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SO(10)-unification in noncommutative geometry revisited

Raimar Wulkenhaar*

Centre de Physique Théorique
CNRS Luminy, Case 907
13288 Marseille Cedex 9, France
e-mail: raimar@cpt.univ-mrs.fr

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Abstract

We investigate the SO(10)-unification model in a Lie algebraic formulation of noncommutative geometry. The SO(10)-symmetry is broken by a **45**-Higgs and the Majorana mass term for the right neutrinos (**126**-Higgs) to the standard model structure group. We study the case that the fermion masses are as general as possible, which leads to two **10**-multiplets, four **120**-multiplets and two additional **126**-multiplets of Higgs fields. This Higgs structure differs considerably from the two Higgs multiplets $\mathbf{16}\otimes\mathbf{16}^*$ and $\mathbf{16}^c\otimes\mathbf{16}^*$ used by Chamseddine and Fröhlich. We find the usual tree-level predictions of noncommutative geometry $m_W = \frac{1}{2}m_t$, $\sin^2\theta_W = \frac{3}{8}$ and $g_2 = g_3$ as well as $m_H \leq m_t$.

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1 Introduction

Noncommutative geometry (NCG) has a long history in mathematics, but it were two discoveries by Alain Connes at the end of the last decade[1] which made NCG so interesting for physicists:

- Dirac operator and smooth functions, both acting on the spinor Hilbert space, contain all metric information of a spin manifold,
- a finite dimensional version of that setting yields some sort of Higgs potential.

In the following years, the idea that one should study the tensor product of ordinary differential geometry and a matrix geometry has prospered. Numerous versions to construct gauge field theories with spontaneous symmetry breaking (in particular the standard model) appeared, all more or less inspired by Connes'

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On leave of absence from Institut für Theoretische Physik, Universität Leipzig, Augustusplatz 10/11, 04109 Leipzig, Germany.

discovery, but different in details. These versions developed, competed with each other, died out – a perfect example of Darwin's theory of evolution. After all, one version survived[2], proposed by Alain Connes who – guided by a deep understanding of mathematics – completed a set of axioms[3]. These axioms have their origin in the classification of classical spin manifolds[4]. It is remarkable that the same axioms which are fruitful in the commutative case of spin manifolds provide the standard model as one of the simplest noncommutative examples.

Sometimes it happens that branch lines of evolution have attractive features, which fascinate mankind long after the dead of the species (think of the dinosaurs). Such a species is grand unification. The axioms of NCG are compatible with the standard model but not with grand unification[5]. Nevertheless, grand unification is such an attractive idea that it will not die very soon, no matter how well the standard model is verified by experiment.

Technically, the essential progress that NCG brings over the traditional formulation of gauge theories is that Yang-Mills and Higgs fields are understood as two complementary parts of one universal gauge potential. It is desirable to extend this unifying feature to grand unified theories (GUTs). The only chance to do so is to allow for a minor modification of the NCG-axioms so that GUTs are accessible as well. An inspiration how to modify the axioms comes from traditional gauge theories. They are formulated in terms of Lie groups or – on infinitesimal level – Lie algebras. Thus, replacing the associative algebra in NCG by a Lie algebra[6], one can expect a formalism (section 2) that is able to produce grand unified models.

In this paper we show that the SO(10)-GUT[7,8] can indeed be formulated with this method. The SO(10)-model has the exceptional property that all known fermions and the supposed right-handed neutrino fit into the irreducible **16**-representation of SO(10) (for each generation). Other GUTs such as SU(5), $\mathrm{SU}(5)\times\mathrm{U}(1)$ and $\mathrm{SU}(4)\times\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ arise as intermediate steps of different SO(10) symmetry breaking chains.

The treatment of the SO(10)-model by NCG-methods is not new. The first approach[9] by Chamseddine and Fröhlich came in the early days of noncommutative geometry. Since then, NCG underwent the development sketched above that singled out the axioms incompatible with grand unification. Our construction differs in its conception and its results from the Chamseddine-Fröhlich approach. The authors of ref. 9 start from the (associative) Clifford algebra of SO(10). The crucial difference of the two approaches lies in the Higgs sector. We denote by $Y=\sum_i Y_i\otimes M^i$ the Yukawa operator. Its part Y_i transforms the **16**-representation into itself or into its charge conjugate $\mathbf{16}^c$. The mass matrices M^i mix the three fermion generations. Thus, for Y_i we have the two possibilities $\mathbf{16}\otimes\mathbf{16}^*$ and $\mathbf{16}^c\otimes\mathbf{16}^*$, which are reducible representations under SO(10) or so(10) and decompose into

$$\mathbf{16} \otimes \mathbf{16}^* = \mathbf{1} \oplus \mathbf{45} \oplus \mathbf{210} \;, \qquad \mathbf{16}^c \otimes \mathbf{16}^* = \mathbf{10} \oplus \mathbf{120} \oplus \mathbf{126} \;. \tag{1}$$

The point is now that these two Y_i-representations are *not* reducible under the Clifford algebra Cliff(SO(10)). That is why the Higgs multiplets in the Chamseddine-Fröhlich model are $\mathbf{16} \otimes \mathbf{16}^*$ and $\mathbf{16}^c \otimes \mathbf{16}$, where only the latter occurs in the fermionic action (due to chirality). The consequence is that there is only one generation matrix M^i in the fermionic action. This leads to the relation $m_e : m_\mu : m_\tau = m_d : m_s : m_b = m_u : m_c : m_t$ between the fermion masses, which is obviously not satisfied. Moreover, the analysis of the Higgs potential shows that also a Kobayashi-Maskawa matrix is not possible, but can be obtained by including an additional fermion in the trivial representation.

In our version based upon the *Lie algebra* so(10) we do have the decomposition (1), and each irreducible representation occurring in (1) is tensorized by its own generation matrix. It is therefore no problem to get the fermion masses we want. For the SO(10)-symmetry breaking we have the **45** and **210**-representations at disposal, which both do not occur in the fermionic action. We employ the **45**-representation to break SO(10) to $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_{B-L}$ in the first step (at about[10] 10^{16}GeV). The corresponding (self-adjoint) generation matrix adds a freedom of 9 real parameters. The other symmetry breaking chain $\mathrm{SO}(10) \to \mathrm{SU}(4)_{PS} \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ mediated by the **210** is possible as well, but we have to make a choice due to the length of the formulae. In the second step this intermediate symmetry is broken by the Majorana mass term for the right-handed neutrinos (**126**-generator) to the standard model symmetry group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ (at about[10] 10^9GeV). We then restrict ourselves to the case that the fermion masses are as general as possible, this implies the Higgs multiplets **10** (twice), **120** (four times) and **126** (twice). The surviving symmetry group is $\mathrm{SU}(3)_C \times \mathrm{U}(1)_{EM}$.

There is also a technical difference to mention. The article ref. 9 was written in the pioneering epoch of noncommutative geometry where auxiliary fields emerged in the action. They eliminate themselves at the end via their equation of motion. Our version is based on a differential calculus and we quotient out the ideal of auxiliary fields before building the action. We think this method is more transparent but the results are independent of the way of eliminating these unphysical degrees of freedom.

Our paper is organized as follows: We review in section 2 our Lie algebraic approach to noncommutative geometry. In section 3 we write down our setting of the so(10)-model, where we acknowledge a lot of inspiration from ref. 9. Section 4 is devoted to the computation of the gauge potential and the Higgs part of the field strength. This is the most cumbersome part, because the extremely rich Higgs structure leads to a big number of terms in the field strength. It remains to calculate some traces to get the bosonic action (section 5) and to implement the various symmetries to get the fermionic action (section 6). We conclude with an outlook towards a minimal SO(10)-model.

2 The Lie algebraic formulation of noncommutative geometry

The starting point is the Lie algebra $\mathfrak{g} = C^\infty(M) \otimes \mathfrak{a}$ acting via a representation $\pi = \mathrm{id} \otimes \hat{\pi}$ on the Hilbert space $\mathcal{H} = L^2(M,S) \otimes \mathbb{C}^F$. Here, $C^\infty(M)$ is the algebra of (real-valued) smooth functions on the (compact Euclidean) spacetime manifold M, $L^2(M,S)$ is the Hilbert space of square integrable bispinors and $\hat{\pi}$ a representation of the semisimple matrix Lie algebra $\mathfrak{a}$ on $\mathbb{C}^F$. The treatment of Abelian factors is possible but more complicated. Moreover, we have the selfadjoint unbounded operator $D = \mathrm{i}\partial\!\!\!/ \otimes 1_F + Y$ on $\mathcal{H}$, with $Y = \gamma^5 \otimes \hat{Y}$ and $\hat{Y} \in \mathrm{M}_F\mathbb{C}$. We also need a $\mathbb{Z}_2$-grading operator $\boldsymbol{\Gamma}$ on $\mathcal{H}$ which commutes with $\pi(\mathfrak{g})$ and anti-commutes with D. In many cases there will exist further discrete symmetries such as the charge conjugation J.

A universal 1-form $\omega^1 \in \Omega^1$ has the structure

$$\omega^1 = \sum_{\alpha,z} [f^z_\alpha \otimes a^z_\alpha, [\dots [f^1_\alpha \otimes a^1_\alpha, \mathrm{d}(f^0_\alpha \otimes a^0_\alpha)] \dots]] ,$$

with $f^i_\alpha \in C^\infty(M)$ and $a^i_\alpha \in \mathfrak{a}$. The commutators should be read as tensor products. The representation π of the universal calculus on $\mathcal{H}$ is obtained by taking π of $f^i_\alpha \otimes a^i_\alpha$ and representing the universal d by the derivation $[-\mathrm{i}D,\ .\ ]$,

$$\begin{aligned}
\rho = \pi(\omega^1) &= \textstyle\sum_{\alpha,z} [f^z_\alpha \otimes \hat{\pi}(a^z_\alpha), [\dots [f^1_\alpha \otimes \hat{\pi}(a^1_\alpha), [-\mathrm{i}D, f^0_\alpha \otimes \hat{\pi}(a^0_\alpha)]] \dots]] \\
&= \textstyle\sum_{\alpha,z} f^z_\alpha \dots f^1_\alpha \partial\!\!\!/(f^0_\alpha) \otimes [\hat{\pi}(a^z_\alpha), [\dots [\hat{\pi}(a^1_\alpha), \hat{\pi}(a^0_\alpha)] \dots]] \qquad \to A \\
&+ \textstyle\sum_{\alpha,z} \gamma^5 f^z_\alpha \dots f^1_\alpha f^0_\alpha \otimes [\hat{\pi}(a^z_\alpha), [\dots [\hat{\pi}(a^1_\alpha), [-\mathrm{i}\hat{Y}, \hat{\pi}(a^0_\alpha)]] \dots]] \quad \to \pi(\eta)
\end{aligned}$$

The second and third line are independent for semisimple $\mathfrak{a}$. In the second line the commutators clearly yield an element of $\hat{\pi}(\mathfrak{a})$ and $f \partial\!\!\!/ f'$ a spacetime 1-form, together a Yang-Mills multiplet represented on $\mathcal{H}$. We decompose the finite dimensional part of the Hilbert space into $\mathbb{C}^F = \bigoplus_i \boldsymbol{n}_i \otimes \mathbb{C}^N$, where $\boldsymbol{n}_i$ are irreducible representations of $\mathfrak{a}$ and N is the number of fermion generations. Then, we have the decomposition $\hat{Y} = \sum \hat{Y}^r_{ij} \otimes M^{ij}_r$, where $\hat{Y}^r_{ij} \in \boldsymbol{n}_i \otimes \boldsymbol{n}^*_j$ is (for each r) a generator of an irreducible representation and $M^{ij}_r \in \mathrm{M}_N\mathbb{C}$ a mass matrix. If we now evaluate the commutators in the third line above, the generators are expanded to irreducible multiplets, and we obtain $\pi(\eta) = \sum \gamma^5 \eta^r_{ij} \otimes M^{ij}_r$, where $\eta^r_{ij} \in C^\infty(M) \otimes \boldsymbol{n}_{ij}$ are function-valued irreducible representations, i.e. Higgs multiplets.

The universal differential of ω^1 is defined as

$$\mathrm{d}\omega^1 = \sum_{\alpha,z} \sum_{y=1}^{z} [f^z_\alpha \otimes a^z_\alpha, [\dots [\mathrm{d}(f^y_\alpha \otimes a^y_\alpha), [\dots [f^1_\alpha \otimes a^1_\alpha, \mathrm{d}(f^0_\alpha \otimes a^0_\alpha)] \dots]] \dots]] .$$

Its representation on $\mathcal{H}$ reads after elementary calculation

$$\begin{aligned}
\pi(\mathrm{d}\omega^1) =& \sum_{\alpha,z}\sum_{y=1}^{z}[f_\alpha^z\otimes\hat\pi(a_\alpha^z),[\dots\{[-\mathrm{i}D,f_\alpha^y\otimes\hat\pi(a_\alpha^y)],[\dots[-\mathrm{i}D,f_\alpha^0\otimes\hat\pi(a_\alpha^0)]\dots]\}\dots]] \\
&= \{-\mathrm{i}D,\rho\}+\textstyle\sum_{\alpha,z}[f_\alpha^z\otimes\hat\pi(a_\alpha^z),[\dots[f_\alpha^1\otimes\hat\pi(a_\alpha^1),[D^2,f_\alpha^0\otimes\hat\pi(a_\alpha^0)]]\dots]] \\
&\equiv \{-\mathrm{i}D,\pi(\omega^1)\}+\sigma(\omega^1) \qquad (2)\\
&= \{\not\partial,A\}+\textstyle\sum_{\alpha,z}[f_\alpha^z\otimes\hat\pi(a_\alpha^z),[\dots[f_\alpha^1\otimes\hat\pi(a_\alpha^1),[-\not\partial^2\otimes 1_F,f_\alpha^0\otimes\hat\pi(a_\alpha^0)]]\dots]] \\
&+ \{\not\partial,\pi(\eta)\}+\{-\mathrm{i}Y,A\}+\{-\mathrm{i}Y,\pi(\eta)\}+\hat\sigma(\eta)\;,
\end{aligned}$$

with

$$\hat\sigma(\eta)=\textstyle\sum_{\alpha,z}f_\alpha^z\dots f_\alpha^1 f_\alpha^0\otimes[\hat\pi(a_\alpha^z),[\dots[\hat\pi(a_\alpha^1),[\hat Y^2,\hat\pi(a_\alpha^0)]]\dots]]\;.$$

After a lengthy calculation[6] one finds

$$\begin{aligned}
\{\not\partial,A\}+\textstyle\sum_{\alpha,z}[f_\alpha^z\otimes\hat\pi(a_\alpha^z),[\dots[f_\alpha^1\otimes\hat\pi(a_\alpha^1),[-\not\partial^2\otimes 1_F,f_\alpha^0\otimes\hat\pi(a_\alpha^0)]]\dots]] \\
= \mathbf{d}A+C^\infty(M)\otimes\{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\}\;,
\end{aligned}$$

where $\mathbf{d}$ is the exterior differential (which anti-commutes with γ^5) and the $\{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\}$-part is independent of A and $\pi(\eta)$. Moreover, we have $\{\not\partial,\pi(\eta)\}=\mathbf{d}\pi(\eta)$. We now decompose $\hat Y^2$ into generators of irreducible representations,

$$\hat Y^2=\hat Y_\parallel^2+\hat Y_\perp^2+\hat\pi(\mathbf{1})\;.$$

Here, $\hat\pi(\mathbf{1})$ contains trivial representations which commute with $\hat\pi(\mathfrak{a})$. Those generators which also occur in $\hat Y$, denoted $\hat Y_\parallel^2$, generate obviously the corresponding representations which occur already in $\pi(\eta)$. The other (non-trivial) generators, denoted $\hat Y_\perp^2$, generate representations independent of $\pi(\eta)$ and A.

It is now crucial to notice that the same ρ can be written in many ways as $\pi(\omega^1)$ so that the definition of a differential "$d\rho=\pi(\mathrm{d}\omega^1)$" is ambiguous. The usual way out is to consider equivalence classes modulo the ideal $\mathcal{J}^2=\pi(\mathrm{d}(\ker\pi\cap\Omega^1))$. We have just shown that if $\pi(\omega^1)=0$ then there remain only the $\{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\}$-part and the representations generated by $\hat Y_\perp^2$, which gives

$$\mathcal{J}^2=C^\infty(M)\otimes\textstyle\sum(\{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\}+[\hat\pi(\mathfrak{a}),[\dots[\hat\pi(\mathfrak{a}),\hat Y_\perp^2]\dots]])\;. \qquad (3)$$

The final formula for the differential of $\rho=A+\pi(\eta)$ is therefore

$$d\rho=\mathbf{d}\rho+\{-\mathrm{i}Y,\rho\}+\hat\sigma(\eta)\quad\mathrm{mod}\;\mathcal{J}^2\;. \qquad (4)$$

In the same way one represents the space Ω^n of universal forms of degree n on $\mathcal{H}$ and determines the corresponding ideal $\mathcal{J}^n=\pi(\mathrm{d}(\ker\pi\cap\Omega^{n-1}))$. The generalization of (2) is

$$d(\pi(\omega^n)+\mathcal{J}^n)=[\![-\mathrm{i}D,\pi(\omega^n)]\!]+\sigma(\omega^n)+\mathcal{J}^{n+1}\;,\qquad \omega^n\in\Omega^n\;, \qquad (5)$$

where $[\![\,.\,,\,.\,]\!]$ is the graded commutator, i.e. the anti-commutator if both entries are odd under $\mathbb{Z}_2$ and the commutator else. This yields the graded differential Lie algebra $\Omega_D = \pi(\Omega)/\mathcal{J}$.

We propose to define the connection ∇ as a generalization of the differential d and the covariant derivative $\mathcal{D}$ as a generalization of the operator D. This means that $\mathcal{D}$ is a linear unbounded selfadjoint operator on $\mathcal{H}$ and odd under $\mathbb{Z}_2$, and $\nabla : \Omega_D^n \to \Omega_D^{n+1}$ is linear. Both $\mathcal{D}$ and ∇ are related via the same formula (5):

$$\nabla(\pi(\omega^n) + \mathcal{J}^n) = [\![-\mathrm{i}\mathcal{D}, \pi(\omega^n)]\!] + \sigma(\omega^n) + \mathcal{J}^{n+1} ,$$

for $\omega^n \in \Omega^n$ and any degree n. The general solution is

$$\mathcal{D} = D + \mathrm{i}\rho , \quad \nabla = d + [\![\rho,\,.\,]\!] , \quad [\![\rho, \pi(\Omega^n)]\!] \subset \pi(\Omega^{n+1}) , \quad [\![\rho, \mathcal{J}^n]\!] \subset \mathcal{J}^{n+1} .$$

One obvious solution is $\rho = A + \pi(\eta) \in \pi(\Omega^1)$. But there are further solutions possible, depending on the setting. These additional solutions allow us to formulate gauge theories with u(1)-factors such as the standard model. Demanding that ρ commutes with functions, we have the decomposition

$$\rho' = \Lambda^1 \otimes \mathbf{r}^0 + \Lambda^0\gamma^5 \otimes \mathbf{r}^1$$

of the additional solutions, where the matrices $\mathbf{r}^i \in \mathrm{M}_F\mathbb{C}$ commute with $\hat{\pi}(\mathfrak{a})$. The essential step is to check $\{\rho', \pi(\Omega^1)\} \subset \pi(\Omega^2)$, which yields several conditions for $\mathbf{r}^0$ and $\mathbf{r}^1$. Finally, one has to verify the compatibility with $\mathcal{J}$.

The curvature is now $\nabla^2 = [\mathcal{F},\,.\,]$, where one finds the usual formula

$$\mathcal{F} = d\rho + \tfrac{1}{2}\{\rho, \rho\}$$

for the field strength. The differential and (anti)commutator are defined via the (graded) Leibniz rule and Jacobi identity; for it one has to enlarge the ideal $\mathcal{J}$ by the graded centralizer $\mathcal{C}$ of $\pi(\Omega)$. Then, the general formula (4) continues to work so that for $\rho = A + \pi(\eta)$ one has

$$\begin{aligned}\mathcal{F} = {} & (\mathbf{d}A + A^2|_{\Lambda^2}) + (\mathbf{d}\pi(\eta) + \{A, (\pi(\eta) - \mathrm{i}Y)\}) \\ & + \big((\pi(\eta))^2 + \{-\mathrm{i}Y, \pi(\eta)\} + \hat{\sigma}(\eta) \mod \mathcal{J}^2 + \mathcal{C}^2\big) . \end{aligned} \tag{6}$$

Here, $A^2|_{\Lambda^2}$ is the restriction of A^2 to the spacetime 2-form part. The bosonic action is defined via the Dixmier trace and can be rewritten as

$$\begin{aligned} S_B &= \tfrac{1}{g^2 F} \int_M dx\ \mathrm{tr}(\mathcal{F}_\perp)^2 = \int_M dx\ (\mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0) \qquad (7) \\ &= \tfrac{1}{g^2 F} \int_M dx\ \Big(\mathrm{tr}((\mathbf{d}A + A^2|_{\Lambda^2})^2) + \mathrm{tr}((\mathbf{d}\pi(\eta) + \{A, (\pi(\eta) - \mathrm{i}Y)\})^2) \\ &\qquad + \mathrm{tr}\big(\big((\pi(\eta))^2 + \{-\mathrm{i}Y, \pi(\eta)\} + \hat{\sigma}(\eta)\big)^2_\perp\big)\Big) . \end{aligned}$$

Here, g is a coupling constant and F the dimension of the matrix part. The trace includes the trace over gamma matrices and $\mathcal{F}_\perp$ is the component of $\mathcal{F}$ orthogonal to $\mathcal{J}^2$. The bosonic action consists of three parts, the Yang-Mills Lagrangian $\mathcal{L}_2$, the covariant derivative $\mathcal{L}_1$ of the Higgs fields and the Higgs potential $\mathcal{L}_0$. The fermionic action is

$$S_F = \tfrac{1}{2^s} \int_M dx\, \boldsymbol{\psi}^* \mathcal{D} \boldsymbol{\psi} = \tfrac{1}{2^s} \int_M dx\, \mathrm{i} \boldsymbol{\psi}^* (\partial\!\!\!/ \otimes \mathbf{1}_F + A + \pi(\eta) - \mathrm{i}Y) \boldsymbol{\psi} \;, \tag{8}$$

where s is the number of discrete symmetries of the setting, and $\boldsymbol{\psi} \in \mathcal{H}$ is invariant under these symmetries (possibly only after passing to Minkowski space). The fermionic action contains the minimal coupling to the Yang-Mills fields A and the Yukawa coupling to the Higgs fields $\pi(\eta)$.

Let us study the part $\theta = \big((\pi(\eta))^2 + \{-\mathrm{i}Y, \pi(\eta)\} + \hat{\sigma}(\eta)\big)_\perp$ of the field strength, whose square gives the Higgs potential. Since the covariant derivative $\mathcal{D}$ can be written as $\mathcal{D} = \mathrm{i}\partial\!\!\!/ \otimes \mathbf{1}_F + \mathrm{i}A + \mathrm{i}(\pi(\eta) - \mathrm{i}Y)$, it is natural to consider $\pi(\tilde{\eta}) = (\pi(\eta) - \mathrm{i}Y)$ as the analogue of a classical Higgs multiplet. This is because $\pi(\tilde{\eta})$ transforms as $\pi(\tilde{\eta}) \mapsto u\pi(\tilde{\eta})u^*$ under gauge transformations $\mathcal{D} \mapsto u\mathcal{D}u^*$, with $u \in C^\infty(M) \otimes \exp(\mathfrak{a} + \mathbf{r}^0)$. In terms of $\tilde{\eta}$ we have

$$\begin{aligned} \theta &= (\pi(\tilde{\eta})^2 + \hat{\sigma}(\eta) + Y^2)_\perp = (\pi(\tilde{\eta})^2 + \hat{\sigma}(\eta) + \pi(\mathbf{1}) + Y_\perp^2 + Y_\parallel^2)_\perp \\ &= (\pi(\tilde{\eta})^2 + \hat{\sigma}(\tilde{\eta}) + \pi(\mathbf{1}))_\perp \;, \end{aligned} \tag{9}$$

because $Y_\perp^2$ has by definition no component orthogonal to $\mathcal{J}^2$ and

$$\begin{array}{ll} \text{if} & \pi(\tilde{\eta}) = -\mathrm{i}Y + \sum_{\alpha,z} [\pi(a_\alpha^z), [\dots[\pi(a_\alpha^1), [-\mathrm{i}Y, \pi(a_\alpha^0)]]\dots]] \\ \text{then} & \hat{\sigma}(\tilde{\eta}) = Y_\parallel^2 + \sum_{\alpha,z} [\pi(a_\alpha^z), [\dots[\pi(a_\alpha^1), [Y_\parallel^2, \pi(a_\alpha^0)]]\dots]] \;, \end{array}$$

with $a_\alpha^i \in \mathfrak{g}$. Thus, θ can be expressed completely in terms of $\tilde{\eta}$ so that gauge invariance of the Higgs potential $V = \mathrm{tr}(\theta^2)$ is obvious. The point is now that we know a priori the Higgs vacuum: it is $\langle \pi(\tilde{\eta}) \rangle_0 = -\mathrm{i}Y$. At this configuration we have $\theta = 0$ and $V = 0$. On the other hand V is non-negative so that $-\mathrm{i}Y$ is a global and local minimum. In the vicinity of $-\mathrm{i}Y$ we have $V = \mathrm{tr}((-\mathrm{i}(Y\pi(\eta) + \pi(\eta)Y) + \hat{\sigma}(\eta))_\perp^2)$, i.e. something bilinear in the physical Higgs fields. The coefficients are the Higgs masses, after diagonalization and rescaling. There are however massless modes, the Goldstone bosons. They are of the form $\pi(\eta) = [\pi(a), -\mathrm{i}Y]$ with $a \in \mathfrak{g}$. In this case we have

$$\begin{aligned} (-\mathrm{i}(Y\pi(\eta) + \pi(\eta)Y) + \hat{\sigma}(\eta))_\perp &= (-[\pi(a), Y^2] + [\pi(a), Y_\parallel^2])_\perp \\ &= (-[\pi(a), Y_\perp^2])_\perp = 0 \;. \end{aligned}$$

The masses of the Yang-Mills fields come from the part $\mathrm{tr}(\{A, -\mathrm{i}Y\})^2 = \mathrm{tr}([\pi(A_\mu), Y][\pi(A^\mu), Y])$ of the Lagrangian $\mathcal{L}_1$. This is a form of the Goldstone-Higgs theorem: The massive Yang-Mills fields are those which do not commute with Y and to each of them there corresponds a massless Goldstone boson. The Higgs mechanism consists in removing the Goldstone bosons by those gauge transformations which do not commute with Y, and which are fixed in this way. The remaining unconstrained gauge degrees of freedom are those which commute with Y, and to each of them there corresponds a massless Yang-Mills field.

3 The so(10)-setting

Let $\Gamma_I \in \mathrm{M}_{32}\mathbb{C}$, $I = 0,\dots,9$, be so(10)-gamma matrices represented in terms of tensor products of five sets of Pauli matrices[9]

$$\begin{aligned} \Gamma_i &= \kappa_1\rho_3\eta_i\;, & \sigma_i &\to 1_2\otimes 1_2\otimes 1_2\otimes 1_2\otimes \sigma_i\;,\\ \Gamma_{i+3} &= \kappa_1\rho_1\sigma_i\;, & \tau_i &\to 1_2\otimes 1_2\otimes 1_2\otimes \tau_i\otimes 1_2\;,\\ \Gamma_{i+6} &= \kappa_1\rho_2\tau_i\;, & \eta_i &\to 1_2\otimes 1_2\otimes \eta_i\otimes 1_2\otimes 1_2\;,\\ \Gamma_0 &= \kappa_2\;, & \rho_i &\to 1_2\otimes \rho_i\otimes 1_2\otimes 1_2\otimes 1_2\;,\\ \Gamma_{11} &= \mathrm{i}\Gamma_0\Gamma_1\cdots\Gamma_9 = \kappa_3\;, & \kappa_i &\to \kappa_i\otimes 1_2\otimes 1_2\otimes 1_2\otimes 1_2\;, \end{aligned} \tag{10}$$

where $i = 1,2,3$. The tensor products are interpreted such that σ_i is 2×2 and κ_i is 32×32.

The matrix Lie algebra is $\mathfrak{a} = \mathrm{so}(10)$ represented as $\mathrm{so}(10) \ni a = a^{IJ}\Gamma_{IJ}$, with $a^{IJ} = -a^{JI} \in \mathbb{R}$, and where $\Gamma_{I_1 I_2\dots I_n} = (1/n!)\Gamma_{[I_1}\Gamma_{I_2}\cdots\Gamma_{I_n]}$ is the completely anti-symmetrized product of Γ-matrices. Summation over equal so(10) indices $I, J, \dots$ from 0 to 9 and over equal spacetime indices $\kappa, \lambda, \dots$ from 0 to 3 is understood. We introduce the projection operators

$$P_\pm = \tfrac{1}{2}(1 \pm \Gamma_{11})\otimes 1_3\;,\quad \mathcal{P} = \mathrm{diag}(1_4\otimes P_+\,,\, 1_4\otimes P_+\,,\, 1_4\otimes P_-\,,\, 1_4\otimes P_-)\;,$$

and the so(10)-conjugation matrix

$$\begin{aligned} B = -\Gamma_1\Gamma_3\Gamma_4\Gamma_6\Gamma_8\otimes 1_3 &= \bar{B} = B^T = B^\dagger\;, & B^2 &= 1_{96}\;,\\ B(\Gamma_I\otimes m)B &= \Gamma_I^T\otimes m \quad \forall m\in \mathrm{M}_3\mathbb{C}\;, & BP_\pm B &= P_\mp\;, \end{aligned}$$

and define the $\mathbb{Z}_2$-grading operator

$$\mathbf{\Gamma} = \mathrm{diag}(-\gamma^5\otimes P_+\,,\, \gamma^5\otimes P_+\,,\, \gamma^5\otimes P_-\,,\, -\gamma^5\otimes P_-)\;.$$

Then, the Hilbert space is

$$\mathcal{H} = \mathcal{P}(L^2(M,S)\otimes\mathbb{C}^{32}\otimes\mathbb{C}^3\otimes\mathbb{C}^4) \cong L^2(M,S)\otimes\mathbb{C}^{192}\;. \tag{11}$$

The representation π of $\mathfrak{g} = C^\infty(M)\otimes\mathfrak{a} \ni f\otimes a$ on $\mathcal{H}$ is defined by

$$\pi(f\otimes a) = f1_4\otimes\begin{pmatrix} P_+(a\otimes 1_3)P_+ & 0 & 0 & 0\\ 0 & P_+(a\otimes 1_3)P_+ & 0 & 0\\ 0 & 0 & P_-(a\otimes 1_3)P_- & 0\\ 0 & 0 & 0 & P_-(a\otimes 1_3)P_- \end{pmatrix}. \tag{12}$$

Note that $P_\pm$ commutes with $a \otimes \mathbf{1}_3$ and that $\mathbf{\Gamma}$ commutes with $\pi(a)$. The selfadjoint Yukawa operator anti-commuting with $\mathbf{\Gamma}$ is

$$Y = \begin{pmatrix} 0 & \gamma^5 \otimes P_+\mathcal{M}P_+ & \gamma^5 \otimes P_+\mathcal{N}P_- & 0 \\ \gamma^{5*} \otimes P_+\mathcal{M}P_+ & 0 & 0 & \gamma^{5*} \otimes P_+\mathcal{N}P_- \\ \gamma^{5*} \otimes P_-\mathcal{N}^\dagger P_+ & 0 & 0 & \gamma^{5*} \otimes P_-B\overline{\mathcal{M}}BP_- \\ 0 & \gamma^5 \otimes P_-\mathcal{N}^\dagger P_+ & \gamma^5 \otimes P_-B\overline{\mathcal{M}}BP_- & 0 \end{pmatrix} . \tag{13}$$

We distinguish explicitly γ^5 and γ^{5*}, which are equal in Euclidean space, in Minkowski space however we have $\gamma^5 = -\gamma^{5*}$. This allows for a parallel development of our model in both Euclidean and Minkowski space. The matrices $\mathcal{M}$ and $\mathcal{N}$ are obtained by tensorizing the generators given in ref. 9 by *independent* generation matrices:

$$\begin{aligned} \mathcal{M} &= -\mathrm{i}(\Gamma_{45} + \Gamma_{78} + \Gamma_{69}) \otimes M_1 \ , \\ \mathcal{N} &= \mathrm{i}\Gamma_0 \otimes M_s + \Gamma_3 \otimes M_p + \Gamma_{120} \otimes M'_a - \mathrm{i}\Gamma_{123} \otimes M_a \\ &\quad + (\Gamma_{450} + \Gamma_{780} + \Gamma_{690}) \otimes M'_b - \mathrm{i}(\Gamma_{453} + \Gamma_{783} + \Gamma_{693}) \otimes M_b \\ &\quad - \mathrm{i}(\Gamma_{01245} + \Gamma_{01278} + \Gamma_{01269}) \otimes M_c - (\Gamma_{31245} + \Gamma_{31278} + \Gamma_{31269}) \otimes M_f \\ &\quad - \tfrac{1}{8}\mathrm{i}(\Gamma_1 - \mathrm{i}\Gamma_2)\Gamma_3(\Gamma_4 - \mathrm{i}\Gamma_5)(\Gamma_6 - \mathrm{i}\Gamma_9)(\Gamma_7 - \mathrm{i}\Gamma_8) \otimes M_2 \ . \end{aligned} \tag{14}$$

Here, M_s, M_p, M_c, M_f, M_2 are symmetric and M_a, M'_a, M_b, M'_b anti-symmetric 3×3-matrices and $M_1 = M_1^\dagger$. That implies $B\mathcal{N}B = \mathcal{N}^T$ and $\mathcal{M} = \mathcal{M}^\dagger$. We stress that here lies the essential difference between our Lie formalism and the algebraic version[9]: There, the matrices M_s, M_p, M_c, M_f, M_2 are all proportional to each other, the same is true for the matrices M_a, M'_a, M_b, M'_b. This is dictated by the fact that $\mathbf{16}^c \otimes \mathbf{16}^*$ is irreducible under the Clifford algebra of SO(10).

The above setting is chosen in such a way that it has two symmetries J and $\mathcal{S}$. First, the charge conjugation is given by

$$J = \mathcal{P} \begin{pmatrix} 0 & 0 & C\otimes B & 0 \\ 0 & 0 & 0 & C\otimes B \\ C\otimes B & 0 & 0 & 0 \\ 0 & C\otimes B & 0 & 0 \end{pmatrix} \mathcal{P} \circ c.c \ ,$$

where C is the spacetime conjugation matrix, $C\overline{\gamma^\mu}C = \gamma^\mu$, and $c.c$ stands for complex conjugation. We use the following convention for Euclidean gamma matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \mathrm{i}\sigma^a \\ -\mathrm{i}\sigma^a & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^2 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad C = \gamma^0\gamma^2 .$$

Our Minkowskian gamma matrices are

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & -\sigma^a \\ \sigma^a & 0 \end{pmatrix}, \quad \gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad C = \gamma^2 .$$

Observe that $J^2 = \mathcal{P}$ in Minkowski space but $J^2 = -\mathcal{P}$ in Euclidean space. Second, we have an exchange symmetry

$$\mathcal{S}_E = \mathcal{P}\begin{pmatrix} 0 & 1_{384} & 0 & 0 \\ 1_{384} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{384} \\ 0 & 0 & 1_{384} & 0 \end{pmatrix}\mathcal{P}, \quad \mathcal{S}_M = \mathcal{P}\begin{pmatrix} 0 & \mathrm{i}1_{384} & 0 & 0 \\ -\mathrm{i}1_{384} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i}1_{384} \\ 0 & 0 & \mathrm{i}1_{384} & 0 \end{pmatrix}\mathcal{P},$$

where $\mathcal{S}_E$ is realized in Euclidean space and $\mathcal{S}_M$ in Minkowski space. This yields in both Euclidean and Minkowski spaces

$$[J,D] = [J,\pi(a)] = [\mathcal{S},D] = [\mathcal{S},\pi(a)] = [J,\mathcal{S}] = 0 ,$$

where $D = \mathcal{P}(\mathrm{i}\partial\!\!\!/ \otimes 1_{32} \otimes 1_3 \otimes 1_4)\mathcal{P} + Y$.

4 The gauge potential and its field strength

The gauge potential $\rho \in \pi(\Omega^1)$ is composed of two parts, of a $\mathfrak{a}$-valued spacetime 1-form $A = \gamma^\mu \pi(A_\mu)$ and (up to γ^5) a $\mathfrak{a}$- representation valued spacetime 0-form $\pi(\eta)$, $\rho = A + \pi(\eta)$. The second part has the general structure

$$\pi(\eta) = \sum_{\alpha,z} [\pi(a^z_\alpha),[\dots,[\pi(a^1_\alpha),[-\mathrm{i}Y,\pi(a^0_\alpha)]]\dots]] ,$$

where $a^i_\alpha \in \mathfrak{g}$. Products of Γ-matrices are generators of irreducible representations, but as some of them occur more than once in Y, we must check that they are linear independent. For instance,

$$-\tfrac{1}{4}\mathrm{ad}_{\Gamma_{01}} \circ \mathrm{ad}_{\Gamma_{01}}(\Gamma_0) \equiv -\tfrac{1}{4}[\Gamma_{01},[\Gamma_{01},\Gamma_0]] = \Gamma_0 , \quad -\tfrac{1}{4}\mathrm{ad}_{\Gamma_{01}} \circ \mathrm{ad}_{\Gamma_{01}}(\Gamma_3) = 0 ,$$

which establishes the independence of the two 10-dimensional representations generated by $\Gamma_0 \otimes M_s$ and $\Gamma_3 \otimes M_p$. Next, application of $-\frac{1}{4}\mathrm{ad}_{\Gamma_{01}} \circ \mathrm{ad}_{\Gamma_{01}}$ to the four 120-dimensional representations generated by Γ_{IJK} establishes the independence of Γ_{123} and $(\Gamma_{450} + \Gamma_{780} + \Gamma_{690})$ from Γ_{120} and $(\Gamma_{453} + \Gamma_{783} + \Gamma_{693})$. Application of $\frac{1}{8}\mathrm{ad}_{\Gamma_{16}} \circ \mathrm{ad}_{\Gamma_{64}} \circ \mathrm{ad}_{\Gamma_{41}}$ leads to independence of all these four 120-dimensional representations. Finally, application of $-\frac{1}{4}\mathrm{ad}_{\Gamma_{69}} \circ \mathrm{ad}_{\Gamma_{69}}$ and $-\frac{1}{4}\mathrm{ad}_{\Gamma_{01}} \circ \mathrm{ad}_{\Gamma_{01}}$ to the three 126-dimensional representations generated by Γ_{IJKLM} shows that they are independent. In conclusion, and using the identity $B(\overline{a \otimes 1_3})B = a \otimes 1_3$, the general form of $\pi(\eta)$ is

$$\pi(\eta) = -\mathrm{i}\mathcal{P}\begin{pmatrix} 0 & \gamma^5\,\eta_{\mathcal{M}} & \gamma^5\,\eta_{\mathcal{N}} & 0 \\ \gamma^{5*}\,\eta_{\mathcal{M}} & 0 & 0 & \gamma^{5*}\,\eta_{\mathcal{N}} \\ \gamma^{5*}\,\eta^\dagger_{\mathcal{N}} & 0 & 0 & \gamma^{5*}\,B\overline{\eta_{\mathcal{M}}}B \\ 0 & \gamma^5\,\eta^\dagger_{\mathcal{N}} & \gamma^5\,B\overline{\eta_{\mathcal{M}}}B & 0 \end{pmatrix}\mathcal{P} , \tag{15}$$

$$\begin{aligned} \eta_{\mathcal{M}} &= -\mathrm{i}\Theta \otimes M_1 \\ \eta_{\mathcal{N}} &= \mathrm{i}\Upsilon_1 \otimes M_s + \Upsilon_2 \otimes M_p + \Phi_1 \otimes M'_a - \mathrm{i}\Phi_2 \otimes M_a + \Phi_3 \otimes M'_b - \mathrm{i}\Phi_4 \otimes M_b \\ &\quad - \mathrm{i}\Psi_1 \otimes M_c - \Psi_2 \otimes M_f - \mathrm{i}\Psi_3 \otimes M_2 , \end{aligned}$$

where $\Theta \in C^\infty(M) \otimes \mathbf{45}$, $\Upsilon_i \in C^\infty(M) \otimes \mathbf{10}$ and $\Phi_i \in C^\infty(M) \otimes \mathbf{120}$, all of them being real representations, and $\Psi_i \in C^\infty(M) \otimes \mathbf{126}$ (complex representation).

The next step is to compute the ideal $\mathcal{J}^2$ and the part $\hat{\sigma}(\eta)$ of the curvature. Both are related to Y^2, decomposed into irreducible representations. Those which occur both in Y and Y^2 contribute to $\hat{\sigma}(\eta)$, the other ones give rise to the ideal. Using $\gamma^5\gamma^{5*} = \epsilon\mathbf{1}_4$, with $\epsilon = 1$ in Euclidean space and $\epsilon = -1$ in Minkowski space, we have

$$Y^2 = \epsilon\mathbf{1}_4 \otimes \begin{pmatrix} P_+\mathcal{Y}_{(1)}P_+ & 0 & 0 & \mathcal{Y}_{(2)}P_- \\ 0 & P_+\mathcal{Y}_{(1)}P_+ & P_+\mathcal{Y}_{(2)}P_- & 0 \\ 0 & P_-\mathcal{Y}^\dagger_{(2)}P_+ & P_-B\overline{\mathcal{Y}_{(1)}}BP_- & 0 \\ P_-\mathcal{Y}^\dagger_{(2)}P_+ & 0 & 0 & P_-B\overline{\mathcal{Y}_{(1)}}BP_- \end{pmatrix}, \qquad (16)$$

$$\mathcal{Y}_1 = \mathcal{M}^2 + \mathcal{N}\mathcal{N}^\dagger \,, \qquad \mathcal{Y}_2 = \mathcal{M}\mathcal{N} + \mathcal{N}B\overline{\mathcal{M}}B \,.$$

In detail, we find

$$\begin{aligned}
\mathcal{Y}_{(1)} = {}& \mathbf{1}_{32} \otimes (3M_1M_1^\dagger + M_sM_s^\dagger + M_pM_p^\dagger + M'_aM'^\dagger_a + M_aM_a^\dagger \\
& \quad + 3M'_bM'^\dagger_b + 3M_bM_b^\dagger + 3M_cM_c^\dagger + 3M_fM_f^\dagger + \tfrac{1}{4}M_2M_2^\dagger) \\
& - \mathrm{i}(\Gamma_{45} + \Gamma_{78} + \Gamma_{69}) \otimes (M_sM'^\dagger_b + M'_bM_s^\dagger + M_pM_b^\dagger + M_bM_p^\dagger + M'_aM_c^\dagger + M_cM'^\dagger_a \\
& \quad + M_aM_f^\dagger + M_fM_a^\dagger + 2M'_bM_f^\dagger + 2M_fM'^\dagger_b + 2M_bM_c^\dagger + 2M_cM_b^\dagger - \tfrac{1}{4}M_2M_2^\dagger) \\
& + \mathrm{i}\Gamma_{03} \otimes \mathcal{Z}_1 - \mathrm{i}\Gamma_{12} \otimes \mathcal{Z}_2 \\
& - (\Gamma_{4578} + \Gamma_{4569} + \Gamma_{7869}) \otimes \mathcal{Z}_3 + (\Gamma_{0345} + \Gamma_{0378} + \Gamma_{0369}) \otimes \mathcal{Z}_4 \\
& - (\Gamma_{1245} + \Gamma_{1278} + \Gamma_{1269}) \otimes \mathcal{Z}_5 + \Gamma_{0123} \otimes \mathcal{Z}_6 \\
& + \tfrac{1}{8}(\Gamma_{2567} - \Gamma_{1467} + \Gamma_{1568} + \Gamma_{2468} - \Gamma_{1489} - \Gamma_{2479} + \Gamma_{2589} - \Gamma_{1579}) \otimes \mathcal{Z}_7 \\
& + \tfrac{1}{8}(\Gamma_{1567} + \Gamma_{2467} - \Gamma_{2568} + \Gamma_{1468} + \Gamma_{2489} - \Gamma_{1479} + \Gamma_{1589} + \Gamma_{2579}) \otimes \mathcal{Z}_8 \,.
\end{aligned}$$

$$\begin{aligned}
\mathcal{Y}_{(2)} = {}& \mathrm{i}\Gamma_0 \otimes (3M_1M'_b - 3M'_b\overline{M}_1) + \Gamma_3 \otimes (3M_1M_b - 3M_b\overline{M}_1) \\
& + \Gamma_{120} \otimes (3M_1M_c - 3M_c\overline{M}_1) - \mathrm{i}\Gamma_{123} \otimes (3M_1M_f - 3M_f\overline{M}_1) \\
& - \mathrm{i}(\Gamma_{453} + \Gamma_{783} + \Gamma_{693}) \otimes (M_1M_p - \mathrm{i}M_p\overline{M}_1 + 2M_1M_c - 2M_c\overline{M}_1) \\
& + (\Gamma_{450} + \Gamma_{780} + \Gamma_{690}) \otimes (M_1M_s - M_s\overline{M}_1 + 2M_1M_f - 2M_fM_1) \\
& - \mathrm{i}(\Gamma_{01245} + \Gamma_{01278} + \Gamma_{01269}) \otimes (M_1M'_a - M'_a\overline{M}_1 + 2M_1M_b - 2M_b\overline{M}_1) \\
& - (\Gamma_{31245} + \Gamma_{31278} + \Gamma_{31269}) \otimes (M_1M_a - M_a\overline{M}_1 + 2M_1M'_b - 2M'_b\overline{M}_1) \\
& - \tfrac{1}{8}\mathrm{i}(\Gamma_1 - \mathrm{i}\Gamma_2)\Gamma_3(\Gamma_4 - \mathrm{i}\Gamma_5)(\Gamma_6 - \mathrm{i}\Gamma_9)(\Gamma_7 - \mathrm{i}\Gamma_8) \otimes (-3M_1M_2 - 3M_2\overline{M}_1) \,,
\end{aligned}$$

where ($h.c$ denotes the Hermitian conjugate of the preceding term)

$$\mathcal{Z}_1 = (M_sM_p^\dagger + M'_aM_a^\dagger + 3M'_bM_b^\dagger + 3M_cM_f^\dagger + \tfrac{1}{8}M_2M_2^\dagger) + \mathrm{h.c}\,,$$
$$\mathcal{Z}_2 = (M_sM'^\dagger_a + M_pM_a^\dagger + 3M'_bM_c^\dagger + 3M_bM_f^\dagger - \tfrac{1}{8}M_2M_2^\dagger) + \mathrm{h.c}\,,$$

$$
\begin{aligned}
\mathcal{Z}_3 &= (M_1M_1^\dagger + \tfrac{1}{8}M_2M_2^\dagger + M_sM_f^\dagger + M_pM_c^\dagger + M_a'M_b^\dagger \\
&\qquad + M_aM_b'^\dagger + M_b'M_b'^\dagger + M_bM_b^\dagger + M_cM_c^\dagger + M_fM_f^\dagger) + \text{h.c}\,, \\
\mathcal{Z}_4 &= (M_sM_b^\dagger + M_pM_b'^\dagger + M_a'M_f^\dagger + M_aM_c^\dagger + 2M_b'M_c^\dagger + 2M_bM_f^\dagger - \tfrac{1}{8}M_2M_2^\dagger) + \text{h.c}\,, \\
\mathcal{Z}_5 &= (M_sM_c{}^\dagger + M_pM_f^\dagger + M_a'M_b'^\dagger + M_aM_b^\dagger + 2M_b'M_b^\dagger + 2M_cM_f^\dagger + \tfrac{1}{8}M_2M_2^\dagger) + \text{h.c}\,, \\
\mathcal{Z}_6 &= (M_sM_a^\dagger + M_pM_a'^\dagger + 3M_b'M_f^\dagger + 3M_bM_c^\dagger - \tfrac{1}{8}M_2M_2^\dagger) + \text{h.c}\,, \\
\mathcal{Z}_7 &= \mathrm{i}(((M_s + M_p + M_a' + M_a + 3M_b' + 3M_b + 3M_c + 3M_f)M_2^\dagger) - \text{h.c})\,, \\
\mathcal{Z}_8 &= ((M_s + M_p + M_a' + M_a + 3M_b' + 3M_b + 3M_c + 3M_f)M_2^\dagger) + \text{h.c}\,.
\end{aligned}
$$

This gives

$$
\hat\sigma(\eta) = \mathcal{P}\Big(\epsilon \mathbf{1}_4 \otimes \begin{pmatrix} \hat\sigma(\eta_{(1)}) & 0 & 0 & \hat\sigma(\eta_{(2)}) \\ 0 & \hat\sigma(\eta_{(1)}) & \overline{\hat\sigma(\eta_{(2)})} & 0 \\ 0 & \hat\sigma(\eta_{(2)})^\dagger & B\overline{\hat\sigma(\eta_{(1)})}B & 0 \\ \hat\sigma(\eta_{(2)})^\dagger & 0 & 0 & B\overline{\hat\sigma(\eta_{(1)})}B \end{pmatrix}\Big)\mathcal{P}\,,
$$

$$
\begin{aligned}
\hat\sigma(\eta_{(1)}) &= -\mathrm{i}\Theta \otimes (M_sM_b'^\dagger + M_b'M_s^\dagger + M_pM_b^\dagger + M_bM_p^\dagger + M_a'M_c^\dagger + M_cM_a'^\dagger \\
&\qquad + M_aM_f^\dagger + M_fM_a^\dagger + 2M_b'M_f^\dagger + 2M_fM_b'^\dagger + 2M_bM_c^\dagger + 2M_cM_b^\dagger - \tfrac{1}{4}M_2M_2^\dagger) \\
\hat\sigma(\eta_{(2)}) &= \mathrm{i}\Upsilon_1 \otimes (3M_1M_b' - 3M_b'\overline{M}_1) + \Upsilon_2 \otimes (3M_1M_b - 3M_b\overline{M}_1) \\
&\quad + \Phi_1 \otimes (3M_1M_c - 3M_c\overline{M}_1) - \mathrm{i}\Phi_2 \otimes (3M_1M_f - 3M_f\overline{M}_1) \\
&\quad + \Phi_3 \otimes (M_1M_s - M_s\overline{M}_1 + 2M_1M_f - 2M_f\overline{M}_1) \\
&\quad - \mathrm{i}\Phi_4 \otimes (M_1M_p - M_p\overline{M}_1 + 2M_1M_c - 2M_c\overline{M}_1) \\
&\quad - \mathrm{i}\Psi_1 \otimes (M_1M_a' - M_a'\overline{M}_1 + 2M_1M_b - 2M_b\overline{M}_1) \\
&\quad - \Psi_2 \otimes (M_1M_a - M_a\overline{M}_1 + 2M_1M_b' - 2M_b'\overline{M}_1) \\
&\quad - \mathrm{i}\Psi_3 \otimes (-3M_1M_2 - 3M_2\overline{M}_1)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{J}^2 &= \textstyle\sum([\pi(\mathfrak{g}), [\dots, [\pi(\mathfrak{g}), Y_\perp^2]]\dots] + \{\pi(\mathfrak{g}), \pi(\mathfrak{g})\}) \\
&= \begin{pmatrix} P_+\mathcal{J}_{(1)}P_+ & 0 & 0 & 0 \\ 0 & P_+\mathcal{J}_{(1)}P_+ & 0 & 0 \\ 0 & 0 & P_-B\overline{\mathcal{J}_{(1)}}BP_- & 0 \\ 0 & 0 & 0 & P_-B\overline{\mathcal{J}_{(1)}}BP_- \end{pmatrix}, \\
\mathcal{J}_{(1)} &= C^\infty(M) \otimes \mathbf{1} \otimes \mathbb{C}\mathbf{1}_3 + C^\infty(M) \otimes \mathrm{i}\mathbf{45} \otimes (\mathbb{C}\mathcal{Z}_1 + \mathbb{C}\mathcal{Z}_2) \\
&\quad + C^\infty(M) \otimes \mathbf{210} \otimes (\mathbb{C}\mathcal{Z}_3 + \mathbb{C}\mathcal{Z}_4 + \mathbb{C}\mathcal{Z}_5 + \mathbb{C}\mathcal{Z}_6 + \mathbb{C}\mathcal{Z}_7 + \mathbb{C}\mathcal{Z}_8 + \mathbb{C}\mathbf{1}_3)\ .
\end{aligned}
$$

The field strength $\mathcal{F}$ of the gauge potential $\rho = A + \pi(\eta)$ is given in (6). Let θ be the spacetime 0-form component of $\mathcal{F}_\perp$ orthogonal to $\mathcal{J}^2$, as given in (9).

Introducing

$$
\begin{aligned}
&\tilde\Theta = \Theta + (\Gamma_{45}+\Gamma_{78}+\Gamma_{69})\;, \qquad \tilde\Upsilon_1 = \Upsilon_1 + \Gamma_0\;, \qquad \tilde\Upsilon_2 = \Upsilon_2 + \Gamma_3\;,\\
&\tilde\Phi_1 = \Phi_1 + \Gamma_{120}\;, \qquad \tilde\Phi_2 = \Phi_2 + \Gamma_{123}\;,\\
&\tilde\Phi_3 = \Phi_3 + (\Gamma_{453}+\Gamma_{783}+\Gamma_{693})\;, \qquad \tilde\Phi_4 = \Phi_4 + (\Gamma_{450}+\Gamma_{780}+\Gamma_{690})\;,\\
&\tilde\Psi_1 = \Psi_1 + (\Gamma_{01245}+\Gamma_{01278}+\Gamma_{01269})\;, \quad \tilde\Psi_2 = \Psi_2 + (\Gamma_{31245}+\Gamma_{31278}+\Gamma_{31269})\;,\\
&\tilde\Psi_3 = \Psi_3 + \tfrac{1}{8}(\Gamma_1 - \mathrm{i}\Gamma_2)\Gamma_3(\Gamma_4-\mathrm{i}\Gamma_5)(\Gamma_6-\mathrm{i}\Gamma_9)(\Gamma_7-\mathrm{i}\Gamma_8)\;,
\end{aligned}
$$

we find after straightforward but apparently lengthy calculation

$$
\theta = \epsilon \mathbf{1}_4 \otimes \begin{pmatrix} P_+\theta_{(1)}P_+ & 0 & 0 & P_+\theta_{(2)}P_- \\ 0 & P_+\theta_{(1)}P_+ & P_+\theta_{(2)}P_- & 0 \\ 0 & P_-\theta^\dagger_{(2)}P_+ & P_-B\overline{\theta_{(1)}}BP_- & 0 \\ P_-\theta^\dagger_{(2)}P_+ & 0 & 0 & P_-B\overline{\theta_{(1)}}BP_- \end{pmatrix},
$$

$$
\begin{aligned}
\theta_{(1)} &= \textstyle\sum_i \theta_1^i \otimes (Q^1_i)^\perp + \sum_j \theta^j_{45}\otimes (Q^{45}_j)_\perp + \sum_k \theta^k_{210}\otimes (Q^{210}_k)_\perp \\
&= \tfrac12(3\,\mathbf{1}_{32} - (\mathrm{i}\tilde\Theta)^2_1)\otimes \tilde M_{\{11\}} \\
&\quad + \tfrac12(\mathbf{1}_{32} - (\tilde\Upsilon_1^2)_1)\otimes \tilde M_{\{ss\}} + \tfrac12(\mathbf{1}_{32} - (\tilde\Upsilon_2^2)_1)\otimes \tilde M_{\{pp\}} \\
&\quad + \tfrac12(\mathbf{1}_{32} - (\mathrm{i}\tilde\Phi_1)^2_1)\otimes \tilde M_{\{a'a'\}} + \tfrac12(\mathbf{1}_{32} - (\mathrm{i}\tilde\Phi_2)^2_1)\otimes \tilde M_{\{aa\}} \\
&\quad + \tfrac12(3\,\mathbf{1}_{32} - (\mathrm{i}\tilde\Phi_3)^2_1)\otimes \tilde M_{\{b'b'\}} + \tfrac12(3\,\mathbf{1}_{32} - (\mathrm{i}\tilde\Phi_4)^2_1)\otimes \tilde M_{\{bb\}} \\
&\quad + \tfrac12(3\,\mathbf{1}_{32} - (\tilde\Psi_1\tilde\Psi_1^\dagger)_1)\otimes \tilde M_{\{cc\}} + \tfrac12(3\,\mathbf{1}_{32} - (\tilde\Psi_2\tilde\Psi_2^\dagger)_1)\otimes \tilde M_{\{ff\}} \\
&\quad + \tfrac12(16\,\mathbf{1}_{32} - (\tilde\Psi_3\tilde\Psi_3^\dagger)_1)\otimes \tilde M_{\{22\}} - (\tilde\Upsilon_1\tilde\Upsilon_2)_1\otimes \tilde M_{[sp]} \\
&\quad + (\tilde\Phi_1\tilde\Phi_2)_1 \otimes \tilde M_{[a'a]} + (\tilde\Phi_1\tilde\Phi_3)_1 \otimes \tilde M_{\{a'b'\}} + (\tilde\Phi_1\tilde\Phi_4)_1 \otimes \tilde M_{[a'b]} \\
&\quad - (\tilde\Phi_2\tilde\Phi_3)_1 \otimes \tilde M_{[ab']} + (\tilde\Phi_2\tilde\Phi_4)_1 \otimes \tilde M_{\{ab\}} + (\tilde\Phi_3\tilde\Phi_4)_1 \otimes \tilde M_{[b'b]} \\
&\quad - \tfrac12\mathrm{i}(\tilde\Psi_1\tilde\Psi_2^\dagger - \tilde\Psi_2\tilde\Psi_1^\dagger)_1 \otimes \tilde M_{\{cf\}} - \tfrac12(\tilde\Psi_1\tilde\Psi_2^\dagger + \tilde\Psi_2\tilde\Psi_1^\dagger)_1 \otimes \tilde M_{[cf]} \\
&\quad + \tfrac12\mathrm{i}(\tilde\Psi_1\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Psi_1^\dagger)_1 \otimes \tilde M_{[c2]} - \tfrac12(\tilde\Psi_1\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Psi_1^\dagger)_1 \otimes \tilde M_{\{c2\}} \\
&\quad + \tfrac12\mathrm{i}(\tilde\Psi_2\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Psi_2^\dagger)_1 \otimes \tilde M_{\{f2\}} + \tfrac12(\tilde\Psi_2\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Psi_2^\dagger)_1 \otimes \tilde M_{[f2]} \\
&\quad - \mathrm{i}(\tilde\Upsilon_1\tilde\Upsilon_2)_{45}\otimes M_{\{sp\}} \\
&\quad + \mathrm{i}(\tilde\Upsilon_1\tilde\Phi_1)_{45}\otimes M_{\{sa'\}} + \mathrm{i}(\tilde\Upsilon_1\tilde\Phi_2)_{45}\otimes M_{[sa]} + \mathrm{i}(\tilde\Upsilon_1\tilde\Phi_3 - \tilde\Theta)_{45}\otimes M_{\{sb'\}} \\
&\quad + \mathrm{i}(\tilde\Upsilon_1\tilde\Phi_4)_{45}\otimes M_{[sb]} - \mathrm{i}(\tilde\Upsilon_2\tilde\Phi_1)_{45}\otimes M_{[pa']} + \mathrm{i}(\tilde\Upsilon_2\tilde\Phi_2)_{45}\otimes M_{\{pa\}} \\
&\quad - \mathrm{i}(\tilde\Upsilon_2\tilde\Phi_3)_{45}\otimes M_{[pb']} + \mathrm{i}(\tilde\Upsilon_2\tilde\Phi_4 - \tilde\Theta)_{45}\otimes M_{\{pb\}} \\
&\quad + \mathrm{i}(\tilde\Phi_1\tilde\Phi_2)_{45}\otimes M_{\{a'a\}} - \mathrm{i}(\tilde\Phi_1\tilde\Phi_3)_{45}\otimes M_{[a'b']} + \mathrm{i}(\tilde\Phi_1\tilde\Phi_4)_{45}\otimes M_{\{a'b\}} \\
&\quad - \tfrac12\mathrm{i}(\tilde\Phi_1\tilde\Psi_1^\dagger + \tilde\Psi_1\tilde\Phi_1 + 2\tilde\Theta)_{45}\otimes M_{\{a'c\}} - \tfrac12(\tilde\Phi_1\tilde\Psi_1^\dagger - \tilde\Psi_1\tilde\Phi_1)_{45}\otimes M_{[a'c]} \\
&\quad - \tfrac12\mathrm{i}(\tilde\Phi_1\tilde\Psi_2^\dagger + \tilde\Psi_2\tilde\Phi_1)_{45}\otimes M_{[a'f]} + \tfrac12(\tilde\Phi_1\tilde\Psi_2^\dagger - \tilde\Psi_2\tilde\Phi_1)_{45}\otimes M_{\{a'f\}} \\
&\quad - \tfrac12\mathrm{i}(\tilde\Phi_1\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Phi_1)_{45}\otimes M_{\{a'2\}} - \tfrac12(\tilde\Phi_1\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Phi_1)_{45}\otimes M_{[a'2]}
\end{aligned}
$$

$$
\begin{aligned}
&- \mathrm{i}(\tilde\Phi_2\tilde\Phi_3)_{45} \otimes M_{\{ab'\}} - \mathrm{i}(\tilde\Phi_2\tilde\Phi_4)_{45} \otimes M_{[ab]} + \mathrm{i}(\tilde\Phi_3\tilde\Phi_4)_{45} \otimes M_{\{b'b\}} \\
&+ \tfrac{1}{2}\mathrm{i}(\tilde\Phi_2\tilde\Psi_1^\dagger + \tilde\Psi_1\tilde\Phi_2)_{45} \otimes M_{[ac]} - \tfrac{1}{2}(\tilde\Phi_2\tilde\Psi_1^\dagger - \tilde\Psi_1\tilde\Phi_2)_{45} \otimes M_{\{ac\}} \\
&- \tfrac{1}{2}\mathrm{i}(\tilde\Phi_2\tilde\Psi_2^\dagger + \tilde\Psi_2\tilde\Phi_2 + 2\tilde\Theta)_{45} \otimes M_{\{af\}} - \tfrac{1}{2}(\tilde\Phi_2\tilde\Psi_2^\dagger - \tilde\Psi_2\tilde\Phi_2)_{45} \otimes M_{[af]} \\
&+ \tfrac{1}{2}\mathrm{i}(\tilde\Phi_2\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Phi_2)_{45} \otimes M_{[a2]} - \tfrac{1}{2}(\tilde\Phi_2\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Phi_2)_{45} \otimes M_{\{a2\}} \\
&- \tfrac{1}{2}\mathrm{i}(\tilde\Phi_3\tilde\Psi_1^\dagger + \tilde\Psi_1\tilde\Phi_3)_{45} \otimes M_{\{b'c\}} - \tfrac{1}{2}(\tilde\Phi_3\tilde\Psi_1^\dagger - \tilde\Psi_1\tilde\Phi_3)_{45} \otimes M_{[b'c]} \\
&- \tfrac{1}{2}\mathrm{i}(\tilde\Phi_3\tilde\Psi_2^\dagger + \tilde\Psi_2\tilde\Phi_3)_{45} \otimes M_{[b'f]} + \tfrac{1}{2}(\tilde\Phi_3\tilde\Psi_2^\dagger - \tilde\Psi_2\tilde\Phi_3 - 4\mathrm{i}\tilde\Theta)_{45} \otimes M_{\{b'f\}} \\
&- \tfrac{1}{2}\mathrm{i}(\tilde\Phi_3\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Phi_3)_{45} \otimes M_{\{b'2\}} - \tfrac{1}{2}(\tilde\Phi_3\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Phi_3)_{45} \otimes M_{[b'2]} \\
&+ \tfrac{1}{2}\mathrm{i}(\tilde\Phi_4\tilde\Psi_1^\dagger + \tilde\Psi_1\tilde\Phi_4)_{45} \otimes M_{[bc]} - \tfrac{1}{2}(\tilde\Phi_4\tilde\Psi_1^\dagger - \tilde\Psi_1\tilde\Phi_4 + 4\mathrm{i}\tilde\Theta)_{45} \otimes M_{\{bc\}} \\
&- \tfrac{1}{2}\mathrm{i}(\tilde\Phi_4\tilde\Psi_2^\dagger + \tilde\Psi_2\tilde\Phi_4)_{45} \otimes M_{\{bf\}} - \tfrac{1}{2}(\tilde\Phi_4\tilde\Psi_2^\dagger - \tilde\Psi_2\tilde\Phi_4)_{45} \otimes M_{[bf]} \\
&+ \tfrac{1}{2}\mathrm{i}(\tilde\Phi_4\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Phi_4)_{45} \otimes M_{[b2]} - \tfrac{1}{2}(\tilde\Phi_4\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Phi_4)_{45} \otimes M_{\{b2\}} \\
&- \tfrac{1}{2}(\tilde\Psi_1\tilde\Psi_1^\dagger)_{45} \otimes M_{\{cc\}} - \tfrac{1}{2}(\tilde\Psi_2\tilde\Psi_2^\dagger)_{45} \otimes M_{\{ff\}} - \tfrac{1}{2}((\tilde\Psi_3\tilde\Psi_3^\dagger)_{45} - 16\mathrm{i}\tilde\Theta) \otimes M_{\{22\}} \\
&- \tfrac{1}{2}\mathrm{i}(\tilde\Psi_1\tilde\Psi_2^\dagger - \tilde\Psi_2\tilde\Psi_1^\dagger)_{45} \otimes M_{\{cf\}} - \tfrac{1}{2}(\tilde\Psi_1\tilde\Psi_2^\dagger + \tilde\Psi_2\tilde\Psi_1^\dagger)_{45} \otimes M_{[cf]} \\
&+ \tfrac{1}{2}\mathrm{i}(\tilde\Psi_1\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Psi_1^\dagger)_{45} \otimes M_{[c2]} - \tfrac{1}{2}(\tilde\Psi_1\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Psi_1^\dagger)_{45} \otimes M_{\{c2\}} \\
&+ \tfrac{1}{2}\mathrm{i}(\tilde\Psi_2\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Psi_2^\dagger)_{45} \otimes M_{\{f2\}} + \tfrac{1}{2}(\tilde\Psi_2\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Psi_2^\dagger)_{45} \otimes M_{[f2]} \\[1ex]
&- \tfrac{1}{2}(\mathrm{i}\tilde\Theta)^2_{210} \otimes \hat M_{\{11\}} \\
&+ (\tilde\Upsilon_1\tilde\Phi_1)_{210} \otimes \hat M_{[sa']} - (\tilde\Upsilon_1\tilde\Phi_2)_{210} \otimes \hat M_{\{sa\}} + (\tilde\Upsilon_1\tilde\Phi_3)_{210} \otimes \hat M_{[sb']} \\
&- (\tilde\Upsilon_1\tilde\Phi_4)_{210} \otimes \hat M_{\{sb\}} + (\tilde\Upsilon_2\tilde\Phi_1)_{210} \otimes \hat M_{\{pa'\}} + (\tilde\Upsilon_2\tilde\Phi_2)_{210} \otimes \hat M_{[pa]} \\
&+ (\tilde\Upsilon_2\tilde\Phi_3)_{210} \otimes \hat M_{\{pb'\}} + (\tilde\Upsilon_2\tilde\Phi_4)_{210} \otimes \hat M_{[pb]} \\
&+ \tfrac{1}{2}(\tilde\Upsilon_1\tilde\Psi_1^\dagger + \tilde\Psi_1\tilde\Upsilon_1)_{210} \otimes \hat M_{\{sc\}} - \tfrac{1}{2}\mathrm{i}(\tilde\Upsilon_1\tilde\Psi_1^\dagger - \tilde\Psi_1\tilde\Upsilon_1)_{210} \otimes \hat M_{[sc]} \\
&+ \tfrac{1}{2}(\tilde\Upsilon_1\tilde\Psi_2^\dagger + \tilde\Psi_2\tilde\Upsilon_1)_{210} \otimes \hat M_{[sf]} - \tfrac{1}{2}\mathrm{i}(\tilde\Upsilon_1\tilde\Psi_2^\dagger - \tilde\Psi_2\tilde\Upsilon_1)_{210} \otimes \hat M_{\{sf\}} \\
&+ \tfrac{1}{2}(\tilde\Upsilon_1\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Upsilon_1)_{210} \otimes \hat M_{\{s2\}} - \tfrac{1}{2}\mathrm{i}(\tilde\Upsilon_1\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Upsilon_1)_{210} \otimes \hat M_{[s2]} \\
&- \tfrac{1}{2}(\tilde\Upsilon_2\tilde\Psi_1^\dagger + \tilde\Psi_1\tilde\Upsilon_2)_{210} \otimes \hat M_{[pc]} - \tfrac{1}{2}\mathrm{i}(\tilde\Upsilon_2\tilde\Psi_1^\dagger - \tilde\Psi_1\tilde\Upsilon_2)_{210} \otimes \hat M_{\{pc\}} \\
&+ \tfrac{1}{2}(\tilde\Upsilon_2\tilde\Psi_2^\dagger + \tilde\Psi_2\tilde\Upsilon_2)_{210} \otimes \hat M_{\{pf\}} - \tfrac{1}{2}\mathrm{i}(\tilde\Upsilon_2\tilde\Psi_2^\dagger - \tilde\Psi_2\tilde\Upsilon_2)_{210} \otimes \hat M_{[pf]} \\
&- \tfrac{1}{2}(\tilde\Upsilon_2\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Upsilon_2)_{210} \otimes \hat M_{[p2]} - \tfrac{1}{2}\mathrm{i}(\tilde\Upsilon_2\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Upsilon_2)_{210} \otimes \hat M_{\{p2\}} \\
&+ \tfrac{1}{2}(\tilde\Phi_1^2)_{210} \otimes \hat M_{\{a'a'\}} + \tfrac{1}{2}(\tilde\Phi_2^2)_{210} \otimes \hat M_{\{aa\}} \\
&+ \tfrac{1}{2}(\tilde\Phi_3^2)_{210} \otimes \hat M_{\{b'b'\}} + \tfrac{1}{2}(\tilde\Phi_4^2)_{210} \otimes \hat M_{\{bb\}} \\
&+ (\tilde\Phi_1\tilde\Phi_2)_{210} \otimes \hat M_{[a'a]} + (\mathrm{i}\tilde\Phi_1\tilde\Phi_2\Gamma_{11})_{210} \otimes \hat M_{\{a'a\}} + (\tilde\Phi_1\tilde\Phi_3)_{210} \otimes \hat M_{\{a'b'\}} \\
&- (\mathrm{i}\tilde\Phi_1\tilde\Phi_3\Gamma_{11})_{210} \otimes \hat M_{[a'b']} + (\tilde\Phi_1\tilde\Phi_4)_{210} \otimes \hat M_{[a'b]} + (\mathrm{i}\tilde\Phi_1\tilde\Phi_4\Gamma_{11})_{210} \otimes \hat M_{\{a'b\}} \\
&- (\tilde\Phi_2\tilde\Phi_3)_{210} \otimes \hat M_{[ab']} - (\mathrm{i}\tilde\Phi_2\tilde\Phi_3\Gamma_{11})_{210} \otimes \hat M_{\{ab'\}} + (\tilde\Phi_2\tilde\Phi_4)_{210} \otimes \hat M_{\{ab\}} \\
&- (\mathrm{i}\tilde\Phi_2\tilde\Phi_4\Gamma_{11})_{210} \otimes \hat M_{[ab]} + (\tilde\Phi_3\tilde\Phi_4)_{210} \otimes \hat M_{[b'b]} + (\mathrm{i}\tilde\Phi_3\tilde\Phi_4\Gamma_{11})_{210} \otimes \hat M_{\{b'b\}} \\
&- \tfrac{1}{2}(\tilde\Phi_1\tilde\Psi_1^\dagger - \tilde\Psi_1\tilde\Phi_1)_{210} \otimes \hat M_{[a'c]} - \tfrac{1}{2}\mathrm{i}(\tilde\Phi_1\tilde\Psi_1^\dagger + \tilde\Psi_1\tilde\Phi_1)_{210} \otimes \hat M_{\{a'c\}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \tfrac{1}{2}(\tilde\Phi_1\tilde\Psi_2^\dagger - \tilde\Psi_2\tilde\Phi_1)_{210}\otimes\hat M_{\{a'f\}} - \tfrac{1}{2}\mathrm{i}(\tilde\Phi_1\tilde\Psi_2^\dagger + \tilde\Psi_2\tilde\Phi_1)_{210}\otimes\hat M_{[a'f]}\\
&- \tfrac{1}{2}(\tilde\Phi_1\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Phi_1)_{210}\otimes\hat M_{[a'2]} - \tfrac{1}{2}\mathrm{i}(\tilde\Phi_1\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Phi_1)_{210}\otimes\hat M_{\{a'2\}}\\
&- \tfrac{1}{2}(\tilde\Phi_2\tilde\Psi_1^\dagger - \tilde\Psi_1\tilde\Phi_2)_{210}\otimes\hat M_{\{ac\}} + \tfrac{1}{2}\mathrm{i}(\tilde\Phi_2\tilde\Psi_1^\dagger + \tilde\Psi_1\tilde\Phi_2)_{210}\otimes\hat M_{[ac]}\\
&- \tfrac{1}{2}(\tilde\Phi_2\tilde\Psi_2^\dagger - \tilde\Psi_2\tilde\Phi_2)_{210}\otimes\hat M_{[af]} - \tfrac{1}{2}\mathrm{i}(\tilde\Phi_2\tilde\Psi_2^\dagger + \tilde\Psi_2\tilde\Phi_2)_{210}\otimes\hat M_{\{af\}}\\
&- \tfrac{1}{2}(\tilde\Phi_2\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Phi_2)_{210}\otimes\hat M_{\{a2\}} + \tfrac{1}{2}\mathrm{i}(\tilde\Phi_2\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Phi_2)_{210}\otimes\hat M_{[a2]}\\
&- \tfrac{1}{2}(\tilde\Phi_3\tilde\Psi_1^\dagger - \tilde\Psi_1\tilde\Phi_3)_{210}\otimes\hat M_{[b'c]} - \tfrac{1}{2}\mathrm{i}(\tilde\Phi_3\tilde\Psi_1^\dagger + \tilde\Psi_1\tilde\Phi_3)_{210}\otimes\hat M_{\{b'c\}}\\
&+ \tfrac{1}{2}(\tilde\Phi_3\tilde\Psi_2^\dagger - \tilde\Psi_2\tilde\Phi_3)_{210}\otimes\hat M_{\{b'f\}} - \tfrac{1}{2}\mathrm{i}(\tilde\Phi_3\tilde\Psi_2^\dagger + \tilde\Psi_2\tilde\Phi_3)_{210}\otimes\hat M_{[b'f]}\\
&- \tfrac{1}{2}(\tilde\Phi_3\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Phi_3)_{210}\otimes\hat M_{[b'2]} - \tfrac{1}{2}\mathrm{i}(\tilde\Phi_3\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Phi_3)_{210}\otimes\hat M_{\{b'2\}}\\
&- \tfrac{1}{2}(\tilde\Phi_4\tilde\Psi_1^\dagger - \tilde\Psi_1\tilde\Phi_4)_{210}\otimes\hat M_{\{bc\}} + \tfrac{1}{2}\mathrm{i}(\tilde\Phi_4\tilde\Psi_1^\dagger + \tilde\Psi_1\tilde\Phi_4)_{210}\otimes\hat M_{[bc]}\\
&- \tfrac{1}{2}(\tilde\Phi_4\tilde\Psi_2^\dagger - \tilde\Psi_2\tilde\Phi_4)_{210}\otimes\hat M_{[bf]} - \tfrac{1}{2}\mathrm{i}(\tilde\Phi_4\tilde\Psi_2^\dagger + \tilde\Psi_2\tilde\Phi_4)_{210}\otimes\hat M_{\{bf\}}\\
&- \tfrac{1}{2}(\tilde\Phi_4\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Phi_4)_{210}\otimes\hat M_{\{b2\}} + \tfrac{1}{2}\mathrm{i}(\tilde\Phi_4\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Phi_4)_{210}\otimes\hat M_{[b2]}\\
&- (\tilde\Psi_1\tilde\Psi_1^\dagger)_{210}\otimes\hat M_{\{cc\}} - (\tilde\Psi_2\tilde\Psi_2^\dagger)_{210}\otimes\hat M_{\{ff\}} - (\tilde\Psi_3\tilde\Psi_3^\dagger)_{210}\otimes\hat M_{\{22\}}\\
&- \tfrac{1}{2}(\tilde\Psi_1\tilde\Psi_2^\dagger + \tilde\Psi_2\tilde\Psi_1^\dagger)_{210}\otimes\hat M_{[cf]} - \tfrac{1}{2}\mathrm{i}(\tilde\Psi_1\tilde\Psi_2^\dagger - \tilde\Psi_2\tilde\Psi_1^\dagger)_{210}\otimes\hat M_{\{cf\}}\\
&- \tfrac{1}{2}(\tilde\Psi_1\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Psi_1^\dagger)_{210}\otimes\hat M_{\{c2\}} + \tfrac{1}{2}\mathrm{i}(\tilde\Psi_1\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Psi_1^\dagger)_{210}\otimes\hat M_{[c2]}\\
&+ \tfrac{1}{2}(\tilde\Psi_2\tilde\Psi_3^\dagger + \tilde\Psi_3\tilde\Psi_2^\dagger)_{210}\otimes\hat M_{[f2]} + \tfrac{1}{2}\mathrm{i}(\tilde\Psi_2\tilde\Psi_3^\dagger - \tilde\Psi_3\tilde\Psi_2^\dagger)_{210}\otimes\hat M_{\{f2\}}\ ,
\end{aligned}
$$

$$
\begin{aligned}
\theta_{(2)} &= \textstyle\sum_i \theta^i_{10}\otimes Q^{10}_i + \sum_j \theta^j_{120}\otimes Q^{120}_j + \sum_k \theta^k_{126}\otimes Q^{126}_k\\
&= -(\tilde\Theta\tilde\Upsilon_1)_{10}\otimes M_{\{1s\}} + \mathrm{i}(\tilde\Theta\tilde\Upsilon_2)_{10}\otimes M_{\{1p\}} + (\tilde\Theta\tilde\Phi_1)_{10}\otimes M_{[1a']}\\
&\quad - \mathrm{i}(\tilde\Theta\tilde\Phi_2)_{10}\otimes M_{[1a]} + ((\tilde\Theta\tilde\Phi_3)_{10} + 3\tilde\Upsilon_1)\otimes M_{[1b']} - \mathrm{i}((\tilde\Theta\tilde\Phi_4)_{10} + 3\tilde\Upsilon_2)\otimes M_{[1b]}\\
&\quad + ((\mathrm{i}\tilde\Theta\tilde\Upsilon_1)_{120} - \mathrm{i}\tilde\Phi_3)\otimes M_{[1s]} - \mathrm{i}((\mathrm{i}\tilde\Theta\tilde\Upsilon_2)_{120} - \mathrm{i}\tilde\Phi_4)\otimes M_{[1p]} + (\mathrm{i}\tilde\Theta\tilde\Phi_1)_{120}\otimes M_{\{1a'\}}\\
&\quad - \mathrm{i}(\mathrm{i}\tilde\Theta\tilde\Phi_2)_{120}\otimes M_{\{1a\}} + (\mathrm{i}\tilde\Theta\tilde\Phi_3)_{120}\otimes M_{\{1b'\}} - \mathrm{i}(\mathrm{i}\tilde\Theta\tilde\Phi_4)_{120}\otimes M_{\{1b\}}\\
&\quad - ((\mathrm{i}\tilde\Theta\tilde\Psi_1)_{120} + 3\mathrm{i}\tilde\Phi_1 + 2\tilde\Phi_4)\otimes M_{[1c]} + \mathrm{i}((\mathrm{i}\tilde\Theta\tilde\Psi_2)_{120} + 3\mathrm{i}\tilde\Phi_2 - 2\tilde\Phi_3)\otimes M_{[1f]}\\
&\quad - (\mathrm{i}\tilde\Theta\tilde\Psi_3)_{120}\otimes M_{[12]}\\
&\quad + ((\tilde\Theta\tilde\Phi_1)_{126} - \tilde\Psi_1)\otimes M_{[1a']} - \mathrm{i}((\tilde\Theta\tilde\Phi_2)_{126} - \tilde\Psi_2)\otimes M_{[1a]}\\
&\quad + ((\tilde\Theta\tilde\Phi_3)_{126} + 2\mathrm{i}\tilde\Psi_2)\otimes M_{[1b']} - \mathrm{i}((\tilde\Theta\tilde\Phi_4)_{126} - 2\mathrm{i}\tilde\Psi_1)\otimes M_{[1b]}\\
&\quad + (\tilde\Theta\tilde\Psi_1)_{126}\otimes M_{\{1c\}} - \mathrm{i}(\tilde\Theta\tilde\Psi_2)_{126}\otimes M_{\{1f\}} + ((\tilde\Theta\tilde\Psi_3)_{126} + 3\mathrm{i}\tilde\Psi_3)\otimes M_{\{12\}}\ ,
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde M_{\{\alpha\beta\}} &= (M_\alpha M_\beta^\dagger + M_\beta M_\alpha^\dagger)^{(\perp)}\ , & \tilde M_{[\alpha\beta]} &= (\mathrm{i}M_\alpha M_\beta^\dagger - \mathrm{i}M_\beta M_\alpha^\dagger)^{(\perp)}\ ,\\
M_{\{\alpha\beta\}} &= (M_\alpha M_\beta^\dagger + M_\beta M_\alpha^\dagger)_\perp\ , & M_{[\alpha\beta]} &= (\mathrm{i}M_\alpha M_\beta^\dagger - \mathrm{i}M_\beta M_\alpha^\dagger)_\perp\ , \quad (\alpha{=}1\Rightarrow\beta{=}1)\\
\hat M_{\{\alpha\beta\}} &= (M_\alpha M_\beta^\dagger + M_\beta M_\alpha^\dagger)^\perp\ , & \hat M_{[\alpha\beta]} &= (\mathrm{i}M_\alpha M_\beta^\dagger - \mathrm{i}M_\beta M_\alpha^\dagger)^\perp\ ,\\
M_{\{1\alpha\}} &= (M_1 M_\alpha + M_\alpha \overline{M}_1)\ , & M_{[1\alpha]} &= (\mathrm{i}M_1 M_\alpha - \mathrm{i}M_\alpha \overline{M}_1)\ , \quad (\alpha \neq 1)
\end{aligned}
$$

$$
\begin{aligned}
&(Q_i^1)^{(\perp)} = Q_i - \tfrac{1}{3}\mathrm{tr}(Q_i)\mathbf{1}_3 \;, \\
&(Q_j^{45})_\perp = Q_j^{45} - \textstyle\sum_{a,b=1}^{2} \mathrm{tr}(Q_j^{45}\mathcal{Z}_a)\mathcal{T}_{ab}\mathcal{Z}_b \;, \qquad \textstyle\sum_{a=1}^{2} \mathcal{T}_{a'a}\mathrm{tr}(\mathcal{Z}_a\mathcal{Z}_b) = \delta_{a'b} \;, \\
&(Q_j^{210})^\perp = Q_j^{210} - \textstyle\sum_{a,b=3}^{9} \mathrm{tr}(Q_j^{210}\mathcal{Z}_a)\mathcal{T}_{ab}\mathcal{Z}_b \;, \qquad \textstyle\sum_{a=3}^{9} \mathcal{T}_{a'a}\mathrm{tr}(\mathcal{Z}_a\mathcal{Z}_b) = \delta_{a'b} \;,
\end{aligned}
$$

with $\mathcal{Z}_9 = \mathbf{1}_3$. We have the following 8 constraints due to the $\mathcal{Z}_i$ of our ideal:

$$
\begin{aligned}
0 &= M_{\{sp\}} + M_{\{a'a\}} + 3M_{\{b'b\}} + 3M_{\{cf\}} + 8M_{\{22\}} \;, \\
0 &= M_{\{sa'\}} + M_{\{pa\}} + 3M_{\{b'c\}} + 3M_{\{bf\}} - 8M_{\{22\}} \;, \\
0 &= \hat{M}_{\{11\}} + 8\hat{M}_{\{22\}} + \hat{M}_{\{sf\}} + \hat{M}_{\{pc\}} + \hat{M}_{\{a'b\}} + \hat{M}_{\{ab'\}} \\
&\quad + \hat{M}_{\{b'b'\}} + \hat{M}_{\{bb\}} + \hat{M}_{\{cc\}} + \hat{M}_{\{ff\}} \;, \\
0 &= \hat{M}_{\{sb\}} + \hat{M}_{\{pb'\}} + \hat{M}_{\{a'f\}} + \hat{M}_{\{ac\}} + 2\hat{M}_{\{b'c\}} + 2\hat{M}_{\{bf\}} - 8\hat{M}_{\{22\}} \;, \qquad (17) \\
0 &= \hat{M}_{\{sc\}} + \hat{M}_{\{pf\}} + \hat{M}_{\{a'b'\}} + \hat{M}_{\{ab\}} + 2\hat{M}_{\{b'b\}} + 2\hat{M}_{\{cf\}} + 8\hat{M}_{\{22\}} \;, \\
0 &= \hat{M}_{\{sa\}} + \hat{M}_{\{pa'\}} + 3\hat{M}_{\{b'f\}} + 3\hat{M}_{\{bc\}} - 8\hat{M}_{\{22\}} \;, \\
0 &= \hat{M}_{[s2]} + \hat{M}_{[p2]} + \hat{M}_{[a'2]} + \hat{M}_{[a2]} + 3\hat{M}_{[b'2]} + 3\hat{M}_{[b2]} + 3\hat{M}_{[c2]} + 3\hat{M}_{[f2]} \;, \\
0 &= \hat{M}_{\{s2\}} + \hat{M}_{\{p2\}} + \hat{M}_{\{a'2\}} + \hat{M}_{\{a2\}} + 3\hat{M}_{\{b'2\}} + 3\hat{M}_{\{b2\}} + 3\hat{M}_{\{c2\}} + 3\hat{M}_{\{f2\}} \;.
\end{aligned}
$$

According to the general theory we have to investigate whether or not the connection form ρ receives an extra contribution $\rho' = \Lambda^1 \otimes \mathbf{r}^0 + \Lambda^0\gamma^5 \otimes \mathbf{r}^1$. Due to the symmetries of our setting and the requirement that the $\mathbf{r}^i$ commute with $\hat{\pi}(\mathrm{so}(10))$, the matrices $\mathbf{r}^i$ have the general form

$$
\mathbf{r}^0 = \begin{pmatrix} \mathrm{i}P_+(\mathbf{1}_{32}\otimes M_0)P_+ & 0 & 0 & 0 \\ 0 & \mathrm{i}P_+(\mathbf{1}_{32}\otimes M_0)P_+ & 0 & 0 \\ 0 & 0 & -\mathrm{i}P_-(\mathbf{1}_{32}\otimes \overline{M_0})P_- & 0 \\ 0 & 0 & 0 & -\mathrm{i}P_-(\mathbf{1}_{32}\otimes \overline{M_0})P_- \end{pmatrix},
$$
$$
\mathbf{r}^1 = \begin{pmatrix} 0 & P_+(\mathbf{1}_{32}\otimes M_0')P_+ & 0 & 0 \\ \epsilon P_+(\mathbf{1}_{32}\otimes M_0')P_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon P_-(\mathbf{1}_{32}\otimes \overline{M_0'})P_- \\ 0 & 0 & P_-(\mathbf{1}_{32}\otimes \overline{M_0'})P_- & 0 \end{pmatrix},
$$

with $M_0 = M_0^\dagger$ and $M_0' = M_0'^\dagger$. The condition $[\mathbf{r}^0, \hat{\pi}(\eta)] \in \hat{\pi}(\Omega^1(\mathfrak{a}))$ yields from the **45**-sector $\mathrm{i}[M_0, M_1] \in \mathbb{R}M_1$, which fixes M_0 up to three parameters. The **10**-sector yields $\mathrm{i}(M_0M_s + M_s\overline{M_0}) \in \mathbb{R}M_s + \mathrm{i}\mathbb{R}M_p$ and $\mathrm{i}(M_0M_p + M_p\overline{M_0}) \in \mathbb{R}M_p + \mathrm{i}\mathbb{R}M_s$. The r.h.s. are two-dimensional so that both $\mathrm{i}(M_0M_s + M_s\overline{M_0})$ and $\mathrm{i}(M_0M_p + M_p\overline{M_0})$ are orthogonal to 7-dimensional spaces. It is clear that there exists no solution for M_0 in general.

The condition $\{\mathbf{r}^1, \hat{\pi}(\eta)\} \in \hat{\pi}(\Omega^2(\mathfrak{a})) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}$ derived from $\{\rho', \pi(\omega^1)\} \in \pi(\Omega^2)$ gives from the **45**-sector no condition at all, because $-\mathrm{i}\Theta\otimes(M_0'M_1 + M_1M_0')$ is contained in $\theta_{(1)}$ for any M_0'. From the **10**-sector we get 2 times $(9-6)$

conditions, from the **120**-sector 4 times $(9-9)$ conditions and from the **10**-sector 3 times $(9-7)$ conditions. This means that there will not exist a solution for M_0' in general. There are no extra contributions to the gauge potential possible.

In the same way one shows that the graded centralizer $\mathcal{C}^2$ is trivial, $\mathcal{C}^2 = C^\infty(M)\mathcal{P} \subset \{\pi(\mathfrak{g}), \pi(\mathfrak{g})\}$. There is also no extra contribution to the ideal $\mathcal{J}^2$.

5 The bosonic action

The bosonic action (7) is now given by

$$S_B = \tfrac{\epsilon}{192g^2} \int_M \mathrm{tr}(\mathcal{F}^2)\, dx = \int_M (\mathcal{L}_2 + \mathcal{L}_1 + V)\, dx \;,$$

where g is the so(10)-coupling constant. In this formula, $V = \frac{\epsilon}{192g^2}\mathrm{tr}(\theta^2)$ is the Higgs potential, which in more detail is given by

$$\begin{aligned} V = \tfrac{\epsilon}{48g^2}\Big(& \sum_{i,i'} \mathrm{tr}(P_+\theta_1^i\theta_1^{i'})\,\mathrm{tr}((Q_i^1)^{(\perp)}(Q_{i'}^1)^{(\perp)}) + \sum_{j,j'} \mathrm{tr}(P_+\theta_{45}^j\theta_{45}^{j'})\,\mathrm{tr}((Q_j^{45})_\perp(Q_{j'}^{45})_\perp) \\ & + \sum_{k,k'} \mathrm{tr}(P_+\theta_{210}^k\theta_{210}^{k'})\,\mathrm{tr}((Q_k^{210})^\perp(Q_{k'}^{210})^\perp) + \sum_{k,k'} \mathrm{tr}(P_+\theta_{126}^k(\theta_{126}^{k'})^\dagger)\,\mathrm{tr}(Q_k^{126}(Q_{k'}^{126})^\dagger) \\ & + \sum_{i,i'} \mathrm{tr}(P_+\theta_{10}^i(\theta_{10}^{i'})^\dagger)\,\mathrm{tr}(Q_i^{10}(Q_{i'}^{10})^\dagger) + \sum_{j,j'} \mathrm{tr}(P_+\theta_{120}^j(\theta_{120}^{j'})^\dagger)\,\mathrm{tr}(Q_j^{120}(Q_{j'}^{120})^\dagger)\Big) \;. \end{aligned}$$

The other parts of the Lagrangian are

$$\mathcal{L}_2 = \tfrac{\epsilon}{192g^2}\mathrm{tr}((\mathbf{d}A + A^2|_{\Lambda^2})^2) \;, \qquad \mathcal{L}_1 = \tfrac{\epsilon}{192g^2}\mathrm{tr}((\mathbf{d}\pi(\eta) + \{A, \pi(\eta) - \mathrm{i}Y\})^2) \;.$$

As there are 23 different **1**-terms, $48-2=46$ different **45**-terms, $70-6=64$ different **210**-terms, 6 different **10**-terms, 9 different **120**-terms and 7 different **10**-terms in θ, there occur $\frac{1}{2}(23\cdot 24 + 46\cdot 47 + 64\cdot 65 + 6\cdot 7 + 9\cdot 10 + 7\cdot 8) = 3531$ different gauge invariant terms in the Higgs potential. All of them are compatible with the configuration $\mathcal{M}$ and $\mathcal{N}$ specified by the Yukawa operator Y as the vacuum. This means that the most general gauge invariant Higgs potential that leads to the desired spontaneous symmetry breaking depends on 3531 parameters. Of course, not all these terms are really necessary, and in the classical formulation one puts most of the coefficients equal to zero. But there is no justification for doing so, the Higgs potential in the classical formulation does contain 3531 parameters. Our theory reduces this huge number to 9, namely the parameters of the unknown matrix M_1. All other matrices occur in the fermionic action and can be measured, in principle. Thus, although our SO(10)-model has more independent parameters than the standard model, the ratio of the fixed parameters to the classical parameters is much better than in corresponding treatments of the standard model.

We now study the Yang-Mills part $\mathcal{L}_2 - \frac{\epsilon}{192g^2}\mathrm{tr}(\{A,Y\}^2)$ of the Lagrangian. For that purpose we introduce chiral Γ-matrices

$$\begin{aligned}
&\Gamma_1^{\pm} = \tfrac{1}{\sqrt{2}}(\Gamma_4 \pm \mathrm{i}\Gamma_5)\,, \quad \Gamma_2^{\pm} = \tfrac{1}{\sqrt{2}}(\Gamma_6 \pm \mathrm{i}\Gamma_9)\,, \quad \Gamma_3^{\pm} = \tfrac{1}{\sqrt{2}}(\Gamma_7 \pm \mathrm{i}\Gamma_8)\,, \\
&\Gamma_4^{\pm} = \tfrac{1}{\sqrt{2}}(\Gamma_0 \pm \mathrm{i}\Gamma_3)\,, \quad \Gamma_5^{\pm} = \tfrac{1}{\sqrt{2}}(\Gamma_1 \pm \mathrm{i}\Gamma_2)\,, \\
&\{\Gamma_i^p,\Gamma_j^q\} = 2\delta_{ij}^{p\bar{q}}\mathbf{1}_{32}\,, \qquad \delta_{ij}^{p\bar{q}} = \delta_{ij}\delta^{p\bar{q}} \qquad \Gamma_{ij}^{pq} = \tfrac{1}{2}[\Gamma_i^p,\Gamma_j^q] = -(\Gamma_{ij}^{\bar{p}\bar{q}})^\dagger\,,
\end{aligned}$$

where $p,q \in \{+,-\}$ and $i,j = 1,\dots,5$. The bar in $\bar{q}$ changes the sign. The Yang-Mills field A can now be decomposed as

$$\begin{aligned}
A &= \mathcal{P}(\tfrac{1}{4\sqrt{2}}gA_{pq,\mu}^{ij}\gamma^\mu \otimes \Gamma_{ij}^{pq} \otimes \mathbf{1}_3 \otimes \mathbf{1}_4) = \mathcal{P}(\gamma^\mu \otimes \mathbb{A}_\mu \otimes \mathbf{1}_3 \otimes \mathbf{1}_4)\,, \qquad (18)\\
\mathbb{A}_\mu &= \tfrac{1}{4\sqrt{2}}A_{pq,\mu}^{ij}\Gamma_{ij}^{pq}\,, \qquad A_{pq,\mu}^{ij} = -A_{qp,\mu}^{ji} = \overline{A_{\bar{p}\bar{q},\mu}^{ij}} \in C^\infty(M)\,,
\end{aligned}$$

where $\mathbf{1}_3$ acts on the generation space and $\mathbf{1}_4$ is the 4×4-matrix structure in (12). Defining $\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ we have

$$\begin{aligned}
\mathbf{d}A &= \mathcal{P}(\tfrac{1}{8\sqrt{2}}g(\partial_\mu A_{pq,\nu}^{ij} - \partial_\nu A_{pq,\mu}^{ij})\gamma^{\mu\nu} \otimes \Gamma_{ij}^{pq} \otimes \mathbf{1}_3 \otimes \mathbf{1}_4)\,,\\
A^2 &= \mathcal{P}(\tfrac{1}{32}g^2 A_{pq,\mu}^{ij}A_{rs,\nu}^{kl}\gamma^\mu\gamma^\nu \otimes \Gamma_{ij}^{pq}\Gamma_{kl}^{rs} \otimes \mathbf{1}_3 \otimes \mathbf{1}_4)\\
&= \mathcal{P}(\tfrac{1}{128}g^2 A_{pq,\mu}^{ij}A_{pq,\nu}^{kl}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) \otimes (\Gamma_{ij}^{pq}\Gamma_{kl}^{rs} + \Gamma_{kl}^{rs}\Gamma_{ij}^{pq}) \otimes \mathbf{1}_3 \otimes \mathbf{1}_4)\\
&+ \mathcal{P}(\tfrac{1}{128}g^2 A_{pq,\mu}^{ij}A_{pq,\nu}^{kl}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \otimes (\Gamma_{ij}^{pq}\Gamma_{kl}^{rs} - \Gamma_{kl}^{rs}\Gamma_{ij}^{pq}) \otimes \mathbf{1}_3 \otimes \mathbf{1}_4)\,.
\end{aligned}$$

This gives with

$$\begin{aligned}
(\Gamma_{ij}^{pq}\Gamma_{kl}^{rs} - \Gamma_{kl}^{rs}\Gamma_{ij}^{pq}) &= 2\,f_{tu,ijkl}^{mn,pqrs}\,\Gamma_{mn}^{tu}\,,\\
f_{tu,ijkl}^{mn,pqrs} &= \delta_{jk}^{q\bar{r}}\delta_{t,i}^{m,p}\delta_{u,l}^{n,s} - \delta_{ik}^{p\bar{r}}\delta_{t,j}^{m,q}\delta_{u,l}^{n,s} - \delta_{jl}^{q\bar{s}}\delta_{t,i}^{m,p}\delta_{u,k}^{n,r} + \delta_{il}^{p\bar{s}}\delta_{t,j}^{m,q}\delta_{u,k}^{n,r}\,,
\end{aligned}$$

where $\delta_{p,j}^{i,q} = \delta_j^i\delta_p^q$, the final results

$$\begin{aligned}
A^2|_{\Lambda^2} &= \mathcal{P}(\tfrac{1}{32}g^2 f_{tu,ijkl}^{mn,pqrs}A_{pq,\mu}^{ij}A_{rs,\nu}^{kl}\gamma^{\mu\nu} \otimes \Gamma_{mn}^{tu} \otimes \mathbf{1}_3 \otimes \mathbf{1}_4)\,,\\
\mathbf{d}A + A^2|_{\Lambda^2} &= \mathcal{P}(\tfrac{1}{8\sqrt{2}}gF_{tu,\mu\nu}^{mn}\gamma^{\mu\nu} \otimes \Gamma_{mn}^{tu} \otimes \mathbf{1}_3 \otimes \mathbf{1}_4)\,,\\
F_{tu,\mu\nu}^{mn} &= \partial_\mu A_{tu,\nu}^{mn} - \partial_\nu A_{tu,\mu}^{mn} + \tfrac{1}{2\sqrt{2}}g f_{tu,ijkl}^{mn,pqrs}A_{pq,\mu}^{ij}A_{rs,\nu}^{kl}\,.
\end{aligned}$$

Using $\mathrm{tr}(\gamma^{\mu\nu}\gamma^{\kappa\lambda}){=}4(g^{\mu\lambda}g^{\nu\kappa}{-}g^{\mu\kappa}g^{\nu\lambda})$ and $\mathrm{tr}(\Gamma_{ij}^{pq}\Gamma_{kl}^{rs}){=}32(\delta_{il}^{p\bar{s}}\delta_{jk}^{q\bar{r}}{-}\delta_{ik}^{p\bar{r}}\delta_{jl}^{q\bar{s}})$, we arrive at

$$\mathcal{L}_2 = \tfrac{\epsilon}{192g^2}\cdot\tfrac{1}{128}g^2 F_{pq,\mu\nu}^{ij}F_{rs,\kappa\lambda}^{kl}\cdot\mathrm{tr}(\gamma^{\mu\nu}\gamma^{\kappa\lambda})\cdot\mathrm{tr}((P_+{+}P_-)\Gamma_{pq}^{ij}\Gamma_{rs}^{kl})\cdot 6 = \tfrac{\epsilon}{8}F_{pq,\mu\nu}^{ij}F_{ij}^{\bar{p}\bar{q},\mu\nu}.$$

It is now convenient to identify the $\mathrm{su}(3)\oplus\mathrm{su}(2)_L\oplus\mathrm{su}(2)_R\oplus\mathrm{u}(1)_{B-L}$ gauge fields $(G^k, V^i, \tilde{V}^i, G^0)$, where from now on $i,j=1,2,3$ and $k=1,\dots,8$:

$$A^{12}_{+-,\mu} = -\tfrac{1}{\sqrt{2}}\mathrm{i}(G^6_\mu - \mathrm{i}G^7_\mu)\,, \quad A^{23}_{+-,\mu} = -\tfrac{1}{\sqrt{2}}(G^4_\mu + \mathrm{i}G^5_\mu)\,, \quad A^{31}_{+-,\mu} = -\tfrac{1}{\sqrt{2}}(G^1_\mu - \mathrm{i}G^2_\mu)\,,$$
$$\tfrac{1}{\sqrt{2}}(A^{33}_{+-,\mu} - A^{11}_{+-,\mu}) = \mathrm{i}G^3_\mu\,, \qquad \tfrac{1}{\sqrt{6}}(A^{33}_{+-,\mu} + A^{11}_{+-,\mu} - 2A^{22}_{+-,\mu}) = \mathrm{i}G^8_\mu\,,$$
$$\tfrac{1}{\sqrt{3}}(A^{11}_{+-,\mu} + A^{22}_{+-,\mu} + A^{33}_{+-,\mu}) = \mathrm{i}G^0_\mu\,,$$
$$A^{45}_{-+,\mu} = -\tfrac{1}{\sqrt{2}}(V^1_\mu - \mathrm{i}V^2_\mu) =: -V^+_\mu = -(V^-_\mu)^\dagger\,, \qquad \tfrac{1}{\sqrt{2}}(A^{44}_{+-,\mu} - A^{55}_{+-,\mu}) = -\mathrm{i}V^3_\mu\,,$$
$$A^{45}_{++,\mu} = \tfrac{1}{\sqrt{2}}(\tilde{V}^1_\mu - \mathrm{i}\tilde{V}^2_\mu) =: \tilde{V}^+_\mu = (\tilde{V}^-_\mu)^\dagger\,, \qquad \tfrac{1}{\sqrt{2}}(A^{44}_{+-,\mu} + A^{55}_{+-,\mu}) = \mathrm{i}\tilde{V}^3_\mu\,.$$

in this notation, the Yang-Mills Lagrangian $\mathcal{L}_2$ takes the form

$$\begin{aligned}\mathcal{L}_2 = {} & \tfrac{\epsilon}{4}\big(G^k_{\mu\nu}G^{\mu\nu}_k + \partial_{[\mu}G^0_{\nu]}\partial^{[\mu}G^{\nu]}_0 + V^i_{\mu\nu}V^{\mu\nu}_i + \tilde{V}^i_{\mu\nu}\tilde{V}^{\mu\nu}_i\big) \\ & + \tfrac{\epsilon}{2}\big(\ \partial_{[\mu}A^{ij}_{++,\nu]}\partial^{[\mu}A^{--,\nu]}_{ij} + \partial_{[\mu}A^{i4}_{++,\nu]}\partial^{[\mu}A^{--,\nu]}_{i4} + \partial_{[\mu}A^{i4}_{-+,\nu]}\partial^{[\mu}A^{+-,\nu]}_{i4} \\ & \qquad + \partial_{[\mu}A^{i5}_{++,\nu]}\partial^{[\mu}A^{--,\nu]}_{i5} + \partial_{[\mu}A^{i5}_{-+,\nu]}\partial^{[\mu}A^{+-,\nu]}_{i5}\big) + I.T\ ,\end{aligned} \tag{19}$$

where $I.T$ stands for interaction terms between 3 or 4 Yang-Mills fields and

$$G^k_{\mu\nu} = \partial_{[\mu}G^k_{\nu]} - g f^k_{k'k''}G^{k'}_\mu G^{k''}_\nu\ ,$$
$$V^i_{\mu\nu} = \partial_{[\mu}V^i_{\nu]} - g\epsilon^i_{jj'}V^j_\mu V^{j'}_\nu\ , \qquad \tilde{V}^i_{\mu\nu} = \partial_{[\mu}\tilde{V}^i_{\nu]} - g\epsilon^i_{jj'}\tilde{V}^j_\mu \tilde{V}^{j'}_\nu\ ,$$

and $f^k_{k'k''}$ and $\epsilon^i_{jj'}$ are su(3) and su(2) structure constants. The lesson is that the choice $\frac{1}{192g^2}$ for the global normalization constant was correct, where g is the coupling constant of su(3) and the two su(2) Lie subalgebras.

It remains to compute the mass terms

$$\begin{aligned}
&-\epsilon\tfrac{1}{192g^2}\mathrm{tr}(\{A,Y\}^2) \\
&\quad = -\epsilon\tfrac{1}{192g^2}\cdot 4\cdot \mathrm{tr}(\gamma^\mu\gamma^{5*}\gamma^\nu\gamma^5 \otimes (P_+[\mathbb{A}_\mu,\mathcal{M}][\mathbb{A}_\nu,\mathcal{M}] + P_+[\mathbb{A}_\mu,\mathcal{N}][\mathbb{A}_\nu,\mathcal{N}^\dagger])) \\
&\quad = \tfrac{1}{12g^2}\cdot(\tfrac{g}{4\sqrt{2}})^2\cdot 4\cdot 16\cdot\big(\\
&\ 4\,(A^{i4}_{-+,\mu}A^{+-,\mu}_{i4} + A^{i4}_{++,\mu}A^{--,\mu}_{i4})\,\mathrm{tr}(2M_1^2 + M'_aM'^\dagger_a + M_aM_a^\dagger + 3M'_bM'^\dagger_b + 3M_bM_b^\dagger \\
&\qquad\qquad + 3M_cM_c^\dagger + 3M_fM_f^\dagger + M_pM_p^\dagger + M_sM_s^\dagger) \\
&\quad + 8\,(A^{i5}_{-+,\mu}A^{+-,\mu}_{i5} + A^{i5}_{++,\mu}A^{--,\mu}_{i5})\,\mathrm{tr}(M_1^2 + M'_aM'^\dagger_a + M_aM_a^\dagger + M'_bM'^\dagger_b \\
&\qquad\qquad + M_bM_b^\dagger + 2M_cM_c^\dagger + 2M_fM_f^\dagger) \\
&\quad + 2\,(A^{i4}_{+-,\mu}A^{-+,\mu}_{i4} + A^{i5}_{++,\mu}A^{--,\mu}_{i5})\,\mathrm{tr}(M_2M_2^\dagger) \\
&\quad + 4\,(A^{i4}_{++,\mu}A^{-+,\mu}_{i4} + A^{i4}_{--,\mu}A^{+-,\mu}_{i4})\,\mathrm{tr}(M'_aM'^\dagger_a - M_aM_a^\dagger + M'_bM'^\dagger_b - M_bM_b^\dagger \\
&\qquad\qquad + M_cM_c^\dagger - M_fM_f^\dagger + M_sM_s^\dagger - M_pM_p^\dagger) \\
&\quad + 4\,(-A^{i4}_{++,\mu}A^{-+,\mu}_{i4} + A^{i4}_{--,\mu}A^{+-,\mu}_{i4})\,\mathrm{tr}(M'_aM_a^\dagger - M_aM'^\dagger_a + M'_bM_b^\dagger - M_bM'^\dagger_b \\
&\qquad\qquad + M_cM_f^\dagger - M_fM_c^\dagger + M_sM_p^\dagger - M_pM_s^\dagger) \\
&\ 8\,(A^{i5}_{++,\mu}A^{--,\mu}_{i5} - A^{i5}_{+-,\mu}A^{-+,\mu}_{i5})\,\mathrm{tr}(M_bM_a^\dagger + M_aM_b^\dagger + M'_aM'^\dagger_b + M'_bM'^\dagger_a \\
&\qquad\qquad + 2M_cM_f^\dagger + 2M_fM_c^\dagger)
\end{aligned}$$

$$
\begin{aligned}
&+ 32\, A^{ij}_{++,\mu} A^{--,\mu}_{ij}\, \mathrm{tr}(M_1^2 + M'_b M'^\dagger_b + M_b M_b^\dagger + M_c M_c^\dagger + M_f M_f^\dagger + \tfrac{1}{16} M_2 M_2^\dagger) \\
&+ 8\mathrm{i}\epsilon^i_{jj'}\, A^{jj'}_{++,\mu} A^{++,\mu}_{i5}\, \mathrm{tr}(M_2 M_f^\dagger + M_2 M_c^\dagger) - 8\mathrm{i}\epsilon^i_{jj'}\, A^{jj'}_{--,\mu} A^{--,\mu}_{i5}\, \mathrm{tr}(M_f M_2^\dagger + M_c M_2^\dagger) \\
&+ 2\,(2 V^+_\mu V^\mu_- + 2\tilde{V}^+_\mu \tilde{V}^\mu_- + (V^3_\mu - \tilde{V}^3_\mu)(V^\mu_3 - \tilde{V}^\mu_3))\, \mathrm{tr}(M'_a M'^\dagger_a + M_a M_a^\dagger + 3 M'_b M'^\dagger_b \\
&\qquad\qquad + 3 M_b M_b^\dagger + 3 M_c M_c^\dagger + 3 M_f M_f^\dagger + M_s M_s^\dagger + M_p M_p^\dagger) \\
&- 4(V^+_\mu \tilde{V}^\mu_- + V^-_\mu \tilde{V}^\mu_+)\, \mathrm{tr}(-M'_a M'^\dagger_a + M_a M_a^\dagger + 3 M'_b M'^\dagger_b - 3 M_b M_b^\dagger \\
&\qquad\qquad - 3 M_c M_c^\dagger + 3 M_f M_f^\dagger + M_s M_s^\dagger - M_p M_p^\dagger) \\
&- 4(V^+_\mu \tilde{V}^\mu_- - V^-_\mu \tilde{V}^\mu_+)\, \mathrm{tr}(M_a M'^\dagger_a - M'_a M_a^\dagger + 3 M'_b M_b^\dagger - 3 M_b M'^\dagger_b \\
&\qquad\qquad + 3 M_f M_c^\dagger - 3 M_c M_f^\dagger + M_s M_p^\dagger - M_p M_s^\dagger) \\
&+ 2(\tilde{V}^+_\mu \tilde{V}^\mu_- + (\sqrt{\tfrac{3}{2}} G^0_\mu + \tilde{V}^3_\mu)(\sqrt{\tfrac{3}{2}} G^\mu_0 + \tilde{V}^\mu_3))\, \mathrm{tr}(M_2 M_2^\dagger)\Big) \;,
\end{aligned}
$$

where we used $\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$ and $\gamma^5\gamma^{5*} = \epsilon \mathbf{1}_4$. The factor 4 comes from the anti-symmetry $A^{ij}_{pq} = -A^{ji}_{qp}$ and the 16 from the trace over P_+ times Γ-matrices.

We anticipate (section 6) the relation between fermion masses and the matrices $M_{s,p,a',a,b',b,c,f}$, which reads ($tp =$ transpose of the preceding term)

$$
\begin{aligned}
M_s &= \tfrac{1}{16}(M_n + 3M_u + M_e + 3M_d) + tp \;, & M_a &= \tfrac{1}{16}(M_n + 3M_u + M_e + 3M_d) - tp \;, \\
M_p &= \tfrac{1}{16}(M_n + 3M_u - M_e - 3M_d) + tp \;, & M'_a &= \tfrac{1}{16}(M_n + 3M_u - M_e - 3M_d) - tp \;, \\
M_c &= \tfrac{1}{16}(M_n - M_u - M_e + M_d) + tp \;, & M_b &= \tfrac{1}{16}(M_n - M_u - M_e + M_d) - tp \;, \\
M_f &= \tfrac{1}{16}(M_n - M_u + M_e - M_d) + tp \;, & M'_b &= \tfrac{1}{16}(M_n - M_u + M_e - M_d) - tp \;. \qquad (20)
\end{aligned}
$$

This gives

$$
\begin{aligned}
&\tfrac{2}{3}\mathrm{tr}(M'_a M'^\dagger_a + M_a M_a^\dagger + 3M'_b M'^\dagger_b + 3M_b M_b^\dagger + 3M_c M_c^\dagger + 3M_f M_f^\dagger + M_s M_s^\dagger + M_p M_p^\dagger) \\
&\quad = \tfrac{1}{12}\mathrm{tr}(M_n M_n^\dagger + M_e M_e^\dagger + 3M_u M_u^\dagger + 3M_d M_d^\dagger) = \mu \;, \\
&\tfrac{4}{3}\mathrm{tr}(-M'_a M'^\dagger_a + M_a M_a^\dagger + 3M'_b M'^\dagger_b - 3M_b M_b^\dagger - 3M_c M_c^\dagger + 3M_f M_f^\dagger + M_s M_s^\dagger - M_p M_p^\dagger) \\
&\quad = \tfrac{1}{6}\mathrm{tr}(M_n M_e^\dagger + M_e M_n^\dagger + 3M_u M_d^\dagger + 3M_d M_u^\dagger) = 2\tilde{\mu} \;, \\
&\tfrac{4}{3}\mathrm{tr}(M_a M'^\dagger_a - M'_a M_a^\dagger + 3M'_b M_b^\dagger - 3M_b M'^\dagger_b + 3M_f M_c^\dagger - 3M_c M_f^\dagger + M_s M_p^\dagger - M_p M_s^\dagger) \\
&\quad = \tfrac{1}{6}\mathrm{tr}(M_e M_n^\dagger - M_n M_e^\dagger + 3M_d M_u^\dagger - 3M_u M_d^\dagger) = 2\mathrm{i}\hat{\mu} \;.
\end{aligned}
$$

The numbers $\mu, \tilde{\mu}, \hat{\mu}$ are not determined by the experimental data because the Dirac mass matrix for the neutrinos M_n is unknown. Let us assume that the largest eigenvalue of M_n is smaller than the mass m_t of the top quark. Then, we have in leading approximation $\mu = \frac{1}{4}m_t^2$ and $\tilde{\mu} = \frac{1}{4}m_t m_b$.

We now diagonalize the V-$\tilde{V}$-G^0 sector. The photon is the massless linear combination $P_\mu = \frac{1}{2}G^0_\mu - \sqrt{\frac{3}{8}}V^3_\mu - \sqrt{\frac{3}{8}}\tilde{V}^3_\mu$, which is perpendicular to the plane spanned by $\frac{1}{\sqrt{2}}(\tilde{V}^3_\mu - V^3_\mu)$ and $\sqrt{\frac{2}{5}}\tilde{V}^3_\mu + \sqrt{\frac{3}{5}}G^0_\mu$. Now, abbreviating $M'^2 = \frac{1}{3}\mathrm{tr}(M_2 M_2^\dagger)$, the mass terms of the V-$\tilde{V}$-G^0 sector are

$$\tfrac{1}{2}\Big(\tfrac{1}{\sqrt{2}}(\tilde{V}^3_\mu - V^3_\mu),\ \tfrac{1}{\sqrt{8}}(\tilde{V}^3_\mu + V^3_\mu) + \tfrac{\sqrt{3}}{2}G^0_\mu\Big)\begin{pmatrix} 2\mu + M'^2 & 2M'^2 \\ 2M'^2 & 4M'^2 \end{pmatrix}\begin{pmatrix} \tfrac{1}{\sqrt{2}}(\tilde{V}^3_\mu - V^3_\mu) \\ \tfrac{1}{\sqrt{8}}(\tilde{V}^3_\mu + V^3_\mu) + \tfrac{\sqrt{3}}{2}G^0_\mu \end{pmatrix}$$
$$+ \Big(V^+_\mu\,,\ \tilde{V}^+_\mu\Big)\begin{pmatrix} \mu & -\tilde{\mu} - \mathrm{i}\hat{\mu} \\ -\tilde{\mu} + \mathrm{i}\hat{\mu} & \mu + M'^2 \end{pmatrix}\begin{pmatrix} V^-_\mu \\ \tilde{V}^-_\mu \end{pmatrix} .$$

After a unitary-orthogonal transformation

$$\begin{pmatrix} Z_\mu \\ \tilde{Z}_\mu \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \tfrac{1}{\sqrt{2}}(V^3_\mu + \tilde{V}^3_\mu) \\ \tfrac{1}{\sqrt{8}}(\tilde{V}^3_\mu - V^3_\mu) + \tfrac{\sqrt{3}}{2}G^0_\mu \end{pmatrix}, \quad \begin{pmatrix} W^-_\mu \\ \tilde{W}^-_\mu \end{pmatrix} = \begin{pmatrix} \cos\chi & -\mathrm{e}^{\mathrm{i}\chi'}\sin\chi \\ \sin\chi & \mathrm{e}^{\mathrm{i}\chi'}\cos\chi \end{pmatrix}\begin{pmatrix} V^-_\mu \\ \tilde{V}^-_\mu \end{pmatrix},$$

where $\mathrm{e}^{\mathrm{i}\chi'} = \frac{\tilde{\mu}+\mathrm{i}\hat{\mu}}{\sqrt{\hat{\mu}^2+\tilde{\mu}^2}}$, the physical particles obtain the following masses:

$$\begin{array}{ll}
m^2_W & \mu + \tfrac{1}{2}M'^2 - \sqrt{\tfrac{1}{4}M'^4 + \hat{\mu}^2 + \tilde{\mu}^2} \approx \mu + (\hat{\mu}^2 + \tilde{\mu}^2)/M'^2 \\
m^2_{\tilde{W}} & \mu + \tfrac{1}{2}M'^2 + \sqrt{\tfrac{1}{4}M'^4 + \hat{\mu}^2 + \tilde{\mu}^2} \approx M'^2 + \mu \\
m^2_Z & \mu + \tfrac{5}{2}M'^2 - \sqrt{\tfrac{25}{4}M'^4 - 3\mu M'^2 + \mu^2} \approx \tfrac{8}{5}\mu - \tfrac{16\,\mu^2}{25\,M'^2} \\
m^2_{\tilde{Z}} & \mu + \tfrac{5}{2}M'^2 + \sqrt{\tfrac{25}{4}M'^4 - 3\mu M'^2 + \mu^2} \approx 5M'^2 + \tfrac{2}{5}\mu
\end{array}$$

This means

$$m_W \approx \tfrac{1}{2}m_t\ , \qquad m_Z = m_W/\cos\theta_W\ , \qquad \sin^2\theta_W \approx \tfrac{3}{8} - \tfrac{m_t^2}{80\,M'^2}\ . \tag{21}$$

We recall that the mass prediction for m_W depends crucially on the assumption that the Dirac mass for the neutrinos can be neglected. But as m_W cannot be much larger than $\frac{1}{2}m_t$, this assumption seems to be correct. For the rotation angles we get $\cot 2\phi = \frac{3}{4} - \frac{\mu}{2M'^2}$ and $\tan 2\chi = -2\sqrt{\tilde{\mu}^2 + \hat{\mu}^2}/M'^2$. Hence, there is a violation of the standard model of the order m_t^2/M'^2, for instance a coupling of the $W^\pm$ bosons to the right-handed fermions, and the Weinberg angle is not universal any more.

We neglect the mixing of the order $\|M_{a,a',b,b',c,f,s,p}\|/\|M_{1,2}\|$ between the very massive leptoquarks. Denoting $M^2 = \frac{4}{3}\mathrm{tr}(M_1^2)$, the masses of the leptoquark are:

A^{i4}_{+-}	A^{i4}_{++}	A^{i5}_{+-}	A^{i5}_{++}	A^{ij}_{++}
$\sqrt{M^2+M'^2}$	M	M	$\sqrt{M^2+M'^2}$	$\sqrt{4M^2+M'^2}$

The renormalization group analysis[10] suggests $M \approx 10^{16}$GeV. Below that energy, the original gauge group SO(10) is broken to the intermediate symmetry group $\mathrm{SU}(3)_C\times\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\times\mathrm{U}(1)_{B-L}$. At the scale $M' \approx 10^9$GeV, the subgroups $\mathrm{SU}(2)_R$ and $\mathrm{U}(1)_{B-L}$ are broken to the standard model symmetry group $\mathrm{SU}(3)_C\times\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$, with the hyper-charge given by $\sqrt{\frac{3}{5}}\tilde{V}^3_\mu-\sqrt{\frac{2}{5}}G^0_\mu$. Finally, the fermion masses break the standard model symmetry at the scale $m_t\approx 10^2\mathrm{GeV}$ to the remaining symmetry $\mathrm{SU}(3)_C\times\mathrm{U}(1)_{EM}$. Hence, the only massless Yang-Mills fields are the photon P_μ and the eight gluons G^a_μ. The Higgs mechanism consists in using the other $45-9=36$ SO(10)-gauge parameters to eliminate the 36 Goldstone bosons $[\pi(a),-\mathrm{i}Y]$, which in turn breaks the symmetry from SO(10) to the fermion symmetry $\mathrm{SU}(3)_C\times\mathrm{U}(1)_{EM}$. Thus, there are $45+2\cdot10+4\cdot120+3\cdot2\cdot126-36=1301-36=1265$ independent Higgs components.

We compute now the upper limit for the mass of the standard model Higgs field. It is obtained by evaluating the Higgs potential V at the configuration

$$\begin{aligned}
&\tilde{\Theta}=(\Gamma_{45}+\Gamma_{78}+\Gamma_{69})\;, \\
&\tilde{\Upsilon}_1=(1+\phi)\Gamma_0\;, && \tilde{\Upsilon}_2=(1+\phi)\Gamma_3\;, \\
&\tilde{\Phi}_1=(1+\phi)\Gamma_{120}\;, && \tilde{\Phi}_2=(1+\phi)\Gamma_{123}\;, \\
&\tilde{\Phi}_3=(1+\phi)(\Gamma_{453}+\Gamma_{783}+\Gamma_{693})\;, && \tilde{\Phi}_4=(1+\phi)(\Gamma_{450}+\Gamma_{780}+\Gamma_{690})\;, \\
&\tilde{\Psi}_1=(1+\phi)(\Gamma_{01245}+\Gamma_{01278}+\Gamma_{01269})\;, && \tilde{\Psi}_2=(1+\phi)(\Gamma_{31245}+\Gamma_{31278}+\Gamma_{31269})\;, \\
&\tilde{\Psi}_3=\tfrac{1}{8}(\Gamma_1-\mathrm{i}\Gamma_2)\Gamma_3(\Gamma_4-\mathrm{i}\Gamma_5)(\Gamma_6-\mathrm{i}\Gamma_9)(\Gamma_7-\mathrm{i}\Gamma_8)\;.
\end{aligned} \tag{22}$$

It is important that ϕ is a real field, because the configuration corresponding to a imaginary part is the Goldstone boson given by the commutator with Γ^{+-}_{44}. One easily finds

$$\begin{aligned}
\theta_{(1)}=&-\phi\mathbf{1}_4\otimes\mathbf{1}_{32}\otimes(\tilde{M}_{\{ss\}}+\tilde{M}_{\{pp\}}+\tilde{M}_{\{a'a'\}}+\tilde{M}_{\{aa\}}+3\tilde{M}_{\{b'b'\}}+3\tilde{M}_{\{bb\}} \\
&\qquad+3\tilde{M}_{\{cc\}}+3\tilde{M}_{\{ff\}}) \\
&+2\phi\mathbf{1}_4\otimes\mathrm{i}(\Gamma_{45}+\Gamma_{78}+\Gamma_{69})\otimes(M_{\{sb'\}}+M_{\{pb\}}+M_{\{a'c\}}+M_{\{af\}}+2M_{\{bc\}}+2M_{\{b'f\}})\;, \\
\theta_{(2)}=&\;0\;,
\end{aligned}$$

up to the Higgs self-interaction ϕ^2. The other **45** and all **210**-contributions are cancelled due to (17). We insert (20) and arrive at

$$\begin{aligned}
\theta_{(1)}=&-\tfrac{1}{16}\phi\mathbf{1}_4\otimes\mathbf{1}_{32}\otimes(\tilde{M}_{\{nn\}}+\tilde{M}_{\{ee\}}+3\tilde{M}_{\{uu\}}+3\tilde{M}_{\{dd\}} \\
&\qquad+\tilde{M}_{\{n^tn^t\}}+\tilde{M}_{\{e^te^t\}}+3\tilde{M}_{\{u^tu^t\}}+3\tilde{M}_{\{d^td^t\}}) \\
&+\tfrac{1}{16}\phi\mathbf{1}_4\otimes\mathrm{i}(\Gamma_{45}+\Gamma_{78}+\Gamma_{69})\otimes(M_{\{nn\}}+M_{\{ee\}}-M_{\{uu\}}-M_{\{dd\}} \\
&\qquad-M_{\{n^tn^t\}}-M_{\{e^te^t\}}+M_{\{u^tu^t\}}+M_{\{d^td^t\}})\;.
\end{aligned}$$

where $\tilde{M}_{\alpha^t\alpha^t}:=(2M^T_\alpha M^{T\dagger}_\alpha)_\perp$. We neglect again M_n and choose $M_u=\mathrm{diag}(m_u,m_c,m_t)$, where the entries are the masses of the u,c,t-quarks. There is a neglectable contribution of θ_{45} to the Higgs potential, and we have in leading approximation

$$\theta_{(1)}=-\tfrac{1}{4}\phi m_t^2\mathbf{1}_4\otimes\mathbf{1}_{32}\otimes\mathrm{diag}(-1,-1,2)\;.$$

This leads to

$$V = \tfrac{\epsilon}{48g^2}\tfrac{1}{16}\phi^2 m_t^4 \cdot 4\cdot 16\cdot 6 = \epsilon\tfrac{1}{2g^2}m_t^4\phi^2 \;.$$

Inserting the configuration (22) into the part $\mathcal{L}_1$ of the Lagrangian we get

$$\begin{aligned}
\mathcal{L}_1 &= \tfrac{\epsilon}{192g^2}\cdot 4\cdot \mathrm{tr}(P_+(\gamma^\mu\partial_\mu\gamma^5\eta_{\mathcal{N}})^*(\gamma^\nu\partial_\nu\gamma^5\eta_{\mathcal{N}})) = \tfrac{1}{12g^2}\mathrm{tr}(P_+\partial_\mu{\eta_{\mathcal{N}}}^\dagger\,\partial^\mu\eta_{\mathcal{N}}) \\
&= \tfrac{1}{12g^2}\partial_\nu\phi\,\partial^\nu\phi\cdot 16\cdot\mathrm{tr}(M_a' {M_a'}^\dagger + M_a M_a^\dagger + 3M_b'{M_b'}^\dagger + 3M_bM_b^\dagger \\
&\qquad\qquad + 3M_cM_c^\dagger + 3M_fM_f^\dagger + M_sM_s^\dagger + M_pM_p^\dagger) \\
&= \tfrac{2}{g^2}\mu\,\partial_\nu\phi\,\partial^\nu\phi \;.
\end{aligned}$$

This means that the physical Higgs boson is obtained by rescaling $\varphi = \frac{g}{2\sqrt{\mu}}\phi = g\phi/m_t$, and it receives the mass

$$m_\varphi = m_t \;. \tag{23}$$

This value is an upper bound for the Higgs mass, because we have only calculated the diagonal matrix element of the whole mass matrix. Due to the off-diagonal matrix elements, the masses = eigenvalues are different from the diagonal matrix elements, and the smallest eigenvalue is smaller than the smallest diagonal matrix element. It is plausible that this smallest diagonal matrix element is just m_φ^2, because any other Higgs configuration obtains a mass contribution from $\theta_{(2)}$ of the order $\mathrm{tr}((M_1M_i \pm M_i\overline{M}_1)^2)/\mathrm{tr}(M_i^2)$.

The computation of the masses of the remaining 1264 Higgs bosons is analogous.

6 The fermionic action

To write down the fermionic action (in Minkowski space!), recall that our setting has two symmetries J and $\mathcal{S}$. It is therefore natural to demand that the fermionic configuration space has the same symmetries,

$$\mathcal{H} = \{\boldsymbol{\psi}\in L^2(M,S)\otimes\mathbb{C}^{384}:\quad \mathcal{P}\boldsymbol{\psi} = J\boldsymbol{\psi} = \mathcal{S}\boldsymbol{\psi} = \boldsymbol{\psi}\}\;.$$

As usual we impose a Weyl condition in Minkowskian case. This means to look for a chirality operator that commutes with both J and $\mathcal{S}$. The unique choice up to the sign is

$$\boldsymbol{\phi} = \boldsymbol{\chi}\boldsymbol{\psi}\;,\quad \boldsymbol{\chi} = \mathcal{P}\mathrm{diag}(-\gamma^5\otimes 1_{96}\,,\,-\gamma^5\otimes 1_{96}\,,\,\gamma^5\otimes 1_{96}\,,\,\gamma^5\otimes 1_{96})\mathcal{P}\;.$$

Thus, elements of $\mathcal{H}$ are of the form

$$\boldsymbol{\psi} = \begin{pmatrix} (\frac{1}{2}(1_4-\gamma^5)\otimes P_+)\tilde{\boldsymbol{\psi}} \\ (-\mathrm{i}\frac{1}{2}(1_4-\gamma^5)\otimes P_+)\tilde{\boldsymbol{\psi}} \\ (\frac{1}{2}(1_4+\gamma^5)\gamma^2\otimes P_-B)\overline{\tilde{\boldsymbol{\psi}}} \\ (\mathrm{i}\frac{1}{2}(1_4+\gamma^5)\gamma^2\otimes P_-B)\overline{\tilde{\boldsymbol{\psi}}} \end{pmatrix}\;,\qquad \tilde{\boldsymbol{\psi}}\in L^2(M,S)\otimes\mathbb{C}^{96}\;.$$

In order to eliminate the projection operators we introduce

$$B = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} , \quad b = \rho_2\eta_2\tau_2\sigma_2 \otimes 1_3 \in \mathbb{C}^{48} ,$$

$$\mathbb{A}_\mu \otimes 1_3 = \begin{pmatrix} \boldsymbol{A}_\mu & 0 \\ 0 & -b\boldsymbol{A}_\mu^T b \end{pmatrix} , \quad \boldsymbol{A}_\mu \in C^\infty(M) \otimes \mathrm{so}(10) \otimes 1_3 ,$$

$$P_+(\eta_{\mathcal{N}} + \mathcal{N})P_- = \begin{pmatrix} 0 & \tilde{H} \\ 0 & 0 \end{pmatrix} ,$$

$$\sigma^0 = \tilde{\sigma}^0 = 1_2 , \quad \sigma^a = -\tilde{\sigma}^a , \quad a = 1,2,3 ,$$

$$(\tfrac{1}{2}(1_4 - \gamma^5) \otimes P_+)\tilde{\boldsymbol{\psi}} = \big(0 ,\ \psi_L ,\ 0 ,\ 0\big)^t , \quad \psi_L \in L^2(M)\mathbb{C}^2 \otimes \mathbb{C}^{16} \otimes \mathbb{C}^3 .$$

Now, the fermionic action (8) is

$$\begin{aligned} S_F &= \int_M dx\ \tfrac{1}{4}\boldsymbol{\psi}^\dagger\gamma^0(D + \mathrm{i}\rho)\boldsymbol{\psi} \\ &= \int_M dx\ \tfrac{1}{2}\Big(\psi_L^\dagger ,\ -\psi_L^T\sigma^2 b\Big)\begin{pmatrix} \mathrm{i}\tilde{\sigma}^\mu(\partial_\mu + \boldsymbol{A}_\mu) & \tilde{H} \\ \tilde{H}^\dagger & \mathrm{i}\sigma^\mu(\partial_\mu - b\boldsymbol{A}_\mu^T b) \end{pmatrix}\begin{pmatrix} \psi_L \\ -\sigma^2 b\overline{\psi_L} \end{pmatrix} \\ &= \int_M dx\ \Big(\mathrm{i}\psi_L^\dagger\tilde{\sigma}^\mu(\partial_\mu + \boldsymbol{A}_\mu)\psi_L + \Big\{\tfrac{1}{2}\psi_L^\dagger\tilde{H}(-\sigma^2 b\overline{\psi_L}) + h.c\Big\}\Big) . \end{aligned} \tag{24}$$

To get the last line one has to take into account that fermions ψ_i are Grassmann-valued, which means $\psi_1^T X^T \overline{\psi_2} = -\psi_2^\dagger X \psi_1$, for any matrix X. The correct fermionic parameterization dictated by the electric charge is

$$\begin{aligned} \psi_L &= \big(n_L, u_L^1, u_L^2, u_L^3, e_L, d_L^1, d_L^2, d_L^3, \\ &\qquad \sigma^2\overline{d_R^3}, -\sigma^2\overline{d_R^2}, -\sigma^2\overline{d_R^1}, \sigma^2\overline{e_R}, -\sigma^2\overline{u_R^3}, \sigma^2\overline{u_R^2}, \sigma^2\overline{u_R^1}, -\sigma^2\overline{\nu_R}\big)^t, \\ -\sigma_2 b\overline{\psi_L} &= \big(-n_R, -u_R^1, -u_R^2, -u_R^3, -e_R, -d_R^1, -d_R^2, -d_R^3, \\ &\qquad \sigma^2\overline{d_L^3}, -\sigma^2\overline{d_L^2}, -\sigma^2\overline{d_L^1}, \sigma^2\overline{e_L}, -\sigma^2\overline{u_L^3}, \sigma^2\overline{u_L^2}, \sigma^2\overline{u_L^1}, -\sigma^2\overline{\nu_L}\big)^t, \end{aligned}$$

where ^{t} does not transpose the entries of the vector. It it now easy to verify that

$$\begin{aligned} \tfrac{1}{2}\psi_L^\dagger\mathcal{N}(-\sigma^2 b\overline{\psi_L}) + h.c = \big\{ &- u_L^\dagger(M_u \otimes 1_3)u_R - d_L^\dagger(M_d \otimes 1_3)d_R \\ &- e_L^\dagger M_e e_R - \nu_L^\dagger M_\nu \nu_R - \nu_R^T\sigma_2 M_2 \nu_R\big\} + h.c , \end{aligned}$$

with the mass matrices $M_{u,d,\nu,e}$ given implicitly in (20). The right neutrinos receive a large Majorana mass of the order $\|M_2\|$ and the see-saw mechanism produces very small masses for the left-handed neutrinos of the order $\|m_n^2\|/\|M_2\|$.

7 Outlook

What we have presented here is a maximal SO(10)-model which allows the fermion masses to be as general as possible. This is in contrast to the original idea of grand unification, namely, to reduce the number of free parameters of the standard model. The number of Higgs multiplets can be reduced by imposing appropriate relations between the fermion masses. For instance, the minimal SO(10)-model containing one complex **10**, one complex **126** and the **45** (or **210**) is obtained by putting $M_s = \lambda_1 M_p$, $M_2 = \lambda_2 M_c = \lambda_3 M_f$ and $M_a = M'_a = M_b = M'_b = 0$, with real parameters λ_i. This model is very predictive in the fermion sector and one can calculate the neutrino masses[10]. However, in our formulation the ideal $\mathcal{J}^2$ becomes so large that the only surviving terms in the Higgs potential are **1** and **10**. This is not sufficient. There seems to be a strong evidence that a **120** representation must be included. This next-to-minimal SO(10)-model will be studied elsewhere.

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