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GRADED DIFFERENTIAL LIE ALGEBRAS AND SU(5) × U(1)-GRAND UNIFICATION

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We formulate the flipped $\mathrm{SU}(5)\times\mathrm{U}(1)$-GUT within a Lie-algebraic approach to non-commutative geometry. It suffices to take the matrix Lie algebra su(5) as the input; the u(1)-part with its representation on the fermions is an algebraic consequence. The occurring Higgs multiplets ($\underline{24},\underline{5},\underline{45},\underline{50}$-representations of su(5)) are uniquely determined by the fermionic mass matrix and the spontaneous symmetry breaking pattern to $\mathrm{SU}(3)_C\times\mathrm{U}(1)_{EM}$. We find the most general gauge invariant Higgs potential that is compatible with the given Higgs vacuum. Our formalism yields tree-level predictions for the masses of all gauge and Higgs bosons. It turns out that the low-energy sector is identical with the standard model. In particular, there exists precisely one light Higgs field, whose upper bound for the mass is $1.45\,m_t$. All remaining 207 Higgs fields are extremely heavy.

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1. Introduction

One of the most important applications of non-commutative geometry (NCG) to physics is a unified description of the standard model. The most elegant version rests upon a K-cycle[1,2] with real structure[3], see Refs. 4,5,6 for details of the construction. The standard model is the only realistic physical model that one can formulate within the most elegant NCG-prescription[7]. On the other hand, there exist good reasons[8] why one could be interested in Grand Unified Theories (GUT's): GUT's explain the quantization of electric charge, yield a fairly well prediction for the Weinberg angle, explain the convergence of running coupling constants at high energies, include massive neutrinos to solve the solar neutrino problem, produce the observed baryon asymmetry of the universe, etc. However, the results of Ref. 7 imply that one needs additional structures or different methods for a NCG-formulation of these models.

Perhaps the most successful NCG-approach towards grand unification was proposed by Chamseddine, Felder and Fröhlich. In the SU(5)-model[9,10], the authors start to construct an auxiliary K-cycle. Within this framework they construct the bosonic sector. Then they interpret some of these bosonic quantities as Lie algebra valued and consider Lie algebra representations on the physical Hilbert space to obtain the fermionic sector. An aesthetic shortcoming of that approach is the auxiliary character of the K-cycle, which of course is inevitable in view of Ref. 7. The SO(10)-model[11] by Chamseddine and Fröhlich fits well[a] into the NCG-scheme. The reason why this model was excluded in Ref. 7 is that only models possessing complex fundamental irreducible representations were admitted in that article.

[a]Nevertheless, the use of Lie algebras instead of algebras could probably justify certain assumptions made in Ref. 11.

The author of this paper has proposed in Refs. 12,13 a modification of noncommutative geometry. In that approach one uses skew-adjoint Lie algebras instead of unital associative $*$-algebras. Differential geometry is formulated in terms of graded differential Lie algebras. The advantage of that framework is that a larger class of physical models can be constructed from the same amount of structures as in the most elegant NCG-prescription. That class includes the standard model[14] and the flipped $\mathrm{SU}(5) \times \mathrm{U}(1)$-GUT as well, as we show in this paper. For the classical treatment of that model see Ref. 15.

We give in Sec. 2 a recipe how to construct classical gauge field theories within our NCG-framework. The arguments why this recipe works can be found in Ref. 13. In Sec. 3 we construct the matrix part of the $\mathrm{SU}(5) \times \mathrm{U}(1)$-model: In Sec. 3.1 we consider relevant su(5)-representations. The remaining ingredients of our scheme are defined in Sec. 3.2. Then it is not difficult to derive in Sec. 3.3 the matrix part of the connection form. Finally, we perform in Sec. 3.6 the factorization of the curvature with respect to a canonically given ideal constructed in Sec. 3.5.

In Sec. 4 we include the space-time part and derive the action for our model: Out of the curvature obtained in Sec. 4.1 we build in Sec. 4.2 the bosonic action. To compare it with usual formulae of gauge field theory we write down this action in terms of local coordinates, see Sec. 4.3. The fermionic action is derived in Sec. 4.4. Comparing it with phenomenology we can identify certain parameters of the generalized Dirac operator with fermion masses and Kobayashi–Maskawa mixing angles. This information is essential for deriving the masses of the Higgs bosons in Sec. 5.

2. NCG with Graded Differential Lie Algebras

The basic object in our approach is an L-cycle $(\mathfrak{g}, h, D, \pi, \Gamma)$, which consists of a $*$-representation π of a skew-adjoint Lie algebra $\mathfrak{g}$ as bounded operators on a Hilbert space h, together with a selfadjoint operator D on h with compact resolvent and a selfadjoint operator Γ on h, $\Gamma^2 = \mathrm{id}_h$, which commutes with $\pi(\mathfrak{g})$ and anticommutes with D. The operator D may be unbounded, but such that $[D, \pi(\mathfrak{g})]$ is bounded. L-cycles are naturally related to physical models on a space-time manifold X if the following input data are given:

1. A unitary matrix Lie group G and its associated gauge group $\mathcal{G} = C^\infty(X) \otimes G$. Here, $C^\infty(X)$ denotes the algebra of real-valued smooth functions on X.
2. Chiral fermions $\boldsymbol{\psi}$ transforming under a representation $\tilde{\pi}_0$ of G. The induced representation of the gauge group $\mathcal{G}$ is $\tilde{\pi} = \mathrm{id} \otimes \tilde{\pi}_0$.
3. The fermionic mass matrix $\widetilde{\mathcal{M}}$ containing fermion masses and generalized Kobayashi–Maskawa matrices.
4. Possibly the spontaneous symmetry breaking pattern of G.

For technical reasons we pass to a compact Euclidian spin manifold X. We take $\mathfrak{g} = C^\infty(X) \otimes \mathfrak{a}$ as the Lie algebra of $\mathcal{G}$. Here, $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}''$ is a skew-adjoint matrix Lie algebra, where $\mathfrak{a}'$ is semisimple and $\mathfrak{a}''$ Abelian. We shall only consider the case that the Abelian part is not present, i.e. $\mathfrak{a} = \mathfrak{a}'$. We choose $h = L^2(X, S) \otimes \mathbb{C}^F$ as the Hilbert space of Euclidian fermions $\boldsymbol{\psi}$. Here, $L^2(X, S)$ is the Hilbert space of square integrable bispinors. We take $\pi = 1 \otimes \hat{\pi}$ as the differential $\tilde{\pi}_*$, where $\hat{\pi}$ is a representation of $\mathfrak{a}$ in $\mathrm{M}_F\mathbb{C}$. We define $D = \mathsf{D} \otimes \mathbf{1}_F + \gamma^5 \otimes \mathcal{M}$, where D is the

Dirac operator associated to the spin connection and $\mathcal{M} \in \mathrm{M}_F\mathbb{C}$. Here, $\gamma^5 \otimes \mathcal{M}$ must coincide with $\widetilde{\mathcal{M}}$ on chiral fermions. The chirality properties of the fermions are encoded in $\Gamma = \gamma^5 \otimes \hat{\Gamma}$, where $\{\hat{\Gamma}, \mathcal{M}\} = 0$ and $[\hat{\Gamma}, \hat{\pi}(\mathfrak{a})] = 0$.

The recipe towards the (classical) gauge field theory associated to the L-cycle is the following: Let $\Omega^1\mathfrak{a}$ be the space of formal commutators

$$\omega^1 = \textstyle\sum_{\alpha,z\geq 0}[a^z_\alpha, [\dots[a^1_\alpha, da^0_\alpha]\dots]] , \quad a^i_\alpha \in \mathfrak{a} . \tag{1}$$

Apply linear mappings $\hat{\pi} : \Omega^1\mathfrak{a} \to \mathrm{M}_F\mathbb{C}$ and $\hat{\sigma} : \Omega^1\mathfrak{a} \to \mathrm{M}_F\mathbb{C}$ defined by

$$\hat{\pi}(\omega^1) := \textstyle\sum_{\alpha,z\geq 0}[\hat{\pi}(a^z_\alpha), [\dots[\hat{\pi}(a^1_\alpha), [-\mathrm{i}\,\mathcal{M}, \hat{\pi}(a^0_\alpha)]]\dots]] , \tag{2a}$$
$$\hat{\sigma}(\omega^1) := \textstyle\sum_{\alpha,z\geq 0}[\hat{\pi}(a^z_\alpha), [\dots[\hat{\pi}(a^1_\alpha), [\mathcal{M}^2, \hat{\pi}(a^0_\alpha)]]\dots]] . \tag{2b}$$

Define $\Omega^n\mathfrak{a} \ni \omega^n = \sum_\alpha[\omega^1_{n,\alpha}, [\omega^1_{n-1,\alpha}, \dots[\omega^1_{2,\alpha}, \omega^1_{1,\alpha}]\dots]]$, where $\omega^1_{i,\alpha} \in \Omega^1\mathfrak{a}$. Extend $\hat{\pi}$ and $\hat{\sigma}$ recursively to $\Omega^n\mathfrak{a}$ by

$$\begin{aligned}
&\hat{\pi}([\omega^1, \omega^k]) := \hat{\pi}(\omega^1)\hat{\pi}(\omega^k) - (-1)^k\hat{\pi}(\omega^k)\hat{\pi}(\omega^1) , \\
&\hat{\sigma}([\omega^1, \omega^k]) := \hat{\sigma}(\omega^1)\hat{\pi}(\omega^k) - \hat{\pi}(\omega^k)\hat{\sigma}(\omega^1) - \hat{\pi}(\omega^1)\hat{\sigma}(\omega^k) - (-1)^k\hat{\sigma}(\omega^k)\hat{\pi}(\omega^1) .
\end{aligned} \tag{3}$$

Define for $n \geq 2$

$$\hat{\pi}(\mathcal{J}^n\mathfrak{a}) := \{\ \hat{\sigma}(\omega^{n-1}) ,\ \ \omega^{n-1} \in \Omega^{n-1}\mathfrak{a} \cap \ker\hat{\pi}\ \} . \tag{4}$$

Define spaces $\boldsymbol{r}^0\mathfrak{a} \subset \mathrm{M}_F\mathbb{C}$ and $\boldsymbol{r}^1\mathfrak{a} \subset \mathrm{M}_F\mathbb{C}$ elementwise by

$$\begin{aligned}
&\boldsymbol{r}^0\mathfrak{a} = -(\boldsymbol{r}^0\mathfrak{a})^* = \hat{\Gamma}(\boldsymbol{r}^0\mathfrak{a})\hat{\Gamma} , &&\boldsymbol{r}^1\mathfrak{a} = -(\boldsymbol{r}^1\mathfrak{a})^* = -\hat{\Gamma}(\boldsymbol{r}^1\mathfrak{a})\hat{\Gamma} , \\
&[\boldsymbol{r}^0\mathfrak{a}, \hat{\pi}(\mathfrak{a})] \subset \hat{\pi}(\mathfrak{a}) , &&[\boldsymbol{r}^0\mathfrak{a}, \hat{\pi}(\Omega^1\mathfrak{a})] \subset \hat{\pi}(\Omega^1\mathfrak{a}) , \\
&\{\boldsymbol{r}^0\mathfrak{a}, \hat{\pi}(\mathfrak{a})\} \subset \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} + \hat{\pi}(\Omega^2\mathfrak{a}) , &&\{\boldsymbol{r}^0\mathfrak{a}, \hat{\pi}(\Omega^1\mathfrak{a})\} \subset \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\Omega^1\mathfrak{a})\} + \hat{\pi}(\Omega^3\mathfrak{a}) , \\
&[\boldsymbol{r}^1\mathfrak{a}, \hat{\pi}(\mathfrak{a})] \subset \hat{\pi}(\Omega^1\mathfrak{a}) , &&\{\boldsymbol{r}^1\mathfrak{a}, \hat{\pi}(\Omega^1\mathfrak{a})\} \subset \hat{\pi}(\Omega^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} .
\end{aligned} \tag{5}$$

Define spaces $\boldsymbol{j}^0\mathfrak{a}, \boldsymbol{j}^1\mathfrak{a}, \boldsymbol{j}^2\mathfrak{a} \subset \mathrm{M}_F\mathbb{C}$ elementwise by

$$\begin{aligned}
&\boldsymbol{j}^0\mathfrak{a} := \boldsymbol{c}^0\mathfrak{a} , &&\boldsymbol{j}^1\mathfrak{a} := \boldsymbol{c}^1\mathfrak{a} , \\
&\boldsymbol{j}^2\mathfrak{a} := \boldsymbol{c}^2\mathfrak{a} + \hat{\pi}(\mathcal{J}^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} , &&\text{where} \\
&\boldsymbol{c}^0\mathfrak{a} = -(\boldsymbol{c}^0\mathfrak{a})^* = \hat{\Gamma}(\boldsymbol{c}^0\mathfrak{a})\hat{\Gamma} , &&\boldsymbol{c}^1\mathfrak{a} = -(\boldsymbol{c}^0\mathfrak{a})^* = -\hat{\Gamma}(\boldsymbol{c}^0\mathfrak{a})\hat{\Gamma} , \\
&\boldsymbol{c}^2\mathfrak{a} = (\boldsymbol{c}^0\mathfrak{a})^* = \hat{\Gamma}(\boldsymbol{c}^0\mathfrak{a})\hat{\Gamma} , && \\
&\boldsymbol{c}^0\mathfrak{a} \cdot \hat{\pi}(\mathfrak{a}) = 0 , &&\boldsymbol{c}^0\mathfrak{a} \cdot \hat{\pi}(\Omega^1\mathfrak{a}) = 0 , \\
&\boldsymbol{c}^1\mathfrak{a} \cdot \hat{\pi}(\mathfrak{a}) = 0 , &&\boldsymbol{c}^1\mathfrak{a} \cdot \hat{\pi}(\Omega^1\mathfrak{a}) = 0 , \\
&[\boldsymbol{c}^2\mathfrak{a}, \hat{\pi}(\mathfrak{a})] = 0 , &&[\boldsymbol{c}^2\mathfrak{a}, \hat{\pi}(\Omega^1\mathfrak{a})] = 0 .
\end{aligned} \tag{6}$$

The connection form ρ has the structure

$$\begin{aligned}
&\rho = \textstyle\sum_\alpha(c^1_\alpha \otimes m^0_\alpha + c^0_\alpha\gamma^5 \otimes m^1_\alpha) , \\
&\quad c^1_\alpha \in \Lambda^1 , \quad c^0_\alpha \in \Lambda^0 , \quad m^0_\alpha \in \boldsymbol{r}^0\mathfrak{a} , \quad m^1_\alpha \in \boldsymbol{r}^1\mathfrak{a} ,
\end{aligned} \tag{7}$$

where Λ^k is the space of differential k-forms represented by gamma matrices. The curvature θ is computed from the connection form ρ by

$$\begin{aligned} \theta &= \mathbf{d}\rho + \rho^2 - \mathrm{i}\{\gamma^5 \otimes \mathcal{M}, \rho\} + \hat{\sigma}_{\mathfrak{g}}(\rho)\gamma^5 + \mathbb{J}^2\mathfrak{g} \;, \\ \mathbb{J}^2\mathfrak{g} &= (\Lambda^2 \otimes \boldsymbol{j}^0\mathfrak{a}) \oplus (\Lambda^1\gamma^5 \otimes \boldsymbol{j}^1\mathfrak{a}) \oplus (\Lambda^0 \otimes \boldsymbol{j}^2\mathfrak{a}) \;, \end{aligned} \tag{8}$$

where $\mathbf{d}$ is the exterior differential and $\hat{\sigma}_{\mathfrak{g}}$ the extension of $\mathrm{id} \otimes \hat{\sigma}$ to elements of the form (7). Select the representative $\mathfrak{e}(\theta)$ orthogonal to $\mathbb{J}^2\mathfrak{g}$, i.e. find $\mathrm{j} \in \mathbb{J}^2\mathfrak{g}$ such that

$$\begin{aligned} &\mathfrak{e}(\theta) = \mathbf{d}\rho + \rho^2 - \mathrm{i}\{\gamma^5 \otimes \mathcal{M}, \rho\} + \hat{\sigma}(\rho)\gamma^5 + \mathrm{j} \;, \\ &\int_X dx \;\; \mathrm{tr}_c(\mathfrak{e}(\theta)\,\mathrm{j}_2) = 0 \;, \quad \forall \mathrm{j}_2 \in \mathbb{J}^2\mathfrak{g} \;. \end{aligned} \tag{9}$$

The trace tr_c includes the trace in $\mathrm{M}_F\mathbb{C}$ and over gamma matrices. Compute the bosonic and fermionic actions

$$S_B = \int_X dx \; \frac{1}{g_0^2\, F} \; \mathrm{tr}_c(\mathfrak{e}(\theta)^2) \;, \qquad\qquad S_F = \int_X dx \; \boldsymbol{\psi}^*(D + \mathrm{i}\,\rho)\boldsymbol{\psi} \;, \tag{10}$$

where g_0 is a coupling constant and $\boldsymbol{\psi} \in h$. At the end, perform a Wick rotation to Minkowski space.

3. The Matrix Part of the Unification Model

3.1. *The representations under consideration*

We shall adapt our notations to the $\mathrm{SU}(5) \times \mathrm{U}(1)$-model. In contrast to what one could expect from the classical treatment[15] of that model, the matrix Lie algebra we use is not $\mathrm{su}(5) \oplus \mathrm{u}(1)$ but $\mathfrak{a} = \mathrm{su}(5)$. In our approach, the $\mathrm{u}(1)$-part is not an input of the model but an algebraic consequence. The internal Hilbert space is

$$\mathbb{C}^{192} = \big(\underline{10} \oplus \underline{5}^* \oplus \underline{1} \oplus \underline{10}^* \oplus \underline{5} \oplus \underline{1}\big) \otimes \mathbb{C}^2 \otimes \mathbb{C}^3 \;, \tag{11}$$

where $\underline{10}, \underline{10}^*, \underline{5}, \underline{5}^*, \underline{1}$ are representations of $\mathrm{su}(5)$. Since we consider linear operators on $\mathbb{C}^{192}$, we need the decomposition rules for homomorphisms between the $\mathrm{su}(5)$-representations occurring in (11):

$$\begin{array}{rllll}
\mathrm{End}(\underline{10}) & = \mathrm{End}(\underline{10}^*) & = \underline{10} \otimes \underline{10}^* & = \underline{1} \oplus \underline{24} \oplus \underline{75} & \text{(12a)} \\
\mathrm{End}(\underline{5}) & = \mathrm{End}(\underline{5}^*) & = \underline{5} \otimes \underline{5}^* & = \underline{1} \oplus \underline{24} & \text{(12b)} \\
\mathrm{End}(\underline{1}) & & & = \underline{1} & \text{(12c)} \\
\mathrm{Hom}(\underline{5}, \underline{10}) & = \mathrm{Hom}(\underline{10}^*, \underline{5}^*) & = \underline{5}^* \otimes \underline{10} & = \underline{5} \oplus \underline{45}^* & \text{(12d)} \\
\mathrm{Hom}(\underline{5}, \underline{10}^*) & = \mathrm{Hom}(\underline{10}, \underline{5}^*) & = \underline{5}^* \otimes \underline{10}^* & = \underline{10} \oplus \underline{40}^* & \text{(12e)} \\
\mathrm{Hom}(\underline{5}^*, \underline{5}) & & = \underline{5} \otimes \underline{5} & = \underline{10} \oplus \underline{15} & \text{(12f)} \\
\mathrm{Hom}(\underline{10}^*, \underline{10}) & & = \underline{10} \otimes \underline{10} & = \underline{5}^* \oplus \underline{45} \oplus \underline{50} & \text{(12g)} \\
\mathrm{Hom}(\underline{1}, \underline{5}) & = \mathrm{Hom}(\underline{5}^*, \underline{1}) & & = \underline{5} & \text{(12h)} \\
\mathrm{Hom}(\underline{1}, \underline{10}) & = \mathrm{Hom}(\underline{10}^*, \underline{1}) & & = \underline{10} & \text{(12i)}
\end{array}$$

We identify the Lie algebra su(5) with its $\underline{24}$-representation. Then, we get a natural representation $\hat{\pi}$ of su(5) in $\mathrm{End}(\mathbb{C}^{192})$ by selecting the $\underline{24}$-representations in (12):

$$\hat{\pi}(a) := \left(\begin{array}{ccc|ccc} \pi_{10}(a) & 0 & 0 & & & \\ 0 & \overline{\pi_5(a)} & 0 & & \mathrm{O} & \\ 0 & 0 & 0_3 & & & \\ \hline & & & \overline{\pi_{10}(a)} & 0 & 0 \\ & \mathrm{O} & & 0 & \pi_5(a) & 0 \\ & & & 0 & 0 & 0_3 \end{array}\right) \otimes \mathbf{1}_6 \ . \tag{13}$$

Here, π_{10} and π_5 denote the embeddings of $\underline{24}$ into the r.h.s. of Eqs. (12).

We define the $\underline{75}$-representation of su(5) occurring in the decomposition (12a) as the set $\mathfrak{v}$ of 10×10-matrices of the form

$$\mathfrak{v} := \{ \ v \in \mathrm{su}(10) \ , \quad \mathrm{tr}(v\, \pi_{10}(a)) = 0 \ \ \forall a \in \mathfrak{a} \ \} \ . \tag{14}$$

The $\underline{5}$-representation is given by

$$\mathfrak{b} = \{ \ b = \mathrm{i}(b_1, b_2, b_3, b_4, b_5)^T \ , \quad b_i \in \mathbb{C} \ \} \ . \tag{15}$$

We define a linear map $\hat{\pi}$ of $\mathfrak{b}$ in $\mathrm{End}(\mathbb{C}^{192})$, putting

$$\hat{\pi}(b) = \left(\begin{array}{ccc|ccc} & & & \pi_{10,10}(b) & \pi_{10,5}(b) & 0 \\ & \mathrm{O} & & \pi_{10,5}(b)^T & 0 & \pi_{5,1}(b) \\ & & & 0 & \pi_{5,1}(b)^T & 0 \\ \hline -\pi_{10,10}(b)^* & -\overline{\pi_{10,5}(b)} & 0 & & & \\ -\pi_{10,5}(b)^* & 0 & -\overline{\pi_{5,1}(b)} & & \mathrm{O} & \\ 0 & -\pi_{5,1}(b)^* & 0 & & & \end{array}\right) \otimes \mathbf{1}_6 \ . \tag{16a}$$

The matrices $\pi_{10,10}(b), \pi_{10,5}(b)$ and $\pi_{5,1}(b)$ are the embeddings of $b \in \underline{5}$ into $\underline{10} \otimes \underline{10}$, $\underline{5}^* \otimes \underline{10}$ and $\underline{1} \otimes \underline{5}^*$, see (12). Observe that

$$[\hat{\pi}(a), \hat{\pi}(b)] = \hat{\pi}(ab) \in \hat{\pi}(\mathfrak{b}) \ , \quad a \in \mathfrak{a} \ , \quad b \in \mathfrak{b} \ . \tag{16b}$$

Due to the first three formulae in (12), the $\underline{24}$-parts and the $\underline{1}$-parts of $\pi_{i,j}(b)\pi_{i,j}(b)^*$, respectively, must be correlated. Indeed, we find with

$$(b,b)' := bb^* - \tfrac{1}{5}\,\mathrm{tr}(bb^*)\mathbf{1}_5 \in \mathrm{i}\,\mathfrak{a} \tag{17a}$$

the identities[16]

$$\begin{aligned} \pi_{10,10}(b)\pi_{10,10}(b)^* &= \mathrm{i}\,\pi_{10}(\mathrm{i}(b,b)') + \tfrac{3}{5}(b^*b)\mathbf{1}_{10} \ , \\ \pi_{10,5}(b)\pi_{10,5}(b)^* &= -\,\mathrm{i}\,\pi_{10}(\mathrm{i}(b,b)') + \tfrac{2}{5}(b^*b)\mathbf{1}_{10} \ , \\ \pi_{10,5}(b)^*\pi_{10,5}(b) &= \mathrm{i}\,\pi_5(\mathrm{i}(b,b)') + \tfrac{4}{5}(b^*b)\mathbf{1}_5 \ , \\ \overline{\pi_{5,1}(b)}\pi_{5,1}(b)^T &= -\,\mathrm{i}\,\pi_5(\mathrm{i}(b,b)') + \tfrac{1}{5}(b^*b)\mathbf{1}_5 \ , \\ \pi_{5,1}(b)^T\overline{\pi_{5,1}(b)} &= (b^*b) \ . \end{aligned} \tag{17b}$$

Moreover, we consider the $\underline{45}$-representation of su(5) occurring in (12d). It is the vector space $\mathfrak{w}$ of 10×5-matrices determined by

$$\mathfrak{w} := \{\; w \in \mathrm{Hom}(\mathbb{C}^5, \mathbb{C}^{10}) \;,\quad \mathrm{tr}(w\,\pi_{10,5}(b)^*) = 0 \;\; \forall b \in \underline{5} \;\} \;. \tag{18a}$$

One has

$$[a, w] := \pi_{10}(a) w - w \pi_5(a) \in \mathfrak{w} \;,\quad w \in \mathfrak{w} \,,\; a \in \mathfrak{a} \,. \tag{18b}$$

Finally, we consider the $\underline{50}$-representation of su(5) occurring in (12g). It is the vector space $\mathfrak{c}$ of symmetric complex 10×10-matrices determined by

$$\mathfrak{c} := \{\; c \in \mathrm{M}_{10}\mathbb{C} \;,\quad c = c^T \;,\quad \mathrm{tr}(c\,\pi_{10,10}(b)^*) = 0 \;\; \forall b \in \underline{5} \;\} \;. \tag{19a}$$

One has

$$[a, c] := \pi_{10}(a) c - c\overline{\pi_{10}(a)} \equiv \pi_{10}(a) c + c \pi_{10}(a)^T \in \mathfrak{c} \;,\quad a \in \mathfrak{a} \,,\; c \in \mathfrak{c} \;. \tag{19b}$$

3.2. *The mass matrix*

Now we define the mass matrix $\mathcal{M}$ of the L-cycle. Let $(E_{ij})_{k \times l}$, $i = 1, \dots, k$, $j = 1, \dots, l$, be the $k \times l$-standard matrix, whose entry at the intersection of the i^{th} row with the j^{th} column is one and whose other entries are zero. Since k, l will not exceed 10 throughout this paper, we write $i, j = 0$ instead of $i, j = 10$. Let

$$\begin{aligned} m \equiv -\mathrm{i}\,\pi_5(\mathrm{i}\,m) &:= \mathrm{diag}(-\tfrac{2}{5}, -\tfrac{2}{5}, -\tfrac{2}{5}, \tfrac{3}{5}, \tfrac{3}{5}) \in \mathrm{i}\,\mathfrak{a} \;, \\ \hat{m} := -\mathrm{i}\,\pi_{10}(\mathrm{i}\,m) &\equiv \mathrm{diag}(\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, -\tfrac{4}{5}, -\tfrac{4}{5}, -\tfrac{4}{5}, \tfrac{6}{5}) \;, \\ n \equiv \overline{-\mathrm{i}\,\pi_{5,1}(\mathrm{i}\,n)} &:= (E_{41})_{5 \times 1} \in \mathrm{i}\,\mathfrak{b} \;, \\ \hat{n} := -\mathrm{i}\,\pi_{10,5}(\mathrm{i}\,n) &\equiv (E_{11} + E_{22} + E_{33} - E_{05})_{10 \times 5} \;, \\ \check{n} := -\mathrm{i}\,\pi_{10,10}(\mathrm{i}\,n) &\equiv (-E_{47} - E_{58} - E_{69} - E_{74} - E_{85} - E_{96})_{10 \times 10} \;, \\ m' &:= (-E_{00})_{10 \times 10} \in \mathrm{i}\,\mathfrak{c} \;, \\ n' &:= (E_{11} + E_{22} + E_{33} + 3E_{05})_{10 \times 5} \in \mathrm{i}\,\mathfrak{w} \;, \\ \check{n}' := -\mathrm{i}\,\pi_{10,10}(\mathrm{i}\,n') &\equiv (2E_{47} + 2E_{58} + 2E_{69} - 2E_{74} - 2E_{85} - 2E_{96})_{10 \times 10} \;. \end{aligned} \tag{20}$$

Then we put

$$\mathcal{M} = \left(\begin{array}{ccc|ccc} \mathcal{M}_{10} & 0 & 0 & \mathcal{M}_{10,10} & \mathcal{M}_{10,5} & 0 \\ 0 & \overline{\mathcal{M}_5} & 0 & \mathcal{M}_{10,5}^T & 0 & \mathcal{M}_{5,1} \\ 0 & 0 & 0 & 0 & \mathcal{M}_{5,1}^T & 0 \\ \hline \mathcal{M}_{10,10}^* & \overline{\mathcal{M}_{10,5}} & 0 & \overline{\mathcal{M}_{10}} & 0 & 0 \\ \mathcal{M}_{10,5}^* & 0 & \overline{\mathcal{M}_{5,1}} & 0 & \mathcal{M}_5 & 0 \\ 0 & \mathcal{M}_{5,1}^* & 0 & 0 & 0 & 0 \end{array} \right) \;, \quad \text{where} \tag{21a}$$

$$\begin{aligned} \mathcal{M}_{10} &= -\hat{m} \otimes M_{10}' \;, & \mathcal{M}_5 &= m \otimes M_5' \;, \\ \mathcal{M}_{10,10} &= -\check{n} \otimes M_d' - m' \otimes M_N' \;, & \mathcal{M}_{5,1} &= -n \otimes M_e' \;, \\ \mathcal{M}_{10,5} &= -\hat{n} \otimes M_{\tilde{u}}' - n' \otimes M_{\tilde{n}}' \;. & & \end{aligned} \tag{21b}$$

Here, $M'_{10}, M'_5, M'_N, M'_{\tilde u}, M'_d, M'_e, M'_{\tilde n}$ are 6×6-matrices of the block structure

$$M'_i = \begin{pmatrix} 0_3 & M_i \\ M_i^* & 0_3 \end{pmatrix} , \qquad\qquad M'_f = \begin{pmatrix} M_f & 0_3 \\ 0_3 & M_f \end{pmatrix} , \tag{22}$$

for $i \in \{5,10\}$ and $f \in \{\tilde u, d, e, \tilde n, N\}$. We demand

$$M_d = M_d^T , \qquad\qquad M_N = M_N^T . \tag{23}$$

The final input of our L-cycle is the grading operator $\hat\Gamma$, which we choose as

$$\hat\Gamma = \begin{pmatrix} -\mathbf{1}_{16}\otimes\hat\Gamma' & 0_{96} \\ 0_{96} & \mathbf{1}_{16}\otimes\hat\Gamma' \end{pmatrix} , \qquad\qquad \hat\Gamma' = \begin{pmatrix} \mathbf{1}_3 & 0 \\ 0 & -\mathbf{1}_3 \end{pmatrix} . \tag{24}$$

One easily verifies $[\hat\Gamma, \pi(\mathfrak{a})] = \{\hat\Gamma, \mathcal{M}] = 0$. Thus, $(\mathfrak{a}, \mathbb{C}^{192}, \mathcal{M}, \hat\pi, \hat\Gamma)$ is an L-cycle.

Hence, we have in terms of 4×4-block matrices with entries in 48×48-matrices:

$$\hat\pi(a) = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & \bar A & 0 \\ 0 & 0 & 0 & \bar A \end{pmatrix} , \quad \mathcal{M} = \begin{pmatrix} 0 & \mathcal{M}_i & \mathcal{M}_f & 0 \\ \mathcal{M}_i^* & 0 & 0 & \mathcal{M}_f \\ \mathcal{M}_f^* & 0 & 0 & \overline{\mathcal{M}_i} \\ 0 & \mathcal{M}_f^* & \mathcal{M}_i^T & 0 \end{pmatrix} , \quad \hat\Gamma = \begin{pmatrix} -\mathbf{1}_{48} & 0 & 0 & 0 \\ 0 & \mathbf{1}_{48} & 0 & 0 \\ 0 & 0 & \mathbf{1}_{48} & 0 \\ 0 & 0 & 0 & -\mathbf{1}_{48} \end{pmatrix} ,$$

$$A := \mathrm{diag}\left(\pi_{10}(a)\otimes\mathbf{1}_3 \;,\; \overline{\pi_5(a)}\otimes\mathbf{1}_3 \;,\; 0_3\right) , \tag{25}$$

$$\mathcal{M}_i := \mathrm{diag}\left(-\hat m\otimes M_{10} \;,\; \overline{m\otimes M_5} \;,\; 0_3\right) ,$$

$$\mathcal{M}_f := \begin{pmatrix} -\check n\otimes M_d - m'\otimes M_N & -\hat n\otimes M_{\tilde u} - n'\otimes M_{\tilde n} & 0 \\ -\hat n^T\otimes M_{\tilde u}^T - n'^T\otimes M_{\tilde n}^T & 0 & -n\otimes M_e \\ 0 & -n^T\otimes M_e^T & 0 \end{pmatrix} \equiv \mathcal{M}_f^T .$$

3.3. *The structure of* $\hat\pi(\Omega^1\mathfrak{a})$ *and* $\hat\pi(\Omega^2\mathfrak{a})$

We recall (2a) that elements $\tau^1 \in \hat\pi(\Omega^1\mathfrak{a})$ are of the form

$$\tau^1 = \textstyle\sum_{\alpha, z\geq 0} [\hat\pi(a_\alpha^z), [\dots[\hat\pi(a_\alpha^1), [-\mathrm{i}\,\mathcal{M}, \hat\pi(a_\alpha^0)]]\dots]] . \tag{26}$$

Using (16b), (18b), (19b) and the fact that $\hat\pi$ is a representation we obtain the explicit structure of elements $\tau^1 \in \hat\pi(\Omega^1\mathfrak{a})$:

$$\tau^1 = \tag{27a}$$

$$\left(\begin{array}{ccc|ccc} \pi_{10}(a)\otimes M'_{10} & 0 & 0 & \begin{bmatrix}\pi_{10,10}(b)\otimes M'_d \\ +c\otimes M'_N\end{bmatrix} & \begin{bmatrix}\pi_{10,5}(b)\otimes M'_{\tilde u} \\ +w\otimes M'_{\tilde n}\end{bmatrix} & 0 \\ 0 & \overline{\pi_5(a)\otimes M'_5} & 0 & \begin{bmatrix}\pi_{10,5}(b)^T\otimes {M'_{\tilde u}}^T \\ +w^T\otimes {M'_{\tilde n}}^T\end{bmatrix} & 0 & \pi_{5,1}(b)\otimes M'_e \\ 0 & 0 & 0 & 0 & \pi_{5,1}(b)^T\otimes {M'_e}^T & 0 \\ \hline \begin{bmatrix}-\pi_{10,10}(b)^*\otimes {M'_d}^* \\ -c^*\otimes M'_N\end{bmatrix} & \begin{bmatrix}-\overline{\pi_{10,5}(b)\otimes M'_{\tilde u}} \\ -\overline{w\otimes M'_{\tilde n}}\end{bmatrix} & 0 & -\overline{\pi_{10}(a)\otimes M'_{10}} & 0 & 0 \\ \begin{bmatrix}-\pi_{10,5}(b)^*\otimes {M'_{\tilde u}}^* \\ -w^*\otimes {M'_{\tilde n}}^*\end{bmatrix} & 0 & -\overline{\pi_{5,1}(b)\otimes M'_e} & 0 & -\pi_5(a)\otimes M'_5 & 0 \\ 0 & -\pi_{5,1}(b)^*\otimes {M'_e}^* & 0 & 0 & 0 & 0 \end{array}\right),$$

$$a = \sum_{\alpha,z\geq 0}[a_\alpha^z,[\dots[a_\alpha^1,[(\mathrm{i}\,m),a_\alpha^0]]\dots]] \in \mathfrak{a} \;, \tag{27b}$$
$$b = \sum_{\alpha,z\geq 0} a_\alpha^z \cdots a_\alpha^1 a_\alpha^0(-\mathrm{i}\,n) \in \mathfrak{b} \;, \tag{27c}$$
$$w = \sum_{\alpha,z\geq 0}[a_\alpha^z,[\dots[a_\alpha^1,[(\mathrm{i}\,n'),a_\alpha^0]]\dots]] \in \mathfrak{w} \;, \tag{27d}$$
$$c = \sum_{\alpha,z\geq 0}[a_\alpha^z,[\dots[a_\alpha^1,[(\mathrm{i}\,m'),a_\alpha^0]]\dots]] \in \mathfrak{c} \;. \tag{27e}$$

Here, the commutators (27d) and (27e) are understood in the sense (18b) and (19b). It is obvious that a, b, c, w are independent as elements of different irreducible representations of su(5) .

Next, we are going to construct $\hat{\pi}(\Omega^2\mathfrak{a})$. According to (3), elements $\tau^2 \in \hat{\pi}(\Omega^2\mathfrak{a})$ are obtained by summing up elements of the type

$$\tau^2 := \tfrac{1}{2}\{\tau^1,\tau^1\} \;, \qquad \tau^1 \in \hat{\pi}(\Omega^1\mathfrak{a}) \;. \tag{28}$$

Thus, using (17b) we get from (27a) the structure

$$\tau^2 = \left(\begin{array}{ccc|ccc} \tau_{10} & \tau_{\widetilde{10,5}} & \tau_{10,1} & \tau_{10,10} & \tau_{10,5} & 0 \\ \tau^*_{\widetilde{10,5}} & \tau_5^T & 0 & \tau^T_{10,5} & 0 & \tau_{5,1} \\ \tau^*_{10,1} & 0 & \tau_1 & 0 & \tau^T_{5,1} & 0 \\ \hline \tau^*_{10,10} & \overline{\tau_{10,5}} & 0 & \tau^T_{10} & \overline{\tau_{\widetilde{10,5}}} & \overline{\tau_{10,1}} \\ \tau^*_{10,5} & 0 & \overline{\tau_{5,1}} & \tau^T_{\widetilde{10,5}} & \tau_5 & 0 \\ 0 & \tau^*_{5,1} & 0 & \tau^T_{10,1} & 0 & \tau_1^T \end{array}\right) \;, \quad \text{where} \tag{29a}$$

$$\begin{aligned}
\tau_{10} &= \mathrm{i}\,\pi_{10}(\mathrm{i}(b,b)') \otimes (M'_{\tilde{u}}M'^{\,*}_{\tilde{u}} - M'_d M'^{\,*}_d) - (b^*b)\mathbf{1}_{10} \otimes (\tfrac{2}{5}M'_{\tilde{u}}M'^{\,*}_{\tilde{u}} + \tfrac{3}{5}M'_d M'^{\,*}_d) \\
&\quad + \tfrac{1}{2}\{\pi_{10}(a),\pi_{10}(a)\} \otimes M'^{\,2}_{10} \\
&\quad - ww^* \otimes M'_{\tilde{n}}M'^{\,*}_{\tilde{n}} - w\pi_{10,5}(b)^* \otimes M'_{\tilde{n}}M'^{\,*}_{\tilde{u}} - \pi_{10,5}(b)w^* \otimes M'_{\tilde{u}}M'^{\,*}_{\tilde{n}} \\
&\quad - cc^* \otimes M'_N M'^{\,*}_N - c\pi_{10,10}(b)^* \otimes M'_N M'^{\,*}_d - \pi_{10,10}(b)c^* \otimes M'_d M'^{\,*}_N \\
\tau_5 &= \mathrm{i}\,\pi_5(\mathrm{i}(b,b)') \otimes (\bar{M}'_e M'^{\,T}_e - M'^{\,*}_{\tilde{u}} M'_{\tilde{u}}) - (b^*b)\mathbf{1}_5 \otimes (\tfrac{4}{5}M'^{\,*}_{\tilde{u}}M'_{\tilde{u}} + \tfrac{1}{5}\bar{M}'_e M'^{\,T}_e) \\
&\quad + \tfrac{1}{2}\{\pi_5(a),\pi_5(a)\} \otimes M'^{\,2}_5 \\
&\quad - w^*w \otimes M'^{\,*}_{\tilde{n}}M'_{\tilde{n}} - w^*\pi_{10,5}(b) \otimes M'^{\,*}_{\tilde{n}}M'_{\tilde{u}} - \pi_{10,5}(b)^*w \otimes M'^{\,*}_u M'_{\tilde{n}} \;, \\
\tau_1 &= -b^*b \otimes M'^{\,T}_e \bar{M}'_e \;, \\
\tau_{10,10} &= \pi_{10,10}(ab) \otimes \tfrac{1}{2}(M'_{10}M'_d + M'_d M'^{\,T}_{10}) \\
&\quad + (\pi_{10}(a)\pi_{10,10}(b) - \pi_{10,10}(b)\pi_{10}(a)^T) \otimes \tfrac{1}{2}(M'_{10}M'_d - M'_d M'^{\,T}_{10}) \\
&\quad + (\pi_{10}(a)c + c\pi_{10}(a)^T) \otimes \tfrac{1}{2}(M'_{10}M'_N + M'_N M'^{\,T}_{10}) \\
&\quad + (\pi_{10}(a)c - c\pi_{10}(a)^T) \otimes \tfrac{1}{2}(M'_{10}M'_N - M'_N M'^{\,T}_{10}) \;, \\
\tau_{10,5} &= \pi_{10}(a)\pi_{10,5}(b) \otimes M'_{10}M'_{\tilde{u}} - \pi_{10,5}(b)\pi_5(a) \otimes M'_{\tilde{u}}M'_5 \\
&\quad + \pi_{10}(a)w \otimes M'_{10}M'_{\tilde{n}} - w\pi_5(a) \otimes M'_{\tilde{n}}M'_5 \;, \\
\tau_{\widetilde{10,5}} &= -\pi_{10,10}(b)\overline{w} \otimes M'_d \bar{M}'_{\tilde{n}} - c\overline{w} \otimes M'_N \bar{M}'_{\tilde{n}} - c\overline{\pi_{10,5}(b)} \otimes M'_N \bar{M}'_{\tilde{u}} \;, \\
\tau_{5,1} &= \pi_{5,1}(ab) \otimes M'^{\,T}_5 M'_e \;, \qquad \tau_{10,1} = -w\overline{\pi_{5,1}(b)} \otimes M'_{\tilde{n}}\bar{M}'_e \;.
\end{aligned} \tag{29b}$$

3.4. *The structure of the connection form*

We know from (7) that for constructing the connection form ρ we need knowledge of the spaces $\boldsymbol{r}^0\mathfrak{a}$ and $\boldsymbol{r}^1\mathfrak{a}$ determined by Eqs. (5). To compute the structure of elements $\eta^0 \in \boldsymbol{r}^0\mathfrak{a}$ we first decompose η^0 according to (12) into irreducible su(5)-representations, each of them tensorized by $\mathrm{M}_6\mathbb{C}$. Then, the condition $[\boldsymbol{r}^0\mathfrak{a},\hat{\pi}(\mathfrak{a})] \subset \hat{\pi}(\mathfrak{a})$ yields the block structure

$$\eta^0 = \hat{\pi}(a) + \mathrm{i}\,\mathrm{diag}(\mathbf{1}_{10}\otimes m'_{10}\,,\; \mathbf{1}_5\otimes m'_{\bar 5}\,,\; m'_1\,,\; \mathbf{1}_{10}\otimes m'_{\widetilde{10}}\,,\; \mathbf{1}_5\otimes m'_5\,,\; m'_{\tilde 1})\,,$$

where $a \in \mathfrak{a}$ and $m'_{10,\tilde 5,1,\widetilde{10},5,\tilde 1}$ are selfadjoint elements of $\mathrm{M}_6\mathbb{C}$. The condition $\boldsymbol{r}^0\mathfrak{a} = \hat{\Gamma}(\boldsymbol{r}^0\mathfrak{a})\hat{\Gamma}$ implies $m'_i = \mathrm{diag}(m_i,\hat{m}_i)$, for $m_i,\hat{m}_i \in \mathrm{M}_3\mathbb{C}$.

We insert this structure into the condition $[\boldsymbol{r}^0\mathfrak{a},\hat{\pi}(\Omega^1\mathfrak{a})] \subset \hat{\pi}(\Omega^1\mathfrak{a})$. Using (27a), (16b), (18b) and (19b) we obtain from the off-diagonal blocks the equations

$$\begin{aligned}
m_{10}M_d - M_d m_{\widetilde{10}} &= -\mathrm{i}\,\bar\alpha M_d\ , & m_{10}M_N - M_N m_{\widetilde{10}} &= -\mathrm{i}\,\bar\alpha' M_N\ ,\\
m_{10}M_{\tilde u} - M_{\tilde u}m_5 &= -\mathrm{i}\,\alpha M_{\tilde u}\ , & m_{10}M_{\tilde n} - M_{\tilde n}m_5 &= -\mathrm{i}\,\alpha'' M_{\tilde n}\ ,\\
m_{\tilde 5}M^T_{\tilde u} - M^T_{\tilde u}m_{\widetilde{10}} &= -\mathrm{i}\,\alpha M^T_{\tilde u}\ , & m_{\tilde 5}M^T_{\tilde n} - M^T_{\tilde n}m_{\widetilde{10}} &= -\mathrm{i}\,\alpha'' M^T_{\tilde n}\ ,\\
m_{\tilde 5}M_e - M_e m_{\tilde 1} &= -\mathrm{i}\,\bar\alpha M_e\ , & m_1 M^T_e - M^T_e m_5 &= -\mathrm{i}\,\bar\alpha M^T_e\ ,
\end{aligned} \tag{30a}$$

for $\alpha,\alpha',\alpha'' \in \mathbb{C}$. The same equations hold for $\hat{m}_i$, with the same parameters α,α',α''. Multiplying the first equation by M^*_d from the right and subtracting the Hermitian conjugate of the resulting equation we get for instance

$$[m_{10}, M_dM^*_d] = -\mathrm{i}(\alpha+\bar\alpha)M_dM^*_d\,.$$

Applying the trace and respecting $\mathrm{tr}(M_dM^*_d) > 0$ we get $\alpha = \mathrm{i}\,\lambda$, for $\lambda \in \mathbb{R}$. Analogously, we have $\alpha' = \mathrm{i}\,\lambda'$ and $\alpha'' = \mathrm{i}\,\lambda''$. Thus, we find the equations

$$[m_{10}, M_dM^*_d] = [m_{10}, M_NM^*_N] = [m_{10}, M_{\tilde u}M^*_{\tilde u}] = [m_{10}, M_{\tilde n}M^*_{\tilde n}] = 0\ . \tag{30b}$$

For generic mass matrices $M_{d,N,\tilde u,\tilde n}$, these equations can only be satisfied for $m_{10} = (\nu - \frac{1}{2}\lambda)\mathbf{1}_3$, for $\nu \in \mathbb{R}$. We assume that $M_{d,\tilde u,e}$ are invertible and find the solution

$$\begin{aligned}
m_{10} &= (\nu-\tfrac{1}{2}\lambda)\mathbf{1}_3\ , & m_5 &= (\nu-\tfrac{3}{2}\lambda)\mathbf{1}_3\ , & m_1 &= (\nu-\tfrac{5}{2}\lambda)\mathbf{1}_3\ ,\\
m_{\widetilde{10}} &= (\nu+\tfrac{1}{2}\lambda)\mathbf{1}_3\ , & m_{\tilde 5} &= (\nu+\tfrac{3}{2}\lambda)\mathbf{1}_3\ , & m_{\tilde 1} &= (\nu+\tfrac{5}{2}\lambda)\mathbf{1}_3\ ,
\end{aligned} \tag{30c}$$

where $\nu,\lambda \in \mathbb{R}$. For $\hat{m}_i$ we get the same equations, with the same λ but possibly a different $\hat\nu$ instead of ν. Inserting this result into the π_{10}-block we get the equations

$$(\nu-\hat\nu)M_{10} = \beta M_{10}\ , \qquad\qquad (\nu-\hat\nu)M^*_{10} = -\beta M^*_{10}\ ,$$

which are only compatible with $\nu = \hat\nu$. Thus, we obtain the preliminary result

$$\eta^0 = \hat{\pi}(a) + \hat{\pi}(\mathrm{u}(1)) + \mathrm{i}\,\nu\mathbf{1}_{192}\ , \tag{31a}$$

$$\hat{\pi}(\mathrm{i}\,\lambda) := \mathrm{i}\,\lambda\ \mathrm{diag}(\,-\tfrac{1}{2}\mathbf{1}_{10}\ ,\ \tfrac{3}{2}\mathbf{1}_5\ ,\ -\tfrac{5}{2}\ ,\ \tfrac{1}{2}\mathbf{1}_{10}\ ,\ -\tfrac{3}{2}\mathbf{1}_5\ ,\ \tfrac{5}{2}\,)\otimes\mathbf{1}_6\ . \tag{31b}$$

Now, one finds[16] that the u(1)-part $\hat{\pi}(\mathrm{u}(1))$ is compatible with the two conditions $\{\boldsymbol{r}^0\mathfrak{a},\hat{\pi}(\mathfrak{a})\} \subset \{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\mathfrak{a})\} + \hat{\pi}(\Omega^2\mathfrak{a})$ and $\{\boldsymbol{r}^0\mathfrak{a},\hat{\pi}(\Omega^1\mathfrak{a})\} \subset \{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\Omega^1\mathfrak{a})\} + \hat{\pi}(\Omega^3\mathfrak{a})$, whereas the identity part $\mathrm{i}\,\nu\mathbf{1}_{192}$ is not. Here, one has to use the following identities:

$$\begin{aligned}
\mathrm{tr}(\pi_{10}(a)\,\pi_{10}(a)) &= \mathrm{tr}(\overline{\pi_{10}(a)}\,\overline{\pi_{10}(a)}) = 3\ \mathrm{tr}(aa)\ ,\\
\mathrm{tr}(\pi_5(a)\,\pi_5(a)) &= \mathrm{tr}(\overline{\pi_5(a)}\,\overline{\pi_5(a)}) \quad = \mathrm{tr}(aa)\ ,
\end{aligned} \tag{32a}$$

$$\mathrm{i}\{\pi_{10}(a),\pi_{10}(a)\}_{\underline{24}} = \tfrac{1}{3}\pi_{10}\big(\,\mathrm{i}\{\pi_5(a),\pi_5(a)\}_{\underline{24}}\big)\ , \tag{32b}$$

$$\begin{aligned}
(\pi_{10}(a)\pi_{10,5}(b))_{\underline{5}} &= \tfrac{3}{4}\pi_{10,5}(ab)\ , & (\pi_{10,5}(b)\pi_5(a))_{\underline{5}} &= -\tfrac{1}{4}\pi_{10,5}(ab)\ ,\\
(\pi_{10}(a)\pi_{10,5}(b))_{\underline{45}} &= (\pi_{10,5}(b)\pi_5(a))_{\underline{45}}\ , & (\pi_{10}(a)w)_{\underline{5}} &= (w\pi_5(a))_{\underline{5}}\ ,
\end{aligned} \tag{32c}$$

for $a \in \mathfrak{a}$, $b \in \mathfrak{b}$ and $w \in \mathfrak{w}$.

The evaluation of the formulae for $\boldsymbol{r}^1\mathfrak{a}$ in (5) yields for a generic choice of the mass matrices $M_{\tilde{u},d,e,\tilde{n},10,5}$ the simple result $\boldsymbol{r}^1\mathfrak{a} = \hat{\pi}(\Omega^1\mathfrak{a})$. Therefore, the connection form has the structure

$$\rho \in \big(\Lambda^1 \otimes (\hat{\pi}(\mathfrak{a}) + \hat{\pi}(\mathrm{u}(1)))\big) \oplus \big(\Lambda^0\gamma^5 \otimes \hat{\pi}(\Omega^1\mathfrak{a})\big) \;. \tag{33}$$

We see that our formalism generates an additional u(1)-part for the connection form and determines uniquely its representation (31b) on the fermionic Hilbert space. Remarkably, this representation is realized in nature!

3.5. *The ideal* $\boldsymbol{j}^2\mathfrak{a}$

We recall (4) that for the analysis of $\hat{\pi}(\mathcal{J}^2\mathfrak{a})$ we must find the space of elements $\hat{\sigma}(\omega^1)$, where $\omega^1 \in \Omega^1\mathfrak{a} \cap \ker\hat{\pi}$. For the computation of $\hat{\sigma}(\omega^1)$ we need knowledge of $\mathcal{M}^2$, see (2b). We define

$$\begin{aligned}
&v_0 := \mathrm{diag}(-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},1\,) \in \mathrm{i}\,\mathfrak{v}\;,\\
&I_3 \equiv -\mathrm{i}\,\pi_5(\mathrm{i}\,I_3) := \mathrm{diag}\,(0,0,0,\tfrac{1}{2},-\tfrac{1}{2})\;, \qquad\qquad (34a)\\
&\hat{I}_3 := -\mathrm{i}\,\pi_{10}(\mathrm{i}\,I_3) \equiv \mathrm{diag}\,(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},0,0,0,0)\\
&M'_u := M'_{\tilde{u}} + M'_{\tilde{n}}\;, \qquad\qquad M'_n := M'_{\tilde{u}} - 3M'_{\tilde{n}}\;, \qquad\qquad (34b)
\end{aligned}$$

analogously for the primeless matrices $M_{u,n,\tilde{u},\tilde{n}}$. Then, using (20) and (21), we find the following formula for $\mathcal{M}^2$:

$$\mathcal{M}^2 = \left(\begin{array}{ccc|ccc}
(\mathcal{M}^2)_{10} & (\mathcal{M}^2)_{\widetilde{10,5}} & 0 & (\mathcal{M}^2)_{10,10} & (\mathcal{M}^2)_{10,5} & 0\\
(\mathcal{M}^2)^*_{\widetilde{10,5}} & (\mathcal{M}^2)^T_5 & 0 & (\mathcal{M}^2)^T_{10,5} & 0 & (\mathcal{M}^2)_{5,1}\\
0 & 0 & (\mathcal{M}^2)_1 & 0 & (\mathcal{M}^2)^T_{5,1} & 0\\
\hline
(\mathcal{M}^2)^*_{10,10} & \overline{(\mathcal{M}^2)_{10,5}} & 0 & (\mathcal{M}^2)^T_{10} & \overline{(\mathcal{M}^2)_{\widetilde{10,5}}} & 0\\
(\mathcal{M}^2)^*_{10,5} & 0 & \overline{(\mathcal{M}^2)_{5,1}} & (\mathcal{M}^2)^T_{\widetilde{10,5}} & (\mathcal{M}^2)_5 & 0\\
0 & (\mathcal{M}^2)^*_{5,1} & 0 & 0 & 0 & (\mathcal{M}^2)^T_1
\end{array}\right), \tag{35a}$$

where

$$\begin{aligned}
(\mathcal{M}^2)_{10} &= \mathbf{1}_{10} \otimes (\tfrac{9}{25}{M'_{10}}^2 + \tfrac{4}{10}M'_{\tilde{u}}{M'_{\tilde{u}}}^* + \tfrac{6}{10}M'_d{M'_d}^* + \tfrac{12}{10}M'_{\tilde{n}}{M'_{\tilde{n}}}^* + \tfrac{1}{10}M'_N{M'_N}^*)\\
&\quad - v_0 \otimes ({M'_{10}}^2 - 2(M'_{\tilde{u}}{M'_{\tilde{n}}}^* + M'_{\tilde{n}}{M'_{\tilde{u}}}^*) + 4M'_{\tilde{n}}{M'_{\tilde{n}}}^* + \tfrac{1}{2}M'_N{M'_N}^*)\\
&\quad + (\tfrac{1}{2}\hat{m} - \hat{I}_3) \otimes (M'_u{M'_u}^* - M'_d{M'_d}^*)\\
&\quad + \tfrac{1}{3}\hat{m} \otimes (\tfrac{1}{5}{M'_{10}}^2 - 4(M'_{\tilde{u}}{M'_{\tilde{n}}}^* + M'_{\tilde{n}}{M'_{\tilde{u}}}^*) + 8M'_{\tilde{n}}{M'_{\tilde{n}}}^* + M'_N{M'_N}^*)\,,\\
(\mathcal{M}^2)_5 &= \mathbf{1}_5 \otimes (\tfrac{6}{25}{M'_5}^2 + \tfrac{12}{5}{M'_{\tilde{n}}}^*M'_{\tilde{n}} + \tfrac{4}{5}{M'_{\tilde{u}}}^*M'_{\tilde{u}} + \tfrac{1}{5}\bar{M}'_e{M'_e}^T)\\
&\quad + (\tfrac{1}{2}m - I_3) \otimes (\bar{M}'_e{M'_e}^T - {M'_n}^*M'_n) \qquad\qquad (35b)\\
&\quad + m \otimes (\tfrac{1}{5}{M'_5}^2 - 4({M'_{\tilde{u}}}^*M'_{\tilde{n}} + {M'_{\tilde{n}}}^*M'_{\tilde{u}}) + 8{M'_{\tilde{n}}}^*M'_{\tilde{n}})\,,\\
(\mathcal{M}^2)_1 &= {M'_e}^T\bar{M}'_e\;,\\
(\mathcal{M}^2)_{10,10} &= -\tfrac{3}{5}\check{n} \otimes \tfrac{1}{2}(M'_{10}M'_d + M_d{M'_{10}}^T)\\
&\quad - \tfrac{1}{2}\check{n}' \otimes \tfrac{1}{2}(M'_{10}M'_d - M_d{M'_{10}}^T) + \tfrac{12}{5}m' \otimes \tfrac{1}{2}(M'_{10}M'_N + M'_N\overline{M'_{10}})\,,
\end{aligned}$$

$$
\begin{aligned}
(\mathcal{M}^2)_{5,1} &= -\tfrac{3}{5} n \otimes M_5'^{\,T} M_e' \;, \\
(\mathcal{M}^2)_{10,5} &= \hat{n} \otimes \big(\tfrac{9}{20} M_{10}' M_{\tilde{u}}' + \tfrac{3}{20} M_{\tilde{u}}' M_5' - \tfrac{3}{4} M_{10}' M_{\tilde{n}}' + \tfrac{3}{4} M_{\tilde{n}}' M_5'\big) \\
&\qquad + n' \otimes \big(-\tfrac{1}{4} M_{10}' M_{\tilde{u}}' + \tfrac{1}{4} M_{\tilde{u}}' M_5' + \tfrac{19}{20} M_{10}' M_{\tilde{n}}' - \tfrac{7}{20} M_{\tilde{n}}' M_5'\big) \;, \\
(\mathcal{M}^2)_{\widetilde{10,5}} &= n'' \otimes M_N' \overline{M_n'} \;.
\end{aligned}
$$

Here, $n'' = (E_{05})_{10\times 5} \in \underline{40}$ is a generator of the $\underline{40}^*$-representation of su(5) occurring in the decomposition (12e).

Due to (4), the ideal $\hat{\pi}(\mathcal{J}^2\mathfrak{a})$ is given as the set of elements j_2 of the form

$$
j_2 = \sum\nolimits_{\alpha, z \geq 0} [\hat{\pi}(a_\alpha^z), [\ldots [\hat{\pi}(a_\alpha^1), [\mathcal{M}^2, \hat{\pi}(a_\alpha^0)]] \ldots]] \;, \quad \text{where} \tag{36a}
$$

$$
0 = \sum\nolimits_{\alpha, z \geq 0} [\hat{\pi}(a_\alpha^z), [\ldots [\hat{\pi}(a_\alpha^1), [-\mathrm{i}\,\mathcal{M}, \hat{\pi}(a_\alpha^0)]] \ldots]] \;. \tag{36b}
$$

Obviously, terms in $\mathcal{M}^2$ proportional to the identities $\mathbf{1}_{10}, \mathbf{1}_5, 1$ do not contribute to j_2 . Next, the term $(\mathcal{M}^2)_{5,1} = -\frac{3}{5} n \otimes M_5'^{\,T} M_e'$ gives a contribution to j_2 , which is $\frac{3\mathrm{i}}{5} \otimes M_5'^{\,T}$ times (from the left) the contribution of $-\mathrm{i}\,\mathcal{M}_{5,1} = \mathrm{i}\, n \otimes M_e'$ to (36b), and hence equals zero. For the same argument, all terms in $(\mathcal{M}^2)_{10,10}$ and $(\mathcal{M}^2)_{10,5}$ do not contribute to j_2 . The same is true for the terms proportional to $\hat{m}$ and m . Thus, there remain only contributions from the terms $(\frac{1}{2}\hat{m} - \hat{I}_3) \otimes M_{A,10}^2$, $(\frac{1}{2} m - I_3) \otimes M_{A,5}^2$, $-v_0 \otimes M_V^2$ and $n'' \otimes M_N' \overline{M_n'}$, where

$$
M_V^2 := M_{10}'^{\,2} - 2(M_{\tilde{u}}' M_{\tilde{n}}'^{\,*} + M_{\tilde{n}}' M_{\tilde{u}}'^{\,*} - 2 M_{\tilde{n}}' M_{\tilde{n}}'^{\,*}) \;, \tag{37a}
$$

$$
M_{A,10}^2 := M_u' M_u'^{\,*} - M_d' M_d'^{\,*} \;, \qquad M_{A,5}^2 := \bar{M}_e' M_e'^{\,T} - M_n'^{\,*} M_n' \;. \tag{37b}
$$

Since the irreducible representations $\underline{24}, \underline{75}, \underline{5}, \underline{45}^*, \underline{50}, \underline{40}^*$ are independent, it is always possible to fulfil (36b) and to generate by the commutators (36a) representations of arbitrary elements of $\underline{75}$ and $\underline{40}^*$. Moreover, it can be checked that the generator $\frac{1}{2} m - I_3$ occurring in $\mathcal{M}^2$ generates independent elements of the $\underline{24}$-representation. Hence, $j_2 \in J_2 := \hat{\pi}(\mathcal{J}^2\mathfrak{a})$ takes the form

$$
j_2 = \tag{38}
$$

$$
\left(\begin{array}{ccc|ccc}
\begin{bmatrix} \mathrm{i}\,\pi_{10}(a) \otimes M_{A,10}^2 \\ + \mathrm{i}\, v \otimes M_V^2 \end{bmatrix} & \mathrm{i}\, c'' \otimes M_N' \overline{M_n'} & 0 & & & \\
(\mathrm{i}\, c'' \otimes M_N' \overline{M_n'})^* & (\mathrm{i}\,\pi_5(a) \otimes M_{A,5}^2)^T & 0 & & \mathrm{O} & \\
0 & 0 & 0 & & & \\
\hline
& & & \begin{bmatrix} (\mathrm{i}\,\pi_{10}(a) \otimes M_{A,10}^2 \\ + \mathrm{i}\, v \otimes M_V^2)^T \end{bmatrix} & \overline{\mathrm{i}\, c'' \otimes M_N' \overline{M_n'}} & 0 \\
& \mathrm{O} & & (\mathrm{i}\, c'' \otimes M_N' \overline{M_n'})^T & \mathrm{i}\,\pi_5(a) \otimes M_{A,5}^2 & 0 \\
& & & 0 & 0 & 0
\end{array}\right) ,
$$

where $a \in \mathfrak{a}$, $v \in \mathfrak{v}$ and $c'' \in \underline{40}^*$.

Let $J_0 := \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}$. From (13) and (32) we conclude that elements $j_0 \in J_0$

are of the form

$$j_0 = \tag{39}$$

$$\left(\begin{array}{ccc|ccc} \frac{3}{5}\alpha\mathbf{1}_{10}+\frac{1}{3}\,\mathrm{i}\,\pi_{10}(a)+\mathrm{i}\,v & 0 & 0 & & & \\ 0 & \frac{2}{5}\alpha\mathbf{1}_5+\mathrm{i}\,\pi_5(a)^T & 0 & & \mathrm{O} & \\ 0 & 0 & 0 & & & \\ \hline & & & \frac{3}{5}\alpha\mathbf{1}_{10}+\frac{1}{3}\,\mathrm{i}\,\pi_{10}(a)^T+\mathrm{i}\,v^T & 0 & 0 \\ & \mathrm{O} & & 0 & \frac{2}{5}\alpha\mathbf{1}_5+\mathrm{i}\,\pi_5(a) & 0 \\ & & & 0 & 0 & 0 \end{array}\right) \otimes \mathbf{1}_6 ,$$

where $\alpha \in \mathbb{R}$, $a \in \mathfrak{a}$ and $v \in \mathfrak{v}$.

It remains to find the spaces $\boldsymbol{j}^0\mathfrak{a}, \boldsymbol{j}^1\mathfrak{a}, \boldsymbol{j}^2\mathfrak{a}$ occurring in (6). For generic mass matrices $M_{\tilde{u},d,e,\tilde{\nu},10,5}$ the result is[16]

$$\boldsymbol{j}^0\mathfrak{a} = \{0\} , \qquad \boldsymbol{j}^1\mathfrak{a} = \{0\} , \qquad \boldsymbol{j}^2\mathfrak{a} = \hat{\pi}(\mathcal{J}^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} + \mathbb{R}\mathbf{1}_{192} . \tag{40}$$

Let $J_3 := \mathbb{R}\mathbf{1}_{192}$. It is advantageous to construct an orthogonal decomposition $J_0 + J_2 + J_3 = (J_0 + J_3) \oplus J_2'$. The result is that elements $j_2' \in J_2'$ are of the form (38) with the replacements

$$\begin{aligned} M^2_{A,10} &\mapsto M^2_{ud} := M^2_{A,10} - \tfrac{1}{24}\,\mathrm{tr}(M^2_{A,10} + M^2_{A,5})\mathbf{1}_6 , \\ M^2_{A,5} &\mapsto M^2_{en} := M^2_{A,5} - \tfrac{1}{8}\,\mathrm{tr}(M^2_{A,10} + M^2_{A,5})\mathbf{1}_6 , \end{aligned} \tag{41a}$$

$$M^2_V \mapsto \tilde{M}^2_V := M^2_V - \tfrac{1}{6}\,\mathrm{tr}(M^2_V)\mathbf{1}_6 . \tag{41b}$$

3.6. *The factorization*

Due to (40), the problem of solving (9) is equivalent to finding for each given $\tau^2 \in \hat{\pi}(\Omega^2\mathfrak{a})$ an element $j \in J$ such that

$$\mathrm{tr}(\tilde{j}^{\,*}(\tau^2 + j)) = 0 , \quad \forall \tilde{j} \in J . \tag{42}$$

Due to the structure of J, the off-diagonal blocks $\tau_{i,j}$ except of $\tau_{\widetilde{10,5}}$ do not contribute to (42). In the parts $\pi_{10;5}(\mathrm{i}(b,b)')$ we can (and must) modulo J_2 replace

$$\begin{aligned} M'_{\tilde{u}}M'^{\,*}_{\tilde{u}} - M'_d M'^{\,*}_d &\mapsto M'_{\tilde{u}}M'^{\,*}_{\tilde{u}} - M'_u M'^{\,*}_u = -M'_{\tilde{u}}M'^{\,*}_{\tilde{n}} - M'_{\tilde{n}}M'^{\,*}_{\tilde{u}} - M'_{\tilde{n}}M'^{\,*}_{\tilde{n}} , \\ \bar{M}'_e M'^{\,T}_e - M'^{\,*}_{\tilde{u}}M'_{\tilde{u}} &\mapsto M'^{\,*}_n M'_n - M'^{\,*}_{\tilde{u}}M'_{\tilde{u}} = -3M'^{\,*}_{\tilde{u}}M'_{\tilde{n}} - 3M'^{\,*}_{\tilde{n}}M'_{\tilde{u}} + 9M'^{\,*}_{\tilde{n}}M'_{\tilde{n}} , \end{aligned} \tag{43}$$

see (34b). In the diagonal part of τ^2 in (29b) let us define

$$\begin{aligned} A^{10} &:= \tfrac{1}{2}\{\pi_{10}(a), \pi_{10}(a)\} , & A^5 &:= \tfrac{1}{2}\{\pi_5(a), \pi_5(a)\} , \\ B &:= -b^*b , & (b,b)' &:= bb^* - \tfrac{1}{5}\,\mathrm{tr}(bb^*)\mathbf{1}_5 , \\ U^{10} &:= -cc^* , & \tilde{U}^{10} &:= -c\pi_{10,10}(b)^* \\ \tilde{V}^{10} &:= -ww^* , & V^{10} &:= \tilde{V}^{10} - \mathrm{i}\,\pi_{10}(\mathrm{i}(b,b)') , \\ \tilde{V}^5 &:= -w^*w , & V^5 &:= \tilde{V}^5 + 9\,\mathrm{i}\,\pi_5(\mathrm{i}(b,b)') , \\ \tilde{W}^{10} &:= -\pi_{10,5}(b)w^* , & W^{10} &:= \tilde{W}^{10} - \mathrm{i}\,\pi_{10}(\mathrm{i}(b,b)') , \\ \tilde{W}^5 &:= -w^*\pi_{10,5}(b) , & W^5 &:= \tilde{W}^5 - 3\,\mathrm{i}\,\pi_5(\mathrm{i}(b,b)') . \end{aligned} \tag{44}$$

It is necessary to split $A^{10}, U^{10}, \tilde{U}^{10}, V^{10}, W^{10}$ according to (12a) and A^5, V^5, W^5 according to (12b) into irreducible components, the non-vanishing of which are

$$
\begin{array}{ll}
A^{10} = A^{10}_{\underline{1}} \oplus A^{10}_{\underline{24}} \oplus A^{10}_{\underline{75}} \;, & A^5 = A^5_{\underline{1}} \oplus A^5_{\underline{24}} \;, \\
U^{10} = U^{10}_{\underline{1}} \oplus U^{10}_{\underline{24}} \oplus U^{10}_{\underline{75}} \;, & \tilde{U}^{10} = \tilde{U}^{10}_{\underline{75}} \;, \\
V^{10} = V^{10}_{\underline{1}} \oplus V^{10}_{\underline{24}} \oplus V^{10}_{\underline{75}} \;, & V^5 = V^5_{\underline{1}} \oplus V^5_{\underline{24}} \;, \\
W^{10} = W^{10}_{\underline{24}} \oplus W^{10}_{\underline{75}} \;, & W^5 = W^5_{\underline{24}} \;.
\end{array}
\tag{45}
$$

For these components we find

$$
\begin{array}{lll}
A^{10}_{\underline{1}} = \frac{3}{10}\,\mathrm{tr}(A^5)\mathbf{1}_{10} \;, & A^5_{\underline{1}} = \frac{1}{5}\,\mathrm{tr}(A^5)\mathbf{1}_5 \;, & A^{10}_{\underline{75}} = A^{10} - A^{10}_{\underline{24}} - A^{10}_{\underline{1}} \;, \\
A^5_{\underline{24}} = A^5 - \frac{1}{5}\,\mathrm{tr}(A^5)\mathbf{1}_5 \;, & & A^{10}_{\underline{24}} = -\frac{1}{3}\,\mathrm{i}\,\pi_{10}(\mathrm{i}\,A^5_{\underline{24}}) \;, \\
U^{10}_{\underline{1}} = \frac{1}{10}\,\mathrm{tr}(U^{10})\mathbf{1}_{10} \;, & W^{10}_{\underline{75}} = W^{10} - W^{10}_{\underline{24}} \;, & U^{10}_{\underline{75}} \equiv U_{10} - U^{10}_{\underline{1}} - U^{10}_{\underline{24}} \;, \\
V^{10}_{\underline{1}} = \frac{1}{10}\,\mathrm{tr}(\tilde{V}^5)\mathbf{1}_{10} \;, & V^5_{\underline{1}} = \frac{1}{5}\,\mathrm{tr}(\tilde{V}^5)\mathbf{1}_5 \;, & V^{10}_{\underline{75}} = V^{10} - V^{10}_{\underline{24}} - V^{10}_{\underline{1}} \;, \\
V^5_{\underline{24}} = \tilde{V}^5 - \frac{1}{5}\,\mathrm{tr}(\tilde{V}^5)\mathbf{1}_5 + 9\,\mathrm{i}\,\pi_5(\mathrm{i}(b,b)') \;, & & V^{10}_{\underline{24}} = -\,\mathrm{i}\,\pi_{10}(\mathrm{i}\,\tilde{V}'_{\underline{24}} + \mathrm{i}(b,b)') \;, \\
W^{10}_{\underline{24}} + W^{10*}_{\underline{24}} = -\frac{1}{3}\,\mathrm{i}\,\pi_{10}(\mathrm{i}\,W^5 + \mathrm{i}\,W^{5*}) \;, & W^{10}_{\underline{24}} - W^{10*}_{\underline{24}} = \frac{1}{3}\pi_{10}(W^5 - W^{5*}) \;, &
\end{array}
\tag{46a}
$$

where, for $\{\beta_j\}_{j=1,\dots,24}$ being an orthonormal basis of $\mathfrak{a}$, i.e. $\mathrm{tr}(\beta_i\beta_j) = -\delta_{ij}$,

$$
\mathrm{i}\,\tilde{V}'_{\underline{24}} = -\tfrac{1}{3}\textstyle\sum_{j=1}^{24} \mathrm{tr}(\mathrm{i}\,\tilde{V}^{10}\pi_{10}(\beta_j))\beta_j \;. \tag{46b}
$$

Due to (37a) we can modulo J_2 replace $A^{10}_{\underline{75}} \otimes {M'_{10}}^2$ by

$$
A^{10}_{\underline{75}} \otimes (2M'_{\tilde{n}}{M'_{\tilde{u}}}^* + 2M'_{\tilde{u}}{M'_{\tilde{n}}}^* - 4M'_{\tilde{n}}{M'_{\tilde{n}}}^* - \tfrac{1}{2}M'_N{M'_N}^*) \;. \tag{47}
$$

Now, it is easy to determine the representative $\hat{\tau}^2 \in \tau^2 + J$ orthogonal to J, see (42):

$$
\begin{aligned}
\hat{\tau}_{10} &= \mathrm{tr}(A^5)\mathbf{1}_{10} \otimes \hat{M}^{10}_{aa} + \mathrm{tr}(U^{10})\mathbf{1}_{10} \otimes \hat{M}^{10}_{cc} + \mathrm{tr}(V^5)\mathbf{1}_{10} \otimes \hat{M}^{10}_{nn} + B\mathbf{1}_{10} \otimes \hat{M}^{10}_{bb} \\
&\quad - \tfrac{1}{3}\,\mathrm{i}\,\pi_{10}(\mathrm{i}\,A^5_{\underline{24}}) \otimes M^{10}_{aa} + U^{10}_{\underline{24}} \otimes M^{10}_{cc} + V^{10}_{\underline{24}} \otimes M^{10}_{nn} - \mathrm{i}\,\pi_{10}(\mathrm{i}\,V^5_{\underline{24}}) \otimes \check{M}^{10}_{nn} \\
&\quad - \tfrac{1}{3}\,\mathrm{i}\,\pi_{10}(\mathrm{i}\,W^5 + \mathrm{i}\,W^{5*}) \otimes M^{10}_{\{un\}} + \tfrac{1}{3}\,\mathrm{i}\,\pi_{10}(W^5 - W^{5*}) \otimes M^{10}_{[un]} \\
&\quad + (V^{10}_{\underline{75}} - 4A^{10}_{\underline{75}}) \otimes \tilde{M}^{10}_{nn} + (U^{10}_{\underline{75}} - \tfrac{1}{2}A^{10}_{\underline{75}}) \otimes \tilde{M}^{10}_{cc} + (\tilde{U}^{10}_{\underline{75}} + \tilde{U}^{10}_{\underline{75}}) \otimes \tilde{M}^{10}_{\{cd\}} \\
&\quad + \mathrm{i}(\tilde{U}^{10}_{\underline{75}} - \tilde{U}^{10}_{\underline{75}}) \otimes \tilde{M}^{10}_{[cd]} + (W^{10}_{\underline{75}} + W^{10*}_{\underline{75}} + 4A^{10}_{\underline{75}}) \otimes \tilde{M}^{10}_{\{un\}} \\
&\quad + \mathrm{i}(W^{10}_{\underline{75}} - W^{10*}_{\underline{75}}) \otimes \tilde{M}^{10}_{[un]} \\
\hat{\tau}_5 &= \mathrm{tr}(A^5)\mathbf{1}_5 \otimes \hat{M}^5_{aa} + \mathrm{tr}(U^{10})\mathbf{1}_5 \otimes \hat{M}^5_{cc} + \mathrm{tr}(V^5)\mathbf{1}_5 \otimes \hat{M}^5_{nn} + B\mathbf{1}_5 \otimes \hat{M}^5_{bb} \\
&\quad + A^5_{\underline{24}} \otimes M^5_{aa} + V^5_{\underline{24}} \otimes \check{M}^5_{nn} - \mathrm{i}\,\pi^{-1}_{10}(\mathrm{i}\,V^{10}_{\underline{24}}) \otimes M^5_{nn} - \mathrm{i}\,\pi^{-1}_{10}(\mathrm{i}\,U^{10}_{\underline{24}}) \otimes M^5_{cc} \\
&\quad + (W^5 + W^{5*}) \otimes M^5_{\{un\}} + \mathrm{i}(W^5 - W^{5*}) \otimes M^5_{[un]} \\
\hat{\tau}_1 &= \mathrm{tr}(A^5) \otimes \hat{M}^1_{aa} + \mathrm{tr}(U^{10}) \otimes \hat{M}^1_{cc} + B \otimes \hat{M}^1_{bb} + \mathrm{tr}(V^5) \otimes \hat{M}^1_{nn} \;, \\
\hat{\tau}_{\widetilde{10,5}} &= -(\pi_{10,10}(b)\overline{w})_{\underline{10}} \otimes M'_d\overline{M_{\tilde{n}'}} - (c\overline{w})_{\underline{10}} \otimes M'_N\overline{M_{\tilde{n}'}} \\
&\quad - (\pi_{10,10}(b)\overline{w})_{\underline{40}} \otimes M'_{d\tilde{n}} - (\tfrac{1}{4}(c\overline{w})_{\underline{40}} + \tfrac{3}{4}c\overline{\pi_{10,5}(b)}) \otimes M'_{Nu} \;,
\end{aligned}
\tag{48}
$$

where the matrices $M^k_{ij}, \tilde{M}^k_{ij}, \hat{M}^k_{ij}$ and $\check{M}^k_{ij}$ are given in Appendix A. The remaining matrix elements $\hat{\tau}_{i,j}$ coincide with $\tau_{i,j}$ given in (29).

The last step before including the function algebra is to apply the map $\hat\sigma\circ\hat\pi^{-1}$ defined in (2b) to elements $\tau^1\in\hat\pi(\Omega^1\mathfrak{a})$. Calculating $\hat\sigma\circ\hat\pi^{-1}(\tau^1)$ means to calculate j_2 in (36a), however with the r.h.s. of (36b) equal to the given element τ^1 and not equal to zero. We have listed the matrix elements of $\mathcal{M}^2$ in (35). Again, terms in $\mathcal{M}^2$ proportional to the identities $\mathbf{1}_{10},\mathbf{1}_5,1$ do not contribute to j_2 . Next, the terms proportional to $-v_0$, $(\frac{1}{2}\hat m-\hat I_3)$, $(\frac{1}{2}m-I_3)$ and n'' contribute to the ideal $\hat\pi(\mathcal{J}^2\mathfrak{a})$, as explained above. Since we regard $\hat\sigma\circ\hat\pi^{-1}(\tau^1)$ modulo $\hat\pi(\mathcal{J}^2\mathfrak{a})$, it is not necessary to consider these terms. Therefore, there remain only the terms

$$\hat m\otimes\tfrac{1}{3}(\tfrac{1}{5}{M'_{10}}^2-4M'_{\tilde u}{M'_{\tilde n}}^*-4M'_{\tilde n}{M'_{\tilde u}}^*+8M'_{\tilde n}{M'_{\tilde n}}^*+M'_N{M'_N}^*)\;, \tag{49a}$$
$$m\otimes(\tfrac{1}{5}{M'_5}^2-4{M'_{\tilde n}}^*M'_{\tilde u}-4{M'_{\tilde u}}^*M'_{\tilde n}+8{M'_{\tilde n}}^*M'_{\tilde n}) \tag{49b}$$

in the diagonal blocks $(\mathcal{M}^2)_{10}$ and $(\mathcal{M}^2)_5$ as well as the off-diagonal blocks $(\mathcal{M}^2)_{i,j}$, which give a contribution to $\hat\sigma\circ\hat\pi^{-1}(\tau^1)$. As we have already noticed, the contribution of $(\mathcal{M}^2)_{5,1}$ is $\frac{3\mathrm{i}}{5}\otimes{M'_5}^T$ times the contribution of $\mathrm{i}\,n\otimes M'_e$ to (36b). We get analogous contributions from the other terms $(\mathcal{M}^2)_{i,j}$ and $(\mathcal{M}^2)_i$. Thus, we obtain in the same notations as in (27a) the formula

$$\hat\sigma\circ\hat\pi^{-1}(\tau^1)=\left(\begin{array}{ccc|ccc} \sigma_{10} & 0 & 0 & \sigma_{10,10} & \sigma_{10,5} & 0\\ 0 & \sigma_5^T & 0 & \sigma_{10,5}^T & 0 & \sigma_{5,1}\\ 0 & 0 & 0 & 0 & \sigma_{5,1}^T & 0\\ \hline \sigma_{10,10}^* & \overline{\sigma_{10,5}} & 0 & \sigma_{10}^T & 0 & 0\\ \sigma_{10,5}^* & 0 & \overline{\sigma_{5,1}} & 0 & \sigma_5 & 0\\ 0 & \sigma_{5,1}^* & 0 & 0 & 0 & 0 \end{array}\right)\,,\quad\text{where} \tag{50}$$

$$\begin{aligned}
\sigma_{10}&=-\mathrm{i}\,\pi_{10}(a)\otimes\tfrac{1}{3}(\tfrac{1}{5}{M'_{10}}^2-4M'_{\tilde u}{M'_{\tilde n}}^*-4M'_{\tilde n}{M'_{\tilde u}}^*+8M'_{\tilde n}{M'_{\tilde n}}^*+M'_N{M'_N}^*)\;,\\
\sigma_5&=-\mathrm{i}\,\pi_5(a)\otimes(\tfrac{1}{5}{M'_5}^2-4{M'_{\tilde n}}^*M'_{\tilde u}-4{M'_{\tilde u}}^*M'_{\tilde n}+8{M'_{\tilde n}}^*M'_{\tilde n})\;,\\
\sigma_{10,10}&=\tfrac{3\mathrm{i}}{5}\pi_{10,10}(b)\otimes\tfrac{1}{2}(M'_{10}M'_d+M'_d{M'_{10}}^T)\\
&\quad+\tfrac{\mathrm{i}}{2}\pi_{10,10}(w)\otimes\tfrac{1}{2}(M'_{10}M'_d-M'_d{M'_{10}}^T)-\tfrac{12\,\mathrm{i}}{5}c\otimes\tfrac{1}{2}(M'_{10}M'_N+M'_N\overline{M'_{10}})\;,\\
\sigma_{5,1}&=\tfrac{3\mathrm{i}}{5}\pi_{5,1}(b)\otimes{M'_5}^TM'_e\;,\\
\sigma_{10,5}&=\mathrm{i}\,\pi_{10,5}(b)\otimes(-\tfrac{9}{20}M'_{10}M'_{\tilde u}-\tfrac{3}{20}M'_{\tilde u}M'_5+\tfrac{3}{4}M'_{10}M'_{\tilde n}-\tfrac{3}{4}M'_{\tilde n}M'_5)\\
&\quad+\mathrm{i}\,w\otimes(\tfrac{1}{4}M'_{10}M'_{\tilde u}-\tfrac{1}{4}M'_{\tilde u}M'_5-\tfrac{19}{20}M'_{10}M'_{\tilde n}+\tfrac{7}{20}M'_{\tilde n}M'_5)\;.
\end{aligned}$$

Now, it remains to perform the factorization in the diagonal blocks (49a) and (49b). The same method as before yields that the representatives orthogonal to J are

$$(49\text{a})\mapsto-\tfrac{1}{3}\,\mathrm{i}\,\pi_{10}(a)\otimes(\tfrac{1}{5}M^{10}_{aa}-8M^{10}_{\{un\}}+8M^{10}_{nn}+24\check M^{10}_{nn}+M^{10}_{cc})\;, \tag{51a}$$
$$(49\text{b})\mapsto-\mathrm{i}\,\pi_5(a)\otimes(\tfrac{1}{5}M^5_{aa}-8M^5_{\{un\}}+\tfrac{8}{3}M^5_{nn}+8\check M^5_{nn}+\tfrac{1}{3}M^5_{cc})\;. \tag{51b}$$

4. The Action of the Unification Model

4.1. *The curvature*

Now we are able to construct the bosonic action of the flipped $\mathrm{SU}(5)\times\mathrm{U}(1)$-grand unification model. We choose X to be a four dimensional Riemannian spin manifold. When using a local basis $\{\gamma^\mu\}_{\mu=1,2,3,4}$ of Λ^1 then the basis elements γ^μ are selfadjoint as complex sections of the Clifford bundle. Elements of Λ^1 are locally represented by real linear combinations of $\{\gamma^\mu\}_{\mu=1,2,3,4}$. The grading operator is $\gamma^5=\gamma^1\gamma^2\gamma^3\gamma^4$.

The first step is to write down the connection form ρ , which has according to (33) the structure

$$
\begin{aligned}
&\rho=\pi(A)+\pi(A'')+\gamma^5\pi(H)\ ,\\
&A\in\Lambda^1\otimes\mathrm{su}(5)\ ,\quad A''\in\Lambda^1\otimes\mathrm{u}(1)\ ,\quad H\in\Lambda^0\otimes\Omega^1\mathfrak{a}\ .
\end{aligned}
\tag{52}
$$

Here, γ^5 acts componentwise and $\pi=\mathrm{id}\otimes\hat\pi$, where the matrix parts of $\pi(A)$ and $\pi(A'')$ are given by (13) and (31b), respectively. Elements of $\hat\pi(\Omega^1\mathfrak{a})$ are specified by elements of $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ and $\mathfrak{w}$, see (27). Thus, we consider H as a sum

$$
\begin{aligned}
&H=\mathrm{i}(\Psi+\Phi+\Xi+\Upsilon)\ ,\\
&\Psi\in\Lambda^0\otimes\mathrm{i}\,\mathfrak{a}\ ,\quad\Phi\in\Lambda^0\otimes\mathrm{i}\,\mathfrak{b}\ ,\quad\Xi\in\Lambda^0\otimes\mathrm{i}\,\mathfrak{c}\ ,\quad\Upsilon\in\Lambda^0\otimes\mathrm{i}\,\mathfrak{w}\ .
\end{aligned}
\tag{53}
$$

Inserting (52) and (53) into Eq. (8) for the curvature, we find with (50)

$$
\begin{aligned}
\theta=\;&\mathbf{d}\pi(A)+\mathbf{d}\pi(A'')+\tfrac12\{\pi(A),\pi(A)\}\\
&-\gamma^5\big(\mathbf{d}\pi(\mathrm{i}(\Psi+\Phi+\Xi+\Upsilon))+[\pi(A)+\pi(A''),\pi(\mathrm{i}(\Psi+\Phi+\Xi+\Upsilon))-\mathrm{i}\,\mathcal{M}]\big)\\
&+\big(\tfrac12\{\pi(\mathrm{i}(\Psi+\Phi+\Xi+\Upsilon)),\pi(\mathrm{i}(\Psi+\Phi+\Xi+\Upsilon))\}\\
&+\{\pi(\mathrm{i}(\Psi+\Phi+\Xi+\Upsilon)),-\mathrm{i}\,\mathcal{M}\}+\hat\sigma_{\mathfrak{g}}(\rho)\mod\Lambda^0\otimes\boldsymbol{j}^2\mathfrak{a}\big)\ ,
\end{aligned}
\tag{54a}
$$

where

$$
\begin{aligned}
\hat\sigma_{\mathfrak{g}}(\rho):=&-\tfrac{12\,\mathrm{i}}{5}\pi(\mathrm{i}\,\Xi)\otimes\tfrac12(M'_{10}M'_N+M'_NM'^{\,T}_{10}))\\
&+\tfrac{\mathrm{i}}{2}\pi(\pi_{10,10}(\mathrm{i}\,\Upsilon)\otimes\tfrac12(M'_{10}M'_d-M'_dM'^{\,T}_{10}))\\
&+\tfrac{3\mathrm{i}}{5}\pi(\pi_{10,10}(\mathrm{i}\,\Phi))\otimes\tfrac12(M'_{10}M'_d+M'_dM'^{\,T}_{10})+\tfrac{3\mathrm{i}}{5}\pi(\pi_{5,1}(\mathrm{i}\,\Phi)\otimes M'^{\,T}_5M'_e)\\
&-\mathrm{i}\,\pi(\pi_{10,5}(\mathrm{i}\,\Phi)\otimes(\tfrac{9}{20}M'_{10}M'_{\check u}+\tfrac{3}{20}M'_{\check u}M'_5-\tfrac34M'_{10}M'_{\check n}+\tfrac34M'_{\check n}M'_5))\\
&-\mathrm{i}\,\pi(\mathrm{i}\,\Upsilon\otimes(-\tfrac14M'_{10}M'_{\check u}+\tfrac14M'_{\check u}M'_5+\tfrac{19}{20}M'_{10}M'_{\check n}-\tfrac{7}{20}M'_{\check n}M'_5))\\
&-\tfrac13\,\mathrm{i}\,\pi(\pi_{10}(\mathrm{i}\,\Psi)\otimes(\tfrac15M^{10}_{aa}-8M^{10}_{\{un\}}+8M^{10}_{nn}+24\check M^{10}_{nn}+M^{10}_{cc}))\\
&-\mathrm{i}\,\pi(\pi_5(\mathrm{i}\,\Psi)\otimes(\tfrac15M^5_{aa}-8M^5_{\{un\}}+\tfrac83M^5_{nn}+8\check M^5_{nn}+\tfrac13M^5_{cc}))\ .
\end{aligned}
\tag{54b}
$$

Here we have denoted by π the embedding of the selected matrix elements of (50) into the matrix (50). We have

$$
\begin{aligned}
&\tfrac12\{\pi(\mathrm{i}(\Psi+\Phi+\Xi+\Upsilon)),\pi(\mathrm{i}(\Psi+\Phi+\Xi+\Upsilon))\}+\{\pi(\mathrm{i}(\Psi+\Phi+\Xi+\Upsilon)),-\mathrm{i}\,\mathcal{M}\}\\
&\quad=\tfrac12\{\pi(\mathrm{i}(\tilde\Psi+\tilde\Phi+\tilde\Xi+\tilde\Upsilon)),\pi(\mathrm{i}(\tilde\Psi+\tilde\Phi+\tilde\Xi+\tilde\Upsilon))\}+\mathcal{M}^2\ ,\quad\text{where}
\end{aligned}
\tag{55a}
$$

$$
\tilde\Psi:=\Psi+m\ ,\qquad\tilde\Phi:=\Phi+n\ ,\qquad\tilde\Xi:=\Xi+m'\ ,\qquad\tilde\Upsilon:=\Upsilon+n'\ .
\tag{55b}
$$

Let

$$
\hat\sigma_{\mathfrak{g}}(\tilde\rho):=\text{ Eq. (54b) with }\ \Psi\mapsto\tilde\Psi\ ,\ \ \Phi\mapsto\tilde\Phi\ ,\ \ \Xi\mapsto\tilde\Xi\ ,\ \ \Upsilon\mapsto\tilde\Upsilon\ .
\tag{56}
$$

Then we obtain from (54a) and (35)

$$
\begin{aligned}
\theta = {} & \mathbf{d}\pi(A) + \tfrac{1}{2}\{\pi(A), \pi(A)\} + \mathbf{d}\pi(A'') \\
& - \gamma^5 \big(\mathbf{d}\pi(\mathrm{i}(\tilde{\Psi} + \tilde{\Phi} + \tilde{\Xi} + \tilde{\Upsilon})) + [\pi(A) + \pi(A''), \pi(\mathrm{i}(\tilde{\Psi} + \tilde{\Phi} + \tilde{\Xi} + \tilde{\Upsilon}))]\big) \\
& + \big(\tfrac{1}{2}\{\pi(\mathrm{i}(\tilde{\Psi} + \tilde{\Phi} + \tilde{\Xi} + \tilde{\Upsilon})), \pi(\mathrm{i}(\tilde{\Psi} + \tilde{\Phi} + \tilde{\Xi} + \tilde{\Upsilon}))\} + \hat{\sigma}_{\mathfrak{g}}(\tilde{\rho}) \\
& + \mathrm{diag}\,\big(\mathbf{1}_{10} \otimes (\tfrac{6}{5}\hat{M}^{10}_{aa} + \hat{M}^{10}_{bb} + 12\hat{M}^{10}_{nn} + \hat{M}^{10}_{cc})\,,\; \mathbf{1}_5 \otimes (\tfrac{6}{5}\hat{M}^{5}_{aa} + \hat{M}^{5}_{bb} + 12\hat{M}^{5}_{nn} + \hat{M}^{5}_{cc})^T, \\
& \qquad \tfrac{6}{5}\hat{M}^{1}_{aa} + \hat{M}^{1}_{bb} + 12\hat{M}^{1}_{nn} + \hat{M}^{1}_{cc}\,,\; \mathbf{1}_{10} \otimes (\tfrac{6}{5}\hat{M}^{10}_{aa} + \hat{M}^{10}_{bb} + 12\hat{M}^{10}_{nn} + \hat{M}^{10}_{cc})^T\,, \\
& \qquad \mathbf{1}_5 \otimes (\tfrac{6}{5}\hat{M}^{5}_{aa} + \hat{M}^{5}_{bb} + 12\hat{M}^{5}_{nn} + \hat{M}^{5}_{cc})\,,\; (\tfrac{6}{5}\hat{M}^{1}_{aa} + \hat{M}^{1}_{bb} + 12\hat{M}^{1}_{nn} + \hat{M}^{1}_{cc})^T \,\big) \\
& \qquad\qquad \mathrm{mod}\ \Lambda^0 \otimes \boldsymbol{j}^2\mathfrak{a}\big)\,.
\end{aligned}
\tag{57}
$$

We define

$$
\begin{aligned}
&\hat{\tilde{\Psi}} := -\mathrm{i}\,\pi_{10}(\mathrm{i}\,\tilde{\Psi})\,, && \hat{\tilde{\Upsilon}} := -\mathrm{i}\,\pi_{10,10}(\mathrm{i}\,\Upsilon)\,, && \hat{\tilde{\Phi}} := -\mathrm{i}\,\pi_{10,5}(\mathrm{i}\,\tilde{\Phi})\,, \\
&\hat{A} := \pi_{10}(A)\,, && \check{\tilde{\Phi}} := -\mathrm{i}\,\pi_{10,10}(\mathrm{i}\,\tilde{\Phi})\,, && (\tilde{\Xi}\tilde{\Xi}^*)' := -\mathrm{i}\,\pi_{10}^{-1}(\mathrm{i}(\tilde{\Xi}\tilde{\Xi}^*)_{\underline{24}})\,.
\end{aligned}
\tag{58}
$$

Using (48) and (29) we obtain the following matrix representation of $\mathfrak{e}(\theta)$:

$$
\mathfrak{e}(\theta) = \left(\begin{array}{ccc|ccc}
\theta_{10} & \theta_{\widetilde{10,5}} & \theta_{10,1} & \theta_{10,10} & \theta_{10,5} & 0 \\
\theta^*_{\widetilde{10,5}} & \theta^T_5 & 0 & \theta^T_{10,5} & 0 & \theta_{5,1} \\
\theta^*_{10,1} & 0 & \theta_1 & 0 & \theta^T_{5,1} & 0 \\
\hline
\theta^*_{10,10} & \overline{\theta_{10,5}} & 0 & \theta^T_{10} & \overline{\theta_{\widetilde{10,5}}} & \overline{\theta_{10,1}} \\
\theta^*_{10,5} & 0 & \overline{\theta_{5,1}} & \theta^T_{\widetilde{10,5}} & \theta_5 & 0 \\
0 & \theta^*_{5,1} & 0 & \theta^T_{10,1} & 0 & \theta^T_1
\end{array}\right), \quad \text{where}
\tag{59a}
$$

$$
\begin{aligned}
\theta_{10} &= (\mathbf{d}\hat{A} + \tfrac{1}{2}\{\hat{A}, \hat{A}\}) \otimes \mathbf{1}_6 - \tfrac{1}{2}\mathbf{d}A''\mathbf{1}_{10} \otimes \mathbf{1}_6 - \mathrm{i}\,\gamma^5(\mathbf{d}\hat{\tilde{\Psi}} + [\hat{A}, \hat{\tilde{\Psi}}]) \otimes M'_{10} \\
&\quad + q_1\mathbf{1}_{10} \otimes \hat{M}^{10}_{aa} + q_2\mathbf{1}_{10} \otimes \hat{M}^{10}_{bb} + q_3\mathbf{1}_{10} \otimes \hat{M}^{10}_{nn} + q_4\mathbf{1}_{10} \otimes \hat{M}^{10}_{cc} \\
&\quad + \tfrac{1}{3}\,\mathrm{i}\,\pi_{10}(\mathrm{i}\,\tilde{q}_1) \otimes M^{10}_{aa} + \mathrm{i}\,\pi_{10}(\mathrm{i}\,\tilde{q}_2) \otimes \check{M}^{10}_{nn} + \mathrm{i}\,\pi_{10}(\mathrm{i}\,\tilde{q}_3) \otimes M^{10}_{nn} \\
&\quad + \tfrac{1}{3}\,\mathrm{i}\,\pi_{10}(\mathrm{i}\,\tilde{q}_4) \otimes M^{10}_{\{un\}} + \tfrac{1}{3}\,\mathrm{i}\,\pi_{10}(\mathrm{i}\,\tilde{q}_5) \otimes M^{10}_{[un]} + \mathrm{i}\,\pi_{10}(\mathrm{i}\,\tilde{q}_6) \otimes M^{10}_{cc} \\
&\quad - \hat{q}_1 \otimes \tilde{M}^{10}_{nn} - \hat{q}_2 \otimes \tilde{M}^{10}_{\{un\}} - \hat{q}_3 \otimes \tilde{M}^{10}_{[un]} - \hat{q}_4 \otimes \tilde{M}^{10}_{cc} - \hat{q}_5 \otimes \tilde{M}^{10}_{\{cd\}} - \hat{q}_6 \otimes \tilde{M}^{10}_{[cd]}\,, \\
\theta_5 &= (\mathbf{d}A + \tfrac{1}{2}\{A, A\}) \otimes \mathbf{1}_6 - \tfrac{3}{2}\mathbf{d}A''\mathbf{1}_5 \otimes \mathbf{1}_6 + \mathrm{i}\,\gamma^5\pi_5(\mathbf{d}\tilde{\Psi} + [A, \tilde{\Psi}]) \otimes M'_5 \\
&\quad + q_1\mathbf{1}_5 \otimes \hat{M}^5_{aa} + q_2\mathbf{1}_5 \otimes \hat{M}^5_{bb} + q_3\mathbf{1}_5 \otimes \hat{M}^5_{nn} + q_4\mathbf{1}_5 \otimes \hat{M}^5_{cc} \\
&\quad - \tilde{q}_1 \otimes M^5_{aa} - \tilde{q}_2 \otimes \check{M}^5_{nn} - \tilde{q}_3 \otimes M^5_{nn} - \tilde{q}_4 \otimes M^5_{\{un\}} - \tilde{q}_5 \otimes M^5_{[un]} - \tilde{q}_6 \otimes M^5_{cc} \\
\theta_1 &= -\tfrac{5}{2}\mathbf{d}A''\mathbf{1}_6 + q_1 \otimes \hat{M}^1_{aa} + q_2 \otimes \hat{M}^1_{bb} + q_3 \otimes \hat{M}^1_{nn} + q_4 \otimes \hat{M}^1_{cc}\,, \qquad\qquad (59\mathrm{b}) \\
\theta_{10,10} &= \mathrm{i}\,\gamma^5(\mathbf{d}\hat{\tilde{\Phi}} + \hat{A}\hat{\tilde{\Phi}} + \hat{\tilde{\Phi}}\hat{A}^T - A''\hat{\tilde{\Phi}}) \otimes M'_d + \mathrm{i}\,\gamma^5(\mathbf{d}\tilde{\Xi} + \hat{A}\tilde{\Xi} + \tilde{\Xi}\hat{A}^T - A''\tilde{\Xi}) \otimes M'_N \\
&\quad - \pi_{10,10}(r) \otimes \tfrac{1}{2}(M'_{10}M'_d + M'_d{M'_{10}}^T) - \hat{r} \otimes \tfrac{1}{2}(M'_{10}M'_N + M'_N{M'_{10}}^T) \\
&\quad - \tilde{r}_1 \otimes \tfrac{1}{2}(M'_{10}M'_d - M'_d{M'_{10}}^T) - \tilde{r}_2 \otimes \tfrac{1}{2}(M'_{10}M'_N - M'_N{M'_{10}}^T)\,, \\
\theta_{10,5} &= -\mathrm{i}\,\gamma^5(\mathbf{d}\hat{\tilde{\Phi}} + \hat{A}\hat{\tilde{\Phi}} - \hat{\tilde{\Phi}}A + A''\hat{\tilde{\Phi}}) \otimes M'_{\tilde{u}} - \mathrm{i}\,\gamma^5(\mathbf{d}\tilde{\Upsilon} + \hat{A}\tilde{\Upsilon} - \tilde{\Upsilon}A + A''\tilde{\Upsilon})) \otimes M'_{\tilde{n}} \\
&\quad - \check{r}_1 \otimes M'_{10}M'_{\tilde{u}} - \check{r}_2 \otimes M'_{\tilde{u}}M'_5 - \check{r}_3 \otimes M'_{10}M'_{\tilde{n}} - \check{r}_4 \otimes M'_{\tilde{n}}M'_5\,, \\
\overline{\theta_{5,1}} &= -\mathrm{i}\,\gamma^5(\mathbf{d}\tilde{\Phi} + A\tilde{\Phi} + A''\tilde{\Phi}) \otimes \overline{M'_e} - r \otimes {M'_5}^*\overline{M'_e}\,,
\end{aligned}
$$

$$
\begin{aligned}
\theta_{10,1} &= -s \otimes M'_{\tilde n} \bar M'_e \;, \\
\theta_{\widetilde{10,5}} &= -\tilde s_1 \otimes M'_d \bar M'_{\tilde n} - \tilde s_2 \otimes M'_N \bar M'_{\tilde n} - \hat s_1 \otimes M'_{d\tilde n} - \hat s_2 \otimes M'_{Nu}
\end{aligned}
$$

and

$$
\begin{aligned}
q_1 &= \tfrac{6}{5} - \mathrm{tr}(\tilde\Psi^2) \;, & q_2 &= 1 - \tilde\Phi^* \tilde\Phi \;, \\
q_3 &= 12 - \mathrm{tr}(\tilde\Upsilon\tilde\Upsilon^*) \;, & q_4 &= 1 - \mathrm{tr}(\tilde\Xi\tilde\Xi^*) \;, \\
\tilde q_1 &= \tilde\Psi^2 - \tfrac{1}{5}\,\mathrm{tr}(\tilde\Psi^2)\mathbf{1}_5 - \tfrac{1}{5}\tilde\Psi \;, \\
\tilde q_2 &= \tilde\Upsilon^*\tilde\Upsilon - \tfrac{1}{5}\,\mathrm{tr}(\tilde\Upsilon^*\tilde\Upsilon)\mathbf{1}_5 - 8\tilde\Psi + 9\tilde\Phi\tilde\Phi - \tfrac{9}{5}\,\mathrm{tr}(\tilde\Phi\tilde\Phi^*)\mathbf{1}_5 \;, \\
\tilde q_3 &= (\tilde\Upsilon\tilde\Upsilon^*)' - \tfrac{8}{3}\tilde\Psi - \tilde\Phi\tilde\Phi^* + \tfrac{1}{5}\,\mathrm{tr}(\tilde\Phi\tilde\Phi^*)\mathbf{1}_5 \;, \\
\tilde q_4 &= \tilde\Upsilon^*\hat{\tilde\Phi} + \hat{\tilde\Phi}^*\tilde\Upsilon + 8\tilde\Psi - 6\tilde\Phi\tilde\Phi^* + \tfrac{6}{5}\,\mathrm{tr}(\tilde\Phi\tilde\Phi^*)\mathbf{1}_5 \;, \\
\tilde q_5 &= \mathrm{i}(\tilde\Upsilon^*\hat{\tilde\Phi} - \hat{\tilde\Phi}^*\tilde\Upsilon) \;, & \tilde q_6 &= (\tilde\Xi\tilde\Xi^*)' - \tfrac{1}{3}\tilde\Psi \;, \\
\hat q_1 &= \tilde\Upsilon\tilde\Upsilon^* - 4\hat{\tilde\Psi}^2 - \tfrac{1}{10}\,\mathrm{tr}(\tilde\Upsilon^*\tilde\Upsilon - 12\tilde\Psi^2)\mathbf{1}_{10} + \mathrm{i}\,\pi_{10}(\mathrm{i}((\tilde\Upsilon\tilde\Upsilon^*)' - \tfrac{4}{3}\tilde\Psi^2 + \tfrac{4}{15}\,\mathrm{tr}(\tilde\Psi^2)\mathbf{1}_5)) \;, \\
\hat q_2 &= \hat{\tilde\Phi}\tilde\Upsilon^* + \tilde\Upsilon\hat{\tilde\Phi}^* + 4\hat{\tilde\Psi}^2 - \tfrac{6}{5}\,\mathrm{tr}(\tilde\Psi^2)\mathbf{1}_{10} + \tfrac{1}{3}\,\mathrm{i}\,\pi_{10}(\mathrm{i}(\tilde\Upsilon^*\hat{\tilde\Phi} + \hat{\tilde\Phi}^*\tilde\Upsilon + 4\tilde\Psi^2 - \tfrac{4}{5}\,\mathrm{tr}(\tilde\Psi^2)\mathbf{1}_5)) \;, \\
\hat q_3 &= \mathrm{i}\left(\hat{\tilde\Phi}\tilde\Upsilon^* - \tilde\Upsilon\hat{\tilde\Phi}^* - \tfrac{1}{3}\pi_{10}(\tilde\Upsilon^*\hat{\tilde\Phi} - \hat{\tilde\Phi}^*\tilde\Upsilon)\right) \;, & & \qquad\qquad (59\mathrm{c}) \\
\hat q_4 &= \tilde\Xi\tilde\Xi^* - \tfrac{1}{2}\hat{\tilde\Psi}^2 - \tfrac{1}{10}(\mathrm{tr}(\tilde\Xi\tilde\Xi^*) - \tfrac{3}{2}\,\mathrm{tr}(\tilde\Psi^2))\mathbf{1}_{10} + \mathrm{i}\,\pi_{10}(\mathrm{i}((\tilde\Xi\tilde\Xi^*)' - \tfrac{1}{6}\tilde\Psi^2 + \tfrac{1}{30}\,\mathrm{tr}(\tilde\Psi^2)\mathbf{1}_5))) \;, \\
\hat q_5 &= \tilde\Xi\check{\tilde\Phi}^* + \check{\tilde\Phi}\tilde\Xi^* \;, & \hat q_6 &= \mathrm{i}(\tilde\Xi\check{\tilde\Phi}^* - \check{\tilde\Phi}\tilde\Xi^*) \;, \\
r &= \tilde\Psi\tilde\Phi - \tfrac{3}{5}\tilde\Phi \;, & \hat r &= \hat{\tilde\Psi}\tilde\Xi + \tilde\Xi\hat{\tilde\Psi}^T - \tfrac{12}{5}\tilde\Xi \;, \\
\tilde r_1 &= \hat{\tilde\Psi}\check{\tilde\Phi} - \check{\tilde\Phi}\hat{\tilde\Psi}^T + \tfrac{1}{2}\hat{\tilde\Upsilon} \;, & \tilde r_2 &= \hat{\tilde\Psi}\tilde\Xi - \tilde\Xi\hat{\tilde\Psi}^T \;, \\
\check r_1 &= \hat{\tilde\Psi}\hat{\tilde\Phi} - \tfrac{9}{20}\hat{\tilde\Phi} + \tfrac{1}{4}\tilde\Upsilon \;, & \check r_2 &= -(\hat{\tilde\Phi}\tilde\Psi + \tfrac{3}{20}\hat{\tilde\Phi} + \tfrac{1}{4}\tilde\Upsilon) \;, \\
\check r_3 &= \hat{\tilde\Psi}\tilde\Upsilon + \tfrac{3}{4}\hat{\tilde\Phi} - \tfrac{19}{20}\tilde\Upsilon \;, & \check r_3 &= -(\tilde\Upsilon\tilde\Psi + \tfrac{3}{4}\hat{\tilde\Phi} - \tfrac{7}{20}\tilde\Upsilon) \;, \\
\tilde s_1 &= (\tilde\Upsilon^*\check{\tilde\Phi})^T_{\underline{10}} \;, & \tilde s_2 &= (\tilde\Upsilon^*\tilde\Xi)^T_{\underline{10}} \;, \\
\hat s_1 &= (\tilde\Upsilon^*\check{\tilde\Phi})^T_{\underline{40}} \;, & \hat s_2 &= (\tfrac{1}{4}(\tilde\Upsilon^*\tilde\Xi)_{\underline{40}} + \tfrac{3}{4}(\hat{\tilde\Phi}\tilde\Xi))^T \;, \\
s &= -\tilde\Upsilon\tilde\Phi \;.
\end{aligned}
$$

4.2. *The bosonic action*

The computation of the bosonic action is not difficult now. All what one needs are the orthogonality of different irreducible representations and the relations

$$
\begin{aligned}
&\mathrm{tr}(\pi_{10}(a)\pi_{10}(\tilde a)) = 3\,\mathrm{tr}(\pi_5(a)\pi_5(\tilde a)) = 3\,\mathrm{tr}(a\tilde a) \;, \\
&\mathrm{tr}\left((A - \tfrac{1}{10}\,\mathrm{tr}(A)\mathbf{1}_{10} - A_{\underline{24}})(\tilde A - \tfrac{1}{10}\,\mathrm{tr}(\tilde A)\mathbf{1}_{10} - \tilde A_{\underline{24}})\right) \\
&\qquad = \mathrm{tr}(A\tilde A) - \tfrac{1}{10}\,\mathrm{tr}(A)\;\mathrm{tr}(\tilde A) - \mathrm{tr}(A_{\underline{24}}\tilde A_{\underline{24}}) \;,
\end{aligned}
\tag{60}
$$

for $a, \tilde a \in \mathfrak{a}$ and skew-adjoint $A, \tilde A \in \mathrm{M}_{10}\mathbb{C}$. We compute the Lagrangian $\mathcal{L} = \frac{1}{192\,g_0^2}\;\mathrm{tr}_c((\mathfrak{e}(\theta))^2)$, where g_0 is a coupling constant and tr_c the combination of the trace over the matrix structure with the trace in the Clifford algebra. For functions

$f \in C^\infty(X)$ we have $\mathrm{tr}_c(f) = 4f$. We find:

$$\tfrac{1}{192\,g_0^2}\ \mathrm{tr}_c((\mathfrak{e}(\theta))^2) = \mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0\ , \tag{61a}$$

$$\mathcal{L}_2 = \tfrac{1}{4\,g_0^2}\ \mathrm{tr}_c((\mathbf{d}A + \tfrac{1}{2}\{A,A\})^2) + \tfrac{5}{4\,g_0^2}\ \mathrm{tr}_c((\mathbf{d}A'')^2)\ , \tag{61b}$$

$$\begin{aligned}\mathcal{L}_1 &= \tfrac{1}{g_0^2}\mu_0\ \mathrm{tr}_c((\mathbf{d}\tilde{\boldsymbol{\Psi}} + [A,\tilde{\boldsymbol{\Psi}}])^2) \\ &+ \tfrac{1}{g_0^2}\mu_1\ \mathrm{tr}_c((\mathbf{d}\tilde{\boldsymbol{\Phi}} + (A + A''\mathbf{1}_5)\tilde{\boldsymbol{\Phi}})^*(\mathbf{d}\tilde{\boldsymbol{\Phi}} + (A + A''\mathbf{1}_5)\tilde{\boldsymbol{\Phi}})) \\ &+ \tfrac{1}{g_0^2}\mu_2\ \mathrm{tr}_c((\mathbf{d}\tilde{\Upsilon} + \hat{A}\tilde{\Upsilon} - \tilde{\Upsilon}A + A''\tilde{\Upsilon})^*(\mathbf{d}\tilde{\Upsilon} + \hat{A}\tilde{\Upsilon} - \tilde{\Upsilon}A + A''\tilde{\Upsilon})) \\ &+ \tfrac{1}{g_0^2}\mu_3\ \mathrm{tr}_c((\mathbf{d}\tilde{\Xi} + \hat{A}\tilde{\Xi} + \tilde{\Xi}\hat{A}^T - A''\tilde{\Xi})^*(\mathbf{d}\tilde{\Xi} + \hat{A}\tilde{\Xi} + \tilde{\Xi}\hat{A}^T - A''\tilde{\Xi}))\,,\end{aligned} \tag{61c}$$

$$\begin{aligned}\mathcal{L}_0 &= \tfrac{1}{24\,g_0^2}\big\{\mu^{\mathrm{a}}q_1^2 + \mu^{\mathrm{b}}q_2^2 + \mu^{\mathrm{c}}q_3^2 + \mu^{\mathrm{d}}q_1q_2 + \mu^{\mathrm{e}}q_1q_3 + \mu^{\mathrm{f}}q_2q_3 \\ &+ \check{\mu}^{\mathrm{a}}q_4^2 + \check{\mu}^{\mathrm{b}}q_1q_4 + \check{\mu}^{\mathrm{c}}q_2q_4 + \check{\mu}^{\mathrm{d}}q_3q_4 \\ &+ \check{\mu}^{\mathrm{e}}\ \mathrm{tr}(\hat{r}\hat{r}^*) + \check{\mu}^{\mathrm{f}}\ \mathrm{tr}(\tilde{r}_2\tilde{r}_2^*) + \check{\mu}^{\mathrm{g}}\,\mathrm{Re}(\mathrm{tr}(\tilde{r}_1\tilde{r}_2^*)) + \check{\mu}^{\mathrm{h}}\,\mathrm{Im}(\mathrm{tr}(\tilde{r}_1\tilde{r}_2^*)) \\ &+ \mu^{\mathrm{g}}\ \mathrm{tr}(rr^*) + \mu^{\mathrm{h}}\ \mathrm{tr}(\tilde{r}_1\tilde{r}_1^*) + \mu^{\mathrm{i}}\ \mathrm{tr}(\check{r}_1^*\check{r}_1) + \mu^{\mathrm{j}}\ \mathrm{tr}(\check{r}_2^*\check{r}_2) + \mu^{\mathrm{k}}\ \mathrm{tr}(\check{r}_3^*\check{r}_3) \\ &+ \mu^{\mathrm{l}}\ \mathrm{tr}(\check{r}_4^*\check{r}_4) + \mu^{\mathrm{m}}\,\mathrm{Re}(\mathrm{tr}(\check{r}_1^*\check{r}_2)) + \mu^{\mathrm{n}}\,\mathrm{Re}(\mathrm{tr}(\check{r}_1^*\check{r}_3)) + \mu^{\mathrm{o}}\,\mathrm{Im}(\mathrm{tr}(\check{r}_1^*\check{r}_3)) \\ &+ \mu^{\mathrm{p}}\,\mathrm{Re}(\mathrm{tr}(\check{r}_1^*\check{r}_4) + \mathrm{tr}(\check{r}_2^*\check{r}_3)) + \mu^{\mathrm{q}}\,\mathrm{Im}(\mathrm{tr}(\check{r}_1^*\check{r}_4) + \mathrm{tr}(\check{r}_2^*\check{r}_3)) \\ &+ \mu^{\mathrm{r}}\,\mathrm{Re}(\mathrm{tr}(\check{r}_2^*\check{r}_4)) + \mu^{\mathrm{s}}\,\mathrm{Im}(\mathrm{tr}(\check{r}_2^*\check{r}_4)) + \mu^{\mathrm{t}}\,\mathrm{Re}(\mathrm{tr}(\check{r}_3^*\check{r}_4)) \\ &+ \mu^{\mathrm{u}}\ \mathrm{tr}(ss^*) + \mu^{\mathrm{v}}\ \mathrm{tr}(\tilde{s}_1\tilde{s}_1^*) + \mu^{\mathrm{w}}\ \mathrm{tr}(\hat{s}_1\hat{s}_1^*) + \check{\mu}^{\mathrm{i}}\ \mathrm{tr}(\tilde{s}_2\tilde{s}_2^*) + \check{\mu}^{\mathrm{j}}\,\mathrm{Re}(\mathrm{tr}(\tilde{s}_1\tilde{s}_2^*)) \\ &+ \check{\mu}^{\mathrm{k}}\,\mathrm{Im}(\mathrm{tr}(\tilde{s}_1\tilde{s}_2^*)) + \check{\mu}^{\mathrm{l}}\ \mathrm{tr}(\hat{s}_2\hat{s}_2^*) + \check{\mu}^{\mathrm{m}}\,\mathrm{Re}(\mathrm{tr}(\hat{s}_2\hat{s}_1^*)) + \check{\mu}^{\mathrm{n}}\,\mathrm{Im}(\mathrm{tr}(\hat{s}_2\hat{s}_1^*)) \\ &+ \tilde{\mu}^{\mathrm{a}}\ \mathrm{tr}(\tilde{q}_1^2) + \tilde{\mu}^{\mathrm{b}}\ \mathrm{tr}(\tilde{q}_2^2) + \tilde{\mu}^{\mathrm{c}}\ \mathrm{tr}(\tilde{q}_3^2) + \tilde{\mu}^{\mathrm{d}}\ \mathrm{tr}(\tilde{q}_4^2) + \tilde{\mu}^{\mathrm{e}}\ \mathrm{tr}(\tilde{q}_5^2) \\ &+ \tilde{\mu}^{\mathrm{f}}\ \mathrm{tr}(\tilde{q}_1\tilde{q}_2) + \tilde{\mu}^{\mathrm{g}}\ \mathrm{tr}(\tilde{q}_1\tilde{q}_3) + \tilde{\mu}^{\mathrm{h}}\ \mathrm{tr}(\tilde{q}_1\tilde{q}_4) + \tilde{\mu}^{\mathrm{i}}\ \mathrm{tr}(\tilde{q}_1\tilde{q}_5) + \tilde{\mu}^{\mathrm{j}}\ \mathrm{tr}(\tilde{q}_2\tilde{q}_3) \\ &+ \tilde{\mu}^{\mathrm{k}}\ \mathrm{tr}(\tilde{q}_2\tilde{q}_4) + \tilde{\mu}^{\mathrm{l}}\ \mathrm{tr}(\tilde{q}_2\tilde{q}_5) + \tilde{\mu}^{\mathrm{m}}\ \mathrm{tr}(\tilde{q}_3\tilde{q}_4) + \tilde{\mu}^{\mathrm{n}}\ \mathrm{tr}(\tilde{q}_3\tilde{q}_5) + \tilde{\mu}^{\mathrm{o}}\ \mathrm{tr}(\tilde{q}_4\tilde{q}_5) \\ &+ \hat{\mu}^{\mathrm{a}}\ \mathrm{tr}(\tilde{q}_6^2) + \hat{\mu}^{\mathrm{b}}\ \mathrm{tr}(\tilde{q}_1\tilde{q}_6) + \hat{\mu}^{\mathrm{c}}\ \mathrm{tr}(\tilde{q}_2\tilde{q}_6) + \hat{\mu}^{\mathrm{d}}\ \mathrm{tr}(\tilde{q}_3\tilde{q}_6) + \hat{\mu}^{\mathrm{e}}\ \mathrm{tr}(\tilde{q}_4\tilde{q}_6) + \hat{\mu}^{\mathrm{f}}\ \mathrm{tr}(\tilde{q}_5\tilde{q}_6) \\ &+ \tilde{\mu}^{\mathrm{p}}\ \mathrm{tr}(\hat{q}_1^2) + \tilde{\mu}^{\mathrm{q}}\ \mathrm{tr}(\hat{q}_2^2) + \tilde{\mu}^{\mathrm{r}}\ \mathrm{tr}(\hat{q}_3^2) + \tilde{\mu}^{\mathrm{w}}\ \mathrm{tr}(\hat{q}_1\hat{q}_2) + \tilde{\mu}^{\mathrm{t}}\ \mathrm{tr}(\hat{q}_1\hat{q}_3) + \tilde{\mu}^{\mathrm{u}}\ \mathrm{tr}(\hat{q}_2\hat{q}_3) \\ &+ \hat{\mu}^{\mathrm{g}}\ \mathrm{tr}(\hat{q}_4^2) + \hat{\mu}^{\mathrm{h}}\ \mathrm{tr}(\hat{q}_5^2) + \hat{\mu}^{\mathrm{i}}\ \mathrm{tr}(\hat{q}_6^2) + \hat{\mu}^{\mathrm{j}}\ \mathrm{tr}(\hat{q}_4\hat{q}_5) + \hat{\mu}^{\mathrm{k}}\ \mathrm{tr}(\hat{q}_4\hat{q}_6) + \hat{\mu}^{\mathrm{l}}\ \mathrm{tr}(\hat{q}_5\hat{q}_6) \\ &+ \hat{\mu}^{\mathrm{m}}\ \mathrm{tr}(\hat{q}_1\hat{q}_4) + \hat{\mu}^{\mathrm{n}}\ \mathrm{tr}(\hat{q}_2\hat{q}_4) + \hat{\mu}^{\mathrm{o}}\ \mathrm{tr}(\hat{q}_3\hat{q}_4) + \hat{\mu}^{\mathrm{p}}\ \mathrm{tr}(\hat{q}_1\hat{q}_5) + \hat{\mu}^{\mathrm{q}}\ \mathrm{tr}(\hat{q}_2\hat{q}_5) \\ &+ \hat{\mu}^{\mathrm{r}}\ \mathrm{tr}(\hat{q}_3\hat{q}_5) + \hat{\mu}^{\mathrm{s}}\ \mathrm{tr}(\hat{q}_1\hat{q}_6) + \hat{\mu}^{\mathrm{t}}\ \mathrm{tr}(\hat{q}_2\hat{q}_6) + \hat{\mu}^{\mathrm{u}}\ \mathrm{tr}(\hat{q}_3\hat{q}_6)\big\}\,,\end{aligned} \tag{61d}$$

where the coefficients μ^i are given in Appendix B.

The group of local gauge transformations associated to our model is

$$\mathcal{U}_0(\mathfrak{g}) = \exp(\pi(C^\infty(X)\otimes(\mathrm{su}(5)\oplus\mathrm{u}(1)))) \cong C^\infty(X)\otimes(\mathrm{SU}(5)\times\mathrm{U}(1))\ . \tag{62}$$

The Lagrangian (61) is invariant under the gauge transformations

$$\begin{aligned}&\gamma_u(A) = u_5\mathbf{d}u_5^* + u_5Au_5^*\,, && \gamma_u(\hat{A}) = u_{10}\mathbf{d}u_{10}^* + u_{10}\hat{A}u_{10}^*\,, \\ &\gamma_u(A'') = u_1\mathbf{d}u_1^* + A''\,, && \\ &\gamma_u(\tilde{\Upsilon}) = u_1u_{10}\tilde{\Upsilon}u_5^*\ , && \gamma_u(\hat{\tilde{\Upsilon}}) = u_1^*u_{10}\hat{\tilde{\Upsilon}}u_{10}^T\ , \\ &\gamma_u(\tilde{\boldsymbol{\Psi}}) = u_5\tilde{\boldsymbol{\Psi}}u_5^*\ , && \gamma_u(\hat{\tilde{\boldsymbol{\Psi}}}) = u_{10}\hat{\tilde{\boldsymbol{\Psi}}}u_{10}^*\ , \\ &\gamma_u(\tilde{\boldsymbol{\Phi}}) = u_1u_5\tilde{\boldsymbol{\Phi}}\ , && \gamma_u(\hat{\tilde{\boldsymbol{\Phi}}}) = u_1u_{10}\hat{\tilde{\boldsymbol{\Phi}}}u_5^*\ , \\ &\gamma_u(\check{\tilde{\boldsymbol{\Phi}}}) = u_1^*u_{10}\check{\tilde{\boldsymbol{\Phi}}}u_{10}^T\ , && \gamma_u(\tilde{\Xi}) = u_1^*u_{10}\tilde{\Xi}u_{10}^T\ , \quad \text{where}\end{aligned} \tag{63a}$$

$$
\begin{aligned}
&u_5 = \exp(t_5)\ , \qquad u_{10} = \exp(\pi_{10}(t_5))\ , \qquad && t_5 \in C^\infty(X)\otimes \mathrm{su}(5)\ ,\\
&u_1 = \exp(t_1)\ , && t_1 \in C^\infty(X)\otimes \mathrm{u}(1)\ .
\end{aligned}
\tag{63b}
$$

To determine the spontaneous symmetry breaking pattern, we must search for a local minimum of the Higgs potential $\mathcal{L}_0$. This problem is easy to solve. We know that, applying the transformation (55b) in the other direction, the Λ^0-part of the curvature $\mathfrak{e}(\theta)$ (and hence the Higgs potential $\mathcal{L}_0$) is zero for

$$
\begin{aligned}
&\Psi = 0\ , \qquad \Phi = 0\ , \qquad \Xi = 0\ , \qquad \Upsilon = 0 \qquad \text{or}\\
&\tilde\Psi = m\ , \qquad \tilde\Phi = n\ , \qquad \tilde\Xi = m'\ , \qquad \tilde\Upsilon = n'\ .
\end{aligned}
\tag{64}
$$

Since the Higgs potential $\mathcal{L}_0$ is not negative as the trace of the square of the Λ^0-part of the selfadjoint matrix $\mathfrak{e}(\theta)$, the point (64) is a global minimum of $\mathcal{L}_0$. But (64) is clearly a local minimum as well: In the vicinity of (64), the Λ^0-part of $\mathfrak{e}(\theta)$ is linear in the components of Ψ, Φ, Ξ and Υ so that the Higgs potential $\mathcal{L}_0$ is in lowest order quadratic in these components.

We underline that, given the fermion masses and the spontaneous symmetry breaking pattern as the input, our formalism provides a straightforward algorithm to determine the occurring Higgs multiplets and their most general gauge invariant Higgs potential.

4.3. *The bosonic Lagrangian in local coordinates*

In this subsection we will write down the Lagrangian (61) in terms of local coordinates. We shall write our formulae in terms of the "physical" fields $\Psi, \Phi, \Xi, \Upsilon$ given by (55b). The subgroup of $C^\infty(X)\otimes(\mathrm{SU}(5)\times \mathrm{U}(1))$, which leaves (64) invariant, is the group $C^\infty(X)\otimes(\mathrm{SU}(3)_C\times \mathrm{U}(1)_{EM})$. The Higgs mechanism consists in reducing the symmetry of the whole theory to the symmetry of the vacuum. This means that we fix the gauge transformations corresponding to

$$
C^\infty(X)\otimes\big((\mathrm{SU}(5)\times \mathrm{U}(1))\,/\,(\mathrm{SU}(3)_C\times \mathrm{U}(1)_{EM})\big)
$$

in such a way that the Higgs multiplets Ψ , Φ and Ξ take the form

$$
\Psi = \begin{pmatrix} -\sqrt{\frac{4}{15}}\mathit{\Psi}_0 \mathbf{1}_3 + \Psi_g & 0 \\ 0 & \sqrt{\frac{3}{5}}\mathit{\Psi}_0\mathbf{1}_2 + \Psi_w \end{pmatrix}\ , \quad \Psi_g = \sum_{a=1}^{8} \mathit{\Psi}_a \lambda^a\ , \ \ \Psi_w = \sum_{a=1}^{3}\mathit{\Psi}'_a \sigma^a\ ,
\tag{65a}
$$

$$
\Phi = \begin{pmatrix} \Phi_g \\ \Phi_w \end{pmatrix}\ , \qquad \Phi_g = \begin{pmatrix} \mathit{\Phi}_1 + \mathrm{i}\,\mathit{\Phi}_4 \\ \mathit{\Phi}_2 + \mathrm{i}\,\mathit{\Phi}_5 \\ \mathit{\Phi}_3 + \mathrm{i}\,\mathit{\Phi}_6 \end{pmatrix}\ , \qquad \Phi_w = \begin{pmatrix} \mathit{\Phi}_0 \\ 0 \end{pmatrix}\ ,
\tag{65b}
$$

$$
\begin{aligned}
\Xi &= \begin{pmatrix} \overline{\Xi_A} & \overline{\Xi_D} - \frac{1}{2}\varepsilon(\overline{\Xi_c}) & (\Xi_E^0)^* & \Xi_a \\ \overline{\Xi_D} + \frac{1}{2}\varepsilon(\overline{\Xi_c}) & \overline{\Xi_B} & (\Xi_F^0)^* & \Xi_b \\ \overline{\Xi_E^0} & \overline{\Xi_F^0} & \Xi_C & \overline{\Xi_c} \\ \Xi_a^T & \Xi_b^T & \Xi_c^* & -\Xi_0 \end{pmatrix} \\
&= \Big(-\mathit{\Xi}_0 E_{00} + \tfrac{1}{\sqrt{3}}(\mathit{\Xi}_1 - \mathrm{i}\,\mathit{\Xi}_{50})(E_{11}+E_{22}+E_{33}) + \tfrac{1}{\sqrt{2}}(\mathit{\Xi}_2 - \mathrm{i}\,\mathit{\Xi}_{51})(E_{11}-E_{22}) \\
&\quad + \tfrac{1}{\sqrt{2}}(\mathit{\Xi}_4 - \mathrm{i}\,\mathit{\Xi}_{53})(E_{12}+E_{21}) + \tfrac{1}{\sqrt{2}}(\mathit{\Xi}_5 - \mathrm{i}\,\mathit{\Xi}_{54})(E_{13}+E_{31}) + \tfrac{1}{\sqrt{2}}(\mathit{\Xi}_6 - \mathrm{i}\,\mathit{\Xi}_{55})(E_{23}+E_{32}) \\
&\quad + \tfrac{1}{\sqrt{6}}(\mathit{\Xi}_3 - \mathrm{i}\,\mathit{\Xi}_{52})(E_{11}+E_{22}-2E_{33}) + \tfrac{1}{\sqrt{3}}(\mathit{\Xi}_7 - \mathrm{i}\,\mathit{\Xi}_{56})(E_{44}+E_{55}+E_{66})
\end{aligned}
\tag{65c}
$$

$$
\begin{aligned}
&+\tfrac{1}{\sqrt{2}}(\Xi_8-\mathrm{i}\,\Xi_{57})(E_{44}-E_{55})+\tfrac{1}{\sqrt{2}}(\Xi_{10}-\mathrm{i}\,\Xi_{59})(E_{45}+E_{54})+\tfrac{1}{\sqrt{2}}(\Xi_{11}-\mathrm{i}\,\Xi_{60})(E_{46}+E_{64})\\
&+\tfrac{1}{\sqrt{6}}(\Xi_9-\mathrm{i}\,\Xi_{58})(E_{44}+E_{55}-2E_{66})+\tfrac{1}{\sqrt{3}}(\Xi_{13}+\mathrm{i}\,\Xi_{62})(E_{77}+E_{88}+E_{99})\\
&+\tfrac{1}{\sqrt{2}}(\Xi_{12}-\mathrm{i}\,\Xi_{61})(E_{56}+E_{65})+\tfrac{1}{\sqrt{2}}(\Xi_{14}+\mathrm{i}\,\Xi_{63})(E_{77}-E_{88})+\tfrac{1}{\sqrt{2}}(\Xi_{16}+\mathrm{i}\,\Xi_{65})(E_{78}+E_{87})\\
&+\tfrac{1}{\sqrt{6}}(\Xi_{15}+\mathrm{i}\,\Xi_{64})(E_{77}+E_{88}-2E_{99})+\tfrac{1}{\sqrt{6}}(\Xi_{19}-\mathrm{i}\,\Xi_{68})(E_{14}+E_{25}+E_{36}+E_{41}+E_{52}+E_{63})\\
&+\tfrac{1}{\sqrt{2}}(\Xi_{17}+\mathrm{i}\,\Xi_{66})(E_{79}+E_{97})+\tfrac{1}{2}(\Xi_{20}-\mathrm{i}\,\Xi_{69})(E_{14}-E_{25}+E_{41}-E_{52})\\
&+\tfrac{1}{\sqrt{2}}(\Xi_{18}+\mathrm{i}\,\Xi_{67})(E_{89}+E_{98})+\tfrac{1}{\sqrt{12}}(\Xi_{21}-\mathrm{i}\,\Xi_{70})(E_{14}+E_{25}-2E_{36}+E_{41}+E_{52}-2E_{63})\\
&+\tfrac{1}{2}(\Xi_{22}-\mathrm{i}\,\Xi_{71})(E_{15}+E_{24}+E_{51}+E_{42})+\tfrac{1}{2}(\Xi_{23}-\mathrm{i}\,\Xi_{72})(E_{16}+E_{34}+E_{61}+E_{43})\\
&+\tfrac{1}{2}(\Xi_{24}-\mathrm{i}\,\Xi_{73})(E_{26}+E_{35}+E_{62}+E_{53})+\tfrac{1}{2}(\Xi_{25}-\mathrm{i}\,\Xi_{74})(E_{18}+E_{27}+E_{81}+E_{72})\\
&-\tfrac{\mathrm{i}}{2}(\Xi_{26}-\mathrm{i}\,\Xi_{75})(E_{18}-E_{27}+E_{81}-E_{72})+\tfrac{1}{2}(\Xi_{27}-\mathrm{i}\,\Xi_{76})(E_{17}-E_{28}+E_{71}-E_{82})\\
&+\tfrac{1}{2}(\Xi_{28}-\mathrm{i}\,\Xi_{77})(E_{19}+E_{37}+E_{91}+E_{73})-\tfrac{\mathrm{i}}{2}(\Xi_{29}-\mathrm{i}\,\Xi_{78})(E_{19}-E_{37}+E_{91}-E_{73})\\
&+\tfrac{1}{2}(\Xi_{30}-\mathrm{i}\,\Xi_{79})(E_{29}+E_{38}+E_{92}+E_{83})-\tfrac{\mathrm{i}}{2}(\Xi_{31}-\mathrm{i}\,\Xi_{80})(E_{29}-E_{38}+E_{92}-E_{83})\\
&+\tfrac{1}{\sqrt{12}}(\Xi_{32}-\mathrm{i}\,\Xi_{81})(E_{17}+E_{28}-2E_{39}+E_{71}+E_{82}-2E_{93})+\tfrac{1}{\sqrt{2}}(\Xi_{41}+\mathrm{i}\,\Xi_{90})(E_{10}+E_{01})\\
&+\tfrac{1}{2}(\Xi_{33}-\mathrm{i}\,\Xi_{82})(E_{48}+E_{57}+E_{84}+E_{75})-\tfrac{\mathrm{i}}{2}(\Xi_{34}-\mathrm{i}\,\Xi_{83})(E_{48}-E_{57}+E_{84}-E_{75})\\
&+\tfrac{1}{2}(\Xi_{35}-\mathrm{i}\,\Xi_{84})(E_{47}-E_{58}+E_{74}-E_{85})+\tfrac{1}{2}(\Xi_{36}-\mathrm{i}\,\Xi_{85})(E_{49}+E_{67}+E_{94}+E_{76})\\
&-\tfrac{\mathrm{i}}{2}(\Xi_{37}-\mathrm{i}\,\Xi_{86})(E_{49}-E_{67}+E_{94}-E_{76})+\tfrac{1}{2}(\Xi_{38}-\mathrm{i}\,\Xi_{87})(E_{59}+E_{68}+E_{95}+E_{86})\\
&+\tfrac{1}{\sqrt{12}}(\Xi_{40}-\mathrm{i}\,\Xi_{89})(E_{47}+E_{58}-2E_{69}+E_{74}+E_{85}-2E_{96})+\tfrac{1}{\sqrt{2}}(\Xi_{43}+\mathrm{i}\,\Xi_{92})(E_{30}+E_{03})\\
&+\tfrac{1}{\sqrt{12}}(\Xi_{47}-\mathrm{i}\,\Xi_{96})(2E_{70}+2E_{07}-E_{26}+E_{35}-E_{62}+E_{53})+\tfrac{1}{\sqrt{2}}(\Xi_{44}+\mathrm{i}\,\Xi_{93})(E_{40}+E_{04})\\
&+\tfrac{1}{\sqrt{12}}(\Xi_{48}-\mathrm{i}\,\Xi_{97})(2E_{80}+2E_{08}+E_{16}-E_{34}+E_{61}-E_{43})+\tfrac{1}{\sqrt{2}}(\Xi_{45}+\mathrm{i}\,\Xi_{94})(E_{50}+E_{05})\\
&+\tfrac{1}{\sqrt{12}}(\Xi_{49}-\mathrm{i}\,\Xi_{98})(2E_{90}+2E_{09}-E_{15}+E_{24}-E_{51}+E_{42})+\tfrac{1}{\sqrt{2}}(\Xi_{46}+\mathrm{i}\,\Xi_{95})(E_{60}+E_{06})\\
&-\tfrac{\mathrm{i}}{2}(\Xi_{39}-\mathrm{i}\,\Xi_{88})(E_{59}-E_{68}+E_{95}-E_{86})+\tfrac{1}{\sqrt{2}}(\Xi_{42}+\mathrm{i}\,\Xi_{91})(E_{20}+E_{02})\Big)_{10\times 10},
\end{aligned}
$$

where $\Xi_0 \in C^\infty(X)$ is a *real* function and $\Psi_a, \Psi'_a, \Phi_i, \Xi_i \in C^\infty(X)$. Here, λ^a are the Gell-Mann matrices and σ^a the Pauli matrices. The matrix Υ is an arbitrary element of $\mathrm{i}\,\mathfrak{w}$, where $\Upsilon_i \in C^\infty(X)$ and $(\varepsilon(A))_{\alpha\beta} = \sum_{\gamma=1}^3 \varepsilon_{\alpha\beta\gamma} A_\gamma$:

$$
\begin{aligned}
\Upsilon &= \begin{pmatrix} \Upsilon_A & \Upsilon_a+\Upsilon_b & \Upsilon_c \\ \Upsilon_B & \Upsilon_d & \Upsilon_a-\Upsilon_b \\ \Upsilon_C-\varepsilon(\Upsilon_a) & \Upsilon_e & \Upsilon_f \\ \Upsilon_g^* & -\mathrm{tr}(\Upsilon_B) & \mathrm{tr}(\Upsilon_A) \end{pmatrix} \qquad (66)\\
&= \Big(\tfrac{1}{\sqrt{6}}(\Upsilon_0+\mathrm{i}\,\Upsilon_{45})(E_{11}+E_{22}+E_{33}+3E_{05})+(\Upsilon_1+\mathrm{i}\,\Upsilon_{46})(E_{12}+E_{21})\\
&-\mathrm{i}(\Upsilon_2+\mathrm{i}\,\Upsilon_{47})(E_{12}-E_{21})+(\Upsilon_3+\mathrm{i}\,\Upsilon_{48})(E_{11}-E_{22})+(\Upsilon_4+\mathrm{i}\,\Upsilon_{49})(E_{13}+E_{31})\\
&-\mathrm{i}(\Upsilon_5+\mathrm{i}\,\Upsilon_{50})(E_{13}-E_{31})+(\Upsilon_5+\mathrm{i}\,\Upsilon_{51})(E_{23}+E_{32})-\mathrm{i}(\Upsilon_7+\mathrm{i}\,\Upsilon_{52})(E_{23}-E_{32})\\
&+\tfrac{1}{\sqrt{3}}(\Upsilon_8+\mathrm{i}\,\Upsilon_{53})(E_{11}+E_{22}-2E_{33})\\
&+\tfrac{1}{\sqrt{2}}(\Upsilon_9+\mathrm{i}\,\Upsilon_{54})(E_{14}+E_{45}-E_{83}+E_{92})+(\Upsilon_{12}+\mathrm{i}\,\Upsilon_{57})(E_{14}-E_{45})\\
&+\tfrac{1}{\sqrt{2}}(\Upsilon_{10}+\mathrm{i}\,\Upsilon_{55})(E_{24}+E_{55}+E_{73}-E_{91})+(\Upsilon_{13}+\mathrm{i}\,\Upsilon_{58})(E_{24}-E_{55})\\
&+\tfrac{1}{\sqrt{2}}(\Upsilon_{11}+\mathrm{i}\,\Upsilon_{56})(E_{34}+E_{65}-E_{72}+E_{81})+(\Upsilon_{14}+\mathrm{i}\,\Upsilon_{59})(E_{34}-E_{65})\\
&+\sqrt{2}(\Upsilon_{15}+\mathrm{i}\,\Upsilon_{60})E_{15}+\sqrt{2}(\Upsilon_{16}+\mathrm{i}\,\Upsilon_{61})E_{25}+\sqrt{2}(\Upsilon_{17}+\mathrm{i}\,\Upsilon_{62})E_{35}\\
&+\tfrac{1}{\sqrt{6}}(\Upsilon_{18}+\mathrm{i}\,\Upsilon_{63})(E_{41}+E_{52}+E_{63}-3E_{04})+(\Upsilon_{19}+\mathrm{i}\,\Upsilon_{64})(E_{42}+E_{51})
\end{aligned}
$$

$$
\begin{aligned}
&-\mathrm{i}(\varUpsilon_{20}+\mathrm{i}\,\varUpsilon_{65})(E_{42}-E_{51})+(\varUpsilon_{21}+\mathrm{i}\,\varUpsilon_{66})(E_{41}-E_{52})+(\varUpsilon_{22}+\mathrm{i}\,\varUpsilon_{67})(E_{43}+E_{61})\\
&-\mathrm{i}(\varUpsilon_{23}+\mathrm{i}\,\varUpsilon_{68})(E_{43}-E_{61})+(\varUpsilon_{24}+\mathrm{i}\,\varUpsilon_{69})(E_{53}+E_{62})-\mathrm{i}(\varUpsilon_{25}+\mathrm{i}\,\varUpsilon_{70})(E_{53}-E_{62})\\
&+\tfrac{1}{\sqrt{3}}(\varUpsilon_{26}+\mathrm{i}\,\varUpsilon_{71})(E_{41}+E_{52}-2E_{63})+\tfrac{1}{\sqrt{3}}(\varUpsilon_{32}+\mathrm{i}\,\varUpsilon_{77})(E_{71}+E_{82}-2E_{93})\\
&+\sqrt{2}(\varUpsilon_{27}+\mathrm{i}\,\varUpsilon_{60})E_{44}+\sqrt{2}(\varUpsilon_{28}+\mathrm{i}\,\varUpsilon_{61})E_{54}+\sqrt{2}(\varUpsilon_{29}+\mathrm{i}\,\varUpsilon_{62})E_{64}\\
&+\tfrac{2}{\sqrt{6}}(\varUpsilon_{30}+\mathrm{i}\,\varUpsilon_{75})(E_{71}+E_{82}+E_{93})+(\varUpsilon_{31}+\mathrm{i}\,\varUpsilon_{76})(E_{71}-E_{82})\\
&+(\varUpsilon_{33}+\mathrm{i}\,\varUpsilon_{78})(E_{72}+E_{81})+(\varUpsilon_{34}+\mathrm{i}\,\varUpsilon_{79})(E_{73}+E_{91})-\mathrm{i}(\varUpsilon_{35}+\mathrm{i}\,\varUpsilon_{80})(E_{83}+E_{94})\\
&+\sqrt{2}(\varUpsilon_{36}-\mathrm{i}\,\varUpsilon_{81})E_{74}+\sqrt{2}(\varUpsilon_{37}-\mathrm{i}\,\varUpsilon_{82})E_{84}+\sqrt{2}(\varUpsilon_{38}-\mathrm{i}\,\varUpsilon_{83})E_{94}\\
&+\sqrt{2}(\varUpsilon_{39}-\mathrm{i}\,\varUpsilon_{84})E_{75}+\sqrt{2}(\varUpsilon_{40}-\mathrm{i}\,\varUpsilon_{85})E_{85}+\sqrt{2}(\varUpsilon_{41}-\mathrm{i}\,\varUpsilon_{86})E_{95}\\
&+\sqrt{2}(\varUpsilon_{42}-\mathrm{i}\,\varUpsilon_{87})E_{01}+\sqrt{2}(\varUpsilon_{43}-\mathrm{i}\,\varUpsilon_{88})E_{02}+\sqrt{2}(\varUpsilon_{44}-\mathrm{i}\,\varUpsilon_{89})E_{03}\Big)_{10\times 5}\ .
\end{aligned}
$$

For A and A'' we make the ansatz

$$
\begin{aligned}
&A=\frac{\mathrm{i}\,g_0}{2}\begin{pmatrix}\sqrt{\frac{4}{15}}A'\mathbf{1}_3+\mathbf{G} & \mathbf{X}\\ \mathbf{X}^* & -\sqrt{\frac{3}{5}}A'\mathbf{1}_2+\mathbf{W}\end{pmatrix}\ ,\quad \mathbf{G}=\sum_{a=1}^{8}G^a\lambda^a\ ,\quad \mathbf{W}=\sum_{a=1}^{3}W^a\sigma^a\ ,\\
&\mathbf{X}=\Big(\mathrm{X}\,,\,\mathrm{Y}\Big)\ ,\quad \mathrm{X}=\begin{pmatrix}X^1-\mathrm{i}\,X^2\\ X^3-\mathrm{i}\,X^4\\ X^5-\mathrm{i}\,X^6\end{pmatrix},\quad \mathrm{Y}=\begin{pmatrix}Y^1-\mathrm{i}\,Y^2\\ Y^3-\mathrm{i}\,Y^4\\ Y^5-\mathrm{i}\,Y^6\end{pmatrix},\qquad (67)\\
&A''=\tfrac{\mathrm{i}\,g_0}{2}\sqrt{\tfrac{2}{5}}\tilde{A}\ ,
\end{aligned}
$$

where $A', G^a, W^a, X^a, Y^a, \tilde{A}\in\Lambda^1$. In terms of the local basis $\{\gamma^\mu\}_{\mu=1,2,3,4}$ of Λ^1 we put

$$
\begin{aligned}
&\mathbf{G}=\mathbf{G}_\mu\gamma^\mu\ ,\quad G^a=G^a_\mu\gamma^\mu\ ,\quad W^a=W^a_\mu\gamma^\mu\ ,\quad A'=A'_\mu\gamma^\mu\ ,\quad \tilde{A}=\tilde{A}_\mu\gamma^\mu\ ,\\
&\mathbf{X}=\mathbf{X}_\mu\gamma^\mu\ ,\quad \mathrm{X}=\mathrm{X}_\mu\gamma^\mu\ ,\quad X^a=X^a_\mu\gamma^\mu\ ,\quad \mathrm{Y}=\mathrm{Y}_\mu\gamma^\mu\ ,\quad Y^a=Y^a_\mu\gamma^\mu\ .
\end{aligned}
$$

Moreover, we introduce the abbreviation $S_{[\mu}T_{\nu]}:=S_\mu T_\nu-S_\nu T_\mu$. Now, a straightforward calculation yields for (61b)

$$
\begin{aligned}
\mathcal{L}_2=\tfrac{1}{4}\delta^{\kappa\mu}\delta^{\lambda\nu}\big(&\textstyle\sum_{a=1}^{8}G^a_{\kappa\lambda}G^a_{\mu\nu}+\sum_{a=1}^{3}W^a_{\kappa\lambda}W^a_{\mu\nu}+A'_{\kappa\lambda}A'_{\mu\nu}+\tilde{A}_{\kappa\lambda}\tilde{A}_{\mu\nu}\\
&\textstyle+\sum_{a=1}^{6}\partial_{[\kappa}X^a_{\lambda]}\,\partial_{[\mu}X^a_{\nu]}+\sum_{a=1}^{6}\partial_{[\kappa}Y^a_{\lambda]}\,\partial_{[\mu}Y^a_{\nu]}\big)+I.T\ ,\qquad (68a)
\end{aligned}
$$

where $I.T$ stands for additional interaction terms involving X^a, Y^a we are not interested in. Moreover, we have put

$$
\begin{aligned}
&G^a_{\mu\nu}=\partial_{[\mu}G^a_{\nu]}-g_0\textstyle\sum_{b,c=1}^{8}f_{abc}G^b_\mu G^c_\nu\ ,\quad W^a_{\mu\nu}=\partial_{[\mu}W^a_{\nu]}-g_0\sum_{b,c=1}^{3}\varepsilon_{abc}W^b_\mu W^c_\nu\ ,\\
&A'_{\mu\nu}=\partial_{[\mu}A'_{\nu]}\ ,\qquad \tilde{A}_{\mu\nu}=\partial_{[\mu}\tilde{A}_{\nu]}\ ,\qquad (68b)
\end{aligned}
$$

where f_{abc} and ε_{abc} are the structure constants of su(3) and su(2).

Next, the Lagrangian $\mathcal{L}_1$ given in (61c) equals

$$\begin{aligned}\mathcal{L}_1 &= \tfrac{8\mu_0}{g_0^2}\delta^{\mu\nu}\big(\textstyle\sum_{a=0}^{8}\partial_\mu\Psi_a\,\partial_\nu\Psi_a + \sum_{a=1}^{3}\partial_\mu\Psi'_a\,\partial_\nu\Psi'_a\big) + \tfrac{4\mu_1}{g_0^2}\delta^{\mu\nu}\ \sum_{a=0}^{6}\partial_\mu\Phi_a\,\partial_\nu\Phi_a \\ &+ \tfrac{8\mu_2}{g_0^2}\delta^{\mu\nu}\ \textstyle\sum_{i=0}^{89}\partial_\mu\Upsilon_i\,\partial_\nu\Upsilon_i + \tfrac{4\mu_3}{g_0^2}\delta^{\mu\nu}\ \sum_{i=0}^{98}\partial_\mu\Xi_i\,\partial_\nu\Xi_i \\ &+ (\mu_1+12\mu_2)\delta^{\mu\nu}\big(W_\mu^1W_\nu^1 + W_\mu^2W_\nu^2 + \\ &\qquad +(W_\mu^3 - \sqrt{\tfrac{3}{5}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu)(W_\nu^3 - \sqrt{\tfrac{3}{5}}A'_\nu + \sqrt{\tfrac{2}{5}}\tilde{A}_\nu)\big) \\ &+ \mu_3\delta^{\mu\nu}(4\sqrt{\tfrac{3}{5}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu)(4\sqrt{\tfrac{3}{5}}A'_\nu + \sqrt{\tfrac{2}{5}}\tilde{A}_\nu) \\ &+ \delta^{\mu\nu}\textstyle\sum_{a=1}^{6}\big((2\mu_0+\mu_1+12\mu_2+2\mu_3)X_\mu^aX_\nu^a + (2\mu_0+32\mu_2+2\mu_3)Y_\mu^aY_\nu^a\big) + I.T\ ,\end{aligned} \tag{69}$$

where $I.T$ stands for tri- and quadrilinear interaction terms.

We perform the orthogonal transformation by Euler angles

$$\begin{pmatrix}Z_\mu\\ Z'_\mu\\ P_\mu\end{pmatrix} = \begin{pmatrix}\cos\phi_E & -\sin\phi_E & 0\\ \sin\phi_E & \cos\phi_E & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta_E & -\sin\theta_E\\ 0 & \sin\theta_E & \cos\theta_E\end{pmatrix}\begin{pmatrix}\cos\psi_E & -\sin\psi_E & 0\\ \sin\psi_E & \cos\psi_E & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}W_\mu^3\\ A'_\mu\\ \tilde{A}_\mu\end{pmatrix}. \tag{70a}$$

The photon P_μ is the massless linear combination, which is perpendicular to the plane spanned by $(W_\mu^3 - \sqrt{\frac{3}{5}}A'_\mu + \sqrt{\frac{2}{5}}\tilde{A}_\mu)$ and $(4\sqrt{\frac{3}{5}}A'_\mu + \sqrt{\frac{2}{5}}\tilde{A}_\mu)$, see (69). Calculating the vector product yields immediately

$$P_\mu = \sqrt{\tfrac{3}{8}}W_\mu^3 + \sqrt{\tfrac{1}{40}}A'_\mu - \sqrt{\tfrac{3}{5}}\tilde{A}_\mu\ , \tag{70b}$$

which implies $\cos\theta_E = -\sqrt{\frac{3}{5}}$, $\sin\theta_E = \sqrt{\frac{2}{5}}$, $\cos\psi_E = \frac{1}{4}$, $\sin\psi_E = \sqrt{\frac{15}{16}}$. The Euler angle ϕ_E is determined by the diagonalization of the mass matrix:

$$\tan 2\phi_E = -\tfrac{3}{4} + \tfrac{5}{2}\sqrt{15}\theta'_W\ , \qquad \theta'_W := \tfrac{(\mu_1+12\mu_2)}{10\sqrt{15}\,\mu_3}\ . \tag{70c}$$

We choose $\cos\phi_E < 0$ and $\sin\phi_E > 0$. Then, the inverse transformation is for $\theta'_W \ll 1$ given by

$$\begin{aligned} W_\mu^3 &= \sqrt{\tfrac{5}{8}}Z_\mu + \sqrt{\tfrac{3}{8}}P_\mu - \sqrt{6}\,\theta'_W Z'_\mu\ , \\ A'_\mu &= -\sqrt{\tfrac{3}{200}}(1 - 32\sqrt{\tfrac{3}{5}}\,\theta'_W)Z_\mu + \sqrt{\tfrac{1}{40}}P_\mu + \sqrt{\tfrac{24}{25}}(1 + \sqrt{\tfrac{3}{20}}\,\theta'_W)Z'_\mu\ , \\ \tilde{A}_\mu &= \tfrac{3}{5}(1 + \tfrac{4}{\sqrt{15}}\,\theta'_W)Z_\mu - \sqrt{\tfrac{3}{5}}P_\mu + \tfrac{1}{5}(1 - 12\sqrt{\tfrac{3}{5}}\,\theta'_W)Z'_\mu\ . \end{aligned} \tag{70d}$$

The Lagrangian (69) requires to perform the reparametrizations

$$\begin{aligned} &\Psi_i = \tfrac{g_0}{\sqrt{16\,\mu_0}}\psi_i\ , && i=0,\dots,8\ , && \Psi'_i = \tfrac{g_0}{\sqrt{16\,\mu_0}}\psi'_i\ , && i=1,\dots,3\ , \\ &\Phi_i = \tfrac{g_0}{\sqrt{8\,\mu_1}}\phi_i\ , && i=0,\dots,6\ , && && \\ &\Upsilon_i = \tfrac{g_0}{\sqrt{16\,\mu_2}}\upsilon_i\ , && i=0,\dots,89\ , && \Xi_i = \tfrac{g_0}{\sqrt{8\,\mu_3}}\xi_i\ , && i=0,\dots,98\ . \end{aligned} \tag{71}$$

It remains to compute the quadratic terms of the Higgs potential (61d). Due to the extremely rich Higgs structure we need computer algebra for that calculation. It is advantageous to perform an orthogonal transformation in the $\{\phi_0, \upsilon_0\}$-sector:

$$\begin{pmatrix} \phi_0 \\ \upsilon_0 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_0' \\ \upsilon_0' \end{pmatrix} , \qquad \tan\alpha = \sqrt{\tfrac{12\mu_2}{\mu_1}} \;. \tag{72}$$

The motivation for this transformation is that the linear combination ϕ_0' receives a much smaller mass than all other Higgs fields, see below. We present the quadratic terms of the Higgs potential in Appendix C.

We perform a Wick rotation from the Riemannian manifold X to the Minkowski space X_M by introduction of a global minus sign in the action and by replacing[b]

$$\delta^{\mu\nu} \mapsto -g^{\mu\nu} \;, \qquad\qquad g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) \;. \tag{73}$$

We define $P_{\mu\nu} := \partial_{[\mu} P_{\nu]}$ and

$$\begin{aligned} m_W^2 &= (2\,\mu_1 + 24\,\mu_2) \;, & m_Z^2 &= \tfrac{1}{\cos^2(\theta_W - \theta_W')} m_W^2 \;, \\ m_{Z'}^2 &= 32\mu_3 \cos^2(\theta_W - \theta_W') \;, & \sin\theta_W &= \sqrt{\tfrac{3}{8}} \;, \\ m_X^2 &= (4\,\mu_0 + 2\,\mu_1 + 24\,\mu_2 + 4\mu_3) \;, & m_Y^2 &= (4\,\mu_0 + 64\,\mu_2 + 4\mu_3) \;. \end{aligned} \tag{74}$$

Now we can write down the final formula for the bosonic Lagrangian:

$$\begin{aligned} \mathcal{L} &= -\tfrac14 g^{\kappa\mu} g^{\lambda\nu} \big(\textstyle\sum_{a=1}^8 (G^a_{\kappa\lambda} G^a_{\mu\nu}) + P_{\kappa\lambda} P_{\mu\nu}\big) \\ &\quad + \textstyle\sum_{a=1}^2 \big(-\tfrac14 g^{\kappa\mu} g^{\lambda\nu} \partial_{[\kappa} W^a_{\lambda]}\, \partial_{[\mu} W^a_{\nu]} + \tfrac12 g^{\mu\nu} m_W^2 W^a_\mu W^a_\nu\big) \\ &\quad + \big(-\tfrac14 g^{\kappa\mu} g^{\lambda\nu} \partial_{[\kappa} Z_{\lambda]}\, \partial_{[\mu} Z_{\nu]} + \tfrac12 g^{\mu\nu} m_Z^2 Z_\mu Z_\nu\big) \\ &\quad + \big(-\tfrac14 g^{\kappa\mu} g^{\lambda\nu} \partial_{[\kappa} Z'_{\lambda]}\, \partial_{[\mu} Z'_{\nu]} + \tfrac12 g^{\mu\nu} m_{Z'}^2 Z'_\mu Z'_\nu\big) + \mathcal{L}_{ew}(P,W,Z,Z') \\ &\quad + \textstyle\sum_{a=1}^6 \big(-\tfrac14 g^{\kappa\mu} g^{\lambda\nu} \partial_{[\kappa} X^a_{\lambda]}\, \partial_{[\mu} X^a_{\nu]} + \tfrac12 g^{\mu\nu} m_X^2 X^a_\mu X^a_\nu\big) \\ &\quad + \textstyle\sum_{a=1}^6 \big(-\tfrac14 g^{\kappa\mu} g^{\lambda\nu} \partial_{[\kappa} Y^a_{\lambda]}\, \partial_{[\mu} Y^a_{\nu]} + \tfrac12 g^{\mu\nu} m_Y^2 Y^a_\mu Y^a_\nu\big) + \mathcal{L}_H + I.T \;, \end{aligned} \tag{75a}$$

$$\begin{aligned} \mathcal{L}_{ew}&(P,W,Z,Z') \\ &= g_0\, g^{\kappa\mu} g^{\lambda\nu} \big(\partial_{[\kappa} W^1_{\lambda]}\, W^2_\mu W^3_\nu + \partial_{[\kappa} W^2_{\lambda]}\, W^3_\mu W^1_\nu + \partial_{[\kappa} W^3_{\lambda]}\, W^1_\mu W^2_\nu\big) \\ &\quad - \tfrac12 g_0^2 (g^{\kappa\mu} g^{\lambda\nu} - g^{\kappa\nu} g^{\lambda\mu}) (W^1_\kappa W^1_\mu W^2_\lambda W^2_\nu + W^1_\kappa W^1_\mu W^3_\lambda W^3_\nu + W^2_\kappa W^2_\mu W^3_\lambda W^3_\nu) \;, \end{aligned} \tag{75b}$$

$$\begin{aligned} \mathcal{L}_H = \tfrac12 g^{\mu\nu} \big(& \textstyle\sum_{i=0}^8 \partial_\mu \psi_i\, \partial_\nu \psi_i + \sum_{i=1}^3 \partial_\mu \psi'_i\, \partial_\nu \psi'_i + \partial_\mu \phi'_0\, \partial_\nu \phi'_0 + \partial_\mu \upsilon'_0\, \partial_\nu \upsilon'_0 \\ & + \textstyle\sum_{i=1}^6 \partial_\mu \phi_i\, \partial_\nu \phi_i + \sum_{i=0}^{98} \partial_\mu \xi_i\, \partial_\nu \xi_i + \sum_{i=1}^{89} \partial_\mu \upsilon_i\, \partial_\nu \upsilon_i\big) - \mathcal{L}_0 \;. \end{aligned} \tag{75c}$$

This is precisely the bosonic Lagrangian of the flipped SU(5) × U(1)-model, where the masses of the gauge bosons are given in (74). The parameters μ_1, μ_2, μ_3 will be determined in Sec. 4.4 when discussing the fermionic action. Within our framework there is no possibility to determine μ_0. However, we will find in Sec. 4.4 that the X and Y bosons lead to proton decay, which is only suppressed if $\mu_0 \gg \max(\mu_1, \mu_2)$. Then, it remains to derive the masses of gauge and Higgs bosons in Sec. 5.

[b] The minus sign in $\delta^{\mu\nu} \mapsto -g^{\mu\nu}$ is due to $(\hat\gamma^5)^* = -\hat\gamma^5$ on the Minkowski space.

4.4. *The fermionic action*

Now we write down the fermionic action S_F defined in (10). However, we pass immediately to the Minkowski space X_M . We denote the gamma matrices in Minkowski space by $\{\hat{\gamma}^\mu\}$ and use the convention

$$\hat{\gamma}^0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix} , \quad \hat{\gamma}^a = \begin{pmatrix} 0 & -\sigma^a \\ \sigma^a & 0 \end{pmatrix} , \quad \hat{\gamma}^5 = \mathrm{i}\,\hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} . \qquad (76)$$

Then, the invariant fermionic action is

$$S_F = \tfrac{1}{4} \int_{X_M} dx\, \boldsymbol{\psi}^* \hat{\gamma}^0 (D + \mathrm{i}\,\rho_M) \boldsymbol{\psi} \ . \qquad (77)$$

The factor $\frac{1}{4}$ additional to (10) occurs because we are going to impose constraints on $\boldsymbol{\psi}$, which require precisely the form (77) for the action, see below. More explicitly, inserting (52) and (53) and using (25) we obtain

$$D + \mathrm{i}\,\rho_M = \qquad (78a)$$
$$\begin{pmatrix} \mathsf{D} + \mathrm{i}\,\tilde{\pi}(A{+}A'') & -\hat{\gamma}^5\tilde{\pi}(\Psi+m) & -\hat{\gamma}^5\tilde{\pi}(\tilde{\Phi}{+}\tilde{\Xi}{+}\tilde{\Upsilon}) & 0 \\ \hat{\gamma}^5\tilde{\pi}(\tilde{\Psi})^* & \mathsf{D} + \mathrm{i}\,\tilde{\pi}(A{+}A'') & 0 & -\hat{\gamma}^5\tilde{\pi}(\overline{\tilde{\Phi}{+}\tilde{\Xi}{+}\tilde{\Upsilon}}) \\ \hat{\gamma}^5\tilde{\pi}(\tilde{\Phi}{+}\tilde{\Xi}{+}\tilde{\Upsilon})^* & 0 & \mathsf{D} - \hat{\gamma}^2\overline{(\mathrm{i}\,\tilde{\pi}(A{+}A''))}\hat{\gamma}^2 & -\hat{\gamma}^5\overline{\tilde{\pi}(\tilde{\Psi})} \\ 0 & \hat{\gamma}^5\tilde{\pi}(\overline{\tilde{\Phi}{+}\tilde{\Xi}{+}\tilde{\Upsilon}})^* & \hat{\gamma}^5\tilde{\pi}(\tilde{\Psi})^T & \mathsf{D} - \hat{\gamma}^2\overline{(\mathrm{i}\,\tilde{\pi}(A{+}A''))}\hat{\gamma}^2 \end{pmatrix} ,$$

where

$$\begin{aligned} \tilde{\pi}(A{+}A'') &:= \mathrm{diag}(\hat{A} - \tfrac{1}{2}A''\mathbf{1}_{10}) \otimes \mathbf{1}_3 \,,\ \hat{\gamma}^2\overline{(A - \tfrac{3}{2}A''\mathbf{1}_5)}\hat{\gamma}^2 \otimes \mathbf{1}_3 \,,\ -\tfrac{5}{2}A'' \otimes \mathbf{1}_3) \,, \\ \tilde{\pi}(\tilde{\Psi}) &:= \mathrm{diag}\big((\hat{\Psi} + \hat{m}) \otimes M_{10} \ ,\ -\overline{(\Psi + m) \otimes M_5} \ ,\ 0_{3\times 3} \big) \ , \qquad (78b) \\ \tilde{\pi}(\tilde{\Phi}{+}\tilde{\Xi}{+}\tilde{\Upsilon}) &:= \begin{pmatrix} \begin{bmatrix} (\hat{\Phi} + \hat{n}) \otimes M_d \\ +(\Xi + m') \otimes M_N \end{bmatrix} & \begin{bmatrix} (\check{\Phi} + \check{n}) \otimes M_{\tilde{u}} \\ +(\Upsilon + n') \otimes M_{\tilde{n}} \end{bmatrix} & 0 \\ \begin{bmatrix} (\check{\Phi} + \check{n})^T \otimes M_{\tilde{u}}^T \\ +(\Upsilon + n')^T \otimes M_{\tilde{n}}^T \end{bmatrix} & 0 & \overline{(\Phi + n)} \otimes M_e \\ 0 & (\Phi + n)^* \otimes M_e^T & 0 \end{pmatrix} . \end{aligned}$$

We have used that within our convention (76) we have $\hat{\gamma}^5 = -(\hat{\gamma}^5)^*$ and $[\mathsf{D}, \bar{f}] = -\hat{\gamma}^2\overline{[\mathsf{D}, f]}\hat{\gamma}^2$. Recall[13] that $\tilde{\pi}(A{+}A'')$ is given by commutators of $\mathsf{D} \otimes \mathbf{1}_{192}$ with an arbitrary number of elements of the form $f \otimes \hat{\pi}(a)$, where $a \in \mathfrak{a}$ and $f \in C^\infty(X)$. This fact and the complex conjugation in (13) are the reasons why terms of the form $[\mathsf{D}, \bar{f}]$ occur in $\tilde{\pi}(A{+}A'')$.

Minkowskian fermions $\boldsymbol{\psi}$ live in the space $h_M = L^2(X_M, S) \otimes \mathbb{C}^{192}$ and have in terms of the decomposition (78a) the form

$$\boldsymbol{\psi} = (\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \boldsymbol{\psi}_3, \boldsymbol{\psi}_4)^T \ , \quad \boldsymbol{\psi}_i \in L^2(X_M, S) \otimes \mathbb{C}^{48} \ . \qquad (79)$$

However, we shall restrict ourselves to the subspace of h_M invariant under the charge conjugation $\mathcal{C}$, the chirality operator $\tilde{\Gamma}$ and a symmetry transformation $\mathcal{S}$

defined in terms of 48×48-blocks by

$$\mathcal{C} := \begin{pmatrix} 0 & 0 & -\hat\gamma^2\otimes\mathbf{1}_{48} & 0 \\ 0 & 0 & 0 & -\hat\gamma^2\otimes\mathbf{1}_{48} \\ -\hat\gamma^2\otimes\mathbf{1}_{48} & 0 & 0 & 0 \\ 0 & -\hat\gamma^2\otimes\mathbf{1}_{48} & 0 & 0 \end{pmatrix} \circ \text{ c.c} \;, \quad \mathcal{S} := \begin{pmatrix} 0 & \mathbf{1}_{48} & 0 & 0 \\ \mathbf{1}_{48} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{48} \\ 0 & 0 & \mathbf{1}_{48} & 0 \end{pmatrix},$$
$$\tilde\Gamma := \mathrm{diag}(-\hat\gamma^5\otimes\mathbf{1}_{48}\;,\;-\hat\gamma^5\otimes\mathbf{1}_{48}\;,\;\hat\gamma^5\otimes\mathbf{1}_{48}\;,\;\hat\gamma^5\otimes\mathbf{1}_{48})\;, \tag{80}$$

where c.c means complex conjugation. Thus, we consider elements $\boldsymbol{\psi}\in h_M$ of the form

$$\boldsymbol{\psi} = \mathcal{C}\boldsymbol{\psi} = \tilde\Gamma\boldsymbol{\psi} = \mathcal{S}\boldsymbol{\psi} = \begin{pmatrix} \frac{1}{2}(1-\hat\gamma^5)\boldsymbol{\psi}_1 \\ \frac{1}{2}(1-\hat\gamma^5)\boldsymbol{\psi}_1 \\ -\frac{1}{2}(1+\hat\gamma^5)\hat\gamma^2\bar{\boldsymbol{\psi}}_1 \\ -\frac{1}{2}(1+\hat\gamma^5)\hat\gamma^2\bar{\boldsymbol{\psi}}_1 \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{2}(1-\hat\gamma^5)\boldsymbol{\psi}_1 \\ \frac{1}{2}(1-\hat\gamma^5)\boldsymbol{\psi}_1 \\ -\hat\gamma^2\overline{\frac{1}{2}(1-\hat\gamma^5)\boldsymbol{\psi}_1} \\ -\hat\gamma^2\overline{\frac{1}{2}(1-\hat\gamma^5)\boldsymbol{\psi}_1} \end{pmatrix}. \tag{81}$$

Observe that the choice (80) for the chirality operator breaks the structure of the model, which is precisely our intention. Since $\tilde\Gamma$ commutes with $\hat\pi(\mathfrak{a})$, the gauge invariance is not destroyed. But $\tilde\Gamma$ no longer anticommutes with the whole D. We see that $D+\mathrm{i}\,\rho_M$, applied on chiral fermions (81), differs from the matrix (78a) by the absence of $\hat\gamma^5\tilde\pi(\tilde\Psi)$. In other words, the choice (80) for the chirality condition eliminates the disturbing terms $\hat\gamma^5\tilde\pi(\tilde\Psi)$ in the fermionic action.

Within our conventions one has the block structure

$$\tfrac{1}{2}(1-\hat\gamma^5)\boldsymbol{\psi}_1 = \begin{pmatrix} 0 \\ \boldsymbol{\psi}_0 \end{pmatrix}, \quad \boldsymbol{\psi}_0 \in L^2(X_M)\otimes\mathbb{C}^2\otimes\mathbb{C}^{48}\;, \tag{82}$$

where $L^2(X_M)$ denotes the space of square integrable functions on the Minkowski space. In local bases we have $\mathsf{D}=\mathrm{i}\,\hat\gamma^\mu\partial_\mu$, $A=A_\mu\hat\gamma^\mu$ and $A''=A''_\mu\hat\gamma^\mu$. We define $\sigma^0=\tilde\sigma^0=\mathbf{1}_2$ and $\tilde\sigma^a=-\sigma^a$, $a=1,2,3$, or in a symbolic notation

$$\sigma^\mu = \{\mathbf{1}_2,\sigma^a\}\;,\quad \tilde\sigma^\mu=\{\mathbf{1}_2,-\sigma^a\}\;,\qquad \mu=0,1,2,3\;,\quad a=1,2,3\;. \tag{83}$$

Then, from (77), (78a), (81) and (76) we get

$$S_F = \tfrac{1}{2}\int_{X_M}\mathrm{v}_M\,\big(\boldsymbol{\psi}_0^*\,,\,\boldsymbol{\psi}_0^T\sigma^2\big)\begin{pmatrix} \mathrm{i}\,\tilde\sigma^\mu(\partial_\mu+\tilde\pi(A_\mu{+}A''_\mu))\;; & -\,\tilde\pi(\tilde\Phi{+}\tilde\Xi{+}\tilde\Upsilon) \\ -\tilde\pi(\tilde\Phi{+}\tilde\Xi{+}\tilde\Upsilon)^*\;; & \mathrm{i}\,\sigma^\mu(\partial_\mu+\overline{\tilde\pi(A_\mu{+}A''_\mu)}) \end{pmatrix}\begin{pmatrix} \boldsymbol{\psi}_0 \\ \sigma^2\overline{\boldsymbol{\psi}_0} \end{pmatrix}. \tag{84}$$

This formula can be further simplified, because we have

$$\begin{aligned} \int_{X_M}\mathrm{v}_M\,\boldsymbol{\psi}_0^T\sigma^2\,\mathrm{i}\,\sigma^\mu(\partial_\mu+\overline{\tilde\pi(A_\mu{+}A''_\mu)})\sigma^2\overline{\boldsymbol{\psi}_0} &= \int_{X_M}\mathrm{v}_M\,\boldsymbol{\psi}_0^T\,\mathrm{i}(\tilde\sigma^\mu)^T(\partial_\mu+\overline{\tilde\pi(A_\mu{+}A''_\mu)})\overline{\boldsymbol{\psi}_0} \\ &= \int_{X_M}\mathrm{v}_M\,\big((-\mathrm{i}\,\partial_\mu\boldsymbol{\psi}_0^T)(\tilde\sigma^\mu)^T\overline{\boldsymbol{\psi}_0} + \boldsymbol{\psi}_0^T(\tilde\sigma^\mu)^T(-\mathrm{i}\,\tilde\pi(A_\mu{+}A''_\mu))^T\overline{\boldsymbol{\psi}_0}\big) \\ &= \int_{X_M}\mathrm{v}_M\,\boldsymbol{\psi}_0^*\,\mathrm{i}\,\tilde\sigma^\mu(\partial_\mu+\tilde\pi(A_\mu{+}A''_\mu))\boldsymbol{\psi}_0\;. \end{aligned} \tag{85}$$

Here, we have integrated by parts and made use of $\tilde{\pi}(A_\mu+A''_\mu) = -\tilde{\pi}(A_\mu+A''_\mu)^*$. In the last step we took into account that in quantum mechanics the fields $\boldsymbol{\psi}_0$ are annihilation operators and the fields $\overline{\boldsymbol{\psi}_0}$ creation operators. In normal ordered products, all creation operators must stand on the left of all annihilation operators. This means that in (85) we have to exchange $\boldsymbol{\psi}_0$ and $\overline{\boldsymbol{\psi}_0}$. But since they represent fermions, which anticommute, this change of order gives a minus sign. Now, (84) takes the form

$$S_F = \int_{X_M} \mathrm{v}_M \,\big(\boldsymbol{\psi}_0^* \,\mathrm{i}\, \tilde{\sigma}^\mu (\partial_\mu + \tilde{\pi}(A_\mu + A''_\mu))\boldsymbol{\psi}_0 - \tfrac{1}{2}(\boldsymbol{\psi}_0^* \tilde{\pi}(\tilde{\Phi}+\tilde{\Xi}+\tilde{\Upsilon})\sigma^2 \overline{\boldsymbol{\psi}_0} + \mathrm{h.c})\big) \;, \tag{86}$$

where h.c denotes the Hermitian conjugate of the preceding term, without change of signs when exchanging fermion fields. For $\boldsymbol{\psi}_0 \in L^2(X_M) \otimes \mathbb{C}^2 \otimes \mathbb{C}^{48}$ we choose the following parametrization:

$$\begin{aligned}
\boldsymbol{\psi}_0 &= \big(u^r_L\,,\, u^b_L\,,\, u^g_L\,,\, d^r_L\,,\, d^b_L\,,\, d^g_L\,, \sigma^2 \bar{d}^r_R\,, \sigma^2 \bar{d}^b_R\,, \sigma^2 \bar{d}^g_R\,,\, \sigma^2 \bar{\nu}_R\,, \\
&\qquad\qquad -\sigma^2 \bar{u}^r_R\,,\, -\sigma^2 \bar{u}^b_R\,,\, -\sigma^2 \bar{u}^g_R\,,\, -e_L\,,\, \nu_L\,,\, \sigma^2 \bar{e}_R\big)^t, \\
\sigma^2 \bar{\boldsymbol{\psi}}_0 &= \big(\sigma^2 \bar{u}^r_L\,, \sigma^2 \bar{u}^b_L\,, \sigma^2 \bar{u}^g_L\,,\, \sigma^2 \bar{d}^r_L\,,\, \sigma^2 \bar{d}^b_L\,,\, \sigma^2 \bar{d}^g_L\,, -d^r_R\,, -d^b_R\,, -d^g_R\,, -\nu_R\,, \\
&\qquad\qquad u^r_R\,,\, u^b_R\,,\, u^g_R\,, -\sigma^2 \bar{e}_L\,,\, \sigma^2 \bar{\nu}_L\,, -e_R\big)^t,
\end{aligned} \tag{87}$$

where $u^r_L, \dots, e_R \in L^2(X_M) \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ and ^{t} means transposition only of the row, without transposing the matrix elements.

Inserting the matrix structures of (65), (66) and (67) into Eqs. (78b), it is straightforward to write down the explicit formula for the fermionic action (86). Here, one must insert the explicit form[16] of the embeddings $\pi_{10}, \pi_{10,10}, \pi_{10,5}$ and $\pi_{5,1}$. The transformation (70d) requires some care. Let us derive the coefficients of P, Z, Z' corresponding to the left electron. From (78b), (67) and (74) we find for $\theta'_W \ll 1$ in good approximation

$$\begin{aligned}
&\pi_{e_L}(A_\mu + A''_\mu) = -\mathrm{i}\,\tfrac{g_0}{2}\big(W^3_\mu - \sqrt{\tfrac{3}{5}}A' - \tfrac{3}{2}\sqrt{\tfrac{2}{5}}\tilde{A}_\mu\big) = -\tfrac{\mathrm{i}\, g_0}{2\cos(\theta_W - 2\theta'_W)}\tilde{Z}_\mu - \mathrm{i}\,\tilde{e}\tilde{Z}'_\mu + \mathrm{i}\, e\tilde{P}_\mu \,, \\
&\tilde{P}_\mu := P_\mu - \tan(\theta_W - 2\theta'_W)Z_\mu + \big(\tfrac{4}{\sqrt{15}} + \tfrac{12}{5}\theta'_W\big)Z'_\mu\,, \quad \tilde{Z}_\mu := Z_\mu - \tfrac{1}{2}(1 + 2\sqrt{15}\theta'_W)Z'_\mu\,, \\
&\tilde{Z}'_\mu := Z'_\mu + 4\theta'_W \tan\theta_W Z_\mu \;, \qquad\qquad e := \sin\theta_W g_0 \;, \qquad \tilde{e} := \cos\theta_W g_0 \;.
\end{aligned} \tag{88}$$

Moreover, we express $\Phi_0, \Phi_g, \Xi_A, \dots, \Xi_c, \Upsilon_A, \dots, \Upsilon_g$ in terms of the physical Higgs bosons $\phi_0, \phi_g, \xi_A, \dots, \xi_c, \upsilon_A, \dots, \upsilon_g$, see (65), (66) and (71). Then we arrive at the following formula for the fermionic Lagrangian:

$$S_F = \int_{X_M} \mathrm{v}_M \,(\mathcal{L}_q + \mathcal{L}_\ell + \mathcal{L}_m + \mathcal{L}_x + \mathcal{L}_h + \mathcal{L}'_h + \mathcal{L}''_h) \;, \qquad \text{where} \tag{89a}$$

$$\begin{aligned}
\mathcal{L}_q = \big(\boldsymbol{u}^*_L\,,\, \boldsymbol{d}^*_L\big)\Big(\tilde{\sigma}^\mu \begin{pmatrix} \begin{bmatrix} \mathrm{i}\,\partial_\mu - \frac{g_0}{2}\mathbf{G}_\mu \\ -\big(\frac{g_0}{2\cos(\theta_W - 2\theta'_W)}\tilde{Z}_\mu \\ +\frac{2}{3}e\tilde{P}_\mu - \frac{1}{3}\tilde{e}\tilde{Z}'_\mu\big)\mathbf{1}_3 \end{bmatrix} & -\frac{g_0}{2}(W^1_\mu - \mathrm{i}\, W^2_\mu)\mathbf{1}_3 \\ -\frac{g_0}{2}(W^1_\mu + \mathrm{i}\, W^2_\mu)\mathbf{1}_3 & \begin{bmatrix} \mathrm{i}\,\partial_\mu - \frac{g_0}{2}\mathbf{G}_\mu \\ -\big(-\frac{g_0}{2\cos(\theta_W - 2\theta'_W)}\tilde{Z}_\mu \\ -\frac{1}{3}e\tilde{P}_\mu - \frac{1}{3}\tilde{e}\tilde{Z}'_\mu\big)\mathbf{1}_3 \end{bmatrix} \end{pmatrix} \otimes \mathbf{1}_3\Big)\begin{pmatrix} \boldsymbol{u}_L \\ \boldsymbol{d}_L \end{pmatrix} \\
+ \boldsymbol{u}^*_R\big(\sigma^\mu(\mathrm{i}\,\partial_\mu - \tfrac{g_0}{2}\mathbf{G}_\mu - (\tfrac{2}{3}e\tilde{P}_\mu - \tfrac{1}{3}\tilde{e}\tilde{Z}'_\mu)\mathbf{1}_3) \otimes \mathbf{1}_3\big)\boldsymbol{u}_R
\end{aligned}$$

$$+\, \boldsymbol{d}_R^*\big(\sigma^\mu(\mathrm{i}\,\partial_\mu - \tfrac{g_0}{2}\mathbf{G}_\mu - (-\tfrac{1}{3}e\tilde{P}_\mu - \tfrac{1}{3}\tilde{e}\tilde{Z}'_\mu)\mathbf{1}_3)\otimes\mathbf{1}_3\big)\boldsymbol{d}_R\,, \tag{89b}$$

$$\mathcal{L}_\ell = \Big(\nu_L^*\,,\, e_L^*\Big)\big(\tilde{\sigma}^\mu \begin{pmatrix} \mathrm{i}\,\partial_\mu - (\frac{g_0}{2\cos(\theta_W - 2\theta'_W)}\tilde{Z}_\mu + \tilde{e}\tilde{Z}'_\mu) & -\frac{g_0}{2}(W_\mu^1 - \mathrm{i}\,W_\mu^2) \\ -\frac{g_0}{2}(W_\mu^1 + \mathrm{i}\,W_\mu^2) & \begin{bmatrix} \mathrm{i}\,\partial_\mu - (-\frac{g_0}{2\cos(\theta_W - 2\theta'_W)}\tilde{Z}_\mu \\ -e\tilde{P}_\mu + \tilde{e}\tilde{Z}'_\mu) \end{bmatrix} \end{pmatrix} \otimes \mathbf{1}_3\big)\begin{pmatrix}\nu_L \\ e_L\end{pmatrix}$$

$$+\, \nu_R^*\big(\sigma^\mu(\mathrm{i}\,\partial_\mu - \tilde{e}\tilde{Z}'_\mu)\otimes\mathbf{1}_3\big)\nu_R + e_R^*\big(\sigma^\mu(\mathrm{i}\,\partial_\mu - (-e\tilde{P}_\mu + \tilde{e}\tilde{Z}'_\mu))\otimes\mathbf{1}_3\big)e_R\,, \tag{89c}$$

$$\begin{aligned}
\mathcal{L}_m = \big(& - \boldsymbol{d}_L^*(\mathbf{1}_3\otimes(M_d + \tfrac{g_0}{\sqrt{8\mu_1}}\phi_0 M_d) - \tfrac{g_0}{\sqrt{8\mu_3}}(\xi_F^0)^*\otimes M_N)\boldsymbol{d}_R \\
& - e_L^*(M_e + \tfrac{g_0}{\sqrt{8\mu_1}}\phi_0 M_e)e_R - \boldsymbol{u}_L^*(\mathbf{1}_3\otimes(M_u + \tfrac{g_0}{\sqrt{8\mu_1}}\phi_0 M_{\tilde{u}}) + \tfrac{g_0}{4\sqrt{\mu_2}}\upsilon_A\otimes M_{\tilde{n}})\boldsymbol{u}_R \\
& - \nu_L^*(M_n^T + \tfrac{g_0}{\sqrt{8\mu_1}}\phi_0 M_{\tilde{u}}^T - \tfrac{3g_0}{4\sqrt{6\mu_2}}(\upsilon_0 + \mathrm{i}\,\upsilon_{45})M_{\tilde{n}}^T)\nu_R \\
& - \boldsymbol{d}_L^*(\tfrac{g_0}{4\sqrt{\mu_2}}\upsilon_B\otimes M_{\tilde{n}})\boldsymbol{u}_R - e_L^*(\tfrac{3g_0}{4\sqrt{6\mu_2}}(\upsilon_{18} + \mathrm{i}\,\upsilon_{63})\otimes M_{\tilde{n}}^T)\nu_R \\
& + \boldsymbol{u}_L^*((\xi_E^0)^*\otimes M_N)\boldsymbol{d}_R - \tfrac{1}{2}\nu_R^T\sigma_2(M_N + \tfrac{g_0}{\sqrt{8\mu_3}}\xi_0 M_N)\nu_R\big) + \mathrm{h.c}\,,
\end{aligned} \tag{89d}$$

$$\begin{aligned}
\mathcal{L}_x = \tfrac{g_0}{2}\big(& - \boldsymbol{u}_L^*(\tilde{\sigma}^\mu\sigma^2\epsilon(\bar{\mathrm{X}}_\mu)\otimes\mathbf{1}_3)\bar{\boldsymbol{d}}_R - \boldsymbol{d}_L^*(\tilde{\sigma}^\mu\sigma^2\epsilon(\bar{\mathrm{Y}}_\mu)\otimes\mathbf{1}_3)\bar{\boldsymbol{d}}_R \\
& + \boldsymbol{u}_L^*(\tilde{\sigma}^\mu\sigma^2\mathrm{Y}_\mu\otimes\mathbf{1}_3)\bar{\nu}_R - \boldsymbol{d}_L^*(\tilde{\sigma}^\mu\sigma^2\mathrm{X}_\mu\otimes\mathbf{1}_3)\bar{\nu}_R \\
& - \nu_L^*(\tilde{\sigma}^\mu\sigma^2\mathrm{Y}_\mu^T\otimes\mathbf{1}_3)\bar{\boldsymbol{u}}_R + e_L^*(\tilde{\sigma}^\mu\sigma^2\mathrm{X}_\mu^T\otimes\mathbf{1}_3)\bar{\boldsymbol{u}}_R\big) + \mathrm{h.c}\,,
\end{aligned} \tag{89e}$$

$$\begin{aligned}
\mathcal{L}_h = \tfrac{g_0}{\sqrt{8\mu_1}}\big(& - \boldsymbol{u}_L^*\{\sigma^2\epsilon(\bar{\phi}_g)\otimes M_d\}\bar{\boldsymbol{d}}_L + \nu_R^T\{\sigma^2\phi_g^*\otimes M_d\}\boldsymbol{d}_R \\
& - \boldsymbol{u}_L^*\{\sigma^2\phi_g\otimes M_{\tilde{u}}\}\bar{e}_L + \boldsymbol{d}_L^*\{\sigma^2\phi_g\otimes M_{\tilde{u}}\}\bar{\nu}_L \\
& + \boldsymbol{d}_R^T\{\sigma^2\epsilon(\phi_g)\otimes M_{\tilde{u}}\}\boldsymbol{u}_R - \boldsymbol{u}_R^T\{\sigma^2\bar{\phi}_g\otimes M_e\}e_R\big) + \mathrm{h.c}\,,
\end{aligned} \tag{89f}$$

$$\begin{aligned}
\mathcal{L}'_h = \tfrac{g_0}{4\sqrt{\mu_2}}\big(& \boldsymbol{u}_L^*(\sigma^2(\upsilon_a + \upsilon_b)\otimes M_{\tilde{n}})\bar{e}_L - \boldsymbol{d}_L^*(\sigma^2(\upsilon_a - \upsilon_b)\otimes M_{\tilde{n}})\bar{\nu}_L \\
& - \boldsymbol{d}_R^T(\sigma^2(\upsilon_C - \epsilon(\upsilon_a))\otimes M_{\tilde{n}})\boldsymbol{u}_R - \boldsymbol{u}_L^*(\sigma^2\upsilon_c\otimes M_{\tilde{n}})\bar{\nu}_L + \boldsymbol{d}_L^*(\sigma^2\upsilon_d\otimes M_{\tilde{n}})\bar{e}_L \\
& - e_L^*(\upsilon_e^*\otimes M_{\tilde{n}}^T)\boldsymbol{d}_R + \nu_L^*(\upsilon_f^*\otimes M_{\tilde{n}}^T)\boldsymbol{d}_R - \nu_R^T(\sigma^2\upsilon_g^*\otimes M_{\tilde{n}})\boldsymbol{u}_R\big) + \mathrm{h.c}\,,
\end{aligned} \tag{89g}$$

$$\begin{aligned}
\mathcal{L}''_h = \tfrac{g_0}{\sqrt{8\mu_3}}\big(& - \tfrac{1}{2}\boldsymbol{u}_L^*(\sigma^2\overline{\xi_A}\otimes M_N)\bar{\boldsymbol{u}}_L - \boldsymbol{u}_L^*(\sigma^2(\overline{\xi_D} - \tfrac{1}{2}\varepsilon(\overline{\xi_c}))\otimes M_N)\bar{\boldsymbol{d}}_L \\
& + \boldsymbol{u}_L^*(\xi_a\otimes M_N)\nu_R - \tfrac{1}{2}\boldsymbol{d}_L^*(\sigma^2\overline{\xi_B}\otimes M_N)\bar{\boldsymbol{d}}_L + \boldsymbol{d}_L^*(\xi_b\otimes M_N)\nu_R \\
& + \tfrac{1}{2}\boldsymbol{d}_R^T(\sigma_2\xi_C\otimes M_N)\boldsymbol{d}_R + \boldsymbol{d}_R^T(\sigma_2\overline{\xi_c}\otimes M_N)\nu_R\big) + \mathrm{h.c}\,.
\end{aligned} \tag{89h}$$

The Lagrangian $\mathcal{L}_q$ contains the kinetic terms and the strong and electroweak interactions of quarks. The Lagrangian $\mathcal{L}_\ell$ contains the kinetic terms and electroweak interactions of leptons. The Lagrangian $\mathcal{L}_m$ contains the mass terms of the fundamental fermions and their interactions with the Higgs fields $\phi_0, \xi_E^0, \xi_F^0, \xi_0, \upsilon_A$ and υ_B . The masses of the u, c, t-quarks, the d, s, b-quarks and the e, μ, τ-leptons are the eigenvalues of M_u, M_d and M_e . The mass Lagrangian of the neutrino sector is

$$-\tfrac{1}{2}\big(-\nu_L^*\,,\,\nu_R^T\sigma_2\big)\begin{pmatrix} 0 & -M_n \\ -M_n & M_N \end{pmatrix}\begin{pmatrix} \sigma_2\bar{\nu}_L \\ \nu_R \end{pmatrix} + \mathrm{h.c}\,. \tag{90}$$

The diagonalization of the mass matrix occurring in (90) yields the masses of the neutrinos. The mixing angles are small for $\|M_N\| \gg \|M_n\|$. In this case, the left-handed neutrinos receive a mass of the order $\frac{\|M_n\|^2}{2\|M_N\|}$ and the right-handed neutrinos a mass of the order $\frac{1}{2}\|M_N\|$. Thus, for $\|M_n\|$ and $\|M_N\|$ being of the order of the mass of the top quark and the unification scale, respectively, we obtain very low masses for the left-handed neutrinos compatible with experiments (seesaw mechanism). Moreover, the matrices M_u, M_d, M_e, M_n and M_N contain mixing angles between the fermions, which constitute generalized Kobayashi–Maskawa matrices. Finally, the Lagrangians $\mathcal{L}_x, \mathcal{L}_h, \mathcal{L}'_h$ and $\mathcal{L}''_h$ describe the coupling of the fermions to the X and Y leptoquarks, the Higgs bosons ϕ_g and the remaining Higgs bosons υ_i and ξ_i. All terms of these Lagrangians contribute to the proton decay.

Observe that the Lagrangians $\mathcal{L}_q$ and $\mathcal{L}_\ell$ differ from the corresponding Lagrangians of the standard model in two aspects: First, there occurs the massive gauge field Z', which of course is not a terrible problem if its mass is sufficiently large. Second, the universal Weinberg angle θ_W of the standard model is modified by angles of the order θ'_W. However, this angle θ'_W is extremely small if $m_{Z'}$ is very large against m_Z. This means that experiments will certainly not detect θ'_W.

5. The Masses of Yang–Mills and Higgs Fields

The final step is to compute the boson masses. For that purpose we must compute the parameters $\mu^i, \tilde{\mu}^i, \check{\mu}^i, \hat{\mu}^i$ of the Higgs potential given in Appendix C, which depend according to Appendix B on the mass matrices occurring in the generalized Dirac operator $\mathcal{M}$. We have found in Sec. 4.4 that the eigenvalues

$$\begin{array}{llllllll} \text{of} & M_u M_u^* & \text{are} & m_u^2, m_c^2, m_t^2\,, & \text{of} & M_d M_d^* & \text{are} & m_d^2, m_s^2, m_b^2\,, \\ \text{of} & M_e M_e^* & \text{are} & m_e^2, m_\mu^2, m_\tau^2\,, & & & & \end{array} \tag{91}$$

referring to the usual names of the fermions. By unitary transformations we can achieve that M_u is diagonal. It is necessary to make several assumptions to simplify the calculation: Since the Kobayashi–Maskawa matrix between M_u and M_d is approximately the identity matrix, let us assume that M_d is diagonal as well:

$$M_u = \operatorname{diag}(m_u, m_c, m_t)\,, \qquad M_d = \operatorname{diag}(m_d, m_s, m_b)\,. \tag{92a}$$

The experimental data show that m_t is much bigger than all other eigenvalues. Among the remaining eigenvalues we neglect all but m_b^2 and m_τ^2. For simplicity we also neglect m_τ^2 against m_b^2, although this is not completely justified. Unfortunately, there are no experimental values for the matrix M_n. Therefore, we can only estimate its contribution: We assume that in the case (92a) we have

$$M_n = \operatorname{diag}(0, 0, \mathrm{e}^{\mathrm{i}\,\chi} m_n)\,. \tag{92b}$$

Quantum corrections suggest that m_n is of the order m_t. Using (34b) we find for the parameters μ_1 and μ_2 given in Appendix B approximately

$$\begin{aligned} \mu_1 &= \tfrac{1}{8} m_b^2 + \tfrac{1}{96}(9m_t^2 + 6 m_t m_n \cos\chi + m_n^2) + \tfrac{1}{24} m_\tau^2\,, \\ \mu_2 &= \tfrac{1}{384}(m_t^2 - 2 m_t m_n \cos\chi + m_n^2)\,, \end{aligned} \tag{93}$$

which yields according to (74) for the mass m_W of the W boson

$$m_W^2 = \tfrac{1}{4}(m_t^2 + m_b^2 + \tfrac{1}{3} m_n^2 + \tfrac{1}{3} m_\tau^2)\,. \tag{94}$$

The comparison with the experimental values for m_t and m_W requires that m_n is small against m_t. Thus, we shall neglect m_n against m_t whenever this is possible.

Since (at energies accessible at present) the standard model is in excellent agreement with experiments, the parameter $\mu_3 \sim \mathrm{tr}(M_N M_N^*)$ must be very large to give a huge mass to the Z' boson. We choose the parametrization

$$M_N = m_N\, U\, \mathrm{diag}(\sin\theta_N \cos\phi_N\,,\, \sin\theta_N \sin\phi_N\,,\, \cos\theta_N) U^T\ , \tag{95}$$

for $U \in \mathrm{U}(3)$, where the parameter $m_N \gg m_t$ determines the mass scale.

The mass of the X and Y bosons must be very large in order to suppress the proton decay. This could be achieved by a sufficiently large μ_3, however, there are also Higgs bosons which induce an insufficient lifetime for the proton if μ_0 is too small. Therefore, we must demand

$$\max(\mathrm{tr}(M_{10}M_{10}^*)\,,\,\mathrm{tr}(M_5 M_5^*)) \gg \mathrm{tr}(M_u M_u^*)\ . \tag{96}$$

We put[c]

$$M_{10} = M\mathbf{1}_3 + m_{10}\ ,\quad M_5 = M\mathbf{1}_3 + m_5\ ,\quad M \in \mathbb{R}\ , \tag{97}$$

where $m_{10}, m_5 \in \mathrm{M}_3\mathbb{C}$ are perturbations, which we consider for the time being as small against $M\mathbf{1}_3$. Thus, we obtain for the parameters $\mu_{0,\dots,3}$ in Appendix B approximately

$$\mu_0 = \tfrac{1}{4}M^2\ ,\qquad \mu_1 = \tfrac{3}{32}m_t^2\ ,\qquad \mu_2 = \tfrac{1}{384}m_t^2\ ,\qquad \mu_3 = \tfrac{1}{48}m_N^2\ . \tag{98}$$

Inserting the leading approximation (97) into the quadratic terms of the Higgs potential given in Appendix C, we can distinguish linear combinations of μ^{i} to μ^{t} that do not depend on M. It turns out that the following combinations are essential:

$$\begin{aligned}
&\tfrac14\mu^{\mathrm{i}}+\tfrac14\mu^{\mathrm{j}}+\tfrac14\mu^{\mathrm{k}}+\tfrac14\mu^{\mathrm{l}}-\tfrac14\mu^{\mathrm{m}}+\tfrac14\mu^{\mathrm{n}}-\tfrac12\mu^{\mathrm{p}}+\tfrac14\mu^{\mathrm{r}}-\tfrac14\mu^{\mathrm{t}} = \tfrac12\,\mathrm{tr}(\tilde M_u\tilde M_u^*+\hat M_u\hat M_u^*) =: \tilde\lambda_1^2 m_t^4,\\
&\tfrac14\mu^{\mathrm{i}}+\tfrac14\mu^{\mathrm{j}}+\tfrac94\mu^{\mathrm{k}}+\tfrac94\mu^{\mathrm{l}}-\tfrac14\mu^{\mathrm{m}}-\tfrac34\mu^{\mathrm{n}}+\tfrac32\mu^{\mathrm{p}}-\tfrac34\mu^{\mathrm{r}}-\tfrac94\mu^{\mathrm{t}}\\
&\qquad\qquad = \tfrac12\,\mathrm{tr}(\tilde M_n\tilde M_n^*+\hat M_n\hat M_n^*) =: \tilde\lambda_2^2 m_t^2 m_n^2\,,\\
&\tfrac12\mu^{\mathrm{i}}+\tfrac12\mu^{\mathrm{j}}-\tfrac32\mu^{\mathrm{k}}-\tfrac32\mu^{\mathrm{l}}-\tfrac12\mu^{\mathrm{m}}-\tfrac12\mu^{\mathrm{n}}+\mu^{\mathrm{p}}-\tfrac12\mu^{\mathrm{r}}+\tfrac32\mu^{\mathrm{t}}\\
&\qquad\qquad = \tfrac12\,\mathrm{tr}(\tilde M_u\tilde M_n^*+\hat M_u\hat M_n^*+\tilde M_n\tilde M_u^*+\hat M_n\hat M_u^*) =: 2\tilde\lambda_3^2 m_t^3 m_n\cos\chi\,,\\
&\mu^{\mathrm{o}}-2\mu^{\mathrm{q}}+\mu^{\mathrm{s}} = \tfrac12\,\mathrm{i}\ \mathrm{tr}(\tilde M_u\tilde M_n^*+\hat M_u\hat M_n^*-\tilde M_n\tilde M_u^*-\hat M_n\hat M_u^*) =: 2\tilde\lambda_4^2 m_t^3 m_n\sin\chi\,,
\end{aligned} \tag{99a}$$

where

$$\begin{aligned}
\tilde M_u &= m_{10}M_u - M_u m_5\ , &\qquad \hat M_u &= m_{10}^*M_u - M_u m_5^*\ ,\\
\tilde M_n &= m_{10}M_n - M_n m_5\ , &\qquad \hat M_n &= m_{10}^*M_n - M_n m_5^*\ ,
\end{aligned} \tag{99b}$$

see (34b) and Appendix B. Within our assumptions (92) we have

$$\tilde\lambda_1 = \tilde\lambda_2 = \tilde\lambda_3 = \tilde\lambda_4 \equiv \lambda < \tfrac{M}{m_t}\ . \tag{99c}$$

The matrices M_{10}' and M_5' enter the matrices in Appendix A only quadratically. Neglecting quadratic terms in m_{10} and m_5 we have

$$M_i^2 = \mathrm{diag}(M^2\mathbf{1}_3 + M(m_i+m_i^*)\,,\, M^2\mathbf{1}_3 + M(m_i+m_i^*))\ ,\quad i = 10,\,5\ .$$

[c] The choice $M_{10} = (M\mathbf{1}_3 + m_{10})\ ,\ \ M_5 = \mathrm{e}^{\mathrm{i}\,\chi_0}(M\mathbf{1}_3 + m_5)$ yields the same results.

Thus, we may assume $m_{10} = m_{10}^*$ and $m_5 = m_5^*$. Moreover, we may assume $\mathrm{tr}(m_{10}) = 0$, because the transformation $m_5 \mapsto m_5 + \nu \mathbf{1}_3$ and $m_{10} \mapsto m_{10} + \nu \mathbf{1}_3$, for $\nu \in \mathbb{R}$, leaves the matrices M^i_{aa} and $\hat{M}^i_{aa}$ invariant. Therefore, we make the ansatz

$$m_5 = \textstyle\sum_{a=1}^8 \nu_a^5 \lambda^a + \frac{1}{\sqrt{3}} \nu_0^5 \;, \qquad m_{10} = \sum_{a=1}^8 \nu_a^{10} \lambda^a \;, \tag{100}$$

where λ^a are the Gell-Mann matrices and $\nu_a^j \in \mathbb{R}$. We introduce the abbreviations

$$\begin{aligned} &\cos^4 \theta_N + \sin^4 \theta_N (\cos^4 \phi_N + \sin^4 \phi_N) \equiv \tfrac{1}{3}(1 + 2\cos^2 \chi_N) \;, \\ &\nu_{10}^2 = 2 \textstyle\sum_{i=1}^8 (\nu_i^{10})^2 \;, \qquad (\nu_1^{10})^2 + (\nu_2^{10})^2 = \frac{1}{\sqrt{2}} \nu_{10} \sin \tilde{\chi} \sin \tilde{\chi}' \cos \tilde{\chi}'' \;. \end{aligned} \tag{101}$$

For physical reasons we assume

$$M, m_N \gg \lambda m_t, \gg m_t \gg m_b, m_n, m_\tau \;. \tag{102}$$

Inserting (92), (95) and (100) into the parameters of Appendix B and this result into the Higgs potential given in Appendix C, we find that – apart from the combinations (99a) – only the following parameters are relevant in leading approximation:

$$\begin{aligned}
&\mu^{\mathrm{b}} = \tfrac{351}{160} m_t^4 \;, && \mu^{\mathrm{c}} = \tfrac{13}{7680} m_t^4 \;, \\
&\mu^{\mathrm{f}} = \tfrac{39}{320} m_t^4 \;, && \mu^{\mathrm{g}} = 12 M^2 m_b^2 \;, \\
&\mu^{\mathrm{h}} = \tfrac{1}{2} m_b^2 \nu_{10}^2 \sin^2 \tilde{\chi} \sin^2 \tilde{\chi}' \sin^2 \tilde{\chi}'' && \mu^{\mathrm{i}} = \mu^{\mathrm{j}} = \tfrac{1}{2} \mu^{\mathrm{m}} = \tfrac{9}{4} M^2 m_t^2 \;, \\
&\mu^{\mathrm{k}} = \mu^{\mathrm{l}} = \tfrac{1}{2} \mu^{\mathrm{t}} = \tfrac{1}{4} M^2 m_t^2 \;, && \mu^{\mathrm{n}} = \mu^{\mathrm{p}} = \mu^{\mathrm{r}} = \tfrac{3}{2} M^2 m_t^2 \;, \\
&\check{\mu}^{\mathrm{a}} = \tfrac{1}{120} m_N^4 (1 + 16 \cos^2 \chi_N) \;, && \check{\mu}^{\mathrm{c}} = \tfrac{3}{10} m_N^2 m_t^2 (|1{+}2\cos \chi_N \cos \hat{\chi}| {-} \tfrac{5}{4}) \;, \\
&\check{\mu}^{\mathrm{d}} = \tfrac{1}{120} m_N^2 m_t^2 (|1{+}2\cos \chi_N \cos \hat{\chi}| {-} \tfrac{5}{4}) \;, && \check{\mu}^{\mathrm{e}} = 2 M^2 m_N^2 \;, \\
&\tilde{\mu}^{\mathrm{b}} = \tfrac{1}{176} m_t^4 \;, && \tilde{\mu}^{\mathrm{d}} = \tfrac{9}{176} m_t^4 \;, \\
&\tilde{\mu}^{\mathrm{k}} = \tfrac{3}{88} m_t^4 \;, && \tilde{\mu}^{\mathrm{p}} = \tfrac{1}{192} m_t^4 \sin^2 \hat{\chi} \;, \\
&\tilde{\mu}^{\mathrm{q}} = \tfrac{3}{64} m_t^4 \sin^2 \hat{\chi} \;, && \tilde{\mu}^{\mathrm{s}} = \tfrac{1}{32} m_t^4 \sin^2 \hat{\chi} \;, \\
&\hat{\mu}^{\mathrm{a}} = 2 m_N^4 ((1 + 2\cos^2 \chi_N) - \tfrac{1}{4}) \cos^2 \hat{\chi}_a \;, && \hat{\mu}^{\mathrm{c}} = \tfrac{1}{88} m_N^2 m_t^2 (|1{+}2\cos \chi_N \cos \hat{\chi}| {-} 1) \;, \\
&\hat{\mu}^{\mathrm{e}} = \tfrac{3}{88} m_N^2 m_t^2 (|1 + 2\cos \chi_N \cos \hat{\chi}| {-} 1) \;.
\end{aligned} \tag{103}$$

The parameters $\hat{\chi}$ and $\hat{\chi}_a$ are complicated functions of the mass matrices. Now we find for (75c) in tree-level approximation

$$\begin{aligned}
\mathcal{L}_H = {} & \tfrac{1}{2} g^{\mu\nu} \big(\partial_\mu \phi_0' \, \partial_\nu \phi_0' + \partial_\mu \upsilon_0' \, \partial_\nu \upsilon_0' + \partial_\mu \upsilon_{45} \, \partial_\nu \upsilon_{45} \\
& + \partial_\mu \psi_0 \, \partial_\nu \psi_0 + \partial_\mu \psi_3' \, \partial_\nu \psi_3' + \partial_\mu \xi_0 \, \partial_\nu \xi_0 \big) \\
& - \tfrac{1}{2} \big(\lambda^2 m_t^2 \upsilon_0^2 + \tfrac{3}{4} \lambda^2 m_t^2 \upsilon_{45}^2 + \tfrac{m_N^4}{M^2} (\tfrac{1}{144} \cos^2 \hat{\chi}_a + \tfrac{1}{54} \cos^2 \chi_N \cos^2 \hat{\chi}_a) {\psi_3'}^2 \\
& + \tfrac{1}{48 M^2} (\tfrac{48}{5} \check{\mu}^{\mathrm{e}} + \tfrac{2}{9} \hat{\mu}^{\mathrm{a}}) \psi_0^2 + \tfrac{1}{2 m_N^2} (4 \check{\mu}^{\mathrm{a}} + \tfrac{8}{15} \hat{\mu}^{\mathrm{a}}) \xi_0^2 \\
& + (\tfrac{207}{110} + \tfrac{2}{9} \sin^2 \hat{\chi}) m_t^2 {\phi_0'}^2 + \tfrac{1}{8\sqrt{15} M m_t} (-\tfrac{32}{3} \hat{\mu}^{\mathrm{c}} + \tfrac{32}{3} \hat{\mu}^{\mathrm{e}}) \psi_0 \phi_0' \\
& + \tfrac{1}{\sqrt{24} m_N m_t} (4 \check{\mu}^{\mathrm{c}} + 48 \check{\mu}^{\mathrm{d}} + \tfrac{64}{5} \hat{\mu}^{\mathrm{c}} - \tfrac{64}{5} \hat{\mu}^{\mathrm{e}}) \phi_0' \xi_0 - \tfrac{2}{9\sqrt{10} M m_N} \hat{\mu}^{\mathrm{a}} \psi_0 \xi_0 \big) \\
& + \tfrac{1}{2} g^{\mu\nu} \big(\partial_\mu \psi_1' \, \partial_\nu \psi_1' + \partial_\mu \psi_2' \, \partial_\nu \psi_2' + \partial_\mu \upsilon_{18} \, \partial_\nu \upsilon_{18} + \partial_\mu \upsilon_{63} \, \partial_\nu \upsilon_{63} \big)
\end{aligned} \tag{104}$$

$$
\begin{aligned}
&\quad -\tfrac{1}{2}\big(\tfrac{m_N^4}{M^2}(\tfrac{1}{144}\cos^2\hat{\chi}_a+\tfrac{1}{54}\cos^2\chi_N\cos^2\hat{\chi}_a)({\psi_1'}^2+{\psi_2'}^2)+\tfrac{3}{4}\lambda^2 m_t^2(\upsilon_{18}^2+\upsilon_{63}^2)\big)\\
&+\tfrac{1}{2}g^{\mu\nu}(\textstyle\sum_{i=1}^{8}\partial_\mu\psi_i\,\partial_\nu\psi_i+\sum_{i=1}^{8}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=46}^{53}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i\\
&\quad +\textstyle\sum_{i=33}^{40}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=82}^{89}\partial_\mu\xi_i\,\partial_\nu\xi_i)\\
&\quad -\tfrac{1}{2}\big(\tfrac{m_N^4}{M^2}(\tfrac{1}{144}\cos^2\hat{\chi}_a+\tfrac{1}{54}\cos^2\chi_N\cos^2\hat{\chi}_a)\textstyle\sum_{i=1}^{8}\psi_i^2+9M^2(\sum_{i=33}^{40}\xi_i^2+\sum_{i=82}^{89}\xi_i^2)\big)\\
&\quad +(\lambda^2 m_n^2+\tfrac{m_b^2\nu_{10}^2}{m_t^2}\sin^2\tilde{\chi}\sin^2\tilde{\chi}'\sin^2\tilde{\chi}'')(\textstyle\sum_{i=1}^{8}\upsilon_i^2+\sum_{i=46}^{53}\upsilon_i^2)\\
&+\tfrac{1}{2}g^{\mu\nu}(\textstyle\sum_{i=19}^{26}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=64}^{71}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=25}^{32}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=74}^{81}\partial_\mu\xi_i\,\partial_\nu\xi_i)\\
&\quad -\tfrac{1}{2}\big((\lambda^2 m_n^2+\tfrac{m_b^2\nu_{10}^2}{m_t^2}\sin^2\tilde{\chi}\sin^2\tilde{\chi}'\sin^2\tilde{\chi}'')(\textstyle\sum_{i=19}^{26}\upsilon_i^2+\sum_{i=64}^{71}\upsilon_i^2)\\
&\quad +9M^2(\textstyle\sum_{i=25}^{32}\xi_i^2+\sum_{i=74}^{81}\xi_i^2)\big)\\
&+\tfrac{1}{2}g^{\mu\nu}(\textstyle\sum_{i=1}^{6}\partial_\mu\phi_i\,\partial_\nu\phi_i+\sum_{i=9}^{14}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=54}^{59}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i\\
&\quad +\textstyle\sum_{i=30}^{35}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=75}^{80}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=39}^{41}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=84}^{86}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i\\
&\quad +\textstyle\sum_{i=19}^{24}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=68}^{73}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=44}^{46}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=93}^{95}\partial_\mu\xi_i\,\partial_\nu\xi_i\\
&\quad +\textstyle\sum_{i=47}^{49}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=96}^{98}\partial_\mu\xi_i\,\partial_\nu\xi_i)\\
&\quad -\tfrac{1}{2}\big(M^2(\textstyle\sum_{i=1}^{6}\phi_i^2+\sum_{i=9}^{14}\upsilon_i^2+\sum_{i=54}^{59}\upsilon_i^2+\sum_{i=30}^{35}\upsilon_i^2+\sum_{i=75}^{80}\upsilon_i^2)\\
&\quad +4M^2(\textstyle\sum_{i=39}^{41}\upsilon_i^2+\sum_{i=84}^{86}\upsilon_i^2+\sum_{i=19}^{24}\xi_i^2+\sum_{i=68}^{73}\xi_i^2+\sum_{i=47}^{49}\xi_i^2+\sum_{i=96}^{98}\xi_i^2)\\
&\quad +(M^2+m_N^2(\tfrac{1}{12}\cos^2\hat{\chi}_a+\tfrac{2}{9}\cos^2\chi_N\cos^2\hat{\chi}_a))(\textstyle\sum_{i=44}^{46}\xi_i^2+\sum_{i=93}^{95}\xi_i^2)\big)\\
&+\tfrac{1}{2}g^{\mu\nu}(\textstyle\sum_{i=7}^{18}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=56}^{67}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=41}^{43}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=90}^{92}\partial_\mu\xi_i\,\partial_\nu\xi_i\\
&\quad +\textstyle\sum_{i=15}^{17}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=60}^{62}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=42}^{44}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=87}^{89}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i\\
&\quad +\textstyle\sum_{i=36}^{38}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=81}^{83}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i)\\
&\quad -\tfrac{1}{2}\big(M^2(\textstyle\sum_{i=15}^{17}\upsilon_i^2+\sum_{i=60}^{62}\upsilon_i^2+\sum_{i=42}^{44}\upsilon_i^2+\sum_{i=87}^{89}\upsilon_i^2)\\
&\quad +4M^2(\textstyle\sum_{i=36}^{38}\upsilon_i^2+\sum_{i=81}^{83}\upsilon_i^2+\sum_{i=7}^{12}\xi_i^2+\sum_{i=56}^{61}\xi_i^2)+16M^2(\sum_{i=13}^{18}\xi_i^2+\sum_{i=62}^{69}\xi_i^2)\\
&\quad +(M^2+m_N^2(\tfrac{1}{12}\cos^2\hat{\chi}_a+\tfrac{2}{9}\cos^2\chi_N\cos^2\hat{\chi}_a))(\textstyle\sum_{i=41}^{43}\xi_i^2+\sum_{i=90}^{92}\xi_i^2)\big)\\
&+\tfrac{1}{2}g^{\mu\nu}(\textstyle\sum_{i=27}^{29}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=72}^{74}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=1}^{6}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=50}^{55}\partial_\mu\xi_i\,\partial_\nu\xi_i)\\
&\quad -\tfrac{1}{2}\big(4M^2(\textstyle\sum_{i=1}^{6}\xi_i^2+\sum_{i=50}^{55}\xi_i^2)+M^2(\sum_{i=27}^{29}\upsilon_i^2+\sum_{i=72}^{74}\upsilon_i^2)\big)\;.
\end{aligned}
$$

It remains to find the eigenvalues of the quadratic form[d] determined by the $\{\phi_0',\psi_0,\xi_0\}$ sector in (104). We use the general result that the smallest (largest) eigenvalue is smaller (larger) than the smallest (largest) diagonal matrix element. This means that the mass of the ϕ_0' Higgs field is smaller than $\sqrt{\frac{2083}{990}}\,m_t\approx 1.45\,m_t$. We assume $\frac{48}{5}\check{\mu}^{\mathrm{e}}\gg\frac{2}{9}\hat{\mu}^{\mathrm{a}}$, or $M^2\gg\frac{55}{864}m_N^2$. Then, the large parameter $\check{\mu}^{\mathrm{e}}$ occurring in the coefficient of ψ_0^2 stabilizes the other two eigenvalues near the diagonal matrix elements $\frac{1}{5M^2}\check{\mu}^{\mathrm{e}}$ and $\frac{1}{m_N^2}(2\check{\mu}^{\mathrm{a}}+\frac{4}{15}\hat{\mu}^{\mathrm{a}})$, respectively.

For convenience we list in Table 1 our tree-level predictions for the masses of the Higgs fields and the masses of the gauge fields derived in Sec. 4.3. We recall that m_t is the mass of the top quark, m_N the mass scale of the right neutrinos and

[d] The corresponding matrix is positive definite by construction. This is not apparent when inserting (103), because there are complicated relations between $\chi_N,\hat{\chi}_a,\hat{\chi}$.

Table 1. The particle masses for the SU(5) × U(1)-model

Particle	Mass	Particle	Mass
1. The completely neutral Higgs fields:			
ϕ'_0	$(0\ldots 1.45)\, m_t$	ξ_0	$(\sqrt{\frac{1}{60}}\cdots\sqrt{\frac{7}{4}})m_N$
υ'_0	λm_t	υ_{45}	$\frac{1}{2}\sqrt{3}\lambda m_t$
ψ_0	$\sqrt{\frac{2}{5}}m_N$	ψ'_3	$(0\ldots\frac{1}{12}\sqrt{\frac{11}{3}})\frac{m_N^2}{M}$
2. The colour-neutral Higgs fields of charge ∓ 1:			
$\frac{1}{\sqrt{2}}(\upsilon_{18}\pm \mathrm{i}\,\upsilon_{63})$	$\frac{1}{2}\sqrt{3}\lambda m_t$	$\frac{1}{\sqrt{2}}(\psi_1\pm \mathrm{i}\,\psi_2)$	$(0\ldots\frac{1}{12}\sqrt{\frac{11}{3}})\frac{m_N^2}{M}$
3. The neutral Higgs fields, for $i=0,\ldots,7$:			
ψ_{1+i}	$(0\ldots\frac{1}{12}\sqrt{\frac{11}{3}})\frac{m_N^2}{M}$		
υ_{1+i}	$(\lambda\ldots\lambda{+}\check{\lambda})m_n$	υ_{45+i}	$(\lambda\ldots\lambda{+}\check{\lambda})m_n$
ξ_{32+i}	$3M$	ξ_{81+i}	$3M$
4. The Higgs fields of charge ∓ 1, for $i=0\ldots 7$:			
$\frac{1}{\sqrt{2}}(\upsilon_{19+i}\pm \mathrm{i}\,\upsilon_{64+i})$	$(\lambda\ldots\lambda{+}\check{\lambda})m_n$	$\frac{1}{\sqrt{2}}(\xi_{25+i}\pm \mathrm{i}\,\xi_{74+i})$	$3M$
5. The Higgs fields of charge $\mp\frac{1}{3}$, for $i=0,1,2$ and $j=0,\ldots,5$:			
$\frac{1}{\sqrt{2}}(\phi_{1+i}\pm \mathrm{i}\,\phi_{4+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{9+i}\pm \mathrm{i}\,\upsilon_{54+i})$	M
$\frac{1}{\sqrt{2}}(\upsilon_{12+i}\pm \mathrm{i}\,\upsilon_{57+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{39+i}\pm \mathrm{i}\,\upsilon_{84+i})$	$2M$
$\frac{1}{\sqrt{2}}(\xi_{44+i}\pm \mathrm{i}\,\xi_{93+i})$	M	$\frac{1}{\sqrt{2}}(\xi_{47+i}\pm \mathrm{i}\,\xi_{96+i})$	$2M$
$\frac{1}{\sqrt{2}}(\xi_{19+j}\pm \mathrm{i}\,\xi_{68+j})$	$2M$	$\frac{1}{\sqrt{2}}(\upsilon_{30+j}\pm \mathrm{i}\,\upsilon_{75+j})$	M
6. The Higgs fields of charge $\pm\frac{2}{3}$, for $i=0,1,2$ and $j=0,\ldots,5$:			
$\frac{1}{\sqrt{2}}(\upsilon_{15+i}\pm \mathrm{i}\,\upsilon_{60+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{36+i}\pm \mathrm{i}\,\upsilon_{81+i})$	$2M$
$\frac{1}{\sqrt{2}}(\upsilon_{42+i}\pm \mathrm{i}\,\upsilon_{87+i})$	M	$\frac{1}{\sqrt{2}}(\xi_{41+i}\pm \mathrm{i}\,\xi_{90+i})$	M
$\frac{1}{\sqrt{2}}(\xi_{7+j}\pm \mathrm{i}\,\xi_{56+j})$	$2M$	$\frac{1}{\sqrt{2}}(\xi_{13+j}\pm \mathrm{i}\,\xi_{62+j})$	$4M$
7. The Higgs fields of charge $\mp\frac{4}{3}$, for $i=0,1,2$ and $j=0,\ldots,5$:			
$\frac{1}{\sqrt{2}}(\upsilon_{27+i}\pm \mathrm{i}\,\upsilon_{72+i})$	M	$\frac{1}{\sqrt{2}}(\xi_{1+j}\pm \mathrm{i}\,\xi_{50+j})$	$2M$
8. The neutral massive gauge fields:			
Z	$\sqrt{\frac{2}{5}}\,m_t$	Z'	$\frac{1}{2}\sqrt{\frac{5}{3}}m_N$
9. The massive gauge fields of charge ± 1:			
$\frac{1}{\sqrt{2}}(W_1\mp \mathrm{i}\,W_2)$	$\frac{1}{2}m_t$	Weinberg angle: $\sin^2\theta_W=\frac{3}{8}$	
10. The leptoquarks leading to proton decay, for $i=0,1,2$:			
$\frac{1}{\sqrt{2}}(X_{1+i}\mp \mathrm{i}\,X_{4+i})$	M	charge: $\mp\frac{1}{3}$	
$\frac{1}{\sqrt{2}}(Y_{1+i}\mp \mathrm{i}\,Y_{4+i})$	M	charge: $\pm\frac{2}{3}$	

M the grand unification scale, where we have assumed $m_N, M \gg m_t$. Moreover, we have introduced the abbreviation

$$\check{\lambda} = \sqrt{\lambda^2 + \tfrac{m_b^2 \nu_{10}^2}{m_t^2 m_n^2}} - \lambda \geq 0 \;.$$

It is interesting to perform the transformation (72) in the Yukawa Lagrangian $\mathcal{L}_m$ of the fermionic action (89). The contribution of the coupling of the ϕ_0' Higgs field to the fermions takes the form

$$\begin{aligned}
\mathcal{L}_{\phi_0'} &= \big(-\boldsymbol{d}_L^*(\mathbf{1}_3 \otimes (M_d + \tfrac{g_0}{\sqrt{8(\mu_1+12\mu_2)}}\phi_0' M_d))\boldsymbol{d}_R - e_L^*(M_e + \tfrac{g_0}{\sqrt{8(\mu_1+12\mu_2)}}\phi_0' M_e)e_R \\
&\quad -\boldsymbol{u}_L^*(\mathbf{1}_3 \otimes (M_u + \tfrac{g_0}{\sqrt{8(\mu_1+12\mu_2)}}\phi_0' M_u))\boldsymbol{u}_R - \nu_L^*(M_n^T + \tfrac{g_0}{\sqrt{8(\mu_1+12\mu_2)}}\phi_0' M_n^T)\nu_R\big) + \text{h.c} \\
&= \big(- \boldsymbol{d}_L^*(\mathbf{1}_3 \otimes (1 + \tfrac{g_0}{m_t}\phi_0')M_d)\boldsymbol{d}_R - e_L^*((1 + \tfrac{g_0}{m_t}\phi_0')M_e)e_R \\
&\quad - \boldsymbol{u}_L^*(\mathbf{1}_3 \otimes (1 + \tfrac{g_0}{m_t}\phi_0')M_u))\boldsymbol{u}_R - \nu_L^*((1 + \tfrac{g_0}{m_t}\phi_0')M_n^T)\nu_R\big) + \text{h.c} \;. \qquad (105)
\end{aligned}$$

Thus, the Higgs field ϕ_0' has the same properties as the standard model Higgs field.

All other Higgs fields are too massive to observe. All Higgs and gauge fields with fractional-valued charge lead to proton decay. Without exception they receive a mass of the order of the grand unification scale M , which must be chosen sufficiently large to ensure the observed stability of matter. The mass of the remaining Higgs fields with integer-valued charge is of the order $M, \lambda m_t, \lambda m_n, m_N$ or $\frac{m_N^2}{M}$. These mass scales are situated somewhere between m_t and M . By assumption, m_N and $\frac{m_N^2}{M}$ are very close to M . Moreover, for generic choices of the mass matrices M_{10} and M_5 , also λm_t and λm_n are close to M .

6. Conclusion

1. We have succeeded in formulating the flipped SU(5) × U(1)-GUT within a NCG-framework based upon graded differential Lie algebras. We have found interesting tree-level relations between fermionic and bosonic parameters: Given the fermionic parameters (fermion masses and Kobayashi–Maskawa mixing angles) and two 3 × 3-matrices determining the unification scale as input, we were able to compute all bosonic quantities:
   - the occurring multiplets of Higgs fields,
   - the spontaneous symmetry breaking pattern,
   - the masses of all Higgs fields,
   - the masses of all Yang–Mills fields,
   - the Weinberg angle.

   However, since not all input parameters are known, we were forced to be satisfied with estimations for some of the masses.
2. The representation of the U(1)-part of the SU(5) × U(1)-model is not an input but an algebraic consequence of the theory. This U(1)-representation is unique and realized in nature.
3. In the SU(5) × U(1)-model there occur Higgs fields in complex $\underline{5}$-, complex $\underline{50}$-, complex $\underline{45}$- and real $\underline{24}$-plets. After the spontaneous symmetry breaking, there survive 12 Higgs fields of the $\underline{24}$-representation, 7 Higgs fields of the

$\underline{5}$-representation, 99 Higgs fields of the $\underline{50}$-representation and 90 Higgs fields of the $\underline{45}$-representation, and 16 gauge fields become massive.

4. There occur three mass scales in the model under consideration:
   - The scale of the fermion masses reaching from the neutrino masses to the mass of the top quark. Moreover, also the electroweak gauge fields Z, W^+, W^- belong to this scale, and – remarkably – one Higgs field as well.
   - The mass of all fields leading to proton decay is of the order of the grand unification scale M.
   - The masses of Higgs fields which do not lead to proton decay lie between the fermions scale and the GUT-scale M, generically close to M.
5. There exists precisely one light Higgs field ϕ_0', which has exactly the same properties as the standard model Higgs field. It couples to a fermion of the mass m_f with the coupling constant $g_0 m_f/m_t$. Moreover, it has the same couplings with the intermediate vector bosons Z, W^+, W^- as the standard model Higgs field. The Higgs field ϕ_0' is a certain linear combination of the $\underline{5}$- and $\underline{45}$-representations. This linear combination is the only one corresponding to a zero mode of the GUT-sector. That the mass of ϕ_0' is generically different from zero is due to the fermion masses. Therefore, the Higgs field ϕ_0' receives a mass of the order m_t: For $m_t = 176\,\mathrm{GeV}$ we have $m_{\phi_0'} \leq 255\,\mathrm{GeV}$. The reason that only an upper bound can be given is the incomplete knowledge of the input parameters. The upper bound is independent of all parameters related to grand unification.
6. The standard model is in perfect agreement with experiment. However, we have shown that the low energy sector of the $\mathrm{SU}(5) \times \mathrm{U}(1)$-GUT is identical with the standard model. Thus, one must be careful with the extrapolation of the standard model to higher energies.

Appendix A. The Generation Space Matrices

$$
\begin{aligned}
\hat{M}^{10}_{aa} &:= \tfrac{3}{10} {M'_{10}}^2 + (\tfrac{3}{5}\alpha_A + \zeta_A)\mathbf{1}_6 \;, & \hat{M}^{10}_{cc} &:= \tfrac{1}{10} M'_N {M'_N}^* + (\tfrac{3}{5}\alpha_U + \zeta_U)\mathbf{1}_6 \;, \\
\hat{M}^{10}_{nn} &:= \tfrac{1}{10} M'_{\tilde{n}} {M'_{\tilde{n}}}^* + (\tfrac{3}{5}\alpha_V + \zeta_V)\mathbf{1}_6 \;, \\
\hat{M}^{10}_{bb} &:= \tfrac{2}{5} M'_{\tilde{u}} {M'_{\tilde{u}}}^* + \tfrac{3}{5} M'_d {M'_d}^* + (\tfrac{3}{5}\alpha_B + \zeta_B)\mathbf{1}_6 \;, \\
M^{10}_{aa} &:= {M'_{10}}^2 + \beta_A \mathbf{1}_6 + 3\delta_A M^2_{ud} \;, & M^{10}_{nn} &:= M'_{\tilde{n}} {M'_{\tilde{n}}}^* + \tfrac{1}{3}\beta_V \mathbf{1}_6 + \delta_V M^2_{ud} \;, \\
M^{10}_{cc} &:= M'_N {M'_N}^* + \tfrac{1}{3}\beta_U \mathbf{1}_6 + \delta_U M^2_{ud} \;, & \check{M}^{10}_{nn} &:= \tfrac{1}{3}\check{\beta}_V \mathbf{1}_6 + \check{\delta}_V M^2_{ud} \;, \\
M^{10}_{\{un\}} &:= \tfrac{1}{2}(M'_{\tilde{u}} {M'_{\tilde{n}}}^* + M'_{\tilde{n}} {M'_{\tilde{u}}}^*) + \beta_W \mathbf{1}_6 + 3\delta_W M^2_{ud} \;, \\
M^{10}_{[un]} &:= \tfrac{1}{2\mathrm{i}}(M'_{\tilde{u}} {M'_{\tilde{n}}}^* - M'_{\tilde{n}} {M'_{\tilde{u}}}^*) + \beta'_W \mathbf{1}_6 + 3\delta'_W M^2_{ud} \;, \\
\tilde{M}^{10}_{nn} &:= M'_{\tilde{n}} {M'_{\tilde{n}}}^* + \gamma_V \mathbf{1}_6 + \epsilon_V \tilde{M}^2_V \;, & \tilde{M}^{10}_{cc} &:= M'_N {M'_N}^* + \gamma_U \mathbf{1}_6 + \epsilon_U \tilde{M}^2_V \;, \\
\tilde{M}^{10}_{\{cd\}} &:= \tfrac{1}{2}(M'_N {M'_d}^* + M'_d {M'_N}^*) + \tilde{\gamma}_U \mathbf{1}_6 + \tilde{\epsilon}_U \tilde{M}^2_V \;, \\
\tilde{M}^{10}_{[cd]} &:= \tfrac{1}{2\mathrm{i}}(M'_N {M'_d}^* - M'_d {M'_N}^*) + \tilde{\gamma}'_U \mathbf{1}_6 + \tilde{\epsilon}'_U \tilde{M}^2_V \;, \\
\tilde{M}^{10}_{\{un\}} &:= \tfrac{1}{2}(M'_{\tilde{u}} {M'_{\tilde{n}}}^* + M'_{\tilde{n}} {M'_{\tilde{u}}}^*) + \gamma_W \mathbf{1}_6 + \epsilon_W \tilde{M}^2_V \;, \\
\tilde{M}^{10}_{[un]} &:= \tfrac{1}{2\mathrm{i}}(M'_{\tilde{u}} {M'_{\tilde{n}}}^* - M'_{\tilde{n}} {M'_{\tilde{u}}}^*) + \gamma'_W \mathbf{1}_6 + \epsilon'_W \tilde{M}^2_V \;, \\
\hat{M}^5_{aa} &:= \tfrac{1}{5} {M'_5}^2 + (\tfrac{2}{5}\alpha_A + \zeta_A)\mathbf{1}_6 \;, & \hat{M}^5_{cc} &:= (\tfrac{2}{5}\alpha_U + \zeta_U)\mathbf{1}_6 \;,
\end{aligned}
$$

$$
\begin{aligned}
\hat{M}^5_{nn} &:= \tfrac{1}{5}M'^*_{\tilde{n}}M'_{\tilde{n}} + (\tfrac{2}{5}\alpha_V+\zeta_V)\mathbf{1}_6 \;,\\
\hat{M}^5_{bb} &:= \tfrac{4}{5}M'^*_{\tilde{u}}M'_{\tilde{u}} + \tfrac{1}{5}\bar{M}'_e M'^T_e + (\tfrac{2}{5}\alpha_B+\zeta_B)\mathbf{1}_6 \;,\\
M^5_{aa} &:= M'^2_5 + \beta_A\mathbf{1}_6 + \delta_A M^2_{en} \;, & \check{M}^5_{nn} &:= M'^*_{\tilde{n}}M'_{\tilde{n}} + \check{\beta}_V\mathbf{1}_6 + \check{\delta}_V M^2_{en} \;,\\
M^5_{nn} &:= \beta_V\mathbf{1}_6 + \delta_V M^2_{en} \;, & M^5_{cc} &:= \beta_U\mathbf{1}_6 + \delta_U M^2_{en} \;,\\
M^5_{\{un\}} &:= \tfrac{1}{2}(M'^*_{\tilde{n}}M'_{\tilde{u}} + M'^*_{\tilde{u}}M'_{\tilde{n}}) + \beta_W\mathbf{1}_6 + \delta_W M^2_{en} \;,\\
M^5_{[un]} &:= \tfrac{1}{2\mathrm{i}}(M'^*_{\tilde{n}}M'_{\tilde{u}} - M'^*_{\tilde{u}}M'_{\tilde{n}}) + \beta'_W\mathbf{1}_6 + \delta'_W M^2_{en} \;,\\
\hat{M}^1_{aa} &:= \zeta_A\mathbf{1}_6 \;, & \hat{M}^1_{nn} &:= \zeta_V\mathbf{1}_6 \;,\\
\hat{M}^1_{cc} &:= \zeta_U\mathbf{1}_6 \;, & \hat{M}^1_{bb} &:= M'^T_e\bar{M}'_e + \zeta_B\mathbf{1}_6 \;,\\
M'_{d\tilde{n}} &:= M'_d\overline{M_{\tilde{n}'}} - \frac{\mathrm{tr}(M'_d\overline{M_{\tilde{n}'}}(M'_N\overline{M'_n})^*)}{\mathrm{tr}((M'_N\overline{M'_n})(M'_N\overline{M'_n})^*)}M'_N\overline{M'_n} \;,\\
M'_{Nu} &:= M'_N\overline{M'_u} - \frac{\mathrm{tr}(M'_N\overline{M'_u}(M'_N\overline{M'_n})^*}{\mathrm{tr}((M'_N\overline{M'_n})(M'_N\overline{M'_n})^*)}M'_N\overline{M'_n} \;.
\end{aligned}
$$

The real constants $\alpha_A,\dots\zeta_V$ are determined by Eq. (42). The solution is

$$
\begin{aligned}
\alpha_A &= -\tfrac{1}{8}\,\mathrm{tr}(M'^{\,2}_{10}) + \tfrac{1}{24}\,\mathrm{tr}(M'^2_5) \;, & \alpha_B &= -\tfrac{1}{4}\,\mathrm{tr}(M'_dM'^*_d) + \tfrac{1}{4}\,\mathrm{tr}(M'_eM'^*_e) \;,\\
\alpha_U &= -\tfrac{1}{24}\,\mathrm{tr}(M'_NM'^*_N) \;, & \alpha_V &= 0 \;,\\
\zeta_A &= \tfrac{1}{32}\,\mathrm{tr}(M'^{\,2}_{10}) - \tfrac{1}{32}\,\mathrm{tr}(M'^2_5) \;, & \zeta_V &= -\tfrac{1}{48}\,\mathrm{tr}(M'_{\tilde{n}}M'^*_{\tilde{n}}) \;,\\
\zeta_B &= -\tfrac{1}{12}\,\mathrm{tr}(M'_{\tilde{u}}M'^*_{\tilde{u}}) + \tfrac{1}{16}\,\mathrm{tr}(M'_dM'^*_d) - \tfrac{7}{48}\,\mathrm{tr}(M'_eM'^*_e) \;,\\
\zeta_U &= \tfrac{1}{96}\,\mathrm{tr}(M'_NM'^*_N) \;,\\
\beta_A &= -\tfrac{1}{8}\,\mathrm{tr}(M'^2_5) - \tfrac{1}{24}\,\mathrm{tr}(M'^{\,2}_{10}) \;, & \delta_A &= -\frac{\mathrm{tr}(M'^{\,2}_{10}M^2_{ud} + M'^2_5M^2_{en})}{\mathrm{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;,\\
\beta_U &= -\tfrac{1}{8}\,\mathrm{tr}(M'_NM'^*_N) \;, & \delta_U &= -3\frac{\mathrm{tr}(M'_NM'^*_NM^2_{ud})}{\mathrm{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;,\\
\beta_V &= -\tfrac{1}{8}\,\mathrm{tr}(M'_{\tilde{n}}M'^*_{\tilde{n}}) \;, & \check{\beta}_V &= -\tfrac{1}{8}\,\mathrm{tr}(M'_{\tilde{n}}M'^*_{\tilde{n}}) \;,\\
\delta_V &= -\frac{\mathrm{tr}(3M'_{\tilde{n}}M'^*_{\tilde{n}}M^2_{ud})}{\mathrm{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;, & \check{\delta}_V &= -\frac{\mathrm{tr}(M'^*_{\tilde{n}}M'_{\tilde{n}}M^2_{en})}{\mathrm{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;,\\
\beta_W &= -\tfrac{1}{12}\,\mathrm{tr}(M'_{\tilde{u}}M'^*_{\tilde{n}} + M'_{\tilde{n}}M'^*_{\tilde{u}}) \;, & \beta'_W &= -\tfrac{1}{12\mathrm{i}}\,\mathrm{tr}(M'_{\tilde{u}}M'^*_{\tilde{n}} - M'_{\tilde{n}}M'^*_{\tilde{u}}) \;,\\
\delta_W &= -\frac{\mathrm{tr}((M'_{\tilde{u}}M'^*_{\tilde{n}} + M'_{\tilde{n}}M'^*_{\tilde{u}})M^2_{ud} + (M'^*_{\tilde{n}}M'_{\tilde{u}} + M'^*_{\tilde{u}}M'_{\tilde{n}})M^2_{en})}{2\;\mathrm{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;,\\
\delta'_W &= -\frac{\mathrm{tr}((M'_{\tilde{u}}M'^*_{\tilde{n}} - M'_{\tilde{n}}M'^*_{\tilde{u}})M^2_{ud} + (M'^*_{\tilde{n}}M'_{\tilde{u}} - M'^*_{\tilde{u}}M'_{\tilde{n}})M^2_{en})}{2\mathrm{i}\;\mathrm{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;,\\
\gamma_V &= -\tfrac{1}{6}\,\mathrm{tr}(M'_{\tilde{n}}M'^*_{\tilde{n}}) \;, & \epsilon_V &= -\frac{\mathrm{tr}(M'_{\tilde{n}}M'^*_{\tilde{n}}\tilde{M}^2_V)}{\mathrm{tr}((\tilde{M}^2_V)^2)} \;,\\
\gamma_W &= -\tfrac{1}{12}\,\mathrm{tr}(M'_{\tilde{u}}M'^*_{\tilde{n}} + M'_{\tilde{n}}M'^*_{\tilde{u}}) \;, & \epsilon_W &= -\frac{\mathrm{tr}((M'_{\tilde{u}}M'^*_{\tilde{n}} + M'_{\tilde{n}}M'^*_{\tilde{u}})\tilde{M}^2_V)}{2\;\mathrm{tr}((\tilde{M}^2_V)^2)} \;,\\
\gamma'_W &= -\tfrac{1}{12\mathrm{i}}\,\mathrm{tr}(M'_{\tilde{u}}M'^*_{\tilde{n}} - M'_{\tilde{n}}M'^*_{\tilde{u}}) \;, & \epsilon'_W &= -\frac{\mathrm{tr}((M'_{\tilde{u}}M'^*_{\tilde{n}} - M'_{\tilde{n}}M'^*_{\tilde{u}})\tilde{M}^2_V)}{2\mathrm{i}\;\mathrm{tr}((\tilde{M}^2_V)^2)} \;,
\end{aligned}
$$

$$\gamma_U = -\tfrac{1}{6}\,\mathrm{tr}(M'_N {M'_N}^*)\;, \qquad \epsilon_U = -\frac{\mathrm{tr}(M'_N {M'_N}^* \tilde{M}_V^2)}{\mathrm{tr}((\tilde{M}_V^2)^2)}\;,$$

$$\tilde{\gamma}_U = -\tfrac{1}{12}\,\mathrm{tr}(M'_N {M'_d}^* + M'_d {M'_N}^*)\;, \qquad \tilde{\epsilon}_U = -\frac{\mathrm{tr}((M'_N {M'_d}^* + M'_d {M'_N}^*)\tilde{M}_V^2)}{2\;\mathrm{tr}((\tilde{M}_V^2)^2)}\;,$$

$$\tilde{\gamma}'_U = -\tfrac{1}{12\,\mathrm{i}}\,\mathrm{tr}(M'_N {M'_d}^* - M'_d {M'_N}^*)\;, \qquad \tilde{\epsilon}'_U = -\frac{\mathrm{tr}((M'_N {M'_d}^* - M'_d {M'_N}^*)\tilde{M}_V^2)}{2\,\mathrm{i}\;\mathrm{tr}((\tilde{M}_V^2)^2)}\;.$$

Appendix B. The Coefficients Occurring in the Higgs Potential

$$\mu_0 = \tfrac{1}{96}\,\mathrm{tr}(3{M'_{10}}^2 + {M'_5}^2)\;, \qquad \mu_2 = \tfrac{1}{48}\,\mathrm{tr}(M'_{\tilde{n}} {M'_{\tilde{n}}}^*)\;,$$

$$\mu_1 = \mathrm{tr}(\tfrac{1}{16} M'_d {M'_d}^* + \tfrac{1}{12} M'_{\tilde{u}} {M'_{\tilde{u}}}^* + \tfrac{1}{48} M'_e {M'_e}^*)\;, \qquad \mu_3 = \tfrac{1}{96}\,\mathrm{tr}(M'_N {M'_N}^*)\;,$$

$$\mu^{\mathrm{a}} = \mathrm{tr}(10(\hat{M}^{10}_{aa})^2 + 5(\hat{M}^5_{aa})^2 + (\hat{M}^1_{aa})^2)\;,$$

$$\mu^{\mathrm{b}} = \mathrm{tr}(10(\hat{M}^{10}_{bb})^2 + 5(\hat{M}^5_{bb})^2 + (\hat{M}^1_{bb})^2)\;,$$

$$\mu^{\mathrm{c}} = \mathrm{tr}(10(\hat{M}^{10}_{nn})^2 + 5(\hat{M}^5_{nn})^2 + (\hat{M}^1_{nn})^2)\;,$$

$$\mu^{\mathrm{d}} = \mathrm{tr}(20\hat{M}^{10}_{aa}\hat{M}^{10}_{bb} + 10\hat{M}^5_{aa}\hat{M}^5_{bb} + 2\hat{M}^1_{aa}\hat{M}^1_{bb})\;,$$

$$\mu^{\mathrm{e}} = \mathrm{tr}(20\hat{M}^1_{aa}\hat{M}^{10}_{nn} + 10\hat{M}^5_{aa}\hat{M}^5_{nn} + 2\hat{M}^1_{aa}\hat{M}^1_{nn})$$

$$\mu^{\mathrm{f}} = \mathrm{tr}(20\hat{M}^{10}_{bb}\hat{M}^{10}_{nn} + 10\hat{M}^5_{bb}\hat{M}^5_{nn} + 2\hat{M}^1_{bb}\hat{M}^1_{nn})\;,$$

$$\mu^{\mathrm{g}} = \mathrm{tr}(\tfrac{3}{2}(M'_{10}M'_d + M'_d {M_{10}}^T)(M'_{10}M'_d + M'_d {M_{10}}^T)^* + 2({M'_5}^T)^2 M'_e {M'_e}^*)\;,$$

$$\mu^{\mathrm{h}} = \tfrac{1}{4}\,\mathrm{tr}((M'_{10}M'_d - M'_d {M'_{10}}^T)(M'_{10}M'_d - M'_d {M'_{10}}^T)^*)\;,$$

$$\mu^{\mathrm{i}} = \mathrm{tr}(2M'_{\tilde{u}}{M'_{\tilde{u}}}^*{M'_{10}}^2)\;, \qquad \mu^{\mathrm{j}} = \mathrm{tr}(2{M'_{\tilde{u}}}^* M'_{\tilde{u}}{M'_5}^2)\;,$$

$$\mu^{\mathrm{k}} = \mathrm{tr}(2M'_{\tilde{n}}{M'_{\tilde{n}}}^*{M'_{10}}^2)\;, \qquad \mu^{\mathrm{l}} = \mathrm{tr}(2{M'_{\tilde{n}}}^* M'_{\tilde{n}}{M'_5}^2)\;,$$

$$\mu^{\mathrm{m}} = \mathrm{tr}(4M'_{\tilde{u}}M'_5{M'_{\tilde{u}}}^*M'_{10})\;, \qquad \mu^{\mathrm{n}} = \mathrm{Re}(\mathrm{tr}(4M'_{\tilde{u}}{M'_{\tilde{n}}}^*{M'_{10}}^2))\;,$$

$$\mu^{\mathrm{o}} = \mathrm{Im}(\mathrm{tr}(4M'_{\tilde{u}}{M'_{\tilde{n}}}^*{M'_{10}}^2))\;, \qquad \mu^{\mathrm{p}} = \mathrm{Re}(\mathrm{tr}(4M'_{\tilde{u}}M'_5{M'_{\tilde{n}}}^*M'_{10}))\;,$$

$$\mu^{\mathrm{q}} = \mathrm{Im}(\mathrm{tr}(4M'_{\tilde{u}}M'_5{M'_{\tilde{n}}}^*M'_{10}))\;, \qquad \mu^{\mathrm{r}} = \mathrm{Re}(\mathrm{tr}(4{M'_{\tilde{n}}}^*M'_{\tilde{u}}{M'_5}^2))\;,$$

$$\mu^{\mathrm{s}} = \mathrm{Im}(\mathrm{tr}(4{M'_{\tilde{n}}}^*M'_{\tilde{u}}{M'_5}^2))\;, \qquad \mu^{\mathrm{t}} = \mathrm{tr}(4M'_{\tilde{n}}M'_5{M'_{\tilde{n}}}^*M'_{10})\;,$$

$$\mu^{\mathrm{u}} = \mathrm{tr}(2{M'_{\tilde{n}}}^*M'_{\tilde{n}}\bar{M}'_e{M'_e}^T)\;, \qquad \mu^{\mathrm{v}} = \mathrm{tr}(2M'_{\tilde{n}}{M'_{\tilde{n}}}^*M'_d{M'_d}^*)\;,$$

$$\mu^{\mathrm{w}} = \mathrm{tr}(2M'_{d\tilde{n}}{M'_{d\tilde{n}}}^*)\;,$$

$$\breve{\mu}^{\mathrm{a}} = \mathrm{tr}(10(\hat{M}^{10}_{cc})^2 + 5(\hat{M}^5_{cc})^2 + (\hat{M}^1_{cc})^2)\;,$$

$$\breve{\mu}^{\mathrm{b}} = \mathrm{tr}(20\hat{M}^{10}_{cc}\hat{M}^{10}_{aa} + 10\hat{M}^5_{cc}\hat{M}^5_{aa} + 2\hat{M}^1_{cc}\hat{M}^1_{aa})\;,$$

$$\breve{\mu}^{\mathrm{c}} = \mathrm{tr}(20\hat{M}^1_{cc}\hat{M}^{10}_{bb} + 10\hat{M}^5_{cc}\hat{M}^5_{bb} + 2\hat{M}^1_{cc}\hat{M}^1_{bb})$$

$$\breve{\mu}^{\mathrm{d}} = \mathrm{tr}(20\hat{M}^{10}_{cc}\hat{M}^{10}_{nn} + 10\hat{M}^5_{cc}\hat{M}^5_{nn} + 2\hat{M}^1_{cc}\hat{M}^1_{nn})\;,$$

$$\breve{\mu}^{\mathrm{e}} = \tfrac{1}{4}\,\mathrm{tr}((M'_{10}M'_N + M'_N{M'_{10}}^T)(M'_{10}M'_N + M'_N{M'_{10}}^T)^*)\;,$$

$$\breve{\mu}^{\mathrm{f}} = \tfrac{1}{4}\,\mathrm{tr}((M'_{10}M'_N - M'_N{M'_{10}}^T)(M'_{10}M'_N - M'_N{M'_{10}}^T)^*)\;,$$

$$\breve{\mu}^{\mathrm{g}} = \tfrac{1}{2}\mathrm{Re}(\mathrm{tr}((M'_{10}M'_N - M'_N{M'_{10}}^T)(M'_{10}M'_d - M'_d{M'_{10}}^T)^*))\;,$$

$$\breve{\mu}^{\mathrm{h}} = \tfrac{1}{2}\mathrm{Im}(\mathrm{tr}((M'_{10}M'_N - M'_N{M'_{10}}^T)(M'_{10}M'_d - M'_d{M'_{10}}^T)^*))\;,$$

$$\breve{\mu}^{\mathrm{i}} = \mathrm{tr}(2M'_{\tilde{n}}{M'_{\tilde{n}}}^*M'_N{M'_N}^*)\;, \qquad \breve{\mu}^{\mathrm{j}} = \mathrm{Re}(\mathrm{tr}(4M'_{\tilde{n}}{M'_{\tilde{n}}}^*M'_N{M'_d}^*))\;,$$

$$\check\mu^{\mathrm{k}} = \mathrm{Im}(\mathrm{tr}(4M'_{\tilde n}M'_{\tilde n}{}^{*}M'_{N}M'_{d}{}^{*}))\ , \qquad \check\mu^{\mathrm{l}} = \mathrm{tr}(2M'_{Nu}M'_{Nu}{}^{*})\ ,$$
$$\check\mu^{\mathrm{m}} = \mathrm{Re}(\mathrm{tr}(4M'_{d\tilde n}M'_{Nu}{}^{*}))\ , \qquad \check\mu^{\mathrm{n}} = \mathrm{Im}(\mathrm{tr}(4M'_{d\tilde n}M'_{Nu}{}^{*}))\ ,$$
$$\tilde\mu^{\mathrm{a}} = \mathrm{tr}(\tfrac{1}{3}(M^{10}_{aa})^2 + (M^{5}_{aa})^2)\ , \qquad \tilde\mu^{\mathrm{b}} = \mathrm{tr}(3(\check M^{10}_{nn})^2 + (\check M^{5}_{nn})^2)\ ,$$
$$\tilde\mu^{\mathrm{c}} = \mathrm{tr}(3(M^{10}_{nn})^2 + (M^{5}_{nn})^2)\ , \qquad \tilde\mu^{\mathrm{d}} = \mathrm{tr}(\tfrac{1}{3}(M^{10}_{\{un\}})^2 + (M^{5}_{\{un\}})^2)\ ,$$
$$\tilde\mu^{\mathrm{e}} = \mathrm{tr}(\tfrac{1}{3}(M^{10}_{[un]})^2 + (M^{5}_{[un]})^2)\ , \qquad \tilde\mu^{\mathrm{f}} = \mathrm{tr}(2M^{10}_{aa}\check M^{10}_{nn} + 2M^{5}_{aa}\check M^{5}_{nn})\ ,$$
$$\tilde\mu^{\mathrm{g}} = \mathrm{tr}(2M^{10}_{aa}M^{10}_{nn} + 2M^{5}_{aa}M^{5}_{nn})\ , \qquad \tilde\mu^{\mathrm{h}} = \mathrm{tr}(\tfrac{2}{3}M^{10}_{aa}M^{10}_{\{un\}} + 2M^{5}_{aa}M^{5}_{\{un\}})\ ,$$
$$\tilde\mu^{\mathrm{i}} = \mathrm{tr}(\tfrac{2}{3}M^{10}_{aa}M^{10}_{[un]} + 2M^{5}_{aa}M^{5}_{[un]})\ , \qquad \tilde\mu^{\mathrm{j}} = \mathrm{tr}(6\check M^{10}_{nn}M^{10}_{nn} + 2\check M^{5}_{nn}M^{5}_{nn})\ ,$$
$$\tilde\mu^{\mathrm{k}} = \mathrm{tr}(2\check M^{10}_{nn}M^{10}_{\{un\}} + 2\check M^{5}_{nn}M^{5}_{\{un\}})\ , \qquad \tilde\mu^{\mathrm{l}} = \mathrm{tr}(2\check M^{10}_{nn}M^{10}_{[un]} + 2\check M^{5}_{nn}M^{5}_{[un]})\ ,$$
$$\tilde\mu^{\mathrm{m}} = \mathrm{tr}(2M^{10}_{nn}M^{10}_{\{un\}} + 2M^{5}_{nn}M^{5}_{\{un\}})\ , \qquad \tilde\mu^{\mathrm{n}} = \mathrm{tr}(2M^{10}_{nn}M^{10}_{[un]} + 2M^{5}_{nn}M^{5}_{[un]})\ ,$$
$$\tilde\mu^{\mathrm{o}} = \mathrm{tr}(\tfrac{2}{3}M^{10}_{\{un\}}M^{10}_{[un]} + 2M^{5}_{\{un\}}M^{5}_{[un]})\ ,$$
$$\tilde\mu^{\mathrm{p}} = \mathrm{tr}((\tilde M^{10}_{nn})^2)\ , \qquad \tilde\mu^{\mathrm{q}} = \mathrm{tr}((\tilde M^{10}_{\{un\}})^2)\ ,$$
$$\tilde\mu^{\mathrm{r}} = \mathrm{tr}((\tilde M^{10}_{[un]})^2)\ , \qquad \tilde\mu^{\mathrm{s}} = \mathrm{tr}(2\tilde M^{10}_{nn}\tilde M^{10}_{\{un\}})\ ,$$
$$\tilde\mu^{\mathrm{t}} = \mathrm{tr}(2\tilde M^{10}_{nn}\tilde M^{10}_{[un]})\ , \qquad \tilde\mu^{\mathrm{u}} = \mathrm{tr}(2\tilde M^{10}_{\{un\}}\tilde M^{10}_{[un]})\ ,$$
$$\hat\mu^{\mathrm{a}} = \mathrm{tr}(3(M^{10}_{cc})^2 + (M^{5}_{cc})^2)\ , \qquad \hat\mu^{\mathrm{b}} = \mathrm{tr}(2M^{10}_{cc}M^{10}_{aa} + 2M^{5}_{cc}M^{5}_{aa})\ ,$$
$$\hat\mu^{\mathrm{c}} = \mathrm{tr}(6M^{10}_{cc}\check M^{10}_{nn} + 2M^{5}_{cc}\check M^{5}_{nn})\ , \qquad \hat\mu^{\mathrm{d}} = \mathrm{tr}(6M^{10}_{cc}M^{10}_{nn} + 2M^{5}_{cc}M^{5}_{nn})\ ,$$
$$\hat\mu^{\mathrm{e}} = \mathrm{tr}(2M^{10}_{cc}M^{10}_{\{un\}} + 2M^{5}_{cc}M^{5}_{\{un\}})\ , \qquad \hat\mu^{\mathrm{f}} = \mathrm{tr}(2M^{10}_{cc}M^{10}_{[un]} + 2M^{5}_{cc}M^{5}_{[un]})\ ,$$
$$\hat\mu^{\mathrm{g}} = \mathrm{tr}((\tilde M^{10}_{cc})^2)\ , \qquad \hat\mu^{\mathrm{h}} = \mathrm{tr}((\tilde M^{10}_{\{cd\}})^2)\ ,$$
$$\hat\mu^{\mathrm{i}} = \mathrm{tr}((\tilde M^{10}_{[cd]})^2)\ , \qquad \hat\mu^{\mathrm{j}} = \mathrm{tr}(2\tilde M^{10}_{cc}\tilde M^{10}_{\{cd\}})\ ,$$
$$\hat\mu^{\mathrm{k}} = \mathrm{tr}(2\tilde M^{10}_{cc}\tilde M^{10}_{[cd]})\ , \qquad \hat\mu^{\mathrm{l}} = \mathrm{tr}(2\tilde M^{10}_{\{cd\}}\tilde M^{10}_{[cd]})\ ,$$
$$\hat\mu^{\mathrm{m}} = \mathrm{tr}(2\tilde M^{10}_{cc}\tilde M^{10}_{nn})\ , \qquad \hat\mu^{\mathrm{n}} = \mathrm{tr}(2\tilde M^{10}_{cc}\tilde M^{10}_{\{un\}})\ ,$$
$$\hat\mu^{\mathrm{o}} = \mathrm{tr}(2\tilde M^{10}_{cc}\tilde M^{10}_{[un]})\ , \qquad \hat\mu^{\mathrm{p}} = \mathrm{tr}(2\tilde M^{10}_{\{cd\}}\tilde M^{10}_{nn})\ ,$$
$$\hat\mu^{\mathrm{q}} = \mathrm{tr}(2\tilde M^{10}_{\{cd\}}\tilde M^{10}_{\{un\}})\ , \qquad \hat\mu^{\mathrm{r}} = \mathrm{tr}(2\tilde M^{10}_{\{cd\}}\tilde M^{10}_{[un]})\ ,$$
$$\hat\mu^{\mathrm{s}} = \mathrm{tr}(2\tilde M^{10}_{[cd]}\tilde M^{10}_{nn})\ , \qquad \hat\mu^{\mathrm{t}} = \mathrm{tr}(2\tilde M^{10}_{[cd]}\tilde M^{10}_{\{un\}})\ ,$$
$$\hat\mu^{\mathrm{u}} = \mathrm{tr}(2\tilde M^{10}_{[cd]}\tilde M^{10}_{[un]})\ .$$

Appendix C. The Quadratic Terms of the Higgs Potential

$$\begin{aligned}
\mathcal{L}_0 = \tfrac{1}{384}\big\{ & \tfrac{1}{12\mu_2+\mu_1}\phi_0'^{\,2}(8\mu^{\mathrm{b}}+1152\mu^{\mathrm{c}}+96\mu^{\mathrm{f}}+\tfrac{3072}{5}\tilde\mu^{\mathrm{b}}+\tfrac{1024}{15}\tilde\mu^{\mathrm{c}}+\tfrac{3072}{5}\tilde\mu^{\mathrm{d}}+\tfrac{1024}{5}\tilde\mu^{\mathrm{j}}-\tfrac{3072}{5}\tilde\mu^{\mathrm{k}}\\
&-\tfrac{1024}{5}\tilde\mu^{\mathrm{m}}+256\tilde\mu^{\mathrm{p}}+256\tilde\mu^{\mathrm{q}}-256\tilde\mu^{\mathrm{s}})\\
&+\sqrt{\tfrac{6}{5}}\tfrac{1}{\sqrt{\mu_0(12\mu_2+\mu_1)}}\psi_0\phi_0'(8\mu^{\mathrm{d}}+96\mu^{\mathrm{e}}-\tfrac{32}{3}\hat\mu^{\mathrm{c}}-\tfrac{32}{9}\hat\mu^{\mathrm{d}}+\tfrac{32}{3}\hat\mu^{\mathrm{e}}-\tfrac{80}{3}\hat\mu^{\mathrm{m}}+\tfrac{80}{3}\hat\mu^{\mathrm{n}}-512\tilde\mu^{\mathrm{b}}-\tfrac{512}{9}\tilde\mu^{\mathrm{c}}\\
&-512\tilde\mu^{\mathrm{d}}+\tfrac{32}{5}\tilde\mu^{\mathrm{f}}+\tfrac{32}{15}\tilde\mu^{\mathrm{g}}-\tfrac{32}{5}\tilde\mu^{\mathrm{h}}-\tfrac{512}{3}\tilde\mu^{\mathrm{j}}+512\tilde\mu^{\mathrm{k}}+\tfrac{512}{3}\tilde\mu^{\mathrm{m}}-\tfrac{1280}{3}\tilde\mu^{\mathrm{p}}-\tfrac{1280}{3}\tilde\mu^{\mathrm{q}}+\tfrac{1280}{3}\tilde\mu^{\mathrm{s}})\\
&+\sqrt{3}\tfrac{1}{12\mu_2+\mu_1}\phi_0'\upsilon_0'(\sqrt{\tfrac{\mu_1}{\mu_2}}(384\mu^{\mathrm{c}}+16\mu^{\mathrm{f}}+\tfrac{448}{5}\tilde\mu^{\mathrm{b}}+\tfrac{1216}{45}\tilde\mu^{\mathrm{c}}+64\tilde\mu^{\mathrm{d}}+\tfrac{832}{15}\tilde\mu^{\mathrm{j}}-\tfrac{384}{5}\tilde\mu^{\mathrm{k}}-\tfrac{256}{5}\tilde\mu^{\mathrm{m}}\\
&+\tfrac{256}{3}\tilde\mu^{\mathrm{p}}+\tfrac{128}{3}\tilde\mu^{\mathrm{q}}-64\tilde\mu^{\mathrm{s}})+\sqrt{\tfrac{\mu_2}{\mu_1}}(-32\mu^{\mathrm{b}}-192\mu^{\mathrm{f}}-\tfrac{6912}{5}\tilde\mu^{\mathrm{b}}+\tfrac{256}{5}\tilde\mu^{\mathrm{c}}-\tfrac{8448}{5}\tilde\mu^{\mathrm{d}}-\tfrac{768}{5}\tilde\mu^{\mathrm{j}}+1536\tilde\mu^{\mathrm{k}}\\
&+\tfrac{1024}{5}\tilde\mu^{\mathrm{m}}-512\tilde\mu^{\mathrm{q}}+256\tilde\mu^{\mathrm{s}}))
\end{aligned}$$

$$
\begin{aligned}
&+\sqrt{3}\frac{1}{\sqrt{\mu_2(12\mu_2+\mu_1)}}\phi_0'\upsilon_{45}\big(-32\tilde\mu^{\rm l}-\tfrac{32}{3}\tilde\mu^{\rm n}+32\tilde\mu^{\rm o}-\tfrac{64}{3}\tilde\mu^{\rm t}+\tfrac{64}{3}\tilde\mu^{\rm u}\big)\\
&+\sqrt{2}\frac{\sqrt{2}}{\sqrt{\mu_3(12\mu_2+\mu_1)}}\phi_0'\xi_0\big(4\check\mu^{\rm c}+48\check\mu^{\rm d}+\tfrac{64}{5}\hat\mu^{\rm c}+\tfrac{64}{15}\hat\mu^{\rm d}-\tfrac{64}{5}\hat\mu^{\rm e}+16\hat\mu^{\rm m}-16\hat\mu^{\rm n}\big)\\
&+\tfrac{1}{12\mu_2+\mu_1}\upsilon_0'^{\,2}\Big(\big(\tfrac{9}{4}\check\mu^{\rm l}-96\mu^{\rm f}+24\mu^{\rm h}+3\mu^{\rm i}+3\mu^{\rm j}+9\mu^{\rm k}+9\mu^{\rm l}-3\mu^{\rm m}-9\mu^{\rm t}+\tfrac{1728}{5}\tilde\mu^{\rm b}+\tfrac{192}{5}\tilde\mu^{\rm c}-192\tilde\mu^{\rm d}\\
&\quad-192\tilde\mu^{\rm j}+\tfrac{384}{5}\tilde\mu^{\rm k}+\tfrac{896}{5}\tilde\mu^{\rm m}-128\tilde\mu^{\rm q}+128\tilde\mu^{\rm s}\big)+\tfrac{\mu_1}{\mu_2}\big(96\mu^{\rm c}+\tfrac{3}{32}\check\mu^{\rm l}+\mu^{\rm h}+\tfrac{1}{8}\mu^{\rm i}+\tfrac{1}{8}\mu^{\rm j}+\tfrac{3}{8}\mu^{\rm k}+\tfrac{3}{8}\mu^{\rm l}\\
&\quad-\tfrac{1}{8}\mu^{\rm m}-\tfrac{3}{8}\mu^{\rm t}+\tfrac{184}{5}\tilde\mu^{\rm b}+\tfrac{376}{45}\tilde\mu^{\rm c}+8\tilde\mu^{\rm d}+\tfrac{88}{15}\tilde\mu^{\rm j}-16\tilde\mu^{\rm k}-\tfrac{16}{3}\tilde\mu^{\rm m}+\tfrac{64}{3}\tilde\mu^{\rm p}+\tfrac{16}{3}\tilde\mu^{\rm q}-\tfrac{32}{3}\tilde\mu^{\rm s}\big)\\
&\quad+\tfrac{\mu_2}{\mu_1}\big(96\mu^{\rm b}+\tfrac{27}{2}\check\mu^{\rm l}+144\mu^{\rm h}+18\mu^{\rm i}+18\mu^{\rm j}+54\mu^{\rm k}+54\mu^{\rm l}-18\mu^{\rm m}-54\mu^{\rm t}\\
&\quad+\tfrac{31104}{5}\tilde\mu^{\rm b}+\tfrac{384}{5}\tilde\mu^{\rm c}+\tfrac{19584}{5}\tilde\mu^{\rm d}-\tfrac{3456}{5}\tilde\mu^{\rm j}-\tfrac{20736}{5}\tilde\mu^{\rm k}+\tfrac{2304}{5}\tilde\mu^{\rm m}+768\tilde\mu^{\rm q}\big)\Big)\\
&+\sqrt{6}\frac{1}{\sqrt{\mu_0(12\mu_2+\mu_1)}}\psi_3'\upsilon_0'\Big(\sqrt{\tfrac{\mu_1}{\mu_2}}+12\sqrt{\tfrac{\mu_2}{\mu_1}}\Big)\big(2\mu^{\rm h}+\tfrac{1}{4}\mu^{\rm i}+\tfrac{1}{4}\mu^{\rm j}-\tfrac{3}{4}\mu^{\rm k}-\tfrac{3}{4}\mu^{\rm l}-\tfrac{1}{4}\mu^{\rm m}-\tfrac{1}{4}\mu^{\rm n}+\tfrac{1}{2}\mu^{\rm p}\\
&\quad-\tfrac{1}{4}\mu^{\rm r}+\tfrac{3}{4}\mu^{\rm t}+48\tilde\mu^{\rm b}-\tfrac{16}{9}\tilde\mu^{\rm c}+16\tilde\mu^{\rm d}-3\tilde\mu^{\rm f}+\tfrac{1}{3}\tilde\mu^{\rm g}+\tilde\mu^{\rm h}+\tfrac{16}{3}\tilde\mu^{\rm j}-32\tilde\mu^{\rm k}+\hat\mu^{\rm c}-\tfrac{1}{9}\hat\mu^{\rm d}-\tfrac{1}{3}\hat\mu^{\rm e}\big)\\
&+\frac{1}{\sqrt{\mu_2(12\mu_2+\mu_1)}}\upsilon_0'\upsilon_{45}\Big(\sqrt{\tfrac{\mu_1}{\mu_2}}\big(-16\tilde\mu^{\rm l}-\tfrac{16}{3}\tilde\mu^{\rm n}+8\tilde\mu^{\rm o}-\tfrac{32}{3}\tilde\mu^{\rm t}+\tfrac{16}{3}\tilde\mu^{\rm u}\big)+\sqrt{\tfrac{\mu_2}{\mu_1}}\big(-96\tilde\mu^{\rm o}-64\tilde\mu^{\rm u}\big)\Big)\\
&+\sqrt{6}\frac{\sqrt{2}}{\sqrt{\mu_3(12\mu_2+\mu_1)}}\xi_0\upsilon_0'\Big(\sqrt{\tfrac{\mu_1}{\mu_2}}\big(8\check\mu^{\rm d}+\tfrac{14}{15}\hat\mu^{\rm c}+\tfrac{38}{45}\hat\mu^{\rm d}-\tfrac{2}{3}\hat\mu^{\rm e}+\tfrac{8}{3}\hat\mu^{\rm m}-\tfrac{4}{3}\hat\mu^{\rm n}\big)\\
&\quad+\sqrt{\tfrac{\mu_2}{\mu_1}}\big(-8\check\mu^{\rm c}-\tfrac{72}{5}\hat\mu^{\rm c}+\tfrac{8}{5}\hat\mu^{\rm d}+\tfrac{88}{5}\hat\mu^{\rm e}+16\hat\mu^{\rm n}\big)\Big)\\
&+\sqrt{\tfrac{2}{5}}\frac{1}{\sqrt{\mu_0(12\mu_2+\mu_1)}}\psi_0\upsilon_0'\Big(\sqrt{\tfrac{\mu_1}{\mu_2}}\big(48\mu^{\rm e}-10\mu^{\rm h}-\tfrac{5}{4}\mu^{\rm i}-\tfrac{5}{4}\mu^{\rm j}+\tfrac{15}{4}\mu^{\rm k}+\tfrac{15}{4}\mu^{\rm l}+\tfrac{5}{4}\mu^{\rm m}+\tfrac{5}{4}\mu^{\rm n}-\tfrac{5}{2}\mu^{\rm p}\\
&\quad+\tfrac{5}{4}\mu^{\rm r}-\tfrac{15}{4}\mu^{\rm t}-112\tilde\mu^{\rm b}-\tfrac{304}{9}\tilde\mu^{\rm c}-80\tilde\mu^{\rm d}+\tfrac{7}{5}\tilde\mu^{\rm f}+\tfrac{19}{15}\tilde\mu^{\rm g}-\tilde\mu^{\rm h}-\tfrac{208}{3}\tilde\mu^{\rm j}+96\tilde\mu^{\rm k}+64\tilde\mu^{\rm m}-\tfrac{640}{3}\tilde\mu^{\rm p}-\tfrac{320}{3}\tilde\mu^{\rm q}\\
&\quad+160\tilde\mu^{\rm s}-\tfrac{7}{3}\hat\mu^{\rm c}-\tfrac{19}{9}\hat\mu^{\rm d}+\tfrac{5}{3}\hat\mu^{\rm e}-\tfrac{40}{3}\hat\mu^{\rm m}+\tfrac{20}{3}\hat\mu^{\rm n}\big)+\sqrt{\tfrac{\mu_1}{\mu_2}}\big(-48\mu^{\rm d}-120\mu^{\rm h}-15\mu^{\rm i}-15\mu^{\rm j}+45\mu^{\rm k}\\
&\quad+45\mu^{\rm l}+15\mu^{\rm m}+15\mu^{\rm n}-30\mu^{\rm p}+15\mu^{\rm r}-45\mu^{\rm t}+1728\tilde\mu^{\rm b}-64\tilde\mu^{\rm c}+2112\tilde\mu^{\rm d}-\tfrac{108}{5}\tilde\mu^{\rm f}+\tfrac{12}{5}\tilde\mu^{\rm g}\\
&\quad+\tfrac{132}{5}\tilde\mu^{\rm h}+192\tilde\mu^{\rm j}-1920\tilde\mu^{\rm k}-256\tilde\mu^{\rm m}+1280\tilde\mu^{\rm q}-640\tilde\mu^{\rm s}+36\hat\mu^{\rm c}-4\hat\mu^{\rm d}-44\hat\mu^{\rm e}-80\hat\mu^{\rm n}\big)\Big)\\
&+\tfrac{1}{\mu_2}\upsilon_{45}^2\big(\tfrac{3}{32}\check\mu^{\rm l}+\mu^{\rm h}+\tfrac{1}{8}\mu^{\rm i}+\tfrac{1}{8}\mu^{\rm j}+\tfrac{3}{8}\mu^{\rm k}+\tfrac{3}{8}\mu^{\rm l}-\tfrac{1}{8}\mu^{\rm m}-\tfrac{3}{8}\mu^{\rm t}+8\tilde\mu^{\rm e}+\tfrac{16}{3}\tilde\mu^{\rm r}\big)+\tfrac{2}{\mu_3}\xi_0^2\big(4\check\mu^{\rm a}+\tfrac{8}{15}\hat\mu^{\rm a}+2\hat\mu^{\rm g}\big)\\
&+\tfrac{1}{\sqrt{\mu_0\mu_2}}\sqrt{\tfrac{2}{5}}\psi_0\upsilon_{45}\big(-\tfrac{7}{2}\mu^{\rm o}+4\mu^{\rm q}-\tfrac{1}{2}\mu^{\rm s}-\tilde\mu^{\rm i}+40\tilde\mu^{\rm l}+\tfrac{40}{3}\tilde\mu^{\rm n}-40\tilde\mu^{\rm o}+\tfrac{160}{3}\tilde\mu^{\rm t}-\tfrac{160}{3}\tilde\mu^{\rm u}+\tfrac{5}{3}\hat\mu^{\rm f}+\tfrac{20}{3}\hat\mu^{\rm o}\big)\\
&+\sqrt{6}\tfrac{1}{\sqrt{\mu_0\mu_2}}\psi_3'\upsilon_{45}\big(-\tfrac{1}{2}\mu^{\rm o}+\tfrac{1}{2}\mu^{\rm s}+\tilde\mu^{\rm i}-8\tilde\mu^{\rm l}-\tfrac{8}{3}\tilde\mu^{\rm n}+8\tilde\mu^{\rm o}-\tfrac{1}{3}\hat\mu^{\rm f}\big)+\sqrt{6}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\xi_0\upsilon_{45}\big(-\tfrac{2}{3}\hat\mu^{\rm f}-\tfrac{4}{3}\hat\mu^{\rm o}\big)\\
&+\tfrac{1}{\mu_0}\psi_0^2\big(\tfrac{48}{5}\mu^{\rm a}+\tfrac{3}{5}\mu^{\rm g}+10\mu^{\rm h}+\tfrac{13}{5}\mu^{\rm i}+\tfrac{7}{5}\mu^{\rm j}+\tfrac{109}{5}\mu^{\rm k}+\tfrac{31}{5}\mu^{\rm l}-\tfrac{4}{5}\mu^{\rm m}-7\mu^{\rm n}+8\mu^{\rm p}-\mu^{\rm r}-\tfrac{52}{5}\mu^{\rm t}+\tfrac{2}{25}\tilde\mu^{\rm a}\\
&\quad+128\tilde\mu^{\rm b}+\tfrac{128}{9}\tilde\mu^{\rm c}+128\tilde\mu^{\rm d}-\tfrac{16}{5}\tilde\mu^{\rm f}-\tfrac{16}{15}\tilde\mu^{\rm g}+\tfrac{16}{5}\tilde\mu^{\rm h}+\tfrac{128}{3}\tilde\mu^{\rm j}-128\tilde\mu^{\rm k}-\tfrac{128}{3}\tilde\mu^{\rm m}+\tfrac{640}{3}\tilde\mu^{\rm p}\\
&\quad+\tfrac{640}{3}\tilde\mu^{\rm q}-\tfrac{640}{3}\tilde\mu^{\rm s}+\tfrac{48}{5}\check\mu^{\rm e}+\tfrac{2}{9}\hat\mu^{\rm a}-\tfrac{2}{15}\hat\mu^{\rm b}+\tfrac{16}{3}\hat\mu^{\rm c}+\tfrac{16}{9}\hat\mu^{\rm d}-\tfrac{16}{3}\hat\mu^{\rm e}+\tfrac{10}{3}\hat\mu^{\rm g}+\tfrac{80}{3}\hat\mu^{\rm m}-\tfrac{80}{3}\hat\mu^{\rm n}\big)\\
&+\tfrac{1}{\mu_0}\textstyle\sum_{i=1}^3\psi_i'^{\,2}\big(\mu^{\rm g}+6\mu^{\rm h}+3\mu^{\rm i}+\mu^{\rm j}+3\mu^{\rm k}+9\mu^{\rm l}+3\mu^{\rm n}-3\mu^{\rm r}+2\tilde\mu^{\rm a}+128\tilde\mu^{\rm b}+\tfrac{128}{9}\tilde\mu^{\rm c}+128\tilde\mu^{\rm d}\\
&\quad-16\tilde\mu^{\rm f}-\tfrac{16}{3}\tilde\mu^{\rm g}+16\tilde\mu^{\rm h}+\tfrac{128}{3}\tilde\mu^{\rm j}-128\tilde\mu^{\rm k}-\tfrac{128}{3}\tilde\mu^{\rm m}+\tfrac{2}{9}\hat\mu^{\rm a}-\tfrac{2}{3}\hat\mu^{\rm b}+\tfrac{16}{3}\hat\mu^{\rm c}+\tfrac{16}{9}\hat\mu^{\rm d}-\tfrac{16}{3}\hat\mu^{\rm e}\big)\\
&+\tfrac{1}{\mu_0}\textstyle\sum_{i=1}^8\psi_i^2\big(16\mu^{\rm h}+2\mu^{\rm i}+2\mu^{\rm j}+2\mu^{\rm k}+2\mu^{\rm l}-2\mu^{\rm m}+2\mu^{\rm n}-4\mu^{\rm p}+2\mu^{\rm r}-2\mu^{\rm t}+2\tilde\mu^{\rm a}+128\tilde\mu^{\rm b}\\
&\quad+\tfrac{128}{9}\tilde\mu^{\rm c}+128\tilde\mu^{\rm d}+16\tilde\mu^{\rm f}+\tfrac{16}{3}\tilde\mu^{\rm g}-16\tilde\mu^{\rm h}+\tfrac{128}{3}\tilde\mu^{\rm j}-128\tilde\mu^{\rm k}-\tfrac{128}{3}\tilde\mu^{\rm m}+\tfrac{256}{3}\tilde\mu^{\rm p}+\tfrac{256}{3}\tilde\mu^{\rm q}-\tfrac{256}{3}\tilde\mu^{\rm s}\\
&\quad+\tfrac{2}{9}\hat\mu^{\rm a}+\tfrac{2}{3}\hat\mu^{\rm b}+\tfrac{16}{3}\hat\mu^{\rm c}+\tfrac{16}{9}\hat\mu^{\rm d}-\tfrac{16}{3}\hat\mu^{\rm e}+\tfrac{4}{3}\hat\mu^{\rm g}+\tfrac{32}{3}\hat\mu^{\rm m}-\tfrac{32}{3}\hat\mu^{\rm n}\big)\\
&+\sqrt{\tfrac{3}{5}}\psi_0\psi_3'\big(2\mu^{\rm g}-20\mu^{\rm h}+2\mu^{\rm i}-2\mu^{\rm j}+2\mu^{\rm k}-18\mu^{\rm l}+4\mu^{\rm m}+2\mu^{\rm n}-8\mu^{\rm p}+6\mu^{\rm r}+20\mu^{\rm t}\big)\\
&+\sqrt{\tfrac{2}{\mu_0\mu_3}}\sqrt{\tfrac{3}{5}}\psi_0\xi_0\big(8\check\mu^{\rm b}-\tfrac{8}{9}\hat\mu^{\rm a}+\tfrac{4}{15}\hat\mu^{\rm b}-\tfrac{32}{3}\hat\mu^{\rm c}-\tfrac{32}{9}\hat\mu^{\rm d}+\tfrac{32}{3}\hat\mu^{\rm e}-\tfrac{20}{3}\hat\mu^{\rm g}-\tfrac{80}{3}\hat\mu^{\rm m}+\tfrac{80}{3}\hat\mu^{\rm n}\big)\\
&+\tfrac{1}{\sqrt{\mu_0\mu_2}}\sqrt{6}\Big(\big(\psi_1'\upsilon_{18}+\psi_2'\upsilon_{63}\big)\big(2\mu^{\rm h}+\tfrac{1}{4}\mu^{\rm i}+\tfrac{1}{4}\mu^{\rm j}-\tfrac{3}{4}\mu^{\rm k}-\tfrac{3}{4}\mu^{\rm l}-\tfrac{1}{4}\mu^{\rm m}-\tfrac{1}{4}\mu^{\rm n}+\tfrac{1}{2}\mu^{\rm p}-\tfrac{1}{4}\mu^{\rm r}+\tfrac{3}{4}\mu^{\rm t}
\end{aligned}
$$

$$
\begin{aligned}
&+48\tilde\mu^{b}-\tfrac{16}{9}\tilde\mu^{c}+16\tilde\mu^{d}-3\tilde\mu^{f}+\tfrac13\tilde\mu^{g}+\tilde\mu^{h}+\tfrac{16}{3}\tilde\mu^{j}-32\tilde\mu^{k}+\hat\mu^{c}-\tfrac19\hat\mu^{d}-\tfrac13\hat\mu^{e})\\
&\quad+(\psi'_1\upsilon_{63}-\psi'_2\upsilon_{18})(-\tfrac12\mu^{o}+\tfrac12\mu^{s}+\tilde\mu^{i}-8\tilde\mu^{l}-\tfrac83\tilde\mu^{n}+8\tilde\mu^{o}-\tfrac13\hat\mu^{f}))\\
&+\tfrac{1}{\sqrt{\mu_0\mu_2}}\textstyle\sum_{i=1}^{8}\big(\psi_i\upsilon_i(8\mu^{h}+\mu^{i}+\mu^{j}-3\mu^{k}-3\mu^{l}-\mu^{m}-\mu^{n}+2\mu^{p}-\mu^{r}+3\mu^{t}-64\tilde\mu^{b}-\tfrac{64}{9}\tilde\mu^{c}+64\tilde\mu^{d}\\
&\quad-4\tilde\mu^{f}-\tfrac43\tilde\mu^{g}-4\tilde\mu^{h}-\tfrac{64}{3}\tilde\mu^{j}-\tfrac{64}{3}\tilde\mu^{p}+\tfrac{64}{3}\tilde\mu^{q}-\tfrac43\hat\mu^{c}-\tfrac49\hat\mu^{d}-\tfrac43\hat\mu^{e}-\tfrac43\hat\mu^{m}-\tfrac43\hat\mu^{n})\\
&\quad+\psi_i\upsilon_{i+45}(-2\mu^{o}+4\mu^{q}-2\mu^{s}-4\tilde\mu^{i}-32\tilde\mu^{l}-\tfrac{32}{3}\tilde\mu^{n}+32\tilde\mu^{o}-\tfrac{32}{3}\tilde\mu^{t}+\tfrac{32}{3}\tilde\mu^{u}-\tfrac43\hat\mu^{f}-\tfrac43\hat\mu^{o})\big)\\
&+\tfrac{\sqrt2}{\sqrt{\mu_0\mu_3}}\textstyle\sum_{i=1}^{8}\big(\psi_i\xi_{i+32}(-4\check\mu^{g}+2\hat\mu^{j}+16\hat\mu^{p}-16\hat\mu^{q})+\psi_i\xi_{i+81}(-4\check\mu^{h}+2\hat\mu^{k}+16\hat\mu^{s}-16\hat\mu^{t})\big)\\
&+\tfrac{2}{\mu_1}\textstyle\sum_{i=1}^{6}\phi_i^2(\mu^{g}+8\mu^{h}+\tfrac{13}{4}\mu^{i}+\tfrac54\mu^{j}+\tfrac94\mu^{k}+\tfrac94\mu^{l}-\tfrac14\mu^{m}-\tfrac94\mu^{n}+\tfrac32\mu^{p}+\tfrac34\mu^{r}-\tfrac94\mu^{t}+\mu^{u}+\tfrac13\mu^{v}\\
&\quad+\tfrac{32}{3}\mu^{w}+162\tilde\mu^{b}+2\tilde\mu^{c}+98\tilde\mu^{d}+2\tilde\mu^{e}-18\tilde\mu^{j}-126\tilde\mu^{k}+14\tilde\mu^{m}+\tfrac{64}{3}\tilde\mu^{q}+\tfrac{64}{3}\tilde\mu^{r}+2\hat\mu^{h}+2\hat\mu^{i})\\
&+\sqrt2\tfrac{2}{\sqrt{\mu_1\mu_3}}\textstyle\sum_{i=1}^{3}\big((\phi_i\xi_{i+43}+\phi_{i+3}\xi_{i+92})(-3\hat\mu^{c}+\tfrac13\hat\mu^{d}+\tfrac73\hat\mu^{e}+2\hat\mu^{h}+2\hat\mu^{i}+\tfrac83\hat\mu^{n})\\
&\quad+(\phi_i\xi_{i+92}-\phi_{i+3}\xi_{i+43}(\tfrac13\hat\mu^{f}+\tfrac83\hat\mu^{o})\big)\\
&+\sqrt3\tfrac{2}{\sqrt{\mu_1\mu_3}}\big((\textstyle\sum_{i=3}^{3}\phi_i\xi_{i+46}+\phi_{i+3}\xi_{i+95})(\tfrac83\check\mu^{g}+\tfrac49\check\mu^{j}-\tfrac49\check\mu^{m}+\tfrac23\hat\mu^{j}+\tfrac83\hat\mu^{q}-\tfrac83\hat\mu^{u})\\
&\quad+(\phi_i\xi_{i+95}-\phi_{i+3}\xi_{i+46})(\tfrac83\check\mu^{h}+\tfrac49\check\mu^{k}+\tfrac49\check\mu^{n}+\tfrac23\hat\mu^{k}+\tfrac83\hat\mu^{r}+\tfrac83\hat\mu^{t})\big)\\
&+\sqrt2\tfrac{\sqrt2}{\sqrt{\mu_1\mu_2}}\textstyle\sum_{i=1}^{3}\big((\phi_i\upsilon_{i+8}+\phi_{i+3}\upsilon_{i+53})(-4\mu^{h}-\tfrac12\mu^{i}-\tfrac12\mu^{j}-\tfrac32\mu^{k}-\tfrac32\mu^{l}+\tfrac12\mu^{m}+2\mu^{n}-2\mu^{p}+\tfrac32\mu^{t}\\
&\quad+\mu^{u}+\tfrac13\mu^{v}-\tfrac{16}{3}\mu^{w}+18\tilde\mu^{b}-\tfrac{10}{3}\tilde\mu^{c}-14\tilde\mu^{d}+2\tilde\mu^{e}+14\tilde\mu^{j}+2\tilde\mu^{k}-\tfrac{38}{3}\tilde\mu^{m}+\tfrac{32}{3}\tilde\mu^{q}-\tfrac{32}{3}\tilde\mu^{r}-\tfrac{16}{3}\tilde\mu^{s})\\
&\quad+(\phi_i\upsilon_{i+53}-\phi_{i+3}\upsilon_{i+8})(2\mu^{o}-2\mu^{q}+8\tilde\mu^{l}-\tfrac83\tilde\mu^{n}-8\tilde\mu^{o}-\tfrac{16}{3}\tilde\mu^{t}+\tfrac{32}{3}\tilde\mu^{u})\big)\\
&+\tfrac{\sqrt2}{\sqrt{\mu_1\mu_2}}\textstyle\sum_{i=1}^{3}\big((\phi_i\upsilon_{i+11}+\phi_{i+3}\upsilon_{i+56})(2\mu^{u}+\tfrac23\mu^{v}+\tfrac{16}{3}\mu^{w}+36\tilde\mu^{b}+4\tilde\mu^{c}-28\tilde\mu^{d}+4\tilde\mu^{e}-20\tilde\mu^{j}\\
&\quad+4\tilde\mu^{k}+12\tilde\mu^{m}-\tfrac{32}{3}\tilde\mu^{q}+\tfrac{32}{3}\tilde\mu^{r}+16\tilde\mu^{s})+(\phi_i\upsilon_{i+56}-\phi_{i+3}\upsilon_{i+11})(16\tilde\mu^{l}-16\tilde\mu^{o}+16\tilde\mu^{t}-\tfrac{32}{3}\tilde\mu^{u})\big)\\
&+\sqrt2\tfrac{\sqrt2}{\sqrt{\mu_1\mu_2}}\textstyle\sum_{i=1}^{3}\big((\phi_i\upsilon_{i+38}+\phi_{i+3}\upsilon_{i+83})(-6\hat\mu^{p}+2\hat\mu^{q}+2\hat\mu^{u})\\
&\quad+(\phi_i\upsilon_{i+83}-\phi_{i+3}\upsilon_{i+38})(-2\hat\mu^{r}-6\hat\mu^{s}+2\hat\mu^{t})\big)\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=1}^{6}\xi_i^2+\sum_{i=50}^{55}\xi_i^2)(4\check\mu^{e}+\check\mu^{l})+\tfrac{2}{\mu_3}(\sum_{i=7}^{12}\xi_i^2+\sum_{i=56}^{61}\xi_i^2)(4\check\mu^{e}+2\hat\mu^{h}+2\hat\mu^{i})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=13}^{18}\xi_i^2+\sum_{i=62}^{67}\xi_i^2)(16\check\mu^{e}+2\hat\mu^{h}+2\hat\mu^{i})+\tfrac{2}{\mu_3}(\sum_{i=19}^{24}\xi_i^2+\sum_{i=68}^{73}\xi_i^2)(4\check\mu^{e}+\tfrac12\check\mu^{l}+\hat\mu^{h}+\hat\mu^{i})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=25}^{32}\xi_i^2+\sum_{i=74}^{81}\xi_i^2)(9\check\mu^{e}+\check\mu^{f}+\tfrac12\check\mu^{l}+\hat\mu^{h}+\hat\mu^{i})+\tfrac{2}{\mu_3}\sum_{i=1}^{8}\xi_{i+32}\xi_{i+81}(4\hat\mu^{l})\\
&+\tfrac{2}{\mu_3}\textstyle\sum_{i=33}^{40}\xi_i^2(9\check\mu^{e}+\check\mu^{f}+4\hat\mu^{h})+\tfrac{2}{\mu_3}(\sum_{i=41}^{43}\xi_i^2+\sum_{i=90}^{92}\xi_i^2)(\check\mu^{e}+\check\mu^{f}+\tfrac83\check\mu^{i}+\tfrac13\check\mu^{l}+\tfrac19\hat\mu^{a}+\tfrac23\hat\mu^{g})\\
&+\tfrac{2}{\mu_3}\textstyle\sum_{i=82}^{89}\xi_i^2(9\check\mu^{e}+\check\mu^{f}+4\hat\mu^{i})+\tfrac{2}{\mu_3}(\sum_{i=44}^{46}\xi_i^2+\sum_{i=93}^{95}\xi_i^2)(\check\mu^{e}+\check\mu^{f}+\tfrac19\hat\mu^{a}+\tfrac23\hat\mu^{g}+\hat\mu^{h}+\hat\mu^{i})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=47}^{49}\xi_i^2+\sum_{i=96}^{98}\xi_i^2)(4\check\mu^{e}+\tfrac83\check\mu^{f}+\tfrac{16}{9}\check\mu^{i}+\tfrac{1}{18}\check\mu^{l}+\tfrac23\hat\mu^{g}+\hat\mu^{h}+\hat\mu^{i})\\
&+\tfrac{2}{\mu_3}\textstyle\sum_{i=1}^{6}\big((\xi_{i+6}\xi_{i+12}+\xi_{i+55}\xi_{i+61})(4\hat\mu^{h}-4\hat\mu^{i})+(\xi_{i+6}\xi_{i+61}-\xi_{i+55}\xi_{i+12})(-4\hat\mu^{l})\big)\\
&+\sqrt6\tfrac{2}{\mu_3}\textstyle\sum_{i=1}^{3}\big((\xi_{i+43}\xi_{i+46}+\xi_{i+92}\xi_{i+95})(\tfrac23\hat\mu^{j})+(\xi_{i+43}\xi_{i+95}-\xi_{i+92}\xi_{i+46})(\tfrac23\hat\mu^{k})\big)\\
&+\tfrac{\sqrt2}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{6}\big((\xi_{i+18}\upsilon_{i+29}+\xi_{i+67}\upsilon_{i+74})(-\check\mu^{m}-2\hat\mu^{p}-2\hat\mu^{q}-2\hat\mu^{u})\\
&\quad+(\xi_{i+18}\upsilon_{i+74}-\xi_{i+67}\upsilon_{i+29})(-\check\mu^{n}-2\hat\mu^{r}+2\hat\mu^{s}+2\hat\mu^{t})\big)\\
&+\tfrac{\sqrt2}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{8}\big((\xi_{i+24}\upsilon_{i+18}+\xi_{i+73}\upsilon_{i+63})(\check\mu^{g}-\check\mu^{m}-2\hat\mu^{p}-2\hat\mu^{q}-2\hat\mu^{u})\\
&\quad+(\xi_{i+24}\upsilon_{i+63}-\xi_{i+73}\upsilon_{i+18})(-\check\mu^{h}-\check\mu^{n}-2\hat\mu^{r}+2\hat\mu^{s}+2\hat\mu^{t})\big)\\
&+\tfrac{\sqrt2}{\sqrt{\mu_2\mu_3}}\big(\textstyle\sum_{i=1}^{8}(\xi_{i+32}\upsilon_i+\xi_{i+81}\upsilon_{i+45})(-\check\mu^{g})+(\xi_{i+32}\upsilon_{i+45}-\xi_{i+81}\upsilon_i)(\check\mu^{h})\big)\\
&+\tfrac{\sqrt2}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{3}\big((\xi_{i+40}\upsilon_{i+14}+\xi_{i+89}\upsilon_{i+59})(\tfrac23\hat\mu^{c}-\tfrac23\hat\mu^{d}+\tfrac23\hat\mu^{e}-4\hat\mu^{m}+\tfrac43\hat\mu^{n})\\
&\quad+(\xi_{i+40}\upsilon_{i+59}-\xi_{i+89}\upsilon_{i+14})(\tfrac23\hat\mu^{f}+\tfrac43\hat\mu^{o})\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{3}\big((\xi_{i+40}\upsilon_{i+35}+\xi_{i+89}\upsilon_{i+80}(-\check{\mu}^{\mathrm{g}}+\tfrac{4}{3}\check{\mu}^{\mathrm{j}}-\tfrac{1}{3}\check{\mu}^{\mathrm{m}})\\
&\quad+(\xi_{i+40}\upsilon_{i+80}-\xi_{i+89}\upsilon_{i+35})(-\check{\mu}^{\mathrm{h}}+\tfrac{4}{3}\check{\mu}^{\mathrm{k}}+\tfrac{1}{3}\check{\mu}^{\mathrm{n}})\big)\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{3}\big((\xi_{i+40}\upsilon_{i+41}+\xi_{i+89}\upsilon_{i+86})(-\tfrac{8}{3}\check{\mu}^{\mathrm{i}}-\tfrac{1}{3}\check{\mu}^{\mathrm{l}}+2\hat{\mu}^{\mathrm{c}}-\tfrac{2}{9}\hat{\mu}^{\mathrm{d}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{e}}-\tfrac{4}{3}\hat{\mu}^{\mathrm{m}}-\tfrac{4}{3}\hat{\mu}^{\mathrm{n}})\\
&\quad+(\xi_{i+40}\upsilon_{i+86}-\xi_{i+89}\upsilon_{i+41})(\tfrac{2}{3}\hat{\mu}^{\mathrm{f}}+\tfrac{4}{3}\hat{\mu}^{\mathrm{o}})\big)\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{3}\big((\xi_{i+43}\upsilon_{i+8}+\xi_{i+92}\upsilon_{i+53})(-\tfrac{1}{3}\hat{\mu}^{\mathrm{c}}-\tfrac{5}{9}\hat{\mu}^{\mathrm{d}}-\tfrac{1}{3}\hat{\mu}^{\mathrm{e}}-\tfrac{4}{3}\hat{\mu}^{\mathrm{m}}+\tfrac{4}{3}\hat{\mu}^{\mathrm{n}})\\
&\quad+(\xi_{i+43}\upsilon_{i+53}-\xi_{i+92}\upsilon_{i+8})(-\tfrac{1}{3}\hat{\mu}^{\mathrm{f}}+\tfrac{4}{3}\hat{\mu}^{\mathrm{o}})\big)\\
&+\sqrt{2}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{3}\big((\xi_{i+43}\upsilon_{i+11}+\xi_{i+92}\upsilon_{i+56})(-\tfrac{1}{3}\hat{\mu}^{\mathrm{c}}+\tfrac{1}{3}\hat{\mu}^{\mathrm{d}}-\tfrac{1}{3}\hat{\mu}^{\mathrm{e}}+2\hat{\mu}^{\mathrm{m}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{n}})\\
&\quad+(\xi_{i+43}\upsilon_{i+56}-\xi_{i+92}\upsilon_{i+11})(-\tfrac{1}{3}\hat{\mu}^{\mathrm{f}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{o}})\big)\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{3}\big((\xi_{i+43}\upsilon_{i+38}+\xi_{i+92}\upsilon_{i+83}(-\check{\mu}^{\mathrm{g}}-6\hat{\mu}^{\mathrm{p}}+2\hat{\mu}^{\mathrm{q}}+2\hat{\mu}^{\mathrm{u}})\\
&\quad+(\xi_{i+43}\upsilon_{i+83}-\xi_{i+92}\upsilon_{i+38})(-\check{\mu}^{\mathrm{h}}-2\hat{\mu}^{\mathrm{r}}-6\hat{\mu}^{\mathrm{s}}+2\hat{\mu}^{\mathrm{t}})\big)\\
&+\sqrt{6}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{3}\big((\xi_{i+46}\upsilon_{i+8}+\xi_{i+95}\upsilon_{i+53})(-\tfrac{2}{3}\check{\mu}^{\mathrm{g}}+\tfrac{2}{9}\check{\mu}^{\mathrm{j}}+\tfrac{1}{9}\check{\mu}^{\mathrm{m}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{p}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{q}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{u}})\\
&\quad+(\xi_{i+46}\upsilon_{i+53}-\xi_{i+95}\upsilon_{i+8})(\tfrac{2}{3}\check{\mu}^{\mathrm{h}}-\tfrac{2}{9}\check{\mu}^{\mathrm{k}}+\tfrac{1}{9}\check{\mu}^{\mathrm{n}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{r}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{s}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{t}})\big)\\
&+\sqrt{3}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{3}\big((\xi_{i+46}\upsilon_{i+11}+\xi_{i+95}\upsilon_{i+56})(\tfrac{4}{9}\check{\mu}^{\mathrm{j}}-\tfrac{1}{9}\check{\mu}^{\mathrm{m}}+2\hat{\mu}^{\mathrm{p}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{q}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{u}})\\
&\quad+(\xi_{i+46}\upsilon_{i+56}-\xi_{i+95}\upsilon_{i+11})(-\tfrac{4}{9}\check{\mu}^{\mathrm{k}}-\tfrac{1}{9}\check{\mu}^{\mathrm{n}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{r}}-2\hat{\mu}^{\mathrm{s}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{t}})\big)\\
&+\sqrt{6}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{3}\big((\xi_{i+46}\upsilon_{i+38}+\xi_{i+95}\upsilon_{i+83})(-2\hat{\mu}^{\mathrm{m}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{n}})+(\xi_{i+46}\upsilon_{i+83}+\xi_{i+95}\upsilon_{i+38})(-\tfrac{2}{3}\hat{\mu}^{\mathrm{o}})\\
&+\tfrac{1}{\mu_2}\textstyle\sum_{i=1}^{8}\upsilon_i^2(\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{9}{8}\mu^{\mathrm{k}}+\tfrac{9}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}-\tfrac{3}{8}\mu^{\mathrm{n}}+\tfrac{3}{4}\mu^{\mathrm{p}}-\tfrac{3}{8}\mu^{\mathrm{r}}-\tfrac{9}{8}\mu^{\mathrm{t}}\\
&\quad+8\tilde{\mu}^{\mathrm{b}}+\tfrac{8}{9}\tilde{\mu}^{\mathrm{c}}+8\tilde{\mu}^{\mathrm{d}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{j}}+8\tilde{\mu}^{\mathrm{k}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{m}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{p}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{s}})\\
&+\tfrac{1}{\mu_2}\textstyle\sum_{i=46}^{53}\upsilon_i^2(\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{9}{8}\mu^{\mathrm{k}}+\tfrac{9}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}-\tfrac{3}{8}\mu^{\mathrm{n}}+\tfrac{3}{4}\mu^{\mathrm{p}}-\tfrac{3}{8}\mu^{\mathrm{r}}-\tfrac{9}{8}\mu^{\mathrm{t}}+8\tilde{\mu}^{\mathrm{e}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{r}})\\
&+\tfrac{1}{\mu_2}\textstyle\sum_{i=1}^{8}\upsilon_i\upsilon_{i+45}(8\tilde{\mu}^{\mathrm{l}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{n}}+8\tilde{\mu}^{\mathrm{o}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{t}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{u}})\\
&+\tfrac{1}{\mu_2}(\textstyle\sum_{i=9}^{11}\upsilon_i^2+\sum_{i=54}^{56}\upsilon_i^2)(\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{29}{8}\mu^{\mathrm{k}}+\tfrac{5}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}-\tfrac{5}{8}\mu^{\mathrm{n}}+\tfrac{3}{4}\mu^{\mathrm{p}}-\tfrac{1}{8}\mu^{\mathrm{r}}-\tfrac{9}{8}\mu^{\mathrm{t}}\\
&\quad+\tfrac{1}{2}\mu^{\mathrm{u}}+\tfrac{1}{6}\mu^{\mathrm{v}}+\tfrac{4}{3}\mu^{\mathrm{w}}+\tilde{\mu}^{\mathrm{b}}+\tfrac{25}{9}\tilde{\mu}^{\mathrm{c}}+\tilde{\mu}^{\mathrm{d}}+\tilde{\mu}^{\mathrm{e}}+\tfrac{5}{3}\tilde{\mu}^{\mathrm{j}}+\tilde{\mu}^{\mathrm{k}}+\tfrac{5}{3}\tilde{\mu}^{\mathrm{m}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{p}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{r}}-\tfrac{8}{3}\tilde{\mu}^{\mathrm{s}})\\
&+\tfrac{1}{\mu_2}(\textstyle\sum_{i=12}^{14}\upsilon_i^2+\sum_{i=57}^{59}\upsilon_i^2)(\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{9}{8}\mu^{\mathrm{k}}+\tfrac{1}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}-\tfrac{3}{8}\mu^{\mathrm{n}}+\tfrac{1}{4}\mu^{\mathrm{p}}+\tfrac{1}{8}\mu^{\mathrm{r}}+\tfrac{3}{8}\mu^{\mathrm{t}}+\mu^{\mathrm{u}}\\
&\quad+\tfrac{1}{3}\mu^{\mathrm{v}}+\tfrac{2}{3}\mu^{\mathrm{w}}+2\tilde{\mu}^{\mathrm{b}}+2\tilde{\mu}^{\mathrm{c}}+2\tilde{\mu}^{\mathrm{d}}+2\tilde{\mu}^{\mathrm{e}}-2\tilde{\mu}^{\mathrm{j}}+2\tilde{\mu}^{\mathrm{k}}-2\tilde{\mu}^{\mathrm{m}}+12\tilde{\mu}^{\mathrm{p}}+\tfrac{4}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{4}{3}\tilde{\mu}^{\mathrm{r}}-4\tilde{\mu}^{\mathrm{s}})\\
&+\tfrac{1}{\mu_2}(\textstyle\sum_{i=15}^{17}\upsilon_i^2+\sum_{i=60}^{62}\upsilon_i^2)(\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{9}{8}\mu^{\mathrm{k}}+\tfrac{1}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}-\tfrac{3}{8}\mu^{\mathrm{n}}+\tfrac{1}{4}\mu^{\mathrm{p}}+\tfrac{1}{8}\mu^{\mathrm{r}}+\tfrac{3}{8}\mu^{\mathrm{t}}\\
&\quad+4\tilde{\mu}^{\mathrm{b}}+4\tilde{\mu}^{\mathrm{c}}+4\tilde{\mu}^{\mathrm{d}}+4\tilde{\mu}^{\mathrm{e}}-4\tilde{\mu}^{\mathrm{j}}+4\tilde{\mu}^{\mathrm{k}}-4\tilde{\mu}^{\mathrm{m}}+24\tilde{\mu}^{\mathrm{p}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{r}}-8\tilde{\mu}^{\mathrm{s}})\\
&+\tfrac{1}{\mu_2}(\upsilon_{18}^2+\upsilon_{63}^2)(\tfrac{3}{32}\check{\mu}^{\mathrm{l}}+\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{3}{8}\mu^{\mathrm{k}}+\tfrac{3}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}-\tfrac{3}{8}\mu^{\mathrm{t}}+\tfrac{3}{2}\mu^{\mathrm{u}}+\tfrac{1}{2}\mu^{\mathrm{v}}+27\tilde{\mu}^{\mathrm{b}}+\tfrac{1}{3}\tilde{\mu}^{\mathrm{c}}\\
&\quad+3\tilde{\mu}^{\mathrm{d}}+3\tilde{\mu}^{\mathrm{e}}-3\tilde{\mu}^{\mathrm{j}}-9\tilde{\mu}^{\mathrm{k}}+\tilde{\mu}^{\mathrm{m}})\\
&+\tfrac{1}{\mu_2}(\textstyle\sum_{i=19}^{26}\upsilon_i^2+\sum_{i=64}^{71}\upsilon_i^2)(\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{9}{8}\mu^{\mathrm{k}}+\tfrac{9}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}-\tfrac{3}{8}\mu^{\mathrm{n}}+\tfrac{3}{4}\mu^{\mathrm{p}}-\tfrac{3}{8}\mu^{\mathrm{r}}-\tfrac{9}{8}\mu^{\mathrm{t}}+2\mu^{\mathrm{w}}\\
&\quad+4\tilde{\mu}^{\mathrm{p}}+4\tilde{\mu}^{\mathrm{q}}+4\tilde{\mu}^{\mathrm{r}}+4\tilde{\mu}^{\mathrm{s}})\\
&+\tfrac{1}{\mu_2}(\textstyle\sum_{i=27}^{29}\upsilon_i^2+\sum_{i=72}^{74}\upsilon_i^2)(\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{9}{8}\mu^{\mathrm{k}}+\tfrac{1}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}-\tfrac{3}{8}\mu^{\mathrm{n}}+\tfrac{1}{4}\mu^{\mathrm{p}}+\tfrac{1}{8}\mu^{\mathrm{r}}+\tfrac{3}{8}\mu^{\mathrm{t}}\\
&\quad+2\mu^{\mathrm{u}}+\tfrac{2}{3}\mu^{\mathrm{v}}+\tfrac{4}{3}\mu^{\mathrm{w}})\\
&+\tfrac{1}{\mu_2}(\textstyle\sum_{i=30}^{35}\upsilon_i^2+\sum_{i=75}^{80}\upsilon_i^2)(\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{49}{8}\mu^{\mathrm{k}}+\tfrac{9}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}-\tfrac{7}{8}\mu^{\mathrm{n}}+\tfrac{5}{4}\mu^{\mathrm{p}}-\tfrac{3}{8}\mu^{\mathrm{r}}-\tfrac{21}{8}\mu^{\mathrm{t}}\\
&\quad+2\mu^{\mathrm{w}}+4\tilde{\mu}^{\mathrm{p}}+4\tilde{\mu}^{\mathrm{q}}+4\tilde{\mu}^{\mathrm{r}}+4\tilde{\mu}^{\mathrm{s}})\\
&+\tfrac{1}{\mu_2}(\textstyle\sum_{i=36}^{38}\upsilon_i^2+\sum_{i=82}^{84}\upsilon_i^2)(\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{49}{8}\mu^{\mathrm{k}}+\tfrac{1}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}-\tfrac{7}{8}\mu^{\mathrm{n}}+\tfrac{3}{4}\mu^{\mathrm{p}}+\tfrac{1}{8}\mu^{\mathrm{r}}+\tfrac{7}{8}\mu^{\mathrm{t}}
\end{aligned}
$$

$$
\begin{aligned}
&\quad +2\mu^{\mathrm{u}}+\tfrac{2}{3}\mu^{\mathrm{v}}+\tfrac{4}{3}\mu^{\mathrm{w}}) \\
&+\tfrac{1}{\mu_2}\big(\textstyle\sum_{i=39}^{41} v_i^2+\sum_{i=84}^{86} v_i^2\big)\big(\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{49}{8}\mu^{\mathrm{k}}+\tfrac{1}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}-\tfrac{7}{8}\mu^{\mathrm{n}}+\tfrac{3}{4}\mu^{\mathrm{p}}+\tfrac{1}{8}\mu^{\mathrm{r}}+\tfrac{7}{8}\mu^{\mathrm{t}} \\
&\quad +2\mu^{\mathrm{w}}+36\tilde{\mu}^{\mathrm{p}}+4\tilde{\mu}^{\mathrm{q}}+4\tilde{\mu}^{\mathrm{r}}-12\tilde{\mu}^{\mathrm{s}}\big) \\
&+\tfrac{1}{\mu_2}\big(\textstyle\sum_{i=42}^{44} v_i^2+\sum_{i=87}^{89} v_i^2\big)\big(\tfrac{2}{3}\check{\mu}^{\mathrm{i}}+\tfrac{1}{12}\check{\mu}^{\mathrm{l}}+\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{1}{8}\mu^{\mathrm{k}}+\tfrac{9}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}+\tfrac{1}{8}\mu^{\mathrm{n}}+\tfrac{1}{4}\mu^{\mathrm{p}}-\tfrac{3}{8}\mu^{\mathrm{r}} \\
&\quad +\tfrac{3}{8}\mu^{\mathrm{t}}+36\tilde{\mu}^{\mathrm{b}}+\tfrac{4}{9}\tilde{\mu}^{\mathrm{c}}+4\tilde{\mu}^{\mathrm{d}}+4\tilde{\mu}^{\mathrm{e}}-4\tilde{\mu}^{\mathrm{j}}-12\tilde{\mu}^{\mathrm{k}}+\tfrac{4}{3}\tilde{\mu}^{\mathrm{m}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{p}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{r}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{s}}\big) \\
&+\sqrt{2}\tfrac{1}{\mu_2}\textstyle\sum_{i=1}^{3}\Big(\big(v_{i+8}v_{i+11}+v_{i+53}v_{i+56}\big)\big(\mu^{\mathrm{u}}+\tfrac{1}{3}\mu^{\mathrm{v}}-\tfrac{4}{3}\mu^{\mathrm{w}}+2\tilde{\mu}^{\mathrm{b}}-\tfrac{10}{3}\tilde{\mu}^{\mathrm{c}}+2\tilde{\mu}^{\mathrm{d}}+2\tilde{\mu}^{\mathrm{e}}+\tfrac{2}{3}\tilde{\mu}^{\mathrm{j}}+2\tilde{\mu}^{\mathrm{k}} \\
&\quad +\tfrac{2}{3}\tilde{\mu}^{\mathrm{m}}-8\tilde{\mu}^{\mathrm{p}}-\tfrac{8}{3}\tilde{\mu}^{\mathrm{q}}-\tfrac{8}{3}\tilde{\mu}^{\mathrm{r}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{s}}\big)+\big(v_{i+8}v_{i+56}-v_{i+53}v_{i+11}\big)\big(\tfrac{8}{3}\tilde{\mu}^{\mathrm{n}}-\tfrac{8}{3}\tilde{\mu}^{\mathrm{t}}\big)\Big) \\
&+\tfrac{1}{\mu_2}\textstyle\sum_{i=1}^{3}\Big(\big(v_{i+14}v_{i+41}+v_{i+59}v_{i+86}\big)\big(24\tilde{\mu}^{\mathrm{b}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{c}}-8\tilde{\mu}^{\mathrm{d}}+8\tilde{\mu}^{\mathrm{e}}-\tfrac{40}{3}\tilde{\mu}^{\mathrm{j}}+8\tilde{\mu}^{\mathrm{k}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{m}}+16\tilde{\mu}^{\mathrm{p}} \\
&\quad -\tfrac{16}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{r}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{s}}\big)+\big(v_{i+14}v_{i+86}-v_{i+59}v_{i+41}\big)\big(-8\tilde{\mu}^{\mathrm{l}}-\tfrac{8}{3}\tilde{\mu}^{\mathrm{n}}+8\tilde{\mu}^{\mathrm{o}}-\tfrac{16}{3}\tilde{\mu}^{\mathrm{t}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{u}}\big)\Big) \\
&+\tfrac{1}{\mu_2}\textstyle\sum_{i=1}^{3}\Big(\big(v_{i+35}v_{i+41}+v_{i+80}v_{i+86}\big)\big(-\tfrac{2}{3}\check{\mu}^{\mathrm{j}}+\tfrac{1}{6}\check{\mu}^{\mathrm{m}}\big) \\
&\quad +\big(v_{i+35}v_{i+86}-v_{i+80}v_{i+41}\big)\big(\tfrac{2}{3}\check{\mu}^{\mathrm{k}}+\tfrac{1}{6}\check{\mu}^{\mathrm{n}}\big)\Big)\Big\}+I.T\;.
\end{aligned}
$$

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