Spectral geometry of the Moyal plane with harmonic propagation

Victor Gayral[1] and Raimar Wulkenhaar[2]

[1]*Laboratoire de Mathématiques de l'Université de Reims and Laboratoire de Mathématiques et Applications de l'Université de Metz, France*

[2]*Mathematisches Institut der Westfälischen Wilhelms-Universität, Münster, Germany*

Abstract

We construct a 'non-unital spectral triple of finite volume' out of the Moyal product and a differential square root of the harmonic oscillator Hamiltonian. We find that the spectral dimension of this triple is d but the KO-dimension is $2d$. We add another Connes-Lott copy and compute the spectral action of the corresponding $U(1)$-Yang-Mills-Higgs model. We find that the 'covariant coordinate' involving the gauge field combines with the Higgs field to a unified potential, yielding a deep unification of discrete and continuous parts of the geometry.

1 Introduction

Unlike the compact (unital) case [7] and until now, there is for complete non-compact Riemannian spin manifolds no proper *reconstruction theorem from a spectral point of view*. Thus the question of the defining 'axioms' for non-unital spectral triples is not yet fully answered. However, the basic and most important ideas of modifications for the locally compact case are clear and appeared already in Connes' founding paper [5]. The case of the

---

[1]victor.gayral@univ-reims.fr

[2]raimar@math.uni-muenster.de

1

ordinary Dirac operator of a locally compact complete Riemannian spin manifold manifests that one cannot assume the resolvent of the Dirac type operator underlying a locally compact (non-unital) spectral triple to be a compact operator. The natural replacement is to ask that the 'localized resolvent', i.e. the resolvent multiplied with an element of the algebra, is a compact operator. Another issue, explained in depth in [12], is the choice of *a unitization* of the algebra. This choice is constrained by the orientability condition, which in the unital case (and with an integral metric dimension) is the question of the existence of an Hochschild cycle defining a volume form through the noncommutative integral given by the Dixmier trace. Again, commutative but locally compact examples show that this has to be a Hochschild cycle on a specific unitization of the algebra we start with, but not on the algebra itself. With these two main modifications (compactness of the localized resolvent and existence of a preferred unitization), most of the conditions for a non-unital spectral triple are easy to spell out. Only the Poincaré duality remains unclear to formulate. To help the reader with this discussion, in the appendix we have reproduced the modified conditions for non-unital spectral triples, as in given in [12] with the only modification that the metric and KO-dimensions do not have to coincide, according to the recent formulation of the standard model [6] and the Podleś quantum sphere [9]. Note that these conditions are not far away from those given in [26]. However, in [12, 26] there is an extra assumption of existence of a system of local (or quasi-local) units, akin to the local structure of a non-compact manifold. These locality assumptions have been fully removed in a more recent joint work of one of us [2]. However, in that work, the focus is on the index theoretical side of the notion of a spectral triple, not on the noncommutative generalization of a spin manifold. The definition for a non-unital spectral triple given in [2] is the minimal one, ensuring a well-posed Fredholm index problem with a numerical index computable by means of a local representative of the Chern character in cyclic cohomology.

The present article is devoted to the study of a situation somehow in between the compact (unital) and non-compact (non-unital) setting. Indeed, our Dirac-type operator has compact resolvent alone, but it does not reflect the metric dimension. It is only the localized resolvent which exhibits the correct metric dimension. We term this weird situation as 'non-unital spectral triple of finite volume'. This has, at least, one very nice feature, namely that the spectral action can be defined and computed in the usual way. The main motivation for this example comes from noncommutative quantum field theory.

Because of easy computability, quantum field theory on the Moyal plane is the most-studied toy model for noncommutative quantum field theories. The ultra-violet/infra-red mixing problems arising in these models have been solved by one of us in [18, 19] by the introduction of a modified propagator associated with the harmonic oscillator Hamiltonian. See also [23, 24, 29] for different renormalization proofs. From a physics point of view, the most fascinating property of this model is the behavior of its β-function [10, 11, 17, 20], which makes it a candidate for non-perturbatively renormalizable quantum field theory in dimension four [22, 25, 28]. We recommend [27] for review and introduction to the literature.

In [21], one of us has sketched a possible spectral triple for Moyal space with harmonic oscillator potential. However, it became clear very soon that working out the mathematical

details is a non-trivial issue so that the simpler commutative case was studied first [31]. In this paper, we achieve the construction of a spectral triple for a suitable algebra of functions on $\mathbb{R}^d$ endowed with the Moyal product, together with a Dirac operator which is a square root of the d-dimensional harmonic oscillator Hamiltonian. Few remarks are in order. Firstly, in the same way as to find a differential (and not pseudo-differential) square root of the ordinary Laplacian on $\mathbb{R}^d$ where one has to go $2^{\lfloor d/2 \rfloor} \times 2^{\lfloor d/2 \rfloor}$ matrices, to find a differential square root of the d-dimensional harmonic oscillator Hamiltonian one has to go $2^d \times 2^d$ matrices—this is the main observation in [21]. The second important remark has to do with the choice of the function algebra with Moyal product. Indeed, there are many non-unital Fréchet algebras of functions with Moyal product that one may use while respecting most of the non-unital spectral triple conditions. But there is only one for which the finiteness axiom is satisfied for a Dirac-type operator given by a square root of the harmonic oscillator Hamiltonian, namely the algebra of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$. A similar phenomenon appeared in [12] where it has been shown that with the ordinary Dirac operator of $\mathbb{R}^d$, there is only one choice of algebra of functions with Moyal product for which the finiteness axiom is satisfied, namely the L^2-Sobolev space $W^{2,\infty}(\mathbb{R}^d)$. Lastly, the construction of a Hochschild cycle satisfying the orientability axiom requires (see also [31]) *two different differential square roots* of the harmonic oscillator Hamiltonian, not only one.

The paper is organized as follow. In Section 2, we introduce two spectral triples with common algebra $\mathcal{A}_\star$ given by the set of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ with Moyal product, and two different differential square roots of the harmonic oscillator Hamiltonian acting densely on $\mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathbb{C}^{2^d}$. The rest of the section is then devoted to prove that these spectral triples are regular, that the metric dimension is d, the KO-dimension is $2d$ and that the dimension spectrum Sd is $d - \mathbb{N}$. In section 3, we specialize to the case $d = 4$ and after having proven a heat-kernel expansion result adapted to our particular situation, we explicitly compute the spectral action for a $U(1)$-Higgs model.

2   The harmonic oscillator spectral triple for Moyal space

We consider two Moyal-type deformations $(\mathcal{A}_\star, \mathcal{D}_\bullet, \mathcal{H})$, $\bullet = 1, 2$, of the (commutative) d-dimensional harmonic oscillator spectral triple introduced in [31]. In order to implement the Moyal product, the dimension d must be even.

2.1   An isospectral deformation

On $L^2(\mathbb{R}^d)$, we introduce the (unbounded) bosonic creation and annihilation operators

$$a_\mu := \partial_\mu + \tilde{\Omega}\, x_\mu \,, \qquad a_\mu^* := -\partial_\mu + \tilde{\Omega}\, x_\mu \,, \qquad \mu = 1, \ldots, d \,,$$

satisfying the commutation relations $[a_\mu, a_\nu] = [a_\mu^*, a_\nu^*] = 0$ and $[a_\mu, a_\nu^*] = 2\tilde{\Omega}\, \delta_{\mu\nu}$. Here, $\tilde{\Omega} > 0$ is a frequency parameter. On the exterior algebra $\bigwedge(\mathbb{C}^d)$, we introduce fermionic partners $b_\mu, b_\mu{}^*$ which fulfill the anticommutation relations $\{b_\mu, b_\nu\} = \{b_\mu{}^*, b_\nu{}^*\} = 0$ and

3

$\{b_\mu, b_\nu{}^*\} = \delta_{\mu\nu}$. Then, on the Hilbert space

$$\mathcal{H} := L^2(\mathbb{R}^d) \otimes \bigwedge(\mathbb{C}^d) \simeq L^2(\mathbb{R}^d) \otimes \mathbb{C}^{2^d} ,$$

these operators give rise to two selfadjoint operators

$$\mathcal{D}_1 := \mathfrak{Q}_1 + \mathfrak{Q}_1^* , \quad \mathcal{D}_2 := i\mathfrak{Q}_2 - i\mathfrak{Q}_2^* , \tag{1}$$

constructed out of the supercharges

$$\mathfrak{Q}_1 := a_\mu \otimes b^{\mu*} , \qquad \mathfrak{Q}_2 := a_\mu \otimes b^\mu ,$$

where Einstein's summation convention is used. Indices are raised or lowered by the Euclidean metric $\delta^{\mu\nu}$ or $\delta_{\mu\nu}$, respectively. The (anti-)commutation relations imply for $\bullet \in \{1, 2\}$

$$\mathcal{D}_\bullet^2 = H \otimes 1 - (-1)^\bullet \tilde{\Omega} \otimes \Sigma , \quad H := \tfrac{1}{2}\{a^\mu, a_\mu^*\} = -\partial_\mu \partial^\mu + \tilde{\Omega}^2 x_\mu x^\mu , \quad \Sigma := [b_\mu^*, b^\mu] . \tag{2}$$

We identify H as the Hamiltonian of the d-dimensional harmonic oscillator with frequency $\tilde{\Omega}$. Its spectrum is $\{\lambda_n = \tilde{\Omega}(\frac{d}{2} + n) , \ n \in \mathbb{N}\}$, where the eigenvalue λ_n appears with multiplicity $\binom{n+d-1}{d-1}$. It then follows that $(|\mathcal{D}_\bullet| + 1)^{-z}$, $\bullet = 1, 2$, is trace-class for $\Re(z) > 2d$.

Remark 1. Our choice of $\mathcal{D}_2$ differs from [31]. One should take $\mathcal{D}_2$ from (1) also for the commutative case to view our spectral triple as isospectral deformation. As seen in the next, the choice (1) is required by the orientability axiom. The commutative version is somehow degenerate and does not detect the sign of the frequency in (2).

We now wish to implement the Moyal product $\star$ in this picture:

$$f \star g(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{dy\, dk}{(2\pi)^d} \ f(x + \tfrac{1}{2}\Theta \cdot k)\, g(x + y)\, \mathrm{e}^{i\langle k, y\rangle} , \tag{3}$$

parametrized by an invertible skew-symmetric matrix $\Theta^t = -\Theta \in M_d(\mathbb{R})$. We first need to find out which algebra $\mathcal{A}_\star$ of functions (or distribution) with Moyal product to use. For that aim, observe that the finiteness condition alone dictates the choice of the topological vector space underlying the algebra $\mathcal{A}_\star$. Indeed, from (2) we conclude for $\bullet = 1, 2$

$$\mathcal{H}_\infty := \bigcap_{m \geq 0} \mathrm{dom}(\mathcal{D}_\bullet^n) = \mathcal{S}(\mathbb{R}^d) \otimes \bigwedge(\mathbb{C}^d) \simeq \mathcal{S}(\mathbb{R}^d, \mathbb{C}^{2^d}) ,$$

which is required to be a finitely generated projective module over the algebra of the spectral triple. Thus this naturally leads us to the choice $\mathcal{A}_\star := (\mathcal{S}(\mathbb{R}^d), \star)$, with even d.

Remark 2. For the ordinary Dirac operator on the trivial spin bundle of $\mathbb{R}^d$, the set of smooth spinors is isomorphic to $W^{2,\infty}(\mathbb{R}^d) \otimes \mathbb{C}^{2^{\lfloor d/2 \rfloor}}$. In this case, the topological vector space underlying the choice of the algebra is the L^2-Sobolev space $W^{2,\infty}(\mathbb{R}^d)$. Note that the latter is stable under the Moyal product too, and that this is the choice made in [12].

4

Here are the main properties of the Moyal product we will use latter on (for more information see [12]). First is strong closedness

$$\int f \star g(x)\, dx = \int f(x)\, g(x)\, dx = \int g \star f(x)\, dx\,, \quad \forall f, g \in L^2(\mathbb{R}^4)\,, \tag{4}$$

then, we have the Leibniz rule

$$\partial_\mu (f \star g) = \partial_\mu f \star g + f \star \partial_\mu g\,, \tag{5}$$

and the following identities

$$\{f, x^\mu\}_\star := x^\mu \star f + f \star x^\mu = 2 x^\mu f\,, \quad [x^\mu, f]_\star := x^\mu \star f - f \star x^\mu = i\Theta^{\mu\nu} \partial_\nu f\,, \tag{6}$$

both holding for $f, g \in \mathcal{A}_\star$. Last is the (non-unique) factorization property [15, p. 877]

$$\forall f \in \mathcal{A}_\star\,, \quad \exists g, h \in \mathcal{A}_\star : f = g \star h\,. \tag{7}$$

Following [12], we then specify the preferred unitization $\mathcal{B}_\star$ of $\mathcal{A}_\star$, as the *space of smooth bounded functions on $\mathbb{R}^d$ with all partial derivatives bounded*. The Moyal product (3) extends to $\mathcal{B}_\star$, and $\mathcal{A}_\star \subset \mathcal{B}_\star$ is an essential two-sided ideal, [12, Theorem 2.21], but is not dense. The reason why we chose this particular unitization is that $\mathcal{B}_\star$ contains the plane waves and constant functions (but no other non-constant polynomials) and this is crucial for the orientability condition (see subsection 2.3). According to [12, Theorem 2.21], the C^*-completion of $\mathcal{B}_\star$ is

$$A_\star := \left\{ T \in \mathcal{S}'(\mathbb{R}^d) \ : \ T \star f \in L^2(\mathbb{R}^d) \text{ for all } f \in L^2(\mathbb{R}^d) \right\}\,.$$

Therefore, $A_\star$ acts on $\mathcal{H}$ by componentwise left Moyal multiplication, that we denote by $L_\star$:

$$L_\star : A_\star \times \mathcal{H} \to \mathcal{H}\,, \qquad (f, \psi \otimes m) \mapsto (f \star \psi) \otimes m\,,$$

for $\psi \in L^2(\mathbb{R}^d)$ and $m \in \bigwedge(\mathbb{C}^d)$. In particular, we have the bounds [12]:

$$\|L_\star(f)\| \leq C_1(\Theta)\,\|f\|_2\,, \quad f \in \mathcal{A}_\star\,, \qquad \|L_\star(f)\| \leq C_2(\Theta) \sup_{|\alpha| \leq d+1} \|\partial^\alpha f\|_\infty\,, \quad f \in \mathcal{B}_\star\,.$$

We also define the (anti-)action $R_\star$ of $A_\star$ on $\mathcal{H}$ by componentwise right Moyal multiplication:

$$R_\star : \mathcal{A}_\star \times \mathcal{H} \to \mathcal{H}\,, \qquad (f, \psi \otimes m) \mapsto (\psi \star f) \otimes m\,.$$

Since the complex conjugation is an involution of the algebra $A_\star$, and from the traciality of the Moyal product (4), we get $L_\star(f)^* = L_\star(\bar{f})$, $R_\star(f)^* = R_\star(\bar{f})$. Moreover, this also shows that the two representations $L_\star$ and $R_\star$ are isometric:

$$\|L_\star(f)\| = \|R_\star(f)\|\,, \quad \forall f \in A_\star\,.$$

To avoid too many notations, $L_\star$, $R_\star$ will also denote the left and right actions of $\mathcal{A}_\star$ and $\mathcal{B}_\star$ on $L^2(\mathbb{R}^d)$.

We now check that our spectral triple $(\mathcal{A}_\star, \mathcal{H}, \mathcal{D}_\bullet)$, $\bullet = 1, 2$, defines a non-unital spectral triple with spectral dimension d and KO-dimension $2d$, in the sense of Definition 25 in the Appendix.

5

2.2 Boundedness and compactness

From (6), we obtain for $f \in \mathcal{B}_\star$ on $\mathrm{dom}(\mathcal{D}_\bullet)$:

$$[\mathcal{D}_1, L_\star(f)] = L_\star(i\partial_\mu f) \otimes \Gamma^\mu , \qquad \Gamma^\mu := (ib^\mu - ib^{*\mu}) - \tfrac{1}{2}\tilde{\Omega}\,\Theta^{\mu\nu}\,(b_\nu + b_\nu^*) ,$$

$$[\mathcal{D}_2, L_\star(f)] = L_\star(i\partial_\mu f) \otimes \Gamma^{\mu+d} , \qquad \Gamma^{\mu+d} := (b^\mu + b^{*\mu}) - \tfrac{1}{2}\tilde{\Omega}\,\Theta^{\mu\nu}\,(ib_\nu - ib_\nu^*) . \qquad (8)$$

As $\partial_\mu f \in \mathcal{B}_\star$, the commutator $[\mathcal{D}_\bullet, L_\star(f)]$ extends to a bounded operator. It is a remarkable property of the Moyal algebra that just the d-dimensional differential of f appears, no x-multiplication.

For the compactness condition, there is not much to say as $(\mathcal{D}_\bullet + \lambda)^{-1}$ is already a compact operator on $\mathcal{H}$. Then, $L_\star(f)(\mathcal{D}_\bullet + \lambda)^{-1}$ is compact for any $f \in \mathcal{A}_\star$, even for $f \in \mathcal{B}_\star$.

2.3 Orientability

Note first that the operators $\Gamma^\mu, \Gamma^{\mu+d}$ defined by (8) satisfy the anticommutation relations

$$\{\Gamma^\mu, \Gamma^\nu\} = \{\Gamma^{\mu+d}, \Gamma^{\nu+d}\} = 2\,(g^{-1})^{\mu\nu} , \quad \{\Gamma^\mu, \Gamma^{\nu+d}\} = 0 ,$$

where the symmetric matrix $g \in \mathrm{GL}(d, \mathbb{R})$ is defined by

$$g := \left(\mathrm{Id}_d - \tfrac{1}{4}\tilde{\Omega}^2\,\Theta^2\right)^{-1} , \qquad (9)$$

and plays the role of a effective metric. Note that $\Theta^2 = -\Theta^t\Theta$ is negative definite so that

$$(1 + \tfrac{1}{4}\tilde{\Omega}^2\|\Theta\|^2)^{-1}\mathrm{Id}_d \leq g \leq \mathrm{Id}_d .$$

We will frequently use that Θ, g, g^{-1} commute with each other. Raising and lowering of summation indices will always be performed with the Euclidean metric $\delta^{\mu\nu}, \delta_{\mu\nu}$.

Thus the $\{\Gamma^1, \ldots, \Gamma^{2d}\}$ generate a Clifford algebra of double dimension $2d$. The inverse transformation of (8) reads

$$ib_\nu - ib_\nu^* = g_{\nu\mu}\left(\Gamma^\mu + \tfrac{1}{2}\tilde{\Omega}\,\Theta^{\mu\rho}\,\Gamma_{\rho+d}\right) , \qquad b_\nu + b_\nu^* = g_{\nu\mu}\left(\Gamma^{\mu+d} + \tfrac{1}{2}\tilde{\Omega}\,\Theta^{\mu\rho}\,\Gamma_\rho\right) .$$

Therefore, we can express $\mathcal{D}_1$, in terms of $\{\Gamma^1, \ldots, \Gamma^{2d}\}$, but not in terms of the half set of operators $\{\Gamma^1, \ldots, \Gamma^d\}$ produced by the commutator of $\mathcal{D}_1$ with $\mathcal{B}_\star$. Similar comments apply for $\mathcal{D}_2$. In conclusion, we get

$$\mathcal{D}_1 = i\partial^\nu \otimes g_{\nu\mu}\left(\Gamma^\mu + \tfrac{1}{2}\tilde{\Omega}\,\Theta^{\mu\rho}\,\Gamma_{\rho+d}\right) + \tilde{\Omega}\,x^\nu \otimes g_{\nu\mu}\left(\Gamma^{\mu+d} + \tfrac{1}{2}\tilde{\Omega}\,\Theta^{\mu\rho}\,\Gamma_\rho\right) ,$$

$$\mathcal{D}_2 = i\partial^\nu \otimes g_{\nu\mu}\left(\Gamma^{\mu+d} + \tfrac{1}{2}\tilde{\Omega}\,\Theta^{\mu\rho}\,\Gamma_\rho\right) + \tilde{\Omega}\,x^\nu \otimes g_{\nu\mu}\left(\Gamma^\mu + \tfrac{1}{2}\tilde{\Omega}\,\Theta^{\mu\rho}\,\Gamma_{\rho+d}\right) . \qquad (10)$$

General results for Clifford algebras then show that any element of the Clifford algebra which anticommutes with *every* Γ^μ and $\Gamma^{\mu+d}$ is a multiple of the anti-symmetrized product of the generators $\{\Gamma^1, \ldots, \Gamma^{2d}\}$. Therefore, a grading operator commuting with $\mathcal{D}_1$ cannot be found in the algebra generated by $L_\star(f), R_\star(f)$ and $[\mathcal{D}_1, L_\star(f)]$, so that an implementation of the orientability axiom requires both Dirac-type operators $\mathcal{D}_1, \mathcal{D}_2$.

Let $u_\mu := e^{-ix_\mu} \in \mathcal{B}_\star$. We know from [12] that the element

$$\boldsymbol{c} := \sum_{\sigma \in S_d} \epsilon(\sigma) \frac{\mathrm{i}^{\frac{d(d-1)}{2}} \sqrt{\det g}}{d!} \left((u_1 \star \cdots \star u_d)^{-1} \otimes 1 \right) \otimes u_{\sigma(1)} \otimes \cdots u_{\sigma(d)} \, ,$$

is a Hochschild d-cycle for the algebra $\mathcal{B}_\star$, with values in $\mathcal{B}_\star \otimes \mathcal{B}_\star^o$. (In the expression of $\boldsymbol{c}$, the inverse is with respect to the $\star$-product and the scaling by $\sqrt{\det g}$ is irrelevant for cyclicity.) For $\pi_{\mathcal{D}_\bullet}$ defined in the Appendix, we then obtain from (8)

$$\boldsymbol{\gamma}_1 := \pi_{\mathcal{D}_1}(\boldsymbol{c}) = \frac{\sqrt{\det g}}{d!} \mathrm{i}^{\frac{d(d-1)}{2}} \sum_{\sigma \in S_d} \epsilon(\sigma) \otimes \Gamma^{\sigma(1)} \cdots \Gamma^{\sigma(d)} \, ,$$

$$\boldsymbol{\gamma}_2 := \pi_{\mathcal{D}_2}(\boldsymbol{c}) = \frac{\sqrt{\det g}}{d!} \mathrm{i}^{\frac{d(d-1)}{2}} \sum_{\sigma \in S_d} \epsilon(\sigma) \otimes \Gamma^{\sigma(1)+d} \cdots \Gamma^{\sigma(d)+d} \, , \tag{11}$$

and they satisfy the relations:

$$\boldsymbol{\gamma}_1^2 = 1 = \boldsymbol{\gamma}_2^2 \, , \qquad \boldsymbol{\gamma}_1^* = \boldsymbol{\gamma}_1 \, , \quad \boldsymbol{\gamma}_2^* = \boldsymbol{\gamma}_2 \, , \qquad \boldsymbol{\gamma}_1 \boldsymbol{\gamma}_2 = (-1)^d \boldsymbol{\gamma}_2 \boldsymbol{\gamma}_1 \, .$$

Thus we define

$$\Gamma := (-i)^d \boldsymbol{\gamma}_1 \boldsymbol{\gamma}_2 \, . \tag{12}$$

Since $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ commute with every element of $\mathcal{A}_\star$ or $\mathcal{B}_\star$, Γ does too and the discussion above shows that

$$\Gamma^2 = 1 \, , \qquad \{\mathcal{D}_\bullet, \Gamma\} = 0 \, , \quad \bullet = 1, 2 \, ,$$

so that Γ defines the grading operator for the two spectral triples $(\mathcal{A}_\star, \mathcal{H}, \mathcal{D}_\bullet)$, $\bullet = 1, 2$. We stress that the necessity of the two Dirac operators $\mathcal{D}_1, \mathcal{D}_2$, is quite different from conventional spectral triples [7] where a single operator is needed.

Note also that from the explicit formulae of $\mathcal{D}_\bullet$, (up to a possible sign) one has the relation $\Gamma = 1 \otimes (-1)^{N_f}$ in terms of the fermionic number operator $N_f = b_\mu^* b^\mu$.

2.4 KO-dimension and other algebraic conditions

The real structure is an anti-linear isometry J on $\mathcal{H}$. We assume that for d even the KO-dimension k is even, too. Then, according to the sign table in the Appendix we have

$$J\mathcal{D}_\bullet = \mathcal{D}_\bullet J \, , \quad \bullet = 1, 2 \, .$$

This is achieved by the following non-trivial action on the matrix part of $\mathcal{H}$:

$$J a_\mu J^{-1} = a_\mu \, , \qquad J a_\mu^* J^{-1} = a_\mu^* \, , \qquad J b_\mu J^{-1} = b_\mu^* \, , \qquad J b_\mu^* J^{-1} = b_\mu \, . \tag{13}$$

In particular, conjugation by J preserves the (anti-)commutation relations. We can view $\bigwedge(\mathbb{C}^d)$ as generated by repeated action of $\{b_\mu^\dagger\}$ on the vacuum vector $|0\rangle$ defined by $b_\mu|0\rangle = 0$.

7

It then follows that, up to a prefactor of modulus 1, which cancels in every relation of the dimension table, J is the Hodge-$*$ operator on $\bigwedge(\mathbb{C}^d)$, i.e. is uniquely defined by

$$J|0\rangle = b_1^* b_2^* \cdots b_d^* |0\rangle \,,$$

together with (13) and the anti-linearity $J(z\psi) = \bar{z}J\psi$. In particular, $J \circ L_\star(f) \circ J^{-1} = R_\star(\overline{f})$, which implements the opposite algebra and achieves the order-one condition:

$$[JL_\star(f_1)J^{-1}, L_\star(f_2)] = 0 \,, \quad [JL_\star(f_1)J^{-1}, [\mathcal{D}_\bullet, L_\star(f_2)]] = 0 \,, \qquad \text{for all } f_1, f_2 \in \mathcal{B}_\star \,. \quad (14)$$

To compute J^2 we consider, for $\mu_1 < \mu_2 < \cdots < \mu_k$,

$$J(b_{\mu_1}^* \cdots b_{\mu_k}^* |0\rangle) = b_{\mu_1} \cdots b_{\mu_k} b_1^* b_2^* \cdots b_d^* |0\rangle = (-1)^{\sum_{j=1}^k (\mu_j - 1)} b_1^* \overset{\mu_1 \ldots \mu_k}{\cdots} b_d^* |0\rangle \,.$$

The notation $\overset{\mu_1 \ldots \mu_k}{\cdots}$ means that $b_{\mu_1}^*, \ldots, b_{\mu_k}^*$ are missing. We apply J again, to get:

$$J^2(b_{\mu_1}^* \cdots b_{\mu_k}^* |0\rangle) = (-1)^{\sum_{j=1}^k (\mu_j - 1)} J(b_1^* \overset{\mu_1 \ldots \mu_k}{\cdots} b_d^* |0\rangle) = (-1)^{\sum_{j=1}^d (j-1)} b_{\mu_1}^* \cdots b_{\mu_k}^* |0\rangle \,,$$

which means

$$J^2 = (-1)^{\frac{d(d-1)}{2}} \,. \quad (15)$$

From (8) it follows that J commutes with Γ^μ and $\Gamma^{\mu+d}$. From (12) we then conclude $J\Gamma = (-1)^d \Gamma J$. Comparing these results with the dimension table in the Appendix, we have proven:

Proposition 3. *The spectral geometries $(\mathcal{A}_\star, \mathcal{H}, \mathcal{D}_\bullet, \Gamma, J)$, $\bullet = 1, 2$, for the d-dimensional Moyal algebra $\mathcal{A}_\star$ are of* KO-*dimension $2d$ mod 8.*

2.5 Metric dimension

Since $\mathcal{D}_\bullet$, $\bullet = 1, 2$, squares (up to matrices) to the d-dimensional harmonic oscillator Hamiltonian, we already now that $(1 + \mathcal{D}_\bullet^2)^{-d}$ belongs to the Dixmier ideal $\mathcal{L}^{1,\infty}(\mathcal{H})$. In this subsection we are going to prove that for the localized operators, the critical dimension is reduced by a factor of 2, that is for all $f \in \mathcal{A}_\star$, the operators $L_\star(f)(1 + \mathcal{D}_\bullet^2)^{-d/2}$ belong to $\mathcal{L}^{1,\infty}(\mathcal{H})$ and that any of its Dixmier traces is a constant multiple of the integral of f. To obtain both Dixmier traceability and the value of the Dixmier trace, we will use the results of [1]. In order to do this, we need some preliminary Lemmas (which will also be needed to check the regularity condition, to obtain the dimension spectrum and to compute the spectral action).

Lemma 4. *Introducing the operators on $L^2(\mathbb{R}^d)$:*

$$\nabla_\mu := \partial_\mu + \tfrac{1}{2}i\, \tilde{\Omega}^2\, \Theta_{\mu\nu}\, x^\nu \,, \qquad \tilde{\nabla}_\mu := \tfrac{1}{2}\big(\partial_\mu - 2i\, (\Theta^{-1})_{\mu\nu}\, x^\nu\big) \,, \qquad \mu = 1, \cdots, d \,,$$

we have the following relations for $f \in \mathcal{B}_\star$:

$$[H, L_\star(f)] = -L_\star\big((g^{-1})^{\mu\nu}\partial_\mu\partial_\nu f\big) - 2L_\star(\partial^\mu f)\nabla_\mu ,$$

$$[\nabla^\mu, L_\star(f)] = L_\star\big((g^{-1})^{\mu\nu}\partial_\nu f\big) , \qquad [\tilde\nabla^\mu, L_\star(f)] = L_\star(\partial_\mu f) ,$$

$$[H, \nabla_\mu] = -2i\tilde\Omega^2\Theta_{\mu\nu}\tilde\nabla^\nu , \qquad [H, \tilde\nabla_\mu] = 2i(\Theta^{-1})_{\mu\nu}\nabla^\nu ,$$

$$[\tilde\nabla_\mu, \tilde\nabla_\nu] = i(\Theta^{-1})_{\mu\nu} , \qquad [\nabla_\mu, \nabla_\nu] = -i\tilde\Omega^2\Theta_{\mu\nu} , \qquad [\nabla^\mu, \tilde\nabla_\nu] = i(g^{-1})^{\mu\rho}(\Theta^{-1})_{\rho\nu} .$$

Proof. This follows from the relations (2), (5) and (6). $\qquad\qquad\square$

Corollary 5. *Let $P_\alpha(\hat\nabla)$ be an element of order α of the polynomial algebra generated by $\nabla, \tilde\nabla$. Then $P_\alpha(\hat\nabla)(1 + H)^{-\alpha/2}$ extends to a bounded operator.*

Proof. From the operator inequalities (no summations on μ but summation on ν)

$$|(1 + H)^{-1/2}\partial_\mu|^2 = -\partial_\mu(1 + H)^{-1}\partial_\mu \le -\partial_\mu(1 - \partial_\nu\partial^\nu)^{-1}\partial_\mu ,$$

and

$$|(1 + H)^{-1/2}x_\mu|^2 = x_\mu(1 + H)^{-1}x_\mu \le x_\mu(1 + \tilde\Omega^2 x_\nu x^\nu)^{-1}x_\mu ,$$

we see that $\hat\nabla_\mu(1 + H)^{-1/2}$ is bounded. Then, the general case follows by induction using

$$\hat\nabla_{\mu_1}\hat\nabla_{\mu_2}(1 + H)^{-1} = \hat\nabla_{\mu_1}(1 + H)^{-1}\hat\nabla_{\mu_2} + \hat\nabla_{\mu_1}[\hat\nabla_{\mu_2}, (1 + H)^{-1}] ,$$

and

$$\hat\nabla_{\mu_1}[\hat\nabla_{\mu_2}, (1 + H)^{-1}] = -\hat\nabla_{\mu_1}(1 + H)^{-1}[\hat\nabla_{\mu_2}, H](1 + H)^{-1} ,$$

which is bounded, too, according to Lemma 4. $\qquad\qquad\square$

The following Proposition will be crucial for the computation of the spectral action, the estimate we need to evaluate the Dixmier trace and to compute the dimension spectrum and the residues of the associated zeta functions.

Proposition 6. *For $f \in \mathcal{B}_\star$, define $\mathcal{T}_{\mu_1\dots\mu_k}(f) := \mathrm{Tr}\big(L_\star(f)\nabla_{\mu_1}\dots\nabla_{\mu_k}e^{-tH}\big)$. Then one has*

$$\mathcal{T}_{\mu_1,\dots\mu_k}(f) = \sum_{1 \le j_1 < j_2 < \cdots < j_{2a} \le k} \left(\frac{\tilde\Omega}{2\pi\sinh(2\tilde\Omega t)}\right)^{\frac{d}{2}}$$

$$\times \int_{\mathbb{R}^d} dz\, \sqrt{\det g}\, f(z)\, e^{-\tilde\Omega\tanh(\tilde\Omega t)\langle z, gz\rangle}\, (\mathcal{Z}_{\mu_1}\overset{j_1\dots j_{2a}}{\dots}\mathcal{Z}_{\mu_k})(\mathcal{N}_{\mu_{j_1}\mu_{j_2}}\cdots\mathcal{N}_{\mu_{j_{2a-1}}\mu_{j_{2a}}}) ,$$

where

$$\mathcal{Z}_\mu := -\tilde\Omega\tanh(\tilde\Omega t)z_\mu + i\tilde\Omega^2(\Theta g z)_\mu ,$$

$$\mathcal{N}_{\mu\nu} := -\tfrac{1}{2}\tilde\Omega\,(\coth(\tilde\Omega t) + \tanh(\tilde\Omega t))(g^{-1})_{\mu\nu} - \tfrac{1}{2}\tilde\Omega^3\coth(\tilde\Omega t)(\Theta g\Theta)_{\mu\nu} - \tfrac{1}{2}i\tilde\Omega^2\,\Theta_{\mu\nu} ,$$

and $\overset{j_1\dots j_{2a}}{\dots}$ means that $\{\mathcal{Z}_{\mu_{j_1}}, \dots \mathcal{Z}_{\mu_{j_{2a}}}\}$ are missing in the product $\mathcal{Z}_{\mu_1}\cdots\mathcal{Z}_{\mu_k}$. (Remember that g is a constant metric so that $\sqrt{\det g}$ can also be taken in front of the integral).

In particular,

$$\mathrm{Tr}\big(e^{-tH}\big) = \big(2\sinh(\tilde\Omega t)\big)^{-d} .$$

Proof. Since $L_\star(f)\nabla_{\mu_1}\ldots\nabla_{\mu_k}e^{-tH}$ is trace-class (because $\nabla_{\mu_1}\ldots\nabla_{\mu_k}e^{-tH/2}$ is bounded by Corollary 5 and $e^{-tH/2}$ is trace-class), the trace can be evaluated as the integral of the kernel on the diagonal. Thus, in integral kernel representation, we have to compute

$$
\mathcal{T}_{\mu_1,\ldots\mu_k}(f) =
$$
$$
\int_{\mathbb{R}^d\times\mathbb{R}^d} dx\,dy\,(L_\star(f))(x,y)\Big(\frac{\partial}{\partial y^{\mu_1}}+\frac{i}{2}\tilde{\Omega}^2\Theta_{\mu_1\nu_1}y^{\nu_1}\Big)\cdots\Big(\frac{\partial}{\partial y^{\mu_k}}+\frac{i}{2}\tilde{\Omega}^2\Theta_{\mu_k\nu_k}y^{\nu_k}\Big)\big(e^{-tH}(y,x)\big)\;.
$$

The operator kernel of e^{-tH} is the Mehler kernel

$$
e^{-tH}(x,y) = \Big(\frac{\tilde{\Omega}}{2\pi\sinh(2\tilde{\Omega}t)}\Big)^{d/2}e^{-\frac{\tilde{\Omega}}{4}\coth(\tilde{\Omega}t)\|x-y\|^2-\frac{\tilde{\Omega}}{4}\tanh(\tilde{\Omega}t)\|x+y\|^2}\;, \tag{16}
$$

while the operator kernel of $L_\star(f)$ is readily identified to be

$$
L_\star(f)(x,y) = \frac{1}{\pi^d\det\Theta}\int dz\,f(z)\,\mathrm{e}^{i\langle x-y,\Theta^{-1}(x+y)\rangle+2i\langle z,\Theta^{-1}(x-y)\rangle}\;. \tag{17}
$$

We introduce $u=x-y$ and $v=x+y$ and

$$
\mathfrak{D}_\mu(u,v) := \frac{\tilde{\Omega}}{2}\coth(\tilde{\Omega}t)u_\mu - \frac{\tilde{\Omega}}{2}\tanh(\tilde{\Omega}t)v_\mu + \frac{i}{4}\tilde{\Omega}^2\Theta_{\mu\alpha}(v^\alpha-u^\alpha)\;,
$$
$$
\mathcal{Y}_{\mu\nu} := -\frac{\tilde{\Omega}}{2}(\coth(\tilde{\Omega}t)+\tanh(\tilde{\Omega}t))\delta_{\mu\nu} - \frac{i}{2}\tilde{\Omega}^2\Theta_{\mu\nu}\;,
$$

to obtain

$$
\mathcal{T}_{\mu_1,\ldots\mu_k}(f)
$$
$$
= \sum_{1\le j_1<j_2<\cdots<j_{2a}\le k}\Big(\frac{\tilde{\Omega}}{2\pi\sinh(2\tilde{\Omega}t)}\Big)^{\frac{d}{2}}\frac{1}{(2\pi)^d\det\Theta}
$$
$$
\times\int du\,dv\,dz\,f(z)\mathfrak{D}_{\mu_1}(u,v)\overset{j_1\ldots j_{2a}}{\cdots}\mathfrak{D}_{\mu_k}(u,v)\mathcal{Y}_{\mu_{j_1}\mu_{j_2}}\cdots\mathcal{Y}_{\mu_{j_{2a-1}}\mu_{j_{2a}}}e^{-\frac{1}{2}\langle(u,v),Q(u,v)\rangle-\langle(u,v),(2i\Theta^{-1}z,0)\rangle}
$$
$$
= \sum_{1\le j_1<j_2<\cdots<j_{2a}\le k}\Big(\frac{\tilde{\Omega}}{2\pi\sinh(2\tilde{\Omega}t)}\Big)^{\frac{d}{2}}\frac{1}{\det\Theta\sqrt{\det Q}}
$$
$$
\times\int_{\mathbb{R}^d} dz\,f(z)\mathfrak{D}_{\mu_1}\big(\tfrac{i\partial}{\partial\xi},\tfrac{i\partial}{\partial\eta}\big)\overset{j_1\ldots j_{2a}}{\cdots}\mathfrak{D}_{\mu_k}\big(\tfrac{i\partial}{\partial\xi},\tfrac{i\partial}{\partial\eta}\big)\mathcal{Y}_{\mu_{j_1}\mu_{j_2}}\cdots\mathcal{Y}_{\mu_{j_{2a-1}}\mu_{j_{2a}}}\mathcal{E}\Big|_{\xi=\eta=0}\;,
$$

where $\mathcal{E}:=e^{-\frac{1}{2}\langle(-2z\Theta^{-1}+\xi,\eta),Q^{-1}(2\Theta^{-1}z+\xi,\eta)\rangle}$ and $Q\in M_{2d}(\mathbb{C})$ is given by

$$
Q = \begin{pmatrix}\frac{\tilde{\Omega}}{2}\coth(\tilde{\Omega}t)\,\mathrm{Id}_d & -i\Theta^{-1}\\ i\Theta^{-1} & \frac{\tilde{\Omega}}{2}\tanh(\tilde{\Omega}t)\,\mathrm{Id}_d\end{pmatrix}\;.
$$

10

Recalling that $g^{-1} = 1 - \frac{\tilde{\Omega}^2}{4}\Theta^2$ and $g\Theta = \Theta g$, we find by writing Q as product of triangle matrices

$$\det Q = \frac{1}{\det g \, (\det \Theta)^2}, \qquad Q^{-1} = \begin{pmatrix} -\frac{\tilde{\Omega}}{2}\tanh(\tilde{\Omega}t)\, g\,\Theta^2 & -ig\,\Theta \\ ig\,\Theta & -\frac{\tilde{\Omega}}{2}\coth(\tilde{\Omega}t)\, g\,\Theta^2 \end{pmatrix},$$

so that

$$\mathcal{E} = \exp\Big\{ -\tilde{\Omega}\tanh(\tilde{\Omega}t)\langle z, gz\rangle - \tilde{\Omega}\tanh(\tilde{\Omega}t)\langle z, g\Theta\xi\rangle - 2i\langle z, g\eta\rangle$$

$$+ \frac{\tilde{\Omega}}{4}\tanh(\tilde{\Omega}t)\langle \xi, \Theta g\Theta\xi\rangle + i\langle \xi, \Theta g\eta\rangle + \frac{\tilde{\Omega}}{4}\coth(\tilde{\Omega}t)\langle \eta, \Theta g\Theta\eta\rangle \Big\},$$

$$\mathfrak{D}_\mu\big(\tfrac{i\partial}{\partial\xi}, \tfrac{i\partial}{\partial\eta}\big)\mathcal{E} = \Big(i\tilde{\Omega}^2(\Theta gz)_\mu - \tilde{\Omega}\tanh(\tilde{\Omega}t)z_\mu + i\frac{\tilde{\Omega}^2}{2}(\Theta g\Theta\xi)_\mu$$

$$- \tilde{\Omega}\coth(\tilde{\Omega}t)(g\Theta\eta)_\mu + \frac{\tilde{\Omega}}{2}\coth(\tilde{\Omega}t)(\Theta\eta)_\mu - \frac{\tilde{\Omega}}{2}\tanh(\tilde{\Omega}t)(\Theta\xi)_\mu\Big)\mathcal{E}\ .$$

Then, the functions

$$\mathcal{Z}_\mu := \mathcal{E}^{-1}\mathfrak{D}_\mu\big(\tfrac{i\partial}{\partial\xi}, \tfrac{i\partial}{\partial\eta}\big)\mathcal{E}\Big|_{\xi=\eta=0},$$

$$\mathcal{N}_{\mu\nu} := \mathcal{Y}_{\mu\nu} + \mathfrak{D}_\mu\big(\tfrac{i\partial}{\partial\xi}, \tfrac{i\partial}{\partial\eta}\big)\big(\mathcal{E}^{-1}\mathfrak{D}_\nu\big(\tfrac{i\partial}{\partial\xi}, \tfrac{i\partial}{\partial\eta}\big)\mathcal{E}\big)\ ,$$

take the values given in the Lemma, and the assertion follows. $\qquad\square$

A very nice feature of the results of [1] is that both the questions of the Dixmier trace-ability and of the value of the Dixmier trace of an operator of the form aG^k are reduced to the value of the Hilbert-Schmidt norm of the heat-type operator $ae^{-tG^{-1}}$. In our context $a = L_\star(f)$, $G = (1 + \mathcal{D}_\bullet^2)^{-1}$, and all we need to do is to evaluate the Hilbert-Schmidt norm of $L_\star(f)e^{-t\mathcal{D}_\bullet^2}$.

Lemma 7. *If $f \in \mathcal{A}_\star$, then we have:*

$$\|L_\star(f)e^{-t\mathcal{D}_\bullet^2}\|_2^2 = \frac{\tilde{\Omega}^{d/2}}{\pi^{d/2}\tanh^{d/2}(2\tilde{\Omega}t)}\int dz\,\sqrt{\det g}\,\,\bar{f}\star f(z)e^{-\tilde{\Omega}\tanh(2\tilde{\Omega}t)\langle z, gz\rangle}\ .$$

Proof. Since $\mathcal{D}_\bullet^2 = H \otimes 1 - (-1)^\bullet\tilde{\Omega}\otimes\Sigma$, we have

$$0 \le e^{-t\mathcal{D}_\bullet^2} = e^{-tH} \otimes e^{(-1)^\bullet t\tilde{\Omega}\Sigma}\ ,$$

and thus

$$\|L_\star(f)e^{-t\mathcal{D}_\bullet^2}\|_2 = \|L_\star(f)e^{-tH}\|_2\,\mathrm{tr}\big(e^{(-1)^\bullet 2t\tilde{\Omega}\Sigma}\big)^{1/2}\ .$$

For the matrix trace, we have

$$\mathrm{tr}\big(e^{(-1)^\bullet t\tilde{\Omega}\Sigma}\big) = \mathrm{tr}\big(e^{(-1)^\bullet t\tilde{\Omega}\sum_{\mu=1}^d(b_\mu^* b_\mu - b_\mu b_\mu^*)}\big) = \mathrm{tr}\Big(\prod_{\mu=1}^d e^{(-1)^\bullet t\tilde{\Omega}(b_\mu^* b_\mu - b_\mu b_\mu^*)}\Big)\ .$$

11

In the basis $|s_1, \ldots, s_d\rangle := (b_1^*)^{s_1} \cdots (b_d^*)^{s_d}|0, \cdots, 0\rangle$ of $\mathbb{C}^{2^d}$, with $s_i \in \{0, 1\}$, we have

$$-(b_\mu^* b_\mu - b_\mu b_\mu^*)|s_1, \ldots, s_d\rangle = (-1)^{s_\mu}|s_1, \ldots, s_d\rangle,$$

and therefore, for both $\bullet = 1, 2$,

$$\text{tr}(e^{(-1)^\bullet t \tilde{\Omega} \Sigma}) = 2^d \cosh^d(\tilde{\Omega}t).$$

The other bit, $\|L_\star(f)e^{-tH}\|_2^2 = \text{Tr}(e^{-tH}L_\star(\bar{f} \star f)e^{-tH}) = \text{Tr}(L_\star(\bar{f} \star f)e^{-2tH})$, has been computed in Proposition 6. $\qquad\square$

Remark 8. Since for $f \in \mathcal{A}_\star$, $\bar{f} \star f$ is a priori *not a positive function*, in the previous Lemma, one may wonder why $\int dz \sqrt{\det g}\, \bar{f} \star f(z) \exp\{-\tilde{\Omega} \tanh(2\tilde{\Omega}t)\langle z, gz\rangle\}$ is positive, as it should be. This follows from the following facts: For A a positive definite matrix commuting with Θ, set $g_A(x) := e^{-<x,Ax>}$. Then a computation gives

$$g_A \star g_A = (\det(1 + \Theta^t A^2 \Theta))^{-1/2} g_B, \quad \text{with} \quad B = \frac{2A}{1 + \Theta^t A^2 \Theta}.$$

It follows

$$\exp\{-\tilde{\Omega} \tanh(2\tilde{\Omega}t)\langle z, gz\rangle\} = (\det(1 + \Theta^t A^2 \Theta))^{1/2} g_A \star g_A,$$

for

$$A = \frac{g^{-1} - (g^{-2} - \tilde{\Omega}^2 \tanh(2\tilde{\Omega}t)^2 \Theta^t \Theta)^{1/2}}{\tilde{\Omega} \tanh(2\tilde{\Omega}t) \Theta^t \Theta}.$$

Note that $g^{-2} - \tilde{\Omega}^2 \Theta^t \Theta = (1 - \tilde{\Omega}^2 \Theta^t \Theta/4)^2$ so that A exists for all t. Using the traciality of the Moyal product (4), we then get for the matrix A given above and up to a positive constant:

$$\int dz\, \bar{f} \star f(z) \exp\{-\tilde{\Omega} \tanh(2\tilde{\Omega}t)\langle z, g^{-1}z\rangle\} = C(\Theta, \tilde{\Omega}, t) \int dz\, \bar{f} \star f(z)\, g_A \star g_A(z)$$

$$= C(\Theta, \tilde{\Omega}, t) \int dz\, \bar{f} \star f \star g_A \star g_A(z)$$

$$= C(\Theta, \tilde{\Omega}, t) \int dz\, \overline{f \star g_A} \star f \star g_A(z)$$

$$= C(\Theta, \tilde{\Omega}, t) \int dz\, \overline{f \star g_A}(z)\, f \star g_A(z)$$

$$= C(\Theta, \tilde{\Omega}, t)\, \|f \star g_A\|_2 \geq 0.$$

Moreover, it explains why $L_\star(f)e^{-t\mathcal{D}_\bullet^2}$ is Hilbert-Schmidt also for f in $\mathcal{B}_\star$, since $\mathcal{A}_\star$ is an ideal of $\mathcal{B}_\star$ and $\mathcal{A}_\star \subset L^2(\mathbb{R}^d)$.

Lemma 9. *For all $t > 0$ and $\bullet = 1, 2$, we have*

$$\text{Tr}(e^{-t\mathcal{D}_\bullet^2}) = \coth^d(\tilde{\Omega}t).$$

Proof. This is a corollary of Lemma 7 and the Remark which follows it, by letting f going to the constant unit function. $\qquad\square$

Lemma 10. *If $f \in \mathcal{A}_\star$ and $t > 0$, then we have the bound*

$$\|L_\star(f)e^{-t\mathcal{D}^2_\bullet}\|_2 \le C\,\|\bar{f} \star f\|_1^{1/2}\,\max(1, t^{-d/4})\,,$$

where the constant depends only on $\tilde{\Omega}$ and Θ.

Proof. This is a direct consequence of Lemma 7. $\qquad\square$

Lemma 11. *There is $C' > 0$ such that for any $f_1, f_2 \in \mathcal{A}_\star$ and $t > 0$ one has*

$$\|L_\star(f_1)[L_\star(f_2), e^{-t\mathcal{D}^2_\bullet}]\|_1 \le C't^{1/2}\sum_{\mu=1}^{d}\|f_1\|_2\,\|L_\star(\partial_\mu f)e^{-t\mathcal{D}^2_\bullet/4}\|_1\,.$$

Proof. By Lemma 9, $e^{-t\mathcal{D}^2_\bullet}$ is trace class for $t > 0$. We use the identity

$$[e^A, B] = \int_0^1 ds \frac{d}{ds}\left(e^{sA}Be^{(1-s)A}\right) = \int_0^1 ds\, e^{sA}[A, B]e^{(1-s)A}\,, \tag{18}$$

to get

$$L_\star(f_1)[L_\star(f_2), e^{-t\mathcal{D}^2_\bullet}] = -tL_\star(f_1)\int_0^1 ds\, e^{-ts\mathcal{D}^2_\bullet}[\mathcal{D}^2_\bullet, L_\star(f_2)]e^{-t(1-s)\mathcal{D}^2_\bullet}\,.$$

Hence we have

$$\|L_\star(f_1)[L_\star(f_2), e^{-t\mathcal{D}^2_\bullet}]\|_1 \le t\|L_\star(f_1)\|\int_0^1 ds\,\Big(\|e^{-ts\mathcal{D}^2_\bullet/2}\mathcal{D}_\bullet\|\,\|e^{-ts\mathcal{D}^2_\bullet/2}[\mathcal{D}_\bullet, L_\star(f_2)]e^{-t(1-s)\mathcal{D}^2_\bullet}\|_1$$
$$+ \|e^{-ts\mathcal{D}^2_\bullet}[\mathcal{D}_\bullet, L_\star(f_2)]e^{-t(1-s)\mathcal{D}^2_\bullet/2}\|_1\|\mathcal{D}_\bullet e^{-t(1-s)\mathcal{D}^2_\bullet/2}\|\Big)\,.$$

By spectral theory, $\|\mathcal{D}_\bullet e^{-t\mathcal{D}^2_\bullet}\| = (2et)^{-1/2}$. Thus, using the relation

$$[\mathcal{D}_\bullet, L_\star(f_2)] = \begin{cases} iL_\star(\partial_\mu f_2) \otimes \Gamma^\mu\,, & \bullet = 1\,, \\ iL_\star(\partial_\mu f_2) \otimes \Gamma^{\mu+4}\,, & \bullet = 2\,, \end{cases}$$

we get with $\|L_\star(f_1)\| \le C\|f_1\|_2$ and $C' := C\sqrt{2}\pi e^{-1}\sup_{\mu=1}^{2d}\|\Gamma^\mu\|$

$$\|L_\star(f_1)[L_\star(f_2), e^{-t\mathcal{D}^2_\bullet}]\|_1$$
$$\le \frac{C'}{\pi}t^{1/2}\|f_1\|_2\sum_{\mu=1}^{d}\int_0^1 ds\, s^{-1/2}(1-s)^{-1/2}\|e^{-ts\mathcal{D}^2_\bullet/2}L_\star(\partial_\mu f_2)e^{-t(1-s)\mathcal{D}^2_\bullet/2}\|_1\,.$$

Estimating

$$\|e^{-ts\mathcal{D}^2_\bullet/2}L_\star(\partial_\mu f_2)e^{-t(1-s)\mathcal{D}^2_\bullet/2}\|_1 \le \begin{cases} \|L_\star(\partial_\mu f_2)e^{-t\mathcal{D}^2_\bullet/4}\|_1\,, & \text{if } s \in [0, 1/2]\,, \\ \|e^{-t\mathcal{D}^2_\bullet/4}L_\star(\partial_\mu f_2)\|_1\,, & \text{if } s \in [1/2, 1]\,, \end{cases}$$

the result follows. $\qquad\square$

Lemma 12. *Let $f \in \mathcal{A}_\star$. Then, there exists a finite constant $C(f)$ such that for all $t > 0$:*

$$\|L_\star(f)e^{-t\mathcal{D}_\bullet^2}\|_1 \leq C(f) \max(t^{-d/2}, t^{d/2}) .$$

Proof. Our strategy is to iterate a combination of the factorization property (7) with Lemma 10 and Lemma 11 far enough so that we can bound $e^{-\epsilon t\mathcal{D}_\bullet^2}$ alone in trace-norm (i.e. without element of the algebra of both sides).

According to (7), for all $f \in \mathcal{A}_\star$ there exist $f_1, f_2 \in \mathcal{A}_\star$ such that $f = f_1 \star f_2$, giving

$$L_\star(f)e^{-t\mathcal{D}_\bullet^2} = L_\star(f_1)e^{-t\mathcal{D}_\bullet^2}L_\star(f_2) + L_\star(f_1)\big[L_\star(f_2), e^{-t\mathcal{D}_\bullet^2}\big] .$$

From Lemma 10 and Lemma 11 we conclude

$$\|L_\star(f)e^{-t\mathcal{D}_\bullet^2}\|_1 \leq \|L_\star(f_1)e^{-t\mathcal{D}_\bullet^2/2}\|_2 \,\|e^{-t\mathcal{D}_\bullet^2/2}L_\star(f_2)\|_2 + \|L_\star(f_1)\big[L_\star(f_2), e^{-t\mathcal{D}_\bullet^2}\big]\|_2 \tag{19}$$

$$\leq \|\bar{f}_1 \star f_1\|_1^{1/2}\|\bar{f}_2 \star f_2\|_1^{1/2} \max(t^{-d/2}, 1) + C't^{1/2}\sum_{\mu=1}^{d} \|f_1\|_2 \,\|L_\star(\partial_\mu f_2)e^{-t\mathcal{D}_\bullet^2/4}\|_1 .$$

Iterating d-times the estimate (19) with the repeated factorization

$$\partial_{\mu_1}f_2 = f_{1,\mu} \star f_{2,\mu} , \quad \dots , \quad \partial_{\mu_{k+1}}f_{2,\mu_1\dots\mu_k} = f_{1,\mu_1\dots\mu_{k+1}} \star f_{2,\mu_1\dots\mu_{k+1}} ,$$

with $f_{1,\mu}, f_{2,\mu}, \dots, f_{1,\mu_1\dots\mu_{k+1}} \star f_{2,\mu_1\dots\mu_{k+1}} \in \mathcal{A}_\star$, we get for some constants $C_0(f), \dots, C_d(f)$ depending on f and on the choice of factorization at each step:

$$\|L_\star(\partial_\mu f)e^{-t\mathcal{D}_\bullet^2}\|_1 \leq \sum_{k=0}^{d-1} C_k(f)t^{k/2}\max(t^{-d/2}, 1) + C_d(f)t^{d/2}\sum_{\mu_1,\dots,\mu_d=1}^{d} \|L_\star(f_{2,\mu_1\dots\mu_d})e^{-t\mathcal{D}_\bullet^2/4^d}\|_1 .$$

Using Lemma 9 we get

$$\|L_\star(f_{2,\mu_1\dots,\mu_d})e^{-t\mathcal{D}_\bullet^2/4^d}\|_1 \leq \|L_\star(f_{2,\mu_1\dots,\mu_d})\| \,\|e^{-t\mathcal{D}_\bullet^2/4^d}\|_1 \leq C''\|f_{2,\mu_1,\dots,\mu_d}\|_2 \max(t^{-d}, 1) ,$$

which completes the proof. $\qquad\square$

Corollary 13. *For any $f \in \mathcal{A}_\star$, the operator $[(\mathcal{D}_\bullet^2 + 1)^{-d/2}, L_\star(f)]$ is of trace class.*

Proof. By factorization $f = f_1 \star f_2$, $f_1, f_2 \in \mathcal{A}_\star$ and Leibniz rule:

$$[(\mathcal{D}_\bullet^2 + 1)^{-d/2}, L_\star(f)] = L_\star(f_1)[(\mathcal{D}_\bullet^2 + 1)^{-d/2}, L_\star(f_2)] - \big(L_\star(\overline{f}_2)[(\mathcal{D}_\bullet^2 + 1)^{-d/2}, L_\star(\overline{f}_1)]\big)^* ,$$

it suffices to show that $L_\star(f_1)[(\mathcal{D}_\bullet^2 + 1)^{-d/2}, L_\star(f_2)]$ is of trace class for arbitrary $f_1, f_2 \in \mathcal{A}_\star$. By spectral theory,

$$L_\star(f_1)[(\mathcal{D}_\bullet^2 + 1)^{-d/2}, L_\star(f_2)] = \frac{1}{\Gamma(d/2)}\int_0^\infty dt\, t^{d/2-1}L_\star(f_1)[e^{-t(\mathcal{D}_\bullet^2+1)}, L_\star(f_2)] .$$

Combining Lemma 11 with Lemma 12 we obtain for a finite constant depending only on f:

$$\|L_\star(f_1)[(\mathcal{D}_\bullet^2 + 1)^{-d/2}, L_\star(f_2)]\|_1 \leq C(f) \int_0^\infty dt\, e^{-t}\, t^{d/2-1} \cdot t^{1/2} \cdot \max(t^{-d/2}, t^{d/2}) .$$

As the integral converges, we are done. $\qquad\square$

14

We have arrived at the main result of this subsection, that the spectral triple $(\mathcal{A}_\star, \mathcal{H}, \mathcal{D}_\bullet)$ has metric dimension d and not $2d$ (remember that d is even).

Theorem 14. *For $f \in \mathcal{A}_\star$, $\bullet = 1, 2$, the operator $L_\star(f)(1 + \mathcal{D}_\bullet^2)^{-d/2}$ belongs to $\mathcal{L}^{1,\infty}(\mathcal{H})$ and for any Dixmier trace Tr_ω, we have*

$$\mathrm{Tr}_\omega\big(L_\star(f)(1 + \mathcal{D}_\bullet^2)^{-d/2}\big) = \frac{1}{\pi^{d/2}(d/2)!} \int dx \, \sqrt{\det g} \, f(x) \, .$$

Proof. We use the factorization property (7) to write $f = f_1 \star f_2$ with $f_1, f_2 \in \mathcal{A}_\star$, which gives

$$L_\star(f)(1 + \mathcal{D}_\bullet^2)^{-d/2} = L_\star(f_1)(1 + \mathcal{D}_\bullet^2)^{-d/2} L_\star(f_2) + L_\star(f_1)\big[L_\star(f_2), (1 + \mathcal{D}_\bullet^2)^{-d/2}\big] \, .$$

By Corollary 13, $L_\star(f_1)[L_\star(f_2), (1 + \mathcal{D}_\bullet^2)^{-d/2}]$ is trace class, and combining Lemma 7 and [1, Proposition 4.8] we get that $L_\star(f_1)(1 + \mathcal{D}_\bullet^2)^{-d/2} L_\star(f_2)$ belongs to $\mathcal{L}^{1,\infty}(\mathcal{H})$. Therefore, $L_\star(f)(1 + \mathcal{D}_\bullet^2)^{-d/2}$ is Dixmier-trace-class too, and any of its Dixmier trace coincides with those of $L_\star(f_1)(1 + \mathcal{D}_\bullet^2)^{-d/2} L_\star(f_2)$.

Using a polarization identity, it suffices to compute $\mathrm{Tr}_\omega\big(L_\star(\bar{f})(1 + \mathcal{D}_\bullet^2)^{-d/2} L_\star(f)\big)$. Note that

$$\lim_{s \to 1^+} (s - 1) \mathrm{Tr}\big(L_\star(\bar{f})(1 + \mathcal{D}_\bullet^2)^{-ds/2} L_\star(f)\big)$$

$$= \lim_{s \to 1^+} \frac{s - 1}{\Gamma(ds/2)} \int_0^\infty dt \, e^{-t} \, t^{ds/2 - 1} \mathrm{Tr}\big(L_\star(\bar{f}) e^{-t\mathcal{D}_\bullet^2} L_\star(f)\big)$$

$$= \lim_{s \to 1^+} \frac{s - 1}{\Gamma(ds/2)} \int_0^\infty dt \, e^{-t} \, t^{d(s-1)/2 - 1} \frac{(\tilde{\Omega} t)^{d/2}}{\pi^{d/2} \tanh^{d/2}(\tilde{\Omega} t)} \int dz \, \sqrt{\det g} \, \bar{f} \star f(z) e^{-\tilde{\Omega} \tanh(\tilde{\Omega} t)\langle z, gz \rangle}$$

$$= \frac{1}{\pi^{d/2}} \lim_{s \to 1^+} \frac{(s - 1)\Gamma(d(s - 1)/2)}{\Gamma(ds/2)} \int dz \, \sqrt{\det g} \, \bar{f} \star f(z) \, .$$

Now [1, Proposition 5.13], which relies on Corollary 13, gives for any Dixmier trace

$$\mathrm{Tr}_\omega\big(L_\star(\bar{f})(1 + \mathcal{D}_\bullet^2)^{-d/2} L_\star(f)\big) = \frac{1}{\pi^{d/2}(d/2)!} \int dx \, \sqrt{\det g} \, \bar{f} \star f(x) \, . \tag{20}$$

This all what we needed to prove. $\qquad\square$

2.6 Regularity and dimension spectrum

Our next task is to check the regularity condition.

Proposition 15. *For any $f \in \mathcal{B}_\star$ and $\bullet = 1, 2$, both $L_\star(f)$ and $[\mathcal{D}_\bullet, L_\star(f)]$ belong to $\bigcap_{n=1}^\infty \mathrm{dom} \, \delta_\bullet^n$, where $\delta_\bullet(T) := [\langle \mathcal{D}_\bullet \rangle, T]$ and $\langle \mathcal{D}_\bullet \rangle := (\mathcal{D}_\bullet^2 + 1)^{\frac{1}{2}}$.*

Proof. It is well known (see for example [2]) that $\bigcap_{n=1}^{\infty} \operatorname{dom} \delta_{\bullet}^n = \bigcap_{n,m=1}^{\infty} \operatorname{dom} R_{\bullet}^m \circ L_{\bullet}^n$ where $R_{\bullet}$ and $L_{\bullet}$ are the unbounded linear operators given by

$$R_{\bullet}(T) := [\mathcal{D}_{\bullet}^2, T]\langle \mathcal{D}_{\bullet}\rangle^{-1}, \quad L_{\bullet}(T) := \langle \mathcal{D}_{\bullet}\rangle^{-1}[\mathcal{D}_{\bullet}^2, T].$$

By an easy inductive argument, we see that for $T \in \mathcal{B}_{\star} \cup [\mathcal{D}_{\bullet}, \mathcal{B}_{\star}]$, we have

$$R_{\bullet}^m \circ L_{\bullet}^n(T) = \langle \mathcal{D}_{\bullet}\rangle^{-n} \left(\operatorname{ad}(\mathcal{D}_{\bullet}^2)\right)^{n+m}(T) \langle \mathcal{D}_{\bullet}\rangle^{-m}.$$

Since $\mathcal{D}_{\bullet}^2 = H - (-1)^{\bullet}\tilde{\Omega}\Sigma$ with Σ bounded and $[H, \Sigma] = 0$, we get

$$R_{\bullet}^m \circ L_{\bullet}^n(T) = \sum_{k=0}^{n+m} \binom{n+m}{k} \left((-1)^{\bullet+1}\tilde{\Omega} \operatorname{ad}(\Sigma)\right)^{n+m-k} \left(\langle \mathcal{D}_{\bullet}\rangle^{-n} \left(\operatorname{ad}(H)\right)^k(T) \langle \mathcal{D}_{\bullet}\rangle^{-m}\right). \quad (21)$$

We treat the worst case only, which is when $k = n + m$ (for the other values of k, one may use similar but simpler arguments). That is, we need to show the boundedness of

$$\langle \mathcal{D}_{\bullet}\rangle^{-n} \left(\operatorname{ad}(H)\right)^{n+m}(T) \langle \mathcal{D}_{\bullet}\rangle^{-m}, \quad n, m \in \mathbb{N}.$$

The preceding expression applied to $T = [\mathcal{D}_1, L_{\star}(f)] = i\Gamma^{\mu}L_{\star}(\partial_{\mu}f)$ gives

$$\langle \mathcal{D}_1\rangle^{-n} \Gamma^{\mu}\left(\operatorname{ad}(H)\right)^{n+m}(L_{\star}(i\partial_{\mu}f)) \langle \mathcal{D}_1\rangle^{-m}$$
$$= \left(\Gamma^{\mu} + \langle \mathcal{D}_1\rangle^{-n}[\Gamma^{\mu}, \langle \mathcal{D}_1\rangle^n]\right)\langle \mathcal{D}_1\rangle^{-n} \left(\operatorname{ad}(H)\right)^{n+m}(L_{\star}(i\partial_{\mu}f)) \langle \mathcal{D}_1\rangle^{-m},$$

and using $[\Gamma^{\mu}, \langle \mathcal{D}_1\rangle^n] = [\Gamma^{\mu}, (\tilde{\Omega}\Sigma)^n]$ it is enough to treat the case $T \in \mathcal{B}_{\star}$. Similarly for $\mathcal{D}_2$. Now, Lemma 4 shows that $(\operatorname{ad}(H))^{n+m}(T)$ can be written as a sum of terms of the form $\hat{\nabla}^k L_{\star}(f)\hat{\nabla}^l$ with $f \in \mathcal{B}_{\star}$ and k (resp. l) not exceeding n (resp. m), where $\hat{\nabla}$ is ∇ or $\tilde{\nabla}$. Then, one concludes using Corollary 5. $\qquad\square$

In the next and in analogy with the regularity condition, we will prove that one can determine the dimension spectrum with the derivations $R_{\bullet}$ and $L_{\bullet}$, instead of $\delta_{\bullet}$.

Proposition 16. *Let b belonging to the polynomial algebra generated by $\delta_{\bullet}^n(\mathcal{A}_{\star})$ and $\delta_{\bullet}^n([\mathcal{D}_{\bullet}, \mathcal{A}_{\star}])$. Let also $\zeta_b(z) := \operatorname{Tr}(b\langle \mathcal{D}_{\bullet}\rangle^{-z})$, defined on the open half plane $\Re(z) > 2d$. Then, for all $M \in \mathbb{R}$, ζ_b is a finite sum of terms of the form $\operatorname{Tr}\left(R_{\bullet}^{n_1}(b_1) \cdots R_{\bullet}^{n_k}(b_k)\langle \mathcal{D}_{\bullet}\rangle^{-z-m}\right)$, $n_j, k, m \in \mathbb{N}$, $b_j \in \mathcal{A}_{\star} \cup [\mathcal{D}_{\bullet}, \mathcal{A}_{\star}]$, plus a function holomorphic on the half plane $\Re(z) > M$.*

Proof. For $\Re(z) > 2d$, $\langle \mathcal{D}_{\bullet}\rangle^{-z}$ is a trace class operator. Since the algebra generated by $\delta_{\bullet}^n(L_{\star}(f))$ and $\delta_{\bullet}^n([\mathcal{D}_{\bullet}, L_{\star}(f)])$ consists by Proposition 15 of *bounded operators*, $b\langle \mathcal{D}_{\bullet}\rangle^{-z}$ is trace class for $\Re(z) > 2d$ and any b in this polynomial algebra.

So let $b \in \mathcal{A}_{\star} \cup [\mathcal{D}_{\bullet}, \mathcal{A}_{\star}]$. From the spectral representation of a positive operator A:

$$A = \frac{1}{\pi} \int_0^{\infty} d\lambda\, \lambda^{-1/2} \frac{A^2}{A^2 + \lambda},$$

we get

$$\delta_\bullet(b) = \frac{1}{\pi} \int_0^\infty d\lambda \, \lambda^{1/2} \frac{1}{\langle \mathcal{D}_\bullet \rangle^2 + \lambda} [\langle \mathcal{D}_\bullet \rangle^2, b] \frac{1}{\langle \mathcal{D}_\bullet \rangle^2 + \lambda} \; .$$

Commuting $[\langle \mathcal{D}_\bullet \rangle^2, b]$ with $(\langle \mathcal{D}_\bullet \rangle^2 + \lambda)^{-1}$ to the left, we get after some re-arrangements and using $\int d\lambda \, \lambda^{1/2} t (t^2 + \lambda)^{-2} = \pi/2$:

$$\delta_\bullet(b) = \tfrac{1}{2} R_\bullet(b) - \frac{1}{\pi} \int_0^\infty d\lambda \, \lambda^{1/2} \frac{1}{\langle \mathcal{D}_\bullet \rangle^2 + \lambda} R_\bullet^2(b) \frac{\langle \mathcal{D}_\bullet \rangle^2}{(\langle \mathcal{D}_\bullet \rangle^2 + \lambda)^2} \; .$$

This suggests to introduce the map $\mathcal{T}_\bullet : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ given by

$$A \mapsto \mathcal{T}_\bullet(A) := \frac{2}{\pi} \int_0^\infty d\lambda \, \lambda^{1/2} \frac{1}{\langle \mathcal{D}_\bullet \rangle^2 + \lambda} A \frac{\langle \mathcal{D}_\bullet \rangle^2}{(\langle \mathcal{D}_\bullet \rangle^2 + \lambda)^2} \; .$$

Note that this operator is contractive. Indeed

$$\|\mathcal{T}_\bullet\| \le \frac{2}{\pi} \int_0^\infty \frac{\lambda^{1/2}}{(1 + \lambda)^2} \, d\lambda = 1 \; .$$

Thus $2\delta_\bullet = R_\bullet - \mathcal{T}_\bullet \circ R_\bullet^2$ and since $\mathcal{T}_\bullet$ commutes with $R_\bullet$ (because $R_\bullet$ commutes with the operators of left and right multiplications by functions of $\langle \mathcal{D}_\bullet \rangle$), we get

$$2^n \delta_\bullet^n = \sum_{k=0}^n \binom{n}{k} (-1)^k \mathcal{T}_\bullet^k \circ R_\bullet^{n+k} \; .$$

Hence, a typical element of the algebra generated by $\delta_\bullet^n(L_\star(f))$ and $\delta_\bullet^n([\mathcal{D}_\bullet, L_\star(f)])$, is a finite sum of elements of the form

$$\prod_{j=1}^k \mathcal{T}_\bullet^{n_j} \circ R_\bullet^{m_j}(b_j), \quad b_j \in \mathcal{A}_\star \cup [\mathcal{D}_\bullet, \mathcal{A}_\star], \quad n_j < m_j \in \mathbb{N} \; .$$

From the same reasoning as at the beginning of the proof, the function

$$\zeta_{R,\mathcal{T}}(b_1, n_1, m_1; \cdots; b_k, n_k, m_k; z) := \mathrm{Tr}\Big(\prod_{j=1}^k \mathcal{T}_\bullet^{n_j} \circ R_\bullet^{m_j}(b_j) \langle \mathcal{D}_\bullet \rangle^{-z} \Big) \; ,$$

is holomorphic on the open half plane $\Re(z) > 2d$. Starting from the definition, we have

$$\prod_{j=1}^k \mathcal{T}_\bullet^{n_j} \circ R_\bullet^{m_j}(b_j) = \int_{[0,\infty]^{|n|}} d\lambda^{|n|} \prod_{j=1}^k \Big(\prod_{r_j=1}^{n_j} \frac{(2/\pi)\lambda_{r_j}^{1/2}}{\langle \mathcal{D}_\bullet \rangle^2 + \lambda_{r_j}} \Big) R_\bullet^{m_j}(b_j) \Big(\prod_{s_j=1}^{n_j} \frac{\langle \mathcal{D}_\bullet \rangle^2}{(\langle \mathcal{D}_\bullet \rangle^2 + \lambda_{s_j})^2} \Big) \; .$$

The next step consists in commuting for each j the $R_\bullet^{m_j}(b_j)$ to the left of $(\langle \mathcal{D}_\bullet \rangle^2 + \lambda_{r_j})^{-1}$:

$$\Big[\frac{1}{\langle \mathcal{D}_\bullet \rangle^2 + \lambda_j}, R_\bullet^{m_j}(b_j) \Big] = -\frac{1}{\langle \mathcal{D}_\bullet \rangle^2 + \lambda_j} R_\bullet^{m_j+1}(b_j) \frac{\langle \mathcal{D}_\bullet \rangle}{\langle \mathcal{D}_\bullet \rangle^2 + \lambda_j}$$

17

$$= \sum_{p_j=1}^{N_j} (-1)^{p_j} R_\bullet^{m_j+p_j}(b_j) \frac{\langle \mathcal{D}_\bullet \rangle^{p_j}}{(\langle \mathcal{D}_\bullet \rangle^2 + \lambda_j)^{p_j+1}}$$

$$+ \frac{(-1)^{N_j+1}}{\langle \mathcal{D}_\bullet \rangle^2 + \lambda_j} R_\bullet^{m_j+N_j+1}(b_j) \left(\frac{\langle \mathcal{D}_\bullet \rangle}{\langle \mathcal{D}_\bullet \rangle^2 + \lambda_j} \right)^{N_j+1} . \qquad (22)$$

Any of the resulting λ-integrals is convergent, and $R_\bullet^n(b)$ is bounded for all $n \in \mathbb{N}$. Choosing the N_j large enough, we generate as much negative powers of $\langle \mathcal{D}_\bullet \rangle$ as necessary to make the product of the remainder with $\langle \mathcal{D}_\bullet \rangle^{-z}$ a trace-class operator for any given z with $\Re(z) > M$ (if M gets more and more negative we need larger and larger N_j). The other terms integrate to

$$\int_{[0,\infty]} d\lambda_j \frac{(2/\pi)\lambda_j^{1/2} \langle \mathcal{D}_\bullet \rangle^{p_j+2}}{(\langle \mathcal{D}_\bullet \rangle^2 + \lambda_{r_j})^{p_j+3}} = \frac{\Gamma(\frac{3}{2}+p_j)\langle \mathcal{D}_\bullet \rangle^{-1-p_j}}{\sqrt{\pi}\Gamma(3+p_j)} ,$$

so that, up to the remainder term, which is easily seen to be holomorphic on the open half plane $\Re(z) > M$, $\prod_{j=1}^k \mathcal{T}_\bullet^{n_j} R_\bullet^{m_j}(b_j)$ is a finite linear combination of

$$R_\bullet^{m_1+q_1}(b_1)\langle \mathcal{D}_\bullet \rangle^{-n_1-q_1} R_\bullet^{m_2+q_2}(b_2)\langle \mathcal{D}_\bullet \rangle^{-n_2-q_2} \cdots R_\bullet^{m_k+q_k}(b_k)\langle \mathcal{D}_\bullet \rangle^{-n_k-q_k} .$$

The final step consists in commuting the $\langle \mathcal{D}_\bullet \rangle^{-n_j-q_j}$ to the right. If $n_j + q_j = 2l_j$ is even, we use the ($\lambda_j = 0$)-case of (22). If $n_j + q_j = 2l_j - 1$ is odd,

$$\left[\langle \mathcal{D}_\bullet \rangle^{1-2l_j}, R_\bullet^{m'_{j+1}}(b_{j+1}) \right] = \langle \mathcal{D}_\bullet \rangle^{-2l_j} \delta_\bullet(R_\bullet^{m'_{j+1}}(b_{j+1})) + \left[\langle \mathcal{D}_\bullet \rangle^{-2l_j}, R_\bullet^{m'_{j+1}}(b_{j+1}) \right]\langle \mathcal{D}_\bullet \rangle .$$

Using $\delta_\bullet = \frac{1}{2}(R_\bullet - \mathcal{T}_\bullet \circ R_\bullet^2)$, this case is reduced to the first one. Eventually, we conclude that, up to a remainder term, which again is easily seen to be holomorphic on the open half plane $\Re(z) > M$, $\prod_{j=1}^k \mathcal{T}_\bullet^{n_j} \circ R_\bullet^{m_j}(b_j)$ is a finite linear combination of

$$R_\bullet^{m'_1}(b_1)R_\bullet^{m'_2}(b_2) \cdots R_\bullet^{m_k}(b_k)\langle \mathcal{D}_\bullet \rangle^{-m} .$$

This concludes the proof. □

We can now state the main result of this section, namely:

Theorem 17. *For $\bullet = 1, 2$, the spectral triple $(\mathcal{A}_\star, \mathcal{H}, \mathcal{D}_\bullet)$ has dimension spectrum* Sd $= d - \mathbb{N}$. *Moreover, all poles of $\zeta_b(z)$ at $z \in$ Sd are simple with local residues, i.e. for $b = \delta_\bullet^{n_1} L_\star(f_1) \cdots \delta_\bullet^{n_k} L_\star(f_k)$, any residue $\mathrm{res}_{z \in \mathrm{Sd}}\zeta_b(z)$ is a finite sum of terms of the form*

$$\int_{\mathbb{R}^d} dx \, x^{\alpha_0} \star (\partial^{\alpha_1}f_1) \star \cdots \star (\partial^{\alpha_k}f_k) ,$$

where $\alpha_i \in \mathbb{N}^d$. An analogous result holds when the $L_\star(f_k)$'s in b are replaced by $[\mathcal{D}_\bullet, L_\star(f_k)]$'s.

Proof. According to Proposition 16, it is equivalent to consider the functions

$$\mathrm{Tr}\big(R_\bullet^{m_1}(b_1) \cdots R_\bullet^{m_k}(b_k)\langle \mathcal{D}_\bullet \rangle^{-z}\big), \quad b_i \in \mathcal{A}_\star \cup [\mathcal{D}_\bullet, \mathcal{A}_\star] ,$$

18

instead of $\zeta_b(z)$. These functions are well defined for $\Re(z) > 2d$, since $R^m_\bullet(b)$ is bounded.

Since $[H, \Sigma] = 0$, we get from (21)

$$R^{m_j}_\bullet(b_j) = \sum_{k=0}^{m_j} \binom{m_j}{k} \big(\mathrm{ad}(H)\big)^k \left(\big(\mathrm{ad}(\tilde{\Omega}\Sigma)\big)^{m_j-k}(b_j) \right) \langle \mathcal{D}_\bullet \rangle^{-m_j} .$$

Since b_j is either $L_\star(f_j)$ or $i\Gamma^\mu L_\star(\partial_\mu f_j)$ for $\bullet = 1$ and $i\Gamma^{\mu+d} L_\star(\partial_\mu f_j)$ for $\bullet = 2$, we get

$$\mathrm{ad}(\tilde{\Omega}\Sigma))^{m_j-k}(b_j) = \begin{cases} \delta_{k,m_j} L_\star(f_j) \\ \mathrm{ad}(\tilde{\Omega}\Sigma))^{m_j-k} i\Gamma^\mu L_\star(\partial_\mu f_j) \\ \mathrm{ad}(\tilde{\Omega}\Sigma))^{m_j-k} i\Gamma^{\mu+d} L_\star(\partial_\mu f_j) . \end{cases}$$

We may therefore assume that $b_j = M_j L_\star(f_j)$ with $f_j \in \mathcal{A}$ and $M_j \in \mathrm{Mat}_{2^d}(\mathbb{C})$. Thus, the product $R^{m_1}_\bullet(b_1) \cdots R^{m_k}_\bullet(b_k)$ can be expressed as a finite sum of terms of the form

$$\big(\mathrm{ad}(H)\big)^{n_1}\big(L_\star(f_1')\big) M_1 \langle \mathcal{D}_\bullet \rangle^{-m_1} \cdots \big(\mathrm{ad}(H)\big)^{n_k}\big(L_\star(f_k')\big) M_k \langle \mathcal{D}_\bullet \rangle^{-m_k}, \quad n_j \le m_j .$$

Using the table given in Lemma 4 we can express $R^{m_1}_\bullet(b_1) \cdots R^{m_k}_\bullet(b_k)\langle \mathcal{D}_\bullet \rangle^{-z}$ as a finite sum of terms

$$L_\star(\partial^{\alpha_1} f_1) M_1 P_{\alpha_1}(\hat{\tilde{\nabla}})\langle \mathcal{D}_\bullet \rangle^{-m_1} \cdots L_\star(\partial^{\alpha_k} f_k) M_k P_{\alpha_k}(\hat{\tilde{\nabla}})\langle \mathcal{D}_\bullet \rangle^{-m_k} \langle \mathcal{D}_\bullet \rangle^{-z}$$

$$= \frac{1}{\Gamma(\frac{m_1}{2}) \cdots \Gamma(\frac{m_{k-1}}{2})\Gamma(\frac{m_k+z}{2})} \int dt_1 \cdots dt_k \, t_1^{\frac{m_1}{2}-1} \cdots t_k^{\frac{m_k+z}{2}-1}$$

$$\times L_\star(\partial^{\alpha_1} f_1) M_1 P_{\alpha_1}(\hat{\tilde{\nabla}}) e^{-t_1(\mathcal{D}_\bullet^2+1)} \cdots L_\star(\partial^{\alpha_k} f_k) M_k P_{\alpha_k}(\hat{\tilde{\nabla}}) e^{-t_k(\mathcal{D}_\bullet^2+1)} ,$$

where $P_{\alpha_j}(\hat{\tilde{\nabla}})$ is a polynomial in $\nabla, \tilde{\nabla}$ of degree $|\alpha_j| \le m_j$ (the α_j are multi-indices).

Using

$$[e^{-t_j(\mathcal{D}_\bullet^2+1)}, T] = -t_j \int_0^1 ds_j e^{-t_j s_j(\mathcal{D}_\bullet^2+1)}[\mathcal{D}_\bullet^2, T] e^{-t_j(1-s_j)(\mathcal{D}_\bullet^2+1)} ,$$

we commute all heat operators $e^{-t_j(\mathcal{D}_\bullet^2+1)}$ to the right, producing in each step a factor of t_j. The commutators $[H, T]$ are expressed by Lemma 4 and produce in each step at most one derivative ∇. In the terms with all heat operators already on the right we then commute the derivatives $\hat{\tilde{\nabla}}$ to the right of all functions f_j but left of all heat operators. The result is a finite sum of terms (with redefined f_j, M_j)

$$X = \frac{\int_{[0,1]^N} ds \, P(s)}{\Gamma(\frac{m_1}{2}) \cdots \Gamma(\frac{m_{k-1}}{2})\Gamma(\frac{m_k+z}{2})} \int dt_1 \cdots dt_k \, t_1^{\frac{m_1}{2}+\beta_1-1} \cdots t_{k-1}^{\frac{m_{k-1}}{2}+\beta_{k-1}-1} t_k^{\frac{m_k+z}{2}+\beta_k-1} \, e^{-t}$$

$$\times \big(M_1 \cdots M_k e^{(-)^\bullet \tilde{\Omega}(t_1+\cdots+t_k)\Sigma} \big) L_\star\big(\partial^{\gamma_1} f_1 \star \cdots \star \partial^{\gamma_k} f_k\big) P_{\gamma_1,\ldots,\gamma_k}(\hat{\tilde{\nabla}}) e^{-(t_1+\cdots+t_k)H} , \quad (23)$$

with $|m + \beta| \ge |\gamma|$ and $|\beta| \le K$ (K can be chosen as big as one wishes by pushing the expansion far enough), plus a finite sum of remainders (with redefined f_j, M_j)

$$Y = \frac{1}{\Gamma(\frac{m_1}{2}) \cdots \Gamma(\frac{m_{k-1}}{2})\Gamma(\frac{m_k+z}{2})} \int dt_1 \cdots dt_k \, t_1^{\frac{m_1}{2}+\beta_1'-1} \cdots t_{k-1}^{\frac{m_{k-1}}{2}+\beta_{k-1}'-1} t_k^{\frac{m_k+z}{2}+\beta_k'-1}$$

$$\times \int_{[0,1]^{N'}} ds \, P'(s) \prod_{j=1}^{k} L_\star(\partial^{\gamma_j'} f_j) M_j P_{\gamma_j'}(\hat{\tilde{\nabla}}) e^{-\tau_j(\mathcal{D}_\bullet^2+1)} , \quad (24)$$

19

with $|\gamma'| \le |m + \beta'|$ and $|\beta'| > K$, where τ_j are positive functions of $\{s\}, t_1, \ldots, t_k$ with $\sum_{j=1}^{k} \tau_j = t_1 + \cdots + t_k$.

In (23) we can use Lemma 4 to express $P_\gamma(\hat{\nabla})$ as a finite sum of $P_{\gamma'}(x)P_{\gamma''}(\nabla)$. The polynomial $P'_\gamma(x)$ can (under the trace) be moved into $L_\star(f)$. We change the variables $t_1 = t(1 - u_2)(1 - u_3) \cdots (1 - u_k)$, $t_2 = tu_2(1 - u_3) \cdots (1 - u_k)$, $t_3 = tu_3(1 - u_4) \cdots (1 - u_k), \ldots t_{k-1} = tu_{k-1}(1 - u_k)$ and $t_k = tu_k$ with Jacobian $t^{k-1}(1 - u_3)(1 - u_4)^2 \cdots (1 - u_k)^{k-2}$ and obtain

$$
\mathrm{Tr}(X) = \frac{\int_{[0,1]^N} ds\, P(s)}{\Gamma(\frac{|m|+z}{2} + |\beta|)} \Big(\prod_{j=1}^{-1} \frac{\Gamma(\frac{m_j}{2} + \beta_j)}{\Gamma(\frac{m_j}{2})} \Big) \frac{\Gamma(\frac{m_k+z}{2} + \beta_k)}{\Gamma(\frac{m_k+z}{2})} \int_0^\infty dt\, e^{-t}\, t^{\frac{|m|+z}{2} + |\beta| - 1} \tag{25}
$$

$$
\times\ \mathrm{tr}_{\mathbb{C}^{2^d}}\big(M_1 \cdots M_k e^{(-1)^\bullet \tilde{\Omega} t \Sigma}\big) \mathrm{Tr}\Big(L_\star\big(x^{\gamma_0} \star \partial^{\gamma_1} f_1 \star \cdots \star \partial^{\gamma_k} f_k\big) P_{\gamma_1, \ldots, \gamma_k}(\nabla)\, e^{-tH}\Big)\,.
$$

Note that $\frac{\Gamma(\frac{m_k+z}{2} + \beta_j)}{\Gamma(\frac{m_k+z}{2})}$ is a polynomial in z of degree β_k.

The traces $\mathrm{Tr}(L_\star(f)\nabla_{\mu_1} \cdots \nabla_{\mu_r} e^{-tH})$ are computed in Proposition 6. Accordingly, they have, up to a remainder which leads to a holomorphic function in z, an asymptotic expansion

$$
\mathrm{Tr}(L_\star(f)\nabla_{\mu_1} \cdots \nabla_{\mu_{|\gamma|}} e^{-tH}) = \sum_{a=0}^{N''} t^{-d/2 - [\frac{|\gamma|}{2}] + a} \int_{\mathbb{R}^d} dx\, f(x) P^a_{|\gamma|}(x)\,,
$$

where $P^a_{|\gamma|}(x)$ is a $\star$-polynomial of degree $\le |\gamma| + 2a$. Inserted into (25), the t-integral of any such term yields (together with $\mathrm{tr}_{\mathbb{C}^{2^d}}\big(M_1 \cdots M_k e^{(-1)^\bullet \tilde{\Omega} t \Sigma}\big)$) a linear combination of

$$
\frac{\Gamma(\frac{|m|+2|\beta|-2[|\gamma|/2]+2a+z-d}{2})}{\Gamma(\frac{|m|+2\beta+z}{2})} \int_{\mathbb{R}^d} dx\, f(x) P^a_{|\gamma|}(x)\,.
$$

As a function of $z \in \mathbb{C}$, the latter is trivially holomorphic in $\mathbb{C} \setminus \mathbb{Z}$. As $|m| + |\beta| \ge |\gamma|$, this function is also holomorphic for $z > d$. If $z = d - N$ there is a finite number of parameters $\beta_j, m_j, a, \gamma_j$ for which $\frac{|m|+2|\beta|-2[|\gamma|/2]+2a-N}{2}$ is a non-positive integer smaller than $\frac{|m|+2\beta+d-N}{2}$. Precisely these parameters yield simple poles at $z = d - N$. Each residue has the claimed structure.

We estimate the remainders (24) in trace norm by:

$$
\frac{\|M_1\| \cdots \|M_k\|}{|\Gamma(\frac{m_1}{2}) \cdots \Gamma(\frac{m_k+z}{2})|} \int dt_1 \cdots dt_k\, t_1^{\frac{m_1}{2} + \beta'_1 - 1} \cdots t_{k-1}^{\frac{m_{k-1}}{2} + \beta'_{k-1} - 1} t_k^{\frac{m_k+z}{2} + \beta'_k - 1}
$$

$$
\times \int_{[0,1]^{N'}} ds\, |P'(s)| \Big\| \prod_{j=1}^{k} L_\star(\partial^{\gamma'_j} f_j) e^{(-1)^\bullet \tau_j \tilde{\Omega} \Sigma} P_{\gamma'_j}(\nabla) e^{-\tau_j(H+1)} \Big\|_1\,.
$$

By spectral theory and Corollary 5, we get for $p \in [1, \infty)$:

$$
\|P_{\gamma'_j}(\nabla) e^{-\tau_j(H+1)}\|_p \le C(\epsilon \tau_j)^{-\gamma'_j/2} \|e^{-\tau'_j(H+1)(1-\epsilon)}\|_p\,.
$$

20

Since the τ_j are linear in t_j with $\sum_{j=1}^{k} \tau_j = \sum_{j=1}^{k} t_j$, setting $t = \sum_{j=1}^{k} t_j$, the Hölder inequality gives

$$\Big\| \prod_{j=1}^{k} L_\star(\partial^{\gamma'_j} f_j) e^{(-1)^\bullet \tau_j \tilde{\Omega}\Sigma} P_{\gamma'_j}(\nabla) e^{-\tau'_j(H+1)} \Big\|_1$$

$$\leq \prod_{j=1}^{k} \|L_\star(\partial^{\gamma'_j} f_j)\| \, \|e^{(-1)^\bullet \tau_j \tilde{\Omega}\Sigma}\|_{t/\tau_j} \|P_{\gamma'_j}(\nabla) e^{-\tau_j(H+1)}\|_{t/\tau'_j}$$

$$\leq C e^{-t(1-\epsilon)} \prod_{j=1}^{k} (\epsilon t_j)^{-\gamma'_j/2} \|L_\star(\partial^{\gamma'_j} f_j)\| \, (2\cosh(\tilde{\Omega}t))^{\frac{d\tau_j}{t}} (2\sinh(\tilde{\Omega}t(1-\epsilon))^{-\frac{d\tau_j}{t}}$$

$$= C e^{-t(1-\epsilon-\tilde{\Omega}\epsilon d)} \Big(\prod_{j=1}^{k} (\epsilon t_j)^{-\gamma'_j/2} \|L_\star(\partial^{\gamma'_j} f_j)\| \Big) \Big(\frac{\cosh(\tilde{\Omega}t)}{\sinh(\tilde{\Omega}t(1-\epsilon))e^{\tilde{\Omega}t\epsilon}} \Big)^d \,,$$

where results of Proposition 6 and Lemma 7 have been used. Hence for $\epsilon' := (1-\epsilon-\tilde{\Omega}\epsilon d) > 0$ the remainders (24) are bounded in trace norm by

$$\frac{C'}{|\Gamma(\frac{m_1}{2}) \cdots \Gamma(\frac{m_k+z}{2})|} \int dt_1 \cdots dt_k \, t_1^{\frac{m_1}{2}+\beta'_1-1-\frac{\gamma'_1}{2}} \cdots t_k^{\frac{m_k+z}{2}+\beta'_k-1-\frac{\gamma'_k}{2}} \, (t_1 + \cdots + t_k)^{-d} e^{-\epsilon'(t_1+\cdots+t_k)} \,.$$

Remember that $|\gamma'| \leq |\beta'| + |m|$ and $|\beta'| > K$ and that K can be chosen as big as one wishes (by pushing the expansion over and over). So given $M \leq 2d$, by choosing $K > M/2 + d$, we see that the remainder terms are well defined as a trace-class operators for $\Re(z) > M$. A similar analysis involving the z-derivative of the remainders can be done, showing that by pushing the expansion far enough, the remainders yield holomorphic contributions for $\Re(z) > M$, with $M \in \mathbb{R}$ arbitrary. $\qquad\square$

3 The spectral action

3.1 Generalities

For a unital spectral triple with real structure $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J)$, according to the spectral action principle [3, 5], the bosonic action should depend only on the spectrum of the fluctuated Dirac operator

$$\mathcal{D} \mapsto \mathcal{D}_A := \mathcal{D} + A + \varepsilon' J A J^{-1} \,,$$

where $A = \sum a_i[\mathcal{D}, b_i]$, $a_i, b_i \in \mathcal{A}$ (finite sum), is a self-adjoint one-form and $\varepsilon' = \pm 1$ depending on the KO-dimension of the triple. Ideally, such an action functional (of $\mathcal{D}$ and of A) should be defined as the number of eigenvalues of $\mathcal{D}_A^2$ smaller than a given scale $\Lambda > 0$:

$$S_\Lambda(\mathcal{D}_A) = \sharp\big\{ \lambda_n \, : \, \lambda_n \in \mathrm{Spect}(\mathcal{D}_A^2), \, \lambda_n \leq \Lambda \big\} \,,$$

or akin to the same and with χ the characteristic function of the interval $[0, 1]$:

$$S_\Lambda(\mathcal{D}_A) = \mathrm{Tr}\big(\chi(\mathcal{D}_A^2/\Lambda^2)\big) \,. \tag{26}$$

Then, the diverging part of $S_\Lambda(\mathcal{D}_A)$ in the limit $\Lambda \to \infty$ should give access to an effective action describing low energy physics. The problem is that with the characteristic function, the expression (26) may not have a well-defined power series expansion in the limit $\Lambda \to \infty$. To overcome this difficulty one uses, instead of the characteristic function, a smooth one approximating it and being the inverse Laplace transform of a Schwartz function on $\mathbb{R}_+^*$. By Laplace transformation one then has

$$S_\Lambda(\mathcal{D}_A) = \int_0^\infty dt \, \mathrm{Tr}(e^{-t\mathcal{D}_A^2/\Lambda^2}) \, \hat{\chi}(t) \,,$$

where $\hat{\chi}$ is the inverse Laplace transform of χ. Assuming that the trace of the heat kernel has an asymptotic expansion

$$\mathrm{Tr}(e^{-t\mathcal{D}_A^2}) = \sum_{k=-n}^\infty a_k(\mathcal{D}_A^2)\, t^k \,, \qquad n \in \mathbb{N} \,,$$

we obtain

$$S_\Lambda(\mathcal{D}_A) = \sum_{k=-n}^\infty a_k(\mathcal{D}_A^2)\, \Lambda^{-2k} \int_0^\infty dt \, t^k \hat{\chi}(t) \,. \tag{27}$$

One easily finds

$$\int_0^\infty dt \, t^k \hat{\chi}(t) = \begin{cases} \frac{1}{\Gamma(-k)} \int_0^\infty ds \, s^{-k-1}\chi(s)\,, & \text{for } k \notin \mathbb{N} \,, \\ (-1)^k \chi^{(k)}(0)\,, & \text{for } k \in \mathbb{N} \,. \end{cases}$$

If one only wants to keep the non-vanishing terms in the power-Λ expansion as $\Lambda \to \infty$, it is therefore sufficient to identify the the non-vanishing terms in the power-t expansion of $\mathrm{Tr}(e^{-t\mathcal{D}_A^2})$ as $t \to 0$.

If the spectral geometry $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J)$ is non-unital, then the expression (26) becomes ill-defined. There are different ways to 'regularize' the spectral action in this case. For instance one may consider instead

$$S_\Lambda(\mathcal{D}_A) := \mathrm{Tr}\big(\chi(\mathcal{D}_A^2/\Lambda^2) - \chi(\mathcal{D}^2/\Lambda^2)\big) \,.$$

But the problem is then that one loses a lot of physical information since one cannot access in this way the Einstein-Hilbert action. Another possibility, used in [13], is to introduce a supplementary (adynamic) scalar field $\rho \in \mathcal{A}$ and to define

$$S_\Lambda(\mathcal{D}_A, \rho) := \mathrm{Tr}\big(\rho\,\chi(\mathcal{D}_A^2/\Lambda^2)\big) \,.$$

The advantage of this scheme is that one keeps the physical interpretation by performing on the field equations the adiabatic limit $\rho \to 1$ and that one can choose the one-form A coming from the unitization of $\mathcal{A}$ and not necessarily from $\mathcal{A}$ itself. However, in the case of the Moyal spectral triple with ordinary Dirac operator, treated in [13], the full computation

22

with the real structure was not possible; only the spectral action for partially fluctuated Dirac operator $\mathcal{D} \mapsto \mathcal{D} + A$ was evaluated. The last possibility spelled out in [4] is to replace the scale Λ by a dilaton field. That is, one performs the replacement

$$\Lambda \mapsto e^{-\phi}, \quad \phi^* = \phi \in \mathcal{A} \, ,$$

and one leaves (26) as it was:

$$S_\phi(\mathcal{D}_A) := \mathrm{Tr}\big(\chi(e^\phi \, \mathcal{D}_A^2 \, e^\phi)\big) \, .$$

Although this expression is analytically well-defined and conceptually perfect, the explicit computation of such a functional seems to be fairly inaccessible, except for the commutative (manifold) case.

In our setting of a spectral triple for Moyal plane with harmonic propagation, the question of the definition and the computation of the spectral action is way more easy. This is because even if the spectral triple $(\mathcal{A}_\star, \mathcal{H}, \mathcal{D}_\bullet)$, $\bullet = 1, 2$, is non-unital, the heat operator $e^{-t\mathcal{D}_\bullet^2}$ is trace-class for all $t > 0$ (see Lemma 9). This means that the definition (26) of the spectral action for unital spectral triple is still adapted to our situation.

In the next subsections we will perform a complete computation of the spectral action for a $U(1)$-Higgs model for $d = 4$. Before this, we will derive a generic heat kernel type expansion when one tensorizes $(\mathcal{A}_\star, \mathcal{H}, \mathcal{D}_\bullet)$ with a finite spectral triple.

3.2 Heat kernel expansion in dimension four

We derive here a short-time heat-kernel expansion for the semi-group generated by the square of a twisted harmonic Dirac operator, for the algebra of Schwartz functions with Moyal product. We start with preliminary results on Schatten norm estimates, using the estimate of Lemma 12 together with complex interpolation methods. Here we specify to the case $d = 4$.

Proposition 18. *Let $f \in \mathcal{A}_\star$. Then for all $1 \le p \le \infty$ and $t \in (0, 1]$, we have*

$$\|L_\star(f) e^{-t\mathcal{D}_\bullet^2}\|_p \le \|f\|_2^{1-1/p} \, C(f)^{1/p} \, p^{-2/p} \, t^{-2/p} \, ,$$

where $C(f)$ is the constant appearing in Lemma 12.

Proof. For $f \in \mathcal{A}_\star$, $t \in (0, 1]$ and $1 \le p < \infty$, consider on the strip $S := \{z \in \mathbb{C} : \Re(z) \in [0, 1]\}$ the operator-valued function

$$F_p : z \mapsto L_\star(f) e^{-tpz\mathcal{D}_\bullet^2} \, .$$

The function F_p is continuous on S, holomorphic on its interior and by Lemma 12 it satisfies for $y \in \mathbb{R}$:

$$\|F_p(iy)\| \le \|f\|_2, \qquad \|F_p(1 + iy)\|_1 \le C(f)(pt)^{-2} \, .$$

Then, by standard complex interpolation methods (see for example [30]) we have $F_p(z) \in \mathcal{L}^{1/\Re(z)}(\mathcal{H})$ for all $z \in S$ with

$$\|F_p(z)\|_{1/\Re(z)} \leq \|F_p(0)\|_\infty^{1-\Re(z)} \|F_p(1)\|_1^{\Re(z)} \leq \|f\|_2^{1-\Re(z)} C(f)^{\Re(z)} (pt)^{-2\Re(z)} \ .$$

Applying this for $z = 1/p$, we get

$$\|L_\star(f) e^{-t\mathcal{D}_\bullet^2}\|_p \leq \|f\|_2^{1-1/p} C(f)^{1/p} p^{-2/p} t^{-2/p} \ ,$$

as needed. $\qquad\square$

Remark 19. *For $f \in \mathcal{A}_\star$, making a recursive chose of factorization as follows:*

$$f = f_1 \star f_2, \quad \partial_{\mu_1} f_2 = f_{1,\mu_1} \star f_{2,\mu_1}, \quad \cdots \quad \partial_{\mu_4} f_{2,\mu_1\mu_2\mu_3} = f_{1,\mu_1\mu_2\mu_3\mu_4} \star f_{2,\mu_1\mu_2\mu_3\mu_4} \ ,$$

the constant $C(f)$ appearing in Lemma 12 (for $d = 4$) and Proposition 18 is a finite multiple (depending only on $\tilde{\Omega}$ and Θ) of

$$\|\bar{f}_1 \star f_1\|_1^{1/2} \|\bar{f}_2 \star f_2\|_1^{1/2} + \|f_1\|_2 \sum_{\mu_1=1}^{4} \left(\|\bar{f}_{1,\mu_1} \star f_{1,\mu_1}\|_1^{1/2} \|\bar{f}_{2,\mu_1} \star f_{2,\mu_1}\|_1^{1/2} + \|f_{1,\mu_1}\|_2 \sum_{\mu_2=1}^{4} \right.$$

$$\|\bar{f}_{1,\mu_1\mu_2} \star f_{1,\mu_1\mu_2}\|_1^{1/2} \|\bar{f}_{2,\mu_1\mu_2} \star f_{2,\mu_1\mu_2}\|_1^{1/2} + \|f_{1,\mu_1\mu_2}\|_2 \sum_{\mu_3=1}^{4} \left(\|\bar{f}_{1,\mu_1\mu_2\mu_3} \star f_{1,\mu_1\mu_2\mu_3}\|_1^{1/2} \right.$$

$$\times \|\bar{f}_{2,\mu_1\mu_2\mu_3} \star f_{2,\mu_1\mu_2\mu_3}\|_1^{1/2} + \|f_{1,\mu_1\mu_2\mu_3}\|_2 \sum_{\mu_4=1}^{4} \left(\|\bar{f}_{1,\mu_1\mu_2\mu_3\mu_4} \star f_{1,\mu_1\mu_2\mu_3\mu_4}\|_1^{1/2} \right.$$

$$\times \|\bar{f}_{2,\mu_1\mu_2\mu_3\mu_4} \star f_{2,\mu_1\mu_2\mu_3\mu_4}\|_1^{1/2} + \|f_{1,\mu_1\mu_2\mu_3\mu_4}\|_2 \|f_{2,\mu_1\mu_2\mu_3\mu_4}\|_2 \Bigg) \Bigg) \Bigg) \Bigg) \ .$$

Lemma 20. *Let $f \in \mathcal{A}_\star$, $1 \leq p \leq \infty$, $t \in (0,1]$ and $k \in \mathbb{N}$. Then, there exists a finite constant $C_{p,k}(f)$ such that*

$$\|L_\star(f) \mathcal{D}_\bullet^k e^{-t\mathcal{D}_\bullet^2}\|_p \leq C_{p,k}(f) t^{-2/p-k/2} \ .$$

Proof. By spectral theory, we have $\|\mathcal{D}_\bullet^k e^{-t\mathcal{D}_\bullet^2}\| = (k/2et)^{k/2}$, so the proof is a consequence of Proposition 18. $\qquad\square$

The next Lemma will explain why there is a major difference in the spectral action when perturbing $\mathcal{D}$ by $A + JAJ^{-1}$ or simply by A.

Lemma 21. *For $f, g \in \mathcal{A}_\star$, the operator $L_\star(f) R_\star(g)$ is of trace class on $\mathcal{H}$.*

Proof. By factorization, we can find $f_1, f_2, g_1, g_2 \in \mathcal{A}_\star$ such that

$$f = f_1 \star f_2, \quad g = g_1 \star g_2 \ .$$

Hence
$$L_\star(f)\,R_\star(g) = L_\star(f_1 \star f_2)\,R_\star(g_1 \star g_2) = L_\star(f_1)L_\star(f_2)\,R_\star(g_1)R_\star(g_2)\ .$$

But since the left and right regular representations commute (by associativity of the Moyal product), we get
$$L_\star(f)\,R_\star(g) = L_\star(f_1)R_\star(g_1)\,L_\star(f_2)R_\star(g_2)\ ,$$

so that it suffices to show that $L_\star(f)\,R_\star(g)$ is Hilbert-Schmidt for all $f, g \in \mathcal{A}_\star$. From the operator kernel formula (17) of $L_\star(f)$, and a similar one for $R_\star(g)$, one easily deduces the operator kernel for the product $L_\star(f)\,R_\star(g)$, and after a few lines of computations, we get for a suitable constant depending only on $\det(\Theta)$:

$$\|L_\star(f)\,R_\star(g)\|_2^2 = \int dx\,dy\,\big|[L_\star(f)\,R_\star(g)](x,y)\big|^2 = C\|f\|_2^2\,\|g\|_2^2\ .$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 22. *Let* ∇_μ^a, $\mu = 1, \cdots, 4$, *be the operators on* $L^2(\mathbb{R}^4)$ *given by*

$$\nabla_\mu^a := i\partial_\mu + a_{\mu\nu}x^\mu\ , \quad a \in M_4(\mathbb{R})\ .$$

Then for $f, g \in \mathcal{A}_\star$, $t \in (0,1]$ *and* $P_\alpha(\hat{\nabla})$ *a polynomial of order* α *in the operators* ∇_μ^a, *there exists a finite constant* $C(f, g, \alpha)$ *such that*

$$\big\|L_\star(f)R_\star(g)P_\alpha(\hat{\nabla})e^{-tH}\big\|_1 \le C(f, g, \alpha)\,t^{-\alpha/2}\ .$$

Proof. From Lemma 21, it suffices to show that $\|P_\alpha(\hat{\nabla})e^{-tH}\| \le Ct^{-\alpha/2}$, which will follow by spectral theory if $P_\alpha(\hat{\nabla})(1 + H)^{-|\alpha|/2}$ is bounded. But this is a slight generalization of Corollary 5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We can now deduce the germ of the asymptotic expansion formula we need.

Proposition 23. *Let* $(\mathcal{A}_\mathbf{f}, \mathcal{H}_\mathbf{f}, \mathcal{D}_\mathbf{f}, J_\mathbf{f})$ *be a finite spectral triple. Let* $\mathcal{D} := \mathcal{D}_\bullet \otimes 1 + \Gamma \otimes \mathcal{D}_\mathbf{f}$, $\bullet = 1, 2$, *be the Dirac operator of the product spectral triple* $(\mathcal{A} \otimes \mathcal{A}_\mathbf{f}, \mathcal{H} \otimes \mathcal{H}_\mathbf{f}, \mathcal{D}_\bullet \otimes 1 + \Gamma \otimes \mathcal{D}_\mathbf{f})$. *Let also* $\mathcal{D}_A := \mathcal{D} + A + \mathbf{J}A\mathbf{J}^{-1}$ *be the fluctuated Dirac operator. Here* $\mathbf{J} := J \otimes J_\mathbf{f}$ *and* A *is a self-adjoint one-form, that is* $A = A^* := \sum_i a_i[\mathcal{D}, b_i]$, *where the sum is finite and* $a_i, b_i \in \mathcal{A} \otimes \mathcal{A}_\mathbf{f}$. *In terms of the decomposition* $\mathcal{D}_A^2 = \mathcal{D}^2 + F_0 + F_1 + \mathbf{J}(F_0 + F_1)\mathbf{J}^{-1} + 2A\mathbf{J}A\mathbf{J}^{-1}$, *where* F_0 *is a bounded operator and* F_1 *is linear in the operators* ∇_μ^a *of Corollary 22, the following holds:*

$$\begin{aligned}
\mathrm{Tr}\big(e^{-t\mathcal{D}_A^2}\big) = \mathrm{Tr}\Big(&\Big\{1 - 2t(F_0 + F_1) + t^2\big(F_0^2 + F_1F_0 + F_0F_1 + F_1^2\big) \\
&- \frac{t^3}{3}\big(F_0[\mathcal{D}^2, F_1] - [\mathcal{D}^2, F_1]F_0 + F_1[\mathcal{D}^2, F_1] + F_0F_1^2 + F_1F_0F_1 + F_1^2F_0 + F_1^3\big) \\
&+ \frac{t^4}{12}\big(F_1[\mathcal{D}^2, [\mathcal{D}^2, F_1]] + 2F_1^2[\mathcal{D}^2, F_1] + F_1[\mathcal{D}^2, F_1]F_1 + F_1^4\big)\Big\}e^{-t\mathcal{D}^2}\Big) + \mathcal{O}(\sqrt{t})\ .
\end{aligned}$$

Proof. First, it is clear that $e^{-t\mathcal{D}_A^2}$ is of trace-class for all $t > 0$. Indeed, since the eigenvalues of $\mathcal{D}_\bullet$ behave like $n^{-1/8}$, its resolvent belongs to the Schatten ideal $\mathcal{L}^{8+\varepsilon}(\mathcal{H})$ for all $\varepsilon > 0$. From the relation

$$\frac{1}{\mathcal{D} + i} = \frac{1}{\mathcal{D}_\bullet \otimes 1 + i}\left(1 - (A + \mathbf{J}A\mathbf{J}^{-1} + \Gamma \otimes \mathcal{D}_\mathbf{f})\frac{1}{\mathcal{D} + i}\right),$$

and the fact that $A + \mathbf{J}A\mathbf{J}^{-1} + \Gamma \otimes \mathcal{D}_\mathbf{f}$ is bounded, we see that the resolvent of $\mathcal{D}$ belongs to $\mathcal{L}^{8+\varepsilon}(\mathcal{H} \otimes \mathcal{H}_\mathbf{f})$ for all $\varepsilon > 0$ too. Accordingly, $e^{-t\mathcal{D}_A^2}$ is of trace-class for all $t > 0$.

Note also that the bounds of Lemmas 10, 12, 20 and Proposition 18, 22 remain valid with $\mathcal{D}$ instead of $\mathcal{D}_\bullet$. Indeed, since $\{\mathcal{D}_\bullet, \Gamma\} = 0$ and $\Gamma^2 = 1$, we get $\mathcal{D}^2 = \mathcal{D}_\bullet^2 \otimes 1 + 1 \otimes \mathcal{D}_\mathbf{f}^2$ and thus

$$\mathcal{D}^k e^{-t\mathcal{D}^2} = \sum_{j=0}^{k} C_{k,j} \Gamma^{k-j} \mathcal{D}_\bullet^j e^{-t\mathcal{D}_\bullet^2} \otimes \mathcal{D}_\mathbf{f}^{k-j} e^{-t\mathcal{D}_\mathbf{f}^2} .$$

This implies for $f, g \in \mathcal{B}_\star$, $a, b \in \mathcal{A}_\mathbf{f}$ and $1 \le p \le \infty$,

$$\|L_\star(f) \otimes a \, \mathbf{J}L_\star(g) \otimes b\mathbf{J}^{-1} \, \mathcal{D}^k e^{-t\mathcal{D}^2}\|_p$$

$$\le \sum_{j=0}^{k} |C_{k,j}| \|L_\star(f) \, R_\star(\overline{g}) \, \mathcal{D}_\bullet^j e^{-t\mathcal{D}_\bullet^2}\|_p \|aJ_\mathbf{f}bJ_\mathbf{f}^{-1}\mathcal{D}_\mathbf{f}^{k-j}e^{-t\mathcal{D}_\mathbf{f}^2}\|_p .$$

Thus, we may assume without loss of generality that there is no finite spectral triple in the picture.

We are going to deduce the expansion from the Duhamel principle:

$$e^{-t(A+B)} = e^{-tA} - t\int_0^1 e^{-st(A+B)} \, B \, e^{-(1-s)tA} \, ds .$$

We write $\mathcal{D}_A^2 = \mathcal{D}^2 + \tilde{F}_0 + \tilde{F}_1$, with $\tilde{F}_0 := F_0 + \mathbf{J}F_0\mathbf{J}^{-1} + 2A\mathbf{J}A\mathbf{J}$ and $\tilde{F}_1 := F_1 + \mathbf{J}F_1\mathbf{J}^{-1}$. The operator $\tilde{F}_0$ is bounded, whereas $\tilde{F}_1$ is unbounded but relatively $\mathcal{D}$-bounded. The Duhamel expansion allows us to write (formally first):

$$e^{-t\mathcal{D}_A^2} = \sum_{j=0}^{\infty} (-t)^j \, E_j(t) , \tag{28}$$

where $E_0(t) := e^{-t\mathcal{D}^2}$ and for $j > 0$:

$$E_j(t) := \sum_{i_1, \cdots, i_j \in \{0,1\}} \int_{\triangle_j} e^{-s_0 t\mathcal{D}^2} \, \tilde{F}_{i_1} \, e^{-s_1 t\mathcal{D}^2} \cdots \tilde{F}_{i_j} \, e^{-s_j t\mathcal{D}^2} \, d^j s ,$$

and $\triangle_j$ denotes the ordinary j-simplex:

$$\triangle_j := \Big\{ s \in \mathbb{R}^{j+1}; \, s_k \ge 0, \sum_{k=0}^{j} s_k = 1 \Big\} .$$

26

We first show that the sum (28) converges in the trace norm for small values of $t > 0$. We only treat the case $j \geq 1$, the case $j = 0$ being covered by Lemma 10. For that we use the Hölder inequality (since $\sum_{k=0}^{j} s_k = 1$):

$$\|E_j(t)\|_1 \leq \sum_{i_1,\cdots,i_j \in \{0,1\}} \int_{\triangle_j} \|e^{-s_0 t \mathcal{D}^2} \, \tilde{F}_{i_1} \, e^{-s_1 t \mathcal{D}^2}\|_{(s_0+s_1)^{-1}} \|\tilde{F}_{i_2} \, e^{-s_2 t \mathcal{D}^2}\|_{s_2^{-1}} \cdots \|\tilde{F}_{i_j} \, e^{-s_j t \mathcal{D}^2}\|_{s_j^{-1}} \, d^j s \; .$$

Then we use the estimate of Proposition 18 and Lemma 21 for $k = 2, \cdots, j$ (see the Remark 19 for the precise value of the constants):

$$\|\tilde{F}_{i_k} \, e^{-s_k t \mathcal{D}^2}\|_{s_k^{-1}} \leq \begin{cases} 4\|\tilde{F}\|^2 C_1(\tilde{F})^{s_k} \, t^{-2s_k} \, , & \text{if} \quad i_k = 0 \, , \\ 2\|\tilde{F}\| C_2(\tilde{F})^{s_k} \, t^{-2s_k} \, (t s_k)^{-1/2} \, , & \text{if} \quad i_k = 1 \, . \end{cases}$$

For the case $i_k = 1$, we need to use the factorization property of the algebra of Schwartz functions with Moyal product (as in the proof of Lemma 12), to expand A as a finite sum of products of elements in $\mathcal{A}_\star \otimes M_{16}(\mathbb{C})$, and then we can proceed as for the other factors. Taking into account that there are 2^j such terms and that

$$\int_{\triangle_j} \prod_{i=0}^{j} s_i^{-1/2} \, d^j s \leq 2^j \; ,$$

we get, since $\sum_{k=0}^{j} s_k = 1$, the rough estimate

$$\|E_j(t)\|_1 \leq 2^j \big(4\|\tilde{F}\|^2 + 4\|\tilde{F}\|\big)^j t^{-j/2-2} \; .$$

Thus the sum $\sum_{j=0}^{\infty} (-t)^j E_j(t)$ converges absolutely in the trace-norm for small values of t. These estimates also show that

$$\left| \mathrm{Tr}\big(e^{-t\mathcal{D}_A^2}\big) - \sum_{j=5}^{\infty} (-t)^j \, \mathrm{Tr}\big(E_j(t)\big) \right| = O(t^{1/2}), \quad t \to 0 \, ,$$

and accordingly, we only need to consider the terms $(-t)^j \mathrm{Tr}\big(E_j(t)\big)$ for $j = 0, 1, 2, 3$ and 4.

Note first that

$$\mathrm{Tr}(-t E_1(t)) = \int_0^1 \mathrm{Tr}\Big(-t \, e^{-st\mathcal{D}^2} \, (\tilde{F}_0 + \tilde{F}_1) \, e^{-(1-s)t\mathcal{D}^2} \Big) \, ds = \mathrm{Tr}\Big(-t \, (\tilde{F}_0 + \tilde{F}_1) \, e^{-t\mathcal{D}^2} \Big) \; .$$

For $j = 2, 3$ and 4, we use the relation (18) to collect the heat operators as follows:

$$E_2(t) = \int_{\triangle_2} e^{-s_1 t \mathcal{D}^2} \, (\tilde{F}_0 + \tilde{F}_1) \, e^{-(s_2-s_1)t\mathcal{D}^2} (\tilde{F}_0 + \tilde{F}_1) \, e^{-(1-s_2)t\mathcal{D}^2} \, ds_1 ds_2$$

$$= \int_{\triangle_2} e^{-s_1 t \mathcal{D}^2} \, (\tilde{F}_0 + \tilde{F}_1)^2 e^{-(1-s_1)t\mathcal{D}^2} \, ds_1 ds_2 - t \int_{\triangle_2} \int_0^1 dr \, (s_2 - s_1) \, e^{-s_1 t \mathcal{D}^2} \, (\tilde{F}_0 + \tilde{F}_1)$$

$$\times\, e^{-(s_2-s_1)rt\mathcal{D}^2}[\mathcal{D}^2,(\tilde{F}_0+\tilde{F}_1)]e^{-(s_2-s_1)(1-r)t\mathcal{D}^2}\,e^{-(1-s_2)t\mathcal{D}^2}\,ds_1ds_2$$

$$=\int_{\triangle_2}ds_1ds_2\,e^{-s_1t\mathcal{D}^2}\left\{(\tilde{F}_0+\tilde{F}_1)^2-t(s_2-s_1)(\tilde{F}_0+\tilde{F}_1)\,[\mathcal{D}^2,(\tilde{F}_0+\tilde{F}_1)]\right\}e^{-(1-s_1)t\mathcal{D}^2}$$

$$+\,t^2\int_{\triangle_2}ds_1ds_2\int_0^1 dr_1dr_2\,r_1(s_2-s_1)^2\,e^{-s_1t\mathcal{D}^2}\,(\tilde{F}_0+\tilde{F}_1)$$

$$\times\, e^{-(s_2-s_1)r_1r_2t\mathcal{D}^2}[\mathcal{D}^2,[\mathcal{D}^2,(\tilde{F}_0+\tilde{F}_1)]]e^{-(s_2-s_1)(1-r_1r_2)t\mathcal{D}^2}e^{-(1-s_2)t\mathcal{D}^2}$$

$$=\int_{\triangle_2}ds_1ds_2\,e^{-s_1t\mathcal{D}^2}\left\{(\tilde{F}_0+\tilde{F}_1)^2-t(s_2-s_1)(\tilde{F}_0+\tilde{F}_1)\,[\mathcal{D}^2,(\tilde{F}_0+\tilde{F}_1)]\right.$$

$$\left.+\frac{t^2}{2}(s_2-s_1)^2(\tilde{F}_0+\tilde{F}_1)[\mathcal{D}^2,[\mathcal{D}^2,(\tilde{F}_0+\tilde{F}_1)]]\right\}e^{-(1-s_1)t\mathcal{D}^2}$$

$$-\,t^3\int_{\triangle_2}ds_1ds_2\int_0^1 dr_1dr_2dr_3\,r_1^2r_2(s_2-s_1)^3\,e^{-s_1t\mathcal{D}^2}\,(\tilde{F}_0+\tilde{F}_1)\,e^{-(s_2-s_1)r_1r_2r_3t\mathcal{D}^2}$$

$$\times\,[\mathcal{D}^2,[\mathcal{D}^2,[\mathcal{D}^2,(\tilde{F}_0+\tilde{F}_1)]]]]e^{-(s_2-s_1)(1-r_1r_2r_3)t\mathcal{D}^2}\,e^{-(1-s_2)t\mathcal{D}^2}\,. \tag{29}$$

Since the principal symbol of $\mathcal{D}^2$ is scalar, we see that $[\mathcal{D}^2,[\mathcal{D}^2,[\mathcal{D}^2,\tilde{F}_1]]]]$ has order 4. Thus, Lemma 20 shows that the last integral (multiplied by its t^2 global prefactor) gives rise to a trace-class operator which trace is of order $t^{1/2}$. Integrating the trace over $\triangle_2$ and disregarding the terms that vanish when $t\to 0$, we find

$$\mathrm{Tr}(t^2E_2(t))=\mathrm{Tr}\Big(\Big\{\frac{t^2}{2}(\tilde{F}_0+\tilde{F}_1)^2-\frac{t^3}{6}\big(\tilde{F}_0[\mathcal{D}^2,\tilde{F}_1]+\tilde{F}_1[\mathcal{D}^2,\tilde{F}_0]+\tilde{F}_1[\mathcal{D}^2,\tilde{F}_1]\big)$$

$$+\frac{t^4}{24}\tilde{F}_1[\mathcal{D}^2,[\mathcal{D}^2,\tilde{F}_1]]\Big\}e^{-t\mathcal{D}^2}\Big)+\mathcal{O}(\sqrt{t})\,.$$

It will be more convenient to write $\mathrm{Tr}\Big(F_1[\mathcal{D}^2,F_0]e^{-t\mathcal{D}^2}\Big)=\mathrm{Tr}\Big(-[\mathcal{D}^2,F_1]F_0e^{-t\mathcal{D}^2}\Big)$. By similar arguments one finds

$$\mathrm{Tr}(-t^3E_3(t))=\mathrm{Tr}\Big(\Big\{-\frac{t^3}{6}(\tilde{F}_0\tilde{F}_1^2+\tilde{F}_1\tilde{F}_0\tilde{F}_1+\tilde{F}_1^2\tilde{F}_0+\tilde{F}_1^3)$$

$$+\frac{t^4}{24}(2\tilde{F}_1^2\,[\mathcal{D}^2,\tilde{F}_1]+\tilde{F}_1\,[\mathcal{D}^2,\tilde{F}_1]\tilde{F}_1)\Big\}e^{-t\mathcal{D}^2}\Big)+\mathcal{O}(\sqrt{t})\,,$$

and lastly

$$\mathrm{Tr}(t^4E_4(t))=\mathrm{Tr}\Big(\frac{t^4}{24}\tilde{F}_1^4\,e^{-t\mathcal{D}^2}\Big)+\mathcal{O}(\sqrt{t})\,.$$

In summary, we have

$$\mathrm{Tr}\big(e^{-t\mathcal{D}_A^2}\big)=\mathrm{Tr}\Big(\Big\{1-t(\tilde{F}_0+\tilde{F}_1)+\frac{t^2}{2}\big(\tilde{F}_0^2+\tilde{F}_1\tilde{F}_0+\tilde{F}_0\tilde{F}_1+\tilde{F}_1^2\big) \tag{30}$$

$$-\frac{t^3}{6}\big(\tilde{F}_0[\mathcal{D}^2,\tilde{F}_1]-[\mathcal{D}^2,\tilde{F}_1]\tilde{F}_0+\tilde{F}_1[\mathcal{D}^2,\tilde{F}_1]+\tilde{F}_0\tilde{F}_1^2+\tilde{F}_1\tilde{F}_0\tilde{F}_1+\tilde{F}_1^2\tilde{F}_0+\tilde{F}_1^3\big)$$

$$+ \frac{t^4}{24} \big(\tilde{F}_1[\mathcal{D}^2,[\mathcal{D}^2,\tilde{F}_1]] + 2\tilde{F}_1^2\,[\mathcal{D}^2,\tilde{F}_1] + \tilde{F}_1\,[\mathcal{D}^2,\tilde{F}_1]\tilde{F}_1 + \tilde{F}_1^4 \big) \Big\} e^{-t\mathcal{D}^2} \Big) + \mathcal{O}(\sqrt{t}) \, .$$

Now, we can take into account the result of Lemma 21, which says in this context that mixed products $F_i \mathbf{J} F_j \mathbf{J}^{-1}$ are already trace-class. Since $JL_\star(g)J^{-1} = R_\star(\bar{g})$, with $R_\star$ the right regular representation, we see by Lemma 4 that all terms in (30) with products of F_i and $\mathbf{J} F_j \mathbf{J}^{-1}$ are (up to matrices) of the form

$$L_\star(f) R_\star(g) \prod_{\mu=1}^{4} (\hat{\nabla}_\mu)^{\alpha_\mu} e^{-tH},$$

with $|\alpha|$ not exceeding the number of F_1 plus the number of commutators by $\mathcal{D}^2$. This argument also relies on the fact that $\mathbf{J}$ commutes with $\hat{\nabla}_\mu$ according to (13). Thus, Corollary 22 shows that the cross-terms, i.e. the terms with powers of both F_i and $\mathbf{J} F_j \mathbf{J}^{-1}$ resulting from products of $\tilde{F}_0 = F_0 + \mathbf{J} F_0 \mathbf{J}^{-1} + 2F_{-1}\mathbf{J}F_{-1}\mathbf{J}^{-1}$ and $\tilde{F}_1 = F_1 + \mathbf{J} F_1 \mathbf{J}^{-1}$, where $F_{-1} := A$, give rise to vanishing contributions in the limit $t \to 0$. Thus, only the terms with either powers of F_i or powers of $\mathbf{J} F_i \mathbf{J}^{-1}$ do contribute to the diverging part of this asymptotic the expansion. Since moreover $\mathbf{J}$ commutes with $\mathcal{D}$ (we are in even KO-dimension), the trace property shows that both terms (with only A or only $\mathbf{J} A \mathbf{J}^{-1}$) give the same contribution and we get the announced result. $\square$

Remark 24. A very important feature of Proposition 23 is that if the heat-trace of the partially fluctuated Dirac operator $\widetilde{\mathcal{D}}_A := \mathcal{D} + A$, $A = \sum_i a_i[\mathcal{D}, b_i]$, has an asymptotic expansion

$$\mathrm{Tr}\big(e^{-t\widetilde{\mathcal{D}}_A^2}\big) = a_0\,t^{-4} + \sum_{k=1}^{4} a_k\,t^{-2+k/2} + \mathcal{O}(\sqrt{t}) \, ,$$

then the heat-trace of the fully fluctuated Dirac operator $\mathcal{D}_A := \mathcal{D} + A + \mathbf{J} A \mathbf{J}^{-1}$ has the asymptotic expansion

$$\mathrm{Tr}\big(e^{-t\mathcal{D}_A^2}\big) = a_0\,t^{-4} + 2\sum_{k=1}^{4} a_k\,t^{-2+k/2} + \mathcal{O}(\sqrt{t}) \, ,$$

for the same coefficients $a_0, \cdots, a_k$. Also, this shows that the asymptotic expansion of the heat-trace of the fully fluctuated Dirac operator is independent of the choice of the real structure $J_{\mathbf{f}}$ of the finite spectral triple $(\mathcal{A}_{\mathbf{f}}, \mathcal{H}_{\mathbf{f}}, \mathcal{D}_{\mathbf{f}})$. This fact holds for Moyal spectral triples with harmonic propagation in any (even) dimension.

3.3 Application: the spectral action for the $U(1)$-Higgs model

In the Connes-Lott spirit [8] we take the tensor product of the 4-dimensional spectral triple $(\mathcal{A}_\star, \mathcal{H}, \mathcal{D}_\bullet, \Gamma, J)$, $\bullet = 1, 2$, with the finite Higgs spectral triple $(\mathbb{C} \oplus \mathbb{C}, \mathbb{C}^2, M\sigma_1, J_{\mathbf{f}})$, where $M > 0$ and $J_{\mathbf{f}}$ is any real structure. The Dirac operator $\mathcal{D} = \mathcal{D}_\bullet \otimes 1 + \Gamma \otimes M\sigma_1$ of the product triple becomes

$$\mathcal{D} = \begin{pmatrix} \mathcal{D}_\bullet & M\Gamma \\ M\Gamma & \mathcal{D}_\bullet \end{pmatrix} \, .$$

In this representation, the algebra is $\mathcal{A}_\star \oplus \mathcal{A}_\star \ni (f, g)$, which acts on $\mathcal{H} \oplus \mathcal{H}$ by diagonal left Moyal multiplication. The commutator of $\mathcal{D}$ with (f, g) is, in case that $\mathcal{D}_1$ is chosen, according to (8) given by

$$[\mathcal{D}, (f, g)] = \begin{pmatrix} i\Gamma^\mu L_\star(\partial_\mu f) & M\Gamma L_\star(g - f) \\ M\Gamma L_\star(f - g) & i\Gamma^\mu L_\star(\partial_\mu g) \end{pmatrix} .$$

(If we choose $\mathcal{D}_2$ instead, then Γ^μ has to be replaced by $\Gamma^{\mu+4}$ everywhere.) This shows that the selfadjoint fluctuation $A = \sum_i a_i[\mathcal{D}, b_i]$ of the fluctuated Dirac operators $\mathcal{D}_A = \mathcal{D} + A + \mathbf{J}A\mathbf{J}^{-1}$, $\mathbf{J} = J \otimes J_\mathbf{f}$, is of the form

$$A = \begin{pmatrix} \Gamma^\mu L_\star(A_\mu) & \Gamma L_\star(\phi) \\ \Gamma L_\star(\bar\phi) & \Gamma^\mu L_\star(B_\mu) \end{pmatrix} ,$$

for real two real one-forms $A_\mu, B_\mu \in \mathcal{A}_\star$ and a one complex field $\phi \in \mathcal{A}_\star$. Again, this holds for $\mathcal{D}_1$; for $\mathcal{D}_2$ we have to replace Γ^μ by $\Gamma^{\mu+4}$.

In terms of the connection introduced in Lemma 4 and using (10) we identify the relevant operators arising in the expansion $\mathcal{D}_A^2 = \mathcal{D}_\bullet^2 + M^2 + (F_0 + F_1) + \mathbf{J}(F_0 + F_1)\mathbf{J}^{-1} + 2A\mathbf{J}A\mathbf{J}^{-1}$ of Proposition 23 as follows:

$$F_0 = \begin{pmatrix} L_\star(V_{A,\phi})1 + \frac{i}{4}[\Gamma^\mu, \Gamma^\nu]L_\star(F_{\mu\nu}^A) & i\Gamma^\mu\Gamma L_\star(D_\mu\phi) \\ i\Gamma^\mu\Gamma L_\star(\overline{D_\mu\phi}) & L_\star(V_{B,\phi})1 + \frac{i}{4}[\Gamma^\mu, \Gamma^\nu]L_\star(F_{\mu\nu}^B) \end{pmatrix} ,$$

$$F_1 = \begin{pmatrix} 2iL_\star(A^\mu)\nabla_\mu & 0 \\ 0 & 2iL_\star(B^\mu)\nabla_\mu \end{pmatrix} ,$$

where

$$V_{A,\phi} := \phi \star \bar\phi + M(\phi + \bar\phi) + (g^{-1})^{\mu\nu}(i\partial_\mu A_\nu + A_\mu \star A_\nu) ,$$
$$V_{B,\phi} := \bar\phi \star \phi + M(\phi + \bar\phi) + (g^{-1})^{\mu\nu}(i\partial_\mu B_\nu + B_\mu \star B_\nu) ,$$
$$F_{\mu\nu}^A := \partial_\mu A_\nu - \partial_\nu A_\mu - i(A_\mu \star A_\nu - A_\nu \star A_\mu) ,$$
$$F_{\mu\nu}^B := \partial_\mu B_\nu - \partial_\nu B_\mu - i(B_\mu \star B_\nu - B_\nu \star B_\mu) ,$$
$$D_\mu\phi := \partial_\mu\phi - iA_\mu \star \phi + i\phi \star B_\mu - iM(A_\mu - B_\mu) .$$

We have used

$$\mathcal{D}_1\Gamma^\sigma L_\star(A_\sigma) + L_\star(A_\sigma)\mathcal{D}_1\Gamma^\sigma = -\Gamma^\sigma[\mathcal{D}_1, L_\star(A_\sigma)] + 2i(g^{-1})^{\mu\nu}L_\star(\partial_\mu A_\nu) + 2iL_\star(A^\mu)\nabla_\mu .$$

According to the general asymptotic expansion we have obtained in Proposition (23), the only further commutators we need are $[\mathcal{D}^2, F_1]$ and $[\mathcal{D}^2, [\mathcal{D}^2, F_1]]$. Their expression will easily follows from the following computation which relies on the relations in Lemma 4:

$$[\mathcal{D}^2, 2iL_\star(A^\mu)\nabla_\mu] = -2iL_\star\big((g^{-1})^{\rho\sigma}\partial_\rho\partial_\sigma A^\mu\big)\nabla_\mu - 4iL_\star(\partial^\nu A^\mu)\nabla_\nu\nabla_\mu + 4\tilde\Omega^2\Theta_{\mu\nu}L_\star(A^\mu)\tilde\nabla^\nu ,$$
$$[\mathcal{D}^2, [\mathcal{D}^2, 2iL_\star(A^\mu)\nabla_\mu] = 8iL_\star(\partial^\rho\partial^\nu A^\mu)\nabla_\rho\nabla_\nu\nabla_\mu + \text{ lower order} ,$$

We have already computed the matrix trace of $e^{-t\tilde{\Omega}\Sigma}$ in Lemma 10, and the result is $16\cosh^4(\tilde{\Omega}t)$. Using

$$-i\Sigma = (ib_\mu - ib_\mu^*)(b^\mu + b^{\mu*}) = g_{\mu\nu}\left(\Gamma^\nu + \frac{\tilde{\Omega}}{2}\Theta^{\nu\rho}\Gamma_{\rho+4}\right)g^{\mu\sigma}\left(\Gamma_{\sigma+4} + \frac{\tilde{\Omega}}{2}\Theta_{\sigma\tau}\Gamma^\tau\right),$$

the other matrix traces follow from the Clifford algebra:

$$\mathrm{tr}_{\mathbb{C}^{16}}\left(\frac{i}{4}[\Gamma^\mu,\Gamma^\nu]\cdot e^{-t\tilde{\Omega}\Sigma}\right) = -8\tilde{\Omega}\Theta^{\mu\nu}t + \mathcal{O}(t^2),$$

$$\mathrm{tr}_{\mathbb{C}^{16}}\left(\frac{i}{4}[\Gamma^\mu,\Gamma^\nu]\cdot\frac{i}{4}[\Gamma^\rho,\Gamma^\sigma]\cdot e^{-t\tilde{\Omega}\Sigma}\right) = 4(g^{-1})^{\mu\rho}(g^{-1})^{\nu\sigma} - 4(g^{-1})^{\mu\sigma}(g^{-1})^{\nu\rho} + \mathcal{O}(t)$$

$$\mathrm{tr}_{\mathbb{C}^{16}}\left(i\Gamma^\mu\Gamma\cdot i\Gamma^\nu\Gamma\cdot e^{-t\tilde{\Omega}\Sigma}\right) = 16(g^{-1})^{\mu\nu} + \mathcal{O}(t).$$

In terms of the functionals $\mathcal{T}_{\mu_1\ldots\mu_k}(f) := \mathrm{Tr}_{L^2(\mathbb{R}^4)}\left(L_\star(f)\nabla_{\mu_1}\ldots\nabla_{\mu_k}e^{-tH}\right)$ on $\mathcal{A}_\star$ introduced and computed in Proposition 6 and the similar functional

$$\tilde{\mathcal{T}}_{\mu\nu}(f) := \mathrm{Tr}_{L^2(\mathbb{R}^4)}\left(L_\star(f)\nabla_\mu\tilde{\nabla}_\nu e^{-tH}\right)$$

we obtain from Proposition 23 the trace $\mathrm{Tr}(e^{-t\mathcal{D}_A^2})$ as follows:

$$\begin{aligned}
\mathrm{Tr}\left(e^{-t\mathcal{D}_A^2}\right) = e^{-tM^2}&\bigg\{16\cosh^4(\tilde{\Omega}t)\mathrm{Tr}(e^{-tH}) - 2t\,\mathcal{T}\left(16V_{A,\phi}\right) - 2t\,\mathcal{T}_\mu\left(32iA^\mu\right)\\
&+ t^2\,\mathcal{T}\left(16V_{A,\phi}\star V_{A,\phi} + 16(g^{-1})^{\mu\nu}D_\mu\phi\star\overline{D_\nu\phi} + 8(g^{-1})^{\mu\rho}(g^{-1})^{\nu\sigma}F^A_{\mu\nu}F^A_{\rho\sigma}\right.\\
&\qquad\left. + 32i(g^{-1})^{\mu\nu}A_\mu\star\partial_\nu V_{A,\phi} + 16\tilde{\Omega}\Theta^{\mu\nu}F^A_{\mu\nu}\right)\\
&+ t^2\,\mathcal{T}_\mu\left(32iA^\mu\star V_{A,\phi} + 32iV_{A,\phi}\star A^\mu - 64(g^{-1})^{\nu\rho}A^\nu\star\partial_\rho A^\mu\right)\\
&+ t^2\,\mathcal{T}_{\mu\nu}\left(-64A^\mu\star A^\nu\right)\\
&- \frac{t^3}{3}\mathcal{T}_{\mu\nu}\bigg(-4i((V_{A,\phi}\star\partial^\mu A^\nu - \partial^\mu A^\nu\star V_{A,\phi})\\
&\qquad + 64A^\mu(g^{-1})^{\rho\sigma}\partial_\rho\partial_\sigma A^\nu + 128A_\rho(g^{-1})^{\rho\sigma}\partial_\sigma\partial^\mu A^\nu\\
&\qquad - 64\left(V_{A,\phi}\star A^\mu\star A^\nu + A^\mu\star V_{A,\phi}\star A^\nu + A^\mu\star A^\nu\star V_{A,\phi}\right)\\
&\qquad - 128i(g^{-1})^{\rho\sigma}\left(A_\rho\star(\partial_\sigma A^\mu)\star A^\nu + A_\rho\star A^\mu\star(\partial_\sigma A^\nu) + A^\mu\star A_\rho\star(\partial_\sigma A^\nu)\right)\bigg)\\
&- \frac{t^3}{3}\mathcal{T}_{\mu\nu\rho}\left(128A^\mu\star\partial^\nu A^\rho - 128iA^\mu\star A^\nu\star A^\rho\right)\\
&- \frac{t^3}{3}\tilde{\mathcal{T}}_{\mu\nu}\left(128i\tilde{\Omega}^2\Theta^{\rho\nu}A^\mu\star A_\rho\right)\\
&+ \frac{t^4}{12}\mathcal{T}_{\mu\nu\rho\sigma}\bigg(-256A^\mu\star\partial^\nu\partial^\rho A^\sigma + 512iA^\mu\star A^\nu\star\partial^\rho A^\sigma\\
&\qquad + 256iA^\mu\star(\partial^\nu A^\rho)\star A^\sigma + 256A^\mu\star A^\nu\star A^\rho\star A^\sigma\bigg)\bigg\}
\end{aligned}$$

$$+\left\{A_\mu \mapsto B_\mu \ , \ F^A_{\mu\nu} \mapsto F^B_{\mu\nu} \ , \ V_{A,\phi} \mapsto V_{B,\phi} \ , \ D_\mu\phi \leftrightarrow \overline{D_\mu\phi}\right\} + \mathcal{O}(\sqrt{t}) \ . \tag{31}$$

The relevant traces have been computed in Proposition 6. A similar procedure gives

$$\tilde{\mathcal{T}}_{\mu\nu}(f) = \left(\frac{\tilde{\Omega}}{2\pi \sinh(2\tilde{\Omega}t)}\right)^2 \int_{\mathbb{R}^4} dz \ \sqrt{\det g} \, f(z)\left(\tilde{\mathcal{N}}_{\mu\nu} + Z_\mu \tilde{Z}_\nu\right) e^{-\tilde{\Omega}\tanh(\tilde{\Omega}t)\langle z,gz\rangle} \ ,$$

with $\tilde{Z}_\nu := -2i(\Theta^{-1}z)_\nu - 2\tilde{\Omega}\tanh(\tilde{\Omega}t)(gz)_\nu$ and $\tilde{\mathcal{N}}_{\mu\nu} := 2i(\Theta^{-1}g^{-1})_{\mu\nu} + i\tilde{\Omega}^2(g\Theta)_{\mu\nu} - \tilde{\Omega}\tanh(\tilde{\Omega}t)\delta_{\mu\nu}$. This shows that the contribution of $\tilde{\mathcal{T}}_{\mu\nu}(f)$ is suppressed with $\mathcal{O}(t)$. Inserting these traces into (31) we arrive at

$$\mathrm{Tr}(e^{-t\mathcal{D}_A^2}) \tag{32}$$
$$= \frac{2}{\tilde{\Omega}^4}t^{-4} - \frac{2M^2}{\tilde{\Omega}^4}t^{-3} + \left(\frac{M^4}{\tilde{\Omega}^4} + \frac{8}{3\tilde{\Omega}^2}\right)t^{-2} - \left(\frac{M^6}{3\tilde{\Omega}^4} + \frac{8M^2}{3\tilde{\Omega}^2}\right)t^{-1} + \left(\frac{52}{45} + \frac{M^8}{12\tilde{\Omega}^4} + \frac{4M^4}{3\tilde{\Omega}^2}\right)$$
$$- t^{-1}\frac{2 - 2M^2 t}{\pi^2} \int d^4 x \sqrt{\det g}\left\{\phi \star \bar{\phi} + M(\phi + \bar{\phi}) + \tilde{\Omega}^2\left(\langle X_A, gX_A\rangle_\star - \langle x, gx\rangle_\star\right)\right.$$
$$\left. + \bar{\phi} \star \phi + M(\phi + \bar{\phi}) + \tilde{\Omega}^2\left(\langle X_B, gX_B\rangle_\star - \langle x, gx\rangle_\star\right)\right\}$$
$$+ \frac{1}{\pi^2}\int d^4 x \sqrt{\det g}\left\{2(g^{-1})^{\mu\nu}D_\mu\phi \star \overline{D_\nu\phi}\right.$$
$$+ \left(\phi \star \bar{\phi} + M(\phi + \bar{\phi}) + \tilde{\Omega}^2\langle X_A, gX_A\rangle_\star\right)^2 - \left(\tilde{\Omega}^2\langle x, gx\rangle_\star\right)^2$$
$$+ \left(\bar{\phi} \star \phi + M(\phi + \bar{\phi}) + \tilde{\Omega}^2\langle X_B, gX_B\rangle_\star\right)^2 - \left(\tilde{\Omega}^2\langle x, gx\rangle_\star\right)^2$$
$$+ \left(\frac{1}{2}(g^{-1})^{\mu\rho}(g^{-1})^{\nu\sigma} - \frac{1}{6}(g^{-1} + \tilde{\Omega}^2\Theta g\Theta)^{\mu\rho}(g^{-1} + \tilde{\Omega}^2\Theta g\Theta)^{\nu\sigma}\right)\left(F^A_{\mu\nu} \star F^A_{\rho\sigma} + F^B_{\mu\nu} \star F^B_{\rho\sigma}\right)\right\}$$
$$+ \mathcal{O}(\sqrt{t}) \ ,$$

where

$$X_A^\mu(x) := x^\mu + \Theta^{\mu\nu}A_\nu \ , \qquad \langle X, gY\rangle_\star := g_{\mu\nu}X^\mu \star Y^\nu \ .$$

To reduce (31) to (32) we have used

- the traciality (4) of the Moyal product and the resulting cyclicity under the integral,
- integration by parts where appropriate,
- $x^\mu \star f = \frac{1}{2}\{x^\mu, f\}_\star + \frac{1}{2}[x^\mu, f]_\star = \frac{1}{2}\{x^\mu, f\}_\star + i\Theta^{\mu\nu}\partial_\nu f$ where appropriate,
- symmetries and antisymmetries in the indices.

The matrix $g^{-1} + \tilde{\Omega}^2\Theta g\Theta$ appearing in front of the curvature term in (32) can be equivalently written as

$$g^{-1} + \tilde{\Omega}^2\Theta g\Theta = g^{-1} + 4g(1 - g^{-1}) = g^{-1}(1 - 2g)^2 \ .$$

Since $g^{-1} \geq 1$ according to (9), hence $0 \leq g \leq 1$, we have $0 \leq g^{-1} + \tilde{\Omega}^2 \Theta g \Theta \leq g^{-1}$, showing that the matrix in front of the curvature term in (32) is strictly positive. It is minimal for $g = \frac{1}{2}$, i.e. $\tilde{\Omega}^2 (\Theta^t \Theta)_{\mu\nu} = 4\delta_{\mu\nu}$. The curvature $F_{\mu\nu}$ can also be expressed in terms of the covariant coordinates, $[X_A^\mu, X_A^\nu]_\star = i\Theta^{\mu\nu} + i\Theta^{\mu\rho}\Theta^{\nu\sigma}F_{\rho\sigma}^A$.

With the moments $\chi_{-n} = \int_0^\infty ds\, s^{n-1}\chi(s)$ of the "characteristic function" and $\chi_0 = \chi(0)$, we identify the spectral action (27) as

$$
\begin{aligned}
S_\Lambda(\mathcal{D}_A) = {}& \frac{2\Lambda^8}{\tilde{\Omega}^4}\chi_{-4} - \frac{2M^2\Lambda^6}{\tilde{\Omega}^4}\chi_{-3} + \left(\frac{M^4\Lambda^4}{\tilde{\Omega}^4} + \frac{8\Lambda^4}{3\tilde{\Omega}^2}\right)\chi_{-2} - \left(\frac{M^6\Lambda^2}{3\tilde{\Omega}^4} + \frac{8M^2\Lambda^2}{3\tilde{\Omega}^2}\right)\chi_{-1} \\
& + \left(\frac{52}{45} + \frac{M^8}{12\tilde{\Omega}^4} + \frac{4M^4}{3\tilde{\Omega}^2}\right)\chi_0 \\
& + \frac{\chi_0}{\pi^2}\int d^4x\sqrt{\det g}\Big\{2(g^{-1})^{\mu\nu}D_\mu\phi \star \overline{D_\nu\phi} \\
& + \left(\phi\star\bar{\phi} + M(\phi+\bar{\phi}) + \tilde{\Omega}^2\langle X_A, gX_A\rangle_\star + M^2 - \frac{\chi_{-1}}{\chi_0}\Lambda^2\right)^2 - \left(\tilde{\Omega}^2\langle x, gx\rangle_\star + M^2 - \frac{\chi_{-1}}{\chi_0}\Lambda^2\right)^2 \\
& + \left(\bar{\phi}\star\phi + M(\phi+\bar{\phi}) + \tilde{\Omega}^2\langle X_B, gX_B\rangle_\star + M^2 - \frac{\chi_{-1}}{\chi_0}\Lambda^2\right)^2 - \left(\tilde{\Omega}^2\langle x, gx\rangle_\star + M^2 - \frac{\chi_{-1}}{\chi_0}\Lambda^2\right)^2 \\
& + \left(\frac{1}{2}(g^{-1})^{\mu\rho}(g^{-1})^{\nu\sigma} - \frac{1}{6}(g^{-1}+\tilde{\Omega}^2\Theta g\Theta)^{\mu\rho}(g^{-1}+\tilde{\Omega}^2\Theta g\Theta)^{\nu\sigma}\right)\left(F_{\mu\nu}^A \star F_{\rho\sigma}^A + F_{\mu\nu}^B \star F_{\rho\sigma}^B\right)\Big\} \\
& + \mathcal{O}(\Lambda^{-1})\,.
\end{aligned}
\tag{33}
$$

The final result (33) for the spectral action agrees, up to typos, with the result obtained in [21]. We recall that with the cumbersome computational method of [21] it was only possible to identify the part of the spectral action at most bilinear in the gauge fields A, B. By gauge-invariant completion it was argued that the total spectral action has to be (33). In [21] the Θ-matrix was chosen as $\Theta = \theta\begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}$. In terms of $\Omega := \frac{\theta\tilde{\Omega}}{2}$ this choice leads to $(g^{-1})^{\mu\nu} = (1+\Omega^2)\delta^{\mu\nu}$ and $\sqrt{\det g} = \frac{1}{(1+\Omega^2)^2}$. Up to the global factor of 2 due to the real structure, the few differences in the prefactors[3] are easily identified as typos in [21]. We finish by a brief discussion of the spectral action:

- The square of covariant coordinates X_A, X_B combines with the Higgs field ϕ to a non-trivial potential. This was not noticed in [14, 16]. We observe here a much deeper unification of the continuous geometry described by Yang-Mills fields and discrete geometry described by the Higgs field than previously in almost-commutative geometry.

- The coefficient in front of the Yang-Mills action is strictly positive for any real-valued $\tilde{\Omega}$. In the bosonic model of [14, 16] there was only the analogue of the negative part, which leads to problems with the field equations.

---

[3]These are $\left(\frac{(1+\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{6(1+\Omega^2)^2}\right)$ versus $\left(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\right)$ in [21] in front of $F_{\mu\nu}F^{\mu\nu}$ and $\frac{(1+\Omega^2)}{2}$ versus $\frac{1}{2}$ in [21] in front of $(D\phi)^2$.

Unlike the scalar model renormalized in [19] where $\Omega = \frac{\theta\tilde{\Omega}}{2}$ can by Langmann-Szabo duality be restricted to $\Omega \in [0, 1]$, the full spectral action (33) does not have a distinguished frequency parameter $\tilde{\Omega} > 0$.

- The action (33) is invariant under gauge transformations

$$\phi + M \mapsto u_A \star (\phi + M) \star \overline{u_B} , \quad X_{A\mu} \mapsto u_A \star X_{A\mu} \star \overline{u_A} , \quad X_{B\mu} \mapsto u_B \star X_{B\mu} \star \overline{u_B} ,$$

where $u_A, u_B \in \mathcal{U}(\mathcal{A}_\star \oplus \mathbb{C})$ are unital elements of the minimal unitization.

- For any value of the free parameter $\frac{M^2 \chi_0}{\Lambda^2 \chi_{-1}}$, the action contains (A, B, ϕ)-linear terms which lead to a complicated vacuum which is *not* attained at vanishing A, B, ϕ. Since A, B, ϕ are Schwartz functions, the *formal* vacuum solution $X_A = 0 = X_B$ and $\phi + M = \sqrt{\frac{\chi_{-1}}{\chi_0}} \Lambda$ is excluded. An enlargement of Schwartz class function to e.g. polynomially bounded functions does not help either, because then we are not allowed to expand the Gaußian $e^{-\tilde{\Omega} \tanh(\tilde{\Omega}t)\langle x, gx \rangle_\star}$ in t, making the spectral action different from (33).

- If we *formally* regard $\phi + M, X_A, X_B$ as dynamical variables of the model, then (33) can be viewed as translation-invariant with respect to

$$\phi(x) + M \mapsto \phi(x+a) + M , \qquad X_A(x) \mapsto X_A(x+a) , \qquad X_B(x) \mapsto X_B(x+a) .$$

This would clear away a frequent objection against the renormalizable ϕ_4^4-models, breaking of translation invariance. However, this transformation leaves the space of Schwartz class functions for A, B, ϕ, so that translation invariance remains broken in the consistent spectral action.

- The vacuum part of the spectral action is finite. In general, the heat kernel expansion for non-compact spectral triples is ill-defined, so that a spatial regularization of the operator trace is unavoidable. See e.g. [13]. The oscillator potential is one of many possibilities. We want to advertise the point of view that if one takes the spectral action principle serious, *the spatial regularization is part of the geometry*. The removal of the spatial regularization must be carefully studied. In general, we should expect that other limiting procedures such as those of quantum field theory make it impossible to remove the regularization (UV/IR).

A Locally compact noncommutative spin manifolds

Definition 25. *A `non-compact spectral triple` is given by the data $(\mathcal{A}, \mathcal{B}, \mathcal{H}, \mathcal{D}, J, \Gamma, \boldsymbol{c})$ satisfying conditions 0-6 given below. The data consist of a non-unital algebra $\mathcal{A}$ acting faithfully (via a representation denoted by π) by bounded operators on the Hilbert space $\mathcal{H}$; a preferred unitization $\mathcal{B}$ of $\mathcal{A}$ acting by bounded operators on the same Hilbert space; and an essentially self-adjoint unbounded operator $\mathcal{D}$ on $\mathcal{H}$ such that $[\mathcal{D}, \pi(a)]$ extends to a bounded operator for any $a \in \mathcal{B}$. The spectral triple is said to be `even` if there exists a $\mathbb{Z}_2$-grading operator Γ on $\mathcal{H}$ satisfying $\Gamma^2 = 1$, for which $\mathcal{B}$ is even and $\mathcal{D}$ is odd. The spectral triple is*

*said to be **real** if there exists an antiunitary operator J on $\mathcal{H}$ which satisfies conditions 4 and 5 below.*

0. **Compactness.**

   The operator $\pi(a)(\mathcal{D} - \lambda)^{-1}$ is compact for all $a \in \mathcal{A}$ and λ in the resolvent set of $\mathcal{D}$. For any $a \in \mathcal{B}$, both $\pi(a)$ and $[\mathcal{D}, \pi(a)]$ belong to $\bigcap_{n=1}^{\infty} \mathrm{dom}(\delta^n)$, with $\delta(T) := [\langle \mathcal{D} \rangle, T]$ and $\langle \mathcal{D} \rangle := (\mathcal{D}^2 + 1)^{\frac{1}{2}}$.

   For any element b of the algebra $\Psi_0(\mathcal{A})$ generated by $\delta^n(\pi(\mathcal{A}))$ and $\delta^n([\mathcal{D}, \pi(\mathcal{A})])$, the function $\zeta_b(z) := \mathrm{Tr}(b\langle \mathcal{D} \rangle^{-z})$ is well defined and holomorphic for $\Re(z)$ large and analytically continues to $\mathbb{C} \setminus \mathrm{Sd}$ for some discrete set $\mathrm{Sd} \subset \mathbb{C}$ (the dimension spectrum). Moreover the dimension spectrum is said to be simple if all the poles are simples, finite if there is $k \in \mathbb{N}$ such that all the poles are order at most k and if not, infinite.

1. **Metric dimension.**

   For the metric dimension $d := \sup\{\Re(z), z \in \mathrm{Sd}\}$, the operator $\pi(a)\langle \mathcal{D} \rangle^{-d}$ belongs to the Dixmier ideal $\mathcal{L}^{1,\infty}(\mathcal{H})$ for any $a \in \mathcal{A}$. Moreover, for any Dixmier trace, the map $\mathcal{A}_+ \ni a \mapsto \mathrm{Tr}_\omega(\pi(a)\langle \mathcal{D} \rangle^{-d})$ is non-vanishing.

2. **Finiteness.**

   The algebra $\mathcal{A}$ and its preferred unitization $\mathcal{B}$ are pre-$C^\star$-algebras, i.e. each one is a $\star$-subalgebra of some C^*-algebra and stable under holomorphic functional calculus.

   The space of smooth spinors $\mathcal{H}^\infty := \bigcap_{k=0}^{\infty} \mathcal{H}^k$, with $\mathcal{H}^k := \mathrm{dom}(\mathcal{D}^k)$ completed with norm $\|\xi\|_k^2 := \|\xi\|^2 + \|\mathcal{D}^k \xi\|^2$, is a finitely generated projective $\mathcal{A}$-module $p\mathcal{A}^m$, for some $m \in \mathbb{N}$ and some projector $p = p^2 = p^* \in M_m(\mathcal{B})$. The composition of the Dixmier trace with the induced hermitian structure $\langle \ , \ \rangle_\mathcal{A} : \mathcal{H}^\infty \times \mathcal{H}^\infty \to \mathcal{A}$ coincides with the scalar product $(\ , \ )$ on $\mathcal{H}^\infty$,

   $$(\xi, \eta) = \mathrm{Tr}_\omega \left(\langle \xi, \eta \rangle_\mathcal{A} \langle \mathcal{D} \rangle^{-d} \right), \qquad \xi, \eta \in \mathcal{H}^\infty .$$

3. **Reality.**

   The operator J defines a real structure of KO-dimension $k \in \mathbb{Z}_8$. This means

   $$J^2 = \varepsilon , \qquad J\mathcal{D} = \varepsilon'\mathcal{D}J , \qquad J\Gamma = \varepsilon''\Gamma J \quad \text{(even case)} \qquad (34)$$

   with signs $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ given as a function of $k \mod 8$ by

k	0	1	2	3	4	5	6	7
ε	1	1	−1	−1	−1	−1	1	1
ε'	1	−1	1	1	1	−1	1	1
ε''	1		−1		1		−1	

   Additionally, the action π of $\mathcal{B}$ on $\mathcal{H}$ satisfies the commutation rule $[\pi(f), \pi^o(g)] = 0$ for all $f, g \in \mathcal{B}$, where $\pi^o(g) = J\pi(g^*)J^{-1}$ is the action of the opposite algebra $\mathcal{B}^o$.

4. *First order.*

$[[\mathcal{D}, \pi(f)], \pi^o(g)] = 0$ *for all* $f, g \in \mathcal{B}$.

5. *Orientability.*

Whenever the metric dimension d is an integer, there is a Hochschild d-cycle $\boldsymbol{c}$ on $\mathcal{B}$ with values in $\mathcal{B} \otimes \mathcal{B}^o$, i.e. a finite sum of terms $(a_0 \otimes b_0) \otimes a_1 \otimes \cdots \otimes a_d$. Its representation $\pi_\mathcal{D}(\boldsymbol{c})$ with $\pi_\mathcal{D}((a_0 \otimes b_0) \otimes a_1 \otimes \cdots \otimes a_d) := \pi(a_0) J \pi(b_0^*) J^{-1} [\mathcal{D}, \pi(a_1)] \cdots [\mathcal{D}, \pi(a_d)]$ satisfies $\pi_\mathcal{D}(\boldsymbol{c})^2 = 1$ and defines the volume form on $\mathcal{A}$, i.e.

$$\phi_{\boldsymbol{c}}(f_0, \ldots, f_d) = \mathrm{Tr}_\omega \big(\pi_\mathcal{D}(\boldsymbol{c}) \pi(f_0) [\mathcal{D}, \pi(f_1)] \cdots [\mathcal{D}, \pi(f_d)] \langle \mathcal{D} \rangle^{-d} \big)$$

provides a non-vanishing Hochschild d-cocycle ϕ_d on $\mathcal{A}$.

Acknowledgments

References

[1] A. Carey, V. Gayral, A. Rennie and F. Sukochev, "Integration on locally compact noncommutative spaces," [arXiv:math.OA/0912.2817].

[2] A. Carey, V. Gayral, A. Rennie and F. Sukochev, "Index theory for locally compact noncommutative geometries," [arXiv:math.OA/1107.0805].

[3] A. H. Chamseddine and A. Connes, "The spectral action principle," Commun. Math. Phys. **186** (1997) 731.

[4] A. H. Chamseddine and A. Connes, "Scale invariance in the spectral action," J. Math. Phys. **47** (2006) 063504.

[5] A. Connes, "Gravity coupled with matter and the foundation of non-commutative geometry," Commun. Math. Phys. **182** (1996) 155.

[6] A. Connes, "Noncommutative geometry and the standard model with neutrino mixing," JHEP **0611** (2006) 081.

[7] A. Connes, "On the spectral characterization of manifolds," [arXiv:math.OA/0810.2088].

[8] A. Connes and J. Lott, "Particle models and noncommutative geometry (expanded version)," Nucl. Phys. Proc. Suppl. **18B** (1991) 29.

[9] L. Dąbrowski and A. Sitarz, "Dirac Operator on the Standard Podleś Quantum Sphere," Center Banach Center Publ. **61** (2003) 49.

[10] M. Disertori and V. Rivasseau, "Two and three loops beta function of noncommutative ϕ_4^4 theory," Eur. Phys. J. C **50** (2007) 661.

[11] M. Disertori, R. Gurau, J. Magnen and V. Rivasseau, "Vanishing of beta function of noncommutative ϕ_4^4 theory to all orders," Phys. Lett. B **649** (2007) 95.

[12] V. Gayral, J. M. Gracia-Bondía, B. Iochum, T. Schücker and J. C. Várilly, "Moyal planes are spectral triples," Commun. Math. Phys. **246** (2004) 569.

[13] V. Gayral and B. Iochum, "The spectral action for Moyal planes," J. Math. Phys. **46** (2005) 043503.

[14] A. de Goursac, J. C. Wallet and R. Wulkenhaar, "Noncommutative induced gauge theory," Eur. Phys. J. C **51** (2007) 977.

[15] J. M. Gracia-Bondía and J. C. Várilly, "Algebras of distributions suitable for phase-space quantum mechanics I," J. Math. Phys. **29** (1988) 869.

[16] H. Grosse and M. Wohlgenannt, "Induced Gauge Theory on a Noncommutative Space," Eur. Phys. J. C **52** (2007) 435.

[17] H. Grosse and R. Wulkenhaar, "The beta-function in duality-covariant noncommutative ϕ^4-theory," Eur. Phys. J. C **35** (2004) 277.

[18] H. Grosse and R. Wulkenhaar, "Power-counting theorem for non-local matrix models and renormalisation," Commun. Math. Phys. **254** (2005) 91.

[19] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base," Commun. Math. Phys. **256** (2005) 305.

[20] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on non-commutative $\mathbb{R}^4$ to all orders," Lett. Math. Phys. **71** (2005) 13.

[21] H. Grosse and R. Wulkenhaar, "8D-spectral triple on 4D-Moyal space and the vacuum of noncommutative gauge theory," [arXiv:hep-th/0709.0095].

[22] H. Grosse and R. Wulkenhaar, "Progress in solving a noncommutative quantum field theory in four dimensions," [arXiv:hep-th/0909.1389].

[23] R. Gurau, J. Magnen, V. Rivasseau and F. Vignes-Tourneret, "Renormalization of non-commutative ϕ_4^4 field theory in x-space," Commun. Math. Phys. **267** (2006) 515.

[24] R. Gurau and V. Rivasseau, "Parametric representation of noncommutative field theory," Commun. Math. Phys. **272** (2007) 811.

[25] J. Magnen and V. Rivasseau, "Constructive ϕ^4 field theory without tears," Ann. Henri Poincaré **9** (2008) 403.

[26] A. Rennie, "Smoothness and locality for nonunital spectral triples," K-theory **28** (2003) 127.

[27] V. Rivasseau, "Non-commutative renormalization," in *Quantum spaces*, Prog. Math. Phys. **53** (2007), Birkhäuser, Basel.

[28] V. Rivasseau, "Constructive Matrix Theory," J. High Energy Phys. **9** (2007) 008.

[29] V. Rivasseau, F. Vignes-Tourneret and R. Wulkenhaar, "Renormalization of noncommutative ϕ^4-theory by multi-scale analysis," Commun. Math. Phys. **262** (2006) 565.

[30] B. Simon, *Trace Ideals and Their Applications*, London Math. Soc. Lecture Notes **35**, Cambridge University Press, Cambridge, 1979.

[31] R. Wulkenhaar, "Non-compact spectral triples with finite volume," in *Quanta of Maths*, Clay Math. Proc. **11** (2010) 617, Amer. Math. Soc., Providence, RI.