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The standard model within non–associative geometry

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Abstract

We present the construction of the standard model within the framework of non–associative geometry. For the simplest scalar product we get the tree–level predictions $m_W = \frac{1}{2}m_t$, $m_H = \frac{3}{2}m_t$ and $\sin^2\theta_W = \frac{3}{8}$. These relations differ slightly from predictions derived in non–commutative geometry.

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1. Introduction

One of the most important applications of non–commutative geometry [1] to physics is a unified description of the standard model. The most elegant version rests upon a K–cycle [2, 1] with real structure [3], see [4, 5] for details and [7, 6] for an older version. There also exist numerous other formulations within non–commutative geometry (NCG), see for instance [9, 8]. The author of this paper has proposed in [10] a modification of non–commutative geometry. In that approach one uses unitary Lie algebras instead of unital associative $*$–algebras. Lie algebras are non–associative algebras – this is the motivation for the working title "non–associative geometry". The only realistic physical model that one can construct within the most elegant NCG–prescription is the standard model [11]. The advantage of non–associative geometry is that a larger class of physical models can be constructed from the same amount of structures as in the most elegant NCG–formulation. That class includes the standard model, as we show in this paper.

We give in Section 2 a recipe how to construct classical gauge field theories within non–associative geometry. The arguments why this recipe works can be found in [10]. Section 3 contains the construction of the standard model. We derive the geometric structures and write down the bosonic action for the simplest scalar product. The fermionic action will not be displayed, because it is identical with the classical formulation.

2. The Recipe of Non–associative Geometry

The basic object in non–associative geometry is an L–cycle $(\mathfrak{g}, h, D, \pi, \Gamma)$, which consists of a $*$–representation π of a unitary Lie algebra $\mathfrak{g}$ in bounded operators on a Hilbert space h, together with a selfadjoint operator D on h with compact resolvent and a selfadjoint operator Γ on h, $\Gamma^2 = \mathrm{id}_h$, which commutes with $\pi(\mathfrak{a})$ and anticommutes with D. The operator D may be unbounded, but such that

$[D,\pi(\mathfrak{g})]$ is bounded. L–cycles are naturally related to physical models if the following input data are given:

1) The (Lie) group of local gauge transformations $\mathscr{G}$.
2) Chiral fermions ψ transforming under a representation $\tilde{\pi}$ of $\mathscr{G}$.
3) The fermionic mass matrix $\widetilde{\mathcal{M}}$, i.e. fermion masses plus generalized Kobayashi–Maskawa matrices.
4) In the case of Grand Unified Theories: The symmetry breaking pattern of $\mathscr{G}$.

Take $\mathfrak{g} = C^\infty(X) \otimes \mathfrak{a}$ as the Lie algebra of $\mathscr{G}$, where $\mathfrak{a}$ is a matrix Lie algebra and $C^\infty(X)$ the algebra of smooth functions on the (compact Euclidian) space–time manifold X. Take $h = L^2(X,S) \otimes \mathbb{C}^F$ as the completion of the Euclidian fermions, where $L^2(X,S)$ is the Hilbert space of square integrable bispinors. Take $\pi = \mathrm{id} \otimes \hat{\pi}$ as the differential $\tilde{\pi}_*$, where $\hat{\pi}$ is a representation of $\mathfrak{a}$ in $\mathrm{M}_F\mathbb{C}$. Put $D = \mathsf{D} \otimes \mathbb{1}_F + \gamma^5 \otimes \mathcal{M}$, where D is the Dirac operator on X and $\mathcal{M} \in \mathrm{M}_F\mathbb{C}$ such that $\gamma^5 \otimes \mathcal{M}$ coincides with $\widetilde{\mathcal{M}}$ on chiral fermions. The chirality properties of the fermions are encoded in $\Gamma = \gamma^5 \otimes \hat{\Gamma}$. In Grand Unification models, additional information on the spontaneous symmetry breaking pattern is contained in the part of the mass matrix $\mathcal{M}$ that vanishes on chiral fermions.

The recipe towards the (classical) gauge field theory associated to the L–cycle is the following: Let $\Omega^1\mathfrak{a}$ be the space of formal commutators

$$\omega^1 = \textstyle\sum_{\alpha,z\geq 0}[a^z_\alpha, \dots [a^1_\alpha, da^0_\alpha]\dots] \;, \quad a^i_\alpha \in \mathfrak{a} \;,$$

where d is the universal differential. Apply linear mappings $\hat{\pi} : \Omega^1\mathfrak{a} \to \mathrm{M}_F\mathbb{C}$ and $\hat{\sigma} : \Omega^1\mathfrak{a} \to \mathrm{M}_F\mathbb{C}$ defined by

$$\hat{\pi}(\omega^1) := \textstyle\sum_{\alpha,z\geq 0}[\hat{\pi}(a^z_\alpha), \dots [\hat{\pi}(a^1_\alpha), [-\mathrm{i}\mathcal{M}, \hat{\pi}(a^0_\alpha)]]\dots] \;, \tag{1}$$

$$\hat{\sigma}(\omega^1) := \textstyle\sum_{\alpha,z\geq 0}[\hat{\pi}(a^z_\alpha), \dots [\hat{\pi}(a^1_\alpha), [\mathcal{M}^2, \hat{\pi}(a^0_\alpha)]]\dots] \;. \tag{2}$$

Define $\Omega^n\mathfrak{a} \ni \omega^n = \sum_\alpha [\omega^1_{n,\alpha}, [\omega^1_{n-1,\alpha}, \dots [\omega^1_{2,\alpha}, \omega^1_{1,\alpha}]\dots]]$, where $\omega^1_{i,\alpha} \in \Omega^1\mathfrak{a}$. Extend $\hat{\pi}$ and $\hat{\sigma}$ recursively to $\Omega^n\mathfrak{a}$ by

$$\hat{\pi}([\omega^1, \omega^k]) := \hat{\pi}(\omega^1)\hat{\pi}(\omega^k) - (-1)^k \hat{\pi}(\omega^k)\hat{\pi}(\omega^1) \;,$$

$$\hat{\sigma}([\omega^1, \omega^k]) := \hat{\sigma}(\omega^1)\hat{\pi}(\omega^k) - \hat{\pi}(\omega^k)\hat{\sigma}(\omega^1) - \hat{\pi}(\omega^1)\hat{\sigma}(\omega^k) - (-1)^k \hat{\sigma}(\omega^k)\hat{\pi}(\omega^1) \;.$$

Define for $n \geq 2$

$$\hat{\pi}(\mathcal{J}^n\mathfrak{a}) := \{\; \hat{\sigma}(\omega^{n-1}) \;,\; \omega^{n-1} \in \Omega^{n-1}\mathfrak{a} \cap \ker\hat{\pi} \;\} \;. \tag{3}$$

Define spaces $\mathbb{r}^0\mathfrak{a} \subset \mathrm{M}_F\mathbb{C}$ and $\mathbb{r}^1\mathfrak{a} \subset \mathrm{M}_F\mathbb{C}$ elementwise by

$$\begin{aligned} &\mathbb{r}^0\mathfrak{a} = -(\mathbb{r}^0\mathfrak{a})^* = \hat{\Gamma}(\mathbb{r}^0\mathfrak{a})\hat{\Gamma} \;, && \mathbb{r}^1\mathfrak{a} = -(\mathbb{r}^1\mathfrak{a})^* = -\hat{\Gamma}(\mathbb{r}^1\mathfrak{a})\hat{\Gamma} \;, \\ &[\mathbb{r}^0\mathfrak{a}, \hat{\pi}(\mathfrak{a})] \subset \hat{\pi}(\mathfrak{a}) \;, && [\mathbb{r}^0\mathfrak{a}, \hat{\pi}(\Omega^1\mathfrak{a})] \subset \hat{\pi}(\Omega^1\mathfrak{a}) \;, \\ &\{\mathbb{r}^0\mathfrak{a}, \hat{\pi}(\mathfrak{a})\} \subset \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} + \hat{\pi}(\Omega^2\mathfrak{a}) \;, && \{\mathbb{r}^0\mathfrak{a}, \hat{\pi}(\Omega^1\mathfrak{a})\} \subset \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\Omega^1\mathfrak{a})\} + \hat{\pi}(\Omega^3\mathfrak{a}) \;, \\ &[\mathbb{r}^1\mathfrak{a}, \hat{\pi}(\mathfrak{a})] \subset \hat{\pi}(\Omega^1\mathfrak{a}) \;, && \{\mathbb{r}^1\mathfrak{a}, \hat{\pi}(\Omega^1\mathfrak{a})\} \subset \hat{\pi}(\Omega^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} \;. \end{aligned} \tag{4}$$

Define spaces $\mathbb{j}^0\mathfrak{a}, \mathbb{j}^1\mathfrak{a}, \mathbb{j}^2\mathfrak{a} \subset \mathrm{M}_F\mathbb{C}$ elementwise by

$$\begin{aligned} &\mathbb{j}^0\mathfrak{a} := \mathbb{c}^0\mathfrak{a} \;, && \mathbb{j}^1\mathfrak{a} := \mathbb{c}^1\mathfrak{a} \;, \\ &\mathbb{j}^2\mathfrak{a} := \mathbb{c}^2\mathfrak{a} + \hat{\pi}(\mathcal{J}^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} \;, && \text{where} \\ &\mathbb{c}^0\mathfrak{a} = -(\mathbb{c}^0\mathfrak{a})^* = \hat{\Gamma}(\mathbb{c}^0\mathfrak{a})\hat{\Gamma} \;, && \mathbb{c}^1\mathfrak{a} = -(\mathbb{c}^0\mathfrak{a})^* = -\hat{\Gamma}(\mathbb{c}^0\mathfrak{a})\hat{\Gamma} \;, \end{aligned} \tag{5}$$

$$
\begin{aligned}
&\mathbb{c}^2\mathfrak{a} = (\mathbb{c}^0\mathfrak{a})^* = \hat\Gamma(\mathbb{c}^0\mathfrak{a})\hat\Gamma \;, && \\
&\mathbb{c}^0\mathfrak{a}\cdot\hat\pi(\mathfrak{a}) = 0 \;, && \mathbb{c}^0\mathfrak{a}\cdot\hat\pi(\Omega^1\mathfrak{a}) = 0 \;, \\
&\mathbb{c}^1\mathfrak{a}\cdot\hat\pi(\mathfrak{a}) = 0 \;, && \mathbb{c}^1\mathfrak{a}\cdot\hat\pi(\Omega^1\mathfrak{a}) = 0 \;, \\
&[\mathbb{c}^2\mathfrak{a},\hat\pi(\mathfrak{a})] = 0 \;, && [\mathbb{c}^2\mathfrak{a},\hat\pi(\Omega^1\mathfrak{a})] = 0 \;.
\end{aligned}
$$

The connection form ρ has the structure

$$
\rho = \textstyle\sum_\alpha (c^1_\alpha\otimes m^0_\alpha + c^0_\alpha\gamma^5\otimes m^1_\alpha) \;, \quad c^1_\alpha\in\Lambda^1 \;,\; c^0_\alpha\in\Lambda^0 \;,\; m^0_\alpha\in\mathbb{r}^0\mathfrak{a} \;,\; m^1_\alpha\in\mathbb{r}^1\mathfrak{a} \;, \tag{6}
$$

where Λ^k is the space of differential k–forms represented by gamma matrices. The curvature θ is computed from the connection form ρ by

$$
\begin{aligned}
\theta &= \mathbf{d}\rho + \rho^2 - \mathrm{i}\{\gamma^5\otimes\mathcal{M},\rho\} + \hat\sigma_{\mathfrak{g}}(\rho)\gamma^5 + \mathbb{J}^2\mathfrak{g} \;, \\
\mathbb{J}^2\mathfrak{g} &= (\Lambda^2\otimes\mathbb{j}^0\mathfrak{a})\oplus(\Lambda^1\gamma^5\otimes\mathbb{j}^1\mathfrak{a})\oplus(\Lambda^0\otimes\mathbb{j}^2\mathfrak{a}) \;,
\end{aligned} \tag{7}
$$

where $\mathbf{d}$ is the exterior differential and $\hat\sigma_{\mathfrak{g}}$ the extension of $\hat\sigma$ to elements of the form (6). Select the representative $\mathfrak{e}(\theta)$ orthogonal to $\mathbb{J}^2\mathfrak{g}$, i.e. find $\mathrm{j}\in\mathbb{J}^2\mathfrak{g}$ such that

$$
\mathfrak{e}(\theta) = \mathbf{d}\rho + \rho^2 - \mathrm{i}\{\gamma^5\otimes\mathcal{M},\rho\} + \hat\sigma(\rho)\gamma^5 + \mathrm{j} \;, \qquad \int_X dx\; \mathrm{tr}(\mathfrak{e}(\theta)\mathrm{j}^2) = 0 \quad \forall \mathrm{j}^2\in\mathbb{J}^2\mathfrak{g} \;.
$$

The trace includes the trace in $\mathrm{M}_F\mathbb{C}$ and over gamma matrices. Compute the bosonic and fermionic actions

$$
S_B = \int_X dx\; \frac{1}{g_0^2 F}\,\mathrm{tr}(\mathfrak{e}(\theta)^2) \;, \qquad\qquad S_F = \int_X dx\; \boldsymbol{\psi}^*(D+\mathrm{i}\rho)\boldsymbol{\psi} \;, \tag{8}
$$

where g_0 is a coupling constant and $\boldsymbol{\psi}\in h$. Finally, perform a Wick rotation to Minkowski space.

3. The Construction

Our constructions requires that the mass matrices of all fermions of the same type (including neutrinos) are different from zero and non–degenerated. In particular, the Kobayashi–Maskawa matrix in both the quark and the lepton sector must be non–trivial. This is necessary to avoid certain degeneracy effects. The matrix Lie algebra of the standard model is

$$
\mathfrak{a} = \mathrm{u}(1)\oplus\mathrm{su}(2)\oplus\mathrm{su}(3) \;.
$$

The Hilbert space is $\mathbb{C}^{48}$, because we need right neutrinos. We label elements of $\mathbb{C}^{48}$ in a suggestive way by the fermions of the first generation:

$$
(\boldsymbol{u}_L\,,\boldsymbol{d}_L\,,\boldsymbol{u}_R\,,\boldsymbol{d}_R\,,\nu_L\,,e_L\,,\nu_R\,,e_R)^T\in\mathbb{C}^{48} \;,
$$

where $\boldsymbol{u}_L,\boldsymbol{d}_L,\boldsymbol{u}_R,\boldsymbol{d}_R\in\mathbb{C}^3\otimes\mathbb{C}^3$ and $\nu_L,e_L,\nu_R,e_R\in\mathbb{C}^3$. The representation $\hat\pi$ of $\mathfrak{a}$ on $\mathbb{C}^{48}$ is

$$\hat\pi((a_1,a_2,a_3)) = \tag{9}$$
$$\mathrm{i}f_0\,\mathrm{diag}\,(\tfrac13 \mathbb{1}_3\otimes\mathbb{1}_3\,,\,\tfrac13 \mathbb{1}_3\otimes\mathbb{1}_3\,,\,\tfrac43 \mathbb{1}_3\otimes\mathbb{1}_3\,,\,-\tfrac23 \mathbb{1}_3\otimes\mathbb{1}_3\,,\,-\mathbb{1}_3\,,\,-\mathbb{1}_3\,,\,0_3\,,\,-2\mathbb{1}_3)+$$
$$\left(\begin{array}{c|c}
\begin{array}{cc|c|c}
(a_3+\mathrm{i}f_3\mathbb{1}_3)\otimes\mathbb{1}_3\,; & \mathrm{i}(f_1-\mathrm{i}f_2)\mathbb{1}_3\otimes\mathbb{1}_3 & 0 & 0 \\
\mathrm{i}(f_1+\mathrm{i}f_2)\mathbb{1}_3\otimes\mathbb{1}_3\,; & (a_3-\mathrm{i}f_3\mathbb{1}_3)\otimes\mathbb{1}_3 & 0 & 0 \\ \hline
0 & 0 & a_3\otimes\mathbb{1}_3 & 0 \\ \hline
0 & 0 & 0 & a_3\otimes\mathbb{1}_3
\end{array} & \mathrm{O} \\ \hline
\mathrm{O} & \begin{array}{cc|c|c}
\mathrm{i}f_3\otimes\mathbb{1}_3\,; & \mathrm{i}(f_1-\mathrm{i}f_2)\otimes\mathbb{1}_3 & 0 & 0 \\
\mathrm{i}(f_1+\mathrm{i}f_2)\otimes\mathbb{1}_3\,; & -\mathrm{i}f_3\otimes\mathbb{1}_3 & 0 & 0 \\ \hline
0 & 0 & 0_3 & 0 \\ \hline
0 & 0 & 0 & 0_3
\end{array}
\end{array}\right).$$

Here, the matrix $a_3\in\mathrm{su}(3)\subset\mathrm{M}_3\mathbb{C}$ is written down in the standard matrix representation, $a_2=\begin{pmatrix}\mathrm{i}f_3\,; & \mathrm{i}(f_1-\mathrm{i}f_2)\\ \mathrm{i}(f_1+\mathrm{i}f_2)\,; & -\mathrm{i}f_3\end{pmatrix}\in\mathrm{su}(2)$, for $f_1,f_2,f_3\in\mathbb{R}$, and $a_1=\mathrm{i}f_0\in\mathrm{u}(1)\equiv\mathrm{i}\mathbb{R}$. The generalized Dirac operator is

$$\mathcal{M}=\left(\begin{array}{c|c}
\begin{array}{cc|c|c}
0 & 0 & \mathbb{1}_3\otimes M_u & 0 \\
0 & 0 & 0 & \mathbb{1}_3\otimes M_d \\ \hline
\mathbb{1}_3\otimes M_u^* & 0 & 0 & 0 \\ \hline
0 & \mathbb{1}_3\otimes M_d^* & 0 & 0
\end{array} & \mathrm{O} \\ \hline
\mathrm{O} & \begin{array}{cc|c|c}
0 & 0 & M_\nu & 0 \\
0 & 0 & 0 & M_e \\ \hline
M_\nu^* & 0 & 0 & 0 \\ \hline
0 & M_e^* & 0 & 0
\end{array}
\end{array}\right),$$

where $M_u,M_d,M_\nu,M_e\in\mathrm{M}_3\mathbb{C}$ are the mass matrices of the fermions. It is easy to see that for $a^i_\alpha=(a^i_{1,\alpha},a^i_{2,\alpha},a^i_{3,\alpha})\in\mathfrak{a}$ one has

$$\tau^1:=\textstyle\sum_{\alpha,z\geq0}[\hat\pi(a^z_\alpha),\dots[\hat\pi(a^1_\alpha),[-\mathrm{i}\mathcal{M},\hat\pi(a^0_\alpha)]]\dots]= \tag{10}$$
$$\mathrm{i}\left(\begin{array}{c|c}
\begin{array}{cc|c|c}
0 & 0 & \bar b_2\mathbb{1}_3\otimes M_u & b_1\mathbb{1}_3\otimes M_d \\
0 & 0 & -\bar b_1\mathbb{1}_3\otimes M_u & b_2\mathbb{1}_3\otimes M_d \\ \hline
b_2\mathbb{1}_3\otimes M_u^*\,; & -b_1\mathbb{1}_3\otimes M_u^* & 0 & 0 \\ \hline
\bar b_1\mathbb{1}_3\otimes M_d^*\,; & \bar b_2\mathbb{1}_3\otimes M_d^* & 0 & 0
\end{array} & \mathrm{O} \\ \hline
\mathrm{O} & \begin{array}{cc|c|c}
0 & 0 & \bar b_2\otimes M_\nu & b_1\otimes M_e \\
0 & 0 & -\bar b_1\otimes M_\nu & b_2\otimes M_e \\ \hline
b_2\otimes M_\nu^*\,; & -b_1\otimes M_\nu^* & 0 & 0 \\ \hline
\bar b_1\otimes M_e^*\,; & \bar b_2\otimes M_e^* & 0 & 0
\end{array}
\end{array}\right),$$

$$\begin{pmatrix}b_1\\ b_2\end{pmatrix}=\sum_{\alpha,z\geq0}a^z_{2,\alpha}a^z_{1,\alpha}\cdots a^1_{2,\alpha}a^1_{1,\alpha}a^0_{2,\alpha}a^0_{1,\alpha}\begin{pmatrix}0\\ 1\end{pmatrix}\in\mathbb{C}^2\;. \tag{11}$$

The matrix (10) is the general form of an element of $\hat\pi(\Omega^1\mathfrak{a})$. The grading operator is

$$\hat\Gamma=\mathrm{diag}\,(-\mathbb{1}_3\otimes\mathbb{1}_3\,,\,-\mathbb{1}_3\otimes\mathbb{1}_3\,,\,\mathbb{1}_3\otimes\mathbb{1}_3\,,\,\mathbb{1}_3\otimes\mathbb{1}_3\,,\,-\mathbb{1}_3\,,\,-\mathbb{1}_3\,,\,\mathbb{1}_3\,,\,\mathbb{1}_3)\;.$$

One has $\hat\Gamma^2=\mathbb{1}_{48}$, $[\hat\Gamma,\hat\pi(\mathfrak{a})]=0$, $\{\hat\Gamma,\mathcal{M}\}=0$ and $\{\hat\Gamma,\hat\pi(\Omega^1\mathfrak{a})\}=0$. Let

$$\begin{aligned}
&\begin{pmatrix}\mathrm{i}f_3\,; & \mathrm{i}(f_1-\mathrm{i}f_2)\\ \mathrm{i}(f_1+\mathrm{i}f_2)\,; & -\mathrm{i}f_3\end{pmatrix}:=\begin{pmatrix}\mathrm{i}(|b_2|^2-|b_1|^2)\,; & -2\mathrm{i}b_1\bar b_2\\ -2\mathrm{i}\bar b_1 b_2\,; & -\mathrm{i}(|b_2|^2-|b_1|^2)\end{pmatrix}\in\mathrm{su}(2)\;,\\
&M_{ud}=M_uM_u^*-M_dM_d^*\;,\;\;M_{\nu e}=M_\nu M_\nu^*-M_eM_e^*\;,\\
&M_{\{ud\}}=M_uM_u^*+M_dM_d^*\;,\;\;M_{\{\nu e\}}=M_\nu M_\nu^*+M_eM_e^*\;.
\end{aligned}$$

Then we have

$$\{\tau^1,\tau^1\} = \mathrm{i}\left(\begin{array}{cc|c|c|cc|c|c} \mathrm{i}f_3\mathbb{1}_3\otimes M_{ud}\,; & \mathrm{i}(f_1-\mathrm{i}f_2)\mathbb{1}_3\otimes M_{ud} & 0 & 0 & & & & \\ \mathrm{i}(f_1+\mathrm{i}f_2)\mathbb{1}_3\otimes M_{ud}\,; & -\mathrm{i}f_3\mathbb{1}_3\otimes M_{ud} & 0 & 0 & & \mathrm{O} & & \\ \hline 0 & 0 & 0_9 & 0 & & & & \\ \hline 0 & 0 & 0 & 0_9 & & & & \\ \hline & & & & \mathrm{i}f_3\otimes M_{\nu e}\,; & \mathrm{i}(f_1-\mathrm{i}f_2)\otimes M_{\nu e} & 0 & 0 \\ & \mathrm{O} & & & \mathrm{i}(f_1+\mathrm{i}f_2)\otimes M_{\nu e}\,; & -\mathrm{i}f_3\otimes M_{\nu e} & 0 & 0 \\ \hline & & & & 0 & 0 & 0_3 & 0 \\ \hline & & & & 0 & 0 & 0 & 0_3 \end{array}\right) \tag{12}$$

$$-(\|b_1\|^2+\|b_2\|^2)\,\mathrm{diag}\big(\mathbb{1}_3\otimes M_{\{ud\}}\,,\,\mathbb{1}_3\otimes M_{\{ud\}}\,,\,\mathbb{1}_3\otimes 2M_u^*M_u\,,\,\mathbb{1}_3\otimes 2M_d^*M_d\,, \\ M_{\{\nu e\}}\,,\,M_{\{\nu e\}}\,,\,2M_\nu^*M_\nu\,,\,2M_e^*M_e\big)\;. \tag{13}$$

Next, for $\tau^1=\hat\pi(\omega^1)$ given by (10) we obtain with (2)

$$\hat\sigma(\omega^1)=\textstyle\sum_{\alpha,z\geq 0}[\hat\pi(a^z_\alpha),\dots[\hat\pi(a^1_\alpha),[\mathcal{M}^2,\hat\pi(a^0_\alpha)]]\dots] = \tag{14}$$

$$\mathrm{i}\left(\begin{array}{cc|c|c|cc|c|c} \mathrm{i}f_3\mathbb{1}_3\otimes M_{ud}\,; & \mathrm{i}(f_1-\mathrm{i}f_2)\mathbb{1}_3\otimes M_{ud} & 0 & 0 & & & & \\ \mathrm{i}(f_1+\mathrm{i}f_2)\mathbb{1}_3\otimes M_{ud}\,; & -\mathrm{i}f_3\mathbb{1}_3\otimes M_{ud} & 0 & 0 & & \mathrm{O} & & \\ \hline 0 & 0 & 0_9 & 0 & & & & \\ \hline 0 & 0 & 0 & 0_9 & & & & \\ \hline & & & & \mathrm{i}f_3\otimes M_{\nu e}\,; & \mathrm{i}(f_1-\mathrm{i}f_2)\otimes M_{\nu e} & 0 & 0 \\ & \mathrm{O} & & & \mathrm{i}(f_1+\mathrm{i}f_2)\otimes M_{\nu e}\,; & -\mathrm{i}f_3\otimes M_{\nu e} & 0 & 0 \\ \hline & & & & 0 & 0 & 0_3 & 0 \\ \hline & & & & 0 & 0 & 0 & 0_3 \end{array}\right),$$

$$\begin{pmatrix}\mathrm{i}f_3\,; & \mathrm{i}(f_1-\mathrm{i}f_2)\\ \mathrm{i}(f_1+\mathrm{i}f_2)\,; & -\mathrm{i}f_3\end{pmatrix} := \sum_{\alpha,z\geq 0}\big[a^z_{2,\alpha},\dots\big[a^1_{2,\alpha},\big[a^0_{2,\alpha},\begin{pmatrix}\frac{\mathrm{i}}{2} & 0\\ 0 & -\frac{\mathrm{i}}{2}\end{pmatrix}\big]\big]\dots\big]\in\mathrm{su}(2)\;.$$

Choosing

$$\omega^1_0 = da^0_2+[a^1_2,[a^1_2,da^0_2]]\;,\quad a^0_2=\begin{pmatrix}0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}\in\mathrm{su}(2)\;,\quad a^1_2=\begin{pmatrix}\mathrm{i} & 0\\ 0 & -\mathrm{i}\end{pmatrix}\in\mathrm{su}(2)\;,$$

we have $\omega^1_0\in\ker\hat\pi$ due to $(a^0_2+a^1_2a^1_2a^0_2)\binom{0}{1}=\binom{0}{0}$, see (11). On the other hand, $\hat\sigma(\omega^1)\neq 0$ is the matrix (14), with $f_1=0, f_2=-3, f_3=0$. Obviously, each matrix of the form (14) can be represented as $\hat\sigma(\omega^1)$, for $\omega^1=\sum_{\alpha,z\geq 0}[a^z_\alpha,\dots[a^0_\alpha,\omega^1_0]\dots]\in\ker\hat\pi$. Therefore, each element of $\hat\pi(\mathcal{J}^2\mathfrak{a})$ is precisely of the form (14), see (3):

$$\hat\sigma(\Omega^1\mathfrak{a})\equiv\hat\pi(\mathcal{J}^2\mathfrak{a})\;. \tag{15}$$

Comparing the results (15) and (14) with (12) and (13) we get

$$\{\tau^1,\tau^1\}=-(\|b_1\|^2+\|b_2\|^2)\,\mathrm{diag}\big(\mathbb{1}_3\otimes M_{\{ud\}}\,,\mathbb{1}_3\otimes M_{\{ud\}}\,,\mathbb{1}_3\otimes 2M_u^*M_u\,,\mathbb{1}_3\otimes 2M_d^*M_d\,, \\ M_{\{\nu e\}}\,,M_{\{\nu e\}}\,,2M_\nu^*M_\nu\,,2M_e^*M_e\big)\quad\mathrm{mod}\;\hat\pi(\mathcal{J}^2\mathfrak{a}) \tag{16}$$

It is clear that the representative chosen in (16) is orthogonal to $\hat\pi(\mathcal{J}^2\mathfrak{a})$. One can prove that

$$\hat\pi(\Omega^n\mathfrak{a})=\hat\pi(\mathcal{J}^n\mathfrak{a})\quad\text{for all } n\geq 3\;. \tag{17}$$

Next, we need the structure of the space $\{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\}$. A simple calculation yields for elements of $\{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\}$ the form

$$\{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\}\ni\mathrm{diag}\,(A_q+\Delta_q\,,\,A_\ell+\Delta_\ell)\;,$$

$$
\begin{aligned}
A_q &= \sum_\alpha \mathrm{i} \left(\begin{array}{cc|c|c}
\begin{bmatrix}(\frac{1}{3}\hat{\lambda}^0_\alpha+\lambda^3_\alpha+\lambda^0_\alpha)a_{3,\alpha} \\ +\frac{1}{3}\mathrm{i}\hat{\lambda}^3_\alpha \mathbb{1}_3\end{bmatrix} & \begin{bmatrix}(\lambda^1_\alpha-\mathrm{i}\lambda^2_\alpha)a_{3,\alpha}+ \\ \frac{1}{3}\mathrm{i}(\hat{\lambda}^1_\alpha-\mathrm{i}\hat{\lambda}^2_\alpha)\mathbb{1}_3\end{bmatrix} & 0 & 0 \\
\begin{bmatrix}(\lambda^1_\alpha+\mathrm{i}\lambda^2_\alpha)a_{3,\alpha}+ \\ \frac{1}{3}\mathrm{i}(\hat{\lambda}^1_\alpha+\mathrm{i}\hat{\lambda}^2_\alpha)\mathbb{1}_3\end{bmatrix} & \begin{bmatrix}(\frac{1}{3}\hat{\lambda}^0_\alpha-\lambda^3_\alpha+\lambda^0_\alpha)a_{3,\alpha} \\ -\frac{1}{3}\mathrm{i}\hat{\lambda}^3_\alpha \mathbb{1}_3\end{bmatrix} & 0 & 0 \\ \hline
0 & 0 & (\lambda^0_\alpha+\frac{4}{3}\hat{\lambda}^0_\alpha)a_{3,\alpha} & 0 \\ \hline
0 & 0 & 0 & (\lambda^0_\alpha-\frac{2}{3}\hat{\lambda}^0_\alpha)a_{3,\alpha}
\end{array}\right) \otimes \mathbb{1}_3 , \\
A_\ell &= \sum_\alpha \mathrm{i} \left(\begin{array}{cc|c|c}
-\mathrm{i}\hat{\lambda}^3_\alpha \mathbb{1}_3 ; & -\mathrm{i}(\hat{\lambda}^1_\alpha - \mathrm{i}\hat{\lambda}^2_\alpha)\mathbb{1}_3 & 0 & 0 \\
-\mathrm{i}(\hat{\lambda}^1_\alpha + \mathrm{i}\hat{\lambda}^2_\alpha)\mathbb{1}_3 & \mathrm{i}\hat{\lambda}^3_\alpha \mathbb{1}_3 & 0 & 0 \\ \hline
0 & 0 & 0_3 & 0 \\ \hline
0 & 0 & 0 & 0_3
\end{array}\right) , \qquad (18) \\
\Delta_q &= \mathrm{diag}\big((\lambda+\tilde{\lambda}+\tfrac{1}{9}\hat{\lambda})\mathbb{1}_3, (\lambda+\tilde{\lambda}+\tfrac{1}{9}\hat{\lambda})\mathbb{1}_3, (\lambda+\tfrac{16}{9}\hat{\lambda})\mathbb{1}_3, (\lambda+\tfrac{4}{9}\hat{\lambda})\mathbb{1}_3\big) \otimes \mathbb{1}_3 , \\
\Delta_\ell &= \mathrm{diag}\big((\tilde{\lambda}+\hat{\lambda})\mathbb{1}_3 , (\tilde{\lambda}+\hat{\lambda})\mathbb{1}_3 , 0_3 , 4\hat{\lambda}\mathbb{1}_3\big) ,
\end{aligned}
$$

where $a_{3,\alpha} \in \mathrm{su}(3)$ and $\lambda^0_\alpha, \lambda^1_\alpha, \lambda^2_\alpha, \lambda^3_\alpha, \hat{\lambda}^0_\alpha, \hat{\lambda}^1_\alpha, \hat{\lambda}^2_\alpha, \hat{\lambda}^3_\alpha, \lambda, \tilde{\lambda}, \hat{\lambda} \in \mathbb{R}$.

In order to write down the structure of the connection form we must find the spaces $\mathbb{r}^0\mathfrak{a}$ and $\mathbb{r}^1\mathfrak{a}$, see (6). The evaluation of (4) yields in the case of generic mass matrices M_u, M_d, M_ν, M_e the simple result

$$
\mathbb{r}^0\mathfrak{a} = \hat{\pi}(\mathfrak{a}) , \qquad\qquad \mathbb{r}^1\mathfrak{a} = \hat{\pi}(\Omega^1\mathfrak{a}) .
$$

In the course of this evaluation it is essential that a non–trivial Kobayashi–Maskawa matrix occurs in both the quark and lepton sectors. Otherwise analogous particles of different generations could have different electric charges. In particular, right neutrinos must exist. Since it is also necessary to invert the mass matrices M_u, M_d, M_ν, M_e, all neutrinos must be massive. For generic mass matrices, equations (5) have the solution $\mathbb{j}^0\mathfrak{a} = 0$, $\mathbb{j}^1\mathfrak{a} = 0$ and

$$
\begin{aligned}
\mathbb{j}^2\mathfrak{a} &= \hat{\pi}(\mathcal{J}^2\mathfrak{a}) \oplus \big(\{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} + \mathrm{diag}\,(\mathbb{R}\mathbb{1}_{36} , \mathbb{R}\mathbb{1}_{12})\big) \\
&\ni J_2 \oplus \mathrm{diag}\,(A_q , A_\ell) \oplus \mathrm{diag}\,(J_q , J_\ell) , \\
J_q &= \mathrm{diag}\big((\lambda_1+\tfrac{1}{9}\lambda_0)\mathbb{1}_3 , (\lambda_1+\tfrac{1}{9}\lambda_0)\mathbb{1}_3 , (\lambda_1+\lambda_3+\tfrac{16}{9}\lambda_0)\mathbb{1}_3 , (\lambda_1+\lambda_3+\tfrac{4}{9}\lambda_0)\mathbb{1}_3\big) \otimes \mathbb{1}_3 , \\
J_\ell &= \mathrm{diag}\big((\lambda_2+\lambda_0)\mathbb{1}_3 , (\lambda_2+\lambda_0)\mathbb{1}_3 , (\lambda_2+\lambda_3)\mathbb{1}_3 , (\lambda_2+\lambda_3+4\lambda_0)\mathbb{1}_3\big) , \qquad (19)
\end{aligned}
$$

for $J_2 \in \hat{\pi}(\mathcal{J}^2\mathfrak{a})$ and $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$.

In order to write down the bosonic action it is necessary to select the representative $\mathfrak{e}(\{\tau^1, \tau^1\})$ of $\{\tau^1, \tau^1\} + \mathbb{j}^2\mathfrak{a}$ orthogonal to $\mathbb{j}^2\mathfrak{a}$. This problem is easy to solve. Let

$$
\begin{aligned}
\tilde{M}_{\{ud\}} &:= M_u M_u^* + M_d M_d^* - \tfrac{1}{3}\mathrm{tr}\,(M_u M_u^* + M_d M_d^*)\mathbb{1}_3 , \\
\tilde{M}_{\{\nu e\}} &:= M_\nu M_\nu^* + M_e M_e^* - \tfrac{1}{3}\mathrm{tr}\,(M_\nu M_\nu^* + M_e M_e^*)\mathbb{1}_3 , \\
\tilde{M}_{uu} &:= M_u^* M_u - \tfrac{1}{24}\mathrm{tr}\,(5M_u M_u^* + 3M_d M_d^* - M_\nu M_\nu^* + M_e M_e^*)\mathbb{1}_3 , \\
\tilde{M}_{dd} &:= M_d^* M_d - \tfrac{1}{24}\mathrm{tr}\,(3M_u M_u^* + 5M_d M_d^* + M_\nu M_\nu^* - M_e M_e^*)\mathbb{1}_3 , \\
\tilde{M}_{\nu\nu} &:= M_\nu^* M_\nu - \tfrac{1}{24}\mathrm{tr}\,(-3M_u M_u^* + 3M_d M_d^* + 7M_\nu M_\nu^* + M_e M_e^*)\mathbb{1}_3 , \\
\tilde{M}_{ee} &:= M_e^* M_e - \tfrac{1}{24}\mathrm{tr}\,(3M_u M_u^* - 3M_d M_d^* + M_\nu M_\nu^* + 7M_e M_e^*)\mathbb{1}_3 .
\end{aligned}
$$

Then, the canonical embedding $\mathfrak{e}(\{\tau^1, \tau^1\})$ of $\{\tau^1, \tau^1\}$ into $\mathrm{M}_{48}\mathbb{C}$ is given by

$$
\begin{aligned}
\mathfrak{e}(\{\tau^1, \tau^1\}) = -(\|b_1\|^2 + \|b_2\|^2)\,\mathrm{diag}\big(&\mathbb{1}_3 \otimes \tilde{M}_{\{ud\}} , \mathbb{1}_3 \otimes \tilde{M}_{\{ud\}} , \mathbb{1}_3 \otimes 2\tilde{M}_{uu} , \mathbb{1}_3 \otimes 2\tilde{M}_{dd} , \\
&\tilde{M}_{\{\nu e\}} , \tilde{M}_{\{\nu e\}} , 2\tilde{M}_{\nu\nu} , 2\tilde{M}_{ee}\big) . \qquad (20)
\end{aligned}
$$

Now we include the four dimensional Riemannian spin manifold X and choose a selfadjoint local basis $\{\gamma^\mu\}_{\mu=1,2,3,4}$ of Λ^1. The connection form ρ has due to (6), (9) and (10) the structure

$$\rho = \begin{pmatrix} \rho_q & 0 \\ 0 & \rho_\ell \end{pmatrix} , \tag{21}$$

$$\rho_q = \left(\begin{array}{cc|c|c} (\mathbf{A}+\mathrm{i}(\frac{1}{3}A^0+A^3)\mathbb{1}_3)\otimes\mathbb{1}_3 ; & \mathrm{i}(A^1-\mathrm{i}A^2)\mathbb{1}_3\otimes\mathbb{1}_3 & -\mathrm{i}\gamma^5\bar{\Phi}_2\mathbb{1}_3\otimes M_u & -\mathrm{i}\gamma^5\Phi_1\mathbb{1}_3\otimes M_d \\ \mathrm{i}(A^1+\mathrm{i}A^2)\mathbb{1}_3\otimes\mathbb{1}_3 ; & (\mathbf{A}+\mathrm{i}(\frac{1}{3}A^0-A^3)\mathbb{1}_3)\otimes\mathbb{1}_3 & \mathrm{i}\gamma^5\bar{\Phi}_1\mathbb{1}_3\otimes M_u & -\mathrm{i}\gamma^5\Phi_2\mathbb{1}_3\otimes M_d \\ \hline -\mathrm{i}\gamma^5\Phi_2\mathbb{1}_3\otimes M_u^* & \mathrm{i}\gamma^5\Phi_1\mathbb{1}_3\otimes M_u^* & (\mathbf{A}+\frac{4}{3}\mathrm{i}A^0\mathbb{1}_3)\otimes\mathbb{1}_3 & 0 \\ \hline -\mathrm{i}\gamma^5\bar{\Phi}_1\mathbb{1}_3\otimes M_d^* & -\mathrm{i}\gamma^5\bar{\Phi}_2\mathbb{1}_3\otimes M_d^* & 0 & (\mathbf{A}-\frac{2}{3}\mathrm{i}A^0\mathbb{1}_3)\otimes\mathbb{1}_3 \end{array}\right),$$

$$\rho_\ell = \left(\begin{array}{cc|c|c} \mathrm{i}(-A^0+A^3)\otimes\mathbb{1}_3 & \mathrm{i}(A^1-\mathrm{i}A^2)\otimes\mathbb{1}_3 & -\mathrm{i}\gamma^5\bar{\Phi}_2\otimes M_\nu & -\mathrm{i}\gamma^5\Phi_1\otimes M_e \\ \mathrm{i}(A^1+\mathrm{i}A^2)\otimes\mathbb{1}_3 & \mathrm{i}(-A^0-A^3)\otimes\mathbb{1}_3 & \mathrm{i}\gamma^5\bar{\Phi}_1\otimes M_\nu & -\mathrm{i}\gamma^5\Phi_2\otimes M_e \\ \hline -\mathrm{i}\gamma^5\Phi_2\otimes M_\nu^* & \mathrm{i}\gamma^5\Phi_1\otimes M_\nu^* & 0_3 & 0 \\ \hline -\mathrm{i}\gamma^5\bar{\Phi}_1\otimes M_e^* & -\mathrm{i}\gamma^5\bar{\Phi}_2\otimes M_e^* & 0 & -2\mathrm{i}A^0\otimes\mathbb{1}_3 \end{array}\right),$$

where $\mathbf{A} \in \Lambda^1\otimes \mathrm{su}(3)$, $\tilde{\mathbf{A}} := \begin{pmatrix} \mathrm{i}A^3 ; & \mathrm{i}(A^1-\mathrm{i}A^2) \\ \mathrm{i}(A^1+\mathrm{i}A^2); & -\mathrm{i}A^3 \end{pmatrix} \in \Lambda^1\otimes\mathrm{su}(2)$, $A^0 \in \Lambda^1$ and $\Phi_1,\Phi_2 \in \Lambda^0\otimes\mathbb{C}$. In formula (7) for the curvature note that $\hat{\sigma}(\omega^1) = 0 \mod \hat{\pi}(\mathcal{J}^2\mathfrak{a})$. Inserting (21) into (7) we obtain for the bosonic action given in (8)

$$\begin{aligned} S_B &= \tfrac{1}{48g_0^2}\int_X dx\; \mathrm{tr}(\mathfrak{e}(\theta)^2) = \int_X dx\; (\mathscr{L}_2+\mathscr{L}_1+\mathscr{L}_0) , \\ \mathscr{L}_2 &= \tfrac{1}{4g_0^2}\mathrm{tr}\big((\mathbf{dA}+\tfrac{1}{2}\{\mathbf{A},\mathbf{A}\})^2\big) + \tfrac{1}{4g_0^2}\mathrm{tr}\big((\mathbf{d}\tilde{\mathbf{A}}+\tfrac{1}{2}\{\tilde{\mathbf{A}},\tilde{\mathbf{A}}\})^2\big) + \tfrac{5}{6g_0^2}\mathrm{tr}\big((\mathbf{d}A^0)^2\big) , \\ \mathscr{L}_1 &= \tfrac{1}{24g_0^2}\mathrm{tr}\big(|\mathbf{d}\Phi_1+\mathrm{i}(A^0+A^3)\Phi_1+\mathrm{i}(A^1-\mathrm{i}A^2)(\Phi_2+1)|^2 + \\ &\qquad +|\mathbf{d}\Phi_2+\mathrm{i}(A^0-A^3)(\Phi_2+1)+\mathrm{i}(A^1+\mathrm{i}A^2)\Phi_1|^2\big)\times \\ &\qquad\qquad \times\mathrm{tr}(3M_uM_u^*+3M_dM_d^*+M_\nu M_\nu^*+M_eM_e^*) , \\ \mathscr{L}_0 &= \tfrac{1}{192g_0^2}(|\Phi_1|^2+|\Phi_2+1|^2-1)^2\mathrm{tr}(1)\times \\ &\qquad\qquad \times\mathrm{tr}(6\tilde{M}^2_{\{ud\}}+12\tilde{M}^2_{uu}+12\tilde{M}^2_{dd}+2\tilde{M}^2_{\{\nu e\}}+4\tilde{M}^2_{\nu\nu}+4\tilde{M}^2_{ee}) . \end{aligned} \tag{22}$$

We perform the reparameterizations

$$\begin{aligned} &\mathbf{A} = \textstyle\sum_{a=1}^8 \frac{\mathrm{i}g_0}{2}G^a_\mu\gamma^\mu\otimes\lambda^a , \quad \tilde{\mathbf{A}} = \sum_{a=1}^3\frac{\mathrm{i}g_0}{2}W^a_\mu\gamma^\mu\otimes\sigma^a , \quad A^0 = \frac{\mathrm{i}g_0}{2}\sqrt{\frac{3}{5}}W^0_\mu\gamma^\mu , \\ &\Phi_i = g_0\phi_i/\sqrt{\mathrm{tr}(M_uM_u^*+M_dM_d^*+\tfrac{1}{3}M_\nu M_\nu^*+\tfrac{1}{3}M_eM_e^*)} , \; i=1,2 , \end{aligned} \tag{23}$$

where $\{\sigma^a\}$ are the Pauli matrices and $\{\lambda^a\}$ the Gell-Mann matrices. Using

$$\mathrm{tr}\big((\gamma^\kappa\wedge\gamma^\lambda)(\gamma^\mu\wedge\gamma^\nu)\big) = 4(\delta^{\lambda\mu}\delta^{\kappa\nu}-\delta^{\kappa\mu}\delta^{\lambda\nu}) , \quad \mathrm{tr}(\gamma^\mu\gamma^\nu) = 4\delta^{\mu\nu} , \quad \mathrm{tr}(1) = 4$$

and performing a Wick rotation to Minkowski space we obtain for (22) precisely the bosonic action of the standard model, see [12]. The Weinberg angle θ_W and the masses m_W, m_Z and m_H of the W, Z and Higgs bosons are given by

$$\begin{aligned} m_W &= \tfrac{1}{2}\sqrt{\mathrm{tr}(M_uM_u^*+M_dM_d^*+\tfrac{1}{3}M_\nu M_\nu^*+\tfrac{1}{3}M_eM_e^*)} = \tfrac{1}{2}m_t , \\ m_Z &= m_W/\cos\theta_W , \qquad \sin^2\theta_W = \tfrac{3}{8} , \\ m_H &= \sqrt{\frac{\mathrm{tr}(\tilde{M}^2_{\{ud\}}+2\tilde{M}^2_{uu}+2\tilde{M}^2_{dd}+\frac{1}{3}\tilde{M}^2_{\{\nu e\}}+\frac{2}{3}M^2_{\nu\nu}+\frac{2}{3}M_{ee})}{\mathrm{tr}(M_uM_u^*+M_dM_d^*+\frac{1}{3}M_\nu M_\nu^*+\frac{1}{3}M_eM_e^*)}} = \tfrac{3}{2}m_t , \end{aligned} \tag{24}$$

where m_t is the mass of the top quark. Here we have neglected the other fermion masses against m_t. The analogous relations in non–commutative geometry read for the simplest scalar product [6]

$$m_W = \tfrac{1}{2} m_t \;, \qquad m_H = \sqrt{\tfrac{69}{28}}\, m_t \;, \qquad \sin^2 \theta_W = \tfrac{12}{29} \;. \tag{25}$$

Inserting (21) and (23) into the fermionic action in (8) we arrive after a Wick rotation to Minkowski space and imposing the chirality condition $\Gamma h = h$ at the usual fermionic action of the standard model [12]. In our model, general scalar products as discussed within NCG for instance in [4] are possible as well. They correspond to different weights of quark and lepton contributions to the entire bosonic action.

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