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On a Certain Construction of Graded Lie Algebras with Derivation

R. Matthes, G. Rudolph and R. Wulkenhaar*
Institut für Theoretische Physik
Universität Leipzig
Augustusplatz 10/11, D–04109 Leipzig, Germany

Abstract

Using a unital associative $*$–algebra $\mathfrak{A}$ over $\mathbb{C}$ and a certain class of Hermitian finite projective modules together with a graded involutive differential algebra, both associated with $\mathfrak{A}$, we develop a procedure for constructing graded Lie algebras with derivation. Taking, in particular, the canonical differential algebra of Connes' theory, related to the simplest two–point K–cycle, we obtain a class of graded Lie algebras with derivation, which as one special case contains the graded Lie algebra used in the Mainz–Marseille approach to model building. Finally, we outline a new derivation of the standard model.

1 Introduction

During the last decade there has been an increasing interest in methods related to non–commutative differential geometric structures. One of the main streams in this field was initiated and mainly developed by A. Connes ([6], [5]). Starting from the observation that the "classical" Dirac K–cycle of a Riemannian manifold X contains all information about this manifold, he invented the abstract notion of a K–cycle over an – in general – non–commutative algebra. This gives the possibility to discuss geometric structures, which – in general – do not possess an underlying "classical" manifold. Connes realized that already slight modifications of the "classical" K–cycle, namely such that the algebra remains commutative, give rise to interesting physical applications. The simplest relevant example of this type [6] is the K–cycle over the algebra $C^\infty(X) \otimes (\mathbb{C} \oplus \mathbb{C})$ leading to a unification of gauge and Higgs bosons. If one takes the tensor product of this algebra with the vector space of fermions, one can derive a version of the classical

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Lagrangian of the Salam–Weinberg model of electroweak interactions with the bosonic sector described in terms of a unified non–commutative gauge field, see [6], [5], [7]. The above algebra is the simplest example of the class of algebras $C^\infty(X)\otimes(\mathrm{M}_k\mathbb{C}\oplus\mathrm{M}_l\mathbb{C})$, which we call two–point algebras. For the derivation of the full (classical) standard model, Connes and Lott [7] proposed to use a K–cycle over the algebra $C^\infty_{\mathbb{R}}(X)\otimes(\mathbb{C}\oplus\mathbb{H})$, where $\mathbb{H}$ denotes the field of quaternions and $C^\infty_{\mathbb{R}}(X)$ the algebra of real smooth functions on X. A detailed presentation of this construction can be found in a series of papers by Kastler ([17], [18], [19], [20]). For an overview over the mathematical background we refer to [26] and for a physicist's review to [9].

There is another approach to model building, proposed by Coquereaux and Scheck and further developed by their groups in Mainz and Marseille, see [12], [11], [8], [10], [13], which at first sight seems to be completely different from that of Connes and Lott. These authors postulate ad hoc a certain graded matrix Lie algebra and consider a generalized connection with values in this algebra. The connection is built both from differential one forms and zero forms, representing the classical gauge fields of the electroweak interaction and the scalar Higgs fields respectively. Adding by hand the gauge bosons of the strong interaction and choosing appropriate fermionic representations, one can derive the classical Lagrangian of the standard model in this way.

The fact that the bosonic sector in this type of models is unified, has nontrivial phenomenological consequences. In particular, in most versions one obtains a prediction of the Higgs mass at tree level. However, there are – from the phenomenological point of view – certain subtle differences between the two above–mentioned approaches. This is mainly related to the fact that within the construction of Connes and Lott one gets additional relations between boson and fermion masses. For a detailed discussion of this aspect we refer to [22].

In this paper we present a rigorous mathematical link between these two approaches. Using results from our previous paper [21] we will prove that given the simplest two–point K–cycle together with the differential algebra Ω^*_D, which is obtained from the universal differential algebra (associated with the algebra of the K–cycle) by factorizing with respect to a canonically given ideal, and taking a finite projective module over the algebra, we are able to construct in a canonical way a graded Lie algebra. Since every finite projective module carries a canonical connection, this graded Lie algebra is naturally endowed with a derivation. If one chooses the module appropriately, then one arrives at the graded Lie algebra used by the Mainz–Marseille group for the derivation of the standard model. This way all structures, ad hoc postulated within this approach, find their natural explanation within the context of Connes' theory.

As a matter of fact, the construction of graded Lie algebras with derivation proposed in this paper is not limited to the case, when a K–cycle together with the canonically associated differential algebra Ω^*_D is given. All we need – in the most general context – is a unital associative algebra $\mathfrak{A}$ over $\mathbb{C}$ (fulfilling a certain technical condition) and a certain graded differential algebra $\Lambda^*_{\mathfrak{A}}$, associated with $\mathfrak{A}$ in a sense defined below. Then taking an arbitrary finite projective module over $\mathfrak{A}$, we can construct a graded Lie algebra with derivation – a fact, which at least from a purely mathematical point of view seems to be of some interest in itself. For physical applications as discussed above one is rather interested in the case, when $\mathfrak{A}$ and $\Lambda^*_{\mathfrak{A}}$ are endowed additionally with an involution and the module carries a Hermitian structure. It will be interesting to apply our general construction to situations more complicated than that of the simplest two–point K–cycle. In particular, a similar analysis for the N–point case would be interesting, because this case seems to be relevant for the construction of grand unified theories, see [2], [3] and [4].

The paper is organized as follows: In subsection 2.1 we present the construction of graded Lie algebras in the general context – as indicated above. In subsection 2.2 we discuss the notion of connections on finite projective modules and show how the canonical connection gives rise to a graded derivation in the graded Lie algebra constructed before. Next, in subsection 2.3 we give a matrix formulation of these structures. In subsection 3.1 we review results [21] on the differential algebra $\Lambda^*_{\mathcal{A}}$ associated canonically with the simplest two–point K–cycle. In subsection 3.2 we consider the graded Lie algebra $\mathcal{H}$ for this case and distinguish a certain graded Lie subalgebra $\mathcal{H}_0$ of $\mathcal{H}$ relevant for model building. In subsection 3.3 we change the standard matrix representation of the structures discussed before. In section 4 we show that the mathematical structures used in the Mainz–Marseille approach are naturally obtained from the framework developed in this paper. More precisely, in subsection 4.1 we derive a slightly generalized version of the graded Lie algebra arising in the Mainz–Marseille approach. In subsection 4.2 we define a projection of the graded Lie algebra of subsection 3.2 to that of subsection 4.1, and we discuss the structure of the projected geometrical objects. Then, in subsection 4.3, we specialize to the original Mainz–Marseille model as described in [12] and [11]. Finally, in section 5 we outline how the standard model can be derived in our scheme.

2 The General Scheme

2.1 Finite Projective Modules with Hermitian Structure and Graded Lie Algebras

Let $\mathfrak{A}$ be a unital associative $*$–algebra over $\mathbb{C}$, so that $a^*a = 0$ iff $a = 0$. Moreover, let $(\Lambda^*_{\mathfrak{A}}, \bullet, *, d)$ be a graded involutive differential algebra associated with $\mathfrak{A}$. That means $\Lambda^*_{\mathfrak{A}} = \bigoplus_{k=0}^{\infty} \Lambda^k_{\mathfrak{A}}$, $\Lambda^0_{\mathfrak{A}} \equiv \mathfrak{A}$. The dot $\bullet$ denotes the multiplication $\Lambda^k_{\mathfrak{A}} \bullet \Lambda^l_{\mathfrak{A}} \subset \Lambda^{k+l}_{\mathfrak{A}}$, d the graded differential $d : \Lambda^k_{\mathfrak{A}} \to \Lambda^{k+1}_{\mathfrak{A}}$, and $*$ is an involution compatible with d,

$$d(\lambda^*) = (-1)^k (d\lambda)^* \,, \quad \lambda \in \Lambda^k_{\mathfrak{A}} \,. \tag{1}$$

Since $\mathfrak{A} \equiv \Lambda^0_{\mathfrak{A}}$, we have a natural $\mathfrak{A}$–bimodule structure on $\Lambda^*_{\mathfrak{A}}$. When multiplying elements of $\mathfrak{A}$ with elements of $\Lambda^*_{\mathfrak{A}}$, we omit the dot for simplicity.

We recall [26] that every finite projective right module $\mathcal{E}$ over $\mathfrak{A}$ has the structure $\mathcal{E} = e\mathfrak{A}^p$, where p is a natural number and $e \in \mathrm{End}_{\mathfrak{A}}(\mathfrak{A}^p)$, with $e^2 = e$. Here, $\mathfrak{A}^p$ is treated as $\mathbb{C}^p \otimes \mathfrak{A}$. Elements $\zeta \in \mathfrak{A}^p$ are of the form $\zeta = \sum_\alpha c_\alpha \otimes a_\alpha$, finite sum, where $c_\alpha \in \mathbb{C}^p$ and $a_\alpha \in \mathfrak{A}$. We shall often write $\zeta = c \otimes a$, with a linear extension to finite sums being understood.

Definition 1 *A Hermitian finite projective right $\mathfrak{A}$–module is a pair $(\mathcal{E}, (\ ,\ )_{\mathcal{E}})$, where $(\ ,\ )_{\mathcal{E}} : \mathcal{E} \times \mathcal{E} \to \mathfrak{A}$ is a sesquilinear, Hermitian, non–degenerate, positive map.*

We define a Hermitian structure on $\mathfrak{A}^p$ by

$$(c \otimes a\, ,\, \tilde{c} \otimes \tilde{a})_{\mathfrak{A}^p} := (c, \tilde{c})_{\mathbb{C}^p}\, a^* \tilde{a}\ , \tag{2}$$

where $(\ ,\ )_{\mathbb{C}^p}$ denotes a scalar product on $\mathbb{C}^p$. The involution of endomorphisms of $\mathfrak{A}^p$ is defined by $(\chi^*\zeta, \tilde{\zeta})_{\mathfrak{A}^p} = (\zeta, \chi\tilde{\zeta})_{\mathfrak{A}^p}$, for $\chi \in \mathrm{End}_{\mathfrak{A}}(\mathfrak{A}^p)$. We assume that e is an orthogonal (Hermitian) projector, $e = e^*$. Restricting the Hermitian structure given by (2) to $\mathcal{E} = e\mathfrak{A}^p$, we get a Hermitian structure on $\mathcal{E}$.

Let us denote the tensor product of the right module $\mathcal{E}$ with the bimodule $\Lambda^k_{\mathfrak{A}}$ over the algebra $\mathfrak{A}$ by $\mathcal{E}^k = \mathcal{E} \otimes_{\mathfrak{A}} \Lambda^k_{\mathfrak{A}}$, $\mathcal{E}^0 := \mathcal{E}$ and $\mathcal{E}^* := \bigoplus_{k \in \mathbb{N}_0} \mathcal{E}^k$. On $\mathcal{E}^*$ we have the natural structure of a right $\Lambda^*_{\mathfrak{A}}$–module inherited from the multiplication in $\Lambda^*_{\mathfrak{A}}$:

$$\mathcal{E}^k \times \Lambda^l_{\mathfrak{A}} \ni (\xi \otimes_{\mathfrak{A}} \lambda\, ,\, \tilde{\lambda}) \mapsto (\xi \otimes_{\mathfrak{A}} \lambda) \bullet \tilde{\lambda} := \xi \otimes_{\mathfrak{A}} (\lambda \bullet \tilde{\lambda}) \in \mathcal{E}^{k+l}\ , \tag{3}$$

for $\xi \in \mathcal{E}$, $\lambda \in \Lambda^k_{\mathfrak{A}}$, $\tilde{\lambda} \in \Lambda^l_{\mathfrak{A}}$. We extend the Hermitian structure on $\mathcal{E}$ to mappings $(\ ,\ )^{k,l}_{\mathcal{E}} : \mathcal{E}^k \times \mathcal{E}^l \to \Lambda^{k+l}_{\mathfrak{A}}$ by

$$(\xi \otimes_{\mathfrak{A}} \lambda\ ,\ \tilde{\xi} \otimes_{\mathfrak{A}} \tilde{\lambda})^{k,l}_{\mathcal{E}} := \lambda^* \bullet (\xi, \tilde{\xi})_{\mathcal{E}} \bullet \tilde{\lambda}\ . \tag{4}$$

Lemma 2

$$\begin{array}{llcll}
(i) & (\xi a, \tilde{\xi}\tilde{a})^{k,l}_{\mathcal{E}} & = & a^*(\xi, \tilde{\xi})^{k,l}_{\mathcal{E}}\tilde{a}\ , & \mathit{for}\ \xi \in \mathcal{E}^k\ ,\ \tilde{\xi} \in \mathcal{E}^l\ ,\ a, \tilde{a} \in \mathfrak{A}\ , \\
(ii) & (\,(\xi, \tilde{\xi})^{k,l}_{\mathcal{E}}\,)^* & = & (\tilde{\xi}, \xi)^{l,k}_{\mathcal{E}}\ , & \mathit{for}\ \xi \in \mathcal{E}^k\ ,\ \tilde{\xi} \in \mathcal{E}^l\ , \\
(iii) & (\xi, \tilde{\xi})^{k,0}_{\mathcal{E}} & = & 0\ \ \forall \tilde{\xi} \in \mathcal{E} & \mathit{iff}\ \xi = 0\ ,\ \xi \in \mathcal{E}^k\ , \\
(iv) & (\xi, \tilde{\xi})^{0,l}_{\mathcal{E}} & = & 0\ \ \forall \xi \in \mathcal{E} & \mathit{iff}\ \tilde{\xi} = 0\ ,\ \tilde{\xi} \in \mathcal{E}^l\ .
\end{array}$$

□

Let $\mathcal{H}^k \equiv \mathrm{Hom}_{\mathfrak{A}}(\mathcal{E}, \mathcal{E}^k)$ be the set of homomorphisms of the right $\mathfrak{A}$–module $\mathcal{E}$ to the right $\mathfrak{A}$–module $\mathcal{E}^k$ and $\mathcal{H} := \bigoplus_{k \in \mathbb{N}_0} \mathcal{H}^k$. Using the right $\Lambda^*_{\mathfrak{A}}$–module structure on $\mathcal{E}^*$, see (3), we get a natural associative multiplication $\bullet$ on $\mathcal{H}$. We define $\bullet : \mathcal{H}^k \times \mathcal{H}^l \to \mathcal{H}^{k+l}$ by

$$(\varrho \bullet \tilde{\varrho})(\xi) := (\,\mathrm{id}_{\mathcal{E}} \otimes_{\mathfrak{A}} \bullet\,) \circ (\,\varrho \otimes_{\mathfrak{A}} \mathrm{id}_{\Lambda^l_{\mathfrak{A}}}\,) \circ \tilde{\varrho}(\xi)\ , \tag{5}$$

for $\varrho \in \mathcal{H}^k$, $\tilde{\varrho} \in \mathcal{H}^l$, $\xi \in \mathcal{E}$. The Hermitian mappings $(\ , \ )_{\mathcal{E}}^{k,0}$ and $(\ , \ )_{\mathcal{E}}^{0,k}$ induce an involution on $\mathcal{H}^k$:

$$(\xi, \varrho^*(\tilde{\xi}))_{\mathcal{E}}^{0,k} := (\varrho(\xi), \tilde{\xi})_{\mathcal{E}}^{k,0} \ , \quad \forall \xi, \tilde{\xi} \in \mathcal{E} \ , \quad \varrho \in \mathcal{H}^k \ . \tag{6}$$

Due to Lemma 2, this involution is well–defined. Moreover, one can show that

$$(\varrho \bullet \tilde{\varrho})^* = \tilde{\varrho}^* \bullet \varrho^* \ , \quad \varrho, \tilde{\varrho} \in \mathcal{H} \ . \tag{7}$$

Thus, $\mathcal{H}$ is an associative, $\mathbb{N}$–graded, unital, involutive algebra over $\mathbb{C}$.

We define

$$[\, \varrho \, , \, \tilde{\varrho} \,]_g := \varrho \bullet \tilde{\varrho} - (-1)^{kl} \tilde{\varrho} \bullet \varrho \ , \quad \varrho \in \mathcal{H}^k \ , \quad \tilde{\varrho} \in \mathcal{H}^l \ . \tag{8}$$

Lemma 3 *With respect to the above bracket, $\mathcal{H}$ is a graded Lie algebra, i.e. we have for $\varrho, \varrho' \in \mathcal{H}^k$, $\tilde{\varrho} \in \mathcal{H}^l$, $\tilde{\tilde{\varrho}} \in \mathcal{H}^m$ and $z, z' \in \mathbb{C}$*

$$\begin{array}{ll} (i) & [\varrho, \tilde{\varrho}]_g = -(-1)^{kl} [\tilde{\varrho}, \varrho]_g \ , \\ (ii) & [z\varrho + z'\varrho', \tilde{\varrho}]_g = z[\varrho, \tilde{\varrho}]_g + z'[\varrho', \tilde{\varrho}]_g \ , \\ (iii) & (-1)^{km}[\varrho, [\tilde{\varrho}, \tilde{\tilde{\varrho}}]_g]_g + (-1)^{lk}[\tilde{\varrho}, [\tilde{\tilde{\varrho}}, \varrho]_g]_g + (-1)^{ml}[\tilde{\tilde{\varrho}}, [\varrho, \tilde{\varrho}]_g]_g = 0 \ . \end{array} \tag{9}$$

□

Finally, we endow $\mathcal{E}^*$ naturally with the structure of a left graded $\mathcal{H}$–module, putting

$$\varrho \bullet \xi = (\, \mathrm{id}_{\mathcal{E}} \otimes_{\mathfrak{A}} \bullet \,) \circ (\, \varrho \otimes_{\mathfrak{A}} \mathrm{id}_{\Lambda^l_{\mathfrak{A}}})(\xi) \ , \tag{10}$$

for $\varrho \in \mathcal{H}^k$ and $\xi \in \mathcal{E}^l$. By construction, we have

$$(\varrho \bullet \tilde{\varrho}) \bullet \xi = \varrho \bullet (\tilde{\varrho} \bullet \xi) \ , \quad \varrho, \tilde{\varrho} \in \mathcal{H} \ , \ \xi \in \mathcal{E}^* \ . \tag{11}$$

Thus, $\mathcal{E}^*$ is a natural representation space of the graded Lie algebra $\mathcal{H}$.

2.2 Connections and Graded Derivations

Now we recall the notion of a connection on $\mathcal{E}$ associated with the differential calculus $(\Lambda^*_{\mathfrak{A}}, \bullet, *, d)$, see [7].

Definition 4 *i) A connection on $\mathcal{E}$ is given by a $\mathbb{C}$–linear map $\nabla : \mathcal{E} \to \mathcal{E}^1$, so that $\nabla(\xi a) = (\nabla \xi)a + \xi \otimes_{\mathfrak{A}} da$, for $\xi \in \mathcal{E}$, $a \in \mathfrak{A}$.*

ii) A connection is compatible (with the Hermitian structure) iff
$(\xi, \nabla\tilde{\xi})_{\mathcal{E}}^{0,1} + (\nabla\xi, \tilde{\xi})_{\mathcal{E}}^{1,0} = d(\xi, \tilde{\xi})_{\mathcal{E}}$ *, for $\xi, \tilde{\xi} \in \mathcal{E}$.*

Definition 5 (*cf.* [7]) *The gauge group $\mathcal{U}(\mathcal{E})$ is the group of unitary automorphisms of $\mathcal{E}$, $\mathcal{U}(\mathcal{E}) := \{ \ u \in \mathrm{End}_{\mathfrak{A}}(\mathcal{E}) \ : \ uu^* = u^*u = \mathrm{id}_{\mathcal{E}} \ \}$, and gauge transformations of the connection ∇ are given by $u \nabla u^*$.*

We extend ∇ uniquely to linear maps $\nabla : \mathcal{E}^n \to \mathcal{E}^{n+1}$ by

$$\nabla(\xi \otimes_{\mathfrak{A}} \lambda) := (\nabla \xi) \bullet \lambda + \xi \otimes_{\mathfrak{A}} d\lambda \ , \quad \xi \in \mathcal{E} \ , \quad \lambda \in \Lambda^n_{\mathfrak{A}} \ , \tag{12}$$

satisfying $\nabla(\xi \bullet \lambda) = (\nabla\xi) \bullet \lambda + (-1)^n \xi \bullet d\lambda$, $\xi \in \mathcal{E}^n$, $\lambda \in \Lambda^*_{\mathfrak{A}}$. The curvature of the connection ∇,

$$\theta := \nabla^2|_{\mathcal{E}} \ , \tag{13}$$

is an element of $\mathcal{H}^2$.

Lemma 6 *There exists a canonical compatible connection ∇_0 on $\mathcal{E}$ given by*

$$\nabla_0(c \otimes a) := e(c \otimes \mathbb{1}_{\mathfrak{A}}) \otimes_{\mathfrak{A}} da \ , \tag{14}$$

with $c \otimes a \in \mathcal{E} \subset \mathfrak{A}^p$ and $\mathbb{1}_{\mathfrak{A}}$ denoting the unit element of $\mathfrak{A}$. □

Lemma 7 *Any compatible connection ∇ on $\mathcal{E}$ has the form*

$$\nabla = \nabla_0 + \rho \ , \quad \textit{with} \quad \rho = -\rho^* \in \mathcal{H}^1 \ . \tag{15}$$

Proof: See [26]. □

The existence of the canonical connection ∇_0 on $\mathcal{E}$ ensures that we have a canonical graded derivation $\mathcal{D} : \mathcal{H}^k \to \mathcal{H}^{k+1}$,

$$(\mathcal{D}\varrho)(\xi) := \nabla_0(\varrho(\xi)) - (-1)^k \varrho \bullet (\nabla_0 \xi) \ , \tag{16}$$

where $\xi \in \mathcal{E}$, $\varrho \in \mathcal{H}^k$. One easily shows that

$$\begin{aligned}
(\mathcal{D}\varrho)(\xi a) &= ((\mathcal{D}\varrho)(\xi))a \ , \\
\mathcal{D}(\varrho \bullet \tilde{\varrho}) &= (\mathcal{D}\varrho) \bullet \tilde{\varrho} + (-1)^k \varrho \bullet \mathcal{D}\tilde{\varrho} \ , \\
\mathcal{D}[\varrho, \tilde{\varrho}]_g &= [\mathcal{D}\varrho, \tilde{\varrho}]_g + (-1)^k [\varrho, \mathcal{D}\tilde{\varrho}]_g \ , \\
(\mathcal{D}\varrho)^* &= (-1)^k \mathcal{D}(\varrho^*) \ ,
\end{aligned} \tag{17}$$

for $\varrho \in \mathcal{H}^k$, $\tilde{\varrho} \in \mathcal{H}^l$ and $a \in \mathfrak{A}$. Note, however, that $\mathcal{D}$ is – in general – *not* a differential of $\mathcal{H}$, because we get from (16)

$$\mathcal{D}^2 \varrho \equiv \theta_0 \bullet \varrho - \varrho \bullet \theta_0 \ , \quad \varrho \in \mathcal{H} \ , \tag{18}$$

where $\theta_0 := \nabla_0^2$ is the curvature of the canonical connection ∇_0. From (16) one also finds

$$\theta = \theta_0 + \mathcal{D}\rho + \rho \bullet \rho \ , \tag{19}$$

and Definition 5 gives the following formulae for gauge transformations:

$$\begin{aligned}
u \nabla u^* &= \nabla_0 + u\mathcal{D}u^* + u\rho u^* \ , \\
\gamma_u(\rho) &= u\mathcal{D}u^* + u\rho u^* \ , \\
\gamma_u(\theta) &= u\theta u^* \ .
\end{aligned} \tag{20}$$

2.3 Matrix Representation

Now we choose the canonical basis $\{\varepsilon_i\}_{i=1,\dots,p}$ in $\mathbb{C}^p$, together with the canonical scalar product. This enables us to embed all structures discussed in the previous two subsections into the tensor product $\Lambda^*_{\mathfrak{A}} \otimes \mathrm{M}_p\mathbb{C}$. Observe that $\{\varepsilon_i \otimes \mathbb{1}_{\mathfrak{A}}\}_{i=1,\dots,p}$ is the canonical basis of the free right $\mathfrak{A}$–module $\mathfrak{A}^p \cong \mathbb{C}^p \otimes \mathfrak{A}$ and

$$e(\varepsilon_i \otimes \mathbb{1}_{\mathfrak{A}}) = \sum_{j=1}^{p} \varepsilon_j \otimes e_{ji} \ . \tag{21}$$

Thus, the projector e is represented by the Hermitian $p \times p$–matrix (e_{ji}), $e_{ji} \in \mathfrak{A}$. Therefore, elements

$$\xi \equiv e\xi = c \otimes a = \sum_{i=1}^{p} \varepsilon_i \otimes c_i a \in \mathcal{E} \ , \quad c = \sum_{i=1}^{p} \varepsilon_i c_i \in \mathbb{C}^p \ , \tag{22}$$

are naturally identified with columns

$$\xi = \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix} \ , \quad a_i = c_i a \in \mathfrak{A} \ . \tag{23}$$

Observe that $e\xi = \xi$ means $\sum_{j=1}^{p} e_{ij} a_j = a_i$. The Hermitian structure on $\mathcal{E}$ takes the form

$$(\xi, \tilde{\xi})_{\mathcal{E}} := \sum_{i=1}^{p} a_i^* \tilde{a}_i \ , \quad \xi, \tilde{\xi} \in \mathcal{E} \ . \tag{24}$$

For $\xi = \tilde{\xi} \otimes_{\mathfrak{A}} \lambda \in \mathcal{E}^k$, with $\tilde{\xi} = \sum_{i=1}^{p} \varepsilon_i \otimes a_i \in \mathcal{E}$ and $\lambda \in \Lambda^k_{\mathfrak{A}}$, we get

$$\xi = \tilde{\xi} \otimes_{\mathfrak{A}} \lambda = \sum_{i=1}^{p} (\varepsilon_i \otimes \mathbb{1}_{\mathfrak{A}}) \otimes_{\mathfrak{A}} a_i \lambda \ . \tag{25}$$

Therefore, elements $\xi \in \mathcal{E}^k$ are naturally identified with columns

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_p \end{pmatrix} \ , \quad \xi_i = a_i \lambda \in \Lambda^k_{\mathfrak{A}} \ . \tag{26}$$

Again, $e\xi = \xi$ means $\sum_{j=1}^{p} e_{ij} \xi_j = \xi_i$. The right $\Lambda^*_{\mathfrak{A}}$–module structure of $\mathcal{E}^*$ is given by

$$\mathcal{E}^k \times \Lambda^l_{\mathfrak{A}} \ni (\xi, \lambda) \mapsto \xi \bullet \lambda = \begin{pmatrix} \xi_1 \bullet \lambda \\ \vdots \\ \xi_p \bullet \lambda \end{pmatrix} \in \mathcal{E}^{k+l} \ . \tag{27}$$

The canonical compatible connection ∇_0 on $\mathcal{E}^*$, see (14) and (12), takes the form

$$\nabla_0\xi = (\nabla_0\tilde{\xi})\bullet\lambda + \tilde{\xi}\otimes_{\mathfrak{A}} d\lambda = \sum_{i,j=1}^{p} (\varepsilon_j \otimes \mathbb{1}_{\mathfrak{A}}) \otimes_{\mathfrak{A}} e_{ji} d(a_i\lambda) \equiv \sum_{i,j=1}^{p} (\varepsilon_j \otimes \mathbb{1}_{\mathfrak{A}}) \otimes_{\mathfrak{A}} e_{ji} d(\xi_i) \ . \tag{28}$$

Thus, $\nabla_0\,\xi \in \mathcal{E}^{k+1}$ can be represented by

$$\nabla_0\,\xi = e \begin{pmatrix} d\xi_1 \\ \vdots \\ d\xi_p \end{pmatrix} \ . \tag{29}$$

Due to (23) and (26), $\varrho \in \mathcal{H}^k$ can be represented by a matrix

$$\varrho = \begin{pmatrix} \varrho_{11} & \dots & \varrho_{1j} & \dots & \varrho_{1p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \varrho_{i1} & \dots & \varrho_{ij} & \dots & \varrho_{ip} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \varrho_{p1} & \dots & \varrho_{pj} & \dots & \varrho_{pp} \end{pmatrix} \ , \quad \varrho_{ij} \in \Lambda^k_{\mathfrak{A}} \ . \tag{30}$$

We have $e\varrho e = \varrho$ or, in matrix representation, $\sum_{i,j,m,n=1}^p e_{im}\varrho_{mn}e_{nj} = \varrho_{ij}$. Moreover, the action of ϱ on $\xi \in \mathcal{E}^l$ and the product $\bullet$ in the algebra $\mathcal{H}$ are represented by matrix multiplication:

$$(\varrho\bullet\xi)_i = \textstyle\sum_{j=1}^p \varrho_{ij}\bullet\xi_j \ , \tag{31}$$

$$(\varrho\bullet\tilde{\varrho})_{ij} = \textstyle\sum_{n=1}^p \varrho_{in}\bullet\tilde{\varrho}_{nj} \ , \tag{32}$$

and the involution (6) is given by

$$(\varrho^*)_{ij} = (\varrho_{ji})^* \ . \tag{33}$$

We observe that $\mathcal{H}$ can be treated as an involutive subalgebra of $\Lambda^*_{\mathfrak{A}} \otimes \mathrm{M}_p\mathbb{C}$.

Using (28) and the above calculus one gets the curvature

$$(\theta_0)_{ij} = \sum_{k,l,m=1}^{p} e_{ik}d(e_{kl})\bullet d(e_{lm})e_{mj} \ , \tag{34}$$

where, in particular, one has to use $\sum_{m,n=1}^p e_{im}d(e_{mn})e_{nj} = 0$. Using (16), (28) and (31) one calculates

$$\begin{aligned}(\mathcal{D}\varrho)(\xi) &= \textstyle\sum_{i,j,n=1}^p(\varepsilon_i\otimes\mathbb{1}_{\mathfrak{A}})\otimes_{\mathfrak{A}}e_{ij}d(\varrho_{jn}a_n) - (-1)^k\sum_{i,j,n=1}^p(\varepsilon_i\otimes\mathbb{1}_{\mathfrak{A}})\otimes_{\mathfrak{A}}\varrho_{ij}e_{jn}d(a_n) \\ &= \textstyle\sum_{i,j,n,m=1}^p(\varepsilon_i\otimes\mathbb{1}_{\mathfrak{A}})\otimes_{\mathfrak{A}}\{e_{ij}d(\varrho_{jn})e_{nm}a_m\} \ .\end{aligned}$$

Thus, $\mathcal{D}\varrho$ can be represented by the following matrix

$$\mathcal{D}\varrho = e\, d(\varrho)\, e \equiv e \begin{pmatrix} d(\varrho_{11}) & \dots & d(\varrho_{1p}) \\ \vdots & \ddots & \vdots \\ d(\varrho_{p1}) & \dots & d(\varrho_{pp}) \end{pmatrix} e \ . \tag{35}$$

For later purposes it is convenient to represent also $\mathcal{E}^*$ and $\Lambda^*_{\mathfrak{A}}$ as subspaces of $\Lambda^*_{\mathfrak{A}} \otimes \mathrm{M}_p\mathbb{C}$. This goes as follows: First, $\mathcal{E}^*$ is embedded as a vector subspace, putting

$$\mathrm{j}_{\mathcal{E}}(\xi) := (\ \underbrace{\xi \,|\, \xi \,|\, \dots \,|\, \xi}_{p}\ )\ , \tag{36}$$

which means building the $p \times p$–block matrix $\mathrm{j}_{\mathcal{E}}(\xi)$ from the $p \times 1$–column $\xi \in \mathcal{E}^*$. To preserve the right $\Lambda^*_{\mathfrak{A}}$–module structure of $\mathcal{E}^*$, we embed $\Lambda^*_{\mathfrak{A}}$ as a subalgebra, putting

$$\mathrm{j}_{\Lambda}(\lambda) := \begin{pmatrix} \lambda & & \mathrm{O} \\ & \ddots & \\ \mathrm{O} & & \lambda \end{pmatrix}\ , \quad \lambda \in \Lambda^*_{\mathfrak{A}}\ . \tag{37}$$

Under this embedding the right module structure and the left action of $\mathcal{H}$ on $\mathcal{E}^*$ are transported as follows:

$$\mathrm{j}_{\mathcal{E}}(\xi \bullet \lambda) \ = \ \mathrm{j}_{\mathcal{E}}(\xi) \bullet \mathrm{j}_{\Lambda}(\lambda)\ , \quad \xi \in \mathcal{E}^*\ , \quad \lambda \in \Lambda^*_{\mathfrak{A}}\ , \tag{38}$$

$$\mathrm{j}_{\mathcal{E}}(\varrho \bullet \xi) \ = \ \varrho \bullet \mathrm{j}_{\mathcal{E}}(\xi)\ , \quad \xi \in \mathcal{E}^*\ , \quad \varrho \in \mathcal{H}\ . \tag{39}$$

3 Application to the Simplest two–point K–cycle and its Associated Differential Algebra Λ^*_A

3.1 The Differential Algebra $\Lambda^*_{\mathcal{A}}$

The construction presented above can be, in particular, applied to the special case of a K–cycle and its canonically associated differential algebra Ω^*_D, see [6], [7]. For the rest of the paper we restrict ourselves to this situation. We consider the simplest two–point K–cycle, whose differential algebra Ω^*_D was analysed in [21]. To keep this paper selfcontained, we review some results obtained there.

Let X be a compact even dimensional Riemannian spin manifold, $\dim(X) =: N$. We denote by $L^2(X,S)$ the Hilbert space of square integrable sections of the spinor bundle over X, by C the Clifford bundle over X, and by C^k the set of those sections of C, whose values at each point $x \in X$ belong to the subspace spanned by products of less than or equal k elements of T^*_xX of the same parity.

We consider the even K–cycle $(\mathcal{A}, h, D, \Gamma)$, see [6], [5], [7]. The Hilbert space h is

$$h := L^2(X, S) \otimes \tilde{F} \ , \tag{40}$$

where $\tilde{F}$ is a finite dimensional Hilbert space, which in physical applications carries fermionic degrees of freedom. We assume that there exists a selfadjoint grading operator Γ acting on h, $\Gamma^2 = \mathrm{id}_h$,

$$\Gamma = \gamma^{N+1} \otimes \tilde{\Gamma} \ , \quad \tilde{\Gamma} \in \mathrm{End}\,(\tilde{F}) \ , \tag{41}$$

with $\gamma^{N+1} := i^{\frac{N}{2}} \gamma^1 \gamma^2 \cdots \gamma^{N-1} \gamma^N$ and $\tilde{\Gamma}$ denoting the grading operators on $L^2(X, S)$ and $\tilde{F}$ respectively. The $\{\gamma^\mu\}_{\mu=1,\ldots,N}$ are chosen as local orthonormal selfadjoint sections of C^1. We have the decomposition

$$\tilde{F} \equiv \frac{1}{2}(\mathrm{id}_{\tilde{F}} + \tilde{\Gamma})\tilde{F} \oplus \frac{1}{2}(\mathrm{id}_{\tilde{F}} - \tilde{\Gamma})\tilde{F} \equiv F_+ \oplus F_- \ . \tag{42}$$

This gives the decomposition $h \equiv h_+ \oplus h_-$, with $h_\pm := L^2(X, S) \otimes F_\pm$. Thus, elements $\psi \in h$ naturally decompose as $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, where $\psi_+ \in h_+$ and $\psi_- \in h_-$. Then, Γ can be represented by $\Gamma = \begin{pmatrix} \gamma^{N+1} \otimes \mathrm{id}_{F_+} & 0 \\ 0 & -\gamma^{N+1} \otimes \mathrm{id}_{F_-} \end{pmatrix}$.

The algebra $\mathcal{A}$ of the K–cycle is

$$\mathcal{A} := C^\infty(X) \otimes (\mathbb{C} \oplus \mathbb{C}) \cong C^\infty(X) \oplus C^\infty(X) \ . \tag{43}$$

We consider the following involutive representation π of $\mathcal{A}$ on h :

$$\pi((f, \tilde{f}))(\psi, \tilde{\psi}) := ((f \otimes \mathrm{id}_{F_+})(\psi), (\tilde{f} \otimes \mathrm{id}_{F_-})(\tilde{\psi})) \ , \tag{44}$$

for $f, \tilde{f} \in C^\infty(X)$ and $\psi \in h_+$, $\tilde{\psi} \in h_-$. This implies that Γ commutes with $\pi(\mathcal{A})$. In the above representation we get

$$\pi(\mathcal{A}) = \{ \ a = \begin{pmatrix} f \otimes \mathrm{id}_{F_+} & 0 \\ 0 & \tilde{f} \otimes \mathrm{id}_{F_-} \end{pmatrix} \ , \quad f, \tilde{f} \in C^0 \cong C^\infty(X) \ \} \ . \tag{45}$$

The selfadjoint generalized Dirac operator D of the K–cycle is

$$D := D^{c\ell} \otimes \mathrm{id}_{\tilde{F}} + \gamma^{N+1} \otimes \mathcal{M} \ , \tag{46}$$

where $D^{c\ell}$ is the classical Dirac operator on $L^2(X, S)$ and $\mathcal{M}$ is an endomorphism of $\tilde{F}$. One demands $D\Gamma + \Gamma D = 0$, which implies $\tilde{\Gamma}\mathcal{M} = -\mathcal{M}\tilde{\Gamma}$. The selfadjointness of D implies $\mathcal{M} = \mathcal{M}^*$. Thus, we have a natural decomposition $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-$,

$$\mathcal{M}_+ := \mathcal{M}\frac{1}{2}(\mathrm{id}_{\tilde{F}} + \tilde{\Gamma}) = \begin{pmatrix} 0 & 0 \\ M^* & 0 \end{pmatrix} \ , \quad \mathcal{M}_- := \mathcal{M}\frac{1}{2}(\mathrm{id}_{\tilde{F}} - \tilde{\Gamma}) = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \ , \tag{47}$$

where $M \in \mathrm{Hom}(F_-, F_+)$. We define

$$
\begin{aligned}
\mathcal{M}_+^0 &= \begin{pmatrix} \mathrm{id}_{F_+} & 0 \\ 0 & 0 \end{pmatrix} , \quad \mathcal{M}_-^0 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id}_{F_-} \end{pmatrix} , \\
\mathcal{M}_+^{2t} &= (\mathcal{M}_-\mathcal{M}_+)^t = \begin{pmatrix} M_1^t & 0 \\ 0 & 0 \end{pmatrix} , \quad \mathcal{M}_-^{2t} = (\mathcal{M}_+\mathcal{M}_-)^t = \begin{pmatrix} 0 & 0 \\ 0 & M_4^t \end{pmatrix} , \\
\mathcal{M}_+^{2t+1} &= \mathcal{M}_+(\mathcal{M}_-\mathcal{M}_+)^t = \begin{pmatrix} 0 & 0 \\ M_3^t & 0 \end{pmatrix} , \qquad (48) \\
\mathcal{M}_-^{2t+1} &= \mathcal{M}_-(\mathcal{M}_+\mathcal{M}_-)^t = \begin{pmatrix} 0 & M_2^t \\ 0 & 0 \end{pmatrix} , \qquad \text{where} \\
& M_1^r := (MM^*)^r\,,\ M_2^r := M(M^*M)^r\,,\ M_3^r := M^*(MM^*)^r\,,\ M_4^r := (M^*M)^r\ .
\end{aligned}
$$

There exists an involutive representation π of the universal differential algebra Ω^* over $\mathcal{A}$ on h , giving the algebra [7]

$$
\begin{aligned}
\pi(\Omega^*) &= \bigoplus_{k=0}^{\infty} \pi(\Omega^k)\ , \quad \pi(\Omega^0) = \pi(\mathcal{A})\ , \qquad (49) \\
\pi(\Omega^k) &= \{\ (-i)^k \sum_\alpha \pi(a_\alpha^0)[D, \pi(a_\alpha^1)] \cdots [D, \pi(a_\alpha^k)]\ ,\quad a_\alpha^i \in \mathcal{A}\ \}\ ,\quad k \geq 1\ .
\end{aligned}
$$

We restrict ourselves to the case $F_+ \cong F_- \equiv F$ and demand additionally $\mathcal{M}^2 \notin \mathbb{C}\,\mathrm{id}_{F\oplus F}$. In this case one can show, see [21], that

$$
\pi(\Omega^k) = \begin{bmatrix} \bigoplus_{t=0}^{m} C^{k-2t} \otimes \mathbb{C}\, M_1^t & ; & \bigoplus_{t=0}^{m} C^{k-2t-1}\gamma^{N+1} \otimes \mathbb{C}\, M_2^t \\ \bigoplus_{t=0}^{m} C^{k-2t-1}\gamma^{N+1} \otimes \mathbb{C}\, M_3^t & ; & \bigoplus_{t=0}^{m} C^{k-2t} \otimes \mathbb{C}\, M_4^t \end{bmatrix} , \qquad (50)
$$

where $m+1$ is the number of linear independent elements $\mathcal{M}_+^{2t}$. We denote $L^n \equiv C^n / C^{n-2}$, for $n \geq 2$, and put $L^0 \equiv C^0$, $L^1 \equiv C^1$ and $L^n = \{0\}$ for $n < 0$. We have $L^n = \{0\}$ for $n > N$. There is a graded algebra $\Lambda_{\mathcal{A}}^*$ associated with $\pi(\Omega^*)$ defined as follows:

$$
\begin{aligned}
\Lambda_{\mathcal{A}}^* &= \bigoplus_{k=0}^{\infty} \Lambda_{\mathcal{A}}^k\ , \\
\Lambda_{\mathcal{A}}^k &\equiv \sigma_k \circ \pi(\Omega^k) := \begin{cases} \pi(\Omega^k)\,/\,\pi(\Omega^{k-2}) & \text{for } k \geq 2\ , \\ \pi(\Omega^k) & \text{for } k = 0, 1\ , \end{cases} \qquad (51)
\end{aligned}
$$

with multiplication

$$
\Lambda_{\mathcal{A}}^k \times \Lambda_{\mathcal{A}}^l \ni (\lambda, \tilde{\lambda}) \mapsto \lambda \bullet \tilde{\lambda} := \sigma_{k+l}(\tau\tilde{\tau}) \in \Lambda_{\mathcal{A}}^{k+l}\ , \qquad (52)
$$

where $\tau \in \pi(\Omega^k)$, $\tilde{\tau} \in \pi(\Omega^l)$, so that $\sigma_k(\tau) = \lambda$, $\sigma_l(\tilde{\tau}) = \tilde{\lambda}$. One can show [21] that

$$\Lambda^k_{\mathcal{A}} \cong \begin{bmatrix} \bigoplus_{t=0}^{m} L^{k-2t} \otimes \mathbb{C}\, M_1^t & ; & \bigoplus_{t=0}^{m} L^{k-2t-1}\gamma^{N+1} \otimes \mathbb{C}\, M_2^t \\ \bigoplus_{t=0}^{m} L^{k-2t-1}\gamma^{N+1} \otimes \mathbb{C}\, M_3^t & ; & \bigoplus_{t=0}^{m} L^{k-2t} \otimes \mathbb{C}\, M_4^t \end{bmatrix} . \tag{53}$$

Elements $\lambda \in \Lambda^k_{\mathcal{A}}$ are of the form

$$\lambda = \begin{pmatrix} \sum_{t=0}^{m} \alpha_1^{k-2t} \otimes M_1^t & ; & \sum_{t=0}^{m} \alpha_2^{k-2t-1}\gamma^{N+1} \otimes M_2^t \\ \sum_{t=0}^{m} \alpha_3^{k-2t-1}\gamma^{N+1} \otimes M_3^t & ; & \sum_{t=0}^{m} \alpha_4^{k-2t} \otimes M_4^t \end{pmatrix} , \quad \alpha_q^n \in L^n . \tag{54}$$

Thus, we see that λ is completely characterized by the sequence of elements α_1^{k-2t}, α_2^{k-2t-1}, α_3^{k-2t-1}, α_4^{k-2t}, where $t = 0, \ldots, m$. Denoting by ι the classical vector space isomorphism $\iota : L^k \equiv C^k/C^{k-2} \to \Lambda^k(X)$, where $\Lambda^k(X)$ is the set of complex–valued k–forms on X, and denoting the transport by the isomorphism ι of the exterior product $\wedge$ in $\Lambda^*(X) = \bigoplus_{k=0}^{N} \Lambda^k(X)$ by the same symbol, we get: If α_q^n, $\tilde{\alpha}_q^n \in L^n$ are the characterizing elements of $\lambda \in \Lambda^k_{\mathcal{A}}$, $\tilde{\lambda} \in \Lambda^l_{\mathcal{A}}$, then the characterizing elements β_q^n of $\lambda \bullet \tilde{\lambda} \in \Lambda^{k+l}_{\mathcal{A}}$ are

$$\begin{aligned} \beta_1^{k+l-2t} &= \textstyle\sum_{r=0}^{t} (\, \alpha_1^{k-2r} \wedge \tilde{\alpha}_1^{l-2(t-r)} + (-1)^{l-1} \alpha_2^{k-2r-1} \wedge \tilde{\alpha}_3^{l-2(t-r)+1} \,) , \\ \beta_2^{k+l-2t-1} &= \textstyle\sum_{r=0}^{t} (\, \alpha_1^{k-2r} \wedge \tilde{\alpha}_2^{l-2(t-r)-1} + (-1)^{l} \alpha_2^{k-2(t-r)-1} \wedge \tilde{\alpha}_4^{l-2r} \,) , \\ \beta_3^{k+l-2t-1} &= \textstyle\sum_{r=0}^{t} (\, \alpha_4^{k-2r} \wedge \tilde{\alpha}_3^{l-2(t-r)-1} + (-1)^{l} \alpha_3^{k-2(t-r)-1} \wedge \tilde{\alpha}_1^{l-2r} \,) , \\ \beta_4^{k+l-2t} &= \textstyle\sum_{r=0}^{t} (\, \alpha_4^{k-2r} \wedge \tilde{\alpha}_4^{l-2(t-r)} + (-1)^{l-1} \alpha_3^{k-2r-1} \wedge \tilde{\alpha}_2^{l-2(t-r)+1} \,) , \end{aligned} \tag{55}$$

where $t = 0, \ldots, m$.

We have an involution on $\Lambda^*_{\mathcal{A}}$ given by $\lambda^* := \sigma_k(\tau^*)$, with $\sigma_k(\tau) = \lambda$. Explicitly, for elements $\lambda \in \Lambda^k_{\mathcal{A}}$ represented as in (54) we find

$$\lambda^* = \begin{pmatrix} \sum_{t=0}^{m} (\alpha_1^{k-2t})^* \otimes M_1^t \, ; & \sum_{t=0}^{m} (-1)^{k-1} (\alpha_3^{k-2t-1})^* \gamma^{N+1} \otimes M_2^t \\ \sum_{t=0}^{m} (-1)^{k-1} (\alpha_2^{k-2t-1})^* \gamma^{N+1} \otimes M_3^t \, ; & \sum_{t=0}^{m} (\alpha_4^{k-2t})^* \otimes M_4^t \end{pmatrix} . \tag{56}$$

We define

$$\begin{aligned} \hat{\mu} &:= -i\gamma^{N+1} \otimes \mathcal{M} \in \Lambda^1_{\mathcal{A}} , & [\hat{\mu}, \lambda]_g &:= \hat{\mu} \bullet \lambda - (-1)^k \lambda \bullet \hat{\mu} , \\ \mathbf{d}\alpha &:= \iota^{-1} \circ \boldsymbol{d} \circ \iota(\alpha) , & \mathbf{d}^* &:= \gamma^{N+1} \mathbf{d} \gamma^{N+1} , \\ \mathbf{D}\lambda &:= pr_{k+1} \circ (\, (\mathbf{d} - \mathbf{d}^*) \otimes \mathrm{id}_{\tilde{F}} \,)(\lambda) , \end{aligned} \tag{57}$$

for $\alpha \in L^k$, $\lambda \in \Lambda^k_{\mathcal{A}}$, where $\boldsymbol{d}$ is the exterior differential on $\Lambda^*(X)$ and pr_{k+1} denotes the projection from $\Lambda^{k+1}_{\mathcal{A}} \oplus \Lambda^{k-1}_{\mathcal{A}}$ onto $\Lambda^{k+1}_{\mathcal{A}}$. One easily proves that $\mathbf{D}$ is a graded differential on $\Lambda^*_{\mathcal{A}}$. Moreover, one shows that

$$\hat{d} := \mathbf{D} + [\hat{\mu}\,,\,.\,]_g \tag{58}$$

is a graded differential on $\Lambda^*_{\mathcal{A}}$, too, which can be characterized as follows: If α^n_q are the characterizing elements of $\lambda \in \Lambda^k_{\mathcal{A}}$, then the characterizing elements β^n_q of $\hat{d}\lambda \in \Lambda^{k+1}_{\mathcal{A}}$ are:

$$\begin{aligned}
\beta_1^{k-2t+1} &= \mathbf{d}\alpha_1^{k-2t} + (-1)^k\, i\,(\,\alpha_2^{k-2t+1} + \alpha_3^{k-2t+1}\,)\;, \\
\beta_2^{k-2t} &= \mathbf{d}\alpha_2^{k-2t-1} + (-1)^k\, i\,(\,\alpha_1^{k-2t} - \alpha_4^{k-2t}\,)\;, \\
\beta_3^{k-2t} &= \mathbf{d}\alpha_3^{k-2t-1} + (-1)^k\, i\,(\,\alpha_4^{k-2t} - \alpha_1^{k-2t}\,)\;, \\
\beta_4^{k-2t+1} &= \mathbf{d}\alpha_4^{k-2t} + (-1)^k\, i\,(\,\alpha_3^{k-2t+1} + \alpha_2^{k-2t+1}\,)\;,
\end{aligned} \tag{59}$$

where $t = 0, \ldots, m$. Relation (1) is fulfilled for the differential algebra $(\Lambda^*_{\mathcal{A}}, \bullet, *, \hat{d})$.

In [21] we have shown that $\Lambda^*_{\mathcal{A}}$ coincides with the differential algebra Ω^*_D of Connes and Lott associated with the even K–cycle $(\mathcal{A}, h, D, \Gamma)$. The result (53) for Ω^*_D can also be obtained from a different procedure presented in [16].

3.2 A certain Lie Subalgebra of $\mathcal{H}$

For the case under consideration, the graded Lie algebra $\mathcal{H}$ can be treated as a subalgebra of $\Lambda^*_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C}$. Thus, it should be possible to define a generalized trace on $\mathcal{H}$ provided that we have a trace on $\Lambda^*_{\mathcal{A}}$. This is the case, indeed.

Proposition 8 *Any linear mapping* $T : \Lambda^*_{\mathcal{A}} \to L^*$ *, which vanishes on graded commutators and which intertwines the differentials, i.e.*

$$T(\lambda \bullet \tilde{\lambda} - (-1)^{kl}\tilde{\lambda} \bullet \lambda) \;=\; 0\;, \quad \lambda \in \Lambda^k_{\mathcal{A}}\,,\; \tilde{\lambda} \in \Lambda^l_{\mathcal{A}}\;, \tag{60}$$

$$T \circ \hat{d} \;=\; \mathbf{d} \circ T\;, \tag{61}$$

has in the representation (54) *the form*

$$T\left(\begin{pmatrix} \sum_{t=0}^{m} \alpha_1^{k-2t} \otimes M_1^t\;; & \sum_{t=0}^{m} \alpha_2^{k-2t-1}\gamma^{N+1} \otimes M_2^t \\ \sum_{t=0}^{m} \alpha_3^{k-2t-1}\gamma^{N+1} \otimes M_3^t\;; & \sum_{t=0}^{m} \alpha_4^{k-2t} \otimes M_4^t \end{pmatrix}\right) = \mathcal{L}_{-1}(\alpha_1^k) + \sum_{t=0}^{m} \mathcal{L}_t(\alpha_1^{k-2t} - \alpha_4^{k-2t})\,, \tag{62}$$

where $\mathcal{L}_t : L^* \to L^*$, $t = -1, 0, \ldots, m$, *are elements of* $\mathrm{End}_{\mathbb{C}}(L^*)$ *commuting with the exterior differential,*

$$\mathbf{d} \circ \mathcal{L}_t = \mathcal{L}_t \circ \mathbf{d}\;, \quad t = -1, 0, \ldots, m\;. \tag{63}$$

Proof: See appendix A. $\square$

Due to (60) we can regard the mapping T as a generalized trace. We restrict ourselves to the simplest case

$$\mathcal{L}_{-1} = 0 \ , \quad \mathcal{L}_t = \mathrm{id}_{L^*} \ , \quad \text{for} \quad t = 0, \dots m \, ,$$

and denote this special trace by T_Λ :

$$T_\Lambda \Big(\begin{pmatrix} \sum_{t=0}^{m} \alpha_1^{k-2t} \otimes M_1^t \ ; & \sum_{t=0}^{m} \alpha_2^{k-2t-1} \gamma^{N+1} \otimes M_2^t \\ \sum_{t=0}^{m} \alpha_3^{k-2t-1} \gamma^{N+1} \otimes M_3^t \ ; & \sum_{t=0}^{m} \alpha_4^{k-2t} \otimes M_4^t \end{pmatrix} \Big) := \sum_{t=0}^{m} (\alpha_1^{k-2t} - \alpha_4^{k-2t}) \ . \tag{64}$$

Now we extend the generalized trace T_Λ to the graded Lie algebra $\mathcal{H}$. Since $\mathcal{H} \subset \Lambda^*_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C}$, we get a generalized trace $T_{\mathcal{H}}$ on $\mathcal{H}$ as the tensor product of the generalized trace T_Λ on $\Lambda^*_{\mathcal{A}}$ and the usual trace on $\mathrm{M}_p\mathbb{C}$. For $\varrho \in \mathcal{H}$ represented by the matrix (30) we define this linear map $T_{\mathcal{H}} : \mathcal{H} \to L^*$ as

$$T_{\mathcal{H}}(\varrho) := \sum_{i=1}^{p} T_\Lambda(\varrho_{ii}) \ . \tag{65}$$

Lemma 9 *For all $\varrho \in \mathcal{H}^k$ and $\tilde{\varrho} \in \mathcal{H}^l$ we have $T_{\mathcal{H}}\left([\varrho, \tilde{\varrho}]_g \right) = 0$.*

Proof: Using formulae (32), (8), (65) and (60) we obtain

$$T_{\mathcal{H}}\left([\varrho, \tilde{\varrho}]_g \right) = \textstyle\sum_{i,j=1}^{p} T_\Lambda \left(\varrho_{ij} \bullet \tilde{\varrho}_{ji} - (-1)^{kl} \tilde{\varrho}_{ji} \bullet \varrho_{ij} \right) = 0 \ . \qquad \square$$

Putting $l = 0$, $\varrho \mapsto u\varrho$, $\tilde{\varrho} = u^*$, for $u \in \mathcal{U}(\mathcal{E})$, in Lemma 9, we get

$$T_{\mathcal{H}}\left(u \varrho u^* \right) = T_{\mathcal{H}}\left(\varrho \right) \ . \tag{66}$$

Thus, $T_{\mathcal{H}}\left(\, . \, \right)$ is invariant under unitary automorphisms of the module. We define

$$\mathcal{H}_0 := \bigoplus_{k=0}^{\infty} \mathcal{H}_0^k \ , \quad \mathcal{H}_0^k := \{ \ \varrho \in \mathcal{H}^k \ : \ T_{\mathcal{H}}\left(\varrho \right) = 0 \ \} \ . \tag{67}$$

Due to Lemma 9, $\mathcal{H}_0$ is a graded Lie subalgebra of $\mathcal{H}$.

We denote by ∇_0 and $\tilde{\nabla}_0$ the canonical compatible connections on $\mathcal{E}$, which are defined according to (14) using the differential $\hat{d}$ respectively $\mathbf{D}$ on $\Lambda^*_{\mathcal{A}}$:

$$\begin{aligned} \nabla_0(c \otimes a) &:= e(c \otimes \mathbb{1}_{\mathcal{A}}) \otimes_{\mathcal{A}} \hat{d}a \ , \\ \tilde{\nabla}_0(c \otimes a) &:= e(c \otimes \mathbb{1}_{\mathcal{A}}) \otimes_{\mathcal{A}} \mathbf{D}a \ , \end{aligned} \tag{68}$$

where $c \in \mathbb{C}^p$ and $a \in \mathcal{A}$. Moreover, we denote by $\mathcal{D}$ and $\tilde{\mathcal{D}}$ the derivations on $\mathcal{H}$ associated to ∇_0 and $\tilde{\nabla}_0$ respectively, see (16):

$$\begin{aligned} (\mathcal{D}\varrho)(\xi) &= \nabla_0(\varrho\xi) - (-1)^k \varrho \bullet (\nabla_0 \, \xi) \ , \\ (\tilde{\mathcal{D}}\varrho)(\xi) &= \tilde{\nabla}_0(\varrho\xi) - (-1)^k \varrho \bullet (\tilde{\nabla}_0 \, \xi) \ , \end{aligned} \tag{69}$$

for $\varrho \in \mathcal{H}^k$. We introduce a special element $\mu \in \mathcal{H}^1$ by

$$\mu(c \otimes a) := e(c \otimes \mathbb{1}_{\mathcal{A}}) \otimes_{\mathcal{A}} \hat{\mu} a \;, \tag{70}$$

where $\hat{\mu}$ was defined in (57). This gives the following matrix form

$$\mu = e(\mathbb{1}_{p \times p} \otimes \hat{\mu}) e \;, \quad \text{or} \quad \mu_{ij} = \sum_{k,l=1}^{p} e_{ik} \delta_{kl} \hat{\mu} e_{lj} \;. \tag{71}$$

Lemma 10 *For the graded Lie algebra $\mathcal{H}$ associated to the differential algebra $\Lambda^*_{\mathcal{A}}$ we have*

$$\mathcal{D}\varrho = \tilde{\mathcal{D}}\varrho + [\mu, \varrho]_g \;, \quad \varrho \in \mathcal{H} \;. \tag{72}$$

Proof: Let $\sum_{i=1}^p \varepsilon_i \otimes a_i \in \mathcal{E}$, with $a_i = \sum_{j=1}^p e_{ij} a_j \in \mathcal{A}$, and $\varrho \in \mathcal{H}^k$ defined by

$\varrho(\sum_{i=1}^p \varepsilon_i \otimes a_i) = \sum_{i,j,n=1}^p (\varepsilon_j \otimes e_{jn}) \otimes_{\mathcal{A}} \varrho_{ni} a_i$,

where $\varrho_{ni} = \sum_{j,m=1}^p e_{nj} \varrho_{jm} e_{mi} \in \Lambda^k_{\mathcal{A}}$. Using (16), (12), (14), (58) and (70) we find

$$\begin{aligned}
(\mathcal{D}\varrho)(\textstyle\sum_{i=1}^p \varepsilon_i \otimes a_i) &= \nabla_0(\varrho(\textstyle\sum_{i=1}^p \varepsilon_i \otimes a_i)) - (-1)^k \varrho \bullet (\nabla_0(\textstyle\sum_{i=1}^p \varepsilon_i \otimes a_i)) \\
&= \textstyle\sum_{i,j,n=1}^p \{(\nabla_0(\varepsilon_j \otimes e_{jn})) \bullet \varrho_{ni} a_i + (\varepsilon_j \otimes e_{jn}) \otimes_{\mathcal{A}} \hat{d}(\varrho_{ni} a_i) \\
&\qquad - (-1)^k (\varepsilon_j \otimes e_{jn}) \otimes_{\mathcal{A}} \varrho_{ni} \bullet \hat{d}(a_i)\} \\
&= \textstyle\sum_{i,j,n=1}^p (\varepsilon_j \otimes e_{jn}) \otimes_{\mathcal{A}} \hat{d}(\varrho_{ni}) a_i \\
&= (\tilde{\mathcal{D}}\varrho)(\textstyle\sum_{i=1}^p \varepsilon_i \otimes a_i) + \textstyle\sum_{i,j,n=1}^p (\varepsilon_j \otimes e_{jn}) \otimes_{\mathcal{A}} (\hat{\mu} \bullet \varrho_{ni} - (-1)^k \varrho_{ni} \bullet \hat{\mu}) a_i \\
&= (\tilde{\mathcal{D}}\varrho)(\textstyle\sum_{i=1}^p \varepsilon_i \otimes a_i) + ([\mu, \varrho]_g)(\textstyle\sum_{i=1}^p \varepsilon_i \otimes a_i) \;. \qquad \square
\end{aligned}$$

Lemma 11 *$\mathcal{D}$ is a graded derivation of $\mathcal{H}_0$.*

Proof: For any $\varrho \in \mathcal{H}_0^k$ we have with (72), Lemma 9, (65) and (35)

$$T_{\mathcal{H}}(\mathcal{D}\varrho) = T_{\mathcal{H}}(\tilde{\mathcal{D}}\varrho + [\mu, \varrho]_g) = T_{\mathcal{H}}(\tilde{\mathcal{D}}\varrho) = \textstyle\sum_{i,j,n=1}^p T_\Lambda(e_{ij} \mathbf{D}(\varrho_{jn}) e_{ni}) \;.$$

We compute the last term using the Leibniz rule for $\mathbf{D}$, the property that e is a projector and, finally, equation (60):

$$\begin{aligned}
&\textstyle\sum_{i,j,n=1}^p T_\Lambda(e_{ij} \mathbf{D}(\varrho_{jn}) e_{ni}) \\
&\quad = \textstyle\sum_{i,n=1}^p T_\Lambda(\mathbf{D}(\varrho_{in}) e_{ni}) - \textstyle\sum_{i,j=1}^p T_\Lambda(\mathbf{D}(e_{ij}) \varrho_{ji}) \\
&\quad = \textstyle\sum_{i,j,n=1}^p T_\Lambda(\mathbf{D}(\varrho_{in}) e_{nj} e_{ji}) - \textstyle\sum_{i,j=1}^p T_\Lambda(\mathbf{D}(e_{ij}) \varrho_{ji}) \\
&\quad = \textstyle\sum_{i,j,n=1}^p T_\Lambda(e_{ji} \mathbf{D}(\varrho_{in}) e_{nj}) - \textstyle\sum_{i,j=1}^p T_\Lambda(\mathbf{D}(e_{ij}) \varrho_{ji})
\end{aligned}$$

This implies $\sum_{i,j=1}^p T_\Lambda(\mathbf{D}(e_{ij}) \varrho_{ji}) = 0$ and $T_{\mathcal{H}}(\mathcal{D}\varrho) = \sum_{i=1}^p T_\Lambda(\mathbf{D}(\varrho_{ii}))$. Finally, formula (61) gives $T_{\mathcal{H}}(\mathcal{D}\varrho) = \mathbf{d}\{\sum_{i=1}^p T_\Lambda(\varrho_{ii})\} = 0$. $\square$

3.3 Changing the Standard Matrix Representation

In this subsection we analyse the matrix structures discussed in subsection 2.3 for the case of the differential algebra presented in subsections 3.1 and 3.2. For this purpose we use the fact that $\Lambda^*_{\mathcal{A}}$ can be treated as a subspace of $L^* \otimes \mathrm{End}\,(F) \otimes \mathrm{M}_2\mathbb{C}$. Of course, elements of the tensor product $\Lambda^*_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C}$, which in subsection 3.2 were treated as $p \times p$–matrices with $L^* \otimes \mathrm{End}\,(F) \otimes \mathrm{M}_2\mathbb{C}$–valued entries, can be treated as 2×2–matrices with $L^* \otimes \mathrm{End}\,(F) \otimes \mathrm{M}_p\mathbb{C}$–valued entries. This natural mapping can be realized as an inner automorphism of $L^* \otimes \mathrm{End}\,(F) \otimes \mathrm{M}_{2p}\mathbb{C}$. It turns out that after applying this automorphism combined with another natural mapping, see subsection 4.2, we find that the image of $\mathcal{H}_0$ coincides with a graded Lie subalgebra of the special graded linear Lie algebra $\Lambda^*(X) \otimes spl(p,p)$. This is the appropriate formulation for deriving the mathematical structure of the Mainz–Marseille approach, as will be shown in subsection 4.3.

Let $W = (W_{ij})_{i,j=1,\ldots,p} \in \mathrm{M}_p\mathbb{C}$ and $w = (w_{AB})_{A,B=1,2} \in \mathrm{M}_2\mathbb{C}$. We denote

$$w \otimes W = \begin{pmatrix} wW_{11} & wW_{12} & \ldots & wW_{1p} \\ wW_{21} & wW_{22} & \ldots & wW_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ wW_{p1} & wW_{p2} & \ldots & wW_{pp} \end{pmatrix} \tag{73}$$

and define

$$\mathfrak{i}_1(w \otimes W) = \begin{pmatrix} Ww_{11} & Ww_{12} \\ Ww_{21} & Ww_{22} \end{pmatrix} . \tag{74}$$

We extend this mapping naturally to the algebra $L^* \otimes \mathrm{End}(F) \otimes \mathrm{M}_2\mathbb{C} \otimes \mathrm{M}_p\mathbb{C}$ and denote it by the same letter, the restriction to the subspace $\Lambda^*_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C}$ will also be denoted by $\mathfrak{i}_1$. It is easy to convince oneself that the mapping (74) can be also realized as an inner automorphism of the algebra $\mathrm{M}_{2p}\mathbb{C}$. This goes as follows:

$$\mathfrak{i}_1(\mathbf{W}) := \mathbf{J}\mathbf{W}\mathbf{J}^{-1} \ , \quad \mathbf{W}, \mathbf{J} \in \mathrm{M}_{2p}\mathbb{C} \ , \quad \mathbf{J}_{ij} = \delta_{j,2i-1} + \delta_{j+2p,2i} \ , \tag{75}$$

for $i, j = 1, \ldots, 2p$. Moreover, it is easy to show that this operation consists in applying the permutation $(1,2,3,4,\ldots,2p-1,2p) \mapsto (1,3,\ldots,2p-1,2,4,\ldots 2p)$ to both rows and columns.

Note that due to (54) after applying the operation (75) to elements of $\Lambda^*_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C}$ the grading operator γ^{N+1} occurs exactly in every component of the two off–diagonal blocks. The next step consists in removing γ^{N+1} from these blocks and applying the classical isomorphism $\iota : L^k \to \Lambda^k(X)$. For this purpose we define the following vector space isomorphism $\mathfrak{i}_2$ from $\mathfrak{i}_1(\Lambda^*_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C})$ onto its image:

$$\mathfrak{i}_2 \begin{pmatrix} \sum_{t=0}^m \boldsymbol{\alpha}_1^{k-2t} \otimes M_1^t & ; & \sum_{t=0}^m \boldsymbol{\alpha}_2^{k-2t-1}\gamma^{N+1} \otimes M_2^t \\ \sum_{t=0}^m \boldsymbol{\alpha}_3^{k-2t-1}\gamma^{N+1} \otimes M_3^t & ; & \sum_{t=0}^m \boldsymbol{\alpha}_4^{k-2t} \otimes M_4^t \end{pmatrix} \tag{76}$$

$$:= \left(\begin{array}{ccc} \sum_{t=0}^{m} \mathbf{a}_1^{k-2t} \otimes M_1^t & ; & \sum_{t=0}^{m} \mathbf{a}_2^{k-2t-1} \otimes M_2^t \\ \sum_{t=0}^{m} \mathbf{a}_3^{k-2t-1} \otimes M_3^t & ; & \sum_{t=0}^{m} \mathbf{a}_4^{k-2t} \otimes M_4^t \end{array} \right) ,$$

where $\boldsymbol{\alpha}_q^n \in L^n \otimes \mathrm{M}_p\mathbb{C}$ and $\mathbf{a}_q^n := \iota(\boldsymbol{\alpha}_q^n) \in \Lambda^n(X) \otimes \mathrm{M}_p\mathbb{C}$. The composition of these two mappings gives the embedding $\mathfrak{i} : \Lambda^*_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C} \to \Lambda^*(X) \otimes \mathrm{End}\,(F) \otimes \mathrm{M}_{2p}\mathbb{C}$,

$$\mathfrak{i} = \mathfrak{i}_2 \circ \mathfrak{i}_1 \ , \tag{77}$$

often we will treat $\mathfrak{i}$ as an isomorphism onto its image.

Now it is easy to characterize elements

$$\left(\begin{array}{ccc} \sum\limits_{t=0}^{m} \mathbf{a}_1^{k-2t} \otimes M_1^t & ; & \sum\limits_{t=0}^{m} \mathbf{a}_2^{k-2t-1} \otimes M_2^t \\ & & \\ \sum\limits_{t=0}^{m} \mathbf{a}_3^{k-2t-1} \otimes M_3^t & ; & \sum\limits_{t=0}^{m} \mathbf{a}_4^{k-2t} \otimes M_4^t \end{array} \right) , \quad \mathbf{a}_q^n \in \Lambda^n(X) \otimes \mathrm{M}_p\mathbb{C} \ , \tag{78}$$

of $\mathcal{H}$, $\mathcal{E}^*$ and $\Lambda^*_{\mathcal{A}}$, see (30), (36) and (37), transported by $\mathfrak{i}$. First, observe that $e \in \mathcal{H}^0$ and, therefore, we have

$$\mathfrak{i}(e) \equiv \mathbf{e} = \left(\begin{array}{cc} \mathbf{e}_1 \otimes \mathrm{id}_F & 0 \\ 0 & \mathbf{e}_4 \otimes \mathrm{id}_F \end{array} \right) , \quad \mathbf{e}_q = \mathbf{e}_q^2 = \mathbf{e}_q^* \in \Lambda^0(X) \otimes \mathrm{M}_p\mathbb{C} \ , \quad q = 1, 4 \ . \tag{79}$$

Since for elements $\varrho \in \mathcal{H}$ we have $e\varrho e = \varrho$, we get for elements $\mathfrak{i}(\varrho) \in \mathfrak{i}(\mathcal{H})$, given in the representation (78),

$$\mathbf{a}_1^n = \mathbf{e}_1 \mathbf{a}_1^n \mathbf{e}_1 \ , \quad \mathbf{a}_2^n = \mathbf{e}_1 \mathbf{a}_2^n \mathbf{e}_4 \ , \quad \mathbf{a}_3^n = \mathbf{e}_4 \mathbf{a}_3^n \mathbf{e}_1 \ , \quad \mathbf{a}_4^n = \mathbf{e}_4 \mathbf{a}_4^n \mathbf{e}_4 \ . \tag{80}$$

Defining $\mathfrak{i}_{\mathcal{E}} : \mathcal{E}^k \to \mathfrak{i}(\Lambda^k_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C})$ and $\mathfrak{i}_\Lambda : \Lambda^k_{\mathcal{A}} \to \mathfrak{i}(\Lambda^k_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C})$ by putting

$$\mathfrak{i}_{\mathcal{E}} := \mathfrak{i} \circ \mathfrak{j}_{\mathcal{E}} \ , \quad \mathfrak{i}_\Lambda := \mathfrak{i} \circ \mathfrak{j}_\Lambda \ , \tag{81}$$

we can represent elements of $\mathcal{E}^*$ and $\Lambda^*_{\mathcal{A}}$ as elements of $\Lambda^*(X) \otimes \mathrm{End}\,(F) \otimes \mathrm{M}_{2p}\mathbb{C}$. For elements of $\mathfrak{i}_{\mathcal{E}}(\mathcal{E}^k)$ we get from the representation (78):

$$\mathbf{a}_q^n \in \Lambda^n(X) \otimes E_{(i)} \ , \quad (i) = 1 \text{ for } q = 1, 2 \, , \ (i) = 2 \text{ for } q = 3, 4 \, , \qquad \text{where} \tag{82}$$

$$E_{(i)} = \mathbf{e}_{(i)} \Big(\mathbb{C} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \oplus \dots \oplus \mathbb{C} \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \Big).$$

Analogously, for elements of $\mathfrak{i}_\Lambda(\Lambda^k_{\mathcal{A}})$ we have

$$\mathbf{a}_q^n \in \Lambda^n(X) \otimes \mathbb{1}_{p \times p} \ , \quad q = 1, \dots, 4 \ . \tag{83}$$

Let us denote the spaces transported via $\mathfrak{i}$ by bold symbols:

$$\begin{aligned}&\boldsymbol{\mathcal{H}} := \mathfrak{i}(\mathcal{H})\ ,\quad \boldsymbol{\mathcal{H}}^k := \mathfrak{i}(\mathcal{H}^k)\ ,\quad \boldsymbol{\mathcal{H}}_0 := \mathfrak{i}(\mathcal{H}_0)\ ,\quad \boldsymbol{\mathcal{H}}_0^k := \mathfrak{i}(\mathcal{H}_0^k)\ ,\quad \boldsymbol{\mathcal{U}}(\boldsymbol{\mathcal{E}}) := \mathfrak{i}(\mathcal{U}(\mathcal{E}))\ ,\\ &\boldsymbol{\mathcal{E}} := \mathfrak{i}_{\mathcal{E}}(\mathcal{E})\ ,\quad \boldsymbol{\mathcal{E}}^k := \mathfrak{i}_{\mathcal{E}}(\mathcal{E}^k)\ ,\quad \boldsymbol{\Lambda}^k_{\boldsymbol{\mathcal{A}}} := \mathfrak{i}_\Lambda(\Lambda^k_{\mathcal{A}})\ ,\quad \boldsymbol{\mathcal{A}} := \mathfrak{i}_\Lambda(\mathcal{A})\ .\end{aligned} \tag{84}$$

We define the multiplication in $\mathfrak{i}(\Lambda^*_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C})$ as the transport of the multiplication $\bullet$ in $\Lambda^*_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C}$ and denote it by the same symbol $\bullet$:

$$\mathfrak{i}(\chi) \bullet \mathfrak{i}(\tilde\chi) := \mathfrak{i}(\chi \bullet \tilde\chi)\ , \tag{85}$$

for $\chi \in \Lambda^k_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C}$, $\tilde\chi \in \Lambda^l_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C}$. Denoting $\mathfrak{i}(\chi) = \boldsymbol{\chi}$ and $\mathfrak{i}(\tilde\chi) = \tilde{\boldsymbol{\chi}}$, which we represent as in (78), and using (55) we get:

$$\boldsymbol{\chi}\bullet\tilde{\boldsymbol{\chi}} = \left(\begin{array}{c|c} \begin{array}{c}\displaystyle\sum_{t=0}^m\sum_{s=0}^t(\,\mathbf{a}_1^{k-2s}\wedge\tilde{\mathbf{a}}_1^{l-2(t-s)}+ \\ (-1)^{l-1}\,\mathbf{a}_2^{k-2s-1}\wedge\tilde{\mathbf{a}}_3^{l-2(t-s)+1})\otimes M_1^t\end{array} & \begin{array}{c}\displaystyle\sum_{t=0}^m\sum_{s=0}^t(\,\mathbf{a}_1^{k-2s}\wedge\tilde{\mathbf{a}}_2^{l-2(t-s)-1}+ \\ (-1)^{l}\,\mathbf{a}_2^{k-2(t-s)-1}\wedge\tilde{\mathbf{a}}_4^{l-2s})\otimes M_2^t\end{array} \\ \hline \begin{array}{c}\displaystyle\sum_{t=0}^m\sum_{s=0}^t(\mathbf{a}_4^{k-2s}\wedge\tilde{\mathbf{a}}_3^{l-2(t-s)-1}+ \\ (-1)^{l}\,\mathbf{a}_3^{k-2(t-s)-1}\wedge\tilde{\mathbf{a}}_1^{l-2s})\otimes M_3^t\end{array} & \begin{array}{c}\displaystyle\sum_{t=0}^m\sum_{s=0}^t(\mathbf{a}_4^{k-2s}\wedge\tilde{\mathbf{a}}_4^{l-2(t-s)}+ \\ (-1)^{l-1}\,\mathbf{a}_3^{k-2s-1}\wedge\tilde{\mathbf{a}}_2^{l-2(t-s)+1})\otimes M_4^t\end{array}\end{array}\right). \tag{86}$$

In particular, we have

$$\mathfrak{i}_{\mathcal{E}}(\varrho \bullet \xi) = \mathfrak{i}(\varrho) \bullet \mathfrak{i}_{\mathcal{E}}(\xi)\ ,\quad \varrho \in \mathcal{H}\ ,\quad \xi \in \mathcal{E}^*\ . \tag{87}$$

Next we transport the remaining structures via $\mathfrak{i}$:

$$\begin{aligned}[\mathfrak{i}(\varrho), \mathfrak{i}(\tilde\varrho)]_g &:= \mathfrak{i}([\varrho, \tilde\varrho]_g)\ , &(88)\\ (\mathfrak{i}(\varrho))^* &:= \mathfrak{i}(\varrho^*)\ , &(89)\\ \boldsymbol{\nabla}(\mathfrak{i}_{\mathcal{E}}(\xi)) &:= \mathfrak{i}_{\mathcal{E}}(\nabla\xi)\ , &(90)\\ \boldsymbol{\mathcal{D}}\mathfrak{i}(\varrho) &:= \mathfrak{i}(\mathcal{D}\varrho)\ , &(91)\end{aligned}$$

where $\varrho, \tilde\varrho \in \mathcal{H}$ and $\xi \in \mathcal{E}^*$. Using (72) we find for $\boldsymbol{\mathcal{D}}\boldsymbol{\varrho}$, $\boldsymbol{\varrho} \in \boldsymbol{\mathcal{H}}$,

$$\begin{aligned}\boldsymbol{\mathcal{D}}\boldsymbol{\varrho} &= \mathbf{e}\,\boldsymbol{d}(\boldsymbol{\varrho})\mathbf{e} + [\boldsymbol{\mu}, \boldsymbol{\varrho}]_g\ ,\quad \boldsymbol{\mu} := \mathfrak{i}(\mu) \equiv \mathbf{eme}\ , &(92)\\ \mathbf{m} &:= \mathfrak{i}(\mathbb{1}_{p\times p} \otimes \hat\mu) = \begin{pmatrix} 0 & -i\,\mathbb{1}_{p\times p} \otimes M_2^0 \\ -i\,\mathbb{1}_{p\times p} \otimes M_3^0 & 0\end{pmatrix}\ ,\end{aligned}$$

where $\boldsymbol{d}$ is the classical exterior differential acting componentwise on $\boldsymbol{\varrho}$. For the involution (89) we get in the representation (78)

$$\begin{pmatrix} \displaystyle\sum_{t=0}^m \mathbf{a}_1^{k-2t} \otimes M_1^t & ; & \displaystyle\sum_{t=0}^m \mathbf{a}_2^{k-2t-1} \otimes M_2^t \\ \displaystyle\sum_{t=0}^m \mathbf{a}_3^{k-2t-1} \otimes M_3^t & ; & \displaystyle\sum_{t=0}^m \mathbf{a}_4^{k-2t} \otimes M_4^t \end{pmatrix}^* \tag{93}$$

$$= \begin{pmatrix} \sum_{t=0}^{m} (\mathbf{a}_1^{k-2t})^* \otimes M_1^t & ; & \sum_{t=0}^{m} (-1)^{k-1} (\mathbf{a}_3^{k-2t-1})^* \otimes M_2^t \\ \sum_{t=0}^{m} (-1)^{k-1} (\mathbf{a}_2^{k-2t-1})^* \otimes M_3^t & ; & \sum_{t=0}^{m} (\mathbf{a}_4^{k-2t})^* \otimes M_4^t \end{pmatrix} .$$

Next, we observe that we can also transport the generalized trace defined in (65):

$$\boldsymbol{T_{\mathcal{H}}}\,(\mathfrak{i}(\varrho)) := \iota(T_{\mathcal{H}}\,(\varrho)) \ , \quad \varrho \in \mathcal{H} \ . \tag{94}$$

For elements $\boldsymbol{\varrho} \in \boldsymbol{\mathcal{H}}^k$ represented as in (78) we get

$$\boldsymbol{T_{\mathcal{H}}}\,(\boldsymbol{\varrho}) = \sum_{t=0}^{m} (\mathrm{tr}\,(\mathbf{a}_1^{k-2t}) - \mathrm{tr}\,(\mathbf{a}_4^{k-2t})) \ . \tag{95}$$

Thus, elements $\boldsymbol{\varrho} \in \boldsymbol{\mathcal{H}}_0$ are characterized by

$$\mathrm{tr}\,(\mathbf{a}_1^n) = \mathrm{tr}\,(\mathbf{a}_4^n) \ , \quad \text{for all } n \ . \tag{96}$$

With the general form $\nabla = \nabla_0 + \rho$ of a connection on $\mathcal{E}$ one finds

$$\begin{aligned} \boldsymbol{\nabla} &= \boldsymbol{\nabla}_0 + \boldsymbol{\rho} \ , \quad \boldsymbol{\rho} := \mathfrak{i}(\rho) \ , \quad \text{where} \\ \boldsymbol{\nabla}_0\, \boldsymbol{\xi} &= \mathfrak{i}_{\mathcal{E}} \circ \nabla_0 \circ \mathfrak{i}_{\mathcal{E}}^{-1}(\boldsymbol{\xi}) = \mathbf{e}(\boldsymbol{d}\boldsymbol{\xi} + [\mathbf{m}, \boldsymbol{\xi}]_g) \ , \quad \boldsymbol{\xi} \in \boldsymbol{\mathcal{E}}^* \ . \end{aligned} \tag{97}$$

Next, using (34), one easily calculates

$$\boldsymbol{\theta}_0 := \mathfrak{i}(\theta_0) = \mathbf{e}(\boldsymbol{d}(\mathbf{e}) + [\mathbf{m}, \mathbf{e}])(\boldsymbol{d}(\mathbf{e}) + [\mathbf{m}, \mathbf{e}])\mathbf{e} \ . \tag{98}$$

Finally, we study the influence of unitary transformations of the module

$$\begin{aligned} &\boldsymbol{\mathcal{E}} \ni \boldsymbol{\xi} \mapsto \boldsymbol{\xi}' := \mathbf{v}\boldsymbol{\xi} \ , \quad \boldsymbol{\mathcal{H}} \ni \boldsymbol{\varrho} \mapsto \boldsymbol{\varrho}' := \mathbf{v}\boldsymbol{\varrho}\mathbf{v}^* \ , \quad \boldsymbol{\Lambda}^*_{\boldsymbol{\mathcal{A}}} \ni \boldsymbol{\lambda} \mapsto \boldsymbol{\lambda}' := \boldsymbol{\lambda} \ , \\ &\mathbf{v} \in \boldsymbol{\mathcal{U}}(\boldsymbol{\mathcal{A}}^p) := \{ \ \tilde{\mathbf{v}} \in \mathrm{End}_{\boldsymbol{\mathcal{A}}}\,(\boldsymbol{\mathcal{A}}^p) \ , \quad \tilde{\mathbf{v}}^*\tilde{\mathbf{v}} = \tilde{\mathbf{v}}\tilde{\mathbf{v}}^* = \mathrm{id}_{\boldsymbol{\mathcal{A}}^p} \ \} \ . \end{aligned} \tag{99}$$

It is easy to show that all formulae in this subsection remain form invariant if we put

$$\mathbf{e}' := \mathbf{v}\mathbf{e}\mathbf{v}^* \ , \ \ \boldsymbol{d}' := \boldsymbol{d} \ , \ \ \mathbf{m}' := \mathbf{v}\mathbf{m}\mathbf{v}^* + \mathbf{v}\boldsymbol{d}(\mathbf{v}^*) \ , \ \ \boldsymbol{\mu}' := \mathbf{v}\boldsymbol{\mu}\mathbf{v}^* + \mathbf{e}'\mathbf{v}\boldsymbol{d}(\mathbf{v}^*)\mathbf{e}' \ . \tag{100}$$

Observe that after such a unitary module transformation the matrices $\boldsymbol{\mu}$ and $\mathbf{m}$ gain – in general – entries in the two diagonal blocks, and the two off–diagonal blocks have no longer the simple form (92).

4 Derivation of the Mathematical Structures Used in the Mainz–Marseille Approach

4.1 The Graded Lie Algebra Used in the Mainz–Marseille Approach

The basic concept used in the Mainz–Marseille approach is that of a graded Lie matrix algebra with values in differential forms. For the sake of completeness, we shortly recall the most important notions in a slightly generalized form.

Defining the grading operator $\Gamma_0 := \begin{pmatrix} \mathbb{1}_{p\times p} & 0 \\ 0 & -\mathbb{1}_{p\times p} \end{pmatrix} \in \mathrm{M}_{2p}\mathbb{C}$, we introduce a $\mathbb{Z}_2$–grading structure in $\mathrm{M}_{2p}\mathbb{C}$ and denote for $\mathbf{M} \in \mathrm{M}_{2p}\mathbb{C}$

$$\mathbf{M}_0 := \frac{1}{2}(\mathbf{M} + \Gamma_0 \mathbf{M} \Gamma_0) \ , \quad \mathbf{M}_1 := \frac{1}{2}(\mathbf{M} - \Gamma_0 \mathbf{M} \Gamma_0) \ . \tag{101}$$

We denote the degree of a matrix $\mathbf{M}$ by $\partial\mathbf{M}$ and define $\partial\mathbf{M}_0 = 0$ and $\partial\mathbf{M}_1 = 1$. Defining the graded commutator

$$[\mathbf{M}, \mathbf{N}]_g := \sum_{i,j=0}^{1} (\, \mathbf{M}_i \, \mathbf{N}_j - (-1)^{\partial\mathbf{M}_i \, \partial\mathbf{N}_j} \, \mathbf{N}_j \, \mathbf{M}_i \,) \ , \quad \mathbf{M}, \mathbf{N} \in \mathrm{M}_{2p}\mathbb{C} \ , \tag{102}$$

we get the structure of a graded Lie algebra on $\mathrm{M}_{2p}\mathbb{C}$, called $pl(p,p)$. There is a non–simple graded Lie subalgebra $spl(p,p) \subset \mathrm{M}_{2p}\mathbb{C}$ of graded–tracefree matrices [24] defined by

$$spl(p,p) := \{ \ \mathbf{M} \in \mathrm{M}_{2p}\mathbb{C} \ : \ \ \mathrm{tr}\,(\Gamma_0 \mathbf{M}) = 0 \ \} \ . \tag{103}$$

In $spl(p,p)$ there exists a differential d_M given by

$$d_M \mathbf{M} := [\mathfrak{m}, \mathbf{M}]_g \ , \quad \mathfrak{m} = z \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix} \in spl(p,p) \ , \tag{104}$$

where u is an arbitrary element of $U(p)$ and $z \in \mathbb{C}$. We choose, however, from the very beginning $u = \mathbb{1}_{p\times p}$ and $z = -i$. The reason for this choice will become clear below.

Now one defines the $\mathbb{Z}_2$–graded algebra $\Lambda^*(X) \otimes \mathrm{M}_{2p}\mathbb{C}$ as the $\mathbb{Z}_2$–graded tensor product of the $\mathbb{Z}_2$–graded algebras $\Lambda^*(X)$ and $\mathrm{M}_{2p}\mathbb{C}$. This means: The total degree of $\mathfrak{b} = \beta \otimes \mathbf{M} \in \Lambda^*(X) \otimes \mathrm{M}_{2p}\mathbb{C}$ is $\partial\mathfrak{b} = (\partial\beta + \partial\mathbf{M}) \pmod 2$, where $\partial\beta$ is the ordinary differential form degree modulo 2. Defining the product $\odot$ in $\Lambda^*(X) \otimes \mathrm{M}_{2p}\mathbb{C}$ by

$$(\beta \otimes \mathbf{M}) \odot (\nu \otimes \mathbf{N}) := (-1)^{\partial\nu \, \partial\mathbf{M}} \, (\beta \wedge \nu) \otimes (\mathbf{M}\,\mathbf{N}) \ , \tag{105}$$

we get the natural graded Lie algebra structure on $\Lambda^*(X) \otimes \mathrm{M}_{2p}\mathbb{C}$:

$$[\mathfrak{b}_1, \mathfrak{b}_2]_g := \mathfrak{b}_1 \odot \mathfrak{b}_2 - (-1)^{\partial\mathfrak{b}_1 \, \partial\mathfrak{b}_2} \, \mathfrak{b}_2 \odot \mathfrak{b}_1 \ . \tag{106}$$

Moreover, $\Lambda^*(X) \otimes \mathrm{M}_{2p}\mathbb{C}$ is a graded involutive differential algebra with differential and involution given by

$$\mathfrak{d}(\beta \otimes \mathbf{M}) \ := \ (\boldsymbol{d}\beta) \otimes \mathbf{M} + (-1)^{\partial\beta}\, \beta \otimes (d_M \mathbf{M}) \ , \tag{107}$$

$$(\beta \otimes \mathbf{M})^* \ := \ (-1)^{\partial\beta\,\partial\mathbf{M}}\, \beta^* \otimes \mathbf{M}^* \ , \tag{108}$$

where $\boldsymbol{d}$ is the exterior differential on $\Lambda^*(X)$ and $(\beta \wedge \nu)^* = \nu^* \wedge \beta^*$. One easily calculates

$$\mathfrak{d}\mathfrak{b} = \boldsymbol{d}\mathfrak{b} + [\mathfrak{m}, \mathfrak{b}]_g \ , \tag{109}$$

where we identified $1 \otimes \mathfrak{m} \equiv \mathfrak{m}$. One finds $([\mathfrak{b}_1, \mathfrak{b}_2]_g)^* = -(-1)^{\partial\mathfrak{b}_1\,\partial\mathfrak{b}_2}\, [\mathfrak{b}_1^*, \mathfrak{b}_2^*]_g$ and $(\mathfrak{d}\mathfrak{b})^* = (-1)^{\partial\mathfrak{b}}\mathfrak{d}\mathfrak{b}^*$ for $\mathfrak{m} = -\mathfrak{m}^*$. In terms of 2×2–block matrices one has

$$\begin{pmatrix} \mathbf{a}_1^{k_1} & \mathbf{a}_2^{k_2} \\ \mathbf{a}_3^{k_3} & \mathbf{a}_4^{k_4} \end{pmatrix}^* = \begin{pmatrix} (\mathbf{a}_1^{k_1})^* & (-1)^{k_3}\,(\mathbf{a}_3^{k_3})^* \\ (-1)^{k_2}\,(\mathbf{a}_2^{k_2})^* & (\mathbf{a}_4^{k_4})^* \end{pmatrix} \ , \tag{110}$$

where $\mathbf{a}_q^n \in \Lambda^n(X) \otimes \mathrm{M}_p\mathbb{C}$. One easily shows that $\Lambda^*(X) \otimes spl(p,p)$ is a graded Lie subalgebra of $\Lambda^*(X) \otimes \mathrm{M}_{2p}\mathbb{C}$. Moreover, the graded differential $\mathfrak{d}$ defined in (107) respects the Lie subalgebra $\Lambda^*(X) \otimes spl(p,p)$.

Using the projection operator

$$\mathfrak{e} = \begin{pmatrix} \mathbf{e}_1 & 0 \\ 0 & \mathbf{e}_4 \end{pmatrix} \ , \tag{111}$$

with $\mathbf{e}_1$ and $\mathbf{e}_4$ fulfilling (79), we define a graded Lie subalgebra of $\Lambda^*(X) \otimes spl(p,p)$:

$$\mathfrak{H}_\mathfrak{e} := \{ \ \mathfrak{b} \in \Lambda^*(X) \otimes spl(p,p) \ \ : \ \ \mathfrak{b} = \mathfrak{e}\,\mathfrak{b}\,\mathfrak{e} \ \ \} \ . \tag{112}$$

We stress that we do not demand that $\mathbf{e}_1$ and $\mathbf{e}_4$ are globally diagonalizable on X. This means that the defining equation $\mathfrak{b} = \mathfrak{e}\,\mathfrak{b}\,\mathfrak{e}$ cannot be globally solved on X. We also underline that – in general – we do not have a differential on $\mathfrak{H}_\mathfrak{e}$. What remains is a derivation $\mathfrak{D} = \mathfrak{e}\mathfrak{d}(\ .\ )\mathfrak{e}$ on $\mathfrak{H}_\mathfrak{e}$. Explicitly, one has

$$\mathfrak{D}\mathfrak{b} = \mathfrak{e}\boldsymbol{d}(\mathfrak{b})\mathfrak{e} + [\mathfrak{e}\mathfrak{m}\mathfrak{e}, \mathfrak{b}]_g \ , \quad \mathfrak{b} \in \mathfrak{H}_\mathfrak{e} \ . \tag{113}$$

4.2 A Projection

Now, recalling the representation (78) for $\mathfrak{i}(\Lambda^*_\mathcal{A} \otimes \mathrm{M}_p\mathbb{C})$, we can define a surjective mapping

$$\mathfrak{p} : \mathfrak{i}(\Lambda^*_\mathcal{A} \otimes \mathrm{M}_p\mathbb{C}) \to \Lambda^*(X) \otimes \mathrm{M}_{2p}\mathbb{C} \ , \tag{114}$$

$$\mathfrak{p} : \begin{pmatrix} \displaystyle\sum_{t=0}^{m} \mathbf{a}_1^{k-2t} \otimes M_1^t & \displaystyle\sum_{t=0}^{m} \mathbf{a}_2^{k-2t-1} \otimes M_2^t \\ \displaystyle\sum_{t=0}^{m} \mathbf{a}_3^{k-2t-1} \otimes M_3^t & \displaystyle\sum_{t=0}^{m} \mathbf{a}_4^{k-2t} \otimes M_4^t \end{pmatrix} \mapsto \begin{pmatrix} \displaystyle\sum_{t=0}^{m} \mathbf{a}_1^{k-2t} & \displaystyle\sum_{t=0}^{m} \mathbf{a}_2^{k-2t-1} \\ \displaystyle\sum_{t=0}^{m} \mathbf{a}_3^{k-2t-1} & \displaystyle\sum_{t=0}^{m} \mathbf{a}_4^{k-2t} \end{pmatrix} .$$

Observe that $\mathfrak{e} = \mathfrak{p}(\mathbf{e}) = \mathrm{diag}\,(\mathbf{e}_1\,,\,\mathbf{e}_4)$, see (79) and (111).

Proposition 12

i) $\mathfrak{p}(\boldsymbol{\mathcal{H}}_0) = \mathfrak{H}_{\mathfrak{e}}$.

ii) $(\mathfrak{p}(\boldsymbol{\varrho}))^* = \mathfrak{p}(\boldsymbol{\varrho}^*)$, $\boldsymbol{\varrho} \in \boldsymbol{\mathcal{H}}_0$.

iii) *For* $k{+}l{\leq}2m{+}1$ *we have* $\mathfrak{p}([\boldsymbol{\varrho}, \tilde{\boldsymbol{\varrho}}]_g) = [\mathfrak{p}(\boldsymbol{\varrho}), \mathfrak{p}(\tilde{\boldsymbol{\varrho}})]_g$, $\boldsymbol{\varrho} \in \boldsymbol{\mathcal{H}}_0^k$, $\tilde{\boldsymbol{\varrho}} \in \boldsymbol{\mathcal{H}}_0^l$.

iv) *For* $k{\leq}2m$ *we have* $\mathfrak{p}(\boldsymbol{\mathcal{D}}\boldsymbol{\varrho}) = \mathfrak{D}(\mathfrak{p}(\boldsymbol{\varrho}))$, $\boldsymbol{\varrho} \in \boldsymbol{\mathcal{H}}_0^k$.

Proof: *i)* From the property (96) of elements of $\boldsymbol{\mathcal{H}}_0$ we obtain immediately

$$\mathrm{tr}\,(\Gamma_0 \circ \mathfrak{p}(\boldsymbol{\varrho})) = 0 \ , \quad \boldsymbol{\varrho} \in \boldsymbol{\mathcal{H}}_0 \ . \tag{115}$$

This together with $\mathfrak{e}\mathfrak{r}\mathfrak{e} = \mathfrak{r}$ for any $\mathfrak{r} = \mathfrak{p}(\boldsymbol{\varrho}) \in \mathfrak{p}(\boldsymbol{\mathcal{H}}_0)$, see (80), means $\mathfrak{p}(\boldsymbol{\mathcal{H}}_0) = \mathfrak{H}_{\mathfrak{e}}$.

ii) follows immediately from (110) and (93).

iii) Using (105) and (86) one can show for $k + l \leq 2m + 1$

$$\mathfrak{p}(\boldsymbol{\chi} \bullet \tilde{\boldsymbol{\chi}}) = \mathfrak{p}(\boldsymbol{\chi}) \odot \mathfrak{p}(\tilde{\boldsymbol{\chi}}) \ , \quad \boldsymbol{\chi} \in \mathfrak{i}(\Lambda^k_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C}) \ , \quad \tilde{\boldsymbol{\chi}} \in \mathfrak{i}(\Lambda^l_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C}) \ . \tag{116}$$

For $k + l > 2m + 1$ certain terms in $\boldsymbol{\chi} \bullet \tilde{\boldsymbol{\chi}}$ disappear, because the summation in (86) only runs from $t = 0$ to $t = m$. These terms will in general not vanish in the product $\odot$ of the projected terms. Then, since for $\boldsymbol{\varrho} \in \boldsymbol{\mathcal{H}}_0^k$ the total degree of $\mathfrak{p}(\boldsymbol{\varrho}) \in \Lambda^*(X) \otimes \mathrm{M}_{2p}\mathbb{C}$ equals k , we find with (8), (106), (85) and (88)

$$\mathfrak{p}(\,[\boldsymbol{\varrho}, \tilde{\boldsymbol{\varrho}}]_g\,) = [\mathfrak{p}(\boldsymbol{\varrho}), \mathfrak{p}(\tilde{\boldsymbol{\varrho}})]_g \ , \quad \boldsymbol{\varrho} \in \boldsymbol{\mathcal{H}}_0^k \ , \quad \tilde{\boldsymbol{\varrho}} \in \boldsymbol{\mathcal{H}}_0^l \ , \quad k + l \leq 2m + 1 \ . \tag{117}$$

Here, on the l.h.s., $[\ ,\ ]_g$ is the graded commutator in $\boldsymbol{\mathcal{H}}_0$, while on the r.h.s., $[\ ,\ ]_g$ is the graded commutator in $\mathfrak{H}_{\mathfrak{e}}$.

iv) Since $\mathfrak{p}(\boldsymbol{\mu}) = \mathfrak{e}\mathfrak{m}\mathfrak{e}$, see (104) and (92), for the choice made for u and z , we obtain *iv)* for $k \leq 2m$ from (91), (92) and (113). The restriction to $k \leq 2m$ is due to the same reasons as in *iii)*, because in $\boldsymbol{\mathcal{D}}\boldsymbol{\varrho}$ there appears a graded commutator. $\square$

The mapping $\mathfrak{p}$ is not injective, because we have $\mathfrak{p} \circ \mathfrak{i}(\Lambda^k_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C}) \subset \mathfrak{p} \circ \mathfrak{i}(\Lambda^{k+2}_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C})$ for $k \leq 2m - 1$. However, we observe that $\mathfrak{p}|_{\mathfrak{i}(\Lambda^k_{\mathcal{A}} \otimes \mathrm{M}_p\mathbb{C})}$ is injective for each fixed k and that $\mathfrak{p}$ restricted to $\boldsymbol{\mathcal{H}}^0$ is an isomorphism of algebras. Since $MM^* \notin \mathbb{C}\,\mathrm{id}_F$, we have $m \geq 1$. Thus, the product of elements of $\boldsymbol{\mathcal{H}}^1$ by elements of $\boldsymbol{\mathcal{H}}^0$ or $\boldsymbol{\mathcal{H}}^1$ is transported via $\mathfrak{p}$ isomorphically. The same is true for the transport of the derivation (113) of elements of $\boldsymbol{\mathcal{H}}^0$ and $\boldsymbol{\mathcal{H}}^1$. We stress that applying $\mathfrak{p}$, one looses[1] the $\mathbb{N}$–grading structure of $\boldsymbol{\mathcal{H}}$. This is inevitable, because on $\mathfrak{H}_{\mathfrak{e}}$ there is only a $\mathbb{Z}_2$–grading structure.

Next, we discuss the transport of the gauge group of the module $\mathcal{E}$, see Definition 5, and the structure of the transported connection form. We have

[1] In some physical models, see section 5, the matrix M contains fermionic mass parameters, which are removed by applying $\mathfrak{p}$

End $(\mathcal{E}) = \mathcal{H}^0$ and, therefore, from (78), (80), (93) and Definition 5 we find

$$\mathfrak{U} := \mathfrak{p}(\boldsymbol{\mathcal{U}}(\boldsymbol{\mathcal{E}})) = \{\ \mathfrak{u} = \begin{pmatrix} \mathbf{u}_1 & 0 \\ 0 & \mathbf{u}_4 \end{pmatrix} , \qquad (118)$$
$$\mathbf{u}_1 = \mathbf{e}_1\mathbf{u}_1\mathbf{e}_1\ ,\ \mathbf{u}_4 = \mathbf{e}_4\mathbf{u}_4\mathbf{e}_4\ ,\ \mathbf{u}_1\mathbf{u}_1^* = \mathbf{u}_1^*\mathbf{u}_1 = \mathbf{e}_1\ ,\ \mathbf{u}_4\mathbf{u}_4^* = \mathbf{u}_4^*\mathbf{u}_4 = \mathbf{e}_4\ \} ,$$

where $\mathbf{u}_1, \mathbf{u}_4 \in \Lambda^0(X) \otimes \mathrm{M}_p\mathbb{C}$.

The transported connection form is a skew–adjoint element of $\mathfrak{p}(\boldsymbol{\mathcal{H}}^1)$ and has according to Lemma 7, (78), (80), (93) and (114) the structure

$$\boldsymbol{\omega} := \mathfrak{p}(\boldsymbol{\rho}) = \begin{pmatrix} \mathbf{r}_1 & \mathbf{r}_2 \\ \mathbf{r}_3 & \mathbf{r}_4 \end{pmatrix} \ ,\quad \mathbf{r}_1 = -\mathbf{r}_1^*\ ,\quad \mathbf{r}_2 = -\mathbf{r}_3^*\ ,\quad \mathbf{r}_4 = -\mathbf{r}_4^*\ , \qquad (119)$$
$$\mathbf{r}_1 = \mathbf{e}_1\mathbf{r}_1\mathbf{e}_1 \in \Lambda^1(X) \otimes \mathrm{M}_p\mathbb{C}\ ,\quad \mathbf{r}_2 = \mathbf{e}_1\mathbf{r}_2\mathbf{e}_4 \in \Lambda^0(X) \otimes \mathrm{M}_p\mathbb{C}\ ,$$
$$\mathbf{r}_3 = \mathbf{e}_4\mathbf{r}_3\mathbf{e}_1 \in \Lambda^0(X) \otimes \mathrm{M}_p\mathbb{C}\ ,\quad \mathbf{r}_4 = \mathbf{e}_4\mathbf{r}_4\mathbf{e}_4 \in \Lambda^1(X) \otimes \mathrm{M}_p\mathbb{C}\ .$$

For physical reasons, see section 5, it is interesting to restrict the connection form $\boldsymbol{\omega}$ to $\mathfrak{p}(\boldsymbol{\mathcal{H}}_0^1)$. This means, see (96),

$$\mathrm{tr}\,(\mathbf{r}_1) = \mathrm{tr}\,(\mathbf{r}_4)\ . \qquad (120)$$

Thus, $\boldsymbol{\omega}$ is a skew–adjoint element of $\mathfrak{H}_\mathfrak{e}$. Using (19), (98), and *iv*) of Proposition 12 one gets for the transported curvature

$$\mathfrak{f} := \mathfrak{p} \circ \mathfrak{i}(\theta) = \mathfrak{e}(\mathfrak{d}\mathfrak{e})(\mathfrak{d}\mathfrak{e})\mathfrak{e} + \mathfrak{D}\,\boldsymbol{\omega} + (1/2)[\boldsymbol{\omega}, \boldsymbol{\omega}]_g\ . \qquad (121)$$

Observe that the curvature – in general – does not take values in $\mathfrak{p}(\boldsymbol{\mathcal{H}}_0)$, because from (98) we get

$$\boldsymbol{T}_{\boldsymbol{\mathcal{H}}}(\boldsymbol{\theta}_0) = \mathrm{tr}\,(\mathbf{e}_1(\boldsymbol{d}\mathbf{e}_1)^2 + \mathbf{e}_1 - \mathbf{e}_4(\boldsymbol{d}\mathbf{e}_4)^2 - \mathbf{e}_4)\ .$$

The transport of the gauge transformed connection form, see (20), is due to Proposition 12 given by

$$\gamma_\mathfrak{u}(\boldsymbol{\omega}) = \mathfrak{u}\mathfrak{D}\mathfrak{u}^* + \mathfrak{u}\boldsymbol{\omega}\mathfrak{u}^*\ , \qquad (122)$$

and in the representation (119) it takes the form

$$\gamma_\mathfrak{u}(\boldsymbol{\omega}) = \left(\begin{array}{c|c} \mathbf{u}_1\boldsymbol{d}(\mathbf{u}_1^*)\mathbf{e}_1 + \mathbf{u}_1\mathbf{r}_1\mathbf{u}_1^* & \mathbf{u}_1(\mathbf{r}_2 - i\mathbf{e}_1\mathbf{e}_4)\mathbf{u}_4^* + i\mathbf{e}_1\mathbf{e}_4 \\ \hline \mathbf{u}_4(\mathbf{r}_3 - i\mathbf{e}_4\mathbf{e}_1)\mathbf{u}_1^* + i\mathbf{e}_4\mathbf{e}_1 & \mathbf{u}_4\boldsymbol{d}(\mathbf{u}_4^*)\mathbf{e}_4 + \mathbf{u}_4\mathbf{r}_4\mathbf{u}_4^* \end{array}\right)\ . \qquad (123)$$

Since $\gamma_\mathfrak{u}(\boldsymbol{\omega})$ must also be an element of $\mathfrak{p}(\boldsymbol{\mathcal{H}}_0^1)$, the group of gauge transformations has to be restricted to

$$\mathfrak{U}_0 := \{\ \mathfrak{u} \in \mathfrak{U}\ :\ \mathrm{tr}\,(\mathbf{u}_1\boldsymbol{d}\mathbf{u}_1^*) = \mathrm{tr}\,(\mathbf{u}_4\boldsymbol{d}\mathbf{u}_4^*)\ \}\ . \qquad (124)$$

Putting $\mathfrak{u} = \mathfrak{e} - \mathfrak{t} + \ldots \in \mathfrak{U}$, with $\mathfrak{t} = -\mathfrak{t}^* \in \mathfrak{p}(\boldsymbol{\mathcal{H}}^0)$, we obtain the infinitesimal version of gauge transformations:

$$\gamma_{\mathfrak{t}}(\boldsymbol{\omega}) = \boldsymbol{\omega} + \mathfrak{D}\mathfrak{t} + [\boldsymbol{\omega}, \mathfrak{t}]_g \ , \tag{125}$$

where we have used $\mathfrak{D}\mathfrak{e} \equiv 0$ and $\mathfrak{t} = \mathfrak{e}\mathfrak{t}\mathfrak{e}$. The condition $\gamma_{\mathfrak{u}}(\boldsymbol{\omega}) \in \mathfrak{p}(\boldsymbol{\mathcal{H}}_0^1)$ gives $\boldsymbol{d}\mathfrak{t} \in \mathfrak{p}(\boldsymbol{\mathcal{H}}_0^1)$. Neglecting global gauge transformations, we integrate $\boldsymbol{d}\mathfrak{t} \in \mathfrak{p}(\boldsymbol{\mathcal{H}}_0^1)$ and obtain for the generator of infinitesimal gauge transformations

$$\mathfrak{t} \in \mathfrak{p}(\boldsymbol{\mathcal{H}}_0^0) \ . \tag{126}$$

Now we give a local description of the gauge groups $\mathfrak{U}$ and $\mathfrak{U}_0$. Since the algebra under consideration is commutative, there corresponds a classical (in general nontrivial) vector bundle E over two copies of X to the Hermitian module $\mathcal{E}$. We choose a covering $\{\mathcal{O}_i\}$ of X, so that E is trivializable over this covering. Then, we can locally – on every $\mathcal{O}_i$ – diagonalize $\mathbf{e}_1$ and $\mathbf{e}_4$, using pointwise unitary matrices $\mathfrak{p}(\mathbf{v}) \in \mathfrak{p}(\boldsymbol{\mathcal{U}}(\boldsymbol{\mathcal{A}}^p))$, see (99). Since $\mathbf{e}_1$ and $\mathbf{e}_4$ are idempotent, we find a unitary module transformation (100), which transforms them locally into the following standard form:

$$\mathbf{e}_1 = \mathrm{diag}\left(\underbrace{1,\ldots,1}_{p_1},\underbrace{0,\ldots,0}_{p-p_1}\right) \ , \quad \mathbf{e}_4 = \mathrm{diag}\left(\underbrace{1,\ldots,1}_{p_4},\underbrace{0,\ldots,0}_{p-p_4}\right) \ . \tag{127}$$

Inserting (127) into (118) we see that the matrices $\mathbf{u}_1$ and $\mathbf{u}_4$ can be locally characterized as follows:

$$\mathbf{u}_1 \in C^\infty_{\mathbb{R}}(\mathcal{O}_i) \otimes U(p_1) \ , \quad \mathbf{u}_4 \in C^\infty_{\mathbb{R}}(\mathcal{O}_i) \otimes U(p_4) \ , \tag{128}$$

where $C^\infty_{\mathbb{R}}(\mathcal{O}_i)$ denotes the algebra of real smooth functions on $\mathcal{O}_i$ and a representation of $U(p_1)$ in $p \times p$–matrices containing $p - p_1$ zero–rows and –columns is used (analogously for $U(p_4)$). This means that the gauge group $\mathfrak{U}$ is locally isomorphic to

$$\mathfrak{U}^i = C^\infty_{\mathbb{R}}(\mathcal{O}_i) \otimes (\, U(p_1) \times U(p_4)\,) \ . \tag{129}$$

There is a natural homeomorphism of $U(n)$ onto $SU(n) \times U(1)$:

$$u = u_0 \begin{pmatrix} \det u & 0 \\ 0 & \mathbb{1}_{(n-1)\times(n-1)} \end{pmatrix} \ , \tag{130}$$

where $u \in U(n)$, $u_0 \in SU(n)$, $\det u \in U(1)$. Extending (130) to $\mathcal{O}_i$ and using $\mathrm{tr}\,(\mathbf{u}_0 \boldsymbol{d}\mathbf{u}_0{}^*) \equiv 0$, for $\mathbf{u}_0 \in C^\infty_{\mathbb{R}}(\mathcal{O}_i) \otimes SU(n)$, we obtain from the condition $\mathrm{tr}\,(\mathbf{u}_1 \boldsymbol{d}\mathbf{u}_1^*) = \mathrm{tr}\,(\mathbf{u}_4 \boldsymbol{d}\mathbf{u}_4^*)$, characterizing elements of $\mathfrak{U}_0$, see (124),

$$\det \mathbf{u}_1 \, \boldsymbol{d}(\det \mathbf{u}_1)^{-1} = \det \mathbf{u}_4 \, \boldsymbol{d}(\det \mathbf{u}_4)^{-1} \ . \tag{131}$$

Integrating this result, we obtain $\det \mathbf{u}_1 = \mathrm{const} \det \mathbf{u}_4$. Since $\mathbf{u}_1$ and $\mathbf{u}_4$ are unitary, the integration constant must be a phase factor, which corresponds to a global $U(1)$–symmetry of the gauge field theory[2]. Here, we are interested only in local gauge groups, so that we put the integration constant equal to one. This shows that we have locally

$$\mathfrak{U}_0^i = C^\infty_{\mathbb{R}}(\mathcal{O}_i) \otimes (\, SU(p_1) \times SU(p_4) \times U(1)\,) \ . \tag{132}$$

Of course, the collection $\{\mathfrak{U}_0^i\}$ can be used to reconstruct the gauge group $\mathfrak{U}_0$ – or in the bundle terminology – the group of vertical automorphisms of the principal bundle associated with E (the group of local gauge transformations).

In particular, for $p_4 = 1$ the group of local gauge transformations is locally given by $\mathfrak{U}_0^i = C^\infty_{\mathbb{R}}(\mathcal{O}_i) \otimes (\, SU(p_1) \times U(1)\,)$ and for $p_1 = 1$ by $\mathfrak{U}_0^i = C^\infty_{\mathbb{R}}(\mathcal{O}_i) \otimes (\, SU(p_4) \times U(1)\,)$. For $p_4 = 0$ the group of local gauge transformations is

$$\mathfrak{U}^i = C^\infty_{\mathbb{R}}(\mathcal{O}_i) \otimes U(p_1) \ , \tag{133}$$

and the group of special local gauge transformations $\mathfrak{U}_0$ is reduced to

$$\mathfrak{U}_0^i = C^\infty_{\mathbb{R}}(\mathcal{O}_i) \otimes SU(p_1) \ . \tag{134}$$

Analogous results can be obtained in the case $p_1 = 0$.

Finally, we comment on the local representation of the connection form $\boldsymbol{\omega}$, see (119). Using the above described local diagonalization procedure for the projection operators $\mathbf{e}_1$ and $\mathbf{e}_4$, one finds local representatives $A_{(i)}$, $B_{(i)}$ and $\Phi_{(i)}$ of $\mathbf{r}_1$, $\mathbf{r}_4$ and $\mathbf{r}_2 = -\mathbf{r}_3^*$ respectively, with $A_{(i)} = -A_{(i)}^*$, $B_{(i)} = -B_{(i)}^*$ and $\mathrm{tr}(A_{(i)}) = \mathrm{tr}(B_{(i)})$. The fields $A_{(i)}$ and $B_{(i)}$ constitute the local representative of a classical gauge connection, that means a classical differential 1–form on $\mathcal{O}_i$ with values in the Lie–algebra of $(\, SU(p_1) \times SU(p_4) \times U(1)\,)$. The field $\Phi_{(i)}$ is a vector–space–valued function on $\mathcal{O}_i$ and can be physically interpreted as a matter field – as done in the next subsection. The fact that two different classical objects are unified in one non–commutative connection form is, of course, due to the fact that we started with a non–commutative differential calculus.

4.3 The Case of the Standard Model

Here we will show that the mathematical structures underlying an approach to the derivation of the standard model, proposed by Coquereaux et al. ([12], [11], [8], [10]), can be obtained as a special case of the structures derived in the previous subsection. Partly our notations and sign conventions differ from the original ones, due to the fact that we started essentially with the conventions of Connes. We put $N = 4$ for the dimension of the manifold X and assume that X is topologically trivial, for many physical applications it has the topology of $\mathbb{R}^4$. In that case all local considerations of the previous subsection concerning the

[2]For the standard model this global symmetry is given by a constant phase transformation only of the right–handed fermions and the Higgs field and not of the left–handed fermions

group of local gauge transformations and the non–commutative connection form $\boldsymbol{\omega}$ become global.

The starting point in the Mainz–Marseille approach is the differential algebra $\Lambda^*(X) \otimes \mathrm{M}_4\mathbb{C}$, or rather [12] $\mathrm{M}_4\mathbb{C} \otimes \Lambda^*(X)$, giving in general a different sign in (105). This means that we put $p = 2$ in formulae of the previous subsection. Putting for $\mathfrak{e}$, see (111), $\mathfrak{e} = \mathrm{diag}(1,1,1,0)$, we get a graded Lie subalgebra of $\Lambda^*(X) \otimes spl(2,2)$, see (112), which we denote by $\Lambda^*(X) \otimes spl(2,1)$. We note that this graded Lie algebra was denoted in [11] by $\Lambda^*(X) \otimes SU(2|1)$.

The authors of [11] formally define a connection putting

$$\nabla = \mathfrak{e}\mathfrak{d} + \boldsymbol{\omega} \ , \tag{135}$$

where $\mathfrak{d}$ is the natural differential on $\Lambda^*(X) \otimes \mathrm{M}_4\mathbb{C}$, see (109). For the gauge potential $\boldsymbol{\omega}$ they postulate the form

$$\boldsymbol{\omega} = -\boldsymbol{\omega}^* = \begin{pmatrix} A_{11} & A_{12} & -i\Phi_1 & 0 \\ A_{21} & A_{22} & -i\Phi_2 & 0 \\ -i\overline{\Phi}_1 & -i\overline{\Phi}_2 & B & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \Lambda^*(X) \otimes spl(2,1) \ , \tag{136}$$

$$A_{ij} = -\overline{A}_{ji} \in \Lambda^1(X) \ , \quad B = -\overline{B} \in \Lambda^1(X) \ , \quad A_{11} + A_{22} = B \ , \quad \Phi_i \in \Lambda^0(X) \ .$$

A certain module, on which this connection can act, was defined in [14]. But a deeper explanation for the choice of the connection form $\boldsymbol{\omega}$ was not given. The curvature of this connection is [11]

$$\mathfrak{f} = \nabla^2 = \mathfrak{e}(\mathfrak{d}\mathfrak{e})(\mathfrak{d}\mathfrak{e})\mathfrak{e} + \mathfrak{D}\,\boldsymbol{\omega} + (1/2)[\boldsymbol{\omega},\boldsymbol{\omega}]_g \ , \tag{137}$$

where $\mathfrak{D}$ is given by (113), and the bosonic action is $S_b = \int_X < \mathfrak{f}, \mathfrak{f} >_0$, with $< \ , \ >_0$ denoting an appropriate product.

It was unclear in this approach what the group of gauge transformations is. Instead of this, only infinitesimal gauge transformations were defined, see [11],

$$\gamma_{\mathfrak{t}}(\boldsymbol{\omega}) := \boldsymbol{\omega} + \mathfrak{D}\,\mathfrak{t} + [\boldsymbol{\omega}, \mathfrak{t}]_g \ , \qquad \mathfrak{t} = -\mathfrak{t}^* \in \Lambda^*(X) \otimes spl(2,1) \ . \tag{138}$$

The authors of [11] notice that for the standard model only those $\mathfrak{t}$ make sense, which have the form

$$\mathfrak{t} = \begin{pmatrix} T_{11} & T_{12} & 0 & 0 \\ T_{21} & T_{22} & 0 & 0 \\ 0 & 0 & T_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \quad T_{ij} = -\overline{T}_{ji} \in \Lambda^0(X) \ , \quad \mathrm{tr}\,(\Gamma_0 \mathfrak{t}) = 0 \ . \tag{139}$$

A deeper explanation why one should restrict $\mathfrak{t}$ to the form (139) was not given. For an extended theory including differential forms of higher degree there were discussed more general “superbosonic” gauge transformations [11].

Finally, we notice that there exists a formulation of the Mainz–Marseille model in terms of 3×3–matrices ([15], [23]), for a parallel treatment of both formulations see [13]. However, in this formulation a field strength was used, which cannot be interpreted as the curvature of a connection, because the term $\mathfrak{e}(\mathfrak{d}\mathfrak{e})(\mathfrak{d}\mathfrak{e})\mathfrak{e}$ occurring in the curvature of a connection on a finite projective module was neglected.

Now we show that all structures occurring here find their natural explanation within the framework developed in the previous subsection. For this purpose we put $p = p_1 = 2$, $p_4 = 1$, see (127).

1. We define the module for the Mainz–Marseille approach as $\mathfrak{p}(\boldsymbol{\mathcal{E}})$, which is a right module over the algebra $\mathfrak{p}(\boldsymbol{\mathcal{A}})$. Next, $\mathfrak{m}$ occurring in formula (109) takes the form

$$\mathfrak{m} = -i \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix} , \tag{140}$$

see also (104). Thus, from (97) and (109) one finds

$$\mathfrak{p}(\boldsymbol{\nabla}_0\, \boldsymbol{\xi}) = \mathfrak{e}\mathfrak{d}\mathfrak{x} \ , \quad \mathfrak{x} = \mathfrak{p}(\boldsymbol{\xi}) \in \mathfrak{p}(\boldsymbol{\mathcal{E}}) \ , \tag{141}$$

and – using (87) and (119) – one gets

$$\mathfrak{p}(\boldsymbol{\nabla}\boldsymbol{\xi}) = \mathfrak{p}(\boldsymbol{\nabla}_0\, \boldsymbol{\xi} + \boldsymbol{\rho}\boldsymbol{\xi}) = \mathfrak{e}\mathfrak{d}\mathfrak{x} + \boldsymbol{\omega} \odot \mathfrak{x} \ , \tag{142}$$

with $\boldsymbol{\omega} \in \Lambda^*(X) \otimes spl(2,1)$. Moreover, $\boldsymbol{\omega}$ given by (119) fulfils additionally (120). Changing the notations $\mathbf{r}_1 = A$, $\mathbf{r}_4 = \tilde{B}$, $\mathbf{r}_2 = -i\Phi$, $\mathbf{r}_3 = -i\Phi^*$, we obtain exactly the form of the gauge potential postulated in the Mainz–Marseille approach, see (136),

$$\boldsymbol{\omega} \ = \ \begin{pmatrix} A & -i\Phi \\ -i\Phi^* & \tilde{B} \end{pmatrix} , \quad \mathrm{tr}\,(A) = \mathrm{tr}\,(\tilde{B}) \ , \tag{143}$$

$$A = -A^* = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} , \quad \tilde{B} = -\tilde{B}^* = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} , \quad \Phi = \begin{pmatrix} \Phi_1 & 0 \\ \Phi_2 & 0 \end{pmatrix} .$$

We note that the transported connection $\nabla_{\mathfrak{p}} \equiv \mathfrak{p}\boldsymbol{\nabla}\mathfrak{p}^{-1} : \mathfrak{p}(\boldsymbol{\mathcal{E}}) \to \mathfrak{p}(\boldsymbol{\mathcal{E}}^1)$ fulfils

$$\nabla_{\mathfrak{p}}(\mathfrak{x}\mathfrak{a}) = (\nabla_{\mathfrak{p}}\, \mathfrak{x})\mathfrak{a} + \mathfrak{x}\mathfrak{d}(\mathfrak{a}) \ , \quad \mathfrak{x} \in \mathfrak{p}(\boldsymbol{\mathcal{E}}) \ , \quad \mathfrak{a} \in \mathfrak{p}(\boldsymbol{\mathcal{A}}) \ , \tag{144}$$

which is exactly the transport of the defining equation of a connection, see Definition 4. Finally, observe that formula (121) for the curvature adapted to the case under consideration gives exactly (137).

2. We define the group of gauge transformations in the Mainz–Marseille model as the group $\mathfrak{U}_0$ of special unitary automorphisms of the module $\mathfrak{p}(\boldsymbol{\mathcal{E}})$ with identity $\mathfrak{p}(\mathbf{e}) = \mathfrak{e}$, see (124). From (132) we find in the case under consideration

$$\mathfrak{U}_0 = C^\infty_{\mathbb{R}}(X) \otimes (\, SU(2) \times U(1)\,) \ , \tag{145}$$

which is just the group of local gauge transformations of the Salam–Weinberg model. Writing down local gauge transformations, see (122), or rather infinitesimal gauge transformations, see (125) and (126), we get exactly (139) postulated in the Mainz–Marseille approach:

$$\begin{aligned} \mathfrak{t} \;&=\; \begin{pmatrix} T_1 & 0 \\ 0 & T_4 \end{pmatrix} \;, \quad \mathrm{tr}\,(T_1) = \mathrm{tr}\,(T_4) \;, \\ T_1 = -T_1^* &= \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \Lambda^0(X) \otimes \mathrm{M}_2\mathbb{C} \;, \\ T_4 = -T_4^* &= \begin{pmatrix} T_{33} & 0 \\ 0 & 0 \end{pmatrix} \in \Lambda^0(X) \otimes \mathbf{e}_4(\mathrm{M}_2\mathbb{C})\mathbf{e}_4 \;, \end{aligned} \tag{146}$$

see (78), (80), (93) and (96). Thus, $\mathfrak{t}$ coincides with (139) of the Mainz–Marseille approach. This justifies the choice of infinitesimal gauge transformations in the model of Coquereaux and Scheck. But extended "superbosonic" gauge transformations $\mathfrak{t} \in \Lambda^*(X) \otimes spl(2,1)$, which were suggested in [11], are within this context not allowed. We stress that – in contrary to classical differential geometry – the Lie algebra of the structure group $SU(2) \times U(1)$ does not coincide with the Lie algebra $spl(2,1)$, where the gauge potential takes its values.

5 Model Building

In this section we outline the derivation of the standard model based on the simplest two–point K–cycle. For a detailed presentation of this approach we refer to [27].

The K–cycle $(\mathcal{A}, h, D)$ reviewed in subsection 3.1, together with the finite projective module $\mathcal{E} = e\mathcal{A}^2$, $e = \mathrm{diag}(1,0,1,1) \otimes \mathrm{id}_F$, was used by Connes in [6], [5] and by Connes and Lott in [7] to obtain a unification of the Salam–Weinberg model – the theory of electroweak interactions of leptons. Using this K–cycle together with the module $\mathcal{E}$ and the canonical prescription for the physical Hilbert space [7], $H = \mathcal{E} \otimes_{\mathcal{A}} h$, it is impossible to derive the full standard model. That is why Connes and Lott proposed a different K–cycle, namely $(\mathcal{A}_s \otimes \mathcal{B}_s \,,\, h_s \,,\, D_s)$, where

$$\begin{aligned} h_s &= L^2(X,S) \otimes (F_- \oplus F_+) \;, \\ F_- &= (\mathbb{C}^2 \oplus (\mathbb{C}^2 \otimes \mathbb{C}^3)) \otimes \mathbb{C}^{N_F} \;, \\ F_+ &= (\mathbb{C} \oplus (\mathbb{C}^2 \otimes \mathbb{C}^3)) \otimes \mathbb{C}^{N_F} \;, \\ \mathcal{A}_s &= C^\infty_{\mathbb{R}}(X) \otimes (\mathbb{C} \oplus \mathbb{H}) \;, \\ \mathcal{B}_s &= C^\infty_{\mathbb{R}}(X) \otimes (\mathbb{C} \oplus \mathrm{M}_3\mathbb{C}) \;. \end{aligned} \tag{147}$$

Here, $N_F = 3$ is the number of generations of fermions and $\mathbb{H}$ is the real algebra of quaternions. All tensor products occurring in (147) are over $\mathbb{R}$, which means in particular that the algebras $\mathcal{A}_s$ and $\mathcal{B}_s$ are real algebras. The differential operator D_s has the same structure as the operator D in (46) for an appropriate choice of $F_\pm$ and $\mathcal{M}$. In this approach one uses a free module, namely $\mathcal{E}_s = \mathcal{A}_s \otimes \mathcal{B}_s$. A detailed exposition of these ideas was presented by Kastler in [18] and [19], see [17] for an earlier version.

It is worthwile to notice that in this approach one obtains certain constraints between the masses of the fermions and the masses of the W–, Z–, and Higgs–bosons. Moreover, one gets a prediction of the Weinberg angle on tree level. In the "grand unification case" [20] Kastler and Schücker obtained

$$(g_3/g_2)^2 = 1\ , \quad \sin^2\theta_W = 3/8\ , \quad m_t/m_W = 2\ , \quad m_H/m_W \approx 3.14\ , \tag{148}$$

where g_2 and g_3 are the coupling constants of the electroweak and strong interactions, θ_W is the Weinberg angle, m_t, m_W and m_H are the masses of the top–quark, the W–boson and the Higgs–boson.

Another way of obtaining the standard model by non–commutative geometry is the Mainz–Marseille approach ([12], [11]), which is based upon the mathematical structures discussed in the previous section. In a first step one writes down the bosonic action of the electroweak sector using the $spl(2,1)$–gauge connection discussed in subsection 4.3, see [12] and [11]. The bosonic action of the chromodynamics sector is added in the same form as in classical gauge field theory. To write down the fermionic sector, one uses the theory of representations of the graded Lie algebra $spl(2,1)$ in a finite dimensional vector space $[y,I]$, see [25] and [11], where y means hypercharge and I isospin. One builds the Hilbert space $L^2(X,S)\otimes[y,I]$, leptons live in $L^2(X,S)\otimes[1,\frac{1}{2}]$ and quarks in $L^2(X,S)\otimes[\frac{1}{3},\frac{1}{2}]\otimes\mathbb{C}^3$.

We note that there do not exist representations of the full graded Lie algebra $\Lambda^*(X)\otimes spl(2,1)$ in these Hilbert spaces. To define a fermionic action one has to define a covariant derivative. For this purpose the connection form $\boldsymbol{\omega}$ has to be considered as an element of $(C^1\oplus C^0)\otimes spl(2,1)$, acting with the first (Clifford) part on $L^2(X,S)$ and with the $spl(2,1)$–part on $[y,I]$. The fermion masses are obtained from free relative normalization constants of $sl(2,\mathbb{C})\oplus gl(1,\mathbb{C})$–subrepresentations. In contrast to the model of Connes, Lott and Kastler, the fermion masses are not related to the masses of the intermediate vector and Higgs–bosons. Using reducible indecomposable representations of $spl(2,1)$ one describes family mixing [11].

It turns out that the combination of these ideas with the scheme developed in this paper leads to a new derivation of the standard model. This derivation starts with the K–cycle of subsection 3.1 over the simplest two–point algebra $\mathcal{A}$ defined in (43), where the vector space F, which plays an auxiliary role, is taken to be $F = \hat{F}\oplus\hat{F}$, $\hat{F} = \mathbb{C}^3$. The first term $\mathbb{C}^3$ stands for the three generations of leptons and the other one for the three generations of quarks. With this K–cycle we associate two finite projective modules: We take for the electroweak interaction part the module $\mathcal{E} = e\mathcal{A}^2$, $e = \mathrm{diag}\,(1,0,1,1)\otimes \mathrm{id}_F$, see also subsection 4.3, and for the chromodynamics part the module $\mathcal{E}_c = e_c\mathcal{A}^3$, $e_c = \mathrm{diag}\,(1,0,1,0,1,0)\otimes \mathrm{id}_F$. As already mentioned above, the module $e\mathcal{A}^2$ can not be used to describe the full electroweak sector if one follows the Connes–Lott prescription. The essential idea, which in our approach makes it possible to build the full electroweak sector out of $e\mathcal{A}^2$, is to consider the graded algebra $\mathcal{H} = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}\otimes_{\mathcal{A}} \Lambda^*_{\mathcal{A}})$ as a graded Lie algebra. For a (graded) Lie algebra there exist representations, which cannot be obtained from representations of a (graded) algebra. The representation describing the electroweak interactions of quarks is of that type.

Using (30) and (54) one obtains in the case of the above module $e\mathcal{A}^2$ for elements $\varrho \in \mathcal{H}_0^k$ the matrix representation

$$\varrho = \sum_{t=0}^{m} \left(\begin{array}{cc|cc} \frac{1}{2}(\alpha_0^{k-2t}+\alpha_3^{k-2t})\otimes M_1^t & 0 & \alpha_-^{k-2t}\otimes M_1^t & \alpha_4^{k-2t-1}\gamma^5\otimes M_2^t \\ 0 & 0 & 0 & 0 \\ \hline \alpha_+^{k-2t}\otimes M_1^t & 0 & \frac{1}{2}(\alpha_0^{k-2t}-\alpha_3^{k-2t})\otimes M_1^t & \alpha_5^{k-2t-1}\gamma^5\otimes M_2^t \\ \alpha_6^{k-2t-1}\gamma^5\otimes M_3^t & 0 & \alpha_7^{k-2t-1}\gamma^5\otimes M_3^t & \alpha_0^{k-2t}\otimes M_4^t \end{array}\right), \tag{149}$$

where $\alpha_f^n \in L^n$, $f = 0,+,-,3,4,5,6,7$. We choose $M = \mathrm{diag}\,(-\mathrm{m}_l, -\mathrm{m}_q)$, where m_l and m_q are real diagonal 3×3–matrices with non–negative entries. The index l stands for lepton and q for quark. Therefore, we have $\varrho \in L^*\otimes \mathrm{M}_4\mathbb{C}\otimes \mathrm{M}_3\mathbb{C}\otimes \mathrm{M}_2\mathbb{C}$. In (149) we considered ϱ as a 4×4–matrix with $L^*\otimes \mathrm{M}_3\mathbb{C}\otimes \mathrm{M}_2\mathbb{C}$–valued entries. Of course, ϱ can also be treated as a 2×2–diagonal matrix with $L^*\otimes \mathrm{M}_4\mathbb{C}\otimes \mathrm{M}_3\mathbb{C}$–valued entries, where the lepton part is in the upper left block and the quark part in the lower right block, see also the beginning of subsection 3.3 for a similar reordering procedure. Next, we define an isomorphism $\mathbf{i} : \mathcal{H}_0 \to \mathbf{i}(\mathcal{H}_0)$, which for the leptons is similar to the $[y=1, I=\frac{1}{2}]$–representation in the Mainz–Marseille model and for the quarks to the $[y=\frac{1}{3}, I=\frac{1}{2}]$–representation:

$$\mathbf{i}(\varrho) := \begin{pmatrix} \mathbf{i}_l(\varrho_l) & 0 \\ 0 & \mathbf{i}_q(\varrho_q)\otimes \mathbb{1}_{3\times 3} \end{pmatrix}, \tag{150}$$

$$\mathbf{i}_l(\varrho_l) := \sum_{r=0}^{m} \begin{pmatrix} \frac{1}{2}(\alpha_0^{k-2r}+\alpha_3^{k-2r})\otimes \mathrm{l}_1^r & \alpha_-^{k-2r}\otimes \mathrm{l}_1^r & \alpha_4^{k-2r-1}\gamma^5\otimes \epsilon \mathrm{l}_2^r \\ \alpha_+^{k-2r}\otimes \mathrm{l}_1^r & \frac{1}{2}(\alpha_0^{k-2r}-\alpha_3^{k-2r})\otimes \mathrm{l}_1^r & \alpha_5^{k-2r-1}\gamma^5\otimes \epsilon \mathrm{l}_2^r \\ \alpha_6^{k-2r-1}\gamma^5\otimes \epsilon^{-1}\mathrm{l}_3^r & \alpha_7^{k-2r-1}\gamma^5\otimes \epsilon^{-1}\mathrm{l}_3^r & \alpha_0^{k-2r}\otimes \mathrm{l}_4^r \end{pmatrix}, \tag{151}$$

$$
\mathbf{i}_q(\varrho_q) := \sum_{r=0}^{m} \left(\begin{array}{c|c|c|c}
\begin{array}{c}(\frac{1}{2}\alpha_3^{k-2r} - \frac{1}{6}\alpha_0^{k-2r}) \\ \otimes \mathrm{q}_1^r\end{array} & \alpha_-^{k-2r} \otimes \mathrm{q}_1^r & \begin{array}{c}\alpha_7^{k-2r-1}\gamma^5 \\ \otimes \beta \sqrt{\frac{2}{3}}\, \mathrm{q}_2^r\end{array} & \begin{array}{c}\alpha_4^{k-2r-1}\gamma^5 \\ \otimes \sqrt{\frac{1}{3}}\, \mathrm{q}_2^r \chi\gamma\end{array} \\ \hline
\alpha_+^{k-2r} \otimes \mathrm{q}_1^r & \begin{array}{c}(-\frac{1}{2}\alpha_3^{k-2r} - \frac{1}{6}\alpha_0^{k-2r}) \\ \otimes \mathrm{q}_1^r\end{array} & \begin{array}{c}-\alpha_6^{k-2r-1}\gamma^5 \\ \otimes \beta \sqrt{\frac{2}{3}}\, \mathrm{q}_2^r\end{array} & \begin{array}{c}\alpha_5^{k-2r-1}\gamma^5 \\ \otimes \sqrt{\frac{1}{3}}\, \mathrm{q}_2^r \chi\gamma\end{array} \\ \hline
\begin{array}{c}-\alpha_5^{k-2r-1}\gamma^5 \\ \otimes \beta^{-1} \sqrt{\frac{2}{3}}\, \mathrm{q}_3^r\end{array} & \begin{array}{c}\alpha_4^{k-2r-1}\gamma^5 \\ \otimes \beta^{-1} \sqrt{\frac{2}{3}}\, \mathrm{q}_3^r\end{array} & \begin{array}{c}-\frac{2}{3}\alpha_0^{k-2r} \\ \otimes \mathrm{q}_4^r\end{array} & 0 \\ \hline
\begin{array}{c}\alpha_6^{k-2r-1}\gamma^5 \\ \otimes (\chi\gamma)^{-1} \sqrt{\frac{1}{3}}\, \mathrm{q}_3^r\end{array} & \begin{array}{c}\alpha_7^{k-2r-1}\gamma^5 \\ \otimes (\chi\gamma)^{-1} \sqrt{\frac{1}{3}}\, \mathrm{q}_3^r\end{array} & 0 & \begin{array}{c}\frac{1}{3}\alpha_0^{k-2r} \\ \otimes \chi^{-1} \mathrm{q}_4^r \chi\end{array}
\end{array}\right), \quad (152)
$$

where $\mathrm{l}_1^r = (\mathrm{m}_l \mathrm{m}_l^*)^r$, $\mathrm{l}_2^r = -\mathrm{m}_l(\mathrm{m}_l^* \mathrm{m}_l)^r$, $\mathrm{l}_3^r = -\mathrm{m}_l^*(\mathrm{m}_l \mathrm{m}_l^*)^r$, $\mathrm{l}_4^r = (\mathrm{m}_l^* \mathrm{m}_l)^r$,
$\mathrm{q}_1^r = (\mathrm{m}_q \mathrm{m}_q^*)^r$, $\mathrm{q}_2^r = -\mathrm{m}_q(\mathrm{m}_q^* \mathrm{m}_q)^r$, $\mathrm{q}_3^r = -\mathrm{m}_q^*(\mathrm{m}_q \mathrm{m}_q^*)^r$, $\mathrm{q}_4^r = (\mathrm{m}_q^* \mathrm{m}_q)^r$.

In the above formulae ϵ and β are invertible diagonal 3×3–matrices which, therefore, commute with $\mathrm{m}_l, \mathrm{m}_l^*, \mathrm{m}_q, \mathrm{m}_q^*$. For the invertible 3×3–matrices γ and χ we have to demand $(\chi\gamma)^{-1}\mathrm{m}_q^*\mathrm{m}_q\chi\gamma = \chi^{-1}\mathrm{m}_q^*\mathrm{m}_q\chi$, which is achieved by taking $\chi\gamma\chi^{-1}$ diagonal. The matrix χ need not to be unitary.

There are essential differences comparing with the representations used in the Mainz–Marseille model: We do not need reducible indecomposable representations to describe family mixing, because the mass matrices m_l and m_q acting on the generation space $\mathbb{C}^3$ are an intrinsic part of the algebra $\mathcal{A}$ and, therefore, of the graded Lie algebra $\mathcal{H}_0$. The existence of the (compared with the Mainz–Marseille scheme additional) $\mathbb{C}^3$–factor leads to the effect that in our model there occur arbitrary 3×3–matrices $\beta, \gamma, \epsilon, \chi$ in the representation, which correspond to the free relative normalization constants of $sl(2,\mathbb{C}) \oplus gl(1,\mathbb{C})$–subrepresentations occurring in the theory of representations of super Lie algebras [25] and in the Mainz–Marseille model. Hence, our model contains a priory a big number of free parameters, namely the free relative normalization matrices $\beta, \gamma, \epsilon, \chi$ as in the Mainz–Marseille scheme and the parameters of the mass matrix M as in the Connes–Lott scheme. However, there is a subtle interplay between these parameters. They occur only in such combinations that, effectively, we end up with one parameter more than in the model of Connes, Lott and Kastler [27].

In order to construct the fermionic action we must take instead of the above defined canonical Hilbert space H the Hilbert space

$$
\hat{H} = L^2(X, S) \otimes (\,[1, \tfrac{1}{2}] \oplus ([\tfrac{1}{3}, \tfrac{1}{2}] \otimes \mathbb{C}^3)\,) \otimes \mathbb{C}^3 .
$$

The last $\mathbb{C}^3$–factor is a representation space of $\mathrm{End}\,(\hat{F})$, labelling the fermion generations. Although there do not exist representations of the full graded Lie algebra $\mathcal{H}_0$ in $\hat{H}$ (just as in the Mainz–Marseille model), one can easily define a canonical action of elements of $\mathcal{H}_0^k$, $k = 0, 1, 2$, on elements of $\hat{H}$ using the representations (151) and (152). Then, the natural fermionic action is

$$
S_F = (1/2) < \Psi, (D^{c\ell} + \mathbf{i}(i\mu + i\rho))\Psi >_{\hat{H}} + h.c. \; , \quad \Psi \in \hat{H} \; , \qquad (153)
$$

where ρ denotes the connection form, $h.c.$ the Hermitian conjugate of the preceding term, $< \, , \, >_{\hat{H}}$ the canonical scalar product on $\hat{H}$ and μ was given in (71). After a Wick rotation to Minkowski space and imposing the usual chirality condition for the fermions we get precisely the fermionic action of the standard model, where the fermionic mass matrices are[3] [27]

$$\mathrm{m}_e=\tfrac{1}{2}(\epsilon^*-\epsilon^{-1})\mathrm{m}_l^* \ , \quad \mathrm{m}_u=\tfrac{1}{2}\sqrt{\tfrac{2}{3}}(\beta^*+\beta^{-1})\mathrm{m}_q^* \ , \quad \mathrm{m}_d'=\tfrac{1}{2}\sqrt{\tfrac{1}{3}}((\chi\gamma)^*-(\chi\gamma)^{-1})\mathrm{m}_q^* \ , \tag{154}$$

with $e \equiv (e,\mu,\tau)^T$, $u \equiv (u,c,t)^T$, $d \equiv (d,s,b)^T$. The occurrence of the γ^5–factor in elements of $\mathcal{H}_0$ leads to the minus signs in the formulae for m_e and m_d' and the plus sign for m_u . In the model of Connes, Lott and Kastler these γ^5–factors are harmful, because they give a wrong sign in some terms of the fermionic Lagrangian. In our model a different sign due to the γ^5–factors is highly desired, because in (154) this leads in the simplest case $\beta,\gamma,\epsilon,\chi \approx \mathbb{1}_{3\times 3}$ to a mass hierarchy in the sense that the top–quark is much heavier than the bottom–quark and the leptons.

In our construction of the standard model one immediately obtains the correct hypercharges of the fermions – for the same reasons as in the Mainz–Marseille model: The $U(1)$–subgroup of $\mathfrak{U}_0$ acts on both the right–handed and the left–handed fermions (see (118) with $\det \mathbf{u}_1 = \det \mathbf{u}_4$), while the $U(1)$–subgroup of Kastler's electroweak gauge group $SU(2) \times U(1)$ acts only on the right–handed fermions. Therefore, in Kastler's version one must include the algebra $\mathcal{B}_s$ and impose a generalized Poincaré duality condition ([18], [20]), which yields a constraint between the three $U(1)$–subgroups of the local gauge group $\mathcal{U}(\mathcal{A}_s \otimes \mathcal{B}_s) = C^\infty_{\mathbb{R}}(X) \otimes (SU(2) \times U(1) \times SU(3) \times U(1) \times U(1))$ giving the local gauge group $C^\infty_{\mathbb{R}}(X) \otimes (SU(2) \times U(1)_Y \times SU(3))$ of the standard model.

To construct the bosonic electroweak action we first transport[4] the curvature θ by $\mathbf{i}$. In a next step we associate to $\mathbf{i}(\theta)$ in a unique way a bounded operator $\tilde{\theta}$ on the Hilbert space $\hat{H}$. This step is completely analogous to the Connes–Lott prescription and uses the Dixmier trace giving a canonical projection procedure. This projection has for our model the same consequences as in the model of Connes, Lott and Kastler: If there was only one generation of fermions then the Higgs potential would vanish – but manifestly we have three generations. Finally, using again the canonical scalar product $< \, , \, >_{B(\hat{H})}$ on $B(\hat{H})$ induced by the Dixmier trace, one defines the bosonic action as $S_B = < \tilde{\theta}, \tilde{\theta} >_{B(\hat{H})}$. After a Wick rotation and certain reparameterizations this action coincides[5] with the

[3] The matrix m_d' is not diagonal, it can be written as $\mathrm{m}_d' = \mathrm{m}_d V$, where m_d is diagonal and V denotes the Kobayashi–Maskawa matrix

[4] There is a subtle point in transporting $\theta_0 \notin \mathcal{H}_0$

[5] There occurs additionally a cosmological constant in the Lagrangian due to the term θ_0 of the curvature (19), which is typical for models with nontrivial projective modules

classical bosonic electroweak action, with the relations [27]

$$
\begin{aligned}
m_W &= \tfrac{1}{2}\sqrt{\tfrac{1}{6}\,\mathrm{tr}((|\epsilon|^2+|\epsilon|^{-2})|\mathrm{m}_l|^2+\{2(|\beta|^2+|\beta|^{-2})+(|\chi\gamma|^2+|\chi\gamma|^{-2})\}|\mathrm{m}_q|^2)}\ ,\\
m_H &= \tfrac{1}{m_W}\sqrt{\mathrm{tr}(\tfrac{5}{9}|\tilde{\mathrm{m}}_q|^4+\tfrac{1}{3}|\tilde{\mathrm{m}}_l|^4)}\ , \qquad m_Z = m_W \,/\cos\theta_W\ , \qquad (155)
\end{aligned}
$$

where $|\mathrm{m}|^2 := \mathrm{mm}^*$, $|\mathrm{m}|^{-2} := (\mathrm{mm}^*)^{-1}$, $|\tilde{\mathrm{m}}|^4 := (\mathrm{mm}^* - (1/3)\mathrm{tr}\,(\mathrm{mm}^*)\mathbb{1}_{3\times 3})^2$, for a 3×3–matrix m. Thus, the fermion masses and the masses of the W–, Z– and Higgs–bosons depend on both the parameters of the mass matrix M, as in the model of Connes, Lott and Kastler, and on the free relative normalization matrices similar to the Mainz–Marseille model. Therefore, we get relations between boson and fermion masses as in the model of Connes, Lott and Kastler, whereas we recall that such relations cannot be obtained within the Mainz–Marseille scheme. From (154) and (155) one obtains

$$
\sqrt{2}\,m_W < m_t \leq \sqrt{\tfrac{8}{3}}\,m_W\ ,\ \ m_H \leq 2.43\,m_W\ . \qquad (156)
$$

Moreover, one has $(g_3/g_2)^2 = 1$ and $\sin^2\theta_W = 3/8$ as in (148). However, we stress that the relations (148) and (156) are on classical (tree) level, they rather do not survive the renormalization procedure. But there seems to be only a weak scale dependence [1]. The construction of the chromodynamics part is, in principle, identical with the classical theory, because elements of the graded Lie algebra $\mathcal{H}_0$ associated to the module $\mathcal{E}_c$ are $su(3)$–valued differential forms.

In conclusion, the K–cycle $(\mathcal{A},h,D)$ of Connes and Lott can be equally well used for a derivation of the standard model as the K–cycle $(\mathcal{A}_s\otimes\mathcal{B}_s,h_s,D_s)$.

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A Proof of Proposition 8

Since the matrices M_q^t are fixed, any linear mapping $T:\Lambda^*_{\mathcal{A}}\to L^*$ has the form

$$
T\Bigg(\begin{pmatrix} \sum\limits_{t=0}^m \alpha_1^{k-2t}\otimes M_1^t\ ; & \sum\limits_{t=0}^m \alpha_2^{k-2t-1}\gamma^{N+1}\otimes M_2^t \\ \sum\limits_{t=0}^m \alpha_3^{k-2t-1}\gamma^{N+1}\otimes M_3^t\ ; & \sum\limits_{t=0}^m \alpha_4^{k-2t}\otimes M_4^t \end{pmatrix}\Bigg) := \sum_{t=0}^m\sum_{q=1}^4 \mathcal{L}_t^q(\alpha_q^{k-2t-\zeta_q})\ , \qquad (157)
$$

where $\mathcal{L}_t^q$ are arbitrary elements of $\mathrm{End}_{\mathbb{C}}(L^*)$ and $\zeta_1=\zeta_4=0\,,\ \zeta_2=\zeta_3=1\,.$

1. Let $\lambda = \begin{pmatrix} 0 & \alpha_2^{k-2t-1}\gamma^{N+1}\otimes M_2^t \\ 0 & 0\end{pmatrix} \in \Lambda^k_{\mathcal{A}}\,,\ \tilde{\lambda} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\alpha}_4^{l-2r}\otimes M_4^r\end{pmatrix} \in \Lambda^l_{\mathcal{A}}\,,$

then we get from (55) for $0 \leq r, t, t+r \leq m$

$$T(\lambda \bullet \tilde{\lambda} - (-1)^{kl} \tilde{\lambda} \bullet \lambda) = (-1)^l \mathcal{L}^2_{t+r} (\alpha_2^{k-2t-1} \wedge \tilde{\alpha}_4^{l-2r}) \; . \tag{158}$$

According to (60), the r.h.s. of formula (158) must be zero for all α_2^{k-2t-1} and $\tilde{\alpha}_4^{l-2r}$, which can be fulfilled only for $\mathcal{L}^2_t = 0$ for all $t = 0, \dots, m$. Analogously, one obtains $\mathcal{L}^3_t = 0$ for all $t = 0, \dots, m$.
2. For both λ and $\tilde{\lambda}$ being block–diagonal, we find $\lambda \bullet \tilde{\lambda} - (-1)^{kl} \tilde{\lambda} \bullet \lambda = 0$, so that we get no additional condition in this case.
3. Let $\lambda = \begin{pmatrix} 0 & \alpha_2^{k-2t-1} \gamma^{N+1} \otimes M_2^t \\ 0 & 0 \end{pmatrix} \in \Lambda^k_{\mathcal{A}}$, $\tilde{\lambda} = \begin{pmatrix} 0 & 0 \\ \tilde{\alpha}_3^{l-2r-1} \gamma^{N+1} \otimes M_3^r & 0 \end{pmatrix} \in \Lambda^l_{\mathcal{A}}$, then we get from (55) for $0 \leq r, t, t+r+1 \leq m$

$$\begin{aligned} & T\,(\lambda \bullet \tilde{\lambda} - (-1)^{kl} \tilde{\lambda} \bullet \lambda) \\ & \quad = (-1)^{l-1} \mathcal{L}^1_{t+r+1} (\alpha_2^{k-2t-1} \wedge \tilde{\alpha}_3^{l-2r-1}) - (-1)^{kl+k-1} \mathcal{L}^4_{t+r+1} (\tilde{\alpha}_3^{l-2r-1} \wedge \alpha_2^{k-2t-1}) \\ & \quad = (-1)^{l-1} (\mathcal{L}^1_{t+r+1} + \mathcal{L}^4_{t+r+1}) (\alpha_2^{k-2t-1} \wedge \tilde{\alpha}_3^{l-2r-1}) \; . \end{aligned} \tag{159}$$

According to (60), the r.h.s. of formula (159) must be zero for all α_2^{k-2t-1} and $\tilde{\alpha}_3^{l-2r-1}$, which can be fulfilled only for $\mathcal{L}^1_t = -\mathcal{L}^4_t$ for all $t = 1, \dots, m$. However, for $t = 0$ there is no condition between $\mathcal{L}^1_0$ and $\mathcal{L}^4_0$, so that we end up with (62), where so far $\mathcal{L}_t$ are arbitrary elements of $\mathrm{End}_{\mathbb{C}}(L^*)$. Inserting this result into (61) and using (59) we get immediately condition (63). □

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