^4-model on noncommutative $\mathbb{R}^4$ [8].

2 ϕ^4-theory on noncommutative $\mathbb{R}^D$

2.1 The regularised action in the matrix base

The noncommutative $\mathbb{R}^D$, $D = 2, 4, 6, \dots$, is defined as the algebra $\mathbb{R}^D_\theta$ which as a vector space is given by the space $\mathcal{S}(\mathbb{R}^D)$ of (complex-valued) Schwartz class functions of rapid decay, equipped with the multiplication rule [7]

$$(a \star b)(x) = \int \frac{d^D k}{(2\pi)^D} \int d^D y \; a(x+\tfrac{1}{2}\theta{\cdot}k)\, b(x+y)\, \mathrm{e}^{\mathrm{i}k\cdot y} \;, \qquad (2.1)$$
$$(\theta{\cdot}k)^\mu = \theta^{\mu\nu} k_\nu \;, \quad k{\cdot}y = k_\mu y^\mu \;, \quad \theta^{\mu\nu} = -\theta^{\nu\mu} \;.$$

The entries $\theta^{\mu\nu}$ in (2.1) have the dimension of an area.

We are going to study a regularised ϕ^4-theory on $\mathbb{R}^D_\theta$ defined by the action

$$S_D[\phi] = \int d^D x \Big(\frac{1}{2} g^{\mu\nu} \big(\partial_\mu \phi \star \partial_\nu \phi + 4\Omega^2 ((\theta^{-1})_{\mu\rho} x^\rho \phi) \star ((\theta^{-1})_{\nu\sigma} x^\sigma \phi) \big) + \frac{1}{2} \mu_0^2 \, \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \Big) \;, \qquad (2.2)$$

which is given by adding a harmonic oscillator potential to the standard ϕ^4-action. The potential beaks translation invariance. We shall learn that the renormalisation of standard ϕ^4-theory has to be performed along a path of actions (2.2).

Our goal is to write the classical action (2.2) in an adapted base. We place ourselves into a coordinate system in which θ has in D dimensions the form

$$\theta_{\mu\nu} = \begin{pmatrix} \boldsymbol{\theta}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{\theta}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{\theta}_{\frac{D}{2}} \end{pmatrix} , \qquad \boldsymbol{\theta}_i = \begin{pmatrix} 0 & \theta_i \\ -\theta_i & 0 \end{pmatrix} . \qquad (2.3)$$

Now an adapted base of $\mathbb{R}^D_\theta$ is

$$b_{mn}(x) = f_{m_1 n_1}(x_1, x_2)\, f_{m_2 n_2}(x_3, x_4) \dots f_{m_{D/2} n_{D/2}}(x_{D-1}, x_D) \;, \qquad (2.4)$$
$$m = (m_1, m_2, \dots, m_{D/2}) \in \mathbb{N}^{\frac{D}{2}} \;, \; n = (n_1, n_2, \dots, n_{D/2}) \in \mathbb{N}^{\frac{D}{2}} \;,$$

where the base $f_{mn}(x_1, x_2) \in \mathbb{R}^2_\theta$ is introduced in (A.6) in Appendix A.

The advantage of this base is that the $\star$-product (2.1) is represented by a product (A.8) of infinite matrices and that the multiplication by x^ρ is easy to realise. This means that expanding the fields according to

$$\phi(x) = \sum_{m,n \in \mathbb{N}^{\frac{D}{2}}} \phi_{mn} b_{mn}(x) \;, \qquad (2.5)$$

the interaction term $\phi \star \phi \star \phi \star \phi$ in (2.2) becomes very simple. The price for this simplification is, however, that the kinetic term given by the first line in (2.2) becomes very complicated. In [4] we have extended the first stept of Polchinski's renormalisation proof [6] of commutative ϕ^4-theory to a renormalisation method suited for dynamical matrix models with arbitrary non-diagonal and non-constant propagators. The kinetic term of the action (2.2) fits precisely into the scope of [4].

2.2 Computation of the propagator in the two-dimensional case

For the remainder of this paper we restrict ourselves to $D = 2$ dimensions. The four-dimensional case will be treated elsewhere [8]. Using the formulae collected in Appendix A we first calculate the kinetic term in two dimensions:

$$\begin{aligned}
G_{mn;kl} &:= \int \frac{d^2x}{2\pi\theta_1} \Big(\big(\partial_1 f_{mn} \star \partial_1 f_{kl} + \partial_2 f_{mn} \star \partial_2 f_{kl} \\
&\qquad + \frac{4\Omega^2}{\theta^2}\big((x_1 f_{mn}) \star (x_1 f_{kl}) + (x_2 f_{mn}) \star (x_2 f_{kl})\big) + \mu_0^2\, f_{mn} \star f_{kl}\Big) \\
&= \int \frac{d^2x}{2\pi\theta_1} \Big(\frac{1{+}\Omega^2}{\theta_1^2} f_{mn} \star (a \star \bar{a} + \bar{a} \star a) \star f_{kl} + \frac{1{+}\Omega^2}{\theta_1^2} f_{kl} \star (a \star \bar{a} + \bar{a} \star a) \star f_{mn} \\
&\qquad - \frac{2(1{+}\Omega^2)}{\theta_1^2} f_{mn} \star a \star f_{kl} \star \bar{a} - \frac{2(1{+}\Omega^2)}{\theta_1^2} f_{kl} \star a \star f_{mn} \star \bar{a} + \mu_0^2\, f_{mn} \star f_{kl}\Big) \\
&= \Big(\mu_0^2 + \frac{2(1{+}\Omega^2)}{\theta_1}(m{+}n{+}1)\Big)\delta_{nk}\delta_{ml} \\
&\qquad - \frac{2(1{-}\Omega^2)}{\theta_1}\sqrt{(n{+}1)(m{+}1)}\delta_{n+1,k}\delta_{m+1,l} - \frac{2(1{-}\Omega^2)}{\theta_1}\sqrt{nm}\delta_{n-1,k}\delta_{m-1,l}\,. \qquad (2.6)
\end{aligned}$$

Defining

$$\mu^2 = \frac{2(1+\Omega^2)}{\theta_1}\,, \qquad \sqrt{\omega} = \frac{1-\Omega^2}{1+\Omega^2}\,, \qquad (2.7)$$

with $-1 < \sqrt{\omega} \leq 1$, we can rewrite (2.6) as

$$\begin{aligned}
G_{mn;kl} &= \big(\mu_0^2 {+} (n{+}m{+}1)\mu^2\big)\delta_{nk}\delta_{ml} \\
&\quad - \mu^2\sqrt{\omega\,(n{+}1)(m{+}1)}\,\delta_{n+1,k}\delta_{m+1,l} - \mu^2\sqrt{\omega\, nm}\,\delta_{n-1,k}\delta_{m-1,l}\,. \qquad (2.8)
\end{aligned}$$

Now the action (2.2) takes the form

$$S_2[\phi] = 2\pi\theta_1 \sum_{m,n,k,l} \Big(\frac{1}{2}\phi_{mn} G_{mn;kl} \phi_{kl} + \frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\Big)\,. \qquad (2.9)$$

Next we are going to invert $G_{mn;kl}$, i.e. we solve in the two-dimensional case

$$\sum_{k,l=0}^{\infty} G_{mn;kl}\Delta_{lk;sr} = \sum_{k,l=0}^{\infty} \Delta_{nm;lk} G_{kl;rs} = \delta_{mr}\delta_{ns}\,. \qquad (2.10)$$

The indices m,n,k,l of each term contributing to (2.8) are restricted by

$$m+k=n+l\;. \tag{2.11}$$

Since the same relation is induced for the propagator $\Delta_{lk;nm}$ as well, the problem to solve (2.10) factorises into the independent equations

$$\sum_{l=0}^{\infty} G_{m,m+\alpha;l+\alpha,l}\Delta_{l,l+\alpha;r+\alpha,r} = \sum_{l=0}^{\infty} \Delta_{m+\alpha,m;l,l+\alpha}G_{l+\alpha,l;r,r+\alpha} = \delta_{mr}\;. \tag{2.12}$$

We define $\Delta_{mn;kl}=0$ and $G_{mn;kl}=0$ if one of the indices m,n,k,l is negative. For each α we have to invert an infinite square matrix. We therefore introduce a cut-off $\mathcal{N}$ with $0\leq m,n,k,l,r,s<\mathcal{N}$ above. Our strategy is to diagonalise the massless kinetic term

$$\begin{aligned}
G^{(\mathcal{N})}_{m,m+\alpha;l+\alpha,l}\Big|_{\mu_0=0} &= \mu^2 \sum_{i=1}^{\mathcal{N}} U^{(\mathcal{N},\alpha,\omega)}_{m+1,i}\, v_i\, U^{(\mathcal{N},\alpha,\omega)*}_{i,l+1}\;, \\
\delta_{ml} = \sum_{i=1}^{\mathcal{N}} U^{(\mathcal{N},\alpha,\omega)}_{mi} U^{(\mathcal{N},\alpha,\omega)*}_{il} &= \sum_{i} U^{(\mathcal{N},\alpha,\omega)*}_{mi} U^{(\mathcal{N},\alpha,\omega)}_{il}\;.
\end{aligned} \tag{2.13}$$

To see what result we can expect let us consider the eigenvalue problem of $\mathcal{N}=4+\alpha$ and $\alpha\geq 0$:

$$\begin{aligned}
&G^{(4)}_{m,m+\alpha;l+\alpha,l}\Big|_{\mu_0=0} - v\mu^2\delta^{(4)}_{ml} \\
&= \mu^2 \begin{pmatrix} \alpha{+}1{-}v & -\sqrt{1(\alpha{+}1)\omega} & 0 & 0 \\ -\sqrt{1(\alpha{+}1)\omega} & \alpha{+}3{-}v & -\sqrt{2(\alpha{+}2)\omega} & 0 \\ 0 & -\sqrt{2(\alpha{+}2)\omega} & \alpha{+}5{-}v & -\sqrt{3(\alpha{+}3)\omega} \\ 0 & 0 & -\sqrt{3(\alpha{+}3)\omega} & \alpha{+}7{-}v \end{pmatrix}_{m+1,l+l} \\
&= \mu^2 \begin{pmatrix} \sqrt{\alpha{+}1}\sqrt{A^{\alpha,\omega}_1(v)} & 0 & 0 & 0 \\ -\sqrt{\frac{1\,\omega}{A^{\alpha,\omega}_1(v)}} & \sqrt{\alpha{+}2}\sqrt{A^{\alpha,\omega}_2(v)} & 0 & 0 \\ 0 & -\sqrt{\frac{2\omega}{A^{\alpha,\omega}_2(v)}} & \sqrt{\alpha{+}3}\sqrt{A^{\alpha,\omega}_3(v)} & 0 \\ 0 & 0 & -\sqrt{\frac{3\omega}{A^{\alpha,\omega}_3(v)}} & \sqrt{\alpha{+}4}\sqrt{A^{\alpha,\omega}_4(v)} \end{pmatrix} \\
&\quad\times \begin{pmatrix} \sqrt{\alpha{+}1}\sqrt{A^{\alpha,\omega}_1(v)} & -\sqrt{\frac{1\,\omega}{A^{\alpha,\omega}_1(v)}} & 0 & 0 \\ 0 & \sqrt{\alpha{+}2}\sqrt{A^{\alpha,\omega}_2(v)} & -\sqrt{\frac{2\omega}{A^{\alpha,\omega}_2(v)}} & 0 \\ 0 & 0 & \sqrt{\alpha{+}3}\sqrt{A^{\alpha,\omega}_3(v)} & -\sqrt{\frac{3\omega}{A^{\alpha,\omega}_3(v)}} \\ 0 & 0 & 0 & \sqrt{\alpha{+}4}\sqrt{A^{\alpha,\omega}_4(v)} \end{pmatrix},
\end{aligned} \tag{2.14}$$

where

$$A^{\alpha,\omega}_n(v) := \frac{1}{\alpha{+}n}\Big(\alpha+2n-1-v-\frac{(n{-}1)\omega}{A^{\alpha,\omega}_{n-1}(v)}\Big)\;, \qquad n\geq 1\;. \tag{2.15}$$

Note that $0 \le \omega := (\sqrt{\omega})^2 \le 1$. With the ansatz

$$A_n^{\alpha,\omega}(v) = \frac{n}{\alpha{+}n} \frac{L_n^{\alpha,\omega}(v)}{L_{n-1}^{\alpha,\omega}(v)} \;, \qquad L_0^{\alpha,\omega}(v) \equiv 1 \;, \tag{2.16}$$

(2.15) can be rewritten as

$$0 = nL_n^{\alpha,\omega}(v) - (\alpha + 2n - 1 - v)L_{n-1}^{\alpha,\omega}(v) + \omega(\alpha + n - 1)L_{n-2}^{\alpha,\omega}(v) \;. \tag{2.17}$$

For $\omega = 1$ we recognise this relation as the recursion relation of Laguerre polynomials [9, §8.971.6]. We thus denote the $L_n^{\alpha,\omega}(v)$ as *deformed Laguerre polynomials*, with $L_n^{\alpha,1}(v) \equiv L_n^{\alpha}(v)$ being the usual Laguerre polynomials.

At given matrix cut-off $\mathcal{N}$ it follows from (2.14) and (2.16) that the eigenvalues v_i are the zeroes of the deformed Laguerre polynomial $L_{\mathcal{N}}^{\alpha,\omega}$:

$$L_{\mathcal{N}}^{\alpha,\omega}\big(v_i^{(\mathcal{N},\alpha,\omega)}\big) = 0 \;, \qquad i = 1, \dots, \mathcal{N} \;, \tag{2.18}$$

$$\begin{aligned} U_{ji}^{(\mathcal{N},\alpha,\omega)} = U_{ij}^{(\mathcal{N},\alpha,\omega)*} &= \sqrt{\frac{\Gamma(\alpha{+}\mathcal{N})\Gamma(j)\omega^{\mathcal{N}}}{\Gamma(\alpha{+}j)\Gamma(\mathcal{N})\omega^j}} \frac{L_{j-1}^{\alpha,\omega}(v_i^{(\mathcal{N},\alpha,\omega)})}{L_{\mathcal{N}-1}^{\alpha,\omega}(v_i^{(\mathcal{N},\alpha,\omega)})} U_{\mathcal{N}i}^{(\mathcal{N},\alpha,\omega)} \\ &= \frac{\sqrt{\frac{\Gamma(j)}{\omega^{j-1}\Gamma(\alpha+j)}} L_{j-1}^{\alpha,\omega}(v_i^{(\mathcal{N},\alpha,\omega)})}{\sqrt{\sum_{h=1}^{\mathcal{N}} \frac{\Gamma(h)}{\omega^{h-1}\Gamma(\alpha+h)} \big(L_{h-1}^{\alpha,\omega}(v_i^{(\mathcal{N},\alpha,\omega)})\big)^2}} \;, \quad j = 1, \dots, \mathcal{N} \;. \end{aligned} \tag{2.19}$$

Inserting (2.19) into (2.13) and (2.12) we obtain for $\alpha = n{-}m = k{-}l \ge 0$ the solutions

$$\delta_{ml}^{(\mathcal{N},\alpha,\omega)} = \sum_{i=1}^{\mathcal{N}} \frac{\sqrt{\frac{m!l!}{\omega^{m+l}\,(m+\alpha)!(l+\alpha)!}} L_m^{\alpha,\omega}(v_i^{(\mathcal{N},\alpha,\omega)}) L_l^{\alpha,\omega}(v_i^{(\mathcal{N},\alpha,\omega)})}{\sum_{h=0}^{\mathcal{N}-1} \frac{h!}{\omega^h(\alpha+h)!} \big(L_h^{\alpha,\omega}(v_i^{(\mathcal{N},\alpha,\omega)})\big)^2} \;, \tag{2.20}$$

$$G_{m,m+\alpha;l+\alpha,l}^{(\mathcal{N},\alpha,\omega)} = \sum_{i=1}^{\mathcal{N}} \frac{\sqrt{\frac{m!l!}{\omega^{m+l}\,(m+\alpha)!(l+\alpha)!}} L_m^{\alpha,\omega}(v_i^{(\mathcal{N},\alpha,\omega)}) L_l^{\alpha,\omega}(v_i^{(\mathcal{N},\alpha,\omega)})}{\sum_{h=0}^{\mathcal{N}-1} \frac{h!}{\omega^h(\alpha+h)!} \big(L_h^{\alpha,\omega}(v_i^{(\mathcal{N},\alpha,\omega)})\big)^2} \Big(\mu_0^2 + \mu^2 v_i^{(\mathcal{N},\alpha,\omega)}\Big) \;, \tag{2.21}$$

$$\Delta_{m+\alpha,m;l,l+\alpha}^{(\mathcal{N},\alpha,\omega)} = \sum_{i=1}^{\mathcal{N}} \frac{\sqrt{\frac{m!l!}{\omega^{m+l}\,(m+\alpha)!(l+\alpha)!}} L_m^{\alpha,\omega}(v_i^{(\mathcal{N},\alpha,\omega)}) L_l^{\alpha,\omega}(v_i^{(\mathcal{N},\alpha,\omega)})}{\sum_{h=0}^{\mathcal{N}-1} \frac{h!}{\omega^h(\alpha+h)!} \big(L_h^{\alpha,\omega}(v_i^{(\mathcal{N},\alpha,\omega)})\big)^2} \frac{1}{\mu_0^2 + \mu^2 v_i^{(\mathcal{N},\alpha,\omega)}} \;. \tag{2.22}$$

Since the kinetic term (2.8) is symmetric in $(m \leftrightarrow n,\, k \leftrightarrow l)$, we obtain the analogue of (2.21) and (2.22) in the case $\alpha = n{-}m = k{-}l \le 0$ by exchanging $(m \leftrightarrow n,\, k \leftrightarrow l)$. Note that the recursion relation (2.17) and the orthogonality (2.20) yield directly the kinetic term (2.8).

2.3 Remarks on the limit $\mathcal{N} \to \infty$

Now we have to take the limit $\mathcal{N} \to \infty$, which can be done explicitly for $\omega = 0$ and $\omega = 1$. For $\omega = 0$ we can invert (2.8) directly:

$$\Delta_{nm;lk}^{(\omega=0)} = \frac{\delta_{ml}\delta_{nk}}{\mu_0^2 + \mu^2(m + n + 1)} \;. \tag{2.23}$$

For $\omega = 1$ the zeroes $v_i^{(\mathcal{N},\alpha,1)}$ of the true Laguerre polynomials L_n^α become continuous variables v, and $\big(\sum_{h=0}^\infty \frac{h!}{(\alpha+h)!}\big(L_h^\alpha(v)\big)^2\big)^{-1}$ is promoted to the measure of integration. This measure is identified by comparison of (2.20) with the standard orthogonality relation [9, §8.904] of Laguerre polynomials

$$\delta_{ml} = \int_0^\infty dv\; v^\alpha\, \mathrm{e}^{-v} \sqrt{\frac{m!l!}{(m+\alpha)!(l+\alpha)!}}\, L_m^\alpha(v) L_l^\alpha(v)\;. \tag{2.24}$$

We thus have to translate (2.22) in the limit $\mathcal{N} \to \infty$ into

$$\Delta^{(\omega=1)}_{nm;lk} = \int_0^\infty dv\; v^{n-m}\, \mathrm{e}^{-v} \sqrt{\frac{m!l!}{n!k!}}\, \frac{L_m^{n-m}(v)\, L_l^{k-l}(v)}{\mu_0^2 + v\mu^2} \delta_{m+k,n+l}\;. \tag{2.25}$$

We have derived the formula (2.25) for $n{-}m = k{-}l \geq 0$ only. However, due to the identity

$$L_{m+\alpha}^{-\alpha}(v) = \frac{m!}{(m+\alpha)!}(-1)^\alpha v^\alpha L_m^\alpha(v) \tag{2.26}$$

it can be transformed into the $(m \leftrightarrow n, l \leftrightarrow k)$-exchanged form so that (2.25) holds actually for any $n{-}m = k{-}l$.

Introducing a Schwinger parameter and using [9, §7.414.4] we can integrate (2.25) to

$$\begin{aligned}\Delta^{(\omega=1)}_{nm;lk} &= \frac{1}{\mu_0^2}\int_0^\infty dt \int_0^\infty dv\; v^{n-m}\, \mathrm{e}^{-v(1+\frac{\mu^2}{\mu_0^2}t)-t} \sqrt{\frac{m!l!}{n!k!}}\, L_m^{n-m}(v)\, L_l^{k-l}(v)\, \delta_{m+k,n+l} \\ &= \frac{1}{\mu_0^2}\sqrt{\frac{(n{+}l)!}{n!l!}\frac{(m{+}k)!}{m!k!}}\, \delta_{m+k,n+l} \int_0^\infty dt\; \frac{\big(\frac{\mu^2}{\mu_0^2}t\big)^{m+l}\, \mathrm{e}^{-t}}{\big(1{+}\frac{\mu^2}{\mu_0^2}t\big)^{n+l+1}}\, F\Big(-m,-l;-n{-}l;1{-}\frac{\mu_0^4}{\mu^4 t^2}\Big)\;.\end{aligned} \tag{2.27}$$

Again, due to the property [9, §9.131.1] of the hypergeometric function the result (2.27) is invariant under the exchange $m \leftrightarrow n$ and $k \leftrightarrow l$.

We recall that in the momentum space version of the ϕ^4-model, the interactions contain oscillating phase factors which to our opinion [4] make a Wilson-Polchinski treatment impossible. Here we use an adapted base which eliminates the phase factors from the interaction. At first sight it seems that these oscillations reappear in the propagator via the Laguerre polynomials. We see, however, from (2.27) that this is not the case. The interpolation of the matrix propagator consists of two monotonous and apparently smooth parts which are glued together at $\alpha = 0$. We show in Figure 1 how $\Delta_{10,10+i;j+i,j}$ depends on the parameters i, j for the indices. The monotonous behaviour is perfect for the renormalisation group approach. One observes that the maximum of $\Delta_{nm;lk}$ for given (large enough) n is found at $m = n = k = l$. The decay rate of $\Delta_{nm;lk}$ for increasing indices decides according to [4] about renormalisability. It turns out that $\Delta^{(\omega=1)}_{nm;lk}$ decays

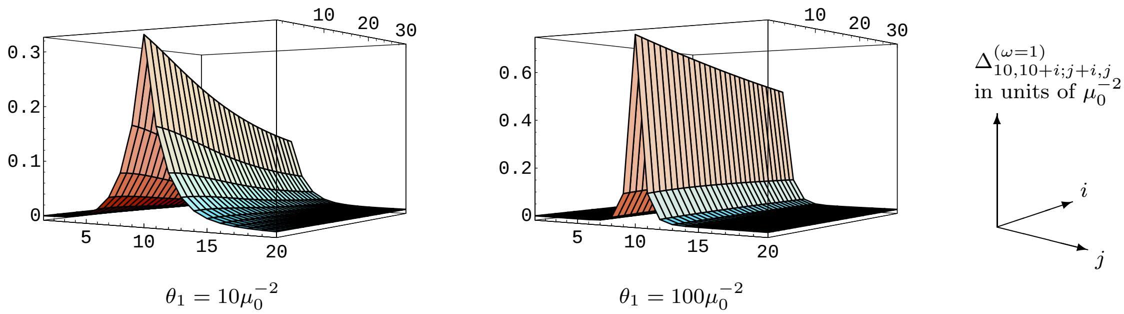


Figure 1: The plot of the propagators $\Delta_{10,10+i;j+i,j}$ over i and j, for two values of θ_1.

too slowly so that we have to pass to $\omega < 1$. For $\theta_1 \to \infty$ one obtains an ordinary matrix model,

$$\lim_{\theta_1 \to \infty} \Delta_{nm;lk} = \frac{1}{\mu_0^2} \delta_{ml} \delta_{nk} . \tag{2.28}$$

This should be compared with [10].

It would be desirable to have an explicit formula as (2.27) for the $\mathcal{N} \to \infty$ limit in case of $\omega < 1$, too. For that purpose a deeper understanding of the deformed Laguerre polynomials is indispensable.

3 The general strategy of renormalisation

3.1 Projection to the irrelevant part

Guided by Wilson's understanding of renormalisation [5] in terms of the scaling of effective Lagrangians, Polchinski has given a very efficient renormalisation proof of commutative ϕ^4-theory in four dimensions [6]. We have adapted in [4] this method to non-local matrix models defined by a kinetic term (Taylor coefficient matrix of the two-point function) which is neither constant nor diagonal. Introducing a cut-off in the measure $\prod_{m,n} d\phi_{mn}$ of the partition function Z, the resulting effect is undone by adjusting the effective action $L[\phi]$ (and other terms which are easy to evaluate). If the cut-off function is a smooth function of the cut-off scale Λ, the adjustment of $L[\phi, \Lambda]$ is described by a differential equation,

$$\Lambda \frac{\partial L[\phi, \Lambda]}{\partial \Lambda} = \sum_{m,n,k,l} \frac{1}{2} \Lambda \frac{\partial \Delta^K_{nm;lk}(\Lambda)}{\partial \Lambda} \bigg(\frac{\partial L[\phi, \Lambda]}{\partial \phi_{mn}} \frac{\partial L[\phi, \Lambda]}{\partial \phi_{kl}} - \frac{1}{\mathcal{V}_D} \Big[\frac{\partial^2 L[\phi, \Lambda]}{\partial \phi_{mn} \, \partial \phi_{kl}} \Big]_\phi \bigg) , \tag{3.1}$$

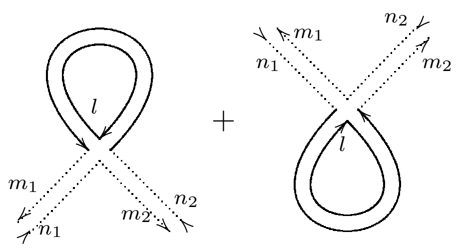
where $\big[F[\phi]\big]_\phi := F[\phi] - F[0]$ and

$$\Delta^K_{nm;lk}(\Lambda) = K[m, n; \Lambda] \Delta_{nm;lk} K[k, l; \Lambda] . \tag{3.2}$$

Here, $K[m, n; \Lambda]$ is the cut-off function which for finite Λ has finite support in m, n and satisfies $K[m, n; \infty] = 1$. By $\mathcal{V}_D$ we denote the volume of an elementary cell.

In [4] we have derived a power-counting theorem for $L[\phi,\Lambda]$ by integrating (3.1) perturbatively between the initial scale Λ_0 and the renormalisation scale $\Lambda_R \ll \Lambda_0$. The power-counting degree is given by topological data of ribbon graphs and two scaling exponents of the (summed and differentiated) cut-off propagator. The power-counting theorem in [4] is model independent. The subtraction of divergences necessary to carry out the limit $\Lambda_0 \to \infty$ has to be worked out model by model.

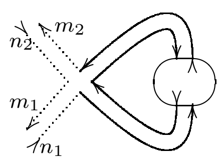
In this paper we will perform the subtraction of divergences for the regularised ϕ^4-model on $\mathbb{R}^2_\theta$. The first step is to extract from the power-counting theorem [4] the set of relevant and marginal interactions. As we will derive in Section 4.1 and Appendix B, there is an infinite number of relevant interactions if the regularisation Ω is not applied. For $\Omega \neq 0$, which means $\omega < 1$, the marginal interaction is (apart from the initial ϕ^4-interaction) given by the planar one-loop two-point function



$$= \rho_{[m_1]}[\Lambda]\,\delta_{m_1 n_2}\delta_{m_2 n_1} + \rho_{[m_2]}[\Lambda]\,\delta_{m_1 n_2}\delta_{m_2 n_1}\;. \tag{3.3}$$

For this graph we have to provide boundary conditions at Λ_R. The simplicity of the divergent sectors makes the renormalisation very easy. On the other hand, the simplicity hides the beauty of renormalisation so that we choose a slightly more general setting to present the strategy.

For presentational reasons let us assume that the divergent graphs have the same structure of external lines as (3.3) but possibly an arbitrary number of vertices,



$$= \rho_{[m_2]}[\Lambda]\,\delta_{m_1 n_2}\delta_{m_2 n_1}\;. \tag{3.4}$$

In this case the corresponding $\rho_{[m]}$-functions for different indices m must be expected to be independent, which means that the model would be determined by an infinite number of free parameters. Since this is not acceptable, we require according to [4] that the parameters $\rho_{[m]}[\Lambda_R]$ are scaled by the same amount to $\rho_{[m]}[\Lambda_0]$ (reduction of couplings [11]). Expanding $\rho_{[m]}[\Lambda]$ as a formal power series in the coupling constant λ, $\rho_{[m]}[\Lambda] = \sum_{V=1}^{\infty}\big(\frac{\lambda}{\mu^2}\big)^V \rho^{(V)}_{[m]}[\Lambda]$ and normalising the renormalised mass μ_0 by $\rho_{[0]}[\Lambda_R] = 0$, we thus demand in general

$$\frac{\frac{\mu_0^2}{\mu^2} + \sum_{V'=1}^{V}\big(\frac{\lambda}{\mu^2}\big)^{V'}\rho^{(V')}_{[m]}[\Lambda_R]}{\frac{\mu_0^2}{\mu^2}} \sim \frac{\frac{\mu_0^2}{\mu^2} + \sum_{V'=1}^{V}\big(\frac{\lambda}{\mu^2}\big)^{V'}\rho^{(V')}_{[m]}[\Lambda_0]}{\frac{\mu_0^2}{\mu^2} + \sum_{V'=1}^{V}\big(\frac{\lambda}{\mu^2}\big)^{V'}\rho^{(V')}_{[0]}[\Lambda_0]} + \mathcal{O}(\lambda^V)\;. \tag{3.5}$$

This leads order by order in λ to relations $a \sim b$ which mean $\lim_{\Lambda_0\to\infty}\frac{a}{b} = 1$.

At the initial scale $\Lambda = \Lambda_0$ the effective action thus reads

$$L[\phi, \Lambda_0, \Lambda_0, \omega, \rho^0] = \frac{\lambda}{4!} \sum_{m,n,k,l} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm} + \frac{1}{2} \sum_{m,n} \rho^0_{[m]} \phi_{mn}\phi_{nm} \;. \tag{3.6}$$

Each summation index runs over $\mathbb{N}$. The solution of (3.1) with initial condition (3.6) will have a completely different form in terms of ϕ_{mn}, but the projection to the same ϕ-structure as in (3.6) can still be defined:

$$\begin{aligned} L[\phi, \Lambda, \Lambda_0, \omega, \rho^0] = \frac{\lambda}{4!} \sum_{m,n,k,l} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm} + \frac{1}{2} \sum_{m,n} \rho_{[m]}[\Lambda, \Lambda_0, \omega, \rho^0] \phi_{mn}\phi_{nm} \\ + \text{ different } \phi\text{-structures} \;. \end{aligned} \tag{3.7}$$

The marginal part of the four-point function will turn out to be scale-independent. We identify $\rho_{[m]}[\Lambda_0, \Lambda_0, \omega, \rho^0] \equiv \rho^0_{[m]}$.

At the end we are interested in the limit $\Lambda_0 \to \infty$. For this purpose we have to admit a Λ_0-dependence of ω and $\rho^0_{[m]}$ the determination of which is the art of renormalisation. For fixed $\Lambda = \Lambda_R$ but variable Λ_0 we consider the identity

$$\begin{aligned} & L[\Lambda_R, \Lambda_0', \omega[\Lambda_0'], \rho^0[\Lambda_0']] - L[\Lambda_R, \Lambda_0'', \omega[\Lambda_0''], \rho^0[\Lambda_0'']] \\ & \equiv \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0} \Big(\Lambda_0 \frac{d}{d\Lambda_0} L[\Lambda_R, \Lambda_0, \omega[\Lambda_0], \rho^0[\Lambda_0]] \Big) \\ & = \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0} \Big(\Lambda_0 \frac{\partial L[\Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \Lambda_0} + \Lambda_0 \frac{d\omega}{d\Lambda_0} \frac{\partial L[\Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \omega} + \Lambda_0 \frac{d\rho^0}{d\Lambda_0} \frac{\partial L[\Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \rho^0} \Big) \;. \end{aligned} \tag{3.8}$$

Here we have omitted for simplicity the dependence of L on ϕ as well as the indices on ρ^0. The model is defined by fixing the boundary condition for the ρ-coefficients at Λ_R, i.e. by keeping $\rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0] = \text{constant}$:

$$\begin{aligned} 0 &= d\rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0] \\ &= \frac{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \Lambda_0} d\Lambda_0 + \frac{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \omega} d\omega + \sum_n \frac{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \rho^0_{[n]}} d\rho^0_{[n]} \;. \end{aligned} \tag{3.9}$$

Assuming that we can invert the matrix $\frac{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \rho^0_{[n]}}$, which is possible in perturbation theory, we get

$$\begin{aligned} \frac{d\rho^0_{[n]}}{d\Lambda_0} = & - \sum_m \frac{\partial \rho^0_{[n]}}{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]} \frac{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \Lambda_0} \\ & - \sum_m \frac{\partial \rho^0_{[n]}}{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]} \frac{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \omega} \frac{d\omega}{d\Lambda_0} \;. \end{aligned} \tag{3.10}$$

Inserting (3.10) into (3.8) we see that the following function[1] will be important:

$$
\begin{aligned}
R[\phi,\Lambda,\Lambda_0,\omega,\rho^0] := {}& \Lambda_0 \frac{\partial L[\phi,\Lambda,\Lambda_0,\omega,\rho^0]}{\partial \Lambda_0} + \frac{\partial L[\phi,\Lambda,\Lambda_0,\omega,\rho^0]}{\partial \omega} \Lambda_0 \frac{d\omega}{d\Lambda_0} \\
& - \sum_{m,n} \frac{\partial L[\phi,\Lambda,\Lambda_0,\omega,\rho^0]}{\partial \rho^0_{[m]}} \frac{\partial \rho^0_{[m]}}{\partial \rho_{[n]}[\Lambda,\Lambda_0,\omega,\rho^0]} \Lambda_0 \frac{\partial \rho_{[n]}[\Lambda,\Lambda_0,\omega,\rho^0]}{\partial \Lambda_0} \\
& - \sum_{m,n} \frac{\partial L[\phi,\Lambda,\Lambda_0,\omega,\rho^0]}{\partial \rho^0_{[m]}} \frac{\partial \rho^0_{[m]}}{\partial \rho_{[n]}[\Lambda,\Lambda_0,\omega,\rho^0]} \frac{\partial \rho_{[n]}[\Lambda,\Lambda_0,\omega,\rho^0]}{\partial \omega} \Lambda_0 \frac{d\omega}{d\Lambda_0} \;.
\end{aligned}
\tag{3.11}
$$

Now we can rewrite (3.8) as

$$
\begin{aligned}
& L[\phi,\Lambda_R,\Lambda_0',\omega[\Lambda_0'],\rho^0[\Lambda_0']] - L[\phi,\Lambda_R,\Lambda_0'',\omega[\Lambda_0''],\rho^0[\Lambda_0'']] \\
& \qquad = \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0} \; R[\phi,\Lambda_R,\Lambda_0,\omega[\Lambda_0],\rho^0[\Lambda_0]] \;.
\end{aligned}
\tag{3.12}
$$

Since R is linear in L, the splitting (3.7) together with (3.11) leads for all Λ to a vanishing projection of R to its ρ-coefficient. In other words, R *projects to the irrelevant part* of the effective action, which is indispensable for the existence of the limit $\Lambda_0 \to \infty$ controlled by (3.12). We have to show, however, that this really eliminates all divergences.

3.2 Flow equations

For this purpose we need estimations for R. This is achieved by computing the Λ-scaling of R:

$$
\begin{aligned}
\Lambda \frac{\partial R}{\partial \Lambda} = {}& \Lambda_0 \frac{\partial}{\partial \Lambda_0}\Big(\Lambda \frac{\partial L}{\partial \Lambda}\Big) + \frac{\partial}{\partial \omega}\Big(\Lambda \frac{\partial L}{\partial \Lambda}\Big) \Lambda_0 \frac{d\omega}{d\Lambda_0} \\
& - \sum_{m,n} \frac{\partial}{\partial \rho^0_{[m]}}\Big(\Lambda \frac{\partial L}{\partial \Lambda}\Big) \frac{\partial \rho^0_{[m]}}{\partial \rho_{[n]}} \Lambda_0 \frac{\partial \rho_{[n]}}{\partial \Lambda_0} - \sum_{m,n} \frac{\partial}{\partial \rho^0_{[m]}}\Big(\Lambda \frac{\partial L}{\partial \Lambda}\Big) \frac{\partial \rho^0_{[m]}}{\partial \rho_{[n]}} \frac{\partial \rho_{[n]}}{\partial \omega} \Lambda_0 \frac{d\omega}{d\Lambda_0} \\
& + \sum_{m,n,k,l} \frac{\partial L}{\partial \rho^0_{[m]}} \frac{\partial \rho^0_{[m]}}{\partial \rho_{[n]}} \frac{\partial}{\partial \rho^0_{[k]}}\Big(\Lambda \frac{\partial \rho_{[n]}}{\partial \Lambda}\Big) \frac{\partial \rho^0_{[k]}}{\partial \rho_{[l]}} \Lambda_0 \frac{\partial \rho_{[l]}}{\partial \Lambda_0} \\
& + \sum_{m,n,k,l} \frac{\partial L}{\partial \rho^0_{[m]}} \frac{\partial \rho^0_{[m]}}{\partial \rho_{[n]}} \frac{\partial}{\partial \rho^0_{[k]}}\Big(\Lambda \frac{\partial \rho_{[n]}}{\partial \Lambda}\Big) \frac{\partial \rho^0_{[k]}}{\partial \rho_{[l]}} \frac{\partial \rho_{[l]}}{\partial \omega} \Lambda_0 \frac{d\omega}{d\Lambda_0} \\
& - \sum_{m,n} \frac{\partial L}{\partial \rho^0_{[m]}} \frac{\partial \rho^0_{[m]}}{\partial \rho_{[n]}} \Lambda_0 \frac{\partial}{\partial \Lambda_0}\Big(\Lambda \frac{\partial \rho_{[n]}}{\partial \Lambda}\Big) - \sum_{m,n} \frac{\partial L}{\partial \rho^0_{[m]}} \frac{\partial \rho^0_{[m]}}{\partial \rho_{[n]}} \frac{\partial}{\partial \omega}\Big(\Lambda \frac{\partial \rho_{[n]}}{\partial \Lambda}\Big) \Lambda_0 \frac{d\omega}{d\Lambda_0} \;.
\end{aligned}
\tag{3.13}
$$

[1]Our function R (for 'renormalised') generalises a function called V in [6]. We use the symbol R in order to avoid confusion with the number V of vertices. Below we shall denote the function B of [6] by H (for having 'holes'), avoiding confusion with the number B of boundary components.

We have omitted the dependencies for simplicity and made use of the fact that the derivatives with respect to $\Lambda, \Lambda_0, \rho^0, \omega$ commute. Using (3.1) we compute the terms on the rhs of (3.13):

$$
\begin{aligned}
&\Lambda_0 \frac{\partial}{\partial \Lambda_0}\Big(\Lambda \frac{\partial L[\phi,\Lambda,\Lambda_0,\omega,\rho^0]}{\partial \Lambda}\Big) \\
&= \sum_{m',n',k',l'} \frac{1}{2}\Lambda \frac{\partial \Delta^K_{n'm';l'k'}(\Lambda)}{\partial \Lambda}\Big(2 \frac{\partial L[\phi,\Lambda,\Lambda_0,\omega,\rho^0]}{\partial \phi_{m'n'}} \frac{\partial}{\partial \phi_{k'l'}}\Big(\Lambda_0 \frac{\partial}{\partial \Lambda_0} L[\phi,\Lambda,\Lambda_0,\omega,\rho^0]\Big) \\
&\qquad - \frac{1}{\mathcal{V}_2}\Big[\frac{\partial^2}{\partial \phi_{m'n'}\,\partial \phi_{k'l'}}\Big(\Lambda_0 \frac{\partial}{\partial \Lambda_0} L[\phi,\Lambda,\Lambda_0,\omega,\rho^0]\Big)\Big]_\phi\Big) \equiv M\Big[L, \Lambda_0 \frac{\partial L}{\partial \Lambda_0}\Big]\,.
\end{aligned}
\tag{3.14}
$$

Similarly we have

$$
\begin{aligned}
\frac{\partial}{\partial \rho^0_{[m]}}\Big(\Lambda \frac{\partial L[\phi,\Lambda,\Lambda_0,\omega,\rho^0]}{\partial \Lambda}\Big) &= M\Big[L, \frac{\partial L}{\partial \rho^0_{[m]}}\Big]\,, \\
\frac{\partial}{\partial \omega}\Big(\Lambda \frac{\partial L[\phi,\Lambda,\Lambda_0,\omega,\rho^0]}{\partial \Lambda}\Big) &= M\Big[L, \frac{\partial L}{\partial \omega}\Big]\,.
\end{aligned}
\tag{3.15}
$$

In the same way as in (3.7) we expand $M[L,\,.\,]$ on the rhs of (3.14) with respect to the ϕ-structures,

$$
M[L,\,.\,] = \frac{1}{2}\sum_{m,n} M_{[m]}[L,\,.\,]\phi_{mn}\phi_{nm} + \text{ different } \phi\text{-structures}\,.
\tag{3.16}
$$

Because of the Λ-derivatives there is no analogue of the initial four-point function. The distinguished expansion coefficients are due to (3.14) and (3.7) identified with

$$
\begin{aligned}
M_{[m]}\Big[L, \Lambda_0 \frac{\partial L}{\partial \Lambda_0}\Big] &= \Lambda_0 \frac{\partial}{\partial \Lambda_0}\Big(\Lambda \frac{\partial \rho_{[m]}[\Lambda,\Lambda_0,\rho^0]}{\partial \Lambda}\Big)\,, \\
M_{[m]}\Big[L, \frac{\partial L}{\partial \omega}\Big] &= \frac{\partial}{\partial \omega}\Big(\Lambda \frac{\partial \rho_{[m]}[\Lambda,\Lambda_0,\rho^0]}{\partial \Lambda}\Big)\,, \\
M_{[m]}\Big[L, \frac{\partial L}{\partial \rho^0_{[n]}}\Big] &= \frac{\partial}{\partial \rho^0_{[n]}}\Big(\Lambda \frac{\partial \rho_{[m]}[\Lambda,\Lambda_0,\rho^0]}{\partial \Lambda}\Big)\,.
\end{aligned}
\tag{3.17}
$$

Using (3.14), (3.15) and (3.17) as well as the linearity of $M[L,\,.\,]$ in the second argument we can rewrite (3.13) as

$$
\Lambda \frac{\partial R}{\partial \Lambda} = M[L,R] - \sum_m \frac{\partial L}{\partial \rho_{[m]}} M_{[m]}[L,R]\,,
\tag{3.18}
$$

where we have defined

$$
\frac{\partial L}{\partial \rho_{[m]}}[\Lambda,\Lambda_0,\omega,\rho^0] := \sum_n \frac{\partial L[\Lambda,\Lambda_0,\omega,\rho^0]}{\partial \rho^0_{[n]}} \frac{\partial \rho^0_{[n]}}{\partial \rho_{[m]}[\Lambda,\Lambda_0,\omega,\rho^0]}\,.
\tag{3.19}
$$

In the same way as for R, the Λ-scaling of (3.19) is computed to

$$\Lambda\frac{\partial}{\partial\Lambda}\Big(\frac{\partial L}{\partial\rho_{[m]}}\Big) = M\Big[L, \frac{\partial L}{\partial\rho_{[m]}}\Big] - \sum_n \frac{\partial L}{\partial\rho_{[n]}} M_{[n]}\Big[L, \frac{\partial L}{\partial\rho_{[m]}}\Big] . \tag{3.20}$$

3.3 Expansion as power series in the coupling constant

Now we expand the functions just introduced as formal power series in the coupling constant λ and with respect to the number of fields ϕ, expressing all dimensionful quantities in terms of the volume $\mathcal{V}_2$ of the elementary cell:

$$L[\phi,\Lambda] = \lambda\sum_{V=1}^{\infty}\big(\lambda\mathcal{V}_2\big)^{V-1}\sum_{N=2}^{\infty}\frac{1}{N!}\sum_{m_i,n_i} A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\phi_{m_1n_1}\cdots\phi_{m_Nn_N} , \tag{3.21}$$

$$R[\phi,\Lambda] = \lambda\sum_{V=1}^{\infty}\big(\lambda\mathcal{V}_2\big)^{V-1}\sum_{N=2}^{\infty}\frac{1}{N!}\sum_{m_i,n_i} R^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\phi_{m_1n_1}\cdots\phi_{m_Nn_N} , \tag{3.22}$$

$$\frac{\partial L}{\partial\rho_{[m]}}[\phi,\Lambda] = \sum_{V=0}^{\infty}\big(\lambda\mathcal{V}_2\big)^{V}\sum_{N=2}^{\infty}\frac{1}{N!}\sum_{m_i,n_i} H^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\phi_{m_1n_1}\cdots\phi_{m_Nn_N} , \tag{3.23}$$

We have suppressed the additional dependence of $L, R, \frac{\partial L}{\partial\rho_{[m]}}, A^{(V)}, R^{(V)}, H^{(V)}$ on Λ_0,ω,ρ^0.

All functions $A^{(V)}, R^{(V)}, H^{(V)}$ have mass dimension zero. The Polchinski equation (3.1) as well as its derived equations (3.20) and (3.18) can now with (3.14) be written as

$$\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda}A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\omega,\rho^0]\\
&\quad= \sum_{N_1=2}^{N}\sum_{V_1=1}^{V-1}\sum_{m,n,k,l}\frac{1}{2}Q_{nm;lk}(\Lambda)A^{(V_1)}_{m_1n_1;\dots;m_{N_1-1}n_{N_1-1};mn}[\Lambda]A^{(V-V_1)}_{m_{N_1}n_{N_1};\dots;m_Nn_N;kl}[\Lambda]\\
&\qquad\qquad\qquad + \Big(\binom{N}{N_1-1}-1\Big)\text{ permutations}\\
&\quad - \sum_{m,n,k,l}\frac{1}{2}Q_{nm;lk}(\Lambda)A^{(V)}_{m_1n_1;\dots;m_Nn_N;mn;kl}[\Lambda] ,
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda}H^{[\hat m](V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\omega,\rho^0]\\
&\quad= \sum_{N_1=2}^{N}\sum_{V_1=1}^{V}\sum_{m,n,k,l}Q_{nm;lk}(\Lambda)A^{(V_1)}_{m_1n_1;\dots;m_{N_1-1}n_{N_1-1};mn}[\Lambda]H^{[\hat m](V-V_1)}_{m_{N_1}n_{N_1};\dots;m_Nn_N;kl}[\Lambda]\\
&\qquad\qquad\qquad + \Big(\binom{N}{N_1-1}-1\Big)\text{ permutations}\\
&\quad - \sum_{m,n,k,l}\frac{1}{2}Q_{nm;lk}(\Lambda)H^{[\hat m](V)}_{m_1n_1;\dots;m_Nn_N;mn;kl}[\Lambda]\\
&\quad + \sum_{\hat n}\sum_{V_1=1}^{V}H^{[\hat n](V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\Big(\sum_{m,n,k,l}\frac{1}{2}Q_{nm;lk}(\Lambda)H^{[\hat m](V_1)}_{m'n';n'm';mn;kl}[\Lambda]\Big)_{[\hat n]} ,
\end{aligned} \tag{3.25}$$

$$\Lambda\frac{\partial}{\partial\Lambda}R^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\omega,\rho^0]$$

$$= \sum_{N_1=2}^{N}\sum_{V_1=1}^{V-1}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) A^{(V_1)}_{m_1n_1;\dots;m_{N_1-1}n_{N_1-1};mn}[\Lambda] R^{(V-V_1)}_{m_{N_1}n_{N_1};\dots;m_Nn_N;kl}[\Lambda] + \Big(\binom{N}{N_1-1}-1\Big) \text{ permutations}$$

$$- \sum_{m,n,k,l}\frac{1}{2}Q_{nm;lk}(\Lambda)R^{(V)}_{m_1n_1;\dots;m_Nn_N;mn;kl}[\Lambda]$$

$$+ \sum_{\hat{n}}\sum_{V_1=1}^{V} H^{[\hat{n}](V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\Big(\sum_{m,n,k,l}\frac{1}{2}Q_{nm;lk}(\Lambda)R^{(V_1)}_{m'n';n'm';mn;kl}[\Lambda]\Big)_{[\hat{n}]}\,, \qquad (3.26)$$

with

$$Q_{nm;lk}(\Lambda) := \frac{1}{\mathcal{V}_2}\Lambda\frac{\partial\Delta^K_{nm;lk}(\Lambda)}{\partial\Lambda}\,. \qquad (3.27)$$

Note that the projection $(\;)_{[m]}$ to the $\rho_{[m]}$-coefficients in (3.25) and (3.26) are due to (3.4) non-zero on the 1PI functions only.

4 Renormalisation of the ϕ^4-model

4.1 Scaling of the cut-off propagator

We have $\mathcal{V}_2 = 2\pi\theta_1 = \frac{8\pi}{(1+\sqrt{\omega})\mu^2}$. We choose the smooth cut-off function

$$K(m,n;\Lambda) = K\Big(\frac{m\mu^2}{\Lambda^2}\Big)K\Big(\frac{n\mu^2}{\Lambda^2}\Big)\,, \qquad \text{where}$$

$$K(x)\in C^\infty(\mathbb{R}^+) \text{ is monotonous with } K(x) = \begin{cases} 1 & \text{for } x\leq 1\,,\\ 0 & \text{for } x\geq 2\,.\end{cases} \qquad (4.1)$$

This choice satisfies the dimensional normalisation

$$\sum_m \text{sign}\Big(\max_{n,l}\big|K(m,n;\Lambda)K(l{+}n{-}m,l;\Lambda)\big|\Big) \leq \sum_{m=0}^{\frac{2\Lambda^2}{\mu^2}-1} 1 = 2\Big(\frac{\Lambda}{\mu}\Big)^2 \qquad (4.2)$$

of a two-dimensional model [4]. We obtain with (3.2)

$$\Lambda\frac{\partial\Delta^K_{nm;lk}(\Lambda)}{\partial\Lambda} = -\sum_{j\in\{m,n,k,l\}}\frac{2j\mu^2}{\Lambda^2}K'\Big(\frac{j\mu^2}{\Lambda^2}\Big)\prod_{i\in\{m,n,k,l\}\setminus\{j\}}K\Big(\frac{i\mu^2}{\Lambda^2}\Big)\Delta_{nm;lk}\,. \qquad (4.3)$$

Since $\text{supp}\,K'(x) = [1,2]$ and $\text{supp}\,K(y) = [0,2]$, (4.3) is non-zero only if the condition

$$\frac{\Lambda^2}{\mu^2} \leq \max(m,n,k,l) \leq \frac{2\Lambda^2}{\mu^2} \qquad (4.4)$$

is satisfied. Note that due to (A.14) and (2.7) this also corresponds to a momentum cut-off $p_{\max} \approx \sqrt{8}\Lambda$. We compute in Appendix B the Λ-dependence of the maximised propagator for selected values of μ_0 and ω, which is extremely well reproduced by (B.2). We thus obtain for the maximum of (3.27)

$$\begin{aligned}\big|Q_{nm;lk}(\Lambda)\big| &\leq \frac{(1+\sqrt{\omega})\mu^2}{8\pi}\,(16\max_x|K'(x)|)\,\big|\Delta^{\mathcal{C}}_{nm;lk}\big|_{\mathcal{C}=\frac{\Lambda^2}{\mu^2}} \\ &\leq \begin{cases} C_0\dfrac{\mu^2}{(1-\omega)^{\frac{1}{2}}\Lambda^2}\,\delta_{m+k,n+l} & \text{for } \omega<1\;, \\ C_0\dfrac{\mu^2}{\mu_0\Lambda}\,\delta_{m+k,n+l} & \text{for } \omega=1\;, \end{cases}\end{aligned} \tag{4.5}$$

where $C_0 = 0.78\, C_0' \max_x |K'(x)|$. The constant $C_0' \gtrapprox 1$ corrects the fact that (B.2) holds asymptotically only. Next, from (B.3) we obtain

$$\begin{aligned}\max_n\sum_k\max_{m,l}\big|Q_{nm;lk}(\Lambda)\big| &\leq \frac{(1+\sqrt{\omega})\mu^2}{8\pi}\,(16\max_x|K'(x)|)\max_n\sum_k\max_{m,l}\big|\Delta^{\mathcal{C}}_{nm;lk}\big|_{\mathcal{C}=\frac{\Lambda^2}{\mu^2}} \\ &\leq \begin{cases} C_1\dfrac{\mu^2}{(1-\omega)\Lambda^2} & \text{for } \omega<1\;, \\ C_1\dfrac{\mu^2}{\mu_0^2} & \text{for } \omega=1\;, \end{cases}\end{aligned} \tag{4.6}$$

where $C_1 = 1.28\, C_1' \max_x |K'(x)|$. We conclude from [4] that the scaling exponents of the propagator are given by

$$\delta_0=\delta_1=2 \qquad \text{for } \omega<1\;, \qquad\qquad \delta_0=1\;,\quad \delta_1=0 \qquad \text{for } \omega=1\;. \tag{4.7}$$

We thus have a regular model for $\omega < 1$ and an anomalous (and not renormalisable) model for $\omega = 1$. We also need the product of (4.5) with (4.2):

$$\begin{aligned}&\max_{m,n,k,l}\big|Q_{nm;lk}(\Lambda)\big|\;\sum_{m'}\mathrm{sign}\Big(\max_{n',l'}|K(m',n';\Lambda)K(l'+n'-m',l';\Lambda)|\Big) \\ &\qquad\leq \begin{cases} 2C_0\dfrac{1}{(1-\omega)^{\frac{1}{2}}} & \text{for } \omega<1\;, \\ 2C_0\dfrac{\Lambda}{\mu_0} & \text{for } \omega=1\;. \end{cases}\end{aligned} \tag{4.8}$$

4.2 Verification of the consistency condition

We first have to verify the consistency condition (3.5), which in the present case simplifies considerably. Since the expansion stops at first order in the coupling constant, $\rho^{(V)}_{[m]} \equiv 0$ for $V > 1$, we get the condition

$$\rho_{[0]}[\Lambda_0] \sim \rho_{[m]}[\Lambda_0] - \rho_{[m]}[\Lambda_R]\;. \tag{4.9}$$

The initial value $\rho_{[m]}[\Lambda_R]$ drops out, and according to (3.3) we have to verify

$$1 = \lim_{\Lambda_0\to\infty} \frac{\displaystyle\int_{\Lambda_R}^{\Lambda_0} \frac{d\Lambda}{\Lambda} \sum_n Q_{nm;mn}(\Lambda)}{\displaystyle\int_{\Lambda_R}^{\Lambda_0} \frac{d\Lambda}{\Lambda} \sum_n Q_{n0;0n}(\Lambda)} \equiv \lim_{\Lambda_0\to\infty} \frac{\sum_n(\Delta^K_{nm;mn}(\Lambda_0) - \Delta^K_{nm;mn}(\Lambda_R))}{\sum_n(\Delta^K_{n0;0n}(\Lambda_0) - \Delta^K_{n0;0n}(\Lambda_R))}\;, \tag{4.10}$$

where we have used (3.27). Let $\Lambda_m \ll \Lambda_0$ be the minimal scale such that for $n \geq \frac{\Lambda_m^2}{\mu^2}$ the value of the propagator $\Delta_{nm;mn}$ lies in the interval formed by the two asymptotics of Figure 5. We have $\Lambda_m^2 \approx 2C_m m\mu^2$ where C_m is of order 1. Then we have with (B.4)

$$\begin{aligned}\sum_{n=0}^{\frac{\Lambda_m^2}{\mu^2}-1} \Delta_{nm;mn} + \sum_{n=\frac{\Lambda_m^2}{\mu^2}}^{\frac{\Lambda_0^2}{\mu^2}-1} \frac{1}{\mu^2(n - \frac{9\omega-5}{4}m + 5)} &< \sum_n \Delta^K_{nm;mn}(\Lambda_0) \\ &< \sum_{n=0}^{\frac{\Lambda_m^2}{\mu^2}-1} \Delta_{nm;mn} + \sum_{n=\frac{\Lambda_m^2}{\mu^2}}^{\frac{2\Lambda_0^2}{\mu^2}-1} \frac{1}{\mu^2(n - \frac{9\omega-5}{4}m - 2)}\;.\end{aligned} \tag{4.11}$$

This shows that $\sum_n \Delta^K_{nm;mn}(\Lambda_0)$ is logarithmically divergent for $\Lambda_0 \to \infty$ and that (4.10) holds independently of the finite quantities $\sum_n \Delta^K_{nm;mn}(\Lambda_R)$ and $\sum_{n=0}^{\frac{\Lambda_m^2}{\mu^2}-1} \Delta_{nm;mn}$ and independently of the cut-off function (4.1).

4.3 Estimations for the interaction coefficients

According to [4] the Polchinski equation (3.24) is solved by ribbon graphs characterised by the number V of vertices, the number V^e of external vertices, the number B of boundary components, the genus $\tilde{g}$ and the segmentation index ι. We also recall that it is necessary to sum over indices of the external legs of ribbon graphs. There are $s \leq V^e + \iota - 1$ summations over different outgoing indices where the corresponding incoming index of the trajectories are kept fixed. We write symbolically $\sum_{\mathcal{E}^s}$ for the index summation.

We can now quote directly the power-counting theorem proven in [4], inserting (4.5), (4.6) and (4.8):

Lemma 1 *The homogeneous parts $A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}$ of the coefficients of the effective action describing a regularised ϕ^4-theory on $\mathbb{R}^2_\theta$ in the matrix base are for $2 \leq N \leq 2V{+}2$ and $\sum_{i=1}^N(m_i{-}n_i) = 0$ bounded by*

$$\begin{aligned}&\sum_{\mathcal{E}^s} \big|A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\omega,\rho_0]\big| \\ &\quad\leq \Big(\frac{\Lambda^2}{\mu^2}\Big)^{2-V-B-2\tilde{g}} \Big(\frac{1}{\sqrt{1{-}\omega}}\Big)^{3V-\frac{N}{2}-1+B+2\tilde{g}-V^e-\iota+s} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\;,\end{aligned} \tag{4.12}$$

where $P^q[X]$ denotes a polynomial in X up to degree q. We have $A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N} \equiv 0$ for $N > 2V+2$ or $\sum_{i=1}^N (m_i - n_i) \neq 0$. $\square$

The choice of the boundary conditions is at the same time determined by (4.12) and required to prove (4.12). We notice that the marginal interaction coefficients are those with $V = B = 1$ (and $\tilde{g} = 0$, but this holds automatically for $V = 1$). We can impose the boundary conditions for $A^{(1,1,1,0,0)}_{m_1n_1;\dots;m_4n_4}$ at Λ_0 whereas for $A^{(1,1,1,0,0)}_{m_1n_1;m_2n_2}$ the limit $\Lambda_0 \to \infty$ later on requires to choose the boundary condition at Λ_R. We thus demand

$$\begin{aligned} A^{(1,1,1,0,0)}_{m_1n_1;\dots;m_4n_4}[\Lambda_0,\Lambda_0,\omega,\rho^0] &= \frac{1}{6}\Big(\delta_{n_1m_2}\delta_{n_2m_3}\delta_{n_3m_4}\delta_{n_4m_1} + 5 \text{ permutations}\Big)\,, \\ A^{(1,1,1,0,0)}_{m_1n_1;m_2n_2}[\Lambda_R,\Lambda_0,\omega,\rho^0] &\equiv (\rho_{[m_1]} + \rho_{[m_2]})[\Lambda_R,\Lambda_0,\omega,\rho^0]\delta_{m_1n_2}\delta_{m_2n_1} = 0\,, \\ A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\omega,\rho^0] &= 0 \qquad \text{for all } V+B > 2\,. \end{aligned} \tag{4.13}$$

We remark that for $\omega = 1$ and an optimal choice of the boundary conditions for $A^{(V;V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}$ in agreement with [4] we would get

$$\begin{aligned} &\sum_{\mathcal{E}^s} \big|A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,1,\rho^0]\big| \\ &\qquad \leq \Big(\frac{\Lambda}{\mu}\Big)^{V-\frac{N}{2}+3-B-2\tilde{g}-V^e-\iota+s} \Big(\frac{\mu}{\mu_0}\Big)^{3V-\frac{N}{2}-1+B+2\tilde{g}-V^e-\iota+s} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,. \end{aligned} \tag{4.14}$$

There would be an infinite number of relevant interaction coefficients, which means that the model is not renormalisable when keeping $\omega = 1$.

For the limit $\Lambda_0 \to \infty$ of the theory we are interested in the functions $R^{(V)}_{m_1n_1;\dots;m_Nn_N}$, see (3.12). The $R^{(V)}_{m_1n_1;\dots;m_Nn_N}$ are the solution of the differential equation (3.26) given again by ribbon graphs. These graphs are identical to the graphs representing the A-functions. The differential equation (3.26) actually simplifies in the model under consideration because for $\omega < 1$ the projection $(\ )_{[\hat{n}]}$ is of at most first order in the coupling constant. This means that

$$\Big(\sum_{m,n,k,l} \frac{1}{2} Q_{nm;lk}(\Lambda) R^{(V_1)}_{m'n';n'm';mn;kl}[\Lambda]\Big)_{[\hat{n}]} = 0 \qquad \text{unless } V_1 = 1 \tag{4.15}$$

in the last line of (3.26). However, the rhs of (3.26) for $V = 1$ and $N = 4$ is identically zero, because $R^{(1)}_{m_1n_1;\dots;m_6n_6} = 0$ by graphical reasons and $H^{(0)}_{m_1n_1;\dots;m_4n_4} = 0$ due to the fact that $A^{(1)}_{m_1n_1;\dots;m_4n_4}[\Lambda] = A^{(1)}_{m_1n_1;\dots;m_4n_4}[\Lambda_0]$ is independent of $\rho^0_{[m]}$, see (3.19) and (4.13). We thus obtain

$$\begin{aligned} &R^{(1)}_{m_1n_1;\dots;m_4n_4}[\Lambda,\Lambda_0,\omega,\rho^0] = R^{(1)}_{m_1n_1;\dots;m_4n_4}[\Lambda_0,\Lambda_0,\omega,\rho^0] \\ &= \Lambda_0 \frac{\partial}{\partial\Lambda_0} A^{(1)}_{m_1n_1;\dots;m_4n_4}[\Lambda,\Lambda_0,\omega,\rho^0]\Big|_{\Lambda=\Lambda_0} + \frac{\partial A^{(1)}_{m_1n_1;\dots;m_4n_4}[\Lambda_0,\Lambda_0,\omega,\rho^0]}{\partial\omega}\Lambda_0\frac{d\omega}{d\Lambda_0} \\ &- \sum_n H^{[n](0)}_{m_1n_1;\dots;m_4n_4}[\Lambda_0,\Lambda_0,\omega,\rho^0]\Big(\Lambda_0\frac{\partial}{\partial\Lambda_0}\rho_{[n]}[\Lambda,\Lambda_0,\omega,\rho^0]\Big|_{\Lambda=\Lambda_0} + \frac{\partial\rho_{[n]}[\Lambda,\Lambda_0,\omega,\rho^0]}{\partial\omega}\Lambda_0\frac{d\omega}{d\Lambda_0}\Big) \\ &= 0\,. \end{aligned} \tag{4.16}$$

We have $R^{(1,1,1,0,0)}_{m_1n_1;m_2n_2} = 0$ by definition (3.11). The conclusion is that (3.26) simplifies to

$$\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda}R^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda] \\
&\qquad = \sum_{N_1=2}^{N}\sum_{V_1=1}^{V-1}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda)A^{(V_1)}_{m_1n_1;\dots;m_{N_1-1}n_{N_1-1};mn}[\Lambda]R^{(V-V_1)}_{m_{N_1}n_{N_1};\dots;m_Nn_N;kl}[\Lambda] \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \Big(\binom{N}{N_1-1}-1\Big) \text{ permutations} \\
&\qquad - \sum_{m,n,k,l}\frac{1}{2}Q_{nm;lk}(\Lambda)R^{(V)}_{m_1n_1;\dots;m_Nn_N;mn;kl}[\Lambda]\;. \qquad (4.17)
\end{aligned}$$

Hence, we do not have to evaluate the H-functions for $\omega < 1$.

Lemma 2 *The homogeneous parts $R^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}$ of the coefficients of the Λ_0-varied effective action describing a regularised ϕ^4-theory on $\mathbb{R}^2_\theta$ in the matrix base are for $2 \le N \le 2V{+}2$ and $\sum_{i=1}^N (m_i{-}n_i) = 0$ bounded by*

$$\begin{aligned}
&\sum_{\mathcal{E}^s}\big|R^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\omega,\rho^0]\big| \\
&\qquad \le \frac{\Lambda^2}{\Lambda_0^2}\Big(\frac{\Lambda^2}{\mu^2}\Big)^{2-V-B-2\tilde{g}}\Big(\frac{1}{\sqrt{1-\omega}}\Big)^{3V-\frac{N}{2}-1+B+2\tilde{g}-V^e-\iota+s} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\;, \qquad (4.18)
\end{aligned}$$

for $V+B > 2$. We have $R^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N} \equiv 0$ for $N > 2V{+}2$, for $V+B = 2$ or for $\sum_{i=1}^N (m_i{-}n_i) \neq 0$.

Proof. We first derive the initial condition. From (4.13) we learn that for $V+B>2$ we have $A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\omega,\rho^0] \equiv 0$ *independent of* Λ_0,ω,ρ^0:

$$\begin{aligned}
0 &= \Lambda_0\frac{\partial}{\partial\Lambda_0}A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\omega,\rho^0] \\
&= \frac{\partial}{\partial\omega}A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\omega,\rho^0] = \frac{\partial}{\partial\rho^0}A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\omega,\rho^0]\;, \qquad (4.19)
\end{aligned}$$

for $V+B>2$. The first line has to be considered with care:

$$\begin{aligned}
0 = \Lambda_0\frac{\partial}{\partial\Lambda_0}A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\omega,\rho^0] &\equiv \Lambda_0\frac{\partial}{\partial\Lambda_0}A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\omega,\rho^0]\Big|_{\Lambda=\Lambda_0} \\
&\quad + \Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\omega,\rho^0]\Big|_{\Lambda=\Lambda_0}\;. \qquad (4.20)
\end{aligned}$$

Inserting (4.12) into (4.20) and further into (4.19) we obtain the initial condition for the functions R defined in (3.11) as

$$\begin{aligned}
&\sum_{\mathcal{E}^s}\big|R^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\omega,\rho^0]\big| \\
&\qquad \le \Big(\frac{\Lambda_0^2}{\mu^2}\Big)^{2-V-B-2\tilde{g}}\Big(\frac{1}{\sqrt{1-\omega}}\Big)^{3V-\frac{N}{2}-1+B+2\tilde{g}-V^e-\iota+s} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\;. \qquad (4.21)
\end{aligned}$$

Because of (4.16) we obtain from (4.17) for the simplest non-vanishing R-functions

$$R^{(1,1,2,0,1)}_{m_1n_1;m_2n_2}[\Lambda] = R^{(1,1,2,0,1)}_{m_1n_1;m_2n_2}[\Lambda_0] \;, \qquad R^{(2,2,1,0,0)}_{m_1n_1;\dots;m_6n_6}[\Lambda] = R^{(2,2,1,0,0)}_{m_1n_1;\dots;m_6n_6}[\Lambda_0] \;, \tag{4.22}$$

which due to (4.21) are in agreement with (4.18). Since (4.17) is a linear differential equation, the factor $\frac{\Lambda^2}{\Lambda_0^2}$ first appearing in (4.22) survives to more complicated graphs. Indeed, the only difference between (4.18) and (4.12) is the factor $\frac{\Lambda^2}{\Lambda_0^2}$, and the structure of the rhs of the differential equation (4.17) and is the same as for (3.24). We can thus repeat the evaluation of the Polchinski equation (3.24) performed in [4] for the similar differential equation (4.17). We find immediately by induction that the rhs of (4.17) is bounded by (4.18) with the degree of the polynomial in $\ln\frac{\Lambda_0}{\Lambda_R}$ reduced by 1. This leads to

$$\begin{aligned}
&\sum_{\mathcal{E}^s} \big|R^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\omega,\rho^0]\big| \\
&\quad\leq \sum_{\mathcal{E}^s} \big|R^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\omega,\rho^0]\big| \\
&\quad+ \int_\Lambda^{\Lambda_0} \frac{d\Lambda'}{\Lambda'}\frac{\Lambda'^2}{\Lambda_0^2}\Big(\frac{\Lambda'^2}{\mu^2}\Big)^{2-V-B-2\tilde{g}}\Big(\frac{1}{\sqrt{1-\omega}}\Big)^{3V-\frac{N}{2}-1+B+2\tilde{g}-V^e-\iota+s} P^{2V-\frac{N}{2}-1}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big] \;.
\end{aligned} \tag{4.23}$$

Since $V+B>2$ the integral is bounded by (4.18). □

We have convinced ourselves that it is crucial to keep $\omega<1$. We are, however, interested in the standard ϕ^4-model given by $\Omega=0$ and thus $\omega=(\frac{1-\Omega^2}{1+\Omega^2})^2=1$. *This model can be achieved in the limit.* For this purpose we have to find a dependence $\omega[\Lambda_0]$ with $\lim_{\Lambda_0\to\infty}\omega[\Lambda_0]=1$ which additionally leads to convergence of (3.12). One choice which meets the criteria is

$$\omega[\Lambda_0] = 1-\Big(1+\ln\frac{\Lambda_0}{\Lambda_R}\Big)^{-2} \;, \quad \Lambda_0\frac{d\omega[\Lambda_0]}{d\Lambda_0} = 2\Big(1+\ln\frac{\Lambda_0}{\Lambda_R}\Big)^{-3} \equiv 2\big(1-\omega[\Lambda_0]\big)^{\frac{3}{2}} \;. \tag{4.24}$$

Theorem 3 *The ϕ^4-model on $\mathbb{R}^2_\theta$ is (order by order in the coupling constant) renormalisable in the matrix base by adjusting the coefficients $\rho^0_{[m]}[\Lambda_0]$ of the initial interaction to give $A^{(1;1;1;0;0)}_{m_1n_1;m_2n_2}[\Lambda_R]=0$ and by performing the limit $\Lambda_0\to\infty$ along the path of regulated models characterised by $\omega[\Lambda_0]=1-(1+\ln\frac{\Lambda_0}{\Lambda_R})^{-2}$. The limit $A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty] := \lim_{\Lambda_0\to\infty} A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\Lambda_0,\omega[\Lambda_0],\rho^0[\Lambda_0]]$ of the expansion coefficients of the effective action $L[\phi,\Lambda_R,\Lambda_0,\omega[\Lambda_0],\rho^0[\Lambda_0]]$, see (3.21), exists and satisfies*

$$\begin{aligned}
&\Big|\lambda\big(\lambda\mathcal{V}_2\big)^{V-1}A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty] - \big(\lambda\mathcal{V}_2\big)^{V-1}A^{(V,V^e,B,\tilde{g},\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\Lambda_0,\omega,\rho^0]\Big|_{\omega=1-(1+\ln\frac{\Lambda_0}{\Lambda_R})^{-2}} \\
&\quad\leq \frac{\Lambda_R^4}{\Lambda_0^2}\Big(\frac{\lambda}{\Lambda_R^2}\Big)^V\Big(\frac{\mu^2(1+\ln\frac{\Lambda_0}{\Lambda_R})}{\Lambda_R^2}\Big)^{B+2\tilde{g}-1} P^{5V-N-V^e-\iota}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big] \;.
\end{aligned} \tag{4.25}$$

Proof. The question is whether $L[\phi,\Lambda_R,\Lambda_0,\omega[\Lambda_0],\rho^0[\Lambda_0]]$ converges to a finite limit when $\Lambda_0\to\infty$. The existence of the limit and its property (4.25) follow from inserting (4.18) and (4.24) into (3.12) and Cauchy's criterion. Note that $\int \frac{dx}{x^3}P^q[\ln x] = \frac{1}{x^2}P'^q[\ln x]$. □

It seems that we can additionally achieve a commutative theory $\theta_1=\frac{2}{\mu^2}\to 0$ in the limit $\Lambda_0\to\infty$ by choosing e.g. $\mu^2=\Lambda_R^2\sqrt{1+\ln\frac{\Lambda_0}{\Lambda_R}}$. (We need $4\Omega/\theta_1=\mu^2\sqrt{1-\omega}\to 0$.) However, this limit is degenerate because due to (4.4) all indices are frozen to zero. A different reference scale than μ would help, but we need precisely the choice (4.4) in order to get the correct momentum cut-off from (A.14). There is additional work necessary to get the commutative limit from (4.25).

5 Conclusion

Using the adapted Wilson-Polchinski approach developed in [4] we have proven that the real ϕ^4-model on $\mathbb{R}^2_\theta$ is perturbatively renormalisable when formulated in the matrix base. It was crucial to define the model at the initial scale Λ_0 by the ϕ^4-action supplemented by a harmonic oscillator potential. The renormalisation is achieved by a suitable Λ_0-dependence of the bare mass and the oscillator frequency. This shows that the limit $\Lambda_0\to\infty$ of our model is different from the subtraction of divergences arising in the naïve Feynman graph approach in momentum space. Whereas the treatment of the oscillator potential is easy in the matrix base, a similar procedure in momentum space will face enormous difficulties. In contrast to the Feynman graph approach, our renormalised Green's functions are bounded.

First calculations of the asymptotic behaviour of the propagator in the four-dimensional case suggest that by the same regulator method it will be possible to renormalise the ϕ^4-model on $\mathbb{R}^4_\theta$ [8].

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A The matrix basis of $\mathbb{R}^2_\theta$

The following is copied from [7], adapted to our notation. The Gaussian

$$f_0(x) = 2\mathrm{e}^{-\frac{1}{\theta_1}(x_1^2+x_2^2)}\;, \tag{A.1}$$

with $\theta_1\equiv\theta^{12}=-\theta^{21}>0$, is an idempotent,

$$(f_0\star f_0)(x) = 4\int d^2y\int\frac{d^2k}{(2\pi)^2}\,\mathrm{e}^{-\frac{1}{\theta_1}(2x^2+y^2+2x\cdot y+x\cdot\theta\cdot k+\frac{1}{4}\theta_1^2k^2)+\mathrm{i}k\cdot y} = f_0(x)\;. \tag{A.2}$$

We consider creation and annihilation operators

$$a = \frac{1}{\sqrt{2}}(x_1 + \mathrm{i}x_2)\ , \qquad \bar{a} = \frac{1}{\sqrt{2}}(x_1 - \mathrm{i}x_2)\ ,$$
$$\frac{\partial}{\partial a} = \frac{1}{\sqrt{2}}(\partial_1 - \mathrm{i}\partial_2)\ , \qquad \frac{\partial}{\partial \bar{a}} = \frac{1}{\sqrt{2}}(\partial_1 + \mathrm{i}\partial_2)\ . \tag{A.3}$$

For any $f \in \mathbb{R}^2_\theta$ we have

$$(a \star f)(x) = a(x)f(x) + \frac{\theta_1}{2}\frac{\partial f}{\partial \bar{a}}(x)\ , \qquad (f \star a)(x) = a(x)f(x) - \frac{\theta_1}{2}\frac{\partial f}{\partial \bar{a}}(x)\ ,$$
$$(\bar{a} \star f)(x) = \bar{a}(x)f(x) - \frac{\theta_1}{2}\frac{\partial f}{\partial a}(x)\ , \qquad (f \star \bar{a})(x) = \bar{a}(x)f(x) + \frac{\theta_1}{2}\frac{\partial f}{\partial a}(x)\ . \tag{A.4}$$

This implies $\bar{a}^{\star m} \star f_0 = 2^m \bar{a}^m f_0$, $f_0 \star a^{\star n} = 2^n a^n f_0$ and

$$a \star \bar{a}^{\star m} \star f_0 = \begin{cases} m\theta_1(\bar{a}^{\star(m-1)} \star f_0) & \text{for } m \geq 1 \\ 0 & \text{for } m = 0 \end{cases}$$
$$f_0 \star a^{\star n} \star \bar{a} = \begin{cases} n\theta_1(f_0 \star a^{\star(n-1)}) & \text{for } n \geq 1 \\ 0 & \text{for } n = 0 \end{cases} \tag{A.5}$$

where $a^{\star n} = a \star a \star \cdots \star a$ (n factors) and similarly for $\bar{a}^{\star m}$. Now, defining

$$f_{mn} := \frac{1}{\sqrt{n!m!\,\theta_1^{m+n}}}\,\bar{a}^{\star m} \star f_0 \star a^{\star n} \tag{A.6}$$
$$= \frac{1}{\sqrt{n!m!\,\theta_1^{m+n}}} \sum_{k=0}^{\min(m,n)} (-1)^k \binom{m}{k}\binom{n}{k} k!\, 2^{m+n-2k}\,\theta_1^k\,\bar{a}^{m-k}\,a^{n-k} f_0\ ,$$

(the second line is proved by induction) it follows from (A.5) and (A.2) that

$$(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x)\ . \tag{A.7}$$

The multiplication rule (A.7) identifies the $\star$-product with the ordinary matrix product:

$$a(x) = \sum_{m,n=0}^{\infty} a_{mn} f_{mn}(x)\ , \qquad b(x) = \sum_{m,n=0}^{\infty} b_{mn} f_{mn}(x)$$
$$\Rightarrow \quad (a \star b)(x) = \sum_{m,n=0}^{\infty} (ab)_{mn} f_{mn}(x)\ , \qquad (ab)_{mn} = \sum_{k=0}^{\infty} a_{mk} b_{kn}\ . \tag{A.8}$$

In order to describe elements of $\mathbb{R}^2_\theta$ the sequences $\{a_{mn}\}$ must be of rapid decay [7]:

$$\sum_{m,n=0}^{\infty} a_{mn} f_{mn} \in \mathbb{R}^2_\theta \qquad \text{iff} \qquad \sum_{m,n=0}^{\infty} \big((2m+1)^{2k}(2n+1)^{2k}|a_{mn}|^2\big)^{\frac{1}{2}} < \infty \quad \text{for all } k\ . \tag{A.9}$$

Finally, using (A.2), the trace property of the integral and (A.5) we compute

$$\int d^2x\, f_{mn}(x) = \frac{1}{\sqrt{m!n!\,\theta_1^{m+n}}} \int d^2x\, \big(\bar{a}^{\star m} \star f_0 \star f_0 \star a^{\star n}\big)(x) = \delta_{mn} \int d^2x f_0(x)$$
$$= 2\pi\theta_1\delta_{mn}\,. \tag{A.10}$$

The functions f_{mn} with $m,n < \mathcal{N}$ provide a cut-off both in position and momentum space. Passing to radial coordinates $x_1 = \rho\cos\varphi$, $x_2 = \rho\sin\varphi$ we can compare (A.6) with the expansion of Laguerre polynomials [9, §8.970.1]:

$$f_{mn}(\rho,\varphi) = 2(-1)^m \sqrt{\tfrac{m!}{n!}} \mathrm{e}^{\mathrm{i}\varphi(n-m)} \Big(\sqrt{\tfrac{2}{\theta_1}}\rho\Big)^{n-m} L_m^{n-m}(\tfrac{2}{\theta_1}\rho^2)\, \mathrm{e}^{-\frac{\rho^2}{\theta_1}}\,. \tag{A.11}$$

The function $L_m^\alpha(z) z^{\alpha/2} \mathrm{e}^{-z/2}$ is rapidly decreasing beyond the last maximum $(z_m^\alpha)_{\max}$. One finds numerically $(z_m^\alpha)_{\max} < 2\alpha + 4m$ and thus the radial cut-off

$$\rho_{\max} \approx \sqrt{2\theta_1\mathcal{N}} \qquad \text{for } m,n < \mathcal{N}\,. \tag{A.12}$$

On the other hand, for $p_1 = -p\sin\psi$, $p_2 = p\cos\psi$ we compute with (A.11), [9, §8.411.1] and [9, §7.421.5]

$$\begin{aligned}\tilde{f}(p,\psi) &:= \int_0^\infty \rho\, d\rho \int_0^{2\pi} d\varphi\; \mathrm{e}^{\mathrm{i}p\rho\sin(\varphi-\psi)} f_{mn}(\rho,\varphi) \\ &= 4\pi(-1)^n \sqrt{\tfrac{m!}{n!}} \mathrm{e}^{\mathrm{i}\psi(n-m)} \int_0^\infty \rho d\rho \Big(\sqrt{\tfrac{2}{\theta_1}}\rho\Big)^{n-m} L_m^{n-m}(\tfrac{2}{\theta_1}\rho^2) J_{n-m}(\rho p)\, \mathrm{e}^{-\frac{\rho^2}{\theta_1}} \\ &= 2\pi\theta_1 \sqrt{\tfrac{m!}{n!}} \mathrm{e}^{\mathrm{i}(\psi+\pi)(n-m)} \Big(\sqrt{\tfrac{\theta_1}{2}}\, p\Big)^{n-m} L_m^{n-m}(\tfrac{\theta_1}{2}p^2) \mathrm{e}^{-\frac{\theta_1}{4}p^2}\,.\end{aligned} \tag{A.13}$$

We thus have

$$p_{\max} \approx \sqrt{\frac{8\mathcal{N}}{\theta_1}} \qquad \text{for } m,n < \mathcal{N}\,. \tag{A.14}$$

B Asymptotic behaviour of the propagator

The crucial question for renormalisation is how fast the propagator $\Delta^K_{nm;lk}(\mu^2,\mu_0^2)$ and a certain summation over its indices decay if the indices m,n,k,l become large. We need two asymptotic formulae which we deduce from the numerical evaluation of the propagator for a representative class of parameters. These formulae involve the cut-off propagator

$$\Delta^{\mathcal{C}}_{nm;lk} := \begin{cases} \Delta_{nm;lk} & \text{for } \mathcal{C} \le \max(m,n,k,l) \le 2\mathcal{C}\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{B.1}$$

which is the restriction of $\Delta_{nm;lk}$ to the support of the cut-off propagator $\Delta^K_{nm;lk}(\Lambda)$ appearing in the Polchinski equation, with $\mathcal{C} = \frac{\Lambda^2}{\mu^2}$.

Formula 1:

$$\max_{m,n,k,l}\Big(\Delta^{\mathcal{C}}_{nm;lk}(\mu^2,\mu_0^2)\Big)\approx\sqrt{\frac{3-2\omega}{\mu_0^4+4\mu_0^2\mu^2\mathcal{C}+4\mu^4(1{-}\omega)\mathcal{C}^2}}\,\delta_{m+k,n+l}\;. \tag{B.2}$$

We demonstrate in Figure 2 that $(\max\Delta^{\mathcal{C}}_{nm;lk})^{-1}$ is asymptotically reproduced by

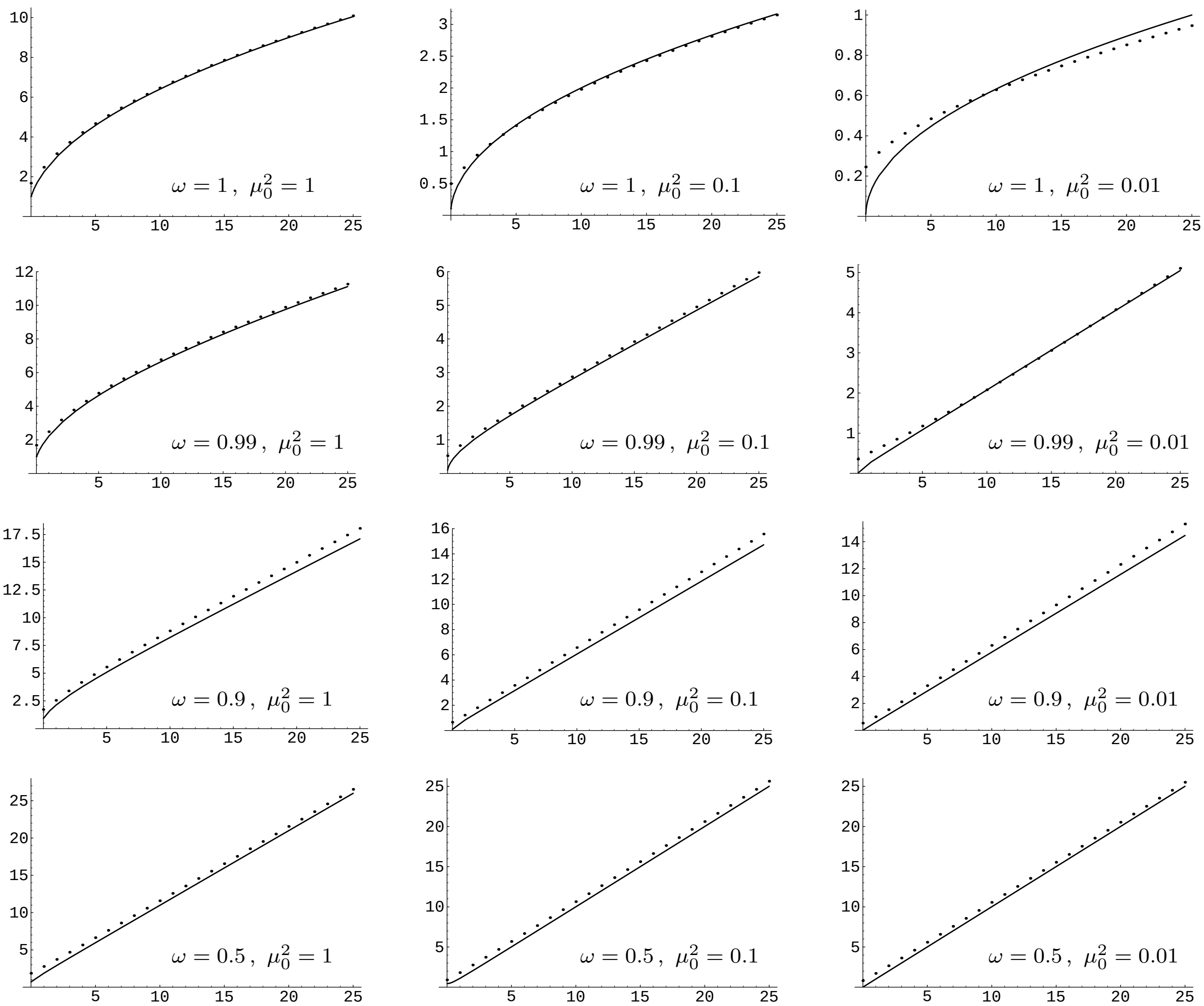

Figure 2: $(\max\Delta^{\mathcal{C}}_{nm;kl}(\mu_0^2,\mu^2))^{-1}$ compared with $((\mu_0^4+2\mu_0^2\mu^2\mathcal{C}+4\mu^4(1{-}\omega)\mathcal{C}^2)/(3-2\omega))^{\frac{1}{2}}$, both plotted over $\mathcal{C}$, for various parameters ω and μ_0^2. We have normalised $\mu^2=1$.

$((\mu_0^4+4\mu_0^2\mu^2\mathcal{C}+4\mu^4(1{-}\omega)\mathcal{C}^2)/(3-2\omega))^{\frac{1}{2}}$. We have evaluated the formula (2.22) for the propagator with $\mathcal{N}=55$. An exception is $\Delta_{nm;lk}$ for $\omega=1$ and $\mu^2\gg\mu_0^2$. Here the choice $\mathcal{N}=55$ in (2.22) is too small, and we have used the numerical evaluation of (2.25) instead. We compare the outcome of (2.22) for $\omega=1$ and (2.25) for various values of μ_0^2 in Figure 3.

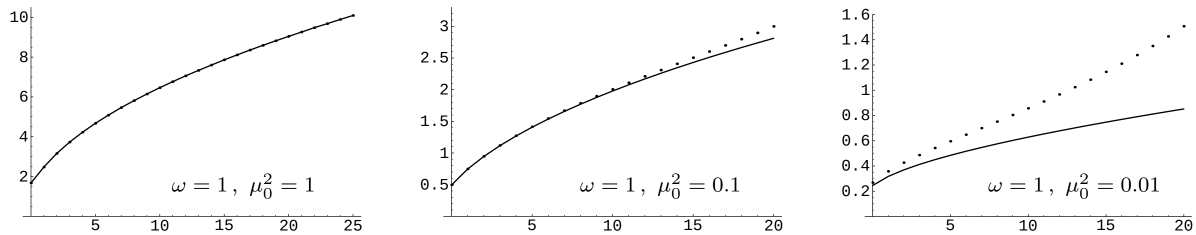

Figure 3: $(\max \Delta^{\mathcal{C}}_{nm;kl}(\mu_0^2, \mu^2))^{-1}$ for $\omega = 1$ computed with (2.22) and $\mathcal{N} = 55$ (dots) and with (2.25) (solid curve), both plotted over $\mathcal{C}$, for various parameters μ_0^2. We have normalised $\mu^2 = 1$. It is apparent that (2.22) converges badly for large $\frac{\mu^2}{\mu_0^2}$.

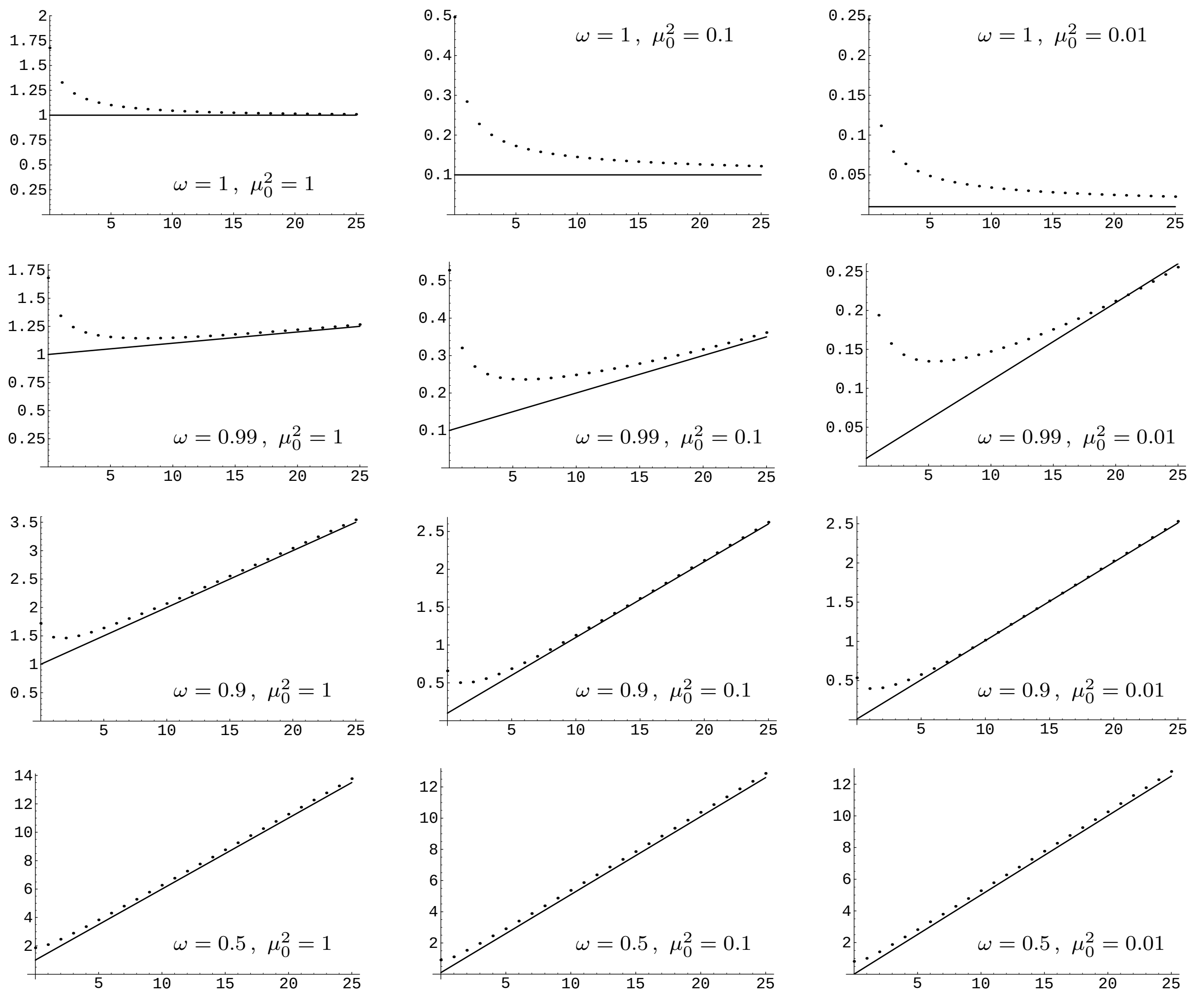

Figure 4: $(\max_n \sum_k \max_{m,l} \Delta^{\mathcal{C}}_{nm;lk})^{-1}$ compared with $\mu_0^2 + \mu^2(1-\omega)\mathcal{C}$, both plotted over $\mathcal{C}$, for various parameters ω and μ_0^2. We have normalised $\mu^2 = 1$.

Formula 2:

$$\max_n \sum_k \max_{m,l} \Delta^{\mathcal{C}}_{nm;lk}(\mu^2,\mu_0^2) \approx \frac{1}{\mu_0^2+\mu^2(1-\omega)\mathcal{C}}\;. \tag{B.3}$$

We demonstrate in Figure 4 that $(\max_n \sum_k \max_{m,l} \Delta^{\mathcal{C}}_{nm;lk})^{-1}$ is asymptotically given by $\mu_0^2+\mu^2(1-\omega)\mathcal{C}$. We have evaluated the formula (2.22) for the propagator with $\mathcal{N}=55$, except for $\omega=1$ and $\mu^2 \gg \mu_0^2$, where (2.25) is used. The crucial observation is that for $\omega=1$ the function $\max_n \sum_k \max_{m,l} \Delta^{\mathcal{C}}_{nm;lk}$ is *increasing* with $\mathcal{C}$ so that $\lim_{\mathcal{C}\to\infty} \max_n \sum_k \max_{m,l} \Delta^{\mathcal{C}}_{nm;lk} = \mu_0^{-2} > 0$.

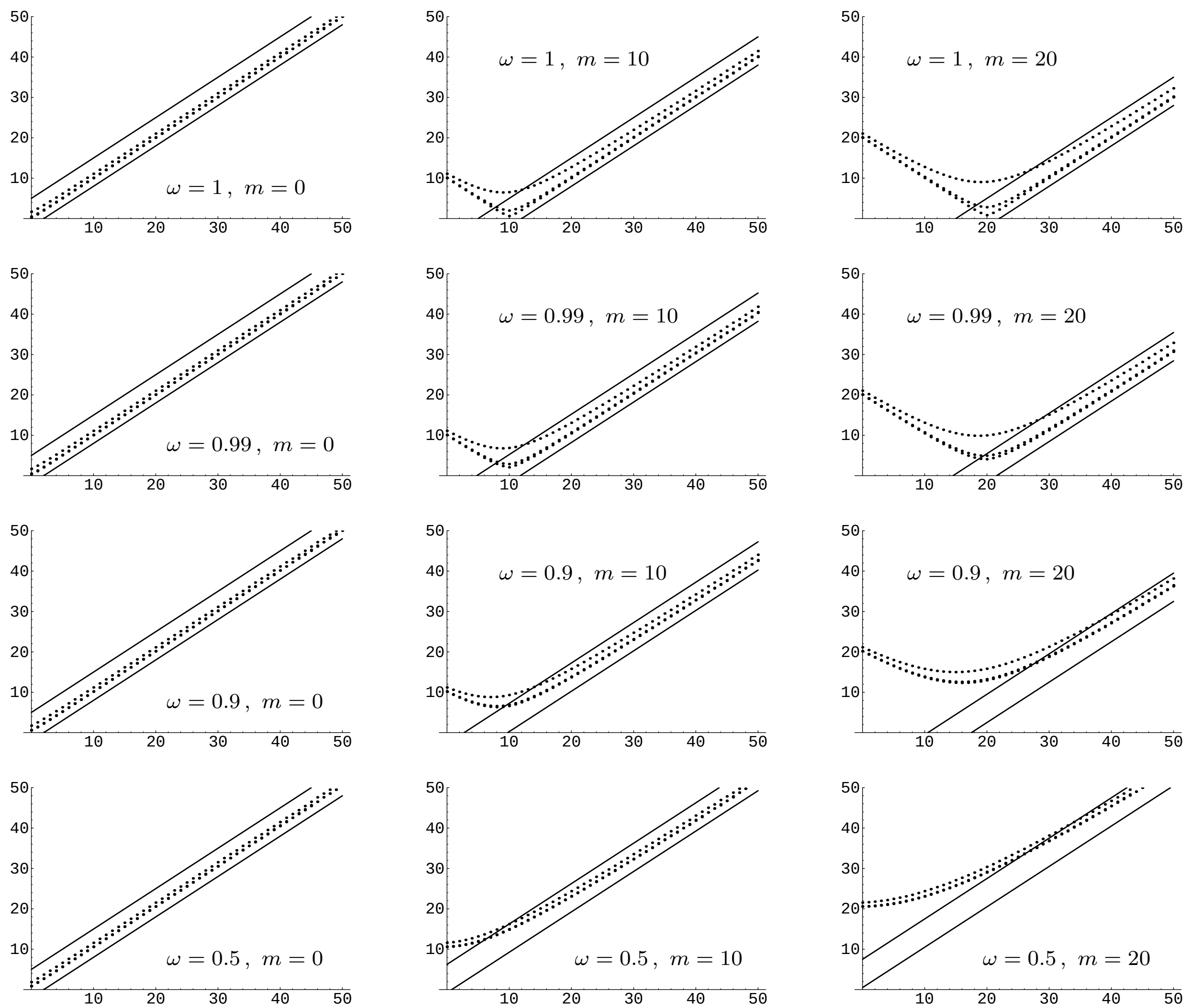

Figure 5: $(\mu^2\Delta_{nm;mn})^{-1}$ compared with $n-\frac{9\omega-5}{4}m+5$ and $n-\frac{9\omega-5}{4}m-2$, both plotted over n, for various parameters ω and m. The dots show $(\mu^2\Delta_{nm;mn})^{-1}$ for three values $\mu_0=\mu$ (upper dots), $\mu_0=0.1\mu$ and $\mu_0=0.01\mu$ (lower dots) of the mass.

Finally, for the verification of (3.5) we need

$$\frac{1}{\mu^2(n-\frac{9\omega-5}{4}m+5)} < \Delta_{nm;mn}(\mu^2,\mu_0^2) < \frac{1}{\mu^2(n-\frac{9\omega-5}{4}m+5)-2} \qquad \text{for } m \ll n\;, \tag{B.4}$$

independent of μ_0. We compare in Figure 5 the inverse of the matrix element $\mu^2\Delta_{nm;mn}(\mu^2,\mu_0^2)$ of the propagator with the asymptotics $n-\frac{9\omega-5}{4}m\{\begin{smallmatrix}+5\\-2\end{smallmatrix}$.

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