hep-th/0401128-v2

Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base

Harald Grosse[1], **Raimar Wulkenhaar**[2]

[1] Institut für Theoretische Physik, Universität Wien, Boltzmanngasse 5, A-1090 Wien, Austria, E-mail: harald.grosse@univie.ac.at

[2] Max-Planck-Institut für Mathematik in den Naturwissenschaften, Inselstraße 22–26, D-04103 Leipzig, Germany, E-mail: raimar.wulkenhaar@mis.mpg.de

Abstract: We prove that the real four-dimensional Euclidean noncommutative ϕ^4-model is renormalisable to all orders in perturbation theory. Compared with the commutative case, the bare action of relevant and marginal couplings contains necessarily an additional term: an harmonic oscillator potential for the free scalar field action. This entails a modified dispersion relation for the free theory, which becomes important at large distances (UV/IR-entanglement). The renormalisation proof relies on flow equations for the expansion coefficients of the effective action with respect to scalar fields written in the matrix base of the noncommutative $\mathbb{R}^4$. The renormalisation flow depends on the topology of ribbon graphs and on the asymptotic and local behaviour of the propagator governed by orthogonal Meixner polynomials.

1. Introduction

Noncommutative ϕ^4-theory is widely believed to be not renormalisable in four dimensions. To underline the belief one usually draws the non-planar one-loop two-point function resulting from the noncommutative ϕ^4-action. The corresponding integral is finite, but behaves $\sim (\theta p)^{-2}$ for small momenta p of the two-point function. The finiteness is important, because the p-dependence of the non-planar graph has no counterpart in the original ϕ^4-action, and thus (if divergent) cannot be absorbed by multiplicative renormalisation. However, if we insert the non-planar graph declared as finite as a subgraph into a bigger graph, one easily builds examples (with an arbitrary number of external legs) where the $\sim p^{-2}$ behaviour leads to non-integrable integrals at small inner momenta. This is the so-called UV/IR-mixing problem [1].

The heuristic argumentation can be made exact: Using a sophisticated mathematical machinery, Chepelev and Roiban have proven a power-counting theorem [2, 3] which relates the power-counting degree of divergence to the topology of *ribbon graphs*. The rough summary of the power-counting theorem is that noncommutative field theories with quadratic divergences become meaningless beyond a certain loop order[1]. For example, in the real noncommutative ϕ^4-model there exist three-loop graphs which cannot be integrated.

In this paper we prove that the real noncommutative ϕ^4-model is *renormalisable to all orders*. At first sight, this seems to be a grave contradiction. However, we do not say that the famous papers [1,2,3] are wrong. In fact, we reconfirm their results, it is only that we take their message serious. The message of the UV/IR-entanglement is that

noncommutativity relevant at very short distances modifies the physics of the model at very large distances.

At large distances we have approximately a free theory. Thus, we have to alter the free theory, whereas the quasi-local ϕ^4-interaction could hopefully be left unchanged.

But how to modify the free action? We found the distinguished modification in the course of a long refinement process of our method. But knowing the result, it can be made plausible. It was pointed out by Langmann and Szabo [4] that the $\star$-product interaction is invariant under a duality transformation between positions and momenta,

$$\hat{\phi}(p) \leftrightarrow \pi^2\sqrt{|\det\theta|}\phi(x) , \qquad p_\mu \leftrightarrow \tilde{x}_\mu := 2(\theta^{-1})_{\mu\nu}x^\nu , \tag{1.1}$$

where $\hat{\phi}(p_a) = \int d^4x_a \, \mathrm{e}^{(-1)^a \mathrm{i} p_{a,\mu} x_a^\mu}\phi(x_a)$. Using the definition of the $\star$-product given in (2.1) and the reality $\phi(x) = \overline{\phi(x)}$ one obtains

$$\begin{aligned} S_{\text{int}} &= \int d^4x \, \frac{\lambda}{4!}(\phi\star\phi\star\phi\star\phi)(x) \\ &= \int \Big(\prod_{a=1}^4 d^4x_a\Big)\, \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\, V(x_1,x_2,x_3,x_4) \\ &= \int \Big(\prod_{a=1}^4 \frac{d^4p_a}{(2\pi)^4}\Big)\, \hat{\phi}(p_1)\hat{\phi}(p_2)\hat{\phi}(p_3)\hat{\phi}(p_4)\, \hat{V}(p_1,p_2,p_3,p_4) , \end{aligned} \tag{1.2}$$

with

$$\begin{aligned} \hat{V}(p_1,p_2,p_3,p_4) &= \frac{\lambda}{4!}(2\pi)^4\delta^4(p_1{-}p_2{-}p_3{+}p_4)\, \cos\Big(\tfrac{1}{2}\theta^{\mu\nu}(p_{1,\mu}p_{2,\nu} + p_{3,\mu}p_{4,\nu})\Big) , \\ V(x_1,x_2,x_3,x_4) &= \frac{\lambda}{4!}\frac{1}{\pi^4\det\theta}\,\delta^4(x_1{-}x_2{+}x_3{-}x_4)\, \cos\Big(2(\theta^{-1})_{\mu\nu}(x_1^\mu x_2^\nu + x_3^\mu x_4^\nu)\Big) . \end{aligned} \tag{1.3}$$

Passing to quantum field theory, $V(x_1,x_2,x_3,x_4)$ and $\hat{V}(p_1,p_2,p_3,p_4)$ become the Feynman rules for the vertices in position space and momentum space, respectively. Multiplicative renormalisability of the four-point function requires that its divergent part has to be self-dual, too. This requires an appropriate Feynman rule for the propagator. Building now two-point functions with these Feynman rules, it is very plausible that if the two-point function is divergent in momentum space, also the duality-transformed two-point function will be divergent. That divergence has to be absorbed in a multiplicative renormalisation of the initial action.

However, the usual free scalar field action is not invariant under that duality transformation and therefore cannot absorb the expected divergence in the two-point function. In order to cure this problem we have to extend the free scalar field action by a harmonic oscillator potential:

$$S_{\text{free}} = \int d^4x \,\Big(\frac{1}{2}(\partial_\mu\phi)\star(\partial^\mu\phi) + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{\mu_0^2}{2}\phi\star\phi\Big)(x) , \tag{1.4}$$

[1] There exist proposals to resum the perturbation series, see [3], but there is no complete proof that this is consistent to all orders.

The action $S_{\text{free}} + S_{\text{int}}$ according to (1.4) and (1.2) is now preserved by duality transformation, up to rescaling. From the previous considerations we can expect that also the renormalisation flow preserves that action $S_{\text{free}} + S_{\text{int}}$. We prove in this paper that this is indeed the case. Thus, the duality-covariance of the action $S_{\text{free}} + S_{\text{int}}$ implements precisely the UV/IR-entanglement.

Of course, we cannot treat the quantum field theory associated with the action (1.4) in momentum space. Fortunately, there is a matrix representation of the noncommutative $\mathbb{R}^D$, where the $\star$-product becomes a simple product of infinite matrices and where the duality between positions and momenta is manifest. The matrix representation plays an important rôle in the proof that the noncommutative $\mathbb{R}^D$ is a spectral triple [5]. It is also crucial for the exact solution of quantum field theories on noncommutative phase space [6,7,8].

Coincidently, the matrix base is also required for another reason. In the traditional Feynman graph approach, the value of the integral associated to non-planar graphs is not unique, because one exchanges the order of integrations in integrals which are not absolutely convergent. To avoid this problem one should use a renormalisation scheme where the various limiting processes are better controlled. The preferred method is the use of flow equations. The idea goes back to Wilson [9]. It was then used by Polchinski [10] to give a very efficient renormalisability proof for commutative ϕ^4-theory. Several improvements have been suggested in [11]. Applying Polchinski's method to the noncommutative ϕ^4-model there is, however, a serious problem in momentum space. We have to guarantee that planar graphs only appear in the distinguished interaction coefficients for which we fix the boundary condition at the renormalisation scale Λ_R. Non-planar graphs have phase factors which involve inner momenta. Polchinski's method consists in taking norms of the interaction coefficients, and these norms ignore possible phase factors. Thus, we would find that boundary conditions for non-planar graphs at Λ_R are required. Since there is an infinite number of different non-planar structures, the model is not renormalisable in this way. A more careful examination of the phase factors is also not possible, because the cut-off integrals prevent the Gaußian integration required for the parametric integral representation [2,3].

As we show in this paper, the bare action $S_{\text{free}} + S_{\text{int}}$ according to (1.4) and (1.2) corresponds to a quantum field theory which is renormalisable to all orders. Together with the duality argument, this is now the conclusive indication that the usual noncommutative ϕ^4-action (with $\Omega = 0$) has to be dismissed in favour of the duality-covariant action of (1.4) and (1.2).

Our proof is very technical. We do not claim that it is the most efficient one. However, it was for us, for the time being, the only possible way. We encountered several "miracles" without which the proof had failed. The first is that the propagator is complicated but numerically accessible. We had thus convinced ourselves that the propagator has such an asymptotic behaviour that all non-planar graphs and all graphs with $N > 4$ external legs are irrelevant according to our general power-counting theorem for dynamical matrix models [12]. However, this still leaves an infinite number of planar two- or four-point functions which would be relevant or marginal according to [12]. In the first versions of [12] we had, therefore, to propose some consistency relations in order to get a meaningful theory.

Miraculously, all this is not necessary. We have further found numerically that the propagator has some universal locality properties suggesting that the infinite number of relevant / marginal planar two- or four-point functions can be decomposed into four relevant / marginal base interactions and an irrelevant remainder. Of course, there must exist a reason for such a coincidence, and the reason are orthogonal polynomials. In our case, it means that the kinetic matrix corresponding to the free action (1.4) written in the matrix base of the noncommutative $\mathbb{R}^D$ is diagonalised by orthogonal Meixner polynomials [13][2] Now, having a closed solution for the free theory in the preferred base of the interaction, the desired local and asymptotic behaviour of the propagator can be derived.

We stress, however, that some of the corresponding estimations of Section 3.4 are, so far, verified numerically only. There is no doubt that the estimations are correct, but for the purists we have to formulate our result as follows: The quantum field theory corresponding to the action (1.4) and (1.2) is renormalisable to all orders provided that the estimations given in Section 3.4 hold. Already this weaker result is a considerable progress, because the elimination of the last possible doubt amounts to estimate a single integral. This estimation will be performed in [17]. The method, further motivation and an outlook to constructive applications are already presented in [18].

Finally, let us recall that the noncommutative ϕ^4-theory in two dimensions is different. We also need the harmonic oscillator potential of (1.4) in all intermediate steps of the renormalisation proof, but at the end it can be switched off with the removal of the cut-off [14]. This is in agreement with the common belief that the UV/IR-mixing problem can be cured in models with only logarithmic divergences.

1.1. Strategy of the proof. As the renormalisation proof is long and technical, we list here the most important steps and the main ideas. A flow chart of these steps is presented in Fig. 1. A more detailed introduction is given in [19].

The first step is to rewrite the ϕ^4-action (1.4)+(1.2) in the harmonic oscillator base of the Moyal plane, see (2.5) and (2.6). The free theory is solved by the propagator (2.7), which we compute in Appendix A using Meixner polynomials in an essential way. The propagator is represented by a finite sum which enables a fast numerical evaluation. Unfortunately, we can offer analytic estimations only in a few special cases.

The propagator is so complicated that a direct calculation of Feynman graphs is not practicable. Therefore, we employ the renormalisation method based on flow equations [10, 11] which we have previously adapted to non-local (dynamical) matrix models [12]. The modification $K[\Lambda]$ of the weights of the matrix indices in the kinetic term is undone in the partition function by a careful adaptation of effective action $L[\phi, \Lambda]$, which is described by the matrix Polchinski equation (3.1). For a modification given by a cut-off function $K[\Lambda]$, renormalisation of the model amounts to prove that the matrix Polchinski equation (3.5) admits a regular solution which depends on a finite number of initial data.

In a perturbative expansion, the matrix Polchinski equation is solved by ribbon graphs drawn on Riemann surfaces. The existence of a regular solution follows from the general power-counting theorem proven in [12] together with the numerical determination of the propagator asymptotics in Appendix C. However, the general proof involved an infinite number of initial conditions, which is physically not acceptable. Therefore, the challenge is to prove the reduction to a finite number of initial data for the renormalisation flow.

The answer is the integration procedure given in Definition 1, Sec. 3.2, which entails mixed boundary conditions for certain planar two- and four-point functions. The idea is to introduce four types of reference graphs with vanishing external indices and to split the integration of the Polchinski equation for the distinguished

[2] In our renormalisation proof [14] of the two-dimensional noncommutative ϕ^4-model we had originally termed these polynomials “deformed Laguerre polynomials”, which we had only constructed via its recursion relation. The closed formula was not known to us. Thus, we are especially grateful to Stefan Schraml who provided us first with [15], from which we got the information that we were using Meixner polynomials, and then with the encyclopaedia [16] of orthogonal polynomials, which was the key to complete the renormalisation proof.

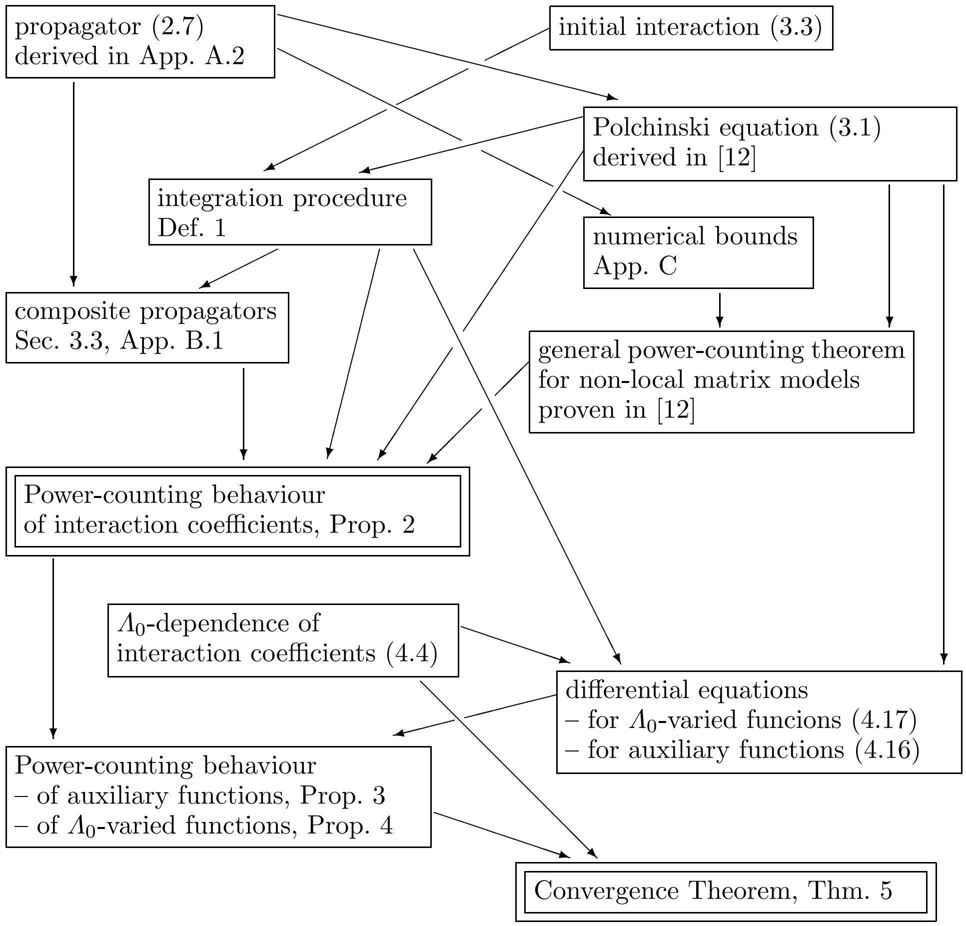


Fig. 1. Relations between the main steps of the proof. The central results are the power-counting behaviour of Proposition 2 and the convergence theorem (Theorem 5). Note that the numerical estimations for the propagator influence the entire chain of the proof.

two- and four-point graphs into an integration of the difference to the reference graphs and a different integration of the reference graphs themselves. The difference between original graph and reference graph is further reduced to differences of propagators, which we call "composite propagators". See Sec. 3.3.

The proof of the power-countung estimations for the interaction coefficients (Proposition 2 in Sec. 3.5) requires the following extensions of the general case treated in [12]:

- We have to prove that graphs where the index jumps along the trajectory between incoming and outgoing indices are suppressed. This leaves 1PI planar four-point functions with constant index along the trajectory and 1PI planar two point functions with in total at most two index jumps along the trajectories as the only graphs which are marginal or relevant.
- For these types of graphs we have to prove that the leading relevant/marginal contribution is captured by reference graphs with vanishing external indices, whereas the difference to the reference graphs is irrelevant. This is the discrete analogue of the BPHZ Taylor subtraction of the expansion coefficients to lowest-order in the external momenta.

Thus, Proposition 2 provides bounds for the interaction coefficients for the effective action at a scale $\Lambda \in [\Lambda_R, \Lambda_0]$. Here, Λ_R is the renormalisation scale where the four reference graphs are normalised, and Λ_0 is the initial scale for the integration which has to be sent to ∞ in order to scale away possible initial conditions for the irrelevant functions. The estimations of Proposition 2 are actually independent of Λ_0 so that the limit $\Lambda_0 \to \infty$ can be taken. This already ensures the renormalisation of the model. However, one would also like to know whether the interaction coefficients converge in the limit $\Lambda_0 \to \infty$ and if so, with which rate. That analysis is performed in Section 4 which culminates in Theorem 5, confirming convergence with a rate Λ_0^{-2}.

2. The duality-covariant noncommutative ϕ^4-action in the matrix base

The noncommutative $\mathbb{R}^4$ is defined as the algebra $\mathbb{R}^4_\theta$ which as a vector space is given by the space $\mathcal{S}(\mathbb{R}^4)$ of (complex-valued) Schwartz class functions of rapid decay, equipped with the multiplication rule [20]

$$(a \star b)(x) = \int \frac{d^4k}{(2\pi)^4} \int d^4y \; a(x+\tfrac{1}{2}\theta{\cdot}k)\, b(x+y)\, \mathrm{e}^{\mathrm{i}k\cdot y} \;, \qquad (2.1)$$
$$(\theta{\cdot}k)^\mu = \theta^{\mu\nu} k_\nu \;, \quad k{\cdot}y = k_\mu y^\mu \;, \quad \theta^{\mu\nu} = -\theta^{\nu\mu} \;.$$

The entries $\theta^{\mu\nu}$ in (2.1) have the dimension of an area. We place ourselves into a coordinate system in which θ has the form

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & \theta_1 & 0 & 0 \\ -\theta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_2 \\ 0 & 0 & -\theta_2 & 0 \end{pmatrix} . \qquad (2.2)$$

We use an adapted base

$$b_{mn}(x) = f_{m^1 n^1}(x^1, x^2)\, f_{m^2 n^2}(x^3, x^4) \;, \qquad m = {m^1 \atop m^2} \in \mathbb{N}^2 \;,\; n = {n^1 \atop n^2} \in \mathbb{N}^2 \;, \qquad (2.3)$$

where the base $f_{m^1 n^1}(x^1, x^2) \in \mathbb{R}^2_\theta$ is given in [14]. This base satisfies

$$(b_{mn} \star b_{kl})(x) = \delta_{nk} b_{ml}(x) \;, \qquad \int d^4x\, b_{mn} = 4\pi^2 \theta_1 \theta_2 \delta_{mn} \;. \qquad (2.4)$$

More information about the noncommutative $\mathbb{R}^D$ can be found in [5,20].

We are going to study a duality-covariant ϕ^4-theory on $\mathbb{R}^4_\theta$. This means that we add a harmonic oscillator potential to the standard ϕ^4-action, which breaks translation invariance but is required for renormalisation. Expanding the scalar field in the matrix base, $\phi(x) = \sum_{m,n\in\mathbb{N}^2} \phi_{mn} b_{mn}(x)$, we have

$$\begin{aligned} S[\phi] &= \int d^4x \Big(\frac{1}{2} g^{\mu\nu} \big(\partial_\mu \phi \star \partial_\nu \phi + 4\Omega^2 ((\theta^{-1})_{\mu\rho} x^\rho \phi) \star ((\theta^{-1})_{\nu\sigma} x^\sigma \phi) \big) + \frac{1}{2} \mu_0^2\, \phi \star \phi \\ &\qquad + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \Big)(x) \\ &= 4\pi^2 \theta_1 \theta_2 \sum_{m,n,k,l \in \mathbb{N}^2} \Big(\frac{1}{2} G_{mn;kl} \phi_{mn} \phi_{kl} + \frac{\lambda}{4!} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} \Big) \;, \end{aligned} \qquad (2.5)$$

where according to [14]

$$\begin{aligned} &G_{mn;kl} \\ &= \big(\mu_0^2+\frac{2}{\theta_1}(1+\Omega^2)(n^1+m^1+1)+\frac{2}{\theta_2}(1+\Omega^2)(n^2+m^2+1)\big)\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2} \\ &-\frac{2}{\theta_1}(1-\Omega^2)\big(\sqrt{k^1l^1}\,\delta_{n^1+1,k^1}\delta_{m^1+1,l^1}+\sqrt{m^1n^1}\,\delta_{n^1-1,k^1}\delta_{m^1-1,l^1}\big)\delta_{n^2k^2}\delta_{m^2l^2} \\ &-\frac{2}{\theta_2}(1-\Omega^2)\big(\sqrt{k^2l^2}\,\delta_{n^2+1,k^2}\delta_{m^2+1,l^2}+\sqrt{m^2n^2}\,\delta_{n^2-1,k^2}\delta_{m^2-1,l^2}\big)\delta_{n^1k^1}\delta_{m^1l^1}\;. \end{aligned} \tag{2.6}$$

We need, in particular, the inverse of the kinetic matrix $G_{mn;kl}$, the propagator $\Delta_{nm;kl}$, which solves the partition function of the free theory ($\lambda=0$) with respect to the preferred base of the interaction. We present the computation of the propagator in Appendix A. The result is

$$\begin{aligned} &\Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}} = \frac{\theta}{2(1+\Omega)^2}\delta_{m^1+k^1,n^1+l^1}\delta_{m^2+k^2,n^2+l^2} \\ &\times \sum_{v^1=\frac{|m^1-l^1|}{2}}^{\frac{\min(m^1+l^1,n^1+k^1)}{2}} \sum_{v^2=\frac{|m^2-l^2|}{2}}^{\frac{\min(m^2+l^2,n^2+k^2)}{2}} B\big(1+\tfrac{\mu_0^2\theta}{8\Omega}+\tfrac{1}{2}(m^1+m^2+k^1+k^2)-v^1-v^2,1+2v^1+2v^2\big) \\ &\times {}_2F_1\Bigg(\begin{matrix}1+2v^1+2v^2\,,\;\frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(m^1+m^2+k^1+k^2)+v^1+v^2\\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(m^1+m^2+k^1+k^2)+v^1+v^2\end{matrix}\Bigg|\frac{(1-\Omega)^2}{(1+\Omega)^2}\Bigg) \\ &\times\prod_{i=1}^2\sqrt{\binom{n^i}{v^i+\frac{n^i-k^i}{2}}\binom{k^i}{v^i+\frac{k^i-n^i}{2}}\binom{m^i}{v^i+\frac{m^i-l^i}{2}}\binom{l^i}{v^i+\frac{l^i-m^i}{2}}}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{2v^i}\;. \end{aligned} \tag{2.7}$$

Here, $B(a,b)$ is the Beta-function and ${}_2F_1\big({a,b \atop c}\big|z\big)$ the hypergeometric function.

3. Estimation of the interaction coefficients

3.1. The Polchinski equation. We have developed in [12] the Wilson-Polchinski renormalisation programme [9,10,11] for non-local matrix models where the kinetic term (Taylor coefficient matrix of the part of the action which is bilinear in the fields) is neither constant nor diagonal. Introducing a cut-off in the measure $\prod_{m,n} d\phi_{mn}$ of the partition function Z, the resulting effect is undone by adjusting the effective action $L[\phi]$ (and other terms which are easy to evaluate). If the cut-off function is a smooth function of the cut-off scale Λ, the adjustment of $L[\phi,\Lambda]$ is described by a differential equation,

$$\Lambda\frac{\partial L[\phi,\Lambda]}{\partial\Lambda} = \sum_{m,n,k,l}\frac{1}{2}\Lambda\frac{\partial\Delta^K_{nm;lk}(\Lambda)}{\partial\Lambda}\Big(\frac{\partial L[\phi,\Lambda]}{\partial\phi_{mn}}\frac{\partial L[\phi,\Lambda]}{\partial\phi_{kl}} - \frac{1}{4\pi^2\theta_1\theta_2}\Big[\frac{\partial^2 L[\phi,\Lambda]}{\partial\phi_{mn}\,\partial\phi_{kl}}\Big]_\phi\Big)\;, \tag{3.1}$$

where $\big[F[\phi]\big]_\phi := F[\phi]-F[0]$ and

$$\Delta^K_{nm;lk}(\Lambda) = \prod_{r=1}^2\Big(\prod_{i^r\in\{m^r,n^r,k^r,l^r\}} K\Big(\frac{i^r}{\theta_r\Lambda^2}\Big)\Big)\Delta_{nm;lk}\;. \tag{3.2}$$

Here, $K(x)$ is a smooth monotonous cut-off function with $K(x)=1$ for $x\leq 1$ and $K(x)=0$ for $x\geq 2$. The differential equation (3.1) is referred to as the Polchinski equation.

In [12] we have derived a power-counting theorem for $L[\phi,\Lambda]$ by integrating (3.1) perturbatively between the initial scale Λ_0 and the renormalisation scale $\Lambda_R \ll \Lambda_0$. The power-counting degree is given by topological data of ribbon graphs and two scaling exponents of the (summed and differentiated) cut-off propagator. The power-counting theorem in [12] is model independent, but it relied on boundary conditions for the integrations which do not correspond to a physically meaningful model.

In this paper we will show that the four-dimensional duality-covariant noncommutative ϕ^4-theory given by the action (2.5) admits an improved power-counting behaviour which only relies on a finite number of physical boundary conditions for the integration. The first step is to extract from the power-counting theorem [12] the set of relevant and marginal interactions, which on the other hand is used as an input to derive the power-counting theorem. To say it differently: One has to be lucky to make the right ansatz for the initial interaction which is then reconfirmed by the power-counting theorem as the set of relevant and marginal interactions. We are going to prove that the following ansatz for the initial interaction is such a lucky choice:

$$
\begin{aligned}
&L[\phi,\Lambda_0,\Lambda_0,\rho^0] \\
&= \sum_{m^1,m^2,n^1,n^2\in\mathbb{N}} \frac{1}{2\pi\theta}\Big(\frac{1}{2}\big(\rho^0_1+(n^1+m^1+n^2+m^2)\rho^0_2\big)\phi_{\substack{m^1\\m^2}\substack{n^1\\n^2}}\phi_{\substack{n^1\\n^2}\substack{m^1\\m^2}} \\
&\qquad\qquad - \rho^0_3\big(\sqrt{n^1m^1}\,\phi_{\substack{m^1\\m^2}\substack{n^1\\n^2}}\phi_{\substack{n^1-1\\n^2}\substack{m^1-1\\m^2}} + \sqrt{n^2m^2}\,\phi_{\substack{m^1\\m^2}\substack{n^1\\n^2}}\phi_{\substack{n^1\\n^2-1}\substack{m^1\\m^2-1}}\big)\Big) \\
&\quad + \sum_{m^1,m^2,n^1,n^2,k^1,k^2,l^1,l^2\in\mathbb{N}} \frac{1}{4!}\rho^0_4\,\phi_{\substack{m^1\\m^2}\substack{n^1\\n^2}}\phi_{\substack{n^1\\n^2}\substack{k^1\\k^2}}\phi_{\substack{k^1\\k^2}\substack{l^1\\l^2}}\phi_{\substack{l^1\\l^2}\substack{m^1\\m^2}}\;.
\end{aligned}
\tag{3.3}
$$

For simplicity we impose a symmetry between the two components m^i of matrix indices $m = \substack{m^1\\m^2} \in \mathbb{N}$, which could be relaxed by taking different ρ-coefficients in front of $m^i{+}n^i$ and $\sqrt{m^in^i}$. Accordingly, we choose the same weights in the noncommutativity matrix, $\theta_1=\theta_2\equiv\theta$.

The differential equation (3.1) is non-perturbatively defined. However, we shall solve it perturbatively as a formal power series in a coupling constant λ which later on will be related to a normalisation condition at $\Lambda=\Lambda_R$, see (3.16). We thus consider the following expansion:

$$
\begin{aligned}
&L[\phi,\Lambda,\Lambda_0,\rho^0] \\
&= \sum_{V=1}^{\infty}\lambda^V \sum_{N=2}^{2V+2}\frac{(2\pi\theta)^{\frac{N}{2}-2}}{N!}\sum_{m_i,n_i} A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho^0]\phi_{m_1n_1}\cdots\phi_{m_Nn_N}\;.
\end{aligned}
\tag{3.4}
$$

Inserting (3.4) into (3.1) we obtain

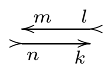
$$
\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda}A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho^0] \\
&= \sum_{N_1=2}^{N}\sum_{V_1=1}^{V-1}\sum_{m,n,k,l\in\mathbb{N}^2}\frac{1}{2}Q_{nm;lk}(\Lambda)A^{(V_1)}_{m_1n_1;\dots;m_{N_1-1}n_{N_1-1};mn}[\Lambda]A^{(V-V_1)}_{m_{N_1}n_{N_1};\dots;m_Nn_N;kl}[\Lambda] \\
&\qquad\qquad\qquad\qquad + \Big(\binom{N}{N_1-1}-1\Big)\text{ permutations} \\
&- \sum_{m,n,k,l\in\mathbb{N}^2}\frac{1}{2}Q_{nm;lk}(\Lambda)A^{(V)}_{m_1n_1;\dots;m_Nn_N;mn;kl}[\Lambda]\;,
\end{aligned}
\tag{3.5}
$$

with

$$Q_{nm;lk}(\Lambda) := \frac{1}{2\pi\theta}\Lambda\frac{\partial\Delta^K_{nm;lk}(\Lambda)}{\partial\Lambda}\;. \tag{3.6}$$

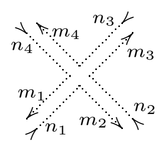
3.2. Integration of the Polchinski equation. We are going to compute the functions $A^{(V)}_{m_1n_1;\dots;m_Nm_N}$ by iteratively integrating the Polchinski equation (3.5) starting from boundary conditions either at Λ_R or at Λ_0. The right choice of the integration direction is an art: The boundary condition influences crucially the estimation, which in turn justifies or discards the original choice of the boundary condition. At the end of numerous trial-and-error experiments with the boundary condition, one convinces oneself that the procedure described in Definition 1 below is—up to finite re-normalisations discussed later—the unique possibility[3] to renormalise the model.

First we have to recall [12] the graphology resulting from the Polchinski equation (3.5). The Polchinski equation is solved by *ribbon graphs* drawn on a Riemann surface of uniquely determined *genus g* and uniquely determined number B of *boundary components* (holes). The ribbons are made of double-line propagators



$$= Q_{mn;kl}(\Lambda) \tag{3.7}$$

attached to vertices



$$= \delta_{n_1m_2}\delta_{n_2m_3}\delta_{n_3m_4}\delta_{n_4m_1}\;. \tag{3.8}$$

Under certain conditions verified by our model, the rough power-counting behaviour of the ribbon graph is determined by the topology g, B of the Riemann surface and the number of vertices V and external legs N. However, in order to prove this behaviour we need some auxiliary notation: the number V^e of external vertices (vertices to which at least one external leg is attached), a certain segmentation index ι and a certain summation of graphs with appropriately varying indices.

We recall in detail the index summation, because we need it for a refinement of the general proof given in [12]. Viewing the ribbon graph as a set of single-lines, we can distinguish closed and open lines. The open lines are called *trajectories* starting at an incoming index n, running through a chain of inner indices k_j and ending at an outgoing index m. Each index belongs to $\mathbb{N}^2$, its components are labelled by superscripts, e.g. $m_j = {}^{m_j^1}_{m_j^2}$. We define $n = \mathfrak{i}[m] = \mathfrak{i}[k_j]$ and $m = \mathfrak{o}[n] = \mathfrak{o}[k_j]$. There is a conservation of the total amount of indices, $\sum_{j=1}^N n_j = \sum_{j=1}^N m_j$ (as vectors in $\mathbb{N}^2$). An index summation $\sum_{\mathcal{E}^s}$ is a summation over the graphs with outgoing indices $\mathcal{E}^s = \{m_1,\dots,m_s\}$ where $\mathfrak{i}[m_1],\dots,\mathfrak{i}[m_s]$ are kept fixed. The number of these summations is restricted by $s \leq V^e + \iota - 1$. Due to the symmetry properties of the propagator one could equivalently sum over n_j where $\mathfrak{o}[n_j]$ are fixed.

A graph γ is produced via a certain *history* of contractions of (in each step) either two smaller subgraphs (with fewer vertices) or a self-contraction of a subgraph with two additional external legs. At a given order V of vertices there are finitely many graphs (distinguished by their topology and the permutation of external indices) contributing the part $A^{(V)\gamma}_{m_1n_1;\dots;m_Nm_N}$ to a function $A^{(V)}_{m_1n_1;\dots;m_Nm_N}$. It is therefore sufficient to prove estimations for each $A^{(V)\gamma}_{m_1n_1;\dots;m_Nm_N}$ separately.

[3] We "only" prove that the method works, not its uniqueness. The reader who doubts uniqueness of the integration procedure is invited to attempt a different way.

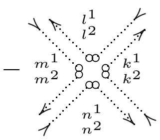
A ribbon graph is called one-particle irreducible (1PI) if it remains connected when removing a single propagator. The first term on the rhs of the Polchinski equation (3.5) leads always to one-particle reducible graphs, because it is left disconnected when removing the propagator $Q_{nm;lk}$ in (3.5).

According to the detailed properties a graph γ, which is possibly a generalisation of the original ribbon graphs as explaned in Section 3.3 below, we define the following recursive procedure (starting with the vertex (3.8) which does not have any subgraphs) to integrate the Polchinski equation (3.5):

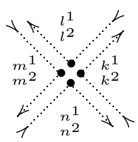
Definition 1 *We consider generalised*[4] *ribbon graphs γ which result via a history of contractions of subgraphs which at each contraction step have already been integrated according to the rules given below.*

1. *Let γ be a planar ($B=1$, $g=0$) one-particle irreducible graph with $N=4$ external legs, where the index along each of its trajectories is constant (this includes the two external indices of a trajectory and the chain of indices at contracting inner vertices in between them). Then, the contribution $A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{m^1\,n^1}{m^2\,n^2};\genfrac{}{}{0pt}{}{n^1\,k^1}{n^2\,k^2};\genfrac{}{}{0pt}{}{k^1\,l^1}{k^2\,l^2};\genfrac{}{}{0pt}{}{l^1\,m^1}{l^2\,m^2}}[\Lambda]$ (using the natural cyclic order of legs of a planar graph) of γ to the effective action is integrated as follows:*

$$A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{m^1\,n^1}{m^2\,n^2};\genfrac{}{}{0pt}{}{n^1\,k^1}{n^2\,k^2};\genfrac{}{}{0pt}{}{k^1\,l^1}{k^2\,l^2};\genfrac{}{}{0pt}{}{l^1\,m^1}{l^2\,m^2}}[\Lambda] := -\int_\Lambda^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \Bigg\{ \widehat{\Lambda' \frac{\partial}{\partial \Lambda'} A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{m^1\,n^1}{m^2\,n^2};\genfrac{}{}{0pt}{}{n^1\,k^1}{n^2\,k^2};\genfrac{}{}{0pt}{}{k^1\,l^1}{k^2\,l^2};\genfrac{}{}{0pt}{}{l^1\,m^1}{l^2\,m^2}}[\Lambda']}$$

$$-$$ 

$$\cdot \left[\widehat{\Lambda' \frac{\partial}{\partial \Lambda'} A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0}}[\Lambda']}\right] \Bigg\}$$

$$+$$ 

$$\left[\int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'} \left(\widehat{\Lambda' \frac{\partial}{\partial \Lambda'} A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0}}[\Lambda']}\right) + A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0}}[\Lambda_R] \right] . \tag{3.9}$$

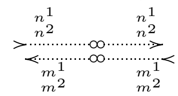
Here (and in the sequel), the wide hat over the Λ'-derivative of an A^γ-function indicates that the rhs of the Polchinski equation (3.5) has to be inserted. The two vertices in the third and fourth lines of (3.9) are identical (both are equal to 1). The four-leg graph in the third line of (3.9) indicates that the graph corresponding to the function in brackets right of it has to be inserted into the holes. The result is a graph with the same topology as the function in the second line, but different indices on inner trajectories. The graph in the fourth line of (3.9) is identical to the original vertex (3.8). The different symbol shall remind us that in the analytic expression for subgraphs containing the vertex of the last line in (3.9) we have to insert the value in brackets right of it.

Remark: We use here (and in all other cases discussed below) the convention (its consistency will be shown later) that at $\Lambda=\Lambda_0$ the contribution to the initial four-point function is independent of the external matrix indices, $A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0}}[\Lambda_0] = A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{m^1\,n^1}{m^2\,n^2};\genfrac{}{}{0pt}{}{n^1\,k^1}{n^2\,k^2};\genfrac{}{}{0pt}{}{k^1\,l^1}{k^2\,l^2};\genfrac{}{}{0pt}{}{l^1\,m^1}{l^2\,m^2}}[\Lambda_0]$. This is not really necessary, we could admit $A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{m^1\,n^1}{m^2\,n^2};\genfrac{}{}{0pt}{}{n^1\,k^1}{n^2\,k^2};\genfrac{}{}{0pt}{}{k^1\,l^1}{k^2\,l^2};\genfrac{}{}{0pt}{}{l^1\,m^1}{l^2\,m^2}}[\Lambda_0] - A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0}}[\Lambda_0] = C_{\genfrac{}{}{0pt}{}{m^1\,n^1}{m^2\,n^2};\genfrac{}{}{0pt}{}{n^1\,k^1}{n^2\,k^2};\genfrac{}{}{0pt}{}{k^1\,l^1}{k^2\,l^2};\genfrac{}{}{0pt}{}{l^1\,m^1}{l^2\,m^2}}[\Lambda_0]$ with $\big|C_{\genfrac{}{}{0pt}{}{m^1\,n^1}{m^2\,n^2};\genfrac{}{}{0pt}{}{n^1\,k^1}{n^2\,k^2};\genfrac{}{}{0pt}{}{k^1\,l^1}{k^2\,l^2};\genfrac{}{}{0pt}{}{l^1\,m^1}{l^2\,m^2}}[\Lambda_0]\big| \leq \frac{\text{const}}{\theta\Lambda_0^2}$.

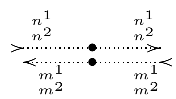
[4] This refers to graphs with composite propagators as defined in Section 3.3.

2. *Let γ be a planar ($B=1$, $g=0$) 1PI graph with $N=2$ external legs, where either the index is constant along each trajectory, or one component of the index jumps*[5] *once by ±1 and back on one of the trajectories, whereas the index along the possible other trajectory remains constant. Then, the contribution $A^{(V)\gamma}_{\substack{m^1\,n^1;n^1\,m^1\\m^2\,n^2;n^2\,m^2}}[\Lambda]$ of γ to the effective action is integrated as follows:*

$$
\begin{aligned}
&A^{(V)\gamma}_{\substack{m^1\,n^1;n^1\,m^1\\m^2\,n^2;n^2\,m^2}}[\Lambda]\\
&:= -\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\;\Bigg\{\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{m^1\,n^1;n^1\,m^1\\m^2\,n^2;n^2\,m^2}}[\Lambda']}
\end{aligned}
$$

$-$ 

$$
\begin{aligned}
&\cdot\Bigg[\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,0;0\,0}}[\Lambda']}\\
&\qquad+m^1\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{1\,0;0\,1\\0\,0;0\,0}}[\Lambda']}-\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,0;0\,0}}[\Lambda']}\Big)\\
&\qquad+m^2\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0;0\,0\\1\,0;0\,1}}[\Lambda']}-\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,0;0\,0}}[\Lambda']}\Big)\\
&\qquad+n^1\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,1;1\,0\\0\,0;0\,0}}[\Lambda']}-\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,0;0\,0}}[\Lambda']}\Big)\\
&\qquad+n^2\;\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,1;1\,0}}[\Lambda']}-\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,0;0\,0}}[\Lambda']}\Big)\Bigg]\Bigg\}
\end{aligned}
$$

$+$ 

$$
\begin{aligned}
&\Bigg[\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\;\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,0;0\,0}}[\Lambda']}\Big)+A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,0;0\,0}}[\Lambda_R]\\
&+m^1\bigg(\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\;\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{1\,0;0\,1\\0\,0;0\,0}}[\Lambda']}-\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,0;0\,0}}[\Lambda']}\Big)\\
&\qquad\qquad+A^{(V)\gamma}_{\substack{1\,0;0\,1\\0\,0;0\,0}}[\Lambda_R]-A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,0;0\,0}}[\Lambda_R]\bigg)\\
&+m^2\bigg(\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\;\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0;0\,0\\1\,0;0\,1}}[\Lambda']}-\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,0;0\,0}}[\Lambda']}\Big)\\
&\qquad\qquad+A^{(V)\gamma}_{\substack{0\,0;0\,0\\1\,0;0\,1}}[\Lambda_R]-A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,0;0\,0}}[\Lambda_R]\bigg)\\
&+n^1\bigg(\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\;\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,1;1\,0\\0\,0;0\,0}}[\Lambda']}-\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,0;0\,0}}[\Lambda']}\Big)\\
&\qquad\qquad+A^{(V)\gamma}_{\substack{0\,1;1\,0\\0\,0;0\,0}}[\Lambda_R]-A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,0;0\,0}}[\Lambda_R]\bigg)\\
&+n^2\bigg(\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\;\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,1;1\,0}}[\Lambda']}-\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,0;0\,0}}[\Lambda']}\Big)\\
&\qquad\qquad+A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,1;1\,0}}[\Lambda_R]-A^{(V)\gamma}_{\substack{0\,0;0\,0\\0\,0;0\,0}}[\Lambda_R]\bigg)\Bigg]\,. \qquad (3.10)
\end{aligned}
$$

[5] A jump forward and backward means the following: Let $k_1,\dots,k_{a-1}$ be the sequence of indices at inner vertices on the considered trajectory $\overrightarrow{nm}$, in correct order between n and m. Then, for either $r=1$ or $r=2$ we require $n^r=k_i^r=m^r$ for all $i\in[1,p{-}1]\cup[q,a{-}1]$ and $k_i^r=n^r\pm1$ (fixed sign) for all $i\in[p,q{-}1]$. The cases $p=1$, $q=p{+}1$ and $q=a$ are admitted. The other index component is constant along the trajectory.

3. *Let γ be a planar ($B=1$, $g=0$) 1PI graph having $N=2$ external legs with external indices $m_1n_1;m_2n_2 = {}^{m^1\pm1}_{m^2}{}^{n^1\pm1}_{n^2};{}^{n^1}_{n^2}{}^{m^1}_{m^2}$ (equal sign) or $m_1n_1;m_2n_2 = {}^{m^1}_{m^2\pm1}{}^{n^1}_{n^2\pm1};{}^{n^1}_{n^2}{}^{m^1}_{m^2}$, with a single[6] jump in the index component of each trajectory. Under these conditions the contribution of γ to the effective action is integrated as follows:*

$$
\begin{aligned}
&A^{(V)\gamma}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda]\\
&:= -\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\;\Bigg\{\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda']}\\
&\qquad -\sqrt{(m^1{+}1)(n^1{+}1)}\;\begin{smallmatrix} n^1+1 & & n^1\\ n^2 & & n^2\\ \succ\cdots\cdots\rhd\!\rhd\cdots\cdots\succ & & \\ \prec\cdots\cdots\rhd\!\rhd\cdots\cdots\prec & & \\ m^1+1 & & m^1\\ m^2 & & m^2\end{smallmatrix}\;\cdot\Bigg[\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{1\\0}\substack{1\\0};\substack{0\\0}\substack{0\\0}}[\Lambda']}\Bigg]\Bigg\}\\
&+\sqrt{(m^1{+}1)(n^1{+}1)}\;\begin{smallmatrix} n^1+1 & & n^1\\ n^2 & & n^2\\ \succ\cdots\cdots\blacktriangleright\cdots\cdots\succ & & \\ \prec\cdots\cdots\blacktriangleright\cdots\cdots\prec & & \\ m^1+1 & & m^1\\ m^2 & & m^2\end{smallmatrix}\;\Bigg[\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\;\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{1\\0}\substack{1\\0};\substack{0\\0}\substack{0\\0}}[\Lambda']}\Big)\\
&\qquad\qquad + A^{(V)\gamma}_{\substack{1\\0}\substack{1\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_R]\Bigg]\,,
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&A^{(V)\gamma}_{\substack{m^1\\m^2+1}\substack{n^1\\n^2+1};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda]\\
&:= -\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\;\Bigg\{\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{m^1\\m^2+1}\substack{n^1\\n^2+1};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda']}\\
&\qquad -\sqrt{(m^2{+}1)(n^2{+}1)}\;\begin{smallmatrix} n^1 & & n^1\\ n^2+1 & & n^2\\ \succ\cdots\cdots\rhd\!\rhd\cdots\cdots\succ & & \\ \prec\cdots\cdots\rhd\!\rhd\cdots\cdots\prec & & \\ m^1 & & m^1\\ m^2+1 & & m^2\end{smallmatrix}\;\cdot\Bigg[\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\\1}\substack{0\\1};\substack{0\\0}\substack{0\\0}}[\Lambda']}\Bigg]\Bigg\}\\
&+\sqrt{(m^2{+}1)(n^2{+}1)}\;\begin{smallmatrix} n^1 & & n^1\\ n^2+1 & & n^2\\ \succ\cdots\cdots\blacktriangleright\cdots\cdots\succ & & \\ \prec\cdots\cdots\blacktriangleright\cdots\cdots\prec & & \\ m^1 & & m^1\\ m^2+1 & & m^2\end{smallmatrix}\;\Bigg[\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\;\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\substack{0\\1}\substack{0\\1};\substack{0\\0}\substack{0\\0}}[\Lambda']}\Big)\\
&\qquad\qquad + A^{(V)\gamma}_{\substack{0\\1}\substack{0\\1};\substack{0\\0}\substack{0\\0}}[\Lambda_R]\Bigg]\,.
\end{aligned}
\tag{3.12}
$$

4. *Let γ be any other type of graph. This includes non-planar graphs ($B>1$ and/or $g>0$), graphs with $N\geq 6$ external legs, one-particle reducible graphs, four-point graphs with non-constant index along at least one trajectory and two-point graphs where the integrated absolute value of the jump along the trajectories is bigger than 2. Then the contribution of γ to the effective action is integrated as follows:*

$$
A^{(V)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda] := -\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\;\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda']}\Big)\,. \tag{3.13}
$$

The previous integration procedure identifies the following distinguished functions $\rho_a[\Lambda,\Lambda_0,\rho^0]$:

$$
\rho_1[\Lambda,\Lambda_0,\rho^0] := \sum_{\gamma\text{ as in Def. 1.2}} A^{\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda,\Lambda_0,\rho^0]\,, \tag{3.14a}
$$

[6] For an index arrangement $m_1n_1;m_2n_2 = {}^{m^1+1}_{m^2}{}^{n^1+1}_{n^2};{}^{n^1}_{n^2}{}^{m^1}_{m^2}$ and sequences $k_1,\dots,k_{a-1}$ ($l_1,\dots,l_{b-1}$) of indices at inner vertices on the trajectory $\overrightarrow{n_1m_2}$ ($\overrightarrow{n_2m_1}$) this means that there exist labels p,q with $n^1{+}1=k^1_i$ for all $i\in[1,p{-}1]$, $n^1=k^1_i$ for all $i\in[p,a{-}1]$ and $n^2=k^2_i$ for all $i\in[1,a{-}1]$ on one trajectory and $m^1{+}1=l^1_j$ for all $j\in[q,b{-}1]$, $m^1=k^1_j$ for all $j\in[1,q{-}1]$ and $m^2=k^2_j$ for all $j\in[1,b{-}1]$ on the other trajectory. The cases $p\in\{1,a\}$ and $q\in\{1,b\}$ are admitted.

$$\rho_2[\Lambda,\Lambda,\rho^0] := \sum_{\gamma \text{ as in Def. 1.2}} \Big(A^{\gamma}_{\substack{1\\0}\substack{0\\0};\substack{0\\0}\substack{1\\0}}[\Lambda,\Lambda_0,\rho^0] - A^{\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda,\Lambda_0,\rho^0]\Big)\;, \tag{3.14b}$$

$$\rho_3[\Lambda,\Lambda_0,\rho^0] := \sum_{\gamma \text{ as in Def. 1.3}} \Big(- A^{\gamma}_{\substack{1\\0}\substack{1\\0};\substack{0\\0}\substack{0\\0}}[\Lambda,\Lambda_0,\rho^0]\Big)\;, \tag{3.14c}$$

$$\rho_4[\Lambda,\Lambda_0,\rho^0] := \sum_{\gamma \text{ as in Def. 1.1}} A^{\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda,\Lambda_0,\rho^0]\;. \tag{3.14d}$$

This identification uses the symmetry properties of the A-functions when summed over all contributing graphs. It follows from Definition 1 and (3.3) that

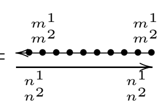
$$\rho_a[\Lambda_0,\Lambda_0,\rho^0] \equiv \rho_a^0\;, \qquad a=1,\dots,4\;. \tag{3.15}$$

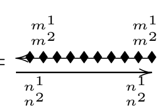
As part of the renormalisation strategy encoded in Definition 1, the coefficients (3.14) are kept constant at $\Lambda=\Lambda_R$. We define

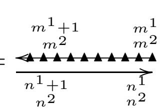
$$\rho_a[\Lambda_R,\Lambda_0,\rho^0]=0 \qquad \text{for } a=1,2,3\;, \qquad \rho_4[\Lambda_R,\Lambda_0,\rho^0]=\lambda\;. \tag{3.16}$$

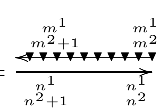
The normalisation (3.16) for ρ_1,ρ_2,ρ_3 identifies $\Delta^K_{nm;lk}(\Lambda_R)$ as the cut-off propagator related to the normalised two-point function at Λ_R. This entails a normalisation of the mass μ_0, the oscillator frequency Ω and the amplitude of the fields ϕ_{mn}. The normalisation condition for $\rho_4[\Lambda_R,\Lambda_0,\rho^0]$ defines the coupling constant used in the expansion (3.4).

3.3. Ribbon graphs with composite propagators. It is convenient to write the linear combination of the functions in braces $\{\ \}$ in (3.9)–(3.12) as a (non-unique) linear combination of graphs in which we find at least one of the following composite propagators:

$$\mathcal{Q}^{(0)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}} := Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}} - Q_{\substack{0\\0}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{0\\0}} =$$  (3.17a)

$$\mathcal{Q}^{(1)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}} := \mathcal{Q}^{(0)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}} - m^1\mathcal{Q}^{(0)}_{\substack{1\\0}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{1\\0}} - m^2\mathcal{Q}^{(0)}_{\substack{0\\1}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{0\\1}} =$$  (3.17b)

$$\mathcal{Q}^{(+\frac{1}{2})}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}} := Q_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}} - \sqrt{m^1+1}\,Q_{\substack{1\\0}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{0\\0}} =$$  (3.17c)

$$\mathcal{Q}^{(-\frac{1}{2})}_{\substack{m^1\\m^2+1}\substack{n^1\\n^2+1};\substack{n^1\\n^2}\substack{m^1\\m^2}} := Q_{\substack{m^1\\m^2+1}\substack{n^1\\n^2+1};\substack{n^1\\n^2}\substack{m^1\\m^2}} - \sqrt{m^2+1}\,Q_{\substack{0\\1}\substack{n^1\\n^2+1};\substack{n^1\\n^2}\substack{0\\0}} =$$  (3.17d)

To obtain the linear combination we recall how the graph γ under consideration is produced via a history of contractions and integrations of subgraphs. For a history

a-b-…-n (a first) we have

$$\begin{aligned} &\Lambda\frac{\partial}{\partial\Lambda}\widehat{A^{(V)\gamma}_{m_1n_1;\dots;m_Nn_N}}[\Lambda] \\ &= \sum_{m_a,n_a,k_a,l_a,\dots,m_n,n_n,k_n,l_n} \int_{(\Lambda)_A}^{(\Lambda)_B}\frac{d\Lambda_n}{\Lambda_n}\int_{(\Lambda_n)_A}^{(\Lambda_n)_B}\frac{d\Lambda_{n-1}}{\Lambda_{n-1}}\dots\int_{(\Lambda_b)_A}^{(\Lambda_b)_B}\frac{d\Lambda_a}{\Lambda_a} \\ &\qquad\times Q_{m_nn_n;k_nl_n}(\Lambda_n)\dots Q_{m_bn_b;k_bl_b}(\Lambda_b)Q_{m_an_a;k_al_a}(\Lambda_a)\,V^{m_an_ak_al_a\dots m_nn_nk_nl_n}_{m_1n_1\dots m_Nn_N}\;, \end{aligned} \tag{3.18}$$

where $V^{m_an_ak_al_a\dots m_nn_nk_nl_n}_{m_1n_1\dots m_Nn_N}$ is the vertex operator and either $(\Lambda_i)_A=\Lambda_i,(\Lambda_i)_B=\Lambda_0$ or $(\Lambda_i)_A=\Lambda_R,(\Lambda_i)_B=\Lambda_i$. The graph $\Lambda\frac{\partial}{\partial\Lambda}\widehat{A^{(V)\gamma}_{00;\dots;00}}[\Lambda]$ is obtained via the same procedure (including the choice of the integration direction), except that we use the vertex operator $V^{m_an_ak_al_a\dots m_nn_nk_nl_n}_{00\dots 00}$. This means that all propagator indices which are not determined by the external indices are the same. Therefore, we can factor out in the difference of graphs all completely inner propagators and the integration operations.

We first consider the difference in (3.9). Since γ is one-particle irreducible with constant index on each trajectory, we get for a certain permutation π ensuring the history of integrations

$$\begin{aligned} &\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{mn;nk;kl;lm}}[\Lambda'] - \Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{00;00;00;00}}[\Lambda'] \\ &= \dots\Big\{\prod_{i=1}^a Q_{m_{\pi_i}k_{\pi_i};k_{\pi_i}m_{\pi_i}}(\Lambda_{\pi_i}) - \prod_{i=1}^a Q_{0k_{\pi_i};k_{\pi_i}0}(\Lambda_{\pi_i})\Big\} \\ &= \dots\Big\{\sum_{b=1}^a\Big(\prod_{i=1}^{b-1}Q_{0k_{\pi_i};k_{\pi_i}0}(\Lambda_{\pi_i})\Big)\mathcal{Q}^{(0)}_{m_{\pi_b}k_{\pi_b};k_{\pi_b}m_{\pi_b}}(\Lambda_{\pi_b}) \\ &\qquad\qquad\times\Big(\prod_{j=b+1}^a Q_{m_{\pi(j)}k_{\pi(j)};k_{\pi(j)}m_{\pi(j)}}(\Lambda_{\pi(j)})\Big)\Big\}\;, \end{aligned} \tag{3.19}$$

where γ contains a propagators with external indices and $m_{\pi_i}\in\{m,n,k,l\}$. The parts of the analytic expression common to both $\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{mn;nk;kl;lm}}[\Lambda']$ and $\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{00;00;00;00}}[\Lambda']$ are symbolised by the dots. The k_{π_i} are inner indices. We thus learn that the difference of graphs appearing in the braces in (3.9) can be written as a sum of graphs each one having a composite propagator (3.17a). Of course, the identity (3.19) is nothing but a generalisation of $a^n-b^n=\sum_{k=0}^{n-1}b^k(a-b)a^{n-k-1}$. There are similar identities for the differences appearing in (3.10)–(3.12). We delegate their derivation to Appendix B.1. In Appendix B.2 we show how the difference operation works for a concrete example of a two-leg graph.

3.4. Bounds for the cut-off propagator. Differentiating the cut-off propagator (3.2) with respect to Λ and recalling that the cut-off function $K(x)$ is constant unless $x\in[1,2]$, we notice that for our choice $\theta_1=\theta_2\equiv\theta$ the indices are restricted as follows:

$$\Lambda\frac{\partial\Delta^K_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(\Lambda)}{\partial\Lambda}=0 \qquad \text{unless}\quad \theta\Lambda^2\leq\max(m^1,m^2,n^1,n^2,k^1,k^2,l^1,l^2)\leq 2\theta\Lambda^2\;. \tag{3.20}$$

In particular, the volume of the support of the differentiated cut-off propagator (3.20) with respect to a single index $m,n,k,l \in \mathbb{N}^2$ equals $4\theta^2\Lambda^4$, which is the correct normalisation of a four-dimensional model [12].

We compute in Appendix C the Λ-dependence of the maximised propagator $\Delta^{\mathcal{C}}_{mn;kl}$, which is the application of the sharp cut-off realising the condition (3.20) to the propagator, for selected values of $\mathcal{C} = \theta\Lambda^2$ and Ω, which is extremely well reproduced by (C.2). We thus obtain for the maximum of (3.6)

$$\max_{m,n,k,l} \big|Q_{mn;kl}(\Lambda)\big| \leq \frac{1}{2\pi\theta}\,(32\max_x |K'(x)|)\, \max_{m,n,k,l} \big|\Delta^{\mathcal{C}}_{mn;kl}\big|_{\mathcal{C}=\Lambda^2\theta} \leq \begin{cases} \dfrac{C_0}{\Omega\Lambda^2\theta}\,\delta_{m+k,n+l} & \text{for } \Omega > 0\;, \\[2ex] \dfrac{C_0}{\sqrt{\Lambda^2\theta}}\,\delta_{m+k,n+l} & \text{for } \Omega = 0\;, \end{cases} \tag{3.21}$$

where $C_0 = C_0' \frac{40}{3\pi} \max_x |K'(x)|$. The constant $C_0' \gtrapprox 1$ corrects the fact that (C.2) holds asymptotically only. Next, from (C.3) we obtain

$$\max_m \Big(\sum_l \max_{n,k} \big|Q_{mn;kl}(\Lambda)\big|\Big) \leq \frac{1}{2\pi\theta}(32\max_x |K'(x)|)\, \max_m \sum_l \max_{n,k} \big|\Delta^{\mathcal{C}}_{mn;kl}\big|_{\mathcal{C}=\Lambda^2\theta} \leq \frac{C_1}{\Omega^2\theta\Lambda^2}\;, \tag{3.22}$$

where $C_1 = 48C_1'/(7\pi)\max_x |K'(x)|$. The product of (3.21) by the volume $4\theta^2\Lambda^4$ of the support of the cut-off propagator with respect to a single index leads to the following bound:

$$\sum_m \Big(\max_{n,k,l} \big|Q_{mn;kl}(\Lambda)\big|\Big) \leq \begin{cases} 4C_0\dfrac{\theta\Lambda^2}{\Omega} & \text{for } \Omega > 0\;, \\[2ex] 4C_0\,\big(\sqrt{\theta}\Lambda\big)^3 & \text{for } \Omega = 0\;. \end{cases} \tag{3.23}$$

According to (A.29) there is the following refinement of the estimation (3.21):

$$\Big|Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1-a^1\\n^2-a^2}\substack{m^1-a^1\\m^2-a^2}}(\Lambda)\Big|_{a^r\geq 0,\, m^r\leq n^r} \leq C_{a^1,a^2}\Big(\frac{m^1}{\theta\Lambda^2}\Big)^{\frac{a_1}{2}}\Big(\frac{m^2}{\theta\Lambda^2}\Big)^{\frac{a_2}{2}}\frac{1}{\Omega\theta\Lambda^2}\;. \tag{3.24}$$

This property will imply that graphs with big total jump along the trajectories are suppressed, provided that the indices on the trajectory are “small”. However, there is a potential danger from the presence of completely inner vertices, where the index summation runs over “large” indices as well. Fortunately, according to (C.4) this case can be controlled by the following property of the propagator:

$$\Big(\sum_{\substack{l\,\in\,\mathbb{N}^2\\ \|m-l\|_1\,\geq\,5}} \max_{k,n\in\mathbb{N}^2} \big|Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(\Lambda)\big|\Big) \leq C_4\Big(\frac{\|m\|_\infty{+}1}{\theta\Lambda^2}\Big)^2\frac{1}{\Omega^2\theta\Lambda^2}\;, \tag{3.25}$$

where we have defined the following norms:

$$\|m-l\|_1 := \sum_{r=1}^2 |m^r - l^r|\;, \qquad \|m\|_\infty := \max(m^1,m^2) \qquad \text{if } m = \substack{m^1\\m^2}\;, \quad l = \substack{l^1\\l^2}\;. \tag{3.26}$$

Moreover, we define

$$\|m_1n_1;\dots;m_Nn_N\|_\infty := \max_{i=1,\dots,N}\big(\|m_i\|_\infty,\|n_i\|_\infty\big)\;. \tag{3.27}$$

Finally, we need estimations for the composite propagators (3.17) and (B.7):

$$\Big|\mathcal{Q}^{(0)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}(\Lambda)\Big| \le C_5\frac{\|m\|_\infty}{\theta\Lambda^2}\frac{1}{\Omega\theta\Lambda^2}\;, \tag{3.28}$$

$$\Big|\mathcal{Q}^{(1)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}(\Lambda)\Big| \le C_6\Big(\frac{\|m\|_\infty}{\theta\Lambda^2}\Big)^2\frac{1}{\Omega\theta\Lambda^2}\;, \tag{3.29}$$

$$\Big|\mathcal{Q}^{(+\frac{1}{2})}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}(\Lambda)\Big| \le C_7\bigg(\frac{\big\|\substack{m^1+1\\m^2}\big\|_\infty}{\theta\Lambda^2}\bigg)^{\frac{3}{2}}\frac{1}{\Omega\theta\Lambda^2}\;. \tag{3.30}$$

These estimations follow from (A.31) and (A.33).

3.5. The power-counting estimation. Now we are going to prove a power-counting theorem for the ϕ^4-model in the matrix base, generalising the theorem proven in [12]. The generalisation concerns 1PI planar graphs and their subgraphs. A subgraph of a planar graph has necessarily genus $g=0$ and an even number of legs on each boundary component. We distinguish one boundary component of the subgraph which after a sequence of contractions will be part of the unique boundary component of an 1PI planar graph. For a trajectory $\overrightarrow{nm}$ on the distinguished boundary component, which passes through the indices $k_1,\dots,k_a$ when going from n to $m=\mathfrak{o}[n]$, we define the total jump as

$$\langle\overrightarrow{nm}\rangle := \|n-k_1\|_1 + \Big(\sum_{c=1}^{a-1}\|k_c-k_{c+1}\|_1\Big) + \|k_a-m\|_1\;. \tag{3.31}$$

Clearly, the jump is additive: if we connect two trajectories $\overrightarrow{nm}$ and $\overrightarrow{mm'}$ to a new trajectory $\overrightarrow{nm'}$, then $\langle\overrightarrow{nm'}\rangle = \langle\overrightarrow{nm}\rangle + \langle\overrightarrow{mm'}\rangle$. We let T be a set of trajectories $\overrightarrow{n_j\mathfrak{o}[n_j]}$ on the distinguished boundary component for which we measure the total jump. By definition, the end points of a trajectory in T cannot belong to $\mathcal{E}^s$.

Moreover, we consider a second set T' of t' trajectories $\overrightarrow{n_j\mathfrak{o}[n_j]}$ of the distinguished boundary component where one of the end points m_j or n_j is kept fixed and the other end point is summed over. However, we require the summation to run over $\langle\overrightarrow{n_j\mathfrak{o}[n_j]}\rangle \ge 5$ only, see (3.25). We let $\sum_{\mathcal{E}^{t'}}$ be the corresponding summation operator.

Additionally, we have to introduce a new notation in order to control

– the behaviour for large indices and given Λ,
– the behaviour for given indices and large Λ.

For this purpose we let $P^a_b\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big]$ denote a function of the indices $m_1,n_1,\dots,m_N,n_N$ and the scale Λ which is bounded as follows:

$$0 \le P^a_b\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] \le \begin{cases} C_aM^a & \text{for } M\ge 1\;,\\ C_bM^b & \text{for } M\le 1\;,\end{cases} \tag{3.32}$$

$$M := \max_{m_i,n_i\notin\mathcal{E}^s,\mathcal{E}^{t'}}\Big(\frac{m_1^r+1}{2\theta\Lambda^2},\frac{n_1^r+1}{2\theta\Lambda^2},\dots,\frac{n_N^r+1}{2\theta\Lambda^2}\Big)\;,$$

for some constants C_a, C_b. The maximisation over the indices m_i^r, n_i^r excludes the summation indices $\mathcal{E}'^t$. Fixing the indices and varying Λ we have

$$P_{b+b'}^{a-a'}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] \le P_b^a\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big]\,, \tag{3.33}$$

for $0 \le a' \le a$ and $b' \ge 0$, assuming appropriate C_a, C_b. Moreover,

$$P_{b_1}^{a_1}\Big[\frac{m_1n_1;\dots;m_{N_1}n_{N_2}}{\theta\Lambda^2}\Big]P_{b_2}^{a_2}\Big[\frac{m_{N_1+1}n_{N_1+1};\dots;m_Nn_N}{\theta\Lambda^2}\Big] \le P_{b_1+b_2}^{a_1+a_2}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big]\,. \tag{3.34}$$

We are going to prove:

Proposition 2 *Let γ be a ribbon graph having N external legs, V vertices, V^e external vertices and segmentation index ι, which is drawn on a genus-g Riemann surface with B boundary components. We require the graph γ to be constructed via a history of subgraphs and an integration procedure according to Definition 1. Then the contribution $A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}$ of γ to the expansion coefficient of the effective action describing a duality-covariant ϕ^4-theory on $\mathbb{R}^4_\theta$ in the matrix base is bounded as follows:*

1. *If γ is as in Definition 1.1, we have*

$$\begin{aligned}\Big|A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1\, n^1\\ m^2\, n^2};\substack{n^1\, k^1\\ n^2\, k^2};\substack{k^1\, l^1\\ k^2\, l^2};\substack{l^1\, m^1\\ l^2\, m^2}}[\Lambda,\Lambda_0,\rho_0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda,\Lambda_0,\rho_0]\Big| \\ \le P_1^{4V-N}\Bigg[\frac{\substack{m^1\, n^1\\ m^2\, n^2};\substack{n^1\, k^1\\ n^2\, k^2};\substack{k^1\, l^1\\ k^2\, l^2};\substack{l^1\, m^1\\ l^2\, m^2}}{\theta\Lambda^2}\Bigg]\Big(\frac{1}{\Omega}\Big)^{3V-2-V^e} P^{2V-2}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,,\end{aligned} \tag{3.35a}$$

$$\Big|A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda,\Lambda_0,\rho_0]\Big| \le \Big(\frac{1}{\Omega}\Big)^{3V-2-V^e} P^{2V-2}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,. \tag{3.35b}$$

2. *If γ is as in Definition 1.2, we have*

$$\begin{aligned}\Big|A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1\, n^1\\ m^2\, n^2};\substack{n^1\, m^1\\ n^2\, m^2}}[\Lambda,\Lambda_0,\rho_0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda,\Lambda_0,\rho_0] \\ - m^1\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{1\,0\\0\,0};\substack{0\,1\\0\,0}}[\Lambda,\Lambda_0,\rho_0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda,\Lambda_0,\rho_0]\big) \\ - n^1\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,1\\0\,0};\substack{1\,0\\0\,0}}[\Lambda,\Lambda_0,\rho_0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda,\Lambda_0,\rho_0]\big) \\ - m^2\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\\1\,0};\substack{0\,0\\0\,1}}[\Lambda,\Lambda_0,\rho_0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda,\Lambda_0,\rho_0]\big) \\ - n^2\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\\0\,1};\substack{0\,0\\1\,0}}[\Lambda,\Lambda_0,\rho_0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda,\Lambda_0,\rho_0]\big)\Big| \\ \le (\theta\Lambda^2)\,P_2^{4V-N}\Bigg[\frac{\substack{m^1\, n^1\\ m^2\, n^2};\substack{n^1\, m^1\\ n^2\, m^2}}{\theta\Lambda^2}\Bigg]\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e} P^{2V-1}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,,\end{aligned} \tag{3.36a}$$

$$\Big|A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda,\Lambda_0,\rho_0]\Big| \le (\theta\Lambda^2)\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e} P^{2V-1}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,, \tag{3.36b}$$

$$\begin{aligned}\Big|A^{(V,V^e,1,0,0)\gamma}_{\substack{1\,0\\0\,0};\substack{0\,1\\0\,0}}[\Lambda,\Lambda_0,\rho_0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[\Lambda,\Lambda_0,\rho_0]\Big| \\ \le \Big(\frac{1}{\Omega}\Big)^{3V-1-V^e} P^{2V-1}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,.\end{aligned} \tag{3.36c}$$

3. *If γ is as in Definition 1.3, we have*

$$\Big|A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda,\Lambda_0,\rho_0] - \sqrt{(m^1{+}1)(n^1{+}1)}A^{(V,V^e,1,0,0)\gamma}_{\substack{1\\0}\substack{1\\0};\substack{0\\0}\substack{0\\0}}[\Lambda,\Lambda_0,\rho_0]\Big|$$
$$\leq (\theta\Lambda^2)\,P_2^{4V-N}\Big[\frac{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}{\theta\Lambda^2}\Big]\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e}P^{2V-1}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,, \tag{3.37a}$$

$$\Big|A^{(V,V^e,1,0,0)\gamma}_{\substack{1\\0}\substack{1\\0};\substack{0\\0}\substack{0\\0}}[\Lambda,\Lambda_0,\rho_0]\Big| \leq \Big(\frac{1}{\Omega}\Big)^{3V-1-V^e}P^{2V-1}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,. \tag{3.37b}$$

4. *If γ is a subgraph of an 1PI planar graph with a selected set T of trajectories on one distinguished boundary component and a second set T' of summed trajectories on that boundary component, we have*

$$\sum_{\mathcal{E}^s}\sum_{\mathcal{E}^{t'}}\big|A^{(V,V^e,B,0,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho_0]\big|$$
$$\leq \big(\theta\Lambda^2\big)^{(2-\frac{N}{2})+2(1-B)}P^{4V-N}_{\big(2t'+\sum_{\overrightarrow{n_j\mathfrak{o}[n_j]}\in T}\min(2,\frac{1}{2}\langle\overrightarrow{n_j\mathfrak{o}[n_j]}\rangle)\big)}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big]$$
$$\times\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}-1+B-V^e-\iota+s+t'}P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,. \tag{3.38}$$

5. *If γ is a non-planar graph, we have*

$$\sum_{\mathcal{E}^s}\big|A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho_0]\big|$$
$$\leq \big(\theta\Lambda^2\big)^{(2-\frac{N}{2})+2(1-B-2g)}\,P_0^{4V-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big]$$
$$\times\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}-1+B+2g-V^e-\iota+s}P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,. \tag{3.39}$$

Proof. We prove the Proposition by induction upward in the vertex order V and for given V downward in the number N of external legs.

5. We start with with the proof for non-planar graphs, noticing that due to (3.33) the estimations (3.35), (3.36), (3.37) and (3.38) can be further bounded by (3.39). The proof of (3.39) reduces to the proof of the general power-counting theorem given in [12], where we have to take for $\big(\frac{\mu}{\Lambda}\big)^{\delta_0}$ $\big(\frac{\mu}{\Lambda}\big)^{\delta_1}$ and $\big(\frac{\Lambda}{\mu}\big)^{\delta_2}$ the estimations (3.21), (3.22) and (3.23) with both their Λ- and Ω-dependence. Independent of the factor (3.32), the non-planarity of the graph guarantees the irrelevance of the corresponding function so that the integration according to Definition 1 agrees with the procedure used in [12]. The dependence on $\frac{m_i^r}{\theta\Lambda^2}, \frac{n_i^r}{\theta\Lambda^2}$ through (3.32) is preserved in its structure, because for $\omega>0$ we have

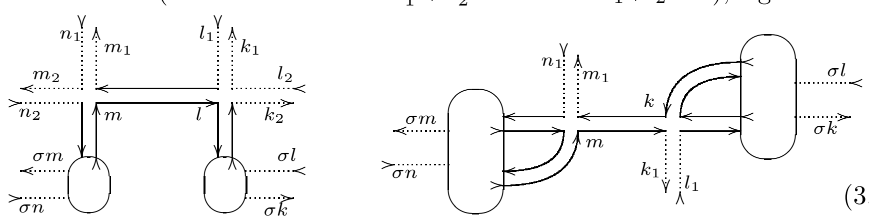
$$\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\frac{C}{\Lambda'^\omega}P_b^a\Big[\frac{m}{\theta\Lambda'^2}\Big] \leq \int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\frac{C}{\Lambda'^\omega}C_b\Big(\frac{m{+}1}{2\theta\Lambda'^2}\Big)^b \leq \frac{1}{\omega{+}2b}\frac{C}{\Lambda^\omega}C_b\Big(\frac{m{+}1}{2\theta\Lambda^2}\Big)^b \tag{3.40a}$$

for $m{+}1\leq 2\theta\Lambda^2$ and

$$\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\frac{C}{\Lambda'^\omega}P_b^a\Big[\frac{m}{\theta\Lambda'^2}\Big]$$
$$\leq \int_{\sqrt{\frac{m+1}{2\theta}}}^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\frac{C}{\Lambda'^\omega}C_b\Big(\frac{m{+}1}{2\theta\Lambda'^2}\Big)^b + \int_\Lambda^{\sqrt{\frac{m+1}{2\theta}}}\frac{d\Lambda'}{\Lambda'}\frac{C}{\Lambda'^\omega}C_a\Big(\frac{m{+}1}{2\theta\Lambda'^2}\Big)^a$$
$$\leq \frac{1}{\omega{+}2a}\frac{C}{\Lambda^\omega}C_a\Big(\frac{m{+}1}{2\theta\Lambda^2}\Big)^a + \frac{(\omega{+}2a)C_b-(\omega{+}2b)C_a}{(\omega{+}2a)(\omega{+}2b)}\frac{C}{\big(\frac{m+1}{2\theta}\big)^{\frac{\omega}{2}}} \tag{3.40b}$$

for $m+1 \geq 2\theta\Lambda^2$. For $(\omega+2b)C_a > (\omega+2a)C_b$ we can omit the last term in the second line of (3.40b), and for $(\omega+2b)C_a < (\omega+2a)C_b$ we estimate it by $\frac{(\omega+2a)C_b-(\omega+2b)C_a}{C_a(\omega+2b)}$ times the first term. Taking a polynomial in $\ln\frac{\Lambda}{\Lambda_R}$ into account, the spirit of (3.40) is unchanged according to [12].
The general power-counting theorem in [12] uses analogues of the bounds (3.21) and (3.22) of the propagator, which do not add factors $\frac{m}{\theta\Lambda^2}$. Since two legs of the subgraph(s) are contracted, the total a-degree of (3.32) becomes $4V-N-2$, which due to (3.33) can be regarded as degree $4V-N$, too.

4. The proof of (3.38) is essentially a repetition of the proof of (3.39), with particular care when contracting trajectories on the distinguished boundary component. The verification of the exponents of $(\theta\Lambda^2)$, $\frac{1}{\Omega}$ and $\ln\frac{\Lambda}{\Lambda_R}$ in (3.38) is identical to the proof of (3.39). It remains to verify the a,b-degrees of the factor (3.32).
We first consider the contraction of two smaller graphs γ_1 (left subgraph) and γ_2 (right subgraph) to the total graph γ.
   (a) We first assume additionally that all indices of the contracting propagator are determined (this is the case for $V_1^e+V_2^e = V^e$ and $\iota_1+\iota_2=\iota$), e.g.



(3.41)

As a subgraph of an 1PI planar graph, at most one side ml or m_1l_1 (mk_1 or m_1k) of the contracting propagator $Q_{m_1m;ll_1}$ ($Q_{m_1m;k_1k}$) can belong to a trajectory in T. In the left graph of (3.41) let us assume that the side $\overrightarrow{ml}$ connects two trajectories $\overrightarrow{\mathfrak{i}[m]m} \in T_1$ and $\overrightarrow{l\mathfrak{o}[l]} \in T_2$ to a new trajectory $\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]} \in T$. The proof for the small-Λ degree $a = 4V-N$ in (3.38) is immediate, because the contraction reduces the number of external legs by 2 and we are free to estimate the contracting propagator by its global maximisation (3.21). Concerning the large-Λ degree b, there is nothing to prove if already $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle + \langle\overrightarrow{l\mathfrak{o}[l]}\rangle \geq 4$. For $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle + \langle\overrightarrow{l\mathfrak{o}[l]}\rangle < 4$ we use the refined estimation (3.24) for the contracting propagator, which gives a relative factor $M^{\frac{1}{2}\langle\overrightarrow{lm}\rangle}$ compared with (3.21), where $M = \max\big(\frac{\|m\|+1}{2\theta\Lambda^2}, \frac{\|l\|+1}{2\theta\Lambda^2}\big)$. Now, the result follows from (3.31). Because of $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle + \langle\overrightarrow{l\mathfrak{o}[l]}\rangle < 4$, the indices m^r, l^r from the propagator can be estimated by $\mathfrak{i}[m]^r$ and $\mathfrak{o}[l]^r$. If the resulting jump leads to $\frac{1}{2}\langle\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]}\rangle > 2$, we use (3.33) to reduce it to 2. In this way we can guarantee that the b-degree does not exceed the a-degree. Alternatively, if $\langle\overrightarrow{lm}\rangle \geq 5$ we can avoid a huge $\langle\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]}\rangle$ by estimating the contracting propagator via (3.25) instead of[7] (3.24), because a single propagator $|Q_{mm_1;l_1l}|$ is clearly smaller than the entire sum over $\|m-l\|_1 \geq 5$.
If $\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]} \in T'$, then the sum over $\mathfrak{o}[l]$ with $\langle\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]}\rangle \geq 5$ can be estimated by the combined sum
   – over finitely many combinations of m,l with $\max(\langle\overrightarrow{\mathfrak{i}[m]m}\rangle, \langle\overrightarrow{ml}\rangle, \langle\overrightarrow{l\mathfrak{o}[l]}\rangle) \leq 4$, which via (3.24) and the induction hypotheses relative to T_1, T_2

[7] In this case there is an additional factor $\frac{1}{\Omega}$ in (3.22) compared with (3.21). It is plausible that this is due to the summation, which we do not need here. However, we do not prove a corresponding formula without summation. In order to be on a safe side, one could replace Ω in the final estimation (4.55) by Ω^2. Since Ω is finite anyway, there is no change of the final result. We therefore ignore the discrepancy in $\frac{1}{\Omega}$.

contributes a factor $M^{\langle\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]}\rangle}$ to the rhs of (3.38), where $M = \max\Big(\frac{\mathfrak{o}[l]^r+1}{\theta\Lambda^2}, \frac{\mathfrak{i}[m]^r+1}{\theta\Lambda^2}, \frac{l^r+1}{\theta\Lambda^2}, \frac{m^r+1}{\theta\Lambda^2}\Big)$. We use (3.33) to reduce the b-degree from $\frac{1}{2}\langle\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]}\rangle$ to 2.

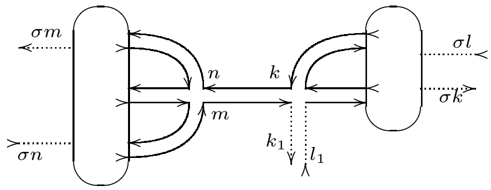
- over m via the induction hypothesis relative to $\overrightarrow{\mathfrak{i}[m]m} \in T_1'$, combined with the usual maximisation (3.21) of the contracting propagator and an estimation of γ_2 where $\overrightarrow{l\mathfrak{o}[l]} \notin T_2, T_2'$,
- over l for fixed $m \approx \mathfrak{i}[m]$, via (3.25), taking $\overrightarrow{\mathfrak{i}[m]m} \notin T_1, T_1'$ and $\overrightarrow{l\mathfrak{o}[l]} \notin T_2, T_2'$.
- over $\mathfrak{o}[l]$ for fixed m and l, with $\mathfrak{i}[m] \approx m \approx l$, via the induction hypothesis relative to $\overrightarrow{l\mathfrak{o}[l]} \in T_2'$, the bound (3.21) of the propagator and $\overrightarrow{\mathfrak{i}[m]m} \notin T_1, T_1'$.

A summation over $\mathfrak{i}[m]$ with given $\mathfrak{o}[l]$ is analogous.
In conclusion, we have proven that the integrand for the graph γ is bounded by (3.38). Since we are dealing with a $N \geq 6$-point function, the total Λ-exponent is negative. Using (3.40) we thus obtain the same bound (3.38) after integration from Λ_0 down to Λ. If $\overrightarrow{m_1 l_1} \in T$ or $\overrightarrow{m_1 l_1} \in T'$ we get (3.38) directly from (3.24) or (3.25).
The discussion of the right graph in (3.41) is similar, showing that the integrand is bounded by (3.38). As long as the integrand is irrelevant (i.e. the total Λ-exponent is negative), we get (3.38) after Λ-integration, too. However, γ might have two legs only with $\langle\overrightarrow{i(m)m}\rangle + \langle\overrightarrow{lo(l)}\rangle \leq 2$. In this case the integrand is marginal or relevant, but according to Definition 1.4 we nonetheless integrate from Λ_0 down to Λ. We have to take into account that the cut-off propagator at the scale Λ vanishes for $\Lambda^2 \geq \|m_1 m; k_1 k\|_\infty/\theta$. Assuming two relevant two-leg subgraphs γ_1, γ_2 bounded by $\theta\Lambda'^2$ times a polynomial in $\ln\frac{\Lambda'}{\Lambda_R}$ each, we have

$$
\begin{aligned}
&\int_\Lambda^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \big|Q_{m_1 m;k_1 k}(\Lambda') A^{\gamma_1}(\Lambda') A^{\gamma_2}(\Lambda')\big| \\
&\qquad \leq \frac{C_0}{\Omega} \int_\Lambda^{\sqrt{\|m_1 m;k_1 k\|_\infty/\theta}} \frac{d\Lambda'}{\Lambda'} (\theta\Lambda'^2) P^{2V-2}\Big[\ln\frac{\Lambda'}{\Lambda_R}\Big] \\
&\qquad \leq \frac{C_0}{\Omega} \|m_1 m; k_1 k\|_\infty P^{2V-2}\Big[\ln\Big(\frac{\|m_1 m; k_1 k\|_\infty}{\theta\Lambda_R^2}\Big)^{\frac{1}{2}}\Big] \\
&\qquad \leq \frac{C_0}{\Omega} (\theta\Lambda^2) P_1^2\Big[\frac{m_1 m; k_1 k}{2\theta\Lambda^2}\Big] P^{2V-2}\Big[\frac{\Lambda}{\Lambda_R}\Big]\,.
\end{aligned}
\tag{3.42}
$$

Here, I have inserted the estimation (3.21) for the propagator, restricted to its support. In the logarithm we expanded $\ln\sqrt{\frac{m}{\theta\Lambda_R^2}} = \ln\sqrt{\frac{m}{\theta\Lambda^2}} + \ln\frac{\Lambda}{\Lambda_R}$ and estimated $\big(\ln\sqrt{\frac{m}{\theta\Lambda^2}}\big)^q < c\frac{m}{2\theta\Lambda^2}$. Thus, the small-$\Lambda$ degree a of the total graph is increased by 2 over the sum of the small-Λ degrees of the subgraphs (taken $= 0$ here), in agreement with (3.38). The estimation for the logarithm is not necessary for the large-Λ degree b in (3.32). Using (3.33) we could reduce that degree to $b = 0$. We would like to underline that the integration of 1PR graphs is one of the sources for the factor (3.32) in the power-counting theorem. Taking the factors (3.32) in the bounds for the subgraphs γ_i into account, the formula modifies accordingly. We confirm (3.38) in any case.
It is clear that all other possibilities with determined propagator indices as discussed in [12] are treated similarly.

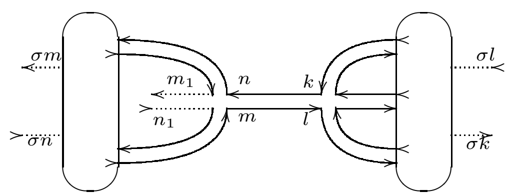
(b) Next, let one index of the contracting propagator be an undetermined summation index, e.g.



(3.43)

Let $\overrightarrow{\mathfrak{i}[k]\mathfrak{o}[n]} \in T$. Then k is determined by the external indices of γ_2. There is nothing to prove for $\langle\overrightarrow{\mathfrak{i}[k]k}\rangle \geq 4$. For $\langle\overrightarrow{\mathfrak{i}[k]k}\rangle < 4$ we partitionate the sum over n into $\langle\overrightarrow{nk}\rangle \leq 4$, where each term yields the integrand (3.38) as before in the case of determined indices (3.41), and the sum over $\langle\overrightarrow{nk}\rangle \geq 5$, which yields the desired factor in (3.38) via (3.25) and the similarity $k \approx \mathfrak{i}[k]$ of indices. As a subgraph of a planar graph, $m \neq \mathfrak{o}[n]$ in γ_1, so that a possible k_1-summation can be transferred to m. If $\overrightarrow{\mathfrak{i}[k]\mathfrak{o}[n]} \in T'$ then in the same way as for (3.41) the summation splits into the four possibilities related to the pieces $\overrightarrow{n\mathfrak{o}[n]}$, $\overrightarrow{kn}$ and $\overrightarrow{\mathfrak{i}[k]k}$, which yield the integrand (3.38) via the induction hypotheses for the subgraphs and via (3.24) or (3.25). The Λ-integration yields (3.38) via (3.40) if the integrand is irrelevant, whereas we have to perform similar considerations as in (3.42) if the integrand is relevant or marginal.

(c) The discussion of graphs with two summation indices on the contracting propagator, such as in



(3.44)

is similar. Note that the planarity requirement implies $m \neq \mathfrak{o}[n]$ and $l \neq \mathfrak{i}[k]$.

(d) Next, we look at self-contractions of the same vertex of a graph. Among the examples discussed in [12] there are only two possibilities which can appear in subgraphs of planar graphs:

(3.45)

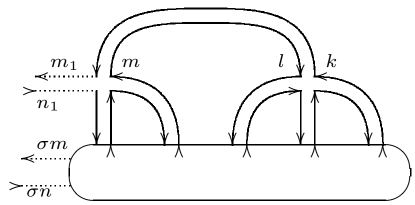
There is nothing to prove for the left graph in (3.45). To verify the large-Λ degree b relative to the right graph, we partitionate the sum over n into

– $\langle\overrightarrow{\mathfrak{i}[n]n}\rangle \leq 4$ and $\langle\overrightarrow{n\mathfrak{o}[n]}\rangle \leq 4$, where each term yields (3.38) via the induction hypothesis for the trajectories $\overrightarrow{\mathfrak{i}[n]n} \in T_1$ and $\overrightarrow{n\mathfrak{o}[n]} \in T_1$ of the subgraph (in the same way as for the examples with determined propagator indices),
– $\langle\overrightarrow{\mathfrak{i}[n]n}\rangle \leq 4$ and $\langle\overrightarrow{n\mathfrak{o}[n]}\rangle \geq 5$, for which the induction hypothesis for $\overrightarrow{\mathfrak{i}[n]n} \notin T_1, T_1'$ and $\overrightarrow{n\mathfrak{o}[n]} \in T_1'$, together with $\mathfrak{i}[n] \approx n$, gives a contribution of 2 to the b-degree in (3.32), and

– $\langle\overrightarrow{\mathfrak{i}[n]n}\rangle \geq 5$, which via the induction hypothesis for $\overrightarrow{\mathfrak{i}[n]n} \in T_1'$ and $\overrightarrow{n\mathfrak{o}[n]} \notin T_1, T_1'$ gives a contribution of 2 to the b-degree in (3.32).

The case $\overrightarrow{\mathfrak{i}[n]\mathfrak{o}[n]} \in T'$ is similar to discuss. At the end we always arrive at the integrand (3.38). If it is irrelevant the integration from Λ_0 down to Λ yields (3.38) according to (3.40). If the integrand is marginal/relevant and γ is one-particle reducible, then the indices of the propagator contracting 1PI subgraphs are of the same order as the incoming and outgoing indices of the trajectories through the propagator (otherwise the 1PI subgraphs are irrelevant). Now a procedure similar to (3.42) yields (3.38) after integration from Λ_0 down to Λ, too. If γ is 1PI and marginal or relevant, it is actually of the type 1–3 of Definition 1 and will be discussed below.

(e) Finally, there will be self-contractions of different vertices of a subgraph, such as in



(3.46)

The vertices to contract have to be situated on the same (distinguished) boundary component, because the contraction of different boundary components increases the genus and for contractions of other boundary components the proof is immediate. Only the large-Λ degree b is questionable.

Let $\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]} \in T$, with $\mathfrak{i}[m] \neq \mathfrak{o}[l]$ due to planarity. Here, m is regarded as a summation index. As before we split that sum over m into a piece with $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle \leq 4$, which yields the b-degree of the integrand (3.38) term by term via the induction hypothesis relative to $\overrightarrow{\mathfrak{i}[m]m}, \overrightarrow{l\mathfrak{o}[l]} \in T_1$ and (3.24) for the contracting propagator, and a piece with $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle \geq 5$, which gives (3.38) via the induction hypothesis relative to $\overrightarrow{\mathfrak{i}[m]m} \in T_1'$ and $\overrightarrow{l\mathfrak{o}[l]} \notin T_1, T_1'$. If $\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]} \in T'$ the sum over $\mathfrak{o}[l]$ with $\overrightarrow{\mathfrak{i}[m]\mathfrak{o}[l]} \geq 5$ is estimated by a finite number of combinations of m, l with $\max(\langle\overrightarrow{\mathfrak{i}[m]m}\rangle, \langle\overrightarrow{ml}\rangle, \langle\overrightarrow{l\mathfrak{o}[l]}\rangle) \leq 4$, which yields the integrand (3.38) via the induction hypothesis for T_1 and (3.24), and the sum over index combinations

– $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle \leq 4$, $\langle\overrightarrow{ml}\rangle \leq 4$, $\langle\overrightarrow{l\mathfrak{o}[l]}\rangle \geq 5$
– $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle \leq 4$, $\langle\overrightarrow{ml}\rangle \geq 5$
– $\langle\overrightarrow{\mathfrak{i}[m]m}\rangle \geq 5$

which is controlled by the induction hypothesis relative to T_1' or (3.25), together with the similarity of trajectory indices at those parts where the jumps is bounded by 4. The case where $\overrightarrow{\mathfrak{i}[k]m_1} \in T$ or $\overrightarrow{\mathfrak{i}[k]m_1} \in T'$ is easier to treat. We thus arrive in any case at the estimation (3.38) for the integrand of γ, which leads to the same estimation (3.38) for γ itself according to the considerations at the end of 4.d. If γ is of type 1–3 of Definition 1 we will treat it below.

The discussion of all other possible self-contractions as listed in [12] is similar.

This finishes the part 4 of the proof of Proposition 2.

1. Now we consider 1PI planar 4-leg graphs γ with constant index on each trajectory. If the external indices are zero, we get (3.35b) directly from the general power-counting theorem [12], because the integration direction used there agrees with Definition 1.1.

For non-zero external indices we decompose the difference (3.35a) according to (3.19) into graphs with composite propagators (3.17a) bounded by (3.28). The composite propagators appear on one of the trajectories of γ, and as such already on the trajectory of a sequence of subgraphs of γ, starting with some minimal subgraph γ_0. The composite propagator is the contracting propagator for γ_0. Now, the integrand of the minimal subgraph γ_0 with composite propagator is bounded by a factor $C_5' \frac{\|m\|}{\Lambda^2\theta}$ times the integrand of the would-be graph γ_0 with ordinary propagator, where m is the index at the trajectory under consideration. If γ_0 is irrelevant, the factor $C_5' \frac{\|m\|}{\Lambda^2\theta}$ of the integrand survives according to (3.40) to the subgraph γ_0 itself. Otherwise, if γ_0 is relevant or marginal, it is decomposed according to 1–3 of Definition 1. Here, the last lines of (3.9)–(3.12) are independent of the external index m so that in the difference relative to the composite propagator these last lines of (3.9)–(3.12) cancel identically. There remains the first part of (3.9)–(3.12), which is integrated from Λ_0 downward and which is irrelevant by induction. Thus, (3.40) applies in this case, too, saving the factor $C_5' \frac{\|m\|}{\Lambda^2\theta}$ to γ_0 in any case. This factor thus appears in the integrand of the subgraph of γ next larger than γ_0. By iteration of the procedure we obtain the additional factor $C_5' \frac{\|m\|}{\Lambda^2\theta}$ in the integrand of the total graph γ with composite propagators, the Λ-degree of which being thus reduced by 2 compared with the original graph γ. Since γ itself is a marginal graph according to the general power-counting behaviour (3.39), the graph with composite propagator is irrelevant and according to Definition 1.1 to be integrated from Λ_0 down to Λ. This explains (3.35a).

3. Similarly, we conclude from the proof of (3.38) that the integrands of graphs γ according to Definition 1.3 are marginal. In particular, we immediately confirm (3.37b). For non-zero external indices we decompose the difference (3.37a) according to (B.1) into graphs either with composite propagators (3.17a) bounded by (3.28) or with composite propagators (3.17c)/(3.17d) bounded by (3.30). In such a graph there are—apart from usual propagators with bound (3.21)/(3.22)—two propagators with $a^1 + a^2 = 1$ in (3.24) and a composite propagator with bound (3.28), or one propagator with $a^1 + a^2 = 1$ in (3.24) and one composite propagator with bound (3.30). In both cases we get a total factor $\frac{\|{m^1+1 \atop m^2}\, {n^1+1 \atop n^2}\|_\infty^2}{(\theta\Lambda^2)^2}$ compared with a general planar two-point graph (3.39). The detailed discussion of the subgraphs is similar as under 1.

2. Finally, we have to discuss graphs γ according to Definition 1.2. We first consider the case that γ has constant index on each trajectory. It is then clear from the proof of (3.39) that (in particular) at vanishing indices the graph γ is relevant, which is expressed by (3.36b). Next, the difference (3.36c) of graphs can as in (3.19) be written as a sum of graphs with one composite propagator (3.17a), the bound of which is given by (3.28). After the treatment of subgraphs as described under 1, the integrand of each term in the linear combination is marginal. According to Definition 1.2 we have to integrate these terms from Λ_R up to Λ which agrees with the procedure in [12] and leads to (3.36c). Finally, according to (B.3) and (B.4), the linear combination constituting the lhs of (3.36a) results in a linear combination of graphs with either one propagator (3.17b) with bound (3.29), or with two propagators (3.17a) with bound (3.28). A similar discussion as under 1 then leads to (3.36a).
The second case is when one index component jumps once on a trajectory and back. According to the proof of (3.38) the integrand of γ at vanishing external indices is marginal. We regard it nevertheless as relevant using the inequality $1 \leq (\theta\Lambda^2)(\theta\Lambda_R^2)^{-1}$, where $(\theta\Lambda_R^2)^{-1}$ is some number kept constant in our renor-

malisation procedure. We now obtain (3.36b). Similarly, the integrand relative to the difference (3.36c) would be irrelevant, but is considered as marginal via the same trick. Finally, the linear combination constituting the lhs of (3.36a) is according to (B.3) and (B.6)–(B.9) a linear combination of graphs having either two propagators with $a^1 + a^2 = 1$ in (3.24) and a composite propagator with bound (3.28), or one propagator with $a^1 + a^2 = 1$ in (3.24) and one composite propagator (3.17c)/(3.17d) with bound (3.30). The discussion as before would lead to an increased large-Λ degrees P_3^{4V-N} instead of P_2^{4V-N} in (3.36a), which can be reduced to P_2^{4V-N} according to (3.33).

This finishes the proof of Proposition 2. □

It is now important to realise [11] that the estimations (3.35)–(3.39) of Proposition 2 do not make any reference to the initial scale Λ_0. Therefore, the estimations (3.35)–(3.39), which give finite bounds for the interaction coefficients with finite external indices, also hold in the limit $\Lambda_0 \to \infty$. This is the renormalisation of the duality-covariant noncommutative ϕ^4-model.

In numerical computations the limit $\Lambda_0 \to \infty$ is difficult to realise. Taking instead a large but finite Λ_0, it is then important to estimate the error and the rate of convergence as Λ_0 approaches ∞. This type of estimations is the subject of the next section.

We finish this section with a remark on the freedom of normalisation conditions. One of the most important steps in the proof is the integration procedure for the Polchinski equation given in Definition 1. For presentational reasons we have chosen the smallest possible set of graphs to be integrated from Λ_R upward. This can easily be generalised. We could admit in (3.9) any planar 1PI four-point graphs for which the incoming index of each trajectory is equal to the outgoing index on that trajectory, but with arbitrary jump along the trajectory. There is no change of the estimation (3.35a), because (according to Proposition 2.4) $\Lambda' \frac{\partial}{\partial \Lambda'} A^{(V,V^e,\widehat{1,0,0)\gamma}}_{{m^1 \atop m^2}{n^1 \atop n^2};{n^1 \atop n^2}{k^1 \atop k^2};{k^1 \atop k^2}{l^1 \atop l^2};{l^1 \atop l^2}{m^1 \atop m^2}}[\Lambda']$ is already irrelevant for these graphs, so is the difference in braces in (3.9). Moreover, integrating such an irrelevant graph according to the last line of (3.9) from Λ_R upward we obtain a bound $\frac{1}{(\Lambda_R^2\theta)} P^{2V-2}[\ln \frac{\Lambda}{\Lambda_R}]$, which agrees with (3.35b), because $\Lambda_R^2\theta$ is finite. Similarly, we can relax the conditions on the jump along the trajectory in (3.10)–(3.12). We would then define the $\rho_a[\Lambda, \Lambda_0, \rho^0]$-functions in (3.14) for that enlarged set of graphs γ.

In a second generalisation we could admit one-particle reducible graphs in 1–3 of Definition 1 and even non-planar graphs with the same condition on the external indices as in 1–3 of Definition 1. Since there is no difference in the power-counting behaviour between non-planar graphs and planar graphs with large jump, the discussion is as before. However, the convergence theorem developed in the next section cannot be adapted in an easy way to normalisation conditions involving non-planar graphs.

In summary, the proposed generalisations constitute different normalisation conditions of the same duality-covariant ϕ^4-model. Passing from one normalisation to another one is a finite re-normalisation. The invariant characterisation of our model is its definition via four independent normalisation conditions for the ρ-functions so that at large scales the effective action approaches (3.3).

4. The convergence theorem

In this section we prove the *convergence* of the coefficients of the effective action in the limit $\Lambda_0 \to \infty$, relative to the integration procedure given in Definition 1. This is a stronger result than the power-counting estimation of Proposition 2, which e.g. would be compatible with bounded oscillations. Additionally, we identify the rate of convergence of the interaction coefficients.

4.1. The Λ_0-dependence of the effective action. We have to control the Λ_0-dependence which enters the effective action via the integration procedure of Definition 1. There is an explicit dependence via the integration domain of irrelevant graphs and an implicit dependence through the normalisation (3.16), which requires a carefully adapted Λ_0-dependence of ρ_a^0. For fixed $\Lambda = \Lambda_R$ but variable Λ_0 we consider the identity

$$\begin{aligned} L[\phi,\Lambda_R,\Lambda_0',\rho^0[\Lambda_0']] - L[\phi,\Lambda_R,\Lambda_0'',\rho^0[\Lambda_0'']] &\equiv \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0}\Big(\Lambda_0 \frac{d}{d\Lambda_0} L[\phi,\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]\Big) \\ = \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0}\Big(\Lambda_0 \frac{\partial L[\phi,\Lambda_R,\Lambda_0,\rho^0]}{\partial\Lambda_0} &+ \sum_{a=1}^4 \Lambda_0 \frac{d\rho_a^0}{d\Lambda_0}\frac{\partial L[\phi,\Lambda_R,\Lambda_0,\rho^0]}{\partial\rho_a^0}\Big)\;. \end{aligned} \tag{4.1}$$

The model is defined by fixing the boundary condition for ρ_b at Λ_R, i.e. by keeping $\rho_b[\Lambda_R,\Lambda_0,\rho^0] = \text{constant}$:

$$0 = d\rho_b[\Lambda_R,\Lambda_0,\rho^0] = \frac{\partial\rho_b[\Lambda_R,\Lambda_0,\rho^0]}{\partial\Lambda_0} d\Lambda_0 + \sum_{a=1}^4 \frac{\partial\rho_b[\Lambda_R,\Lambda_0,\rho^0]}{\partial\rho_a^0}\frac{d\rho_a^0}{d\Lambda_0} d\Lambda_0\;. \tag{4.2}$$

Assuming that we can invert the matrix $\frac{\partial\rho_b[\Lambda_R,\Lambda_0,\rho^0]}{\partial\rho_a^0}$, which is possible in perturbation theory, we get

$$\frac{d\rho_a^0}{d\Lambda_0} = -\sum_{b=1}^4 \frac{\partial\rho_a^0}{\partial\rho_b[\Lambda_R,\Lambda_0,\rho^0]}\frac{\partial\rho_b[\Lambda_R,\Lambda_0,\rho^0]}{\partial\Lambda_0}\;. \tag{4.3}$$

Inserting (4.3) into (4.1) we obtain

$$L[\phi,\Lambda_R,\Lambda_0',\rho^0[\Lambda_0']] - L[\phi,\Lambda_R,\Lambda_0'',\rho^0[\Lambda_0'']] = \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0}\; R[\phi,\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]\;, \tag{4.4}$$

with

$$\begin{aligned} R[\phi,\Lambda,\Lambda_0,\rho^0] := \Lambda_0 &\frac{\partial L[\phi,\Lambda,\Lambda_0,\rho^0]}{\partial\Lambda_0} \\ &- \sum_{a,b=1}^4 \frac{\partial L[\phi,\Lambda,\Lambda_0,\rho^0]}{\partial\rho_a^0}\frac{\partial\rho_a^0}{\partial\rho_b[\Lambda,\Lambda_0,\rho^0]}\Lambda_0\frac{\partial\rho_b[\Lambda,\Lambda_0,\rho^0]}{\partial\Lambda_0}\;. \end{aligned} \tag{4.5}$$

Following [10] we differentiate (4.5) with respect to Λ:

$$\begin{aligned} \Lambda\frac{\partial R}{\partial\Lambda} &= \Lambda_0\frac{\partial}{\partial\Lambda_0}\Big(\Lambda\frac{\partial L}{\partial\Lambda}\Big) - \sum_{a,b=1}^4 \frac{\partial}{\partial\rho_a^0}\Big(\Lambda\frac{\partial L}{\partial\Lambda}\Big)\frac{\partial\rho_a^0}{\partial\rho_b}\Lambda_0\frac{\partial\rho_b}{\partial\Lambda_0} \\ &+ \sum_{a,b,c,d=1}^4 \frac{\partial L}{\partial\rho_a^0}\frac{\partial\rho_a^0}{\partial\rho_b}\frac{\partial}{\partial\rho_c^0}\Big(\Lambda\frac{\partial\rho_b}{\partial\Lambda}\Big)\frac{\partial\rho_c^0}{\partial\rho_d}\Lambda_0\frac{\partial\rho_d}{\partial\Lambda_0} - \sum_{a,b=1}^4 \frac{\partial L}{\partial\rho_a^0}\frac{\partial\rho_a^0}{\partial\rho_b}\Lambda_0\frac{\partial}{\partial\Lambda_0}\Big(\Lambda\frac{\partial\rho_b}{\partial\Lambda}\Big)\;. \end{aligned} \tag{4.6}$$

We have omitted the dependencies for simplicity and made use of the fact that the derivatives with respect to $\Lambda, \Lambda_0, \rho^0$ commute. Using (3.1), with $\theta_1 = \theta_2 \equiv \theta$, we compute the terms on the rhs of (4.6):

$$\begin{aligned}
&\Lambda_0 \frac{\partial}{\partial \Lambda_0}\Big(\Lambda \frac{\partial L[\phi,\Lambda,\Lambda_0,\rho^0]}{\partial \Lambda}\Big) \\
&\quad = \sum_{m,n,k,l} \frac{1}{2} \Lambda \frac{\partial \Delta^K_{nm;lk}(\Lambda)}{\partial \Lambda} \Big(2 \frac{\partial L[\phi,\Lambda,\Lambda_0,\rho^0]}{\partial \phi_{mn}} \frac{\partial}{\partial \phi_{kl}} \Big(\Lambda_0 \frac{\partial}{\partial \Lambda_0} L[\phi,\Lambda,\Lambda_0,\rho^0]\Big) \\
&\qquad\qquad - \frac{1}{(2\pi\theta)^2} \Big[\frac{\partial^2}{\partial \phi_{mn}\, \partial \phi_{kl}} \Big(\Lambda_0 \frac{\partial}{\partial \Lambda_0} L[\phi,\Lambda,\Lambda_0,\rho^0]\Big)\Big]_\phi \Big) \\
&\quad \equiv M\Big[L, \Lambda_0 \frac{\partial L}{\partial \Lambda_0}\Big] \,.
\end{aligned} \tag{4.7}$$

Similarly, we have

$$\frac{\partial}{\partial \rho^0_a}\Big(\Lambda \frac{\partial L[\phi,\Lambda,\Lambda_0,\rho^0]}{\partial \Lambda}\Big) = M\Big[L, \frac{\partial L}{\partial \rho^0_a}\Big] \,. \tag{4.8}$$

For (4.6) we also need the function $\Lambda_0 \frac{\partial}{\partial \Lambda_0}\big(\Lambda \frac{\partial \rho_b}{\partial \Lambda}\big)$, which is obtained from (4.7) by first expanding L on the lhs according to (3.4) and by further choosing the indices at the A-coefficients according to (3.14). Applying these operations to the rhs of (4.7), we obtain for $U \mapsto \Lambda_0 \frac{\partial L}{\partial \Lambda_0}$ or $U \mapsto \frac{\partial L}{\partial \rho^0_a}$ the expansions

$$M[L,U] = \sum_{N=2}^{\infty} \sum_{m_i,n_i \in \mathbb{N}^2} \frac{1}{N!} M_{m_1n_1;\dots;m_Nn_N}[L,U] \phi_{m_1n_1} \cdots \phi_{m_Nn_N} \tag{4.9}$$

and the projections

$$M_1[L,U] := \sum_{\gamma \text{ as in Def. 1.2}} M^\gamma_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[L,U] \,, \tag{4.10a}$$

$$M_2[L,U] := \sum_{\gamma \text{ as in Def. 1.2}} \big(M^\gamma_{\substack{1\,0\\0\,0};\substack{0\,1\\0\,0}}[L,U] - M^\gamma_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[L,U]\big) \,, \tag{4.10b}$$

$$M_3[M,U] := \sum_{\gamma \text{ as in Def. 1.3}} \big(- M^\gamma_{\substack{1\,1\\0\,0};\substack{0\,0\\0\,0}}[L,U]\big) \,, \tag{4.10c}$$

$$M_4[L,U] := \sum_{\gamma \text{ as in Def. 1.1}} M^\gamma_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}[L,U] \,. \tag{4.10d}$$

Since the graphs γ in (4.10) are one-particle irreducible, only the third line of (4.7) can contribute[8] to M_a. Using (4.7), (4.8) and (4.10) as well as the linearity of $M[L,U]$ in the second argument we can rewrite (4.6) as

$$\Lambda \frac{\partial R}{\partial \Lambda} = M[L,R] - \sum_{a=1}^{4} \frac{\partial L}{\partial \rho_a} M_a[L,R] \,, \tag{4.11}$$

where the function

$$\frac{\partial L}{\partial \rho_a}[\Lambda,\Lambda_0,\rho^0] := \sum_{b=1}^{4} \frac{\partial L[\Lambda,\Lambda_0,\rho^0]}{\partial \rho^0_b} \frac{\partial \rho^0_b}{\partial \rho_a[\Lambda,\Lambda_0,\rho^0]} \tag{4.12}$$

[8] If one-particle reducible graphs are included in the normalisation conditions as discussed at the end of Section 3, also the second line of (4.7) must be taken into account.

scales according to

$$\Lambda\frac{\partial}{\partial\Lambda}\Big(\frac{\partial L}{\partial\rho_a}\Big) = M\Big[L, \frac{\partial L}{\partial\rho_a}\Big] - \sum_{b=1}^{4} \frac{\partial L}{\partial\rho_b} M_b\Big[L, \frac{\partial L}{\partial\rho_a}\Big]\;, \tag{4.13}$$

as a similar calculation shows.

Next, we also expand (4.5) and (4.12) as power series in the coupling constant:

$$\begin{aligned}
&\frac{\partial L}{\partial\rho_a}[\phi,\Lambda,\Lambda_0,\rho^0] \\
&= \sum_{V=0}^{\infty}\lambda^V \sum_{N=2}^{2V+4} \frac{(2\pi\theta)^{\frac{N}{2}-2}}{N!} \sum_{m_i,n_i} H^{a(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho^0]\phi_{m_1n_1}\cdots\phi_{m_Nn_N}\;, \qquad (4.14)\\
&R[\phi,\Lambda,\Lambda_0,\rho^0] \\
&= \sum_{V=1}^{\infty}\lambda^V \sum_{N=2}^{2V+2} \frac{(2\pi\theta)^{\frac{N}{2}-2}}{N!} \sum_{m_i,n_i} R^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho^0]\phi_{m_1n_1}\cdots\phi_{m_Nn_N}\;. \qquad (4.15)
\end{aligned}$$

The differential equations (4.13) and (4.11) can now with (4.9) and (4.10) be written as

$$\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda} H^{a(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho^0] \\
&= \Big\{ \sum_{N_1=2}^{N}\sum_{V_1=1}^{V}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) A^{(V_1)}_{m_1n_1;\dots;m_{N_1-1}n_{N_1-1};mn}[\Lambda] H^{a(V-V_1)}_{m_{N_1}n_{N_1};\dots;m_Nn_N;kl}[\Lambda] \\
&\qquad + \Big(\binom{N}{N_1-1}-1\Big) \text{ permutations } \Big\} - \sum_{m,n,k,l}\frac{1}{2} Q_{nm;lk}(\Lambda) H^{a(V)}_{m_1n_1;\dots;m_Nn_N;mn;kl}[\Lambda] \\
&- \sum_{V_1=0}^{V} H^{1(V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda] \Big\{ -\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) H^{a(V_1)}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};mn;kl}[\Lambda]\Big\}_{\text{[Def. 1.2]}} \\
&- \sum_{V_1=0}^{V} H^{2(V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda] \\
&\qquad\qquad \times \Big\{ -\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda)\Big(H^{a(V_1)}_{\substack{1\,0\\0\,0};\substack{0\,1\\0\,0};mn;kl}[\Lambda] - H^{a(V_1)}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};mn;kl}[\Lambda]\Big)\Big\}_{\text{[Def. 1.2]}} \\
&+ \sum_{V_1=0}^{V} H^{3(V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda] \Big\{ -\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) H^{a(V_1)}_{\substack{1\,1\\0\,0};\substack{0\,0\\0\,0};mn;kl}[\Lambda]\Big\}_{\text{[Def. 1.3]}} \\
&- \sum_{V_1=1}^{V} H^{4(V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda] \Big\{ -\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) H^{a(V_1)}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};mn;kl}[\Lambda]\Big\}_{\text{[Def. 1.1]}}\;,
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda} R^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho^0] \\
&= \Big\{ \sum_{N_1=2}^{N}\sum_{V_1=1}^{V-1}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) A^{(V_1)}_{m_1n_1;\dots;m_{N_1-1}n_{N_1-1};mn}[\Lambda] R^{(V-V_1)}_{m_{N_1}n_{N_1};\dots;m_Nn_N;kl}[\Lambda] \\
&\qquad + \Big(\binom{N}{N_1-1}-1\Big) \text{ permutations } \Big\} - \sum_{m,n,k,l}\frac{1}{2} Q_{nm;lk}(\Lambda) R^{(V)}_{m_1n_1;\dots;m_Nn_N;mn;kl}[\Lambda]
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{V_1=1}^{V} H^{1(V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\Big\{ -\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) R^{(V_1)}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};mn;kl}[\Lambda]\Big\}_{\text{[Def. 1.2]}}\\
&- \sum_{V_1=1}^{V} H^{2(V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\\
&\qquad\times \Big\{ -\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda)\Big(R^{(V_1)}_{{}^{1\,0}_{0\,0};{}^{0\,1}_{0\,0};mn;kl}[\Lambda] - R^{(V_1)}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};mn;kl}[\Lambda]\Big)\Big\}_{\text{[Def. 1.2]}}\\
&+ \sum_{V_1=1}^{V} H^{3(V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\Big\{ -\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) R^{(V_1)}_{{}^{1\,1}_{0\,0};{}^{0\,0}_{0\,0};mn;kl}[\Lambda]\Big\}_{\text{[Def. 1.3]}}\\
&- \sum_{V_1=1}^{V} H^{4(V-V_1)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\Big\{ -\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) R^{(V_1)}_{{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};{}^{0\,0}_{0\,0};mn;kl}[\Lambda]\Big\}_{\text{[Def. 1.1]}} .
\end{aligned}
\tag{4.17}
$$

We have used several times symmetry properties of the expansion coefficients and of the propagator and the fact that to the 1PI projections (4.10) only the last line of (4.9) can contribute. By $\{\dots\}_{\text{[Def. 1.?.]}}$ we understand the restriction to H-graphs and R-graphs, respectively, which satisfy the index criteria on the trajectories as given in Definition 1. The H-graphs will be constructed later in Section 4.2. The R-graphs are in their structure identical to the previously considered graphs for the A-functions, but have a different meaning. See Section 4.4.

4.2. Initial data and graphs for the auxiliary functions. Next, we derive the bounds for the H-functions. Inserting (3.3) into the definition (4.12) we obtain immediately the initial condition at $\Lambda = \Lambda_0$:

$$
H^{1(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\rho^0] = \delta_{N2}\delta^{V0}\delta_{n_1m_2}\delta_{n_2m_1} , \tag{4.18}
$$

$$
H^{2(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\rho^0] = \delta_{N2}\delta^{V0}(m^1_1{+}n^1_1{+}m^2_1{+}n^2_1)\delta_{n_1m_2}\delta_{n_2m_1} , \tag{4.19}
$$

$$
\begin{aligned}
&H^{3(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\rho^0]\\
&= -\delta_{N2}\delta^{V0}\Big(\big(\sqrt{n^1_1m^1_1}\delta_{n^1_1,m^1_2+1}\delta_{n^1_2+1,m^1_1} + \sqrt{n^1_2m^1_2}\delta_{n^1_1,m^1_2-1}\delta_{n^1_2-1,m^1_1}\big)\delta_{n^2_1,m^2_2}\delta_{n^2_2,m^2_1}\\
&+ \big(\sqrt{n^2_1m^2_1}\delta_{n^2_1,m^2_2+1}\delta_{n^2_2+1,m^2_1} + \sqrt{n^2_2m^2_2}\delta_{n^2_1,m^2_2-1}\delta_{n^2_2-1,m^2_1}\big)\delta_{n^1_1,m^1_2}\delta_{n^1_2,m^1_1}\Big) ,
\end{aligned}
\tag{4.20}
$$

$$
H^{4(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\rho^0] = \delta_{N4}\delta^{V0}\Big(\frac{1}{6}\delta_{n_1m_2}\delta_{n_2m_3}\delta_{n_3m_4}\delta_{n_4m_1} + 5\text{ permutations}\Big) . \tag{4.21}
$$

We first compute $H^{a(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda]$ for $a \in \{1,2,3\}$. Since there is no 6-point function at order 0 in V, the differential equation (4.16) reduces to

$$
\begin{aligned}
&\Lambda\frac{\partial H^{a(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda]}{\partial\Lambda}\\
&= \sum_{b=1}^{3}\sum_{m,n,k,l,m',n',k',l'} C_b^{m'n';k'l'} H^{b(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda]\Big(Q_{nm;lk}(\Lambda) H^{a(0)}_{m'n';k'l';mn;kl}[\Lambda]\Big) ,
\end{aligned}
\tag{4.22}
$$

for certain coefficients $C_b^{m'n';k'l'}$. The solution is due to (3.6) given by

$$\begin{aligned}
&H^{a(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda] = H^{a(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda_0] \\
&+\sum_{b=1}^{3}\sum_{m,n,k,l,m',n',k',l'} C_b^{m'n';k'l'} H^{b(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda_0] \\
&\quad\times\Big(\big(\Delta^K_{nm;lk}(\Lambda)-\Delta^K_{nm;lk}(\Lambda_0)\big)H^{a(0)}_{m'n';k'l';mn;kl}[\Lambda_0]\Big) \\
&+\sum_{b=1}^{3}\sum_{m,n,k,l,m',n',k',l'} C_b^{m'n';k'l'} H^{b(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda_0] \\
&\quad\times\Big(\big(\Delta^K_{nm;lk}(\Lambda)-\Delta^K_{nm;lk}(\Lambda_0)\big) \\
&\qquad\quad\times\sum_{b'=1}^{3}\sum_{m'',n'',k'',l'',m''',n''',k''',l'''} C_{b'}^{m'''n''';k'''l'''} H^{b'(0)}_{m'n';k'l';mn;kl}[\Lambda_0]\Big) \\
&\qquad\times\Big(\big(\Delta^K_{n''m'';l''k''}(\Lambda)-\Delta^K_{n''m'';l''k''}(\Lambda_0)\big)H^{a(0)}_{m'''n''';k'''l''';m''n'';k''l''}[\Lambda_0]\Big) \\
&\quad+\dots\,.
\end{aligned} \tag{4.23}$$

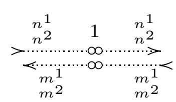
With the initial conditions (4.18)–(4.20) we get

$$H^{a(0)}_{m_1n_1;\dots;m_4n_4}[\Lambda,\Lambda_0,\rho^0] \equiv 0 \qquad \text{for } a\in\{1,2,3\}\,. \tag{4.24}$$

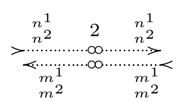
Inserting (4.24) into the rhs of (4.16) we see that $H^{a(0)}_{m_1n_1;m_2n_2}$ for $a\in\{1,2,3\}$ and $H^{4(0)}_{m_1n_1;\dots;m_4n_4}$ are constant, which means that the relations (4.18)–(4.21) hold actually at any value Λ and not only at $\Lambda=\Lambda_0$.

We need a graphical notation for the H-functions. We represent the base functions (4.18)–(4.21), valid for any Λ, as follows:

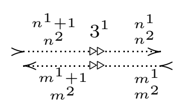
$$H^{1(0)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda] = \tag{4.25}$$



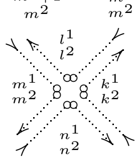
$$H^{2(0)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda] = \tag{4.26}$$



$$H^{3(0)}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda] = \tag{4.27}$$



$$H^{4(0)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{k^1\\k^2};\substack{k^1\\k^2}\substack{l^1\\l^2};\substack{l^1\\l^2}\substack{m^1\\m^2}}[\Lambda] = \frac{1}{6} \quad + \text{ 5 permutations}\,. \tag{4.28}$$

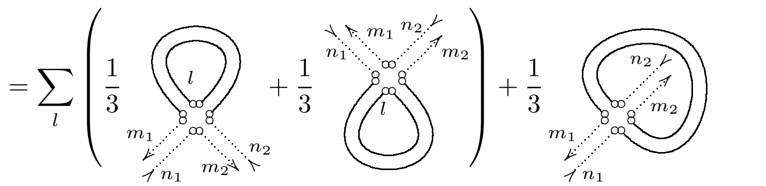


The special vertices stand for some sort of hole into which we can insert planar two- or four-point functions at vanishing external indices. However, the graph remains connected at these holes, in particular, there is index conservation at the hole ∞ and a jump by $\substack{1\\0}$ or $\substack{0\\1}$ at the hole $\triangleright\!\triangleright$.

By repeated contraction with A-graphs and self-contractions we build out of (4.25)–(4.28) more complicated graphs with holes. We use this method to compute

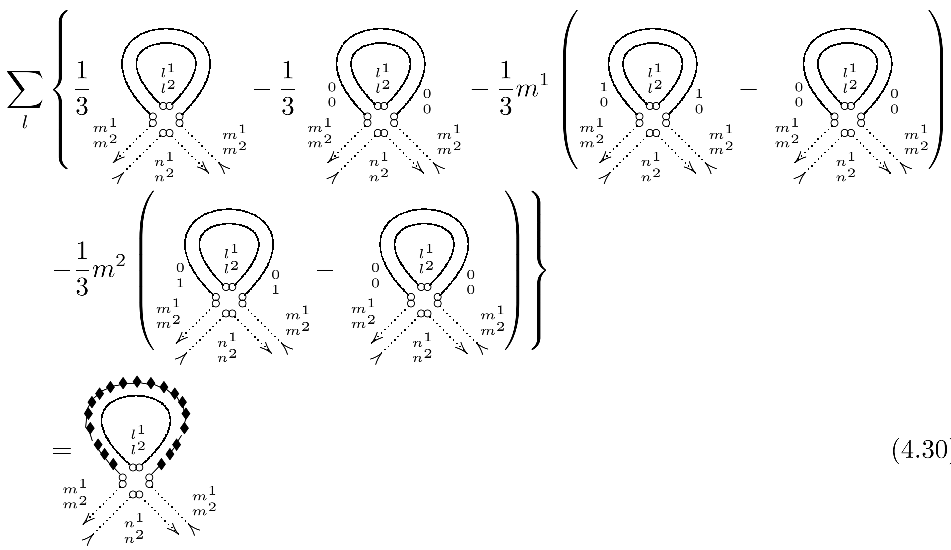
$H^{4(0)}_{m_1n_1;m_2n_2}[\Lambda]$. In this case, we need the planar and non-planar self-contractions of (4.28):

$$\sum_{m,n,k,l} Q_{mn;kl} H^{4(0)}_{m_1n_1;m_2n_2;mn;kl}$$



(4.29)

These contractions correspond (with a factor $-\frac{1}{2}$) to last term in the third line of (4.16), for $a = 4$ and $N = 2$. We also have to subtract (again up to the factor $-\frac{1}{2}$) the fourth to last lines of (4.16). For instance, the fourth line amounts to insert the *planar graphs* of (4.29) with $m_1 = n_1 = m_2 = n_2 = {}^0_0$ into (4.25). The total contribution to the rhs of (4.16) corresponding to the first graph in (4.29), with $m_1 = n_2 = {}^{m^1}_{m^2}$ and $n_1 = m_2 = {}^{n^1}_{n^2}$, reads



(4.30)

The second graph in the first line of (4.30) corresponds to the fourth line of (4.16). The second line of (4.30) represents the fifth/sixth lines of (4.16), undoing the symmetry properties of the upper and lower component used in (4.16). The difference of graphs corresponding to the ${}^{n^1}_{n^2}$ component vanishes, because the value of the graph is independent of ${}^{n^1}_{n^2}$. There is no planar contribution from the last two lines of (4.16). In total, we get the projection (3.29) to the irrelevant part of the graph. The same procedure leads to the irrelevant part of the second graph in (4.29).

With these considerations, the differential equation (4.16) takes for $N = 2$ and $a = 4$ the form

$$\Lambda \frac{\partial}{\partial \Lambda} H^{4(0)}_{m_1n_1;m_2n_2}[\Lambda] = -\frac{1}{6} \delta_{n_1m_2} \delta_{n_2m_1} \Big(\sum_{l \in \mathbb{N}^2} \mathcal{Q}^{(1)}_{n_1l;ln_1}(\Lambda) + \sum_{l \in \mathbb{N}^2} \mathcal{Q}^{(1)}_{n_2l;ln_2}(\Lambda) \Big) - \frac{1}{6} Q_{m_1n_1;m_2n_2}(\Lambda) \,. \tag{4.31}$$

The first line comes from the planar graphs in (4.29) and the subtraction terms according to (4.30), whereas the second line of (4.31) is obtained from the last (non-planar) graph in (4.29). Using the initial condition (4.21) at $\Lambda=\Lambda_0$, the bounds (3.21) and (3.29) combined with the volume factor $(C_2\theta\Lambda^2)^2$ for the l-summation we get

$$|H^{4(0)}_{m_1n_1;m_2n_2}[\Lambda]| \leq \frac{C\big(\|m_1\|^2_\infty+\|n_1\|^2_\infty\big)}{\Omega\theta\Lambda^2}\,\delta_{n_1m_2}\delta_{n_2m_1} + \frac{C_0}{\Omega\theta\Lambda^2}\,\delta_{n_1m_2}\delta_{n_2m_1}\;. \tag{4.32}$$

It is extremely important here that the irrelevant projection $\mathcal{Q}^{(1)}_{nl;ln}$ and not the propagator $Q_{nl;ln}$ itself appears in the first line of (4.31).

4.3. The power-counting behaviour of the auxiliary functions. The example suggests that similar cancellations of relevant and marginal parts appear in general, too. Thus, we expect all H-functions to be irrelevant. This is indeed the case:

Proposition 3 *Let γ be a ribbon graph with holes having N external legs, V vertices, V^e external vertices and segmentation index ι, which is drawn on a genus-g Riemann surface with B boundary components. Then, the contribution $H^{a(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}$ of γ to the expansion coefficient of the auxiliary function of a duality-covariant ϕ^4-theory on $\mathbb{R}^4_\theta$ in the matrix base is bounded as follows:*

1. *For γ according to Definition 1.1 we have*

$$\begin{aligned}&\Big|\sum_{\gamma\text{ as in Def. 1.1}} H^{a(V,V^e,1,0,0)\gamma}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{k^1\\k^2};\substack{k^1\\k^2}\substack{l^1\\l^2};\substack{l^1\\l^2}\substack{m^1\\m^2}}[\Lambda,\Lambda_0,\rho_0]\Big|\\ &\leq \big(\theta\Lambda^2\big)^{-\delta^{1a}} P^{4V-2+2\delta^{a4}}_{1-\delta^{V0}}\left[\frac{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{k^1\\k^2};\substack{k^1\\k^2}\substack{l^1\\l^2};\substack{l^1\\l^2}\substack{m^1\\m^2}}{\theta\Lambda^2}\right]\Big(\frac{1}{\Omega}\Big)^{3V-1+\delta^{a4}-V^e}\\ &\qquad\times P^{2V-1+\delta^{a4}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,,\end{aligned}\tag{4.33}$$

   where all vertices on the trajectories contribute to V^e.

2. *For γ according to Definition 1.2 we have*

$$\begin{aligned}&\Big|\sum_{\gamma\text{ as in Def. 1.2}} H^{a(V,V^e,1,0,0)\gamma}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda,\Lambda_0,\rho_0]\Big|\\ &\leq \big(\theta\Lambda^2\big)^{1-\delta^{1a}} P^{4V+2\delta^{a4}}_{2-\delta_{V0}(2\delta^{a1}+\delta^{a2})}\left[\frac{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}{\theta\Lambda^2}\right]\Big(\frac{1}{\Omega}\Big)^{3V+\delta^{a4}-V^e} P^{2V+\delta^{a4}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,,\end{aligned}\tag{4.34}$$

   where all vertices on the trajectories contribute to V^e.

3. *For γ according to Definition 1.3 we have*

$$\begin{aligned}&\Big|\sum_{\gamma\text{ as in Def. 1.3}} H^{a(V,V^e,1,0,0)}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda,\Lambda_0,\rho_0]\Big|\\ &\leq \big(\theta\Lambda^2\big)^{1-\delta^{1a}} P^{4V+2\delta^{a4}}_{2-\delta_{V0}}\left[\frac{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}{\theta\Lambda^2}\right]\Big(\frac{1}{\Omega}\Big)^{3V+\delta^{a4}-V^e} P^{2V+\delta^{a4}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,,\end{aligned}\tag{4.35}$$

   where all vertices on the trajectories contribute to V^e.

4. *If γ is a subgraph of an 1PI planar graph with a selected set T of trajectories on one distinguished boundary component and a second set T' of summed trajectories on that boundary component, we have*

$$\sum_{\mathcal{E}^s}\sum_{\mathcal{E}^{t'}}\big|H^{a(V,V^e,B,0,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho_0]\big| \leq \big(\theta\Lambda^2\big)^{(2-\delta^{1a}-\frac{N}{2})+2(1-B)}P^{4V+2+2\delta^{a4}-N}_{\big(2t'+\sum_{\overrightarrow{n_j\circ[n_j]}\in T}\min(2,\frac{1}{2}\langle\overrightarrow{n_j\circ[n_j]}\rangle)\big)}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] \times\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}+\delta^{a4}+B-V^e-\iota+s+t'}P^{2V+1+\delta^{a4}-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]. \tag{4.36}$$

The number of summations is now restricted by $s+t'\leq V^e+\iota$.

5. *If γ is a non-planar graph or a graph with $N>4$ external legs, we have*

$$\sum_{\mathcal{E}^s}\big|H^{a(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho_0]\big| \leq \big(\theta\Lambda^2\big)^{(2-\delta^{a1}-\frac{N}{2})+2(1-B-2g)}P_0^{4V+2+2\delta^{a4}-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] \times\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}+\delta^{a4}+B+2g-V^e-\iota+s}P^{2V+1+\delta^{a4}-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]. \tag{4.37}$$

The number of summations is now restricted by $s\leq V^e+\iota$.

Proof. The Proposition will be proven by induction upward in the number V of vertices and for given V downward in the number N of external legs.

5. Taking (3.33) into account, the estimations (4.33)–(4.36) are further bound by (4.37). In particular, the inequality (4.37) correctly reproduces the bounds for $V=0$ derived in Section 4.2. By comparison with (3.39), the estimation (4.37) follows immediately for the H-*linear* parts on the rhs of (4.16) which contribute to the integrand of $H^{a(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$. Since planar two- and four-point functions are preliminarily excluded, the Λ-integration (from Λ_0 down to Λ) confirms (4.37) for those contributions which arise from H-linear terms on the rhs of (4.16) that are non-planar or have $N>4$ external legs.
We now consider in the H-bilinear part on the rhs of (4.16) the contributions of non-planar graphs or graphs with $N>4$ external legs. We start with the fourth line in (4.16), with the first term being a non-planar H-function which (apart from the number of vertices and the hole label a) has the same topological data as the total H-graph to estimate. From the induction hypothesis it is clear that the term in braces $\{\ \}$ is bounded by the planar unsummed version ($B_1=1,g_1=0,\iota_1=0,s_1=0$) of (4.36), with $N_1=2$ and $T=T'=\emptyset$, and with a reduction of the degree of the polynomial in $\ln\frac{\Lambda}{\Lambda_R}$ by 1:

$$\Big|\Big\{-\frac{1}{2}\sum_{m,n,k,l}Q_{nm;lk}(\Lambda)H^{a(V_1)}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};mn;kl}[\Lambda]\Big\}_{\text{Def. 1.2}}\Big| \leq\big(\theta\Lambda^2\big)^{(1-\delta^{a1})}\Big(\frac{1}{\Omega}\Big)^{3V_1+\delta^{a4}-V_1^e}P^{2V_1-1+\delta^{a4}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big], \tag{4.38a}$$

$$\sum_{\mathcal{E}^s}\big|H^{1(V-V_1,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \leq\big(\theta\Lambda^2\big)^{(1-\frac{N}{2})+2(1-B-2g)}P_0^{4(V-V_1)+2-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] \times\Big(\frac{1}{\Omega}\Big)^{3(V-V_1)-\frac{N}{2}+B+2g-V^e-\iota+s}P^{2(V-V_1)+1-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]. \tag{4.38b}$$

We can ignore the term $P^a_b[\,]$, see (3.32), in (4.38a) because the external indices of that part are zero. In the first step we exclude $a=4$ so that the sum over V_1 in (4.16) starts due to (4.18)–(4.20) at $V_1=1$. For $V_1=V$ there is a contribution to (4.38b) with $N=2$ only, where (4.38a) can be regarded as known by induction. Since the factor $(\frac{1}{\Omega})^{-V_1^e}$ can safely be absorbed in the polynomial $P[\ln\frac{\Lambda}{\Lambda_R}]$, the product of (4.38a) and (4.38b) confirms the bound (4.37) for the integrand under consideration, preliminarily for $a\neq 4$. In the next step we repeat the argumentation for $a=4$, where (4.38b), with $V_1=0$, is known from the first step.

Second, we consider the fifth/sixth lines in (4.16). The difference of functions in braces $\{\ \}$ involves graphs with constant index along the trajectories. We have seen in Section 3.3 that such a difference can be written as a sum of graphs each having a composite propagator (3.28) at a trajectory. As such the $(\theta\Lambda^2)$-degree of the part in braces $\{\ \}$ is reduced[9] by 1 compared with planar analogues of (4.37) for $N=2$. The difference of functions in braces $\{\ \}$ involves also graphs where the index along one of the trajectories jumps once by $\genfrac{}{}{0pt}{}{1}{0}$ or $\genfrac{}{}{0pt}{}{0}{1}$ and back. For these graphs we conclude from (3.24) (and the fact that the maximal index along the trajectory is 2) that the $(\theta\Lambda^2)$-degree of the part in braces $\{\ \}$ is also reduced by 1:

$$\begin{aligned}
&\Big|\Big\{-\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda)\Big(H^{a(V_1)}_{\genfrac{}{}{0pt}{}{1\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,1}{0\,0};mn;kl}[\Lambda]-H^{a(V_1)}_{\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};mn;kl}[\Lambda]\Big)\Big\}_{[\text{Def. 1.2}]}\Big|\\
&\qquad\leq \big(\theta\Lambda^2\big)^{(-\delta^{a1})}\Big(\frac{1}{\Omega}\Big)^{3V_1+\delta^{a4}(1-V_1^e)}\,P^{2V_1-1+\delta^{a4}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,,
\end{aligned}\tag{4.39a}$$

$$\begin{aligned}
&\sum_{\mathcal{E}^s}\big|H^{2(V-V_1,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big|\\
&\quad\leq \big(\theta\Lambda^2\big)^{(2-\frac{N}{2})+2(1-B-2g)}\,P_0^{4(V-V_1)+2-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big]\\
&\qquad\times\Big(\frac{1}{\Omega}\Big)^{3(V-V_1)-\frac{N}{2}+1+B+2g-V^e-\iota+s}\,P^{2(V-V_1)+1-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,.
\end{aligned}\tag{4.39b}$$

Again we have to exclude $a=4$ in the first step, which then confirms the bound (4.37) for the integrand under consideration. In the second step we repeat the argumentation for $a=4$.

Third, the discussion of the seventh line of (4.16) is completely similar, because there the index on each trajectory jumps once by $\genfrac{}{}{0pt}{}{1}{0}$ or $\genfrac{}{}{0pt}{}{0}{1}$. This leads again to a reduction by 1 of the $(\theta\Lambda^2)$-degree of the part in braces $\{\ \}$ compared with planar analogues of (4.37) for $N=2$.

Finally, the part in braces in the last line of (4.16) can be estimated by a planar $N=4$ version of (4.37), again with a reduction by 1 of the degree of $P[\ln\frac{\Lambda}{\Lambda_R}]$:

$$\begin{aligned}
&\Big|\Big\{-\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda)H^{a(V_1)}_{\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};\genfrac{}{}{0pt}{}{0\,0}{0\,0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 1.1}]}\Big|\\
&\quad\leq \big(\theta\Lambda^2\big)^{-\delta^{a1}}\Big(\frac{1}{\Omega}\Big)^{3V_1-1+\delta^{a4}-V_1^e}\,P^{2V_1-2+\delta^{a4}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,,
\end{aligned}\tag{4.40a}$$

[9] The origin of the reduction is the term $P^a_b[\,]$ introduced in (3.32), with $b=1$ in presence of a composite propagator (3.28). The argument in the brackets of $P^a_b[\,]$ is the ratio of the maximal external index to the reference scale $\theta\Lambda^2$. Since the maximal index along the trajectory is 1, we can globally estimate in this case $P^a_1[\,]$ by a constant times $(\theta\Lambda^2)^{-1}$.

$$
\begin{aligned}
&\sum_{\mathcal{E}^s} \big|H^{4(V-V_1,B,g,V^e,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \\
&\leq \big(\theta\Lambda^2\big)^{(2-\frac{N}{2})+2(1-B-2g)}\, P_0^{4(V-V_1)+4-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\Big] \\
&\qquad \times\Big(\frac{1}{\Omega}\Big)^{3(V-V_1)-\frac{N}{2}+1+B+2g-V^e-\iota+s} P^{2(V-V_1)+2-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,. \qquad (4.40b)
\end{aligned}
$$

We confirm again the bound (4.37) for the integrand under consideration.
Since we have assumed that the total H-graph is non-planar or has $N > 4$ external legs, the integrand (4.37) is irrelevant so that we obtain after integration from Λ_0 down to Λ (and use of the initial conditions (4.18)–(4.21)) the same bound (4.37) for the graph, too.

4. According to Section 4.2, the inequality (4.36) is correct for $V = 0$. By comparison with (3.38), the estimation (4.36) follows immediately for the H-*linear* parts on the rhs of (4.16) which contribute to the integrand of $H^{a(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$. Excluding planar two- and four-point functions with constant index on the trajectory or with limited jump according to 1–3 of Definition 1, the Λ-integration confirms (4.36) for those contributions which arise from H-linear terms on the rhs of (4.16) that correspond to subgraphs of planar graphs (subject to the above restrictions). The proof of (4.36) for the H-bilinear terms in (4.16) is completely analogous to the non-planar case. We only have to replace (4.38b), (4.39b) and (4.40b) by the adapted version of (4.36). In particular, the distinguished trajectory with its subsets T, T' of indices comes exclusively from the (4.36)-analogues of (4.38b), (4.39b) and (4.40b) and not from the terms in braces in (4.16).

1. We first consider $a \neq 4$. Then, according to (4.18)–(4.20) we need $V \geq 1$ in order to have a non-vanishing contribution to (4.33). Since according to Definition 1.1 the index along each trajectory of the (planar) graph γ is constant, we have

$$
H^{a(V,V^e,1,0,0)\gamma}_{m_1n_1;m_2n_2;m_3n_3;m_4n_4}[\Lambda] = \frac{1}{6} H^{a(V,V^e,1,0,0)}_{m_1m_2;m_2m_3;m_3m_4;m_4n_1}[\Lambda] + 5 \text{ permutations}\,. \qquad (4.41)
$$

Then, using (4.18)–(4.21) and the fact that γ is 1PI, the differential equation (4.16) reduces to

$$
\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda}\Big(\sum_{\gamma \text{ as in Def. 1.1}} H^{a(V,V^e,1,0,0)\gamma}_{m_1n_1;m_2n_2;m_3n_3;m_4n_4}[\Lambda,\Lambda_0,\rho^0]\Big)_{a\neq 4} \\
&= \Big(-\frac{1}{12}\Big\{\sum_{m,n,k,l} Q_{nm;lk}(\Lambda)\Big(H^{a(V,V^e,B,0,\iota)}_{m_1m_2;m_2m_3;m_3m_4;m_4n_1;mn;kl}[\Lambda] \\
&\qquad\qquad - H^{a(V,V^e,B,0,\iota)}_{00;00;00;00;mn;kl}[\Lambda]\Big)\Big\}_{[\text{Def. 1.1}]} + 5 \text{ permutations}\Big) \\
&\quad + \text{the } 4^{\text{th}} \text{ to last lines of (4.16) with} \sum_{V_1=0}^{V} \mapsto \sum_{V_1=1}^{V-1}\,. \qquad (4.42)
\end{aligned}
$$

Here, the term $H^{a(V,V^e,B,0,\iota)}_{00;00;00;00;mn;kl}[\Lambda]$ in the second line of (4.42) comes from the ($V_1 = V$)-contribution of the last line in (4.16), together with (4.21). In the same way as in Section 3.3 we conclude that the second line of (4.42) can be written as a linear combination of graphs having a composite propagator (3.17a) on one of the trajectories. As such we have to replace the bound (3.22) relative to the contribution of an ordinary propagator by (3.28). For the total graph

this amounts to multiply the corresponding estimation (4.36) of ordinary H-graphs with $N=4$ by a factor $\frac{\max\|m_i\|}{\theta\Lambda^2}$, which yields the subscript 1 of the part $P_1^{4V-2+2\delta^{a4}}[\;]$ of the integrand (4.33), for the time being restricted to the second line of (4.42). Since the resulting integrand is irrelevant, we also obtain (4.33) after Λ-integration from Λ_0 down to Λ. Clearly, this is the only contribution for $V=1$ so that (4.33) is proven for $V=1$ and $a\neq 4$.
In the second step we use this result to extend the proof to $V=1$ and $a=4$. Now the differential equation (4.16) reduces to the second line of (4.42), with $a=4$, and the fourth to sixth lines of (4.16) with $V=1$ and $V_1=0$. There is no contribution from the seventh line of (4.16) for $V_1=0$, because the part in braces would be non-planar, which is excluded in Definition 1.3. Inserting (4.21) we obtain the composite propagator (3.17b) in the part in braces $\{\;\}$ of the fifth line of (4.16). Together with (4.33) for $V=1$ and $a\neq 4$ already proven we verify the integrand (4.33) for $V=1$ and $a=4$. After Λ-integration we thus obtain (4.33) for $V=1$ and any a.
This allows us to use (4.33) as induction hypothesis for the remaining contributions in the last line of (4.42). This is similar to the procedure in 5, we only have to replace (4.38b), (4.39b) and (4.40b) by the according parametrisation of (4.33). We thus prove (4.33) to all orders.

2. We first consider $a\neq 4$. Then, according to (4.18)–(4.20) only terms with $V_1\geq 1$ contribute to (4.16). Using (4.18)–(4.21) and the fact that γ is 1PI, the differential equation (4.16) reduces to

$$
\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda}\Big(\sum_{\gamma\text{ as in Def. 1.2}} H^{a(V,V^e,1,0,0)\gamma}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda,\Lambda_0,\rho^0]\Big)_{a\neq 4}\\
&= -\frac{1}{2}\Big\{\sum_{m,n,k,l} Q_{nm;lk}(\Lambda)\Big(H^{a(V,V^e,B,0,\iota)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2};mn;kl}[\Lambda] - H^{a(V,V^e,B,0,\iota)}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};mn;kl}[\Lambda]\\
&\qquad - m^1\big(H^{a(V,V^e,B,0,\iota)}_{\substack{1\\0}\substack{0\\0};\substack{0\\0}\substack{1\\0};mn;kl}[\Lambda] - H^{a(V,V^e,B,0,\iota)}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};mn;kl}[\Lambda]\big)\\
&\qquad - n^1\big(H^{a(V,V^e,B,0,\iota)}_{\substack{0\\0}\substack{1\\0};\substack{1\\0}\substack{0\\0};mn;kl}[\Lambda] - H^{a(V,V^e,B,0,\iota)}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};mn;kl}[\Lambda]\big)\\
&\qquad - m^2\big(H^{a(V,V^e,B,0,\iota)}_{\substack{0\\1}\substack{0\\0};\substack{0\\0}\substack{0\\1};mn;kl}[\Lambda] - H^{a(V,V^e,B,0,\iota)}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};mn;kl}[\Lambda]\big)\\
&\qquad - n^2\big(H^{a(V,V^e,B,0,\iota)}_{\substack{0\\0}\substack{0\\1};\substack{0\\1}\substack{0\\0};mn;kl}[\Lambda] - H^{a(V,V^e,B,0,\iota)}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};mn;kl}[\Lambda]\big)\Big)\Big\}_{[\text{Def. 1.2}]}
\end{aligned}
\tag{4.43a}
$$

$$
- H^{4(0)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda]\Big\{-\frac{1}{2}\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) H^{a(V)}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 1.1}]}
\tag{4.43b}
$$

$$
+\text{ the }4^{\text{th}}\text{ to last lines of (4.16) with }\sum_{V_1=0}^{V}\mapsto\sum_{V_1=1}^{V-1}\;.
\tag{4.43c}
$$

If the graphs have constant indices along the trajectories, we conclude in the same way as in Appendix B.1 that the part (4.43a) can be written as a linear combination of graphs having either a composite propagator (3.17b) or two composite propagators (3.17a) on the trajectories. As such we have to replace the bound (3.22) relative to the contribution of an ordinary propagator by (3.29) or twice (3.22) by (3.28). For the total graph this amounts to multiply the corresponding estimation (4.36) of ordinary H-graphs with $N=2$ by a factor $\big(\frac{\max(m^r,n^r)}{\theta\Lambda^2}\big)^2$, which yields the subscript 2 of the part $P_2^{4V+2\delta^{a4}}[\;]$ of the integrand (4.34), for

the time being restricted to the part (4.43a). For graphs with index jump in Definition 1.2 we obtain according to Appendix B.1 the same improvement by $\big(\frac{\max(m^r,n^r)}{\theta\Lambda^2}\big)^2$. Next, the product of (4.32) with (4.40a) gives for (4.43b) the same bound (4.34) for the integrand. Since the resulting integrand is irrelevant, we also obtain (4.34) after Λ-integration. Clearly, this is the only contribution for $V=1$ so that (4.34) is proven for $V=1$ and $a\neq 4$.
In the second step we use this result to extend the proof to $V=1$ and $a=4$. Now the differential equation (4.16) reduces to the sum of (4.43a) and (4.43b), with $a=4$, and the fourth to sixth lines of (4.16) with $V=1$ and $V_1=0$. There is again no contribution of the seventh line of (4.16) for $V_1=0$. Inserting (4.21) we obtain the composite propagators (3.17b) in the fifth/sixth lines of (4.16), which together with (4.34) for $V=1$ and $a\neq 4$ already proven verifies the integrand (4.34) for $V=1$ and $a=4$. After Λ-integration we thus obtain (4.34) for $V=1$ and any a.
This allows us to use (4.34) as induction hypothesis for the remaining contributions (4.43c) for $V>1$. This is similar to the procedure in 5, we only have to replace (4.38b), (4.39b) and (4.40b) by the according parametrisation of (4.34). We thus prove (4.34) to all orders.

3. The proof of (4.35) is performed along the same lines as the proof of (4.33) and (4.34). There is one factor $\frac{\max(m^r,n^r)}{\theta\Lambda^2}$ from $\langle\overrightarrow{n_1\mathfrak{o}[n_1]}\rangle+\langle\overrightarrow{n_2\mathfrak{o}[n_2]}\rangle=2$ in (4.36) and a second factor from the composite propagator (3.28) or (3.30) appearing according to Appendix B.1 in the $(V_1=V)$-contribution to (4.16).

This finishes the proof of Proposition 3. $\square$

4.4. The power-counting behaviour of the Λ_0-varied functions. The estimations in Propositions 2 and 3 allow us to estimate the R-functions by integrating the differential equation (4.17). Again, the R-functions are expanded in terms of ribbon graphs. Let us look at R-ribbon graphs of the type described in Definition 1.1. Since $\sum_{\gamma\text{ as in Def. 1.1}} A^{(V)\gamma}_{{0\atop 0}{0\atop 0};{0\atop 0}{0\atop 0};{0\atop 0}{0\atop 0};{0\atop 0}{0\atop 0}}[\Lambda,\Lambda_0,\rho^0]\equiv\rho_4[\Lambda,\Lambda_0,\rho^0]$, we can rewrite the expansion coefficients of (4.5) as follows:

$$\begin{aligned}
&\sum_{\gamma\text{ as in Def. 1.1}} R^{(V)\gamma}_{{m^1\atop m^2}{n^1\atop n^2};{n^1\atop n^2}{k^1\atop k^2};{k^1\atop k^2}{l^1\atop l^2};{l^1\atop l^2}{m^1\atop m^2}}[\Lambda,\Lambda_0,\rho^0]\\
&=\Lambda_0\frac{\partial}{\partial\Lambda_0}\sum_{\gamma\text{ as in Def. 1.1}}\Big(A^{(V)\gamma}_{{m^1\atop m^2}{n^1\atop n^2};{n^1\atop n^2}{k^1\atop k^2};{k^1\atop k^2}{l^1\atop l^2};{l^1\atop l^2}{m^1\atop m^2}}[\Lambda,\Lambda_0,\rho^0]\\
&\qquad\qquad\qquad\qquad - A^{(V)\gamma}_{{0\atop 0}{0\atop 0};{0\atop 0}{0\atop 0};{0\atop 0}{0\atop 0};{0\atop 0}{0\atop 0}}[\Lambda,\Lambda_0,\rho^0]\Big)\\
&-\sum_{a,b=1}^4\frac{\partial}{\partial\rho^0_a}\sum_{\gamma\text{ as in Def. 1.1}}\Big(A^{(V)\gamma}_{{m^1\atop m^2}{n^1\atop n^2};{n^1\atop n^2}{k^1\atop k^2};{k^1\atop k^2}{l^1\atop l^2};{l^1\atop l^2}{m^1\atop m^2}}[\Lambda,\Lambda_0,\rho^0]\\
&\qquad\qquad\qquad\qquad - A^{(V)\gamma}_{{0\atop 0}{0\atop 0};{0\atop 0}{0\atop 0};{0\atop 0}{0\atop 0};{0\atop 0}{0\atop 0}}[\Lambda,\Lambda_0,\rho^0]\Big)\\
&\qquad\times\frac{\partial\rho^0_a}{\partial\rho_b[\Lambda,\Lambda_0,\rho^0]}\Lambda_0\frac{\partial}{\partial\Lambda_0}\rho_b[\Lambda,\Lambda_0,\rho^0]\,.
\end{aligned}\tag{4.44}$$

This means that (by construction) only the (Λ_0,ρ^0)-derivatives of the projection to the irrelevant part (3.35a) of the planar four-point function contributes to R. Similarly, only the (Λ_0,ρ^0)-derivatives of the irrelevant parts (3.36a) and (3.37a) of the planar two-point function contribute to R. According to the initial condition

(3.3), these projections and the other functions given in Definition 1.4 vanish at $\Lambda = \Lambda_0$ *independently* of Λ_0 or ρ_a^0:

$$\begin{aligned}
0 &= \Lambda_0 \frac{\partial}{\partial \Lambda_0}\Big(A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{k^1\\k^2};\substack{k^1\\k^2}\substack{l^1\\l^2};\substack{l^1\\l^2}\substack{m^1\\m^2}}[\Lambda_0,\Lambda_0,\rho^0] \\
&\qquad - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0]\Big)\Big|_{\gamma \text{ as in Def. 1.1}} \\
&= \frac{\partial}{\partial \rho_a^0}\Big(A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{k^1\\k^2};\substack{k^1\\k^2}\substack{l^1\\l^2};\substack{l^1\\l^2}\substack{m^1\\m^2}}[\Lambda_0,\Lambda_0,\rho^0] \\
&\qquad - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0]\Big)\Big|_{\gamma \text{ as in Def. 1.1}}\,,
\end{aligned} \tag{4.45a}$$

$$\begin{aligned}
0 &= \Lambda_0 \frac{\partial}{\partial \Lambda_0}\Big(A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0] \\
&\quad - m^1\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{1\\0}\substack{0\\0};\substack{0\\0}\substack{1\\0}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0]\big) \\
&\quad - n^1\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{1\\0};\substack{1\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0]\big) \\
&\quad - m^2\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\1}\substack{0\\0};\substack{0\\0}\substack{0\\1}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0]\big) \\
&\quad - n^2\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\1};\substack{0\\1}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0]\big)\Big)\Big|_{\gamma \text{ as in Def. 1.2}} \\
&= \frac{\partial}{\partial \rho_a^0}\Big(A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0] \\
&\quad - m^1\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{1\\0}\substack{0\\0};\substack{0\\0}\substack{1\\0}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0]\big) \\
&\quad - n^1\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{1\\0};\substack{1\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0]\big) \\
&\quad - m^2\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\1}\substack{0\\0};\substack{0\\0}\substack{0\\1}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0]\big) \\
&\quad - n^2\big(A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\1};\substack{0\\1}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\substack{0\\0}\substack{0\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0]\big)\Big)\Big|_{\gamma \text{ as in Def. 1.2}}\,,
\end{aligned} \tag{4.45b}$$

$$\begin{aligned}
0 &= \Lambda_0 \frac{\partial}{\partial \Lambda_0}\Big(A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda_0,\Lambda_0,\rho^0] \\
&\quad - \sqrt{(m^1{+}1)(n^1{+}1)}\,A^{(V,V^e,1,0,0)\gamma}_{\substack{1\\0}\substack{1\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0]\Big)\Big|_{\gamma \text{ as in Def. 1.3}} \\
&= \frac{\partial}{\partial \rho_a^0}\Big(A^{(V,V^e,1,0,0)\gamma}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}[\Lambda_0,\Lambda_0,\rho^0] \\
&\quad - \sqrt{(m^1{+}1)(n^1{+}1)}\,A^{(V,V^e,1,0,0)\gamma}_{\substack{1\\0}\substack{1\\0};\substack{0\\0}\substack{0\\0}}[\Lambda_0,\Lambda_0,\rho^0]\Big)\Big|_{\gamma \text{ as in Def. 1.3}}\,,
\end{aligned} \tag{4.45c}$$

$$\begin{aligned}
0 &= \Lambda_0 \frac{\partial}{\partial \Lambda_0} A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\rho^0]\Big|_{\gamma \text{ as in Def. 1.4}} \\
&= \frac{\partial}{\partial \rho_a^0} A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\rho^0]\Big|_{\gamma \text{ as in Def. 1.4}}\,.
\end{aligned} \tag{4.45d}$$

The Λ_0-derivative at $\Lambda = \Lambda_0$ has to be considered with care:

$$\begin{aligned}
0 &= \Lambda_0 \frac{\partial}{\partial \Lambda_0} A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\rho^0] \\
&= \Big(\Lambda \frac{\partial}{\partial \Lambda} A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho^0]\Big)_{\Lambda=\Lambda_0} + \Big(\Lambda_0 \frac{\partial}{\partial \Lambda_0} A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho^0]\Big)_{\Lambda=\Lambda_0},
\end{aligned} \tag{4.46}$$

and similarly for (4.45a)–(4.45c). Inserting (4.44), (4.45), (4.46) and according formulae into the Taylor expansion of (4.5) we thus have

$$
\begin{aligned}
&\sum_{\gamma\text{ as in Def. 1.1}} R^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{m^1}{m^2}\genfrac{}{}{0pt}{}{n^1}{n^2};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{k^1}{k^2};\genfrac{}{}{0pt}{}{k^1}{k^2}\genfrac{}{}{0pt}{}{l^1}{l^2};\genfrac{}{}{0pt}{}{l^1}{l^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}[\Lambda_0,\Lambda_0,\rho^0] \\
&= -\sum_{\gamma\text{ as in Def. 1.1}} \Big(\Lambda\frac{\partial}{\partial\Lambda}\Big(A^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{m^1}{m^2}\genfrac{}{}{0pt}{}{n^1}{n^2};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{k^1}{k^2};\genfrac{}{}{0pt}{}{k^1}{k^2}\genfrac{}{}{0pt}{}{l^1}{l^2};\genfrac{}{}{0pt}{}{l^1}{l^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}[\Lambda,\Lambda_0,\rho^0] \\
&\qquad\qquad - A^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda,\Lambda_0,\rho^0]\Big)\Big)_{\Lambda=\Lambda_0}\;,
\end{aligned}
\tag{4.47a}
$$

$$
\begin{aligned}
&\sum_{\gamma\text{ as in Def. 1.2}} R^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{m^1}{m^2}\genfrac{}{}{0pt}{}{n^1}{n^2};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}[\Lambda_0,\Lambda_0,\rho^0] \\
&= -\sum_{\gamma\text{ as in Def. 1.2}} \Big(\Lambda\frac{\partial}{\partial\Lambda}\Big(A^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{m^1}{m^2}\genfrac{}{}{0pt}{}{n^1}{n^2};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}[\Lambda,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda,\Lambda_0,\rho^0] \\
&\qquad - m^1\big(A^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{1}{0}\genfrac{}{}{0pt}{}{0}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{1}{0}}[\Lambda,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda,\Lambda_0,\rho^0]\big) \\
&\qquad - n^1\big(A^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{1}{0};\genfrac{}{}{0pt}{}{1}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda,\Lambda_0,\rho^0]\big) \\
&\qquad - m^2\big(A^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{0}{1}\genfrac{}{}{0pt}{}{0}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{1}}[\Lambda,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda,\Lambda_0,\rho^0]\big) \\
&\qquad - n^2\big(A^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{1};\genfrac{}{}{0pt}{}{0}{1}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda,\Lambda_0,\rho^0] - A^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda,\Lambda_0,\rho^0]\big)\Big)\Big)_{\Lambda=\Lambda_0}\;,
\end{aligned}
\tag{4.47b}
$$

$$
\begin{aligned}
&\sum_{\gamma\text{ as in Def. 1.3}} R^{(V,2,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{m^1+1}{m^2}\genfrac{}{}{0pt}{}{n^1+1}{n^2};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}[\Lambda_0,\Lambda_0,\rho^0] \\
&= -\sum_{\gamma\text{ as in Def. 1.3}} \Big(\Lambda\frac{\partial}{\partial\Lambda}\Big(A^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{m^1+1}{m^2}\genfrac{}{}{0pt}{}{n^1+1}{n^2};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}[\Lambda,\Lambda_0,\rho^0] \\
&\qquad - \sqrt{(m^1{+}1)(n^1{+}1)}\,A^{(V,V^e,1,0,0)\gamma}_{\genfrac{}{}{0pt}{}{1}{0}\genfrac{}{}{0pt}{}{1}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}[\Lambda,\Lambda_0,\rho^0]\Big)\Big)_{\Lambda=\Lambda_0}\;,
\end{aligned}
\tag{4.47c}
$$

$$
\begin{aligned}
&R^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda_0,\Lambda_0,\rho^0]\Big|_{\gamma\text{ as in Def. 1.4}} \\
&= -\Big(\Lambda\frac{\partial}{\partial\Lambda}A^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho^0]\Big|_{\gamma\text{ as in Def. 1.4}}\Big)_{\Lambda=\Lambda_0}\;.
\end{aligned}
\tag{4.47d}
$$

In particular,

$$
R^{(1,1,1,0,0)}_{m_1n_1;\dots;m_4n_4}[\Lambda,\Lambda_0,\rho^0] \equiv 0\;. \tag{4.48}
$$

We first get (4.48) at $\Lambda = \Lambda_0$ from (4.47a). Since the rhs of (4.17) vanishes for $V = 1$ and $N = 4$, we conclude (4.48) for any Λ.

Proposition 4 *Let γ be an R-ribbon graph having N external legs, V vertices, V^e external vertices and segmentation index ι, which is drawn on a genus-g Riemann surface with B boundary components. Then the contribution $R^{(V,V^e,B,g,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}$ of γ to the expansion coefficient of the Λ_0-varied effective action describing a duality-covariant ϕ^4-theory on $\mathbb{R}^4_\theta$ in the matrix base is bounded as follows:*

1. *If γ is of the type described under 1–3 of Definition 1, we have*

$$\sum_{\gamma \text{ as in Def. 1.1}} \Big|R^{(V,V^e,1,0,0)\gamma}_{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{n^1 \\ n^2}\substack{k^1 \\ k^2};\substack{k^1 \\ k^2}\substack{l^1 \\ l^2};\substack{l^1 \\ l^2}\substack{m^1 \\ m^2}}[\Lambda,\Lambda_0,\rho_0]\Big| \\ \leq \Big(\frac{\Lambda^2}{\Lambda_0^2}\Big) P_1^{4V-4}\left[\frac{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{n^1 \\ n^2}\substack{k^1 \\ k^2};\substack{k^1 \\ k^2}\substack{l^1 \\ l^2};\substack{l^1 \\ l^2}\substack{m^1 \\ m^2}}{\theta\Lambda^2}\right]\Big(\frac{1}{\Omega}\Big)^{3V-2-V^e} P^{2V-2}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,, \tag{4.49}$$

$$\sum_{\gamma \text{ as in Def. 1.2}} \Big|R^{(V,V^e,1,0,0)\gamma}_{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{n^1 \\ n^2}\substack{m^1 \\ m^2}}[\Lambda,\Lambda_0,\rho_0]\Big| \\ \leq \Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)(\theta\Lambda^2) P_2^{4V-2}\left[\frac{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{n^1 \\ n^2}\substack{m^1 \\ m^2}}{\theta\Lambda^2}\right]\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e} P^{2V-1}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,, \tag{4.50}$$

$$\sum_{\gamma \text{ as in Def. 1.3}} \Big|R^{(V,V^e,1,0,0)}_{\substack{m^1+1 \\ m^2}\substack{n^1+1 \\ n^2};\substack{n^1 \\ n^2}\substack{m^1 \\ m^2}}[\Lambda,\Lambda_0,\rho_0]\Big| \\ \leq \Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)(\theta\Lambda^2) P_2^{4V-2}\left[\frac{\substack{m^1+1 \\ m^2}\substack{n^1+1 \\ n^2};\substack{n^1 \\ n^2}\substack{m^1 \\ m^2}}{\theta\Lambda^2}\right]\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e} P^{2V-1}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,, \tag{4.51}$$

2. *If γ is a subgraph of an 1PI planar graph with a selected set T of trajectories on one distinguished boundary component and a second set T' of summed trajectories on that boundary component, we have*

$$\sum_{\mathcal{E}^s}\sum_{\mathcal{E}^{t'}} \big|R^{(V,V^e,B,0,\iota)\gamma}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho_0]\big| \\ \leq \Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)\big(\theta\Lambda^2\big)^{(2-\frac{N}{2})+2(1-B)}\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}-1+B+2g-V^e-\iota+s+t'} \\ \times P^{4V-N}_{\big(2t'+\sum_{\overrightarrow{n_j\mathfrak{o}[n_j]}\in T}\min(2,\frac{1}{2}\langle\overrightarrow{n_j\mathfrak{o}[n_j]}\rangle)\big)}\left[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\right] P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,. \tag{4.52}$$

3. *If γ is a non-planar graph, we have*

$$\sum_{\mathcal{E}^s} \big|R^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\rho_0]\big| \\ \leq \Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)\big(\theta\Lambda^2\big)^{(2-\frac{N}{2})+2(1-B-2g)} P_0^{4V-N}\left[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda^2}\right] \\ \times\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}-1+B+2g-V^e-\iota+s} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,. \tag{4.53}$$

We have $R^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}\equiv 0$ for $N>2V{+}2$ or $\sum_{i=1}^N(m_i{-}n_i)\neq 0$.

Proof. Inserting the estimations of Proposition 2 into (4.47) we confirm Proposition 4 for $\Lambda=\Lambda_0$, which serves as initial condition for the Λ-integration of (4.17). This entails the polynomial in $\ln\frac{\Lambda_0}{\Lambda_R}$ instead of $\ln\frac{\Lambda}{\Lambda_R}$ appearing in Propositions 2 and 3. Accordingly, when using Propositions 2 and 3 as the input for (4.17), we will further bound these estimations by replacing $\ln\frac{\Lambda}{\Lambda_R}$ by $\ln\frac{\Lambda_0}{\Lambda_R}$.

Due to (4.48) the rhs of (4.17) vanishes for $N=2, V=1$ and for $N=6, V=2$. This means that the corresponding R-functions are constant in Λ so that the

Proposition holds for $R^{(1,1,1,0,0)}_{m_1n_1;m_2n_2}[\Lambda]$, $R^{(1,1,2,0,1)}_{m_1n_1;m_2n_2}[\Lambda]$ and $R^{(2,2,1,0,0)}_{m_1n_1;\dots;m_2n_2}[\Lambda]$. Since (4.17) is a linear differential equation, the factor $\frac{\Lambda^2}{\Lambda_0^2}$ relative to the estimation of the A-functions of Proposition 2, first appearing in $R^{(1,1,1,0,0)}_{m_1n_1;m_2n_2}[\Lambda]$, $R^{(1,1,2,0,1)}_{m_1n_1;m_2n_2}[\Lambda]$ and $R^{(2,2,1,0,0)}_{m_1n_1;\dots;m_2n_2}[\Lambda]$, survives to more complicated graphs, provided that none of the R-functions is relevant in Λ.

For graphs according to Definition 1.4, the first two lines on the rhs of (4.17) yield in the same way as in the proof of (3.39) the integrand (4.53), with the degree of the polynomial in $\ln \frac{\Lambda_0}{\Lambda_R}$ lowered by 1. Since under the given conditions an A-graph would be irrelevant, an R-graph with the additional factor $\frac{\Lambda^2}{\Lambda_0^2}$ is relevant or marginal. Thus, the Λ-integration of the first two lines on the rhs of (4.17) can be estimated by the integrand and a factor $P^1[\ln \frac{\Lambda_0}{\Lambda_R}]$, in agreement with (4.53). In the same way we verify (4.52) for the first two lines on the rhs of (4.17).

In the remaining lines of (4.17) we get by induction the following estimation:

$$\begin{aligned}
&\Big|\Big\{\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) R^{(V_1)}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 1.2}]}\Big| \\
&\qquad \leq \Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)(\theta\Lambda^2)\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e} P^{2V-2}\Big[\ln \frac{\Lambda_0}{\Lambda_R}\Big]\,, \qquad (4.54\text{a})\\
&\Big|\Big\{\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) \Big(R^{(V_1)}_{\substack{1\,0\\0\,0};\substack{0\,1\\0\,0};mn;kl}[\Lambda] - R^{(V_1)}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};mn;kl}[\Lambda]\Big)\Big\}_{[\text{Def. 1.2}]}\Big| \\
&\qquad \leq \Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e} P^{2V-2}\Big[\ln \frac{\Lambda_0}{\Lambda_R}\Big]\,, \qquad (4.54\text{b})\\
&\Big|\Big\{\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) R^{(V_1)}_{\substack{1\,1\\0\,0};\substack{0\,0\\0\,0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 1.3}]}\Big| \\
&\qquad \leq \Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)\Big(\frac{1}{\Omega}\Big)^{3V-1-V^e} P^{2V-2}\Big[\ln \frac{\Lambda_0}{\Lambda_R}\Big]\,, \qquad (4.54\text{c})\\
&\Big|\Big\{\sum_{m,n,k,l} Q_{nm;lk}(\Lambda) R^{(V_1)}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};mn;kl}[\Lambda]\Big\}_{[\text{Def. 1.1}]}\Big| \\
&\qquad \leq \Big(\frac{\Lambda^2}{\Lambda_0^2}\Big)\Big(\frac{1}{\Omega}\Big)^{3V-2-V^e} P^{2V-3}\Big[\ln \frac{\Lambda_0}{\Lambda_R}\Big]\,. \qquad (4.54\text{d})
\end{aligned}$$

These estimations are obtained in a similar way as (4.38a), (4.39a) and (4.40a). In particular, the improvement by $(\theta\Lambda^2)^{-1}$ in (4.54b) is due to the difference of graphs which according to Section 3.3 yield a composite propagator (3.17a). To obtain (4.54c) we have to use (4.52) with $\langle\overrightarrow{n_1 o(n_1)}\rangle + \langle\overrightarrow{n_2 o(n_2)}\rangle = 2$, which for the graphs under consideration is known by induction.

Multiplying (4.54) by versions of Proposition 3 according to (4.17), for $V_1 < V$, we obtain again (4.53) or (4.52), with the degree of the polynomial in $\ln \frac{\Lambda_0}{\Lambda_R}$ lowered by 1, for the integrand. Then the Λ-integration proves (4.53) and (4.52).

For graphs as in 1–3 of Definition 1 one shows in the same way as in the proof of 1–3 of Proposition 3 that the last term in the third line of (4.17) and the $(V_1 = V)$-terms in the remaining lines project to the irrelevant part of these R-functions, i.e. lead to (4.49)–(4.51). This was already clear from (4.44). For the remaining $(V_1 < V)$-terms in the fourth to last lines of (4.17) we obtain (4.49)–(4.51) from (4.54) and (4.33)–(4.35). This finishes the proof. □

4.5. Finishing the convergence and renormalisation theorem. We return now to the starting point of the entire estimation procedure—the identity (4.4). We put $\Lambda = \Lambda_R$ in Proposition 4 and perform the Λ_0-integration in (4.4):

Theorem 5 *The ϕ^4-model on $\mathbb{R}^4_\theta$ is (order by order in the coupling constant) renormalisable in the matrix base by adjusting the coefficients $\rho^0_a[\Lambda_0]$ defined in (3.15) and (3.14) of the initial interaction (3.3) to give (3.16) and by integrating the the Polchinski equation according to Definition 1.*

The limit $A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty] := \lim_{\Lambda_0\to\infty} A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]$ of the expansion coefficients of the effective action $L[\phi,\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]$, see (3.4), exists and satisfies

$$\begin{aligned}
&\Big|(2\pi\theta)^{\frac{N}{2}-2} A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty] - (2\pi\theta)^{\frac{N}{2}-2} A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\Lambda_0,\rho^0]\Big| \\
&\quad\leq \frac{\Lambda_R^{6-N}}{\Lambda_0^2}\Big(\frac{1}{\Omega\theta\Lambda_R^2}\Big)^{2(B+2g-1)} \\
&\quad\quad\times P_0^{4V-N}\Big[\frac{m_1n_1;\dots;m_Nn_N}{\theta\Lambda_R^2}\Big]\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}-V^e-\iota} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,. \qquad (4.55)
\end{aligned}$$

Proof. We insert Proposition 4, taken at $\Lambda = \Lambda_R$, into (4.4). We also use (3.33) in Proposition 4.1. Now, the existence of the limit and its property (4.55) are a consequence of Cauchy's criterion. Note that $\int \frac{dx}{x^3}\,P^q[\ln x] = \frac{1}{x^2}P'^q[\ln x]$. □

5. Conclusion

In this paper we have proven that the real ϕ^4-model on (Euclidean) noncommutative $\mathbb{R}^4$ is renormalisable to all orders in perturbation theory. The bare action of relevant and marginal couplings of the model is parametrised by four (divergent) quantities which require normalisation to the experimental data at a physical renormalisation scale. The corresponding physical parameters which determine the model are the mass, the field amplitude (to be normalised to 1), the coupling constant and (in addition to the commutative version) the frequency of an harmonic oscillator potential. The appearance of the oscillator potential is not a bad trick but a true physical effect. It is the self-consistent solution of the UV/IR-mixing problem found in the traditional noncommutative ϕ^4-model in momentum space. It implements the duality (see also [4]) that *noncommutativity relevant at short distances goes hand in hand with a modified structure of space relevant at large distances.*

Such a modified structure of space at very large distances seems to be in contradiction with experimental data. But this is not true. Neither position space nor momentum space are the adapted frames to interpret the model. An invariant characterisation of the model is the spectrum of the Laplace-like operator which defines the free theory. Due to the link to Meixner polynomials, the spectrum is discrete. Comparing (A.1) with (A.11) and (A.12) we see that the spectrum of the squared momentum variable has an equidistant spacing of $\frac{4\Omega}{\theta}$. Thus, $\sqrt{\frac{4\Omega}{\theta}}$ is the minimal (non-vanishing) momentum of the scalar field which is allowed in the noncommutative universe. We can thus identify the parameter $\sqrt{\Omega}$ with the ratio of the Planck length to the size of the (finite!) universe. Thus, for typical momenta on earth, the discretisation is not visible. However, there should be an observable effect at extremely huge scales. Indeed, there is some evidence of discrete momenta in the spectrum of the cosmic microwave background [21][10].

[10] According to the main purpose of [21] one should also discuss other topologies than the noncommutative $\mathbb{R}^D$.

Of course, when we pass to a frame where the propagator becomes $\frac{1}{\mu_0^2+p^2}$, with p now being discrete, we also have to transform the interactions. We thus have to shift the unitary matrices $U_m^{(\alpha)}$ appearing in (A.1) from the kinetic matrix or the propagator into the vertex. The properties of that dressed (physical) vertex will be studied elsewhere.

Another interesting exercise is the evaluation of the β-function of the duality-covariant ϕ^4-model [22]. It turns out that the one-loop β-function for the coupling constant remains non-negative and and vanishes for the self-dual case $\Omega = 1$. Moreover, the limit $\Omega \to 0$ exists at the one-loop level. This is related to the fact that the UV/IR-mixing in momentum space becomes problematic only at higher loop order.

Of particular interest would be the limit $\theta \to 0$. In the developed approach, θ defines the reference size of an elementary cell in the Moyal plane. All dimensionful quantities, in particular the energy scale Λ, are measured in units of (appropriate powers of) θ. In the final result of Theorem 5, these mass dimensions are restored. Then, we learn from (4.55) that a finite θ regularises the non-planar graphs. This means that for given Λ_0 and Λ_R the limit $\theta \to 0$ cannot be taken.

On the other hand, there could be a chance to let θ depend on Λ_0 in the same way as in the two-dimensional case [14] the oscillator frequency Ω was switched off with the limit $\Lambda_0 \to \infty$. However, this does not work. The point is that taking in (4.7) instead of the Λ_0-derivative the θ-derivative, there is now a contribution from the θ-dependence of the propagator. This leads in the analogue of the differential equation (4.11) to a term bilinear in L. Looking at the proof of Proposition 4, we see that this L-bilinear term will remove the factor Λ_0^{-2}.

Thus, the limit $\theta \to 0$ is singular. This is not surprising. In the limit $\theta \to 0$ the distinction between planar and non-planar graphs disappears (which is immediately clear in momentum space). Then, non-planar two- and four-point functions should yield the same divergent values as their planar analogues. Whereas the bare divergences in the planar sector are avoided by the mixed boundary conditions in 1-3 of Definition 1, the naïve initial condition in Definition 1.4 for non-planar graphs leaves the bare divergences in the limit $\theta \to 0$.

The next goal must be to generalise the renormalisation proof to gauge theories. This requires probably a gauge-invariant extension of the harmonic osillator potential. The result should be compared with string theory, because gauge theory on the Moyal plane arises in the zero-slope limit of string theory in presence of a Neveu-Schwarz B-field [23]. As renormalisation requires an appropriate structure of the space at very large distances, the question arises whether the oscillator potential has a counterpart in string theory. In this respect, it is tempting[11] to relate the oscillator potential to the maximally supersymmetric pp-wave background metric of type IIB string theory found in [24],

$$ds^2 = 2dx^+dx^- - 4\lambda^2 \sum_{i=1}^{8} (x^i)^2(dx^-)^2 + \sum_{i=1}^{8} (dx^i)^2 , \tag{5.1}$$

for $dx^\pm = \frac{1}{\sqrt{2}}(dx^9 \pm dx^{10})$, which solves Einstein's equations for an energy-momentum tensor relative to the 5-form field strength

$$F_5 = \lambda dx^- \big(dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8\big) . \tag{5.2}$$

Acknowledgement. We are indebted to Stefan Schraml for providing us with the references to orthogonal polynomials, without which the completion of the proof would have been impossible. We

[11] We would like to thank G. Bonelli for this interesting remark.

had stimulating discussions with Edwin Langmann, Vincent Rivasseau and Harold Steinacker. We are grateful to Christoph Kopper for indicating to us a way to reduce in our original power-counting estimation the polynomial in $\big(\ln\frac{\Lambda_0}{\Lambda_R}\big)$ to a polynomial in $\big(\ln\frac{\Lambda}{\Lambda_R}\big)$, thus permitting immediately the limit $\Lambda_0\to\infty$. We would like to thank the Max-Planck-Institute for Mathematics in the Sciences (especially Eberhard Zeidler), the Erwin-Schrödinger-Institute and the Institute for Theoretical Physics of the University of Vienna for the generous support of our collaboration.

A. Evaluation of the propagator

A.1. Diagonalisation of the kinetic matrix via Meixner polynomials. Our goal is to diagonalise the (four-dimensional) kinetic matrix $G_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}$ given in (2.6), making use of the angular momentum conservation $\alpha^r=n^r-m^r=k^r-l^r$ (which is due to the $SO(2)\times SO(2)$-symmetry of the action. For $\alpha^r\geq 0$ we thus look for a representation

$$G_{\substack{m^1\\m^2}\substack{m^1+\alpha^1\\m^2+\alpha^2};\substack{l^1+\alpha^1\\l^2+\alpha^2}\substack{l^1\\l^2}}=\sum_{i^1,i^2}U^{(\alpha^1)}_{m^1i^1}U^{(\alpha^2)}_{m^2i^2}\big(\tfrac{2}{\theta_1}v_{i^1}+\tfrac{2}{\theta_2}v_{i^2}+\mu_0^2\big)U^{(\alpha^1)}_{i^1l^1}U^{(\alpha^2)}_{i^2l^2}\;, \tag{A.1}$$

$$\delta_{ml}=\sum_i U^{(\alpha)}_{mi}U^{(\alpha)}_{il}\;. \tag{A.2}$$

The sum over i^1,i^2 would be an integration for continuous eigenvalues v_{i^r}. Comparing this ansatz with (2.6) we obtain, eliminating i in favour of v, the recurrence relation

$$\begin{aligned}(1-\Omega^2)\sqrt{m(\alpha+m)}U^{(\alpha)}_{m-1}(v)+\big(v-(1+\Omega^2)(\alpha+1+2m)\big)U^{(\alpha)}_m(v)&\\ +(1-\Omega^2)\sqrt{(m+1)(\alpha+m+1)}U^{(\alpha)}_{m+1}(v)=0&\end{aligned} \tag{A.3}$$

to determine $U^{(\alpha)}_m(v)$ and v. We are interested in the case $\Omega>0$. In order to make contact with standard formulae we put

$$U^{(\alpha)}_m(v)=f^{(\alpha)}(v)\,\frac{1}{\tau^m}\sqrt{\frac{(\alpha+m)!}{m!}}V^{(\alpha)}_m(v)\;,\qquad v=\nu x+\rho\;. \tag{A.4}$$

We obtain after division by $f^{(\alpha)}(v)$

$$\begin{aligned}0&=\frac{(1-\Omega^2)}{\tau^{m-1}}\sqrt{\frac{m^2(\alpha+m)!}{m!}}V^{(\alpha)}_{m-1}(\nu x+\rho)\\ &-\frac{1}{\tau^m}\big((1+\Omega^2)(\alpha+1+2m)-\rho-\nu x\big)\sqrt{\frac{(\alpha+m)!}{m!}}V^{(\alpha)}_m(\nu x+\rho)\\ &+\frac{(1-\Omega^2)}{\tau^{m+1}}\sqrt{\frac{(\alpha+m+1)^2(\alpha+m)!}{m!}}V^{(\alpha)}_{m+1}(\nu x+\rho)\;,\end{aligned} \tag{A.5}$$

i.e.

$$\begin{aligned}-\frac{\nu}{\tau(1-\Omega^2)}xV^{(\alpha)}_m(\nu x+\rho)=mV^{(\alpha)}_{m-1}(\nu x+\rho)-\frac{(1+\Omega^2)(\alpha+1+2m)-\rho}{\tau(1-\Omega^2)}V^{(\alpha)}_m(\nu x+\rho)&\\ +\frac{1}{\tau^2}(\alpha+m+1)V^{(\alpha)}_{m+1}(\nu x+\rho)\;,&\end{aligned} \tag{A.6}$$

Now we put

$$1+\alpha=\beta\;,\quad \frac{1}{\tau^2}=c\;,\quad \frac{2(1+\Omega^2)}{\tau(1-\Omega^2)}=1+c\;,\quad \frac{(1+\Omega^2)\beta-\rho}{\tau(1-\Omega^2)}=\beta c\;,\quad \frac{\nu}{(1-\Omega^2)\tau}=1-c \tag{A.7}$$

and

$$V_n^{(\alpha)}(\nu x+\rho) = M_n(x;\beta,c) \;, \tag{A.8}$$

which yields the recursion relation for the Meixner polynomials [16]:

$$\begin{aligned}(c-1)xM_m(x;\beta,c) &= c(m+\beta)M_{m+1}(x;\beta,c) \\ &\quad - (m+(m+\beta)c)M_m(x;\beta,c) + mM_{m-1}(x;\beta,c) \;. \end{aligned} \tag{A.9}$$

The solution of (A.7) is

$$\tau = \frac{(1\pm\Omega)^2}{1-\Omega^2} \equiv \frac{1\pm\Omega}{1\mp\Omega}\;, \qquad c = \frac{(1\mp\Omega)^2}{(1\pm\Omega)^2}\;, \qquad \nu = \pm 4\Omega\;, \qquad \rho = \pm 2\Omega(1+\alpha)\;. \tag{A.10}$$

We have to chose the upper sign, because the eigenvalues v are positive. We thus obtain

$$\begin{aligned} U_m^{(\alpha)}(v_x) &= f^{(\alpha)}(x)\sqrt{\frac{(\alpha+n)!}{n!}}\Big(\frac{1-\Omega}{1+\Omega}\Big)^m M_m\Big(x;1+\alpha,\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)\;, \\ v_x &= 2\Omega(2x+\alpha+1)\;. \end{aligned} \tag{A.11}$$

The function $f^{(\alpha)}(x)$ is identified by comparison of (A.2) with the orthogonality relation of Meixner polynomials [16],

$$\sum_{x=0}^{\infty} \frac{\Gamma(\beta+x)c^x}{\Gamma(\beta)x!} M_m(x;\beta,c)M_n(x;\beta,c) = \frac{c^{-n}n!\Gamma(\beta)}{\Gamma(\beta+n)(1-c)^\beta}\delta_{mn}\;. \tag{A.12}$$

The result is

$$U_m^{(\alpha)}(v_x) = \sqrt{\binom{\alpha+m}{m}\binom{\alpha+x}{x}}\Big(\frac{2\sqrt{\Omega}}{1+\Omega}\Big)^{\alpha+1}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{m+x} M_m\Big(x;1+\alpha,\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)\;. \tag{A.13}$$

The Meixner polynomials can be represented by hypergeometric functions [16]

$$M_m\Big(x;1+\alpha,\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big) = {}_2F_1\Big(\begin{matrix}-m,-x\\ 1+\alpha\end{matrix}\Big|-\frac{4\Omega}{(1-\Omega)^2}\Big)\;. \tag{A.14}$$

This shows that the matrices $U_{ml}^{(\alpha)}$ in (A.1) and (A.2) are symmetric in the lower indices.

A.2. Evaluation of the propagator. Now we return to the computation of the propagator, which is obtained by sandwiching the inverse eigenvalues $(\frac{2}{\theta_1}v_{i^1} + \frac{2}{\theta_2}v_{i^2} + \mu_0^2)$ between the unitary matrices $U^{(\alpha)}$. With (A.11) and the use of Schwinger's trick

$\frac{1}{A} = \int_0^\infty dt\, \mathrm{e}^{-tA}$ we have for $\theta_1=\theta_2=\theta$

$$\begin{aligned}
&\Delta_{\substack{m^1\\m^2}\substack{m^1+\alpha^1\\m^2+\alpha^2};\substack{l^1+\alpha^1\\l^2+\alpha^2}\substack{l^1\\l^2}} \\
&= \frac{\theta}{8\Omega}\int_0^\infty dt \sum_{x^1,x^2=0}^\infty \mathrm{e}^{-\frac{t}{4\Omega}(v_{x^1}+v_{x^2}+\theta\mu_0^2/2)} U_{m^1}^{(\alpha^1)}(v_{x^1}) U_{m^2}^{(\alpha^2)}(v_{x^2}) U_{l^1}^{(\alpha^1)}(v_{x^1}) U_{l^2}^{(\alpha^2)}(v_{x^2}) \\
&= \frac{\theta}{8\Omega}\int_0^\infty dt\ \mathrm{e}^{-t(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2))} \\
&\qquad \times \prod_{i=1}^2 \Bigg\{ \sqrt{\binom{\alpha^i+m^i}{m^i}\binom{\alpha^i+l^i}{l^i}} \Big(\frac{4\Omega}{(1+\Omega)^2}\Big)^{\alpha^i+1} \Big(\frac{1-\Omega}{1+\Omega}\Big)^{m^i+l^i} \\
&\qquad\qquad \times \sum_{x^i=0}^\infty \bigg\{ \frac{(\alpha^i+x^i)!}{x^i!\alpha^i!} \Big(\frac{\mathrm{e}^{-t}(1-\Omega)^2}{(1+\Omega)^2}\Big)^{x^i} \\
&\qquad\qquad \times {}_2F_1\Big(\begin{matrix}-m^i,-x^i\\1+\alpha^i\end{matrix}\Big| -\frac{4\Omega}{(1-\Omega)^2}\Big) {}_2F_1\Big(\begin{matrix}-l^i,-x^i\\1+\alpha^i\end{matrix}\Big| -\frac{4\Omega}{(1-\Omega)^2}\Big)\bigg\}\Bigg\}\;. \qquad \text{(A.15)}
\end{aligned}$$

We use the following identity for hypergeometric functions,

$$\begin{aligned}
&\sum_{x=0}^\infty \frac{(\alpha+x)!}{x!\alpha!} a^x\ {}_2F_1\Big(\begin{matrix}-m,-x\\1+\alpha\end{matrix}\Big| b\Big)\ {}_2F_1\Big(\begin{matrix}-l,-x\\1+\alpha\end{matrix}\Big| b\Big) \\
&\qquad = \frac{(1-(1-b)a)^{m+l}}{(1-a)^{\alpha+m+l+1}}\,{}_2F_1\Big(\begin{matrix}-m\,,\ -l\\1+\alpha\end{matrix}\Big| \frac{ab^2}{(1-(1-b)a)^2}\Big)\;, \qquad |a|<1\;. \qquad \text{(A.16)}
\end{aligned}$$

The identity (A.16) is probably known, but because it is crucial for the solution of the free theory, we provide the proof in Section A.4. We insert the rhs of (A.16), expanded as a finite sum, into (A.15), where we also put $z=\mathrm{e}^{-t}$:

$$\begin{aligned}
&\Delta_{\substack{m^1\\m^2}\substack{m^1+\alpha^1\\m^2+\alpha^2};\substack{l^1+\alpha^1\\l^2+\alpha^2}\substack{l^1\\l^2}} \\
&= \frac{\theta}{8\Omega}\sum_{u^1=0}^{\min(m^1,l^1)}\sum_{u^2=0}^{\min(m^2,l^2)} \int_0^1 dz\ \frac{z^{\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+u^1+u^2}(1-z)^{m^1+m^2+l^1+l^2-2u^1-2u^2}}{\big(1-\frac{(1-\Omega)^2}{(1+\Omega)^2}z\big)^{\alpha^1+\alpha^2+m^1+m^2+l^1+l^2+2}} \\
&\times \prod_{i=1}^2\bigg\{\Big(\frac{4\Omega}{(1+\Omega)^2}\Big)^{\alpha^i+2u^i+1}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{m^i+l^i-2u^i} \frac{\sqrt{m^i!(\alpha^i+m^i)!l^i!(\alpha^i+l^i)!}}{(m^i-u^i)!(l^i-u^i)!(\alpha^i+u^i)!u^i!}\bigg\}\;. \\
&\qquad\qquad \text{(A.17)}
\end{aligned}$$

This formula tells us the important property

$$0 \le \Delta_{\substack{m^1\\m^2}\substack{m^1+\alpha^1\\m^2+\alpha^2};\substack{l^1+\alpha^1\\l^2+\alpha^2}\substack{l^1\\l^2}} \le \Delta_{\substack{m^1\\m^2}\substack{m^1+\alpha^1\\m^2+\alpha^2};\substack{l^1+\alpha^1\\l^2+\alpha^2}\substack{l^1\\l^2}}\Big|_{\mu_0^2=0}\;, \qquad \text{(A.18)}$$

i.e. all matrix elements of the propagator are positive and majorised by the massless matrix elements. The representation (A.17) seems to be the most convenient one for analytical estimations of the propagator. The strategy[12] would be to divide the integration domain into slices and to maximise the individual z-dependent terms of the integrand over the slice, followed by resummation [18,17].

[12] We are grateful to Vincent Rivasseau for this idea.

The z-integration in (A.17) leads according to [25, §9.111] again to a hypergeometric function:

$$\begin{aligned}
&\Delta_{\substack{m^1\\m^2}\substack{m^1+\alpha^1\\m^2+\alpha^2};\substack{l^1+\alpha^1\\l^2+\alpha^2}\substack{l^1\\l^2}} = \frac{\theta}{8\Omega} \\
&\times \sum_{u^1=0}^{\min(m^1,l^1)} \sum_{u^2=0}^{\min(m^2,l^2)} \frac{\Gamma\big(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+u^1+u^2\big)(m^1+m^2+l^1+l^2-2u^1-2u^2)!}{\Gamma\big(2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2+l^1+l^2-u^1-u^2\big)} \\
&\times {}_2F_1\left(\begin{matrix}1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+u^1+u^2\,,\; 2+m^1+m^2+l^1+l^2+\alpha^1+\alpha^2\\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2+l^1+l^2-u^1-u^2\end{matrix}\,\middle|\,\frac{(1-\Omega)^2}{(1+\Omega)^2}\right) \\
&\times \prod_{i=1}^{2}\left\{\Big(\frac{4\Omega}{(1+\Omega)^2}\Big)^{\alpha^i+2u^i+1}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{m^i+l^i-2u^i}\frac{\sqrt{m^i!(\alpha^i+m^i)!l^i!(\alpha^i+l^i)!}}{(m^i-u^i)!(l^i-u^i)!(\alpha^i+u^i)!u^i!}\right\} \\
&= \frac{\theta}{2(1+\Omega)^2}\sum_{u^1=0}^{\min(m^1,l^1)} \sum_{u^2=0}^{\min(m^2,l^2)} \prod_{i=1}^{2}\sqrt{\binom{\alpha^i+m^i}{\alpha^i+u^i}\binom{\alpha^i+l^i}{\alpha^i+u^i}\binom{m^i}{u^i}\binom{l^i}{u^i}} \\
&\times \Big(\frac{1-\Omega}{1+\Omega}\Big)^{m^i+l^i-2u^i} B\big(1+\tfrac{\mu_0^2\theta}{8\Omega}+\tfrac{1}{2}(\alpha^1+\alpha^2)+u^1+u^2,\, 1+m^1+m^2+l^1+l^2-2u^1-2u^2\big) \\
&\times {}_2F_1\left(\begin{matrix}1+m^1+m^2+l^1+l^2-2u^1-2u^2\,,\; \frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(\alpha^1+\alpha^2)-u^1-u^2\\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2+l^1+l^2-u^1-u^2\end{matrix}\,\middle|\,\frac{(1-\Omega)^2}{(1+\Omega)^2}\right).
\end{aligned} \tag{A.19}$$

We have used [25, §9.131.1] to obtain the last line. The form (A.19) will be useful for the evaluation of special cases and of the asymptotic behaviour. In the main part, for presentational purposes, α^i is eliminated in favour of k^i, n^i and the summation variable $v^i := m^i + l^i - 2u^i$ is used. The final result is given in (2.7).

For $\mu_0 = 0$ we can in few cases evaluate the sum over u^i exactly. First, for $l^i = 0$ we also have $u^i = 0$. If additionally $\alpha^i = 0$ we get

$$\Delta_{\substack{m^1\\m^2}\substack{m^1\\m^2};\substack{0\\0}\substack{0\\0}}\Big|_{\mu_0=0} = \frac{\theta}{2(1+\Omega)^2(1+m^1+m^2)}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{m^1+m^2}. \tag{A.20}$$

One should notice here the exponential decay for $\Omega > 0$. It can be seen numerically that this is a general feature of the propagator: Given m^i and α^i, the maximum of the propagator is attained at $l^i = m^i$. Moreover, the decay with $|l^i - m^i|$ is exponentially so that the sum

$$\sum_{l^1,l^2} \Delta_{\substack{m^1\\m^2}\substack{m^1+\alpha^1\\m^2+\alpha^2};\substack{l^1+\alpha^1\\l^2+\alpha^2}\substack{l^1\\l^2}} \tag{A.21}$$

converges. We confirm this argumentation numerically in (C.3).

It turns out numerically that the maximum of the propagator for indices restricted by $\mathcal{C} \leq \max(m^1, m^2, n^1, n^2, k^1, k^2, l^1, l^2) \leq 2\mathcal{C}$ is found in the subclass $\Delta_{\substack{m^1\\0}\substack{n^1\\0};\substack{n^1\\0}\substack{m^1\\0}}$ of propagators. Coincidently, the computation in case of $m^2 = l^2 = \alpha^2 = 0$ simplifies considerably. If additionally $m^1 = n^1$ we obtain a closed result:

$$\begin{aligned}
\Delta_{\substack{m\\0}\substack{m\\0};\substack{m\\0}\substack{m\\0}} &= \frac{\theta}{2(1+\Omega)^2}\sum_{u=0}^{m}\frac{(m!)^2(2u)!}{(m-u)!(u!)^2(1+m+u)!} \\
&\qquad\times {}_2F_1\left(\begin{matrix}1+2u\,,\; u-m\\ 2+m+u\end{matrix}\,\middle|\,\frac{(1-\Omega)^2}{(1+\Omega)^2}\right)\Big(\frac{1-\Omega}{1+\Omega}\Big)^{2u}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\theta}{2(1+\Omega)^2} \sum_{u=0}^{m} \sum_{s=0}^{m-u} (-1)^s \frac{(m!)^2 (2u+s)!}{(m-u-s)!(u!)^2(1+m+u+s)!s!} \Big(\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)^{u+s} \\
&= \frac{\theta}{2(1+\Omega)^2} \sum_{u=0}^{m} \sum_{r=u}^{m} (-1)^{u+r} \frac{(m!)^2 (r+u)!}{(m-r)!(u!)^2(1+m+r)!(r-u)!} \Big(\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)^{r} \\
&= \frac{\theta}{2(1+\Omega)^2} \sum_{r=0}^{m} (-1)^r \frac{(m!)^2}{(m-r)!(1+m+r)!}\; {}_2F_1\Big(\begin{matrix} r+1\,,\; -r \\ 1 \end{matrix}\Big| 1\Big) \Big(\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)^{r} \\
&= \frac{\theta}{2(1+\Omega)^2} \sum_{r=0}^{m} \frac{(m!)^2}{(m-r)!(1+m+r)!} \Big(\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)^{r} \\
&= \frac{\theta}{2(1+\Omega)^2(m+1)}\; {}_2F_1\Big(\begin{matrix} 1\,,\; -m \\ m+2 \end{matrix}\Big| -\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big) \\
&\sim \begin{cases} \dfrac{\theta}{8\Omega(m+1)} & \text{for } \Omega > 0\,,\; m \gg 1\,, \\[2ex] \dfrac{\sqrt{\pi}\theta}{4\sqrt{m+\frac{3}{4}}} & \text{for } \Omega = 0\,,\; m \gg 1\,. \end{cases}
\end{aligned}
\tag{A.22}
$$

We see a crucial difference in the asymptotic behaviour for $\Omega > 0$ versus $\Omega = 0$. The slow decay with $m^{-\frac{1}{2}}$ of the propagator is responsible for the non-renormalisability of the ϕ^4-model in case of $\Omega = 0$. The numerical result (C.2) shows that the maximum of the propagator for indices restricted by $\mathcal{C} \leq \max(m^1, m^2, n^1, n^2, k^1, k^2, l^1, l^2) \leq 2\mathcal{C}$ is very close to the result (A.22), for $m = \mathcal{C}$. For $\Omega = 0$ the maximum is exactly given by the 6^{th} line of (A.22).

A.3. Asymptotic behaviour of the propagator for large α^i. We consider various limiting cases of the propagator, making use of the asymptotic expansion (Stirling's formula) of the Γ-function,

$$
\Gamma(n+1) \sim \Big(\frac{n}{\mathrm{e}}\Big)^n \sqrt{2\pi(n+\tfrac{1}{6})} + \mathcal{O}(n^{-2})\,. \tag{A.23}
$$

This implies

$$
\frac{\Gamma(n+1+a)}{\Gamma(n+1+b)} \sim n^{a-b}\Big(1 + \frac{(a-b)(a+b+1)}{2n} + \mathcal{O}(n^{-2})\Big)\,. \tag{A.24}
$$

We rewrite the propagator (A.19) in a manner where the large-α^i behaviour is easier to discuss:

$$
\begin{aligned}
&\Delta_{\substack{m^1 \\ m^2} \substack{m^1+\alpha^1 \\ m^2+\alpha^2} ; \substack{l^1+\alpha^1 \\ l^2+\alpha^2} \substack{l^1 \\ l^2}} \\
&= \sum_{u^1=0}^{\min(m^1,l^1)} \sum_{u^2=0}^{\min(m^2,l^2)} \frac{\theta}{2(1+\Omega)^2(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2+l^1+l^2-u^1-u^2)} \\
&\times \frac{(m^1+m^2+l^1+l^2-2u^1-2u^2)!\sqrt{m^1!l^1!m^2!l^2!}}{(m^1-u^1)!(l^1-u^1)!u^1!(m^2-u^2)!(l^2-u^2)!u^2!} \Big(\frac{1-\Omega}{1+\Omega}\Big)^{m^1+m^2+l^1+l^2-2u^1-2u^2}
\end{aligned}
$$

$$\times \left\{ \frac{\Gamma\big(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+u^1+u^2\big)}{\Gamma\big(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2+l^1+l^2-u^1-u^2\big)} \right.$$
$$\left. \times \sqrt{\frac{(\alpha^1+m^1)!(\alpha^1+l^1)!}{(\alpha^1+u^1)!(\alpha^1+u^1)!}}\sqrt{\frac{(\alpha^2+m^2)!(\alpha^2+l^2)!}{(\alpha^2+u^2)!(\alpha^2+u^2)!}} \right\}$$
$$\times\, {}_2F_1\Big(\begin{matrix} 1+m^1+m^2+l^1+l^2-2u^1-2u^2\,,\ \frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(\alpha^1+\alpha^2)-u^1-u^2 \\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2+l^1+l^2-u^1-u^2 \end{matrix}\Big| \frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)\,. \tag{A.25}$$

We assume $\frac{1}{2}(\alpha_1+\alpha_2) \geq \max(\frac{\mu_0^2\theta}{8\Omega}, m, l)$. The term in braces $\{\ \}$ in (A.25) behaves like

$$\begin{aligned}\{\dots\} \sim{}& \big(\tfrac{1}{2}(\alpha^1+\alpha^2)\big)^{2u^1+2u^2-m^1-m^2-l^2-l^2} (\alpha^1)^{\frac{1}{2}(m^1+l^1-2u^1)}(\alpha^2)^{\frac{1}{2}(m^2+l^2-2u^1)} \\ &\times \Big(1+\frac{(2u^1+2u^2-m^1-m^2-l^2-l^2)(m^1+m^2+l^1+l^2+\frac{\mu_0^2\theta}{4\Omega}+1)}{(\alpha^1+\alpha^2)} \\ &\qquad + \mathcal{O}\big((\alpha^1+\alpha^2)^{-2}\big)\Big) \\ &\times \Big(1+\frac{(m^1-u^1)(m^1+u^1+1)}{4\alpha^1}+\frac{(l^1-u^1)(l^1+u^1+1)}{4\alpha^1}+\mathcal{O}\big((\alpha^1)^{-2}\big)\Big) \\ &\times \Big(1+\frac{(m^2-u^2)(m^2+u^2+1)}{4\alpha^2}+\frac{(l^2-u^2)(l^2+u^2+1)}{4\alpha^1}+\mathcal{O}\big((\alpha^2)^{-2}\big)\Big)\,. \end{aligned} \tag{A.26}$$

We look for the maximum of the propagator under the condition $\mathcal{C} \leq \max(\alpha^1,\alpha^2) \leq 2\mathcal{C}$. Defining $s^i = m^i + l^i - 2u^i$ and $s = s^1 + s^2$, the dominating term in (A.26) is

$$\frac{(\alpha^1)^{\frac{s^1}{2}}(\alpha^2)^{\frac{s^2}{2}}}{\big(\frac{1}{2}(\alpha^1+\alpha^2)\big)^s}\bigg|_{\mathcal{C}\leq\max(\alpha^1,\alpha^2)\leq 2\mathcal{C}} \leq \frac{\max\Big(\frac{(\frac{s^1}{s^1+2s^2})^{\frac{s^1}{2}}}{(\frac{s^1+s^2}{s^1+2s^2})^{s^1+s^2}}, \frac{(\frac{s^2}{s^2+2s^1})^{\frac{s^2}{2}}}{(\frac{s^1+s^2}{s^2+2s^1})^{s^1+s^2}}\Big)}{\mathcal{C}^{\frac{s}{2}}}\,. \tag{A.27}$$

The maximum is attained at $(\alpha^1,\alpha^2) = (\frac{s^1\mathcal{C}}{s^1+2s^2}, \mathcal{C})$ for $s^1 \leq s^2$ and at $(\alpha^1,\alpha^2) = (\mathcal{C}, \frac{s^2\mathcal{C}}{s^2+2s^1})$ for $s^1 \geq s^2$. Thus, the leading contribution to the propagator will come from the summation index $u^i = \min(m^i, l^i)$.

Next we evaluate the leading contribution of the hypergeometric function:

$$\begin{aligned} &{}_2F_1\Big(\begin{matrix} 1+m^1+m^2+l^1+l^2-2u^1-2u^2\,,\ \frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(\alpha^1+\alpha^2)-u^1-u^2 \\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2+l^1+l^2-u^1-u^2 \end{matrix}\Big| \frac{(1-\Omega)^2}{(1+\Omega)^2}\Big) \\ &\sim \sum_{k=0}^\infty \frac{(m^1+m^2+l^1+l^2-2u^1-2u^2+k)!}{(m^1+m^2+l^1+l^2-2u^1-2u^2)!}\frac{\big(-\frac{(1-\Omega)^2}{(1+\Omega)^2}\big)^k}{k!}\Big(1+\frac{k(2u^1+2u^2-\frac{\mu_0^2\theta}{4\Omega}+1-k)}{\alpha^1+\alpha^2} \\ &\qquad - \frac{k(3+2m^1+2m^2+2l^1+2l^2-2u^1-2u^2+\frac{\mu_0^2\theta}{4\Omega}+k)}{\alpha^1+\alpha^2}+\mathcal{O}\big((\alpha^1+\alpha^2)^{-2}\big)\Big) \\ &= \sum_{k=0}^\infty \frac{(s+k)!}{s!}\Big(1-\frac{2k(1+s+\frac{\mu_0^2\theta}{4\Omega}+k)}{\alpha^1+\alpha^2}\Big)\frac{\big(-\frac{(1-\Omega)^2}{(1+\Omega)^2}\big)^k}{k!}+\mathcal{O}\big((\alpha^1+\alpha^2)^{-2}\big) \\ &= \Big(\frac{(1+\Omega)^2}{2(1+\Omega^2)}\Big)^{1+s}\Big(1+\frac{\frac{(1-\Omega)^2}{1+\Omega^2}(1+s)}{(\alpha^1+\alpha^2)}\Big(1+\frac{\mu_0^2\theta}{4\Omega}+\frac{s}{2}+(s+2)\frac{\Omega}{(1+\Omega^2)}\Big)\Big) \\ &\quad + \mathcal{O}\big((\alpha^1+\alpha^2)^{-2}\big)\,. \end{aligned} \tag{A.28}$$

Assuming $s^1 \leq s^2$, we obtain from (A.23), (A.27) and (A.28) the following leading contribution to the propagator (A.25)

$$\Delta_{\substack{m^1\\m^2}\substack{m^1+\alpha^1\\m^2+\alpha^2};\substack{l^1+\alpha^1\\l^2+\alpha^2}\substack{l^1\\l^2}}\Big|_{\max(m^1,m^2,l^1,l^2)\ll\mathcal{C}\leq\max(\alpha^1,\alpha^2)\leq 2\mathcal{C}}$$
$$= \frac{\theta\big(\max(m^1,l^1)\big)^{\frac{s^1}{2}}\big(\max(m^2,l^2)\big)^{\frac{s^2}{2}}}{(1+\Omega)^2\mathcal{C}^{1+\frac{s^1+s^2}{2}}}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{s^1+s^2}\Big(\frac{(1+\Omega)^2}{2(1+\Omega^2)}\Big)^{1+s^1+s^2}$$
$$\times\frac{(s^1+s^2)^{s^1+s^2}\sqrt{2\pi(s^1+s^2)}}{(s^1)^{s^1}(s^2)^{s^2}2\pi\sqrt{s^1s^2}}\frac{\big(\frac{s^1}{s^1+2s^2}\big)^{\frac{s^1}{2}}}{\big(\frac{s^1+s^2}{s^1+2s^2}\big)^{1+s^1+s^2}}\Big(1+\mathcal{O}(\mathcal{C}^{-1})\Big)\Big|_{s^i:=|m^i-l^i|}. \tag{A.29}$$

The numerator comes from $\sqrt{\frac{m!}{l!}} \leq m^{\frac{m-l}{2}}$ for $m \geq l$. The estimation (A.29) is the explanation of (3.24).

Let us now look at propagators with $m^i = l^i$ and $m^i \ll \mathcal{C} \leq \max(\alpha^1,\alpha^2) \leq 2\mathcal{C}$:

$$\Delta_{\substack{m^1\\m^2}\substack{m^1+\alpha^1\\m^2+\alpha^2};\substack{m^1+\alpha^1\\m^2+\alpha^2}\substack{m^1\\m^2}}$$
$$= \frac{\theta}{2(1+\Omega)^2\big(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2\big)}$$
$$\quad\times\bigg(\frac{(1+\Omega)^2}{2(1+\Omega^2)}+\frac{\big(\frac{1-\Omega^2}{1+\Omega^2}\big)^2}{2(\alpha_1+\alpha_2)}\Big(1+\frac{\mu_0^2\theta}{4\Omega}+\frac{2\Omega}{(1+\Omega^2)}\Big)\bigg)$$
$$+\frac{\theta}{2(1+\Omega)^2\big(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2)+m^1+m^2+1\big)}\Big(\frac{1-\Omega^2}{1+\Omega^2}\Big)^2\frac{(1+\Omega)^2}{1+\Omega^2}\frac{m^1\alpha^1+m^2\alpha^2}{(\alpha^1+\alpha^2)^2}$$
$$+\mathcal{O}\big((\alpha^1+\alpha^2)^{-3}\big)\,. \tag{A.30}$$

This means

$$\Delta_{\substack{m_1\\m_2}\substack{m_1+\alpha_1\\m_2+\alpha_2};\substack{m_1+\alpha_1\\m_2+\alpha_2}\substack{m_1\\m_2}}-\Delta_{\substack{0\\0}\substack{m^1+\alpha_1\\m^2+\alpha_2};\substack{m^1+\alpha_1\\m^2+\alpha_2}\substack{0\\0}}$$
$$= -(m_1+m_2)\frac{\theta}{8(1+\Omega^2)\big(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2+m^1+m^2)\big)^2}$$
$$+\frac{\theta}{2(1+\Omega^2)\big(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1+\alpha^2+m^1+m^2)\big)}\Big(\frac{1-\Omega^2}{1+\Omega^2}\Big)^2\frac{m^1(\alpha^1+m^1)+m^2(\alpha^2+m^2)}{(\alpha^1+\alpha^2+m^1+m^2)^2}$$
$$+\mathcal{O}\Big(\frac{1}{(\alpha^1+\alpha^2+m^1+m^2)}\frac{(m^1+m^2)^2}{(\alpha^1+\alpha^2+m^1+m^2)^2}\Big)$$
$$= m_1\Big(\Delta_{\substack{1\\0}\substack{m_1+\alpha_1\\m_2+\alpha_2};\substack{m_1+\alpha_1\\m_2+\alpha_2}\substack{1\\0}}-\Delta_{\substack{0\\0}\substack{m_1+\alpha_1\\m_2+\alpha_2};\substack{m_1+\alpha_1\\m_2+\alpha_2}\substack{0\\0}}\Big)$$
$$+m_2\Big(\Delta_{\substack{0\\1}\substack{m_1+\alpha_1\\m_2+\alpha_2};\substack{m_1+\alpha_1\\m_2+\alpha_2}\substack{0\\1}}-\Delta_{\substack{0\\0}\substack{m_1+\alpha_1\\m_2+\alpha_2};\substack{m_1+\alpha_1\\m_2+\alpha_2}\substack{0\\0}}\Big)$$
$$+\mathcal{O}\Big(\frac{1}{(\alpha^1+\alpha^2+m^1+m^2)}\frac{(m^1+m^2)^2}{(\alpha^1+\alpha^2+m^1+m^2)^2}\Big)\,. \tag{A.31}$$

The second and third line of (A.31) explains the estimation (3.28). Clearly, the next term in the expansion is of the order $\frac{(m^1+m^2)^2}{(\alpha^1+\alpha^2+m^1+m^2)^3}$, which explains the estimation (3.29).

For $m_1=l_1+1$ and $m_2=l_2$ we have

$$\Delta_{\substack{l_1+1\\l_2}\substack{l_1+1+\alpha_1\\l_2+\alpha_2};\substack{l_1+\alpha_1\\l_2+\alpha_2}\substack{l_1\\l_2}} = \frac{\theta}{2(1{+}\Omega)^2\big(\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\alpha^1{+}\alpha^2)+l^1{+}l^2{+}2\big)} \times \frac{1{-}\Omega^2}{1{+}\Omega^2}\frac{\sqrt{(l^1{+}1)(l^1{+}\alpha^1{+}1)}}{\alpha^1{+}\alpha^2}\Big(1+\mathcal{O}\big((\alpha_1{+}\alpha_2)^{-1}\big)\Big)\,. \tag{A.32}$$

This yields

$$\Delta_{\substack{l_1+1\\l_2}\substack{l_1+1+\alpha_1\\l_2+\alpha_2};\substack{l_1+\alpha_1\\l_2+\alpha_2}\substack{l_1\\l_2}} - \sqrt{l_1{+}1}\Delta_{\substack{1\\0}\substack{l_1+1+\alpha_1\\l_2+\alpha_2};\substack{l_1+\alpha_1\\l_2+\alpha_2}\substack{0\\0}} = \mathcal{O}\bigg(\frac{1}{(\alpha^1{+}\alpha^2{+}l^1{+}l^2)}\sqrt{\frac{l^1{+}1}{\alpha^1{+}\alpha^2{+}l^1{+}l^2}}\frac{(l^1{+}1)}{(\alpha^1{+}\alpha^2{+}l^1{+}l^2)}\bigg)\,, \tag{A.33}$$

which explains the estimation (3.30). Similarly, we have

$$\Delta_{\substack{1\\1}\substack{1+\alpha_1\\1+\alpha_2};\substack{\alpha_1\\1+\alpha_2}\substack{0\\1}} - \Delta_{\substack{1\\0}\substack{1+\alpha_1\\1+\alpha_2};\substack{\alpha_1\\1+\alpha_2}\substack{0\\0}} = \mathcal{O}\bigg(\frac{\theta\sqrt{\alpha^1{+}1}}{(\alpha^1{+}\alpha^2{+}1)^3}\bigg)\,, \tag{A.34}$$

which shows that the norm of (B.7) is of the same order as (3.30).

A.4. An identity for hypergeometric functions. For terminating hypergeometric series ($m,l\in\mathbb{N}$) we compute the sum in the last line of (A.15):

$$\begin{aligned}
&\sum_{x=0}^\infty \frac{(\alpha{+}x)!}{x!\alpha!}a^x\,{}_2F_1\Big(\begin{matrix}-m,-x\\1{+}\alpha\end{matrix}\Big|b\Big)\,{}_2F_1\Big(\begin{matrix}-l,-x\\1{+}\alpha\end{matrix}\Big|b\Big)\\
&=\sum_{x=0}^\infty\sum_{r=0}^{\min(x,m)}\sum_{s=0}^{\min(x,l)}\frac{(\alpha{+}x)!}{x!\alpha!}a^x\frac{m!x!\alpha!}{(m{-}r)!(x{-}r)!(\alpha{+}r)!r!}b^r\frac{l!x!\alpha!}{(m{-}s)!(x{-}s)!(\alpha{+}s)!s!}b^s\\
&=\sum_{r=0}^m\sum_{s=0}^l\sum_{x=\max(r,s)}^\infty(\alpha{+}x)!x!\alpha!m!l!a^x\frac{b^{r+s}}{(m{-}r)!(x{-}r)!(\alpha{+}r)!r!(l{-}s)!(x{-}s)!(\alpha{+}s)!s!}\\
&=\sum_{r=0}^m\sum_{s=0}^l\frac{\alpha!m!l!}{(m{-}r)!(\alpha{+}r)!r!(m{-}s)!(\alpha{+}s)!s!}a^{\max(r,s)}b^{r+s}\\
&\qquad\times\sum_{y=0}^\infty\frac{(\alpha{+}y{+}\max(r,s))!(y{+}\max(r,s))!}{(y{+}|r{-}s|)!y!}a^y\\
&=\sum_{r=0}^m\sum_{s=0}^l\frac{\alpha!m!l!}{(m{-}r)!(\alpha{+}r)!r!(l{-}s)!(\alpha{+}s)!s!}a^{\max(r,s)}b^{r+s}\\
&\qquad\times\frac{(\alpha{+}\max(r,s))!(\max(r,s))!}{(|r{-}s|)!}\,{}_2F_1\Big(\begin{matrix}\alpha{+}\max(r,s){+}1\,,\;\max(r,s){+}1\\|r{-}s|{+}1\end{matrix}\Big|a\Big)\\
&=^*\sum_{r=0}^m\sum_{s=0}^l\frac{\alpha!m!l!}{(m{-}r)!(\alpha{+}r)!r!(l{-}s)!(\alpha{+}s)!s!}\frac{a^{\max(r,s)}\,b^{r+s}}{(1-a)^{\alpha+r+s+1}}\\
&\qquad\times\frac{(\alpha{+}\max(r,s))!(\max(r,s))!}{(|r{-}s|)!}\,{}_2F_1\Big(\begin{matrix}-\min(\alpha{+}r,\alpha{+}s)\,,\;-\min(r,s)\\|r{-}s|{+}1\end{matrix}\Big|a\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{r=0}^{m}\sum_{s=0}^{l} \frac{\alpha!m!l!}{(m-r)!(\alpha+r)!r!(l-s)!(\alpha+s)!s!}\,\frac{a^{\max(r,s)}\,b^{r+s}}{(1-a)^{\alpha+r+s+1}} \\
&\qquad \times \sum_{u'=0}^{\min(r,s)} \frac{(\alpha+\max(r,s))!(\max(r,s))!(\alpha+\min(r,s))!(\min(r,s))!}{(|r-s|+u')!(\min(r,s)-u')!(\alpha+\min(r,s)-u')!u'!}a^{u'} \\
&= \sum_{r=0}^{m}\sum_{s=0}^{l}\sum_{u=0}^{\min(r,s)} \frac{\alpha!m!l!}{(m-r)!(r-u)!(l-s)!(s-u)!(\alpha+u)!u!}\,\frac{a^{r+s-u}\,b^{r+s}}{(1-a)^{\alpha+r+s+1}} \\
&= \sum_{u=0}^{\min(m,l)}\sum_{r=u}^{m}\sum_{s=u}^{l} \frac{\alpha!m!l!}{(m-r)!(r-u)!(l-s)!(s-u)!(\alpha+u)!u!}\,\frac{a^{r+s-u}\,b^{r+s}}{(1-a)^{\alpha+r+s+1}} \\
&= \sum_{u=0}^{\min(m,l)} \frac{\alpha!m!l!}{(m-u)!(l-u)!(\alpha+u)!u!}\Big(\frac{ab}{1-a}\Big)^{2u}\Big(1+\frac{ab}{1-a}\Big)^{m+l-2u}\frac{1}{a^u(1-a)^{\alpha+1}} \\
&= \frac{(1-a+ab)^{m+l}}{(1-a)^{m+l+\alpha+1}}\,{}_2F_1\Big(\begin{matrix}-m\,,\ -l\\ \alpha+1\end{matrix}\Big|\frac{ab^2}{(1-a+ab)^2}\Big)\,. \qquad\qquad (A.35)
\end{aligned}
$$

In the step denoted by $=^*$ we have used [25, §9.131.1]. All other transformations should be self-explaining.

B. On composite propagators

B.1. Identities for differences of ribbon graphs. We continue here the discussion of Section 3.3 on composite propagators generated by differences of interaction coefficients.

After having derived (3.19), we now have a look at (3.11). Since γ is one-particle irreducible, we get for a certain permutation π ensuring the history of integrations the following linear combination:

$$
\begin{aligned}
&\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{m^1+1}{m^2}\genfrac{}{}{0pt}{}{n^1+1}{n^2};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}}[\Lambda'] - \sqrt{(m^1+1)(n^1+1)}\,\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{\genfrac{}{}{0pt}{}{1}{0}\genfrac{}{}{0pt}{}{1}{0};\genfrac{}{}{0pt}{}{0}{0}\genfrac{}{}{0pt}{}{0}{0}}}[\Lambda'] \\
&= \dots\Big\{\prod_{i=1}^{p-1} Q_{\genfrac{}{}{0pt}{}{n^1+1}{n^2}k_{\pi_i};k_{\pi_i}\genfrac{}{}{0pt}{}{n^1+1}{n^2}}(\Lambda_{\pi_i})\,Q_{\genfrac{}{}{0pt}{}{n^1+1}{n^2}(k_{\pi_p}+1^+);k_{\pi_p}\genfrac{}{}{0pt}{}{n^1}{n^2}}(\Lambda_{\pi_p}) \\
&\qquad\times \prod_{i=p+1}^{a} Q_{\genfrac{}{}{0pt}{}{n^1}{n^2}k_{\pi_i};k_{\pi_i}\genfrac{}{}{0pt}{}{n^1}{n^2}}(\Lambda_{\pi_i}) \prod_{j=1}^{q-1} Q_{\genfrac{}{}{0pt}{}{m^1}{m^2}l_{\pi_j};l_{\pi_j}\genfrac{}{}{0pt}{}{m^1}{m^2}}(\Lambda_{\pi_j}) \\
&\qquad\times Q_{\genfrac{}{}{0pt}{}{m^1}{m^2}l_{\pi_q};(l_{\pi_q}+1^+)\genfrac{}{}{0pt}{}{m^1+1}{m^2}}(\Lambda_{\pi_q}) \prod_{j=q+1}^{b} Q_{\genfrac{}{}{0pt}{}{m^1+1}{m^2}l_{\pi_j};l_{\pi_j}\genfrac{}{}{0pt}{}{m^1+1}{m^2}}(\Lambda_{\pi_j}) \\
&- \sqrt{(m^1+1)(n^1+1)}\prod_{i=1}^{p-1} Q_{\genfrac{}{}{0pt}{}{1}{0}k_{\pi_i};k_{\pi_i}\genfrac{}{}{0pt}{}{1}{0}}(\Lambda_{\pi_i})\,Q_{\genfrac{}{}{0pt}{}{1}{0}(k_{\pi_p}+1^+);k_{\pi_p}\genfrac{}{}{0pt}{}{0}{0}}(\Lambda_{\pi_p}) \prod_{i=p+1}^{a} Q_{\genfrac{}{}{0pt}{}{0}{0}k_{\pi_i};k_{\pi_i}\genfrac{}{}{0pt}{}{0}{0}}(\Lambda_{\pi_i}) \\
&\qquad\times \prod_{j=1}^{q} Q_{\genfrac{}{}{0pt}{}{0}{0}l_{\pi_j};l_{\pi_j}\genfrac{}{}{0pt}{}{0}{0}}(\Lambda_{\pi_j})\,Q_{\genfrac{}{}{0pt}{}{0}{0}l_{\pi_q};(l_{\pi_q}+1^+)\genfrac{}{}{0pt}{}{1}{0}}(\Lambda_{\pi_q}) \prod_{j=q+1}^{b} Q_{\genfrac{}{}{0pt}{}{1}{0}l_{\pi_j};l_{\pi_j}\genfrac{}{}{0pt}{}{1}{0}}(\Lambda_{\pi_j})\Big\}
\end{aligned}
$$

$$
\begin{aligned}
= \dots \Bigg\{ \Bigg(& \prod_{i=1}^{p-1} Q_{{}^{n^1+1}_{n^2} k_{\pi_i};k_{\pi_i} {}^{n^1+1}_{n^2}}(\Lambda_{\pi_i}) \\
& \qquad \times Q_{{}^{n^1+1}_{n^2} (k_{\pi_p}+1^+);k_{\pi_p} {}^{n^1}_{n^2}}(\Lambda_{\pi_p}) \prod_{i=p+1}^{a} Q_{{}^{n^1}_{n^2} k_{\pi_i};k_{\pi_i} {}^{n^1}_{n^2}}(\Lambda_{\pi_i}) \\
& - \sqrt{n^1+1} \prod_{i=1}^{p-1} Q_{{}^1_0 k_{\pi_i};k_{\pi_i} {}^1_0}(\Lambda_{\pi_i}) Q_{{}^1_0 (k_{\pi_p}+1^+);k_{\pi_p} {}^0_0}(\Lambda_{\pi_p}) \prod_{i=p+1}^{a} Q_{{}^0_0 k_{\pi_i};k_{\pi_i} {}^0_0}(\Lambda_{\pi_i}) \Bigg) \\
& \times \prod_{j=1}^{q-1} Q_{{}^{m^1}_{m^2} l_{\pi_j};l_{\pi_j} {}^{m^1}_{m^2}}(\Lambda_{\pi_j}) Q_{{}^{m^1}_{m^2} l_{\pi_q};(l_{\pi_q}+1^+) {}^{m^1+1}_{m^2}}(\Lambda_{\pi_q}) \prod_{j=q+1}^{b} Q_{{}^{m^1+1}_{m^2} l_{\pi_j};l_{\pi_j} {}^{m^1+1}_{m^2}}(\Lambda_{\pi_j})
\end{aligned}
\tag{B.1a}
$$

$$
\begin{aligned}
+ \sqrt{n^1+1} & \prod_{i=1}^{p-1} Q_{{}^1_0 k_{\pi_i};k_{\pi_i} {}^1_0}(\Lambda_{\pi_i}) Q_{{}^1_0 (k_{\pi_p}+1^+);k_{\pi_p} {}^0_0}(\Lambda_{\pi_p}) \prod_{i=p+1}^{a} Q_{{}^0_0 k_{\pi_i};k_{\pi_i} {}^0_0}(\Lambda_{\pi_i}) \\
\times \Bigg(& \prod_{j=1}^{q-1} Q_{{}^{m^1}_{m^2} l_{\pi_j};l_{\pi_j} {}^{m^1}_{m^2}}(\Lambda_{\pi_j}) Q_{{}^{m^1}_{m^2} l_{\pi_q};(l_{\pi_q}+1^+) {}^{m^1+1}_{m^2}}(\Lambda_{\pi_q}) \prod_{i=q+1}^{b} Q_{{}^{m^1+1}_{m^2} l_{\pi_j};l_{\pi_j} {}^{m^1+1}_{m^2}}(\Lambda_{\pi_j}) \\
& - \sqrt{(m^1+1)} \prod_{j=1}^{q} Q_{{}^0_0 l_{\pi_j};l_{\pi_j} {}^0_0}(\Lambda_{\pi_j}) Q_{{}^0_0 l_{\pi_q};(l_{\pi_q}+1^+) {}^1_0}(\Lambda_{\pi_q}) \prod_{j=q+1}^{b} Q_{{}^1_0 l_{\pi_j};l_{\pi_j} {}^1_0}(\Lambda_{\pi_j}) \Bigg) \Bigg\}\,,
\end{aligned}
\tag{B.1b}
$$

with $1^+ := {}^1_0$. We further analyse the the first three lines of (B.1a):

$$
\begin{aligned}
& \Bigg(\prod_{i=1}^{p-1} Q_{{}^{n^1+1}_{n^2} k_{\pi_i};k_{\pi_i} {}^{n^1+1}_{n^2}}(\Lambda_{\pi_i}) Q_{{}^{n^1+1}_{n^2} (k_{\pi_p}+1^+);k_{\pi_p} {}^{n^1}_{n^2}}(\Lambda_{\pi_p}) \prod_{i=p+1}^{a} Q_{{}^{n^1}_{n^2} k_{\pi_i};k_{\pi_i} {}^{n^1}_{n^2}}(\Lambda_{\pi_i}) \\
& \qquad - \sqrt{n^1+1} \prod_{i=1}^{p-1} Q_{{}^1_0 k_{\pi_i};k_{\pi_i} {}^1_0}(\Lambda_{\pi_i}) Q_{{}^1_0 (k_{\pi_p}+1^+);k_{\pi_p} {}^0_0}(\Lambda_{\pi_p}) \prod_{i=p+1}^{a} Q_{{}^0_0 k_{\pi_i};k_{\pi_i} {}^0_0}(\Lambda_{\pi_i}) \Bigg) \\
& = \Bigg(\Bigg(\prod_{i=1}^{p-1} Q_{{}^{n^1+1}_{n^2} k_{\pi_i};k_{\pi_i} {}^{n^1+1}_{n^2}}(\Lambda_{\pi_i}) - \prod_{i=1}^{p-1} Q_{{}^1_0 k_{\pi_i};k_{\pi_i} {}^1_0}(\Lambda_{\pi_i}) \Bigg) \\
& \qquad \times Q_{{}^{n^1+1}_{n^2} (k_{\pi_p}+1^+);k_{\pi_p} {}^{n^1}_{n^2}}(\Lambda_{\pi_p}) \prod_{i=p+1}^{a} Q_{{}^{n^1}_{n^2} k_{\pi_i};k_{\pi_i} {}^{n^1}_{n^2}}(\Lambda_{\pi_i}) \Bigg) && \text{(B.2a)} \\
& + \prod_{i=1}^{p-1} Q_{{}^1_0 k_{\pi_i};k_{\pi_i} {}^1_0}(\Lambda_{\pi_i}) \Big(\mathcal{Q}^{(+\frac{1}{2})}_{{}^{n^1+1}_{n^2} (k_{\pi_p}+1^+);k_{\pi_p} {}^{n^1}_{n^2}}(\Lambda_{\pi_p}) \Big) \prod_{i=p+1}^{a} Q_{{}^{n^1}_{n^2} k_{\pi_i};k_{\pi_i} {}^{n^1}_{n^2}}(\Lambda_{\pi_i}) && \text{(B.2b)} \\
& + \sqrt{n^1+1} \Bigg(\prod_{i=1}^{p-1} Q_{{}^1_0 k_{\pi_i};k_{\pi_i} {}^1_0}(\Lambda_{\pi_i}) Q_{{}^1_0 (k_{\pi_p}+1^+);k_{\pi_p} {}^0_0}(\Lambda_{\pi_p}) \\
& \qquad \times \Bigg(\prod_{i=p+1}^{a} Q_{{}^{n^1}_{n^2} k_{\pi_i};k_{\pi_i} {}^{n^1}_{n^2}}(\Lambda_{\pi_i}) - \prod_{i=p+1}^{a} Q_{{}^0_0 k_{\pi_i};k_{\pi_i} {}^0_0}(\Lambda_{\pi_i}) \Bigg) \Bigg)\,. && \text{(B.2c)}
\end{aligned}
$$

According to (3.19), the two lines (B.2a) and (B.2c) yield graphs having one composite propagator (3.17a), whereas the line (B.2b) yields a graph having one composite propagator[13] (3.17c). In total, we get from (B.1) $a+b$ graphs with composite propagator (3.17a) or (3.17c). The treatment of (3.12) is similar.

[13] Note that the estimation (3.24) yields $\sqrt{n^1+1}|Q_{{}^1_0 (k_{\pi_p}+1^+);k_{\pi_p} {}^0_0}(\Lambda_{\pi_p})| \leq \frac{C}{\Omega\theta\Lambda^2}\big(\frac{n^1+1}{\theta\Lambda^2}\big)^{\frac{1}{2}}$. Therefore, the prefactor $\sqrt{n^1+1}$ in (B.2c) combines actually to the ratio $\big(\frac{n^1+1}{\theta\Lambda^2}\big)^{\frac{1}{2}}$ which is required for the (3.32)-term in Proposition 2.3.

Second, we treat that contribution to (3.10) which consists of graphs with constant index along the trajectories:

$$
\begin{aligned}
&\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{{}^{m^1}_{m^2}{}^{n^1}_{n^2};{}^{n^1}_{n^2}{}^{m^1}_{m^2}}}[\Lambda'] \\
&\quad-\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{{}^{0}_{0}{}^{0}_{0};{}^{0}_{0}{}^{0}_{0}}}[\Lambda'] - m^1\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{{}^{1}_{0}{}^{0}_{0};{}^{0}_{0}{}^{1}_{0}}}[\Lambda'] - \Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{{}^{0}_{0}{}^{0}_{0};{}^{0}_{0}{}^{0}_{0}}}[\Lambda']\Big)\\
&\quad-m^2\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{{}^{0}_{1}{}^{0}_{0};{}^{0}_{0}{}^{0}_{1}}}[\Lambda'] - \Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{{}^{0}_{0}{}^{0}_{0};{}^{0}_{0}{}^{0}_{0}}}[\Lambda']\Big) - n^1\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{{}^{0}_{0}{}^{1}_{0};{}^{1}_{0}{}^{0}_{0}}}[\Lambda']\\
&\quad-\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{{}^{0}_{0}{}^{0}_{0};{}^{0}_{0}{}^{0}_{0}}}[\Lambda']\Big) - n^2\Big(\Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{{}^{0}_{0}{}^{0}_{1};{}^{0}_{1}{}^{0}_{0}}}[\Lambda'] - \Lambda'\frac{\partial}{\partial\Lambda'}\widehat{A^{(V)\gamma}_{{}^{0}_{0}{}^{0}_{0};{}^{0}_{0}{}^{0}_{0}}}[\Lambda']\Big)\\
&= \dots\Big\{\prod_{i=1}^{a} Q_{{}^{n^1}_{n^2}k_{\pi_i};k_{\pi_i}{}^{n^1}_{n^2}}(\Lambda_{\pi_i})\prod_{j=1}^{b} Q_{{}^{m^1}_{m^2}l_{\pi_j};l_{\pi_j}{}^{m^1}_{m^2}}(\Lambda_{\pi_j})\\
&\qquad\qquad -\prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\\
&\quad -\prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\Big(m^1\Big(\prod_{j=1}^{b} Q_{{}^{1}_{0}l_{\pi_j};l_{\pi_j}{}^{1}_{0}}(\Lambda_{\pi_j}) - \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big)\\
&\qquad\qquad + m^2\Big(\prod_{j=1}^{b} Q_{{}^{0}_{1}l_{\pi_j};l_{\pi_j}{}^{0}_{1}}(\Lambda_{\pi_j}) - \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big)\Big)\\
&\quad -\Big(n^1\Big(\prod_{i=1}^{a} Q_{{}^{1}_{0}k_{\pi_i};k_{\pi_i}{}^{1}_{0}}(\Lambda_{\pi_i}) - \prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\Big)\\
&\qquad\qquad + n^2\Big(\prod_{i=1}^{a} Q_{{}^{0}_{1}k_{\pi_i};k_{\pi_i}{}^{0}_{1}}(\Lambda_{\pi_i}) - \prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\Big)\Big)\prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big\}\\
&= \dots\Big\{\Big(\Big(\prod_{i=1}^{a} Q_{{}^{n^1}_{n^2}k_{\pi_i};k_{\pi_i}{}^{n^1}_{n^2}}(\Lambda_{\pi_i}) - \prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\Big)\\
&\qquad\qquad \times\Big(\prod_{j=1}^{b} Q_{{}^{m^1}_{m^2}l_{\pi_j};l_{\pi_j}{}^{m^1}_{m^2}}(\Lambda_{\pi_j}) - \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big)\Big) &\text{(B.3a)}\\
&\quad +\prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\Big(\Big(\prod_{j=1}^{b} Q_{{}^{m^1}_{m^2}l_{\pi_j};l_{\pi_j}{}^{m^1}_{m^2}}(\Lambda_{\pi_i}) - \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big)\\
&\qquad\qquad - m^1\Big(\prod_{j=1}^{b} Q_{{}^{1}_{0}l_{\pi_j};l_{\pi_j}{}^{1}_{0}}(\Lambda_{\pi_j}) - \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big)\\
&\qquad\qquad - m^2\Big(\prod_{j=1}^{b} Q_{{}^{0}_{1}l_{\pi_j};l_{\pi_j}{}^{0}_{1}}(\Lambda_{\pi_j}) - \prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big)\Big) &\text{(B.3b)}\\
&\quad +\Big(\Big(\prod_{i=1}^{a} Q_{{}^{n^1}_{n^2}k_{\pi_i};k_{\pi_i}{}^{n^1}_{n^2}}(\Lambda_{\pi_i}) - \prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\Big)\\
&\qquad\qquad - n^1\Big(\prod_{i=1}^{a} Q_{{}^{1}_{0}k_{\pi_i};k_{\pi_i}{}^{1}_{0}}(\Lambda_{\pi_i}) - \prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\Big)\\
&\qquad\qquad - n^2\Big(\prod_{i=1}^{a} Q_{{}^{0}_{1}k_{\pi_i};k_{\pi_i}{}^{0}_{1}}(\Lambda_{\pi_i}) - \prod_{i=1}^{a} Q_{{}^{0}_{0}k_{\pi_i};k_{\pi_i}{}^{0}_{0}}(\Lambda_{\pi_i})\Big)\Big)\prod_{j=1}^{b} Q_{{}^{0}_{0}l_{\pi_j};l_{\pi_j}{}^{0}_{0}}(\Lambda_{\pi_j})\Big\}. &\text{(B.3c)}
\end{aligned}
$$

It is clear from (3.19) that the part corresponding to (B.3a) can be written as a sum of graphs containing (at different trajectories) two composite propagators $\mathcal{Q}^{(0)}_{\substack{n^1\\n^2}k_{\pi_i};k_{\pi_i}\substack{n^1\\n^2}}(\Lambda_{\pi_i})$ and $\mathcal{Q}^{(0)}_{\substack{m^1\\m^2}l_{\pi_j};l_{\pi_j}\substack{m^1\\m^2}}$ of type (3.17a). We further analyse (B.3b):

$$
\begin{aligned}
&\prod_{j=1}^{b} Q_{\substack{m^1\\m^2}l_{\pi_j};l_{\pi_j}\substack{m^1\\m^2}}(\Lambda_{\pi_j}) - \prod_{j=1}^{b} Q_{\substack{0\\0}l_{\pi_j};l_{\pi_j}\substack{0\\0}}(\Lambda_{\pi_j}) - m^1\Big(\prod_{j=1}^{b} Q_{\substack{1\\0}l_{\pi_j};l_{\pi_j}\substack{1\\0}}(\Lambda_{\pi_j}) \\
&\qquad - \prod_{j=1}^{b} Q_{\substack{0\\0}l_{\pi_j};l_{\pi_j}\substack{0\\0}}(\Lambda_{\pi_j})\Big) - m^2\Big(\prod_{j=1}^{b} Q_{\substack{0\\1}l_{\pi_j};l_{\pi_j}\substack{0\\1}}(\Lambda_{\pi_j}) - \prod_{j=1}^{b} Q_{\substack{0\\0}l_{\pi_j};l_{\pi_j}\substack{0\\0}}(\Lambda_{\pi_j})\Big) \\
&= \mathcal{Q}^{(1)}_{\substack{m^1\\m^2}l_{\pi_1};l_{\pi_1}\substack{m^1\\m^2}}(\Lambda_{\pi_1}) \prod_{j=2}^{b} Q_{\substack{0\\0}l_{\pi_j};l_{\pi_j}\substack{0\\0}}(\Lambda_{\pi_j}) \qquad\qquad \text{(B.4a)} \\
&+ \bigg(\mathcal{Q}^{(0)}_{\substack{m^1\\m^2}l_{\pi_1};l_{\pi_1}\substack{m^1\\m^2}}(\Lambda_{\pi_1})\Big(\prod_{j=2}^{b} Q_{\substack{m^1\\m^2}l_{\pi_j};l_{\pi_j}\substack{m^1\\m^2}}(\Lambda_{\pi_j}) - \prod_{j=2}^{b} Q_{\substack{0\\0}l_{\pi_j};l_{\pi_j}\substack{0\\0}}(\Lambda_{\pi_j})\Big) \\
&\qquad - m^1\mathcal{Q}^{(0)}_{\substack{1\\0}l_{\pi_1};l_{\pi_1}\substack{1\\0}}(\Lambda_{\pi_1})\Big(\prod_{j=2}^{b} Q_{\substack{1\\0}l_{\pi_j};l_{\pi_j}\substack{1\\0}}(\Lambda_{\pi_j}) - \prod_{j=2}^{b} Q_{\substack{0\\0}l_{\pi_j};l_{\pi_j}\substack{0\\0}}(\Lambda_{\pi_j})\Big) \\
&\qquad - m^2\mathcal{Q}^{(0)}_{\substack{0\\1}l_{\pi_1};l_{\pi_1}\substack{0\\1}}(\Lambda_{\pi_1})\Big(\prod_{j=2}^{b} Q_{\substack{0\\1}l_{\pi_j};l_{\pi_j}\substack{0\\1}}(\Lambda_{\pi_j}) - \prod_{j=2}^{b} Q_{\substack{0\\0}l_{\pi_j};l_{\pi_j}\substack{0\\0}}(\Lambda_{\pi_j})\Big)\bigg) \qquad \text{(B.4b)} \\
&+ Q_{\substack{0\\0}l_{\pi_1};l_{\pi_1}\substack{0\\0}}(\Lambda_{\pi_1})\bigg(\Big(\prod_{j=2}^{b} Q_{\substack{m^1\\m^2}l_{\pi_j};l_{\pi_j}\substack{m^1\\m^2}}(\Lambda_{\pi_j}) - \prod_{j=2}^{b} Q_{\substack{0\\0}l_{\pi_j};l_{\pi_j}\substack{0\\0}}(\Lambda_{\pi_j})\Big) \\
&\qquad - m^1\Big(\prod_{j=2}^{b} Q_{\substack{1\\0}l_{\pi_j};l_{\pi_j}\substack{1\\0}}(\Lambda_{\pi_j}) - \prod_{j=2}^{b} Q_{\substack{0\\0}l_{\pi_j};l_{\pi_j}\substack{0\\0}}(\Lambda_{\pi_j})\Big) \\
&\qquad - m^2\Big(\prod_{j=2}^{b} Q_{\substack{0\\1}l_{\pi_j};l_{\pi_j}\substack{0\\1}}(\Lambda_{\pi_j}) - \prod_{j=2}^{b} Q_{\substack{0\\0}l_{\pi_j};l_{\pi_j}\substack{0\\0}}(\Lambda_{\pi_j})\Big)\bigg)\,. \qquad\qquad \text{(B.4c)}
\end{aligned}
$$

The part (B.4a) gives rise to graphs with one propagator (3.17b). Due to (3.19) the part (B.4b) yields graphs with two propagators[14] (3.17a) appearing on the same trajectory. Finally, the part (B.4c) has the same structure as the lhs of the equation, now starting with $j = 2$. After iteration we obtain further graphs of the type (B.4a) and (B.4b).

Finally, we look at that contribution to (3.10) which consists of graphs where one index component jumps forward and backward in the n^1-component. We can directly use the decomposition derived in (B.3) regarding, if the n^1-index jumps up,

$$
\begin{aligned}
&\prod_{i=1}^{a} Q_{\substack{n^1\\n^2}k_{\pi_i};k_{\pi_i}\substack{n^1\\n^2}}(\Lambda_{\pi_i}) \\
&\quad\mapsto \prod_{i=1}^{p-1} Q_{\substack{n^1\\n^2}k_{\pi_i};k_{\pi_i}\substack{n^1\\n^2}}(\Lambda_{\pi_i}) Q_{\substack{n^1\\n^2}k_{\pi_p};(k_{\pi_p}+1)\substack{n^1+1\\n^2}}(\Lambda_{\pi_p}) \prod_{i=p+1}^{q-1} Q_{\substack{n^1+1\\n^2}k_{\pi_i};k_{\pi_i}\substack{n^1+1\\n^2}}(\Lambda_{\pi_i}) \\
&\qquad\quad \times Q_{\substack{n^1+1\\n^2}(k_{\pi_q}+1^+);k_{\pi_q}\substack{n^1\\n^2}}(\Lambda_{\pi_q}) \prod_{i=q+1}^{a} Q_{\substack{n^1\\n^2}k_{\pi_i};k_{\pi_i}\substack{n^1\\n^2}}(\Lambda_{\pi_i})\,. \qquad\qquad \text{(B.5)}
\end{aligned}
$$

---

[14] Note that the product $m^1\mathcal{Q}^{(0)}_{\substack{1\\0}l_{\pi_1};l_{\pi_1}\substack{1\\0}}(\Lambda_{\pi_1})$ is according to (3.28) bounded by $\frac{C}{\Omega\theta\Lambda^2}\big(\frac{m^1}{\theta\Lambda^2}\big)$. This means that the prefactors m^1, m^2 in (B.4b) combine actually to the ratio $\frac{m^r}{\theta\Lambda^2}$ which is required for the (3.32)-term in Proposition 2.2.

This requires to process (B.3) slightly differently. The two parts (B.3a) and (B.3b) need no further discussion, as they lead to graphs having a composite propagator (3.17a) on the m-trajectory. We write (B.3c) as follows:

$$
\begin{aligned}
\text{(B.3c)} = \Big(\Big(&\prod_{i=1}^{a} Q_{\genfrac{}{}{0pt}{}{n^1}{n^2} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{n^1}{n^2}}(\Lambda_{\pi_i}) - (n^1{+}1)\prod_{i=1}^{a} Q_{\genfrac{}{}{0pt}{}{0}{0} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{0}{0}}(\Lambda_{\pi_i})\Big) && \text{(B.6a)}\\
&- n^1\Big(\prod_{i=1}^{a} Q_{\genfrac{}{}{0pt}{}{1}{0} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{1}{0}}(\Lambda_{\pi_i}) - 2\prod_{i=1}^{a} Q_{\genfrac{}{}{0pt}{}{0}{0} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{0}{0}}(\Lambda_{\pi_i})\Big) && \text{(B.6b)}\\
&- n^2\Big(\prod_{i=1}^{a} Q_{\genfrac{}{}{0pt}{}{0}{1} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{0}{1}}(\Lambda_{\pi_i}) - \prod_{i=1}^{a} Q_{\genfrac{}{}{0pt}{}{0}{0} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{0}{0}}(\Lambda_{\pi_i})\Big)\Big) \prod_{j=1}^{b} Q_{\genfrac{}{}{0pt}{}{0}{0} l_{\pi_j}; l_{\pi_j} \genfrac{}{}{0pt}{}{0}{0}}(\Lambda_{\pi_j})\;. && \text{(B.6c)}
\end{aligned}
$$

The part (B.6c) leads according to (3.19) and (B.5) to graphs either with composite propagators (3.17a) or with propagators

$$
Q_{\genfrac{}{}{0pt}{}{1}{1} \genfrac{}{}{0pt}{}{l^1+1}{l^2}; \genfrac{}{}{0pt}{}{l^1}{l^2} \genfrac{}{}{0pt}{}{0}{1}} - Q_{\genfrac{}{}{0pt}{}{1}{0} \genfrac{}{}{0pt}{}{l^1+1}{l^2}; \genfrac{}{}{0pt}{}{l^1}{l^2} \genfrac{}{}{0pt}{}{0}{0}}\;. \tag{B.7}
$$

Inserting (B.5) into (B.6a) we have

$$
\begin{aligned}
&\Big(\prod_{i=1}^{a} Q_{\genfrac{}{}{0pt}{}{n^1}{n^2} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{n^1}{n^2}}(\Lambda_{\pi_i}) - (n^1{+}1)\prod_{i=1}^{a} Q_{\genfrac{}{}{0pt}{}{0}{0} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{0}{0}}(\Lambda_{\pi_i})\Big)\\
&\stackrel{(B.5)}{\longmapsto} \Big(\prod_{i=1}^{p-1} Q_{\genfrac{}{}{0pt}{}{n^1}{n^2} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{n^1}{n^2}}(\Lambda_{\pi_i}) - \prod_{i=1}^{p-1} Q_{\genfrac{}{}{0pt}{}{0}{0} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{0}{0}}(\Lambda_{\pi_i})\Big) Q_{\genfrac{}{}{0pt}{}{n^1}{n^2} k_{\pi_p}; (k_{\pi_p}+1) \genfrac{}{}{0pt}{}{n^1+1}{n^2}}(\Lambda_{\pi_p})\\
&\qquad \times \prod_{i=p+1}^{q-1} Q_{\genfrac{}{}{0pt}{}{n^1+1}{n^2} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{n^1+1}{n^2}}(\Lambda_{\pi_i}) Q_{\genfrac{}{}{0pt}{}{n^1+1}{n^2} (k_{\pi_q}+1^+); k_{\pi_q} \genfrac{}{}{0pt}{}{n^1}{n^2}}(\Lambda_{\pi_q}) \prod_{i=q+1}^{a} Q_{\genfrac{}{}{0pt}{}{n^1}{n^2} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{n^1}{n^2}}(\Lambda_{\pi_i}) \qquad \text{(B.8a)}\\
&+ \prod_{i=1}^{p-1} Q_{\genfrac{}{}{0pt}{}{0}{0} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{0}{0}}(\Lambda_{\pi_i}) \mathcal{Q}^{(+\frac{1}{2})}_{\genfrac{}{}{0pt}{}{n^1}{n^2} k_{\pi_p}; (k_{\pi_p}+1) \genfrac{}{}{0pt}{}{n^1+1}{n^2}}(\Lambda_{\pi_p}) \prod_{i=p+1}^{q-1} Q_{\genfrac{}{}{0pt}{}{n^1+1}{n^2} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{n^1+1}{n^2}}(\Lambda_{\pi_i})\\
&\qquad \times Q_{\genfrac{}{}{0pt}{}{n^1+1}{n^2} (k_{\pi_q}+1^+); k_{\pi_q} \genfrac{}{}{0pt}{}{n^1}{n^2}}(\Lambda_{\pi_q}) \prod_{i=q+1}^{a} Q_{\genfrac{}{}{0pt}{}{n^1}{n^2} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{n^1}{n^2}}(\Lambda_{\pi_i}) \qquad \text{(B.8b)}\\
&+ \sqrt{n^1{+}1} \prod_{i=1}^{p-1} Q_{\genfrac{}{}{0pt}{}{0}{0} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{0}{0}}(\Lambda_{\pi_i}) Q_{\genfrac{}{}{0pt}{}{0}{0} k_{\pi_p}; (k_{\pi_p}+1) \genfrac{}{}{0pt}{}{1}{0}}(\Lambda_{\pi_p})\\
&\qquad \Big(\Big(\prod_{i=p+1}^{q-1} Q_{\genfrac{}{}{0pt}{}{n^1+1}{n^2} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{n^1+1}{n^2}}(\Lambda_{\pi_i}) - \prod_{i=p+1}^{q-1} Q_{\genfrac{}{}{0pt}{}{0}{0} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{0}{0}}(\Lambda_{\pi_i})\Big)\\
&\qquad\quad - \Big(\prod_{i=p+1}^{q-1} Q_{\genfrac{}{}{0pt}{}{1}{0} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{1}{0}}(\Lambda_{\pi_i}) - \prod_{i=p+1}^{q-1} Q_{\genfrac{}{}{0pt}{}{0}{0} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{0}{0}}(\Lambda_{\pi_i})\Big)\Big)\\
&\qquad\quad \times Q_{\genfrac{}{}{0pt}{}{n^1+1}{n^2} (k_{\pi_q}+1^+); k_{\pi_q} \genfrac{}{}{0pt}{}{n^1}{n^2}}(\Lambda_{\pi_q}) \prod_{i=q+1}^{a} Q_{\genfrac{}{}{0pt}{}{n^1}{n^2} k_{\pi_i}; k_{\pi_i} \genfrac{}{}{0pt}{}{n^1}{n^2}}(\Lambda_{\pi_i}) \qquad \text{(B.8c)}
\end{aligned}
$$

$$+\sqrt{n^1{+}1}\prod_{i=1}^{p-1}Q_{{}^0_0k_{\pi_i};k_{\pi_i}{}^0_0}(\Lambda_{\pi_i})Q_{{}^0_0k_{\pi_p};(k_{\pi_p}+1){}^1_0}(\Lambda_{\pi_p})\prod_{i=p+1}^{q-1}Q_{{}^1_0k_{\pi_i};k_{\pi_i}{}^1_0}(\Lambda_{\pi_i})$$

$$\times\,\mathcal{Q}^{(+\frac{1}{2})}_{{}^{n^1}_{n^2}(k_{\pi_q}+1^+);k_{\pi_q}{}^{n^1}_{n^2}}(\Lambda_{\pi_q})\prod_{i=q+1}^{a}Q_{{}^{n^1}_{n^2}k_{\pi_i};k_{\pi_i}{}^{n^1}_{n^2}}(\Lambda_{\pi_i}) \qquad \text{(B.8d)}$$

$$+\,(n^1{+}1)\prod_{i=1}^{p-1}Q_{{}^0_0k_{\pi_i};k_{\pi_i}{}^0_0}(\Lambda_{\pi_i})Q_{{}^0_0k_{\pi_p};(k_{\pi_p}+1){}^1_0}(\Lambda_{\pi_p})$$

$$\times\prod_{i=p+1}^{q-1}Q_{{}^1_0k_{\pi_i};k_{\pi_i}{}^1_0}(\Lambda_{\pi_i})\,Q_{{}^1_0(k_{\pi_q}+1^+);k_{\pi_q}{}^0_0}(\Lambda_{\pi_q})$$

$$\times\Big(\prod_{i=q+1}^{a}Q_{{}^{n^1}_{n^2}k_{\pi_i};k_{\pi_i}{}^{n^1}_{n^2}}(\Lambda_{\pi_i})-\prod_{i=q+1}^{a}Q_{{}^0_0k_{\pi_i};k_{\pi_i}{}^0_0}(\Lambda_{\pi_i})\Big)\,. \qquad \text{(B.8e)}$$

Thus, we obtain (recall also (3.19)) a linear combination of graphs either with composite propagator (3.17a) or with composite propagator (3.17c). In power-counting estimations, the prefactors $\sqrt{n^1{+}1}$ combine according to footnote 13 to the required ratio with the scale $\theta\Lambda^2$. The part (B.6b) is nothing but (B.6a) with $n^1 = 1$ and $n^2 = 0$.

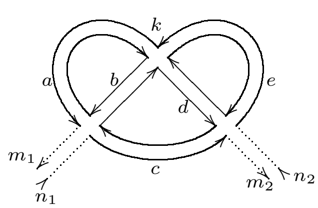
If the index jumps down from n^1 to $n^1 - 1$, then the graph with $n^1 = 0$ does not exist. There is no change of the discussion of (B.3a) and (B.3b), but now (B.3c) becomes

$$\text{(B.3c)} = \Big(\prod_{i=1}^{a}Q_{{}^{n^1}_{n^2}k_{\pi_i};k_{\pi_i}{}^{n^1}_{n^2}}(\Lambda_{\pi_i}) - n^1\prod_{i=1}^{a}Q_{{}^1_0k_{\pi_i};k_{\pi_i}{}^1_0}(\Lambda_{\pi_j})\Big)\prod_{j=1}^{b}Q_{{}^0_0l_{\pi_j};l_{\pi_j}{}^0_0}(\Lambda_{\pi_j})\,. \qquad \text{(B.9)}$$

Using the same steps as in (B.8) we obtain the desired representation through graphs either with composite propagator (3.17a) or with composite propagator (3.17c).

We show in Appendix B.2 how the decomposition works in a concrete example.

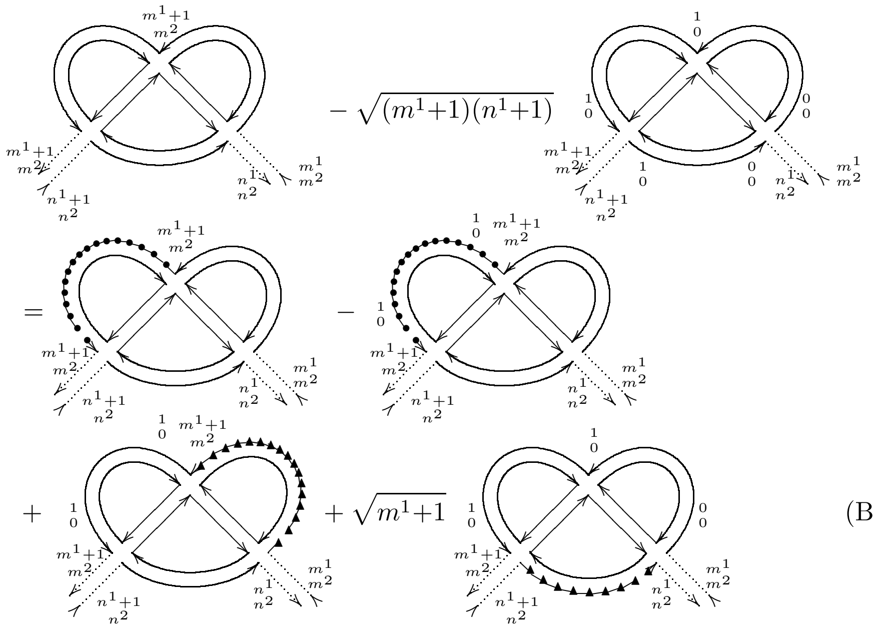
B.2. Example of a difference operation for ribbon graphs. To make the considerations in Section 3.3 and Appendix B.1 about differences of graphs and composite propagators understandable, we look at the following example of a planar two-leg graph:



(B.10)

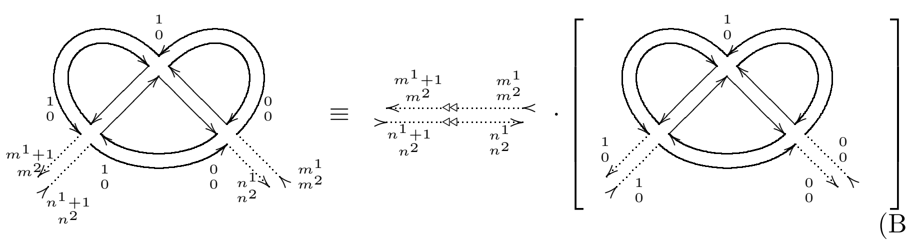
According to Proposition 2 it depends on the indices m_1, n_1, m_2, n_2, k whether this graph is irrelevant, marginal, or relevant. It depends on the history of contraction of subgraphs whether there are marginal subgraphs or not.

Let us consider $m_1 = k = \genfrac{}{}{0pt}{}{m^1+1}{m^2}$, $n_2 = \genfrac{}{}{0pt}{}{m^1}{m^2}$, $n_1 = \genfrac{}{}{0pt}{}{n^1+1}{n^2}$ and $m_2 = \genfrac{}{}{0pt}{}{n^1}{n^2}$ and the history *a*-*c*-*d*-*e*-*b* of contraction. Then, all resulting subgraphs are irrelevant and the total graph is marginal, which leads us to consider the following difference of graphs:



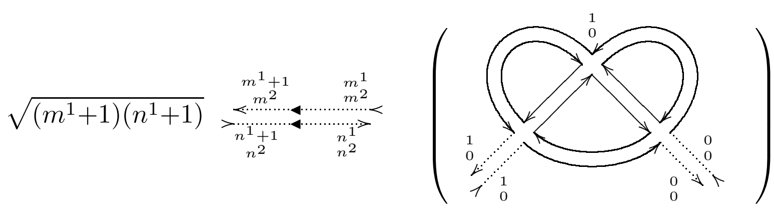
(B.11)

It is important to understand that according to (3.11) the indices at the external lines of the reference graph (with zero-indices) are adjusted to the external indices of the original (leftmost) graph:



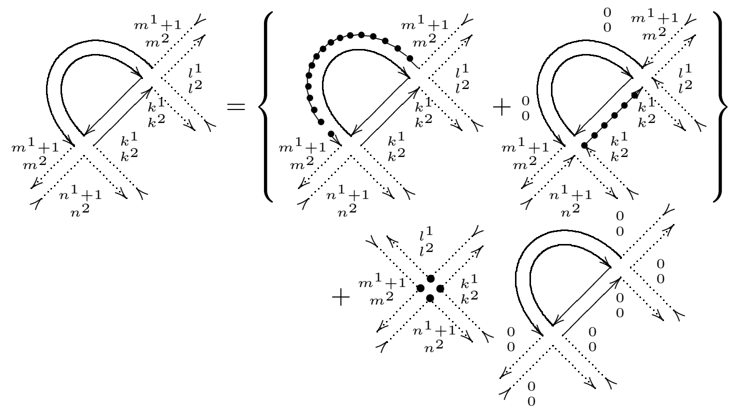
(B.12)

Thus, all graphs with composite propagators have the same index structure at the external legs. When further contracting these graphs, the contracting propagator matches the external indices of the original graph. The argumentation in the proof of Proposition 2.3 should be transparent now. In particular, it becomes understandable why the difference (B.11) is irrelevant and can be integrated from Λ_0 down to Λ. On the other hand, the reference graph to be integrated from Λ_R up to Λ becomes



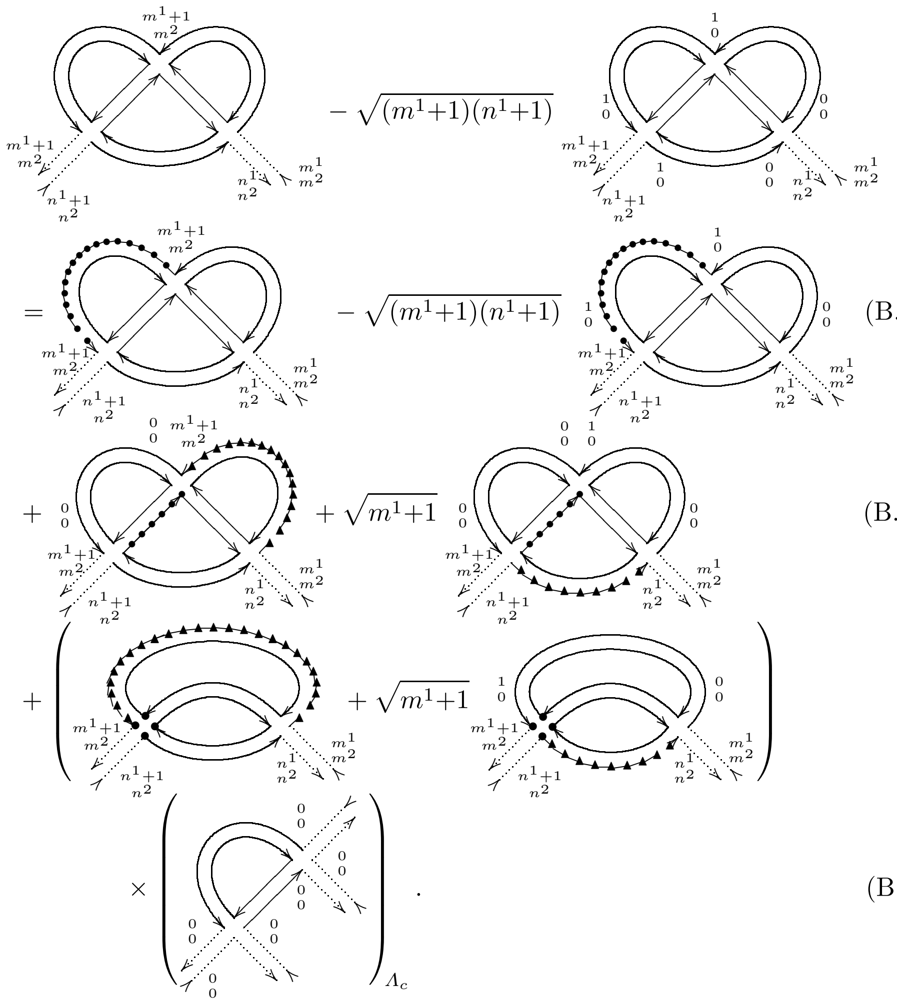
(B.13)

We cannot use the same procedure for the history *a*-*b*-*c*-*d*-*e* of contractions in (B.10), because we end up with a marginal subgraph after the *a*-*b* contractions. According to Definition 1.1 we have to decompose the *a*-*b* subgraph into an irrelevant (according to Proposition 2.1) difference and a marginal reference graph:



(B.14)

The two graphs in braces { } are irrelevant and integrated from Λ_0 down to Λ_c. The remaining piece can be written as the original ϕ^4-vertex times a graph with vanishing external indices, which is integrated from Λ_R up to Λ_c and can be bounded by $C \ln \frac{\Lambda_c}{\Lambda_R}$. Inserting the decomposition (B.14) into (B.10) we obtain the following decomposition valid for the history *a*-*b*-*c*-*d*-*e*:



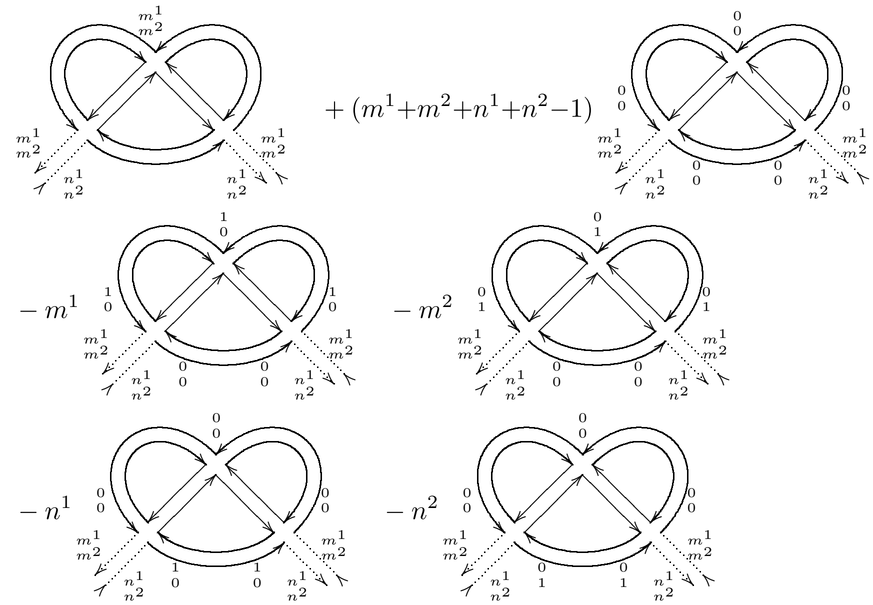
(B.15a)

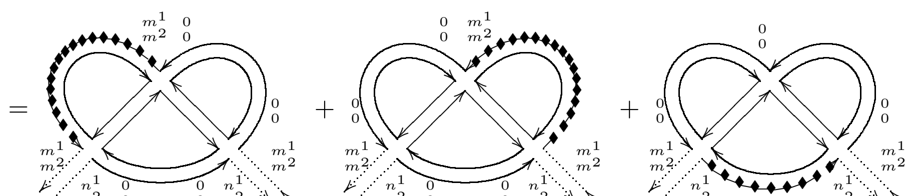
(B.15b)

(B.15c)

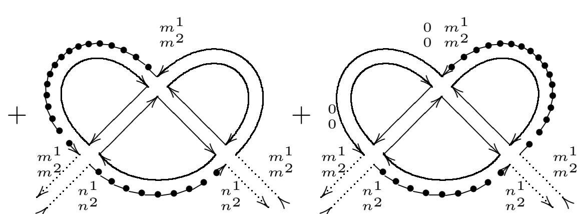
The line (B.15a) corresponds to the first graph in the braces { } of (B.14) for both graphs on the lhs of (B.15). These graphs are already irrelevant[15] so that no further decomposition is necessary. The second graph in the braces { } of (B.14), inserted into the lhs of (B.15), yields the line (B.15b). Finally, the last part of (B.14) leads to the line (B.15c).

Let us also look at the relevant contribution $m_1 = k = n_2 = {}^{m^1}_{m^2}$, $n_1 = m_2 = {}^{n^1}_{n^2}$ of the graph (B.10). The history a-c-d-e-b contains irrelevant subgraphs only:

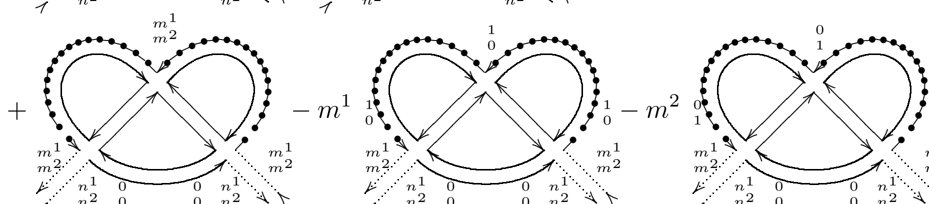




(B.16a)



(B.16b)



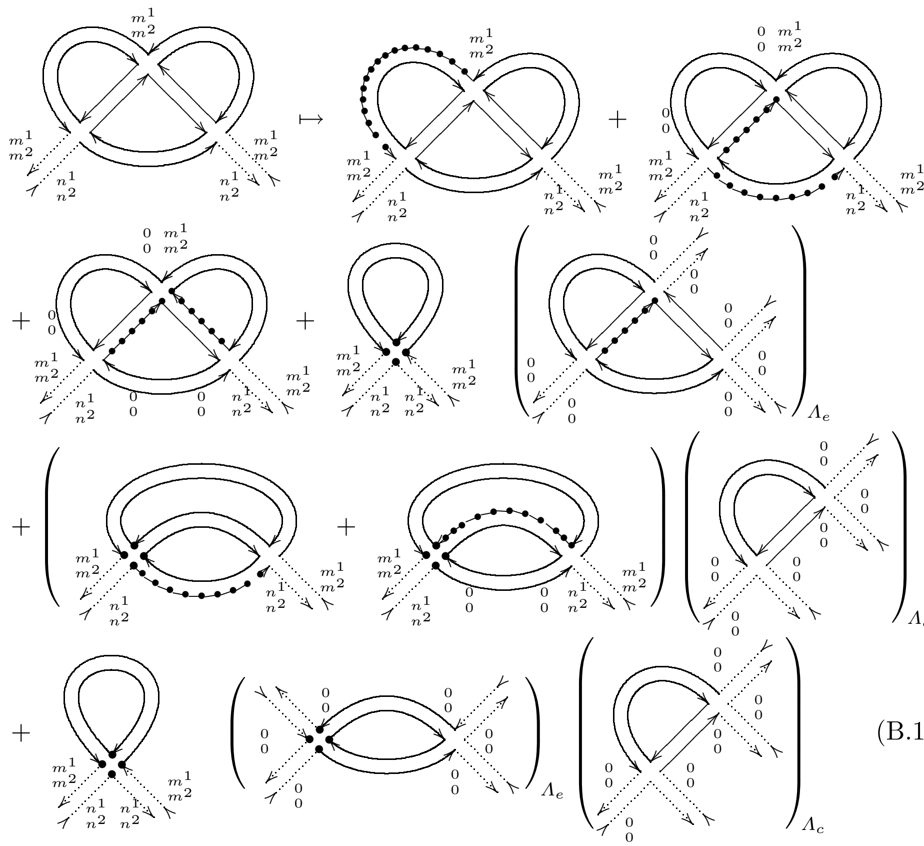
(B.16c)

---

[15] In the right graph (B.15a) the composite propagator is according to (3.28) bounded by $\frac{C}{\Omega\theta\Lambda^2}\frac{1}{\theta\Lambda^2}$ so that the combination with the prefactor $\sqrt{(m^1{+}1)(n^1{+}1)}$ leads to the ratio $\sqrt{(\frac{m^1+1}{\theta\Lambda^2})(\frac{m^1+1}{\theta\Lambda^2})}$ by which (B.15a) is suppressed over the first graph on the lhs of (B.15).

The line (B.16a) corresponds to (B.4a), the line (B.16b) to (B.3a) and the line (B.16c) to (B.4b).

If the history of contractions contains relevant or marginal subgraphs, we first have to decompose the subgraphs into the reference function with vanishing external indices and an irrelevant remainder. For instance, the decomposition relative to the history *a*-*b*-*c*-*d*-*e* would be



(B.17)

C. Asymptotic behaviour of the propagator

For the power-counting theorem we need asymptotic formulae about the scaling behaviour of the cut-off propagator $\Delta^K_{nm;lk}$ and certain index summations. We shall restrict ourselves to the case $\theta_1 = \theta_2 = \theta$ and deduce these formulae from the numerical evaluation of the propagator for a representative class of parameters and special choices of the parameters where we can compute the propagator exactly. These formulae involve the cut-off propagator

$$\Delta^{\mathcal{C}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}} := \begin{cases} \Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}} & \text{for } \mathcal{C} \leq \max(m^1,m^2,n^1,n^2,k^1,k^2,l^1,l^2) \leq 2\mathcal{C}\;, \\ 0 & \text{otherwise}\;, \end{cases} \tag{C.1}$$

which is the restriction of $\Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}$ to the support of the cut-off propagator $\Lambda\frac{\partial}{\partial\Lambda}\Delta^K_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(\Lambda)$ appearing in the Polchinski equation, with $\mathcal{C} = \theta\Lambda^2$.

Formula 1:

$$\max_{m^r,n^r,k^r,l^r} \Big|\Delta^{\mathcal{C}}_{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{k^1 \\ k^2}\substack{l^1 \\ l^2}}\Big|_{\mu_0=0} \approx \frac{\theta\ \delta_{m+k,n+l}}{\sqrt{\frac{1}{\pi}(16\,\mathcal{C}{+}12)} + \frac{6\Omega}{1+2\Omega^3+2\Omega^4}\mathcal{C}} \tag{C.2}$$

We demonstrate in Figure 2 for selected values of the parameters $\Omega, \mathcal{C}$ that $\theta/(\max \Delta^{\mathcal{C}}_{mn;kl})$ at $\mu_0 = 0$ is asymptotically reproduced by $\sqrt{\frac{1}{\pi}(16\,\mathcal{C}{+}12)} + \frac{6\Omega}{1+2\Omega^3+2\Omega^4}\mathcal{C}$.

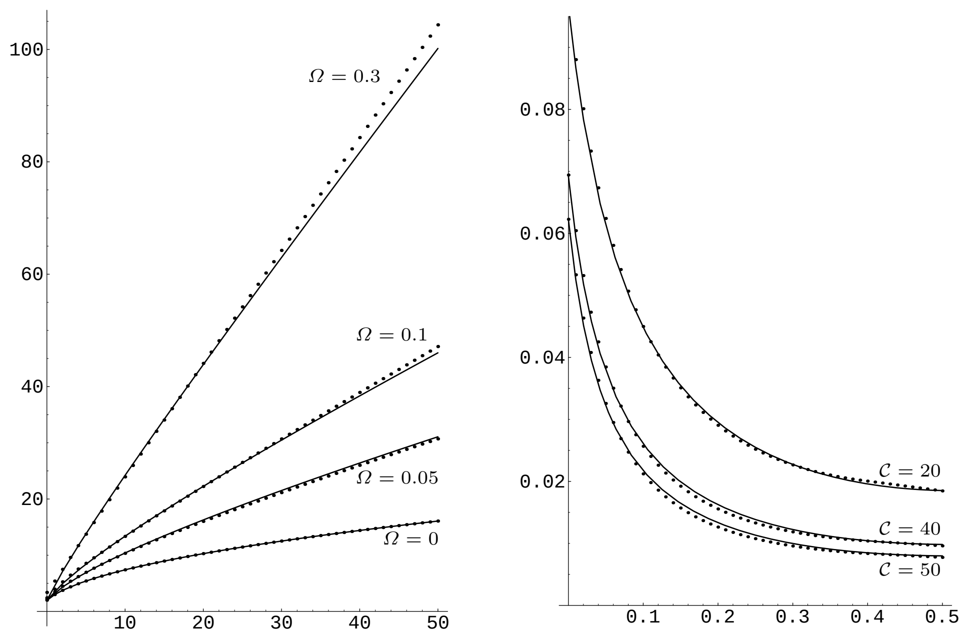


Fig. 2. Comparison of $\max \Delta^{\mathcal{C}}_{mn;kl}/\theta$ at $\mu_0 = 0$ (dots) with $\big(\sqrt{\frac{1}{\pi}(16\,\mathcal{C}{+}12)} + \frac{6\Omega}{1+2\Omega^3+2\Omega^4}\mathcal{C}\big)^{-1}$ (solid line). The left plot shows the inverses of both the propagator and its approximation over $\mathcal{C}$ for various values of Ω. The right plot shows the propagator and its approximation over Ω for various values of $\mathcal{C}$.

Formula 2:

$$\max_{m^r} \sum_{l^1,l^2\in\mathbb{N}} \max_{k^r,n^r} \Big|\Delta^{\mathcal{C}}_{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{k^1 \\ k^2}\substack{l^1 \\ l^2}}\Big|_{\mu_0=0} \approx \frac{\theta\,(1{+}2\Omega^3)}{7\Omega^2(\mathcal{C}+1)}\ . \tag{C.3}$$

We demonstrate in Figure 3 that $\theta/\big(\max_m \sum_l \max_{n,k} |\Delta^{\mathcal{C}}_{mn;kl}|\big)$ is for $\mu_0 = 0$ asymptotically given by $7\Omega^2(\mathcal{C}+1)/(1{+}2\Omega^3)$.

Formula 3:

$$\sum_{\substack{l^1,l^2\in\mathbb{N} \\ \|m-l\|_1\geq 5}} \max_{k^r,n^r} \Big|\Delta^{\mathcal{C}}_{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{k^1 \\ k^2}\substack{l^1 \\ l^2}}\Big|_{\mu_0=0} \leq \frac{\theta\,(1{-}\Omega)^4\big(15 + \frac{4}{5}\|m\|_\infty + \frac{1}{25}\|m\|_\infty^2\big)}{\Omega^2(\mathcal{C}{+}1)^3}\ . \tag{C.4}$$

We verify (C.4) for several choices of the parameters in Figures 4, 5 and 6.

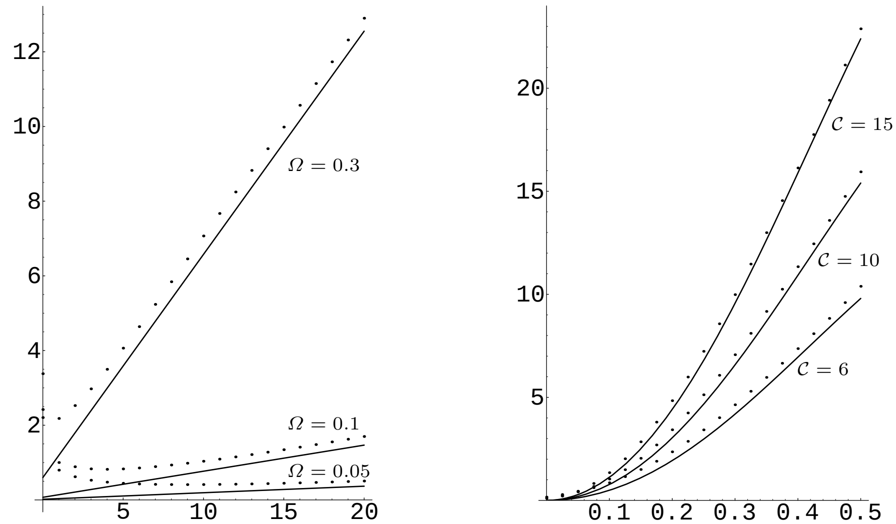


Fig. 3. Comparison of $\theta/\big(\max_m \sum_l \max_{n,k} |\Delta^{\mathcal{C}}_{mn;kl}|\big)$ at $\mu_0 = 0$ (dots) with $7\Omega^2(\mathcal{C}{+}1)/(1{+}2\Omega^2)$ (solid line). The left plot shows the inverse propagator and its approximation over $\mathcal{C}$ for three values of Ω, whereas the right plot shows the inverse propagator and its approximation over Ω for three values of $\mathcal{C}$.

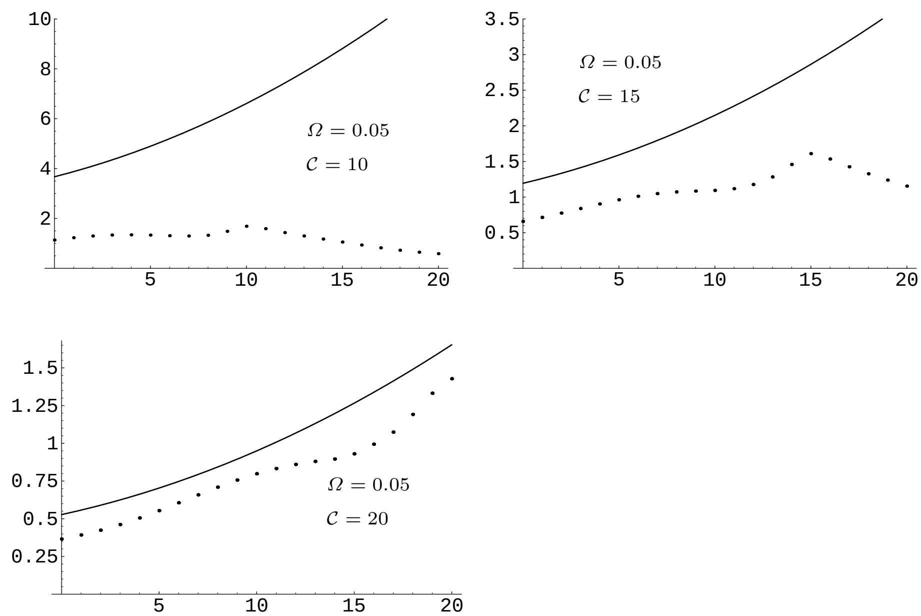


Fig. 4. The index summation $\dfrac{1}{\theta}\Big(\displaystyle\sum_{l\;,\;\|m-l\|_1\geq 5} \max_{k,r} \big|\Delta^{\mathcal{C}}_{mn;kl}\big|\Big)$ of the cut-off propagator at $\mu_0 = 0$ (dots) compared with $\frac{\theta\,(1-\Omega)^4\big(15+\frac{4}{5}\|m\|_\infty+\frac{1}{25}\|m\|^2_\infty\big)}{\Omega^2(\mathcal{C}+1)^3}$ (solid line), both plotted over $\|m\|_\infty$.

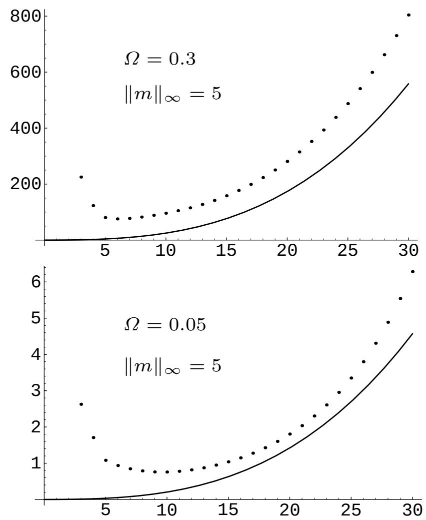


Fig. 5. The inverse $\theta\Big(\sum_{l\,,\ \|m-l\|_1\geq 5} \max_{k,r} \big|\Delta^{\mathcal{C}}_{mn;kl}\big|\Big)^{-1}$ of the summed propagator at $\mu_0 = 0$ (dots) compared with $\frac{\Omega^2(\mathcal{C}+1)^3}{(1-\Omega)^4\big(15+\frac{4}{5}\|m\|_\infty+\frac{1}{25}\|m\|^2_\infty\big)}$ (solid line), both plotted over $\mathcal{C}$.

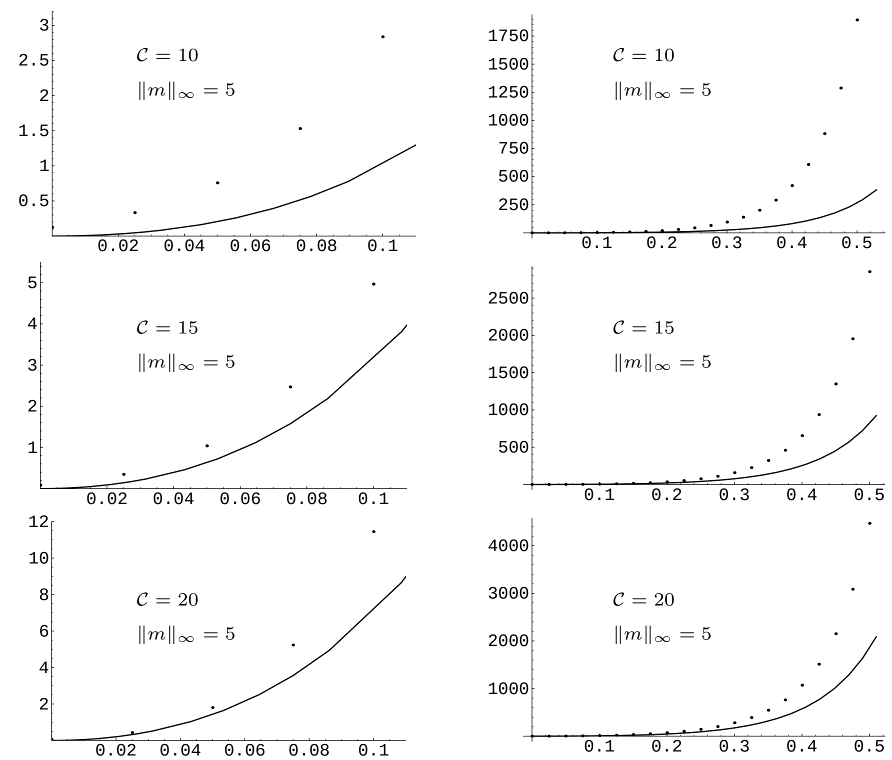


Fig. 6. The inverse $\theta\Big(\sum_{l\,,\ \|m-l\|_1\geq 5} \max_{k,r} \big|\Delta^{\mathcal{C}}_{mn;kl}\big|\Big)^{-1}$ of the summed propagator at $\mu_0 = 0$ (dots) compared with $\frac{\Omega^2(\mathcal{C}+1)^3}{(1-\Omega)^4\big(15+\frac{4}{5}\|m\|_\infty+\frac{1}{25}\|m\|^2_\infty\big)}$ (solid line), both plotted over Ω.

References

1. S. Minwalla, M. Van Raamsdonk and N. Seiberg, "Noncommutative perturbative dynamics," JHEP **0002** (2000) 020 [arXiv:hep-th/9912072].
2. I. Chepelev and R. Roiban, "Renormalization of quantum field theories on noncommutative $\mathbb{R}^d$. I: Scalars," JHEP **0005** (2000) 037 [arXiv:hep-th/9911098].
3. I. Chepelev and R. Roiban, "Convergence theorem for non-commutative Feynman graphs and renormalization," JHEP **0103** (2001) 001 [arXiv:hep-th/0008090].
4. E. Langmann and R. J. Szabo, "Duality in scalar field theory on noncommutative phase spaces," Phys. Lett. B **533** (2002) 168 [arXiv:hep-th/0202039].
5. V. Gayral, J. M. Gracia-Bondía, B. Iochum, T. Schücker and J. C. Várilly, "Moyal planes are spectral triples," Commun. Math. Phys. **246** (2004) 569 [arXiv:hep-th/0307241] .
6. E. Langmann, "Interacting fermions on noncommutative spaces: Exactly solvable quantum field theories in 2n+1 dimensions," Nucl. Phys. B **654** (2003) 404 [arXiv:hep-th/0205287].
7. E. Langmann, R. J. Szabo and K. Zarembo, "Exact solution of noncommutative field theory in background magnetic fields," Phys. Lett. B **569** (2003) 95 [arXiv:hep-th/0303082].
8. E. Langmann, R. J. Szabo and K. Zarembo, "Exact solution of quantum field theory on noncommutative phase spaces," JHEP **0401** (2004) 017 [arXiv:hep-th/0308043].
9. K. G. Wilson and J. B. Kogut, "The Renormalization Group And The Epsilon Expansion," Phys. Rept. **12** (1974) 75.
10. J. Polchinski, "Renormalization And Effective Lagrangians," Nucl. Phys. B **231** (1984) 269.
11. G. Keller, C. Kopper and M. Salmhofer, "Perturbative renormalization and effective Lagrangians in ϕ^4_4," Helv. Phys. Acta **65** (1992) 32.
12. H. Grosse and R. Wulkenhaar, "Power-counting theorem for non-local matrix models and renormalisation," arXiv:hep-th/0305066, to appear in Commun. Math. Phys.
13. J. Meixner, "Orthogonale Polynomsysteme mit einer besonderen Gestalt der erzeugenden Funktion," J. London Math. Soc. **9** (1934) 6.
14. H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4 theory on noncommutative $\mathbb{R}^2$ in the matrix base," JHEP **0312** (2003) 019 [arXiv:hep-th/0307017].
15. D. R. Masson and J. Repka, "Spectral theory of Jacobi matrices in $\ell^2(\mathbb{Z})$ and the $\mathfrak{su}(1,1)$ Lie algebra, SIAM J. Math. Anal. **22** (1991) 1131.
16. R. Koekoek and R. F. Swarttouw, "The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue," arXiv:math.CA/9602214.
17. V. Rivasseau, F. Vignes-Tourneret and R. Wulkenhaar, work in progress.
18. V. Rivasseau and F. Vignes-Tourneret, "Non-Commutative Renormalization," arXiv:hep-th/0409312.
19. H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4 theory on noncommutative $\mathbb{R}^4$ to all orders," arXiv:hep-th/0403232, to appear in Lett. Math. Phys.
20. J. M. Gracia-Bondía and J. C. Várilly, "Algebras Of Distributions Suitable For Phase Space Quantum Mechanics. 1," J. Math. Phys. **29** (1988) 869.
21. J. P. M. Luminet, J. Weeks, A. Riazuelo, R. Lehoucq and J. P. Uzan, "Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background," Nature **425** (2003) 593 [arXiv:astro-ph/0310253].
22. H. Grosse & R. Wulkenhaar, "The β-function in duality-covariant noncommutative ϕ^4-theory," Eur. Phys. J. C **35** (2004) 277–282 [hep-th/0402093].
23. N. Seiberg and E. Witten, "String theory and noncommutative geometry," JHEP **9909** (1999) 032 [arXiv:hep-th/9908142].
24. M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, "A new maximally supersymmetric background of IIB superstring theory," JHEP **0201**, 047 (2002) [arXiv:hep-th/0110242].
25. I. S. Gradshteyn and I. M. Ryzhik, "Tables of Series, Produces, and Integrals. Sixth Edition," Academic Press, San Diego (2000).