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On the vacuum states for non-commutative gauge theory[a]

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Abstract. Candidates for renormalizable gauge theory models on Moyal spaces constructed recently have non-trivial vacua. We show that these models support vacuum states that are invariant under both global rotations and symplectic isomorphisms which form a global symmetry group for the action. We compute the explicit expression in position space for these vacuum configurations in two and four dimensions.

1 Introduction

A new family of non-commutative (NC) field theories [1–3] (for details on non-commutative geometry, see [4, 5]) came under increasing scrutiny after 1998 when it was realized [6, 7] that string theory seems to have some effective regimes described by non-commutative field theories (NCFT) defined on a simple NC version of flat four-dimensional space. This latter is the so-called Moyal–Weyl space (for a mathematical description see e.g. [8, 9]), which has constant commutators between space coordinates. NCFT on Moyal spaces were also shown to be the ones whose non-relativistic counterparts correspond to many body quantum theory in strong external magnetic field (see e.g. [10, 11]).

This growing interest received however a blow when it was noticed [12, 13] that the simplest NC φ^4 model (φ real-valued) on four-dimensional Moyal space is not renormalizable due to the occurrence of a phenomenon called ultraviolet/infrared mixing [12–14]. This phenomenon basically results from the existence of some non-planar diagrams which are ultraviolet finite but nevertheless develop infrared singularities which, when inserted into higher order diagrams, are not of the renormalizable type [1–3]. A possible solution to this problem, hereafter called the "harmonic solution", was proposed in 2004 [15, 16] (see also [17–19]); it basically amounts to supplementing the initial action with a simple harmonic oscillator term leading to a fully renormalizable NCFT. For recent reviews, see e.g. [20, 21]. This result seems to be related to the covariance of the model under a new symmetry, called Langmann–Szabo duality [22], which, roughly speaking, exchanges coordinates and momenta. Other renormalizable non-commutative matter field theories on Moyal spaces have subsequently been identified [23–26] and some studies of the properties of the corresponding renormalization group flows have been carried out [27–29], exhibiting in particular the vanishing of the β-function to all orders for the φ^4_4 model [30].

So far, the construction of a fully renormalizable gauge theory on four-dimensional Moyal spaces remains a challenging problem. Recall that the naive non-commutative version of the pure Yang–Mills action on Moyal spaces given by $S_0 = \frac{1}{4}\int d^4x(F_{\mu\nu}\star F_{\mu\nu})(x)$ (in the standard notation and conventions recalled in Sect. 2) suffers from UV/IR mixing, which makes its renormalizability unlikely. This basically stems from the occurrence of an IR singularity in the one-loop polarization tensor $\omega_{\mu\nu}(p)$ (p is some external momentum). Indeed, by standard calculation one easily infers that

$$\omega_{\mu\nu}(p) \sim \frac{(D-2)}{4}\Gamma\left(\frac{D}{2}\right)\frac{\tilde{p}_\mu\tilde{p}_\nu}{\pi^{D/2}\left(\tilde{p}^2\right)^{D/2}} + \cdots \quad p\to 0\,, \tag{1}$$

where $\tilde{p}_\mu \equiv 2\Theta^{-1}_{\mu\nu}p_\nu$ and $\Gamma(z)$ denotes the Euler function. Notice that this singularity, albeit obviously transverse in the sense of the Slavnov–Taylor–Ward identities, does not correspond to some gauge invariant term. This implies that the recent alternative solution to the UV/IR mixing proposed within the NC φ^4 model in [31], which roughly amounts to balancing the IR singularity through a counterterm having a similar form, cannot be extended straightforwardly (if possible at all) to the case of gauge theories.

It turns out that the extension of the harmonic solution to the case of gauge theories has been achieved recently in [32, 33] (see also [34, 35]), starting basically from a computation of the one-loop effective gauge action obtained by integrating out the matter degree of freedom of a NC matter field theory with harmonic term similar to the one used in [15, 16] minimally coupled to an external gauge potential. Both analyses have singled out, as a

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possible candidate for renormalizable gauge theory defined on (four-dimensional) Moyal space, the following generic action:

$$S=\int \mathrm{d}^4x\left(\frac{1}{4}F_{\mu\nu}\star F_{\mu\nu}+\frac{\Omega^2}{4}\{\mathcal{A}_\mu,\mathcal{A}_\nu\}^2_\star+\kappa\mathcal{A}_\mu\star\mathcal{A}_\mu\right). \tag{2}$$

Here, $\mathcal{A}_\mu$ denotes the so-called covariant coordinates defined in Sect. 2, a natural gauge covariant tensorial form stemming from the existence of a canonical gauge invariant connection within the present NC framework (for more details on the relevant mathematical structures see e.g. in [21, 36, 37]. In (2), the additional second term may be viewed as a "gauge counterpart" of the harmonic term introduced in [15]. This action, which has been shown to be related to a spectral triple [38], exhibits interesting properties [32, 33] that deserve further studies. For instance, gauge invariant mass terms for the gauge fields are allowed even in the absence of some Higgs mechanism. However, the presence of additional terms implies in general a non-vanishing vacuum expectation value for the gauge potential. Somewhat similar non-trivial vacuum configurations also occur within NC scalar models with harmonic term as shown and studied recently in [39]. It turns out that the explicit determination of the relevant vacuum for any gauge theory model of the type (2) is a necessary step to be reached before the study of its renormalizability can be undertaken. Indeed, a reliable perturbative analysis in the present situation can only be defined after the action is expanded around the non-trivial vacuum which actually demands the explicit expression for that vacuum. This should then be followed by suitable gauge fixing, presumably inherited from background field methods.

The purpose of the present paper is the determination of the vacuum configurations for the gauge theories generically defined by (2), for $D=2$ and $D=4$ dimensions. The paper is organized as follows. In Sect. 2, we fix the notation and collect the main features of the non-commutative geometry framework that will be used throughout the analysis. The relevant symmetries for the vacua are examined in Sect. 3, focusing on vacuum configurations that are invariant under both rotations and symplectic isomorphisms. Sections 4 and 5 are devoted to the determination of these symmetric vacuum configurations for $D=2$ and $D=4$. It turns out that the use of the matrix basis formalism proves convenient to obtain explicit expressions for the relevant solutions of the equation of motion from which vacuum solutions can be obtained. Finally, we summarize and discuss our results in Sect. 6, and we draw conclusions.

2 Basic features

In this section, we collect the main ingredients that will be needed in the subsequent discussion. Some detailed studies of the Moyal NC algebra are carried out e.g. in [8, 9], while mathematical descriptions of the NC framework underlying the present study can be found in [21, 37].

2.1 The non-commutative gauge theory

The Moyal space, on which the gauge theory considered here is constructed can be defined as an algebra of tempered distributions on $\mathbb{R}^D$ endowed with the Moyal product [8, 9], hereafter denoted by the $\star$-symbol. Indeed, the latter can be defined on $\mathcal{S}=\mathcal{S}(\mathbb{R}^D)$, the space of complex-valued Schwartz functions, by

$$\forall f,h\in\mathcal{S}$$
$$(f\star h)(x)=\frac{1}{\pi^D\theta^D}\int \mathrm{d}^Dy\,\mathrm{d}^Dz\, f(x+y)h(x+z)\mathrm{e}^{-\mathrm{i}y\wedge z}, \tag{3}$$

where $x\wedge y=2x_\mu\Theta^{-1}_{\mu\nu}y_\nu$ and

$$\Theta_{\mu\nu}=\theta\begin{pmatrix}0&-1&0&0&\\1&0&0&0&\\0&0&0&-1&\ddots\\0&0&1&0&\\&&\ddots&&\end{pmatrix}, \tag{4}$$

where θ has mass dimension -2. Notice that $\mathcal{S}$ is stable by the product $\star$. Then the Moyal product can actually be extended by duality to the following subalgebra of tempered distributions $\mathcal{S}'(\mathbb{R}^D)$:

$$\mathcal{M}_\theta=\{T\in\mathcal{S}'(\mathbb{R}^D),\ \forall f\in\mathcal{S}\quad T\star f\in\mathcal{S}\text{ and } f\star T\in\mathcal{S}\}. \tag{5}$$

The Moyal space $(\mathcal{M}_\theta,\star,\dagger)$ is a unital involutive associative algebra (where $\dagger$ denotes complex conjugation), and it involves in particular the "coordinate functions" x_μ, satisfying the following commutation relation: $[x_\mu,x_\nu]_\star=x_\mu\star x_\nu-x_\nu\star x_\mu=\mathrm{i}\Theta_{\mu\nu}$. From this relation, and defining $\tilde{x}_\mu=2\Theta^{-1}_{\mu\nu}x_\nu$, we deduce some useful properties of $\mathcal{M}_\theta$:

$$\forall f,h\in\mathcal{M}_\theta,\quad \partial_\mu(f\star h)=\partial_\mu f\star h+f\star\partial_\mu h, \tag{6a}$$
$$\int \mathrm{d}^4x f\star h=\int \mathrm{d}^4x fh, \tag{6b}$$
$$[\tilde{x}_\mu,f]_\star=2\mathrm{i}\partial_\mu f, \tag{6c}$$
$$\{\tilde{x}_\mu,f\}_\star=\tilde{x}_\mu\star f+f\star\tilde{x}_\mu=2\tilde{x}_\mu f. \tag{6d}$$

In the present non-commutative framework, the Yang–Mills theory can be built from real-valued gauge potentials A_μ defined on $\mathcal{M}_\theta$, stemming from the very definition of non-commutative connections. For more mathematical details, see [4, 21, 32, 36, 37]. Recall that the group of gauge transformations acts on the gauge potential as

$$A^g_\mu=g\star A_\mu\star g^\dagger+\mathrm{i}g\star\partial_\mu g^\dagger, \tag{7}$$

where $g\in\mathcal{M}_\theta$ is the gauge function and satisfies $g^\dagger\star g=g\star g^\dagger=\mathbb{I}$. Recall also that there exists a special gauge potential in the Moyal space defined by $(-\frac{1}{2}\tilde{x}_\mu)$. One can check that $\left(-\frac{1}{2}\tilde{x}_\mu\right)^g=-\frac{1}{2}\tilde{x}_\mu$ holds, stemming from the existence of a gauge invariant connection [21, 36, 37], whose occurrence is implied by the fact that all derivations on $\mathcal{M}_\theta$

are inner. From this gauge invariant potential, we construct the covariant coordinates [1]:

$$\mathcal{A}_\mu = A_\mu + \frac{1}{2}\tilde{x}_\mu \,, \tag{8}$$

which transform covariantly:

$$\mathcal{A}_\mu^g = g \star \mathcal{A}_\mu \star g^\dagger \,. \tag{9}$$

Then the curvature $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star$, transforming as $F^g_{\mu\nu} = g \star F_{\mu\nu} \star g^\dagger$ can be reexpressed in terms of $\mathcal{A}_\mu$ as

$$F_{\mu\nu} = \Theta^{-1}_{\mu\nu} - \mathrm{i}[\mathcal{A}_\mu, \mathcal{A}_\nu]_\star \,. \tag{10}$$

It is known that the naive non-commutative extension of the Yang–Mills action is plagued by UV/IR mixing [14], which renders its renormalizability very unlikely, unless it is suitably modified. It turns out that the extension of the harmonic solution proposed in [15] to the case of gauge theories has been achieved recently [32, 33]. This singled out a class of potentially renormalizable theories on $\mathcal{M}_\theta$ whose action can be generically written as

$$S = \int \mathrm{d}^D x \left(\frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} + \frac{\Omega^2}{4} \{\mathcal{A}_\mu, \mathcal{A}_\nu\}^2_\star + \kappa \mathcal{A}_\mu \star \mathcal{A}_\mu \right) , \tag{11}$$

in which Ω and κ are real parameters, with mass dimensions respectively given by $[\Omega] = 0$ and $[\kappa] = 2$. This has been further shown to be related to a spectral triple [38].

2.2 The matrix basis

It will be convenient to represent elements on $\mathcal{M}_\theta$ with the help of the "so-called" matrix basis [23]. Recall that its elements $(b^{(D)}_{mn}(x))$ in D dimensions are eigenfunctions of the harmonic oscillator Hamiltonian $H = \frac{x^2}{2}$:

$$H \star b^{(D)}_{mn} = \theta \left(|m| + \frac{1}{2} \right) b^{(D)}_{mn} \,,$$
$$b^{(D)}_{mn} \star H = \theta \left(|n| + \frac{1}{2} \right) b^{(D)}_{mn} \,, \tag{12}$$

where $m,\, n \in \mathbb{N}^{\frac{D}{2}}$ and $|m| = \sum_{i=1}^{\frac{D}{2}} m_i$. In two dimensions, the expression of the elements $\left(b^{(2)}_{mn}\right) = (f_{mn})$ of the matrix basis in polar coordinates,

$$x_1 = r\cos(\varphi) \,, \quad x_2 = r\sin(\varphi) \,, \tag{13}$$

is given by

$$f_{mn}(x) = 2(-1)^m \sqrt{\frac{m!}{n!}} \mathrm{e}^{\mathrm{i}(n-m)\varphi} \left(\frac{2r^2}{\theta} \right)^{\frac{n-m}{2}} \times L^{n-m}_m \left(\frac{2r^2}{\theta} \right) \mathrm{e}^{-\frac{r^2}{\theta}} \,, \tag{14}$$

where the $L^k_n(x)$ are the associated Laguerre polynomials. The extension in four dimensions is straightforward. Namely, one has $m = (m_1, m_2)$, $n = (n_1, n_2)$ and

$$b^{(4)}_{mn}(x) = f_{m_1,n_1}(x_1, x_2) f_{m_2,n_2}(x_3, x_4) \,. \tag{15}$$

The matrix basis has the following properties:

$$\left(b^{(D)}_{mn} \star b^{(D)}_{kl} \right)(x) = \delta_{nk} b^{(D)}_{ml}(x) \,, \tag{16}$$

$$\int \mathrm{d}^D x\, b^{(D)}_{mn}(x) = (2\pi\theta)^{\frac{D}{2}} \delta_{mn} \,, \tag{17}$$

$$\left(b^{(D)}_{mn} \right)^\dagger (x) = b^{(D)}_{nm}(x) \,. \tag{18}$$

We recall that this basis defines therefore an isomorphism between the unital involutive Moyal algebra and a subalgebra of the unital involutive algebra of complex infinite-dimensional matrices. Indeed, for all $g \in \mathcal{M}_\theta$, there is a unique matrix (g_{mn}) satisfying

$$\forall x \in \mathbb{R}^D \quad g(x) = \sum_{m,n \in \mathbb{N}^{\frac{D}{2}}} g_{mn} b^{(D)}_{mn}(x) \,. \tag{19}$$

Then this matrix is given by

$$g_{mn} = \frac{1}{(2\pi\theta)^{\frac{D}{2}}} \int \mathrm{d}^D x\, g(x) b^{(D)}_{mn}(x) \,. \tag{20}$$

3 Symmetries of the vacua

In the following, we will have to solve the equation of motion. This is a difficult task. In this respect, it is convenient to exhibit some symmetries of the theory that will be used to constrain the expression for the solutions we look for. In fact, the group of symmetries of the euclidean Moyal algebra $\mathcal{M}_\theta$ in D dimensions is

$$G_D = \mathrm{SO}(D) \cap \mathrm{Sp}(D) \,, \tag{21}$$

where $\mathrm{SO}(D)$ is the group of rotations and $\mathrm{Sp}(D)$ is the group of symplectic isomorphisms. G_D acts on the field A_μ or $\mathcal{A}_\mu$ by

$$\forall \Lambda \in G_D \,, \quad A^\Lambda_\mu(x) = \Lambda_{\mu\nu} A_\nu(\Lambda^{-1}x) \,. \tag{22}$$

The action $S(\mathcal{A})$ (11) is of course invariant under G_D. We further require that the new action $\tilde{S}(\mathcal{A}^0_\mu, \delta\mathcal{A}_\mu) = S(\mathcal{A}^0_\mu + \delta\mathcal{A}_\mu)$, obtained from the expansion of S around a non-trivial vacuum $\mathcal{A}^0_\mu$, is also invariant under G_D. This means that $\tilde{S}(\mathcal{A}^0_\mu, \delta\mathcal{A}^\Lambda_\mu) = \tilde{S}(\mathcal{A}^0_\mu, \delta\mathcal{A}_\mu)$, where the $\Lambda \in G_D$ do not affect the vacuum $\mathcal{A}^0_\mu$. Since $S(\mathcal{A}_\mu)$ is invariant under G_D, this is equivalent to

$$\forall \Lambda \in G_D \,, \quad \tilde{S}\left(\left(\mathcal{A}^0_\mu\right)^\Lambda, \delta\mathcal{A}_\mu \right) = \tilde{S}\left(\mathcal{A}^0_\mu, \delta\mathcal{A}_\mu \right) . \tag{23}$$

This relation implies that the vacuum is invariant under G_D. Indeed, using (30) for the action given in the next section, the part of (23) quadratic in $\delta\mathcal{A}_\mu$ can be written as

$$\begin{aligned}
&\int \mathrm{d}^D x\Big(-\frac{(1-\Omega^2)}{2}\big(2\mathcal{A}^0_\mu\star\mathcal{A}^0_\nu\star\delta\mathcal{A}_\mu\star\delta\mathcal{A}_\nu \\
&\quad +2\mathcal{A}^0_\mu\star\mathcal{A}^0_\nu\star\delta\mathcal{A}_\nu\star\delta\mathcal{A}_\mu+2\mathcal{A}^0_\mu\star\delta\mathcal{A}_\nu\star\mathcal{A}^0_\mu\star\delta\mathcal{A}_\nu\big) \\
&\quad +\frac{(1+\Omega^2)}{2}\big(2\mathcal{A}^0_\mu\star\mathcal{A}^0_\mu\star\delta\mathcal{A}_\nu\star\delta\mathcal{A}_\nu \\
&\quad +2\mathcal{A}^0_\mu\star\mathcal{A}^0_\nu\star\delta\mathcal{A}_\nu\star\delta\mathcal{A}_\mu \\
&\quad +\mathcal{A}^0_\mu\star\delta\mathcal{A}_\mu\star\mathcal{A}^0_\nu\star\delta\mathcal{A}_\nu+\delta\mathcal{A}_\mu\star\mathcal{A}^0_\mu\star\delta\mathcal{A}_\nu\star\mathcal{A}^0_\nu\big)\Big) \\
&=\int \mathrm{d}^D x\Big(-\frac{(1-\Omega^2)}{2}\big(2\big(\mathcal{A}^0_\mu\big)^\Lambda\star\big(\mathcal{A}^0_\nu\big)^\Lambda\star\delta\mathcal{A}_\mu\star\delta\mathcal{A}_\nu \\
&\quad +2(\mathcal{A}^0)^\Lambda_\mu\star\big(\mathcal{A}^0_\nu\big)^\Lambda\star\delta\mathcal{A}_\nu\star\delta\mathcal{A}_\mu \\
&\quad +2\big(\mathcal{A}^0_\mu\big)^\Lambda\star\delta\mathcal{A}_\nu\star\big(\mathcal{A}^0_\mu\big)^\Lambda\star\delta\mathcal{A}_\nu\big) \\
&\quad +\frac{(1+\Omega^2)}{2}\Big(2\big(\mathcal{A}^0_\mu\big)^\Lambda\star\big(\mathcal{A}^0_\mu\big)^\Lambda\star\delta\mathcal{A}_\nu\star\delta\mathcal{A}_\nu \\
&\quad +2\big(\mathcal{A}^0_\mu\big)^\Lambda\star\big(\mathcal{A}^0_\nu\big)^\Lambda\star\delta\mathcal{A}_\nu\star\delta\mathcal{A}_\mu \\
&\quad +\big(\mathcal{A}^0_\mu\big)^\Lambda\star\delta\mathcal{A}_\mu\star\big(\mathcal{A}^0_\nu\big)^\Lambda\star\delta\mathcal{A}_\nu \\
&\quad +\delta\mathcal{A}_\mu\star\big(\mathcal{A}^0_\mu\big)^\Lambda\star\delta\mathcal{A}_\nu\star\big(\mathcal{A}^0_\nu\big)^\Lambda\Big)\Big)\,. \qquad (24)
\end{aligned}$$

This relation is true for all the fluctuations $\delta\mathcal{A}_\mu$, so it holds at the level of the lagrangians involved in the integrals. Assuming now $\delta\mathcal{A}_\mu(x)=\delta_{\mu\rho}$, for some fixed ρ, we obtain from (24)

$$2\Omega^2\Big(\mathcal{A}^0_\mu\mathcal{A}^0_\mu+2\big(\mathcal{A}^0_\rho\big)^2-\big(\mathcal{A}^0_\mu\big)^\Lambda\big(\mathcal{A}^0_\mu\big)^\Lambda-2\Big(\big(\mathcal{A}^0_\rho\big)^\Lambda\Big)^2\Big)=0\,, \qquad (25)$$

where the index ρ is not summed over. It is now easy to get $\big((\mathcal{A}^0_\rho)^\Lambda\big)^2=\big(\mathcal{A}^0_\rho\big)^2$, and since G_D is a connected Lie group,

$$\big(\mathcal{A}^0_\rho\big)^\Lambda(x)=\mathcal{A}^0_\rho(x)\,. \qquad (26)$$

The vacuum is invariant under G_D, i.e., dropping the superscript 0 from now on in $\mathcal{A}^0_\mu$, we have $\mathcal{A}_\mu(x)=\Lambda_{\mu\nu}\mathcal{A}_\nu(\Lambda^{-1}x)$ for all $\Lambda\in G_D$. Since the identity matrix and $2\Theta^{-1}$ are the only matrices up to a scalar multiplication that commute with G_D, we can write

$$\mathcal{A}_\mu(x)=\phi_1(x)x_\mu+\phi_2(x)2\Theta^{-1}_{\mu\nu}x_\nu=\phi_1(x)x_\mu+\phi_2(x)\tilde{x}_\mu\,, \qquad (27)$$

where ϕ_1 and ϕ_2 are two scalar fields invariant under G_D. Then, G_D is isomorphic to $\mathrm{U}(\frac{D}{2})$ and the isomorphism is described by associating to each coefficient u_{ij} of a matrix of $\mathrm{U}(\frac{D}{2})$, the submatrix

$$\begin{pmatrix}\mathrm{Re}(u_{ij}) & -\mathrm{Im}(u_{ij})\\ \mathrm{Im}(u_{ij}) & \mathrm{Re}(u_{ij})\end{pmatrix}, \qquad (28)$$

in the (i,j) entry of the matrix of G_D. From the theory of the invariants of $\mathrm{U}(\frac{D}{2})$ [40], one infers that ϕ_1 and ϕ_2 are therefore functions only on x^2. Then, the general expression for $\mathcal{A}_\mu$ can be written as

$$\mathcal{A}_\mu(x)=\Phi_1(x^2)x_\mu+\Phi_2(x^2)\tilde{x}_\mu\,. \qquad (29)$$

This form will be extensively used to solve the equation of motion in the following section.

4 Solving the equation of motion

By using the definition of the covariant coordinate (8), namely $\mathcal{A}_\mu=A_\mu+\frac{1}{2}\tilde{x}_\mu$, the action (11) can be rewritten as

$$\begin{aligned}
S=\int \mathrm{d}^D x\Big(&-\frac{(1-\Omega^2)}{2}\mathcal{A}_\mu\star\mathcal{A}_\nu\star\mathcal{A}_\mu\star\mathcal{A}_\nu \\
&+\frac{(1+\Omega^2)}{2}\mathcal{A}_\mu\star\mathcal{A}_\mu\star\mathcal{A}_\nu\star\mathcal{A}_\nu+\kappa\mathcal{A}_\mu\star\mathcal{A}_\mu\Big)\,. \qquad (30)
\end{aligned}$$

Then, the corresponding equation of motion $\frac{\delta S}{\delta\mathcal{A}_\mu(x)}=0$ is given by

$$\begin{aligned}
&-2(1-\Omega^2)\mathcal{A}_\nu\star\mathcal{A}_\mu\star\mathcal{A}_\nu+(1+\Omega^2)\mathcal{A}_\mu\star\mathcal{A}_\nu\star\mathcal{A}_\nu \\
&\quad+(1+\Omega^2)\mathcal{A}_\nu\star\mathcal{A}_\nu\star\mathcal{A}_\mu+2\kappa\mathcal{A}_\mu \\
&=0\,. \qquad (31)
\end{aligned}$$

Due to the very structure of the Moyal product, this is a complicated integro-differential equation for which no known algorithm to solve it does exist so far. Notice that it supports the trivial solution $\mathcal{A}_\mu(x)=0$, which, however, is not so interesting since expanding the action around it gives rise to a non-dynamical matrix model, as already noted in [34]. It turns out that (31) supports other non-trivial solutions. These can be conveniently determined for $D=2$ and $D=4$ using the matrix basis (14) and (15) as we now show in the rest of this section.

4.1 The case $D=2$

When $D=2$, it is convenient to define

$$Z(x)=\frac{\mathcal{A}_1(x)+\mathrm{i}\mathcal{A}_2(x)}{\sqrt{2}}\,,\qquad Z^\dagger(x)=\frac{\mathcal{A}_1(x)-\mathrm{i}\mathcal{A}_2(x)}{\sqrt{2}}\,. \qquad (32)$$

Then the action can be expressed as

$$\begin{aligned}
S=\int \mathrm{d}^2x((&-1+3\Omega^2)Z\star Z\star Z^\dagger\star Z^\dagger \\
&+(1+\Omega^2)Z\star Z^\dagger\star Z\star Z^\dagger+2\kappa Z\star Z^\dagger)\,, \qquad (33)
\end{aligned}$$

so that the equation of motion takes the form

$$\begin{aligned}
&(3\Omega^2-1)(Z^\dagger\star Z\star Z+Z\star Z\star Z^\dagger) \\
&\quad+2(1+\Omega^2)Z\star Z^\dagger\star Z+2\kappa Z \\
&=0\,. \qquad (34)
\end{aligned}$$

Expressing now $Z(x)$ in the matrix basis, namely

$$Z(x)=\sum_{m,n=0}^{\infty} Z_{mn} f_{mn}(x)\,, \tag{35}$$

(34) becomes a cubic infinite-dimensional matrix equation. In view of the discussion in Sect. 3, we now look for the symmetric solutions of the form given by (29), namely

$$Z(x)=\Phi_1(x^2)\left(\frac{x_1+\mathrm{i}x_2}{\sqrt{2}}\right)+\Phi_2(x^2)\left(\frac{\tilde{x}_1+\mathrm{i}\tilde{x}_2}{\sqrt{2}}\right). \tag{36}$$

To translate (36) into the matrix basis, we first note that the expression of the matrix coefficients of $Z(x)$ is given by (20)

$$Z_{mn}=\frac{1}{2\pi\theta}\int \mathrm{d}^2x Z(x) f_{nm}(x)\,. \tag{37}$$

In polar coordinates (r,φ), we have

$$x_1+\mathrm{i}x_2=r\mathrm{e}^{\mathrm{i}\varphi}\,,\quad \tilde{x}_1+\mathrm{i}\tilde{x}_2=-\frac{2\mathrm{i}}{\theta}r\mathrm{e}^{\mathrm{i}\varphi}\,, \tag{38}$$

and

$$\begin{aligned}Z_{mn}=&\frac{(-1)^n}{2\pi\theta\sqrt{2}}\sqrt{\frac{n!}{m!}}\int r\,\mathrm{d}r\,\mathrm{d}\varphi \mathrm{e}^{\mathrm{i}(m-n)\varphi}\left(\frac{2r^2}{\theta}\right)^{\frac{m-n}{2}}\\ &\times L_n^{m-n}\left(\frac{2r^2}{\theta}\right)\mathrm{e}^{-\frac{r^2}{\theta}}\left(\Phi_1(r^2)r\mathrm{e}^{\mathrm{i}\varphi}-\Phi_2(r^2)\frac{2\mathrm{i}}{\theta}r\mathrm{e}^{\mathrm{i}\varphi}\right).\end{aligned} \tag{39}$$

By performing the integration over φ, we easily find that

$$\begin{aligned}Z_{mn}=&\frac{(-1)^n}{\theta\sqrt{2}}\sqrt{\frac{n!}{m!}}\int r^2\,\mathrm{d}r\left(\frac{2r^2}{\theta}\right)^{\frac{m-n}{2}}\\ &\times L_n^{m-n}\left(\frac{2r^2}{\theta}\right)\mathrm{e}^{-\frac{r^2}{\theta}}\left(\Phi_1(r^2)-\Phi_2(r^2)\frac{2\mathrm{i}}{\theta}\right)\delta_{m+1,n}\,.\end{aligned} \tag{40}$$

Then, defining $z=\frac{2r^2}{\theta}$ and

$$\begin{aligned}a_m=&\frac{(-1)^{m+1}}{4}\sqrt{\frac{m+1}{\theta}}\\ &\times\int \mathrm{d}z L_{m+1}^{-1}(z)\mathrm{e}^{-\frac{z}{2}}\left(\Phi_2\left(\frac{\theta z}{2}\right)+\frac{\mathrm{i}\theta}{2}\Phi_1\left(\frac{\theta z}{2}\right)\right),\end{aligned} \tag{41}$$

we obtain

$$Z_{mn}=-\mathrm{i}a_m\delta_{m+1,n}\,. \tag{42}$$

This, inserted into (34), yields

$$\begin{aligned}&\forall m\in\mathbb{N}\,,\\ &a_m\left((3\Omega^2-1)\left(|a_{m-1}|^2+|a_{m+1}|^2\right)+2(1+\Omega^2)|a_m|^2+2\kappa\right)\\ &\quad=0\,,\end{aligned} \tag{43}$$

where it is understood that $a_{-1}=0$. Then (43) implies

$$\begin{aligned}&\forall m\in\mathbb{N}\,,\\ &(i)\ \ a_m=0\quad\text{or}\\ &(ii)\ \ (3\Omega^2-1)\left(|a_{m-1}|^2+|a_{m+1}|^2\right)+2(1+\Omega^2)|a_m|^2+2\kappa\\ &\qquad=0\,.\end{aligned} \tag{44}$$

From now on, we will focus only on the second condition (ii) (this will be discussed in more detail in Sect. 6; see the hypothesis $(\mathfrak{H})$):

$$\begin{aligned}&\forall m\in\mathbb{N}\,,\\ &(3\Omega^2-1)\left(|a_{m-1}|^2+|a_{m+1}|^2\right)+2(1+\Omega^2)|a_m|^2+2\kappa=0\,.\end{aligned} \tag{45}$$

Upon setting $u_{m+1}=|a_m|^2$, (45) becomes

$$\begin{aligned}&\forall m\in\mathbb{N}\,,\\ &(3\Omega^2-1)(u_m+u_{m+2})+2(1+\Omega^2)u_{m+1}+2\kappa=0\,,\end{aligned} \tag{46}$$

which is a non-homogeneous linear iterative equation of second order with the boundary condition $u_0=0$. This will be solved in Sect. 5.1.

4.2 The case $D=4$

The case $D=4$ can be straightforwardly adapted from the two-dimensional case. Owing to the fact that two symplectic pairs are now involved in the four-dimensional Moyal space, we define two complex quantities, namely

$$Z_1(x)=\frac{\mathcal{A}_1(x)+\mathrm{i}\mathcal{A}_2(x)}{\sqrt{2}}\,,\qquad Z_2(x)=\frac{\mathcal{A}_3(x)+\mathrm{i}\mathcal{A}_4(x)}{\sqrt{2}}\,. \tag{47}$$

Then, using these new variables, (30) can be conveniently reexpressed as

$$\begin{aligned}S=\int \mathrm{d}^4x\big(&(-1+3\Omega^2)Z_1\star Z_1\star Z_1^\dagger\star Z_1^\dagger\\ &+(1+\Omega^2)Z_1\star Z_1^\dagger\star Z_1\star Z_1^\dagger+2\kappa Z_1\star Z_1^\dagger\\ &+(-1+3\Omega^2)Z_2\star Z_2\star Z_2^\dagger\star Z_2^\dagger\\ &+(1+\Omega^2)Z_2\star Z_2^\dagger\star Z_2\star Z_2^\dagger+2\kappa Z_2\star Z_2^\dagger\\ &-2(1-\Omega^2)Z_1\star Z_2\star Z_1^\dagger\star Z_2^\dagger\\ &-2(1-\Omega^2)Z_2\star Z_1\star Z_2^\dagger\star Z_1^\dagger\\ &+(1+\Omega^2)Z_1\star Z_1^\dagger\star Z_2\star Z_2^\dagger\\ &+(1+\Omega^2)Z_2\star Z_1\star Z_1^\dagger\star Z_2^\dagger\\ &+(1+\Omega^2)Z_1\star Z_2\star Z_2^\dagger\star Z_1^\dagger\\ &+(1+\Omega^2)Z_1^\dagger\star Z_1\star Z_2^\dagger\star Z_2\big)\,.\end{aligned} \tag{48}$$

From (48), we derive the equations of motion

$$\begin{aligned}&(3\Omega^2-1)\big(Z_1^\dagger\star Z_1\star Z_1+Z_1\star Z_1\star Z_1^\dagger\big)\\&\quad+(1+\Omega^2)\big(2Z_1\star Z_1^\dagger\star Z_1+Z_2\star Z_2^\dagger\star Z_1\\&\quad+Z_2^\dagger\star Z_2\star Z_1+Z_1\star Z_2\star Z_2^\dagger+Z_1\star Z_2^\dagger\star Z_2\big)\\&\quad-2(1-\Omega^2)\big(Z_2^\dagger\star Z_1\star Z_2+Z_2\star Z_1\star Z_2^\dagger\big)+2\kappa Z_1\\&=0\,,\end{aligned}\tag{49a}$$

$$\begin{aligned}&(3\Omega^2-1)\big(Z_2^\dagger\star Z_2\star Z_2+Z_2\star Z_2\star Z_2^\dagger\big)\\&\quad+(1+\Omega^2)\big(2Z_2\star Z_2^\dagger\star Z_2+Z_1\star Z_1^\dagger\star Z_2\\&\quad+Z_1^\dagger\star Z_1\star Z_2+Z_2\star Z_1\star Z_1^\dagger+Z_2\star Z_1^\dagger\star Z_1\big)\\&\quad-2(1-\Omega^2)\big(Z_1^\dagger\star Z_2\star Z_1+Z_1\star Z_2\star Z_1^\dagger\big)+2\kappa Z_2\\&=0\,.\end{aligned}\tag{49b}$$

Notice that (49a) and (49b) are exchanged upon performing the exchange $Z_1\rightleftarrows Z_2$. Now we again specialize to the symmetric solutions of the form (29), namely

$$\begin{aligned}Z_1(x)&=\frac{1}{\sqrt2}\left(\Phi_1(x^2)(x_1+\mathrm{i}x_2)+\Phi_2(x^2)(\tilde x_1+\mathrm{i}\tilde x_2)\right),\\Z_2(x)&=\frac{1}{\sqrt2}\left(\Phi_1(x^2)(x_3+\mathrm{i}x_4)+\Phi_2(x^2)(\tilde x_3+\mathrm{i}\tilde x_4)\right).\end{aligned}\tag{50}$$

Then, in view of (15) and (20), one has

$$\begin{aligned}(Z_1)_{m,n}=&\frac{1}{(2\pi\theta)^2}\int\mathrm{d}^4xZ_1(x)f_{n_1,m_1}(x_1,x_2)\\&\times f_{n_2,m_2}(x_3,x_4)\,.\end{aligned}\tag{51}$$

Let us introduce the polar coordinates associated to each symplectic pair:

$$\begin{aligned}&x_1=r_1\cos(\varphi_1)\,,\quad x_2=r_1\sin(\varphi_1)\,,\\&x_3=r_2\cos(\varphi_2)\,,\quad x_4=r_2\sin(\varphi_2)\,,\\&r^2=r_1^2+r_2^2\,.\end{aligned}\tag{52}$$

Then, by using (14) and integrating over the two angular variables φ_1 and φ_2, one obtains

$$\begin{aligned}(Z_1)_{m,n}=&\frac{2(-1)^{m_1+m_2+1}}{\theta}\sqrt{\frac{m_1+1}{\theta}}\\&\times\int r_1\,\mathrm{d}r_1r_2\,\mathrm{d}r_2\left(\Phi_1(r^2)-\frac{2i}{\theta}\Phi_2(r^2)\right)\\&\times\mathrm{e}^{-\frac{r^2}{\theta}}L^{-1}_{m_1+1}\left(\frac{2r^2}{\theta}\right)L^0_{m_2}\left(\frac{2r^2}{\theta}\right)\\&\times\delta_{m_1+1,n_1}\delta_{m_2,n_2}\,.\end{aligned}\tag{53}$$

Let us now integrate this expression by parts. Defining $z_1=\frac{2r_1^2}{\theta}$, $z_2=\frac{2r_2^2}{\theta}$, $z=z_1+z_2$, and denoting by $F(z)$ one primitive function of $\big(\Phi_2\big(\frac{\theta z}{2}\big)+\frac{\mathrm{i}\theta}{2}\Phi_1\big(\frac{\theta z}{2}\big)\big)\mathrm{e}^{-\frac{z}{2}}$, (53) leads to

$$\begin{aligned}(Z_1)_{m,n}=&-\mathrm{i}\frac{(-1)^{m_1+m_2+1}}{4}\sqrt{\frac{m_1+1}{\theta}}\\&\times\int\mathrm{d}z_1\,\mathrm{d}z_2F(z)L^0_{m_1}(z_1)L^0_{m_2}(z_2)\\&\times\delta_{m_1+1,n_1}\delta_{m_2,n_2}\,,\end{aligned}\tag{54}$$

where we have used

$$\frac{\mathrm{d}}{\mathrm{d}x}L^{-1}_{m_1+1}(x)=-L^0_{m_1}(x)\,.\tag{55}$$

A similar derivation holds for $(Z_2)_{m,n}$. Finally, using the symmetry argument developed in Sect. 3, we find

$$(Z_1)_{m,n}=-\mathrm{i}a_{m_1m_2}\sqrt{m_1+1}\delta_{m_1+1,n_1}\delta_{m_2,n_2}\,,\tag{56}$$

$$(Z_2)_{m,n}=-\mathrm{i}a_{m_1m_2}\sqrt{m_2+1}\delta_{m_1,n_1}\delta_{m_2+1,n_2}\,,\tag{57}$$

where $a_{m_1,m_2}\in\mathbb{C}$ is symmetric upon the exchange of m_1 and m_2. Then, (49a) and (49b) become

$$\begin{aligned}&(3\Omega^2-1)\big(m_1|a_{m_1-1,m_2}|^2+(m_1+2)|a_{m_1+1,m_2}|^2\big)a_{m_1,m_2}\\&\quad+(1+\Omega^2)\big(2(m_1+1)|a_{m_1,m_2}|^2\\&\quad+(m_2+1)|a_{m_1,m_2}|^2+m_2|a_{m_1,m_2-1}|^2\\&\quad+(m_2+1)|a_{m_1+1,m_2}|^2+m_2|a_{m_1+1,m_2-1}|^2\big)a_{m_1,m_2}\\&\quad-2(1-\Omega^2)\big(m_2|a_{m_1,m_2-1}|^2a_{m_1+1,m_2-1}\\&\quad+(m_2+1)\overline{a}_{m_1+1,m_2}a_{m_1,m_2+1}a_{m_1,m_2}\big)+2\kappa a_{m_1,m_2}\\&=0\,,\end{aligned}\tag{58a}$$

$$\begin{aligned}&(3\Omega^2-1)\big(m_2|a_{m_1,m_2-1}|^2+(m_2+2)|a_{m_1,m_2+1}|^2\big)a_{m_1,m_2}\\&\quad+(1+\Omega^2)\big(2(m_2+1)|a_{m_1,m_2}|^2\\&\quad+(m_1+1)|a_{m_1,m_2}|^2+m_1|a_{m_1-1,m_2}|^2\\&\quad+(m_1+1)|a_{m_1,m_2+1}|^2+m_1|a_{m_1-1,m_2+1}|^2\big)a_{m_1,m_2}\\&\quad-2(1-\Omega^2)\big(m_1|a_{m_1-1,m_2}|^2a_{m_1-1,m_2+1}\\&\quad+(m_1+1)\overline{a}_{m_1,m_2+1}a_{m_1+1,m_2}a_{m_1,m_2}\big)+2\kappa a_{m_1,m_2}\\&=0\,.\end{aligned}\tag{58b}$$

As we did for the case $D=2$, we assume now that $a_{m_1,m_2}\neq0$ (for a more detailed discussion of this point, see Sect. 6; this will be called hypothesis $(\mathfrak{H})$). Combining this latter assumption with (58), it can be shown that a_{m_1,m_2} depends only on $m=m_1+m_2$. The corresponding proof is presented in the appendix. Now, if we define $v_{m_1+m_2+1}=|a_{m_1,m_2}|^2$, (58) is equivalent to

$$\begin{aligned}\forall m\in\mathbb{N}\,,\quad&(3\Omega^2-1)(mv_m+(m+3)v_{m+2})\\&\quad+(1+\Omega^2)(2m+3)v_{m+1}+2\kappa\\&=0\,,\end{aligned}\tag{59}$$

which is a non-homogeneous linear iterative equation of second order with non-constant coefficients, with the boundary condition $v_0=0$. Notice also that this equation is very close to this of the case $D=2$, defining $v_m=\frac{u_m}{m}$ in (46).

5 Solutions

In this section, we solve (46) and (59) to obtain the vacuum configurations for $D=2$ and $D=4$.

5.1 The case $D=2$

Let us consider (46):

$$\forall m\in\mathbb{N}\,,\quad (3\Omega^2-1)(u_m+u_{m+2})+2(1+\Omega^2)u_{m+1}+2\kappa =0\,,\qquad u_0=0\,,\quad u_m\geq 0\,. \tag{60}$$

For $\Omega^2\neq\frac{1}{3}$, we define

$$r=\frac{1+\Omega^2+\sqrt{8\Omega^2(1-\Omega^2)}}{1-3\Omega^2}\,, \tag{61}$$

and one has $\frac{1}{r}=\frac{1+\Omega^2-\sqrt{8\Omega^2(1-\Omega^2)}}{1-3\Omega^2}$. Then, it is easy to realize that (60) supports different types of solutions according to the range for the values taken by Ω. Namely, one has

- for $\Omega^2=0$, $\kappa=0$, $u_m=\alpha m$ and $\alpha\geq 0$;
- for $0<\Omega^2<\frac{1}{3}$, $u_m=\alpha(r^m-r^{-m})-\frac{\kappa}{4\Omega^2}(1-r^{-m})$, $\alpha\geq 0$ and $r>1$;
- for $\Omega^2=\frac{1}{3}$, $\kappa\leq 0$ and $u_m=-\frac{3\kappa}{4}$;
- for $\frac{1}{3}<\Omega^2<1$, $\kappa\leq 0$, $u_m=-\frac{\kappa}{4\Omega^2}(1-r^{-m})$ and $r<-1$;
- for $\Omega^2=1$, $\kappa\leq 0$ and $u_m=-\frac{\kappa}{4}(1-(-1)^{-m})$.

Notice that the solution for $\Omega=0$ corresponds to the commutative case: $u_m=\frac{m}{\theta}$ is equivalent to a vacuum $\mathcal{A}_\mu(x)=\frac{1}{2}\tilde{x}_\mu$ or $A_\mu=0$. Then it is possible to choose α depending on Ω, so that the solution is continuous in Ω near 0. With the Taylor expansions

$$\begin{aligned} r^m-r^{-m}&=4\sqrt{2}m\Omega+O(\Omega^3)\,,\\ 1-r^{-m}&=2\sqrt{2}m\Omega-4m^2\Omega^2+O(\Omega^3)\,,\\ \alpha(\Omega)&=\frac{1}{\Omega}(\alpha_0+O(\Omega))\,, \end{aligned} \tag{62}$$

one deduces that κ must have the same asymptotic behavior as Ω near 0. If $\kappa=\kappa_0\Omega+O(\Omega)$,

$$u_m=4\sqrt{2}m\alpha_0-\frac{\sqrt{2}m}{2}\kappa_0+O(\Omega)\,. \tag{63}$$

For $\alpha_0=\frac{\kappa_0}{8}+\frac{4}{4\theta\sqrt{2}}$, we find the commutative limit for the vacuum: $A_\mu=0$, and the gauge potential is massless ($\lim_{\Omega\to 0}\kappa=0$).

Consider now the asymptotic behavior of the vacuum in the configuration space for $x^2\to\infty$. If $0<\Omega^2<\frac{1}{3}$ and $\alpha\neq 0$, then $\sqrt{u_m}\sim_{m\to\infty} r^{\frac{m}{2}}$ and $r>1$. As a consequence [8], the solution $\mathcal{A}_\mu(x)$ of the equation of motion does not belong to the Moyal algebra. So we require that $\alpha=0$. Then, for $\Omega\neq 0$, κ has to be negative and u_m has a finite limit. This indicates that $\mathcal{A}_\mu(x)$ has a constant limit as $x^2\to\infty$.

Let us try to obtain an expression for the vacuum in the configuration space. Using the variable (32), the solution is

$$\begin{aligned} Z(x)&=\sum_{m,n=0}^{\infty}-\mathrm{i}a_m\delta_{m+1,n}f_{m,n}(x)\\ &=-\mathrm{i}\sum_{m=0}^{\infty}a_mf_{m,m+1}(x)\,, \end{aligned} \tag{64}$$

with $a_m=\mathrm{e}^{\mathrm{i}\xi_m}\sqrt{u_m}$ and $\xi_m\in\mathbb{R}$ an arbitrary phase. Using (14), we obtain

$$Z(x)=-2\mathrm{i}\sqrt{z}\mathrm{e}^{\frac{z}{2}}\mathrm{e}^{\mathrm{i}\varphi}\sum_{m=0}^{\infty}\frac{(-1)^m}{\sqrt{m+1}}a_mL_m^1(z)\,, \tag{65}$$

where $z=\frac{2r^2}{\theta}$. Then use of the property

$$L_m^k(z)=\frac{\mathrm{e}^z z^{-\frac{k}{2}}}{m!}\int_0^\infty \mathrm{d}t\mathrm{e}^{-t}t^{m+\frac{k}{2}}J_k(2\sqrt{tz}) \tag{66}$$

permits one to express $Z(x)$ as

$$Z(x)=-2\mathrm{i}\mathrm{e}^{\frac{z}{2}}\mathrm{e}^{\mathrm{i}\varphi}\int_0^\infty \mathrm{d}t\mathrm{e}^{-t}\sqrt{t}J_1(2\sqrt{tz})\sum_{m=0}^{\infty}\frac{(-1)^m a_m}{m!\sqrt{m+1}}t^m\,, \tag{67}$$

with $J_k(x)$ the kth Bessel function of the first kind. Since

$$\begin{aligned} -\mathrm{i}\sqrt{\frac{2z}{\theta}}\mathrm{e}^{\mathrm{i}(\varphi+\xi_m)}&=(\tilde{x}_1+\mathrm{i}\tilde{x}_2)\cos(\xi_m)\\ &\quad+\frac{2}{\theta}(x_1+\mathrm{i}x_2)\sin(\xi_m)\,, \end{aligned}$$

one can deduce the expression of the vacuum in two dimensions:

$$\begin{aligned} \mathcal{A}_\mu(x)&=2\sqrt{\theta}\frac{\mathrm{e}^{\frac{z}{2}}}{\sqrt{z}}\int_0^\infty \mathrm{d}t\mathrm{e}^{-t}\sqrt{t}J_1(2\sqrt{tz})\\ &\quad\times\sum_{m=0}^{\infty}\frac{(-1)^m\sqrt{u_{m+1}}}{m!\sqrt{m+1}}t^m\\ &\quad\times\left(\tilde{x}_\mu\cos(\xi_m)+\frac{2}{\theta}x_\mu\sin(\xi_m)\right)\,, \end{aligned} \tag{68}$$

with u_m given above.

5.2 A special case

In this subsection, we consider the case $\Omega^2=\frac{1}{3}$ and $\kappa<0$ in two dimensions. We have found in the previous subsection that the solution is given by $u_m=-\frac{3\kappa}{4}$ and $a_m=\mathrm{e}^{\mathrm{i}\xi_m}\sqrt{u_m}$. Now we set $\xi_m=0$; let us check that this solution of the equation of motion is a minimum of the action. The action can be expanded around the vacuum $Z(x)$ for these values of parameters and its quadratic part is given by

$$\begin{aligned} \tilde{S}_{\mathrm{quadr}}=\int \mathrm{d}^2x(&2\kappa\delta Z\star\delta Z^\dagger+4Z\star Z^\dagger\star\delta Z\star\delta Z^\dagger\\ &+4Z^\dagger\star Z\star\delta Z^\dagger\star\delta Z+2Z\star\delta Z^\dagger\star Z\star\delta Z^\dagger\\ &+2Z^\dagger\star\delta Z\star Z^\dagger\star\delta Z)\,, \end{aligned} \tag{69}$$

where $\delta Z(x)$ is the fluctuation. Denoting $\alpha=-\pi\theta\kappa>0$, (69) can be reexpressed in the matrix basis as

$$\begin{aligned} \tilde{S}_{\mathrm{quadr}}=&4\alpha\delta Z_{m,n}\delta Z^\dagger_{n,m}-2\alpha\delta Z^\dagger_{m+1,n}\delta Z^\dagger_{n+1,m}\\ &-2\alpha\delta Z_{m,n+1}\delta Z_{n,m+1}\,. \end{aligned} \tag{70}$$

Using (32), one can find that

$$\begin{aligned}\tilde{S}_{\text{quadr}} = 2\alpha(\delta\mathcal{A}_1)_{m,n}(\delta\mathcal{A}_1)_{n,m} - \alpha(\delta\mathcal{A}_1)_{m,n}(\delta\mathcal{A}_1)_{n+1,m-1} \\ - \alpha(\delta\mathcal{A}_1)_{m,n}(\delta\mathcal{A}_1)_{n-1,m+1} \\ + 2\alpha(\delta\mathcal{A}_2)_{m,n}(\delta\mathcal{A}_2)_{n,m} \\ + \alpha(\delta\mathcal{A}_2)_{m,n}(\delta\mathcal{A}_2)_{n+1,m-1} \\ + \alpha(\delta\mathcal{A}_2)_{m,n}(\delta\mathcal{A}_2)_{n-1,m+1} \\ + 2\mathrm{i}\alpha(\delta\mathcal{A}_1)_{m,n}(\delta\mathcal{A}_2)_{n+1,m-1} \\ - 2\mathrm{i}\alpha(\delta\mathcal{A}_1)_{m,n}(\delta\mathcal{A}_2)_{n-1,m+1}\,. \end{aligned} \tag{71}$$

By defining the variable

$$X_{m,n} = \begin{pmatrix}(\delta\mathcal{A}_1)_{m,n} \\ (\delta\mathcal{A}_2)_{m,n}\end{pmatrix}, \tag{72}$$

we find the following expression for (69):

$$\begin{aligned}\tilde{S}_{\text{quadr}} = 2\alpha X^T_{m,n}\begin{pmatrix}1&0\\0&1\end{pmatrix}X_{n,m} + \alpha X^T_{m,n}\begin{pmatrix}-1&\mathrm{i}\\\mathrm{i}&1\end{pmatrix}X_{n+1,m-1} \\ + \alpha X^T_{m,n}\begin{pmatrix}-1&-\mathrm{i}\\-\mathrm{i}&1\end{pmatrix}X_{n-1,m+1}\,.\end{aligned} \tag{73}$$

The operator involved in (73) is

$$\begin{aligned}G_{m,n;k,l} = 2\alpha\begin{pmatrix}1&0\\0&1\end{pmatrix}\delta_{n,k}\delta_{m,l} + \alpha\begin{pmatrix}-1&\mathrm{i}\\\mathrm{i}&1\end{pmatrix}\delta_{n+1,k}\delta_{m,l+1} \\ + \alpha\begin{pmatrix}-1&-\mathrm{i}\\-\mathrm{i}&1\end{pmatrix}\delta_{n,k+1}\delta_{m+1,l}\,.\end{aligned} \tag{74}$$

The above solution is a minimum for the action provided (74) is a positive operator. This can be shown indeed to be the case once it is realized that $G_{m,n;k,l}$ depends actually only on two indices, since the following identity among the indices holds here: $m+n=k+l$. It follows that $G_{m,\gamma-m;\gamma-l,l}$ with $\gamma=m+n=k+l$ does not depend on γ and therefore $G_{m,\gamma-m;\gamma-l,l}=G_{m,l}$. Then, one has

$$\begin{aligned}G_{ml} = 2\alpha\begin{pmatrix}1&0\\0&1\end{pmatrix}\delta_{m,l} + \alpha\begin{pmatrix}-1&\mathrm{i}\\\mathrm{i}&1\end{pmatrix}\delta_{m,l+1} \\ + \alpha\begin{pmatrix}-1&-\mathrm{i}\\-\mathrm{i}&1\end{pmatrix}\delta_{m+1,l}\,.\end{aligned} \tag{75}$$

This operator can be represented by an infinite-dimensional matrix. Let us set a cut-off N on the dimension of this matrix: $m,l\leq N$. We have

$$G^{(N)} = \alpha\begin{pmatrix} 2 & 0 & -1 & -\mathrm{i} & 0 & 0 & \\ 0 & 2 & -\mathrm{i} & 1 & 0 & 0 & \\ -1 & \mathrm{i} & 2 & 0 & -1 & -\mathrm{i} & \dots \\ \mathrm{i} & 1 & 0 & 2 & -\mathrm{i} & 1 & \\ 0 & 0 & -1 & \mathrm{i} & 2 & 0 & \\ 0 & 0 & \mathrm{i} & 1 & 0 & 2 & \\ & & & \vdots & & & \ddots \end{pmatrix} \tag{76}$$

which is now a diagonalizable $2N\times 2N$ matrix. Indeed:

- 2α is a two-fold degenerate eigenvalue, with $(\mathrm{i},1,0,\dots,0)$ and $(0,\dots,0,-\mathrm{i},1)$ as associated eigenvectors.
- 0 is a $(N-1)$-fold degenerate eigenvalue, with $(\mathrm{i},-1,\mathrm{i},1,0,\dots,0)$, $(0,0,\mathrm{i},-1,\mathrm{i},1,0,\dots,0)$, $\dots$, $(0,\dots,0,\mathrm{i},-1,\mathrm{i},1,0,\dots,0)$, $\dots$, $(0,\dots,0,\mathrm{i},-1,\mathrm{i},1)$ as associated eigenvectors.
- 4α is a $(N-1)$-fold degenerate eigenvalue, with $(-\mathrm{i},1,\mathrm{i},1,0,\dots,0)$, $(0,0,-\mathrm{i},1,\mathrm{i},1,0,\dots,0)$, $\dots$, $(0,\dots,0,-\mathrm{i},1,\mathrm{i},1,0,\dots,0)$, $\dots$, $(0,\dots,0,-\mathrm{i},1,\mathrm{i},1)$ as associated eigenvectors.

Since $\alpha=-\pi\theta\kappa>0$, $Z_{mn}=-\mathrm{i}\sqrt{-\frac{3\kappa}{4}}\delta_{m+1,n}$, or equivalently

$$\begin{aligned}\mathcal{A}_\mu(x) = \sqrt{-3\kappa\theta}\left(\frac{\mathrm{e}^{\frac{z}{2}}}{\sqrt{z}}\int_0^\infty \mathrm{d}t\mathrm{e}^{-t}\sqrt{t}J_1(2\sqrt{tz}) \right. \\ \left. \times\sum_{m=0}^\infty\frac{(-1)^m t^m}{m!\sqrt{m+1}}\right)\tilde{x}_\mu\end{aligned} \tag{77}$$

is a degenerate minimum of the action (11) for $\Omega^2=\frac{1}{3}$, where $\kappa<0$ and $z=\frac{2x^2}{\theta}$.

5.3 The case $D=4$

Equation (59) looks like (46), but the non-triviality of the coefficients of this linear iterative equation makes it much more difficult to solve. Let us introduce the following auxiliary function:

$$y(x) = \sum_{m=1}^\infty v_m x^m\,, \tag{78}$$

since $v_0=0$. Then (59) is equivalent to

$$\begin{aligned}((3\Omega^2-1)(1+x^2)+2(1+\Omega^2)x)y'(x) \\ + \left(\frac{3\Omega^2-1}{x}+1+\Omega^2\right)y(x) \\ = 2(3\Omega^2-1)v_1 - \frac{2\kappa x}{1-x}\,.\end{aligned} \tag{79}$$

This first-order linear differential equation near $x=0$ can be solved whenever $0<\Omega^2<\frac{1}{3}$. Similar considerations apply for $\frac{1}{3}\leq\Omega^2\leq 1$. One obtains

$$\begin{aligned}y(x) = \frac{\sqrt{(1-3\Omega^2)(1+x^2)-2(1+\Omega^2)x}}{x}K \\ + \frac{(1-3\Omega^2)v_1}{4\Omega^2(1-\Omega^2)x}(1-3\Omega^2-(1+\Omega^2)x) \\ + \frac{\kappa\sqrt{2}}{16\Omega^3 x}\arctan\left(\frac{\Omega\sqrt{2}(1+x)}{\sqrt{(1-3\Omega^2)(1+x^2)-2(1+\Omega^2)x}}\right) \\ \times\sqrt{(1-3\Omega^2)(1+x^2)-2(1+\Omega^2)x} \\ + \frac{\kappa(1-3\Omega^2+(\Omega^2-3)x)}{8\Omega^2(1-\Omega^2)x}\,,\end{aligned} \tag{80}$$

where K is a constant. As (v_m) is given by the expansion of the solution (80) near $x=0$, it has to be continuous in

$x=0$. This fixes the value for K. It is given by

$$K=-\frac{\sqrt{1-3\Omega^2}}{8\Omega^2(1-\Omega^2)}\left(2(1-3\Omega^2)v_1+\kappa\right) -\frac{\kappa\sqrt{2}}{16\Omega^3}\arctan\left(\frac{\Omega\sqrt{2}}{\sqrt{1-3\Omega^2}}\right). \quad (81)$$

For $0<\Omega^2<\frac{1}{3}$, it is possible to write down the general solution for the (v_m). Here we will assume that $\kappa=0$ for the sake of simplicity. From the relation

$$\sqrt{1-2\alpha x+x^2} =\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{3}{2}\right)}{k!\Gamma(n+1-2k)\Gamma\left(\frac{3}{2}-n+k\right)}(-2\alpha)^{n-2k}\right)x^n, \quad (82)$$

we have

$$\forall n\geq 2\quad v_n=-\frac{(1-3\Omega^2)^2v_1}{4\Omega^2(1-\Omega^2)}\sum_{k=0}^{\infty}\frac{(-2)^{n+1-2k}\Gamma\left(\frac{3}{2}\right)}{k!\Gamma(n+2-2k)\Gamma\left(\frac{1}{2}-n+k\right)} \times\left(\frac{1+\Omega^2}{1-3\Omega^2}\right)^{n+1-2k}. \quad (83)$$

Using now the definition for the hypergeometric function

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+k)}\frac{z^k}{k!}, \quad (84)$$

we can conclude that

$$\forall m\geq 0\quad v_{m+1}=\frac{(1+\Omega^2)^2v_1}{4\sqrt{\pi}\Omega^2(1-\Omega^2)}\frac{\Gamma(3/2)\Gamma(m+3/2)}{\Gamma(m/2+3/2)\Gamma(m/2+2)} \times\left(\frac{1+\Omega^2}{1-3\Omega^2}\right)^m \times{}_2F_1\left(-\frac{m}{2}-\frac{1}{2},-\frac{m}{2}-1;-m-\frac{1}{2};\frac{(1-3\Omega^2)^2}{(1+\Omega^2)^2}\right). \quad (85)$$

Notice that for $\frac{(1-3\Omega^2)^2}{(1+\Omega^2)^2}$ small enough this expression is positive if we choose $v_1\geq 0$. Furthermore, ${}_2F_1\big(-\frac{m}{2}-\frac{1}{2}, -\frac{m}{2}-1;-m-\frac{1}{2};z\big)$ is a polynomial of degree $\lfloor\frac{m+2}{2}\rfloor$ in z, so that

$$\begin{aligned} v_2&=\frac{1+\Omega^2}{1-3\Omega^2}v_1,\\ v_3&=\frac{(1+4\Omega^2-4\Omega^4)}{(1-3\Omega^2)^2}v_1,\\ v_4&=\frac{(1+\Omega^2)(1+8\Omega^2-5\Omega^4)}{(1-3\Omega^2)^3}v_1,\\ v_5&=\frac{(1+16\Omega^2+26\Omega^4-24\Omega^6-3\Omega^8)}{(1-3\Omega^2)^4}v_1,\\ v_6&=\frac{(1+\Omega^2)(1+24\Omega^2+66\Omega^4-96\Omega^6+21\Omega^8)}{(1-3\Omega^2)^5}v_1,\ldots \end{aligned} \quad (86)$$

Notice also that using (80), the commutative limit for $D=4$ can be obtained in a way similar to what has been done for the two-dimensional case. The solution (v_m) is indeed continuous in $\Omega=0$ for the well-chosen coefficient $v_1=\frac{1}{\theta}+O(\Omega)$ and with $\kappa=O(\Omega)$. It can be realized that the sequence given by (85) is divergent, since it behaves like an exponential so that it does not belong to the Moyal algebra. However, as in the two-dimensional case, it is possible to calculate in (80) the contribution for $\kappa\neq 0$ and to set to zero the coefficient of the divergent part for the solution (v_m), so that the resulting vacuum will again belong to the Moyal algebra with suitable asymptotic behavior.

In the general case, for $\Omega^2\in[0,1]$ and $\kappa\neq 0$, we can express the general solution in the configuration space.

$$Z_1(x)=-\mathrm{i}\sum_{m_1,m_2=0}^{\infty}\sqrt{m_1+1}a_{m_1,m_2}f_{m_1,m_1+1}(x_1,x_2) \times f_{m_2,m_2}(x_3,x_4), \quad (87)$$

where $a_{m_1,m_2}=\mathrm{e}^{\mathrm{i}\xi_m}\sqrt{v_{m+1}}$, $m=m_1+m_2$ and $\xi_m\in\mathbb{R}$ is an arbitrary phase. Using the polar coordinates (52) and the expression (14), one has

$$Z_1(x)=-4\mathrm{i}\sqrt{\frac{2}{\theta}}r_1\mathrm{e}^{\mathrm{i}\varphi_1}\mathrm{e}^{-\frac{r^2}{\theta}}\sum_{m=0}^{\infty}\sum_{m_1=0}^{m}\mathrm{e}^{\mathrm{i}\xi_m}(-1)^m \times\sqrt{v_{m+1}}L^1_{m_1}\left(\frac{2r_1^2}{\theta}\right)L^0_{m-m_1}\left(\frac{2r_2^2}{\theta}\right). \quad (88)$$

Since the identity

$$\sum_{k=0}^{m}L^{\alpha}_k(x)L^{\beta}_{m-k}(y)=L^{\alpha+\beta+1}_m(x+y) \quad (89)$$

is verified, and with (66), we find

$$\begin{aligned} Z_1(x)&=4\sqrt{\frac{\theta}{2}}\frac{\mathrm{e}^{\frac{z}{2}}}{z}\int_0^{\infty}\mathrm{d}t\mathrm{e}^{-t}J_2(2\sqrt{tz}) \times\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sqrt{v_{m+1}}t^{m+1} \times\left((\tilde{x}_1+\mathrm{i}\tilde{x}_2)\cos(\xi_m)+\frac{2}{\theta}(x_1+\mathrm{i}x_2)\sin(\xi_m)\right),\\ Z_2(x)&=4\sqrt{\frac{\theta}{2}}\frac{\mathrm{e}^{\frac{z}{2}}}{z}\int_0^{\infty}\mathrm{d}t\mathrm{e}^{-t}J_2(2\sqrt{tz}) \times\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sqrt{v_{m+1}}t^{m+1} \times\left((\tilde{x}_3+\mathrm{i}\tilde{x}_4)\cos(\xi_m)+\frac{2}{\theta}(x_3+\mathrm{i}x_4)\sin(\xi_m)\right), \end{aligned} \quad (90)$$

where $z=\frac{2x^2}{\theta}$, v_{m+1} is determined above, and $Z_2(x)$ is computed in the same way. The vacuum of the covariant coordinates can therefore be written

$$\mathcal{A}_\mu(x) = 2\sqrt{2\theta}\frac{e^{\frac{z}{2}}}{z}\int_0^\infty dt e^{-t} J_2(2\sqrt{tz}) \times \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\sqrt{v_{m+1}} t^{m+1} \times \left(\tilde{x}_\mu \cos(\xi_m) + \frac{2}{\theta} x_\mu \sin(\xi_m)\right). \quad (91)$$

6 Discussion

Recent attempts to extend the harmonic solution proposed in [15] to the case of gauge theories defined on Moyal spaces have singled out a class of gauge theory models generically described by the action given in (2), for which the gauge potential has a non-vanishing expectation value signaling therefore a non-trivial vacuum. In this paper, we have performed a detailed study of the corresponding vacuum states, focusing on those configurations that are invariant under both rotation and symplectic isomorphisms, i.e. invariant under $G_D = \mathrm{SO}(D) \cap \mathrm{Sp}(D)$, which is a symmetry group for the action, as discussed in Sect. 3. Recall that the explicit determination of these vacua is a necessary step to be reached before the study of its renormalizability can be undertaken, since a reliable perturbative analysis in the present situation can only be defined after the action is expanded around the non-trivial vacuum. The use of the matrix basis for both $D = 2$ and $D = 4$ dimensions proved very convenient when solving the relevant equations of motion in order to obtain rather tractable expressions, written first in the matrix basis and turned back to the position space when necessary. Notice that the technical machinery we set-up in Sects. 4 and 5 of this paper provides, as a byproduct, a rather simple algorithm to solve the equation of motion that involves the Moyal product together with (star)-polynomial interactions.

As the main result of this paper, we have found that the vacuum configurations in the $D = 2$- and $D = 4$-dimensional position space are generically given

$$\mathcal{A}^{2D}_\mu(x) = 2\sqrt{\theta}\frac{e^{\frac{z}{2}}}{\sqrt{z}}\int_0^\infty dt e^{-t}\sqrt{t} J_1(2\sqrt{tz}) \times \sum_{m=0}^{\infty} \frac{(-1)^m \sqrt{u_{m+1}}}{m!\sqrt{m+1}} t^m \times \left(\tilde{x}_\mu \cos(\xi_m) + \frac{2}{\theta} x_\mu \sin(\xi_m)\right), \quad (92)$$

$$\mathcal{A}^{4D}_\mu(x) = 2\sqrt{2\theta}\frac{e^{\frac{z}{2}}}{z}\int_0^\infty dt e^{-t} t J_2(2\sqrt{tz}) \times \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\sqrt{v_{m+1}} t^m \times \left(\tilde{x}_\mu \cos(\xi_m) + \frac{2}{\theta} x_\mu \sin(\xi_m)\right), \quad (93)$$

where ξ_m, (u_m) and (v_m) have been defined in Sects. 5.1 and 5.3 and $z = \frac{2x^2}{\theta}$. Note that these solutions do not correspond to the whole set of G_D-invariant solutions for the equation of motion. Indeed, we have made in Sect. 4 the assumption $(\mathfrak{H})$ that the coefficients a_m and a_{m_1,m_2} are non-zero. Let us now discuss this assumption. In fact, it is tempting to conjecture that requiring the hypothesis $(\mathfrak{H})$ permits one to select only the minima of the action among all the solutions of the equation of motion. This is somewhat supported by the scalar case studied in [39], for which the equation of motion (again a Moyal cubic equation) bears some similarity with the one considered in this paper. In that scalar case, it has been shown that the minima are obtained for a maximal use of the assumption $(\mathfrak{H})$. Moreover, this is also verified for the two special cases $\Omega = 0$ and $\Omega^2 = \frac{1}{3}$. Indeed, the solution for $\Omega = 0$ corresponds to the usual vacuum $A_\mu = 0$, which is of course a minimum of the action, while the case $\Omega^2 = \frac{1}{3}$ in two dimensions has been treated in Sect. 5.2. This unfortunately is much more difficult to verify when Ω is arbitrary, because the operator involved in the quadratic part of the action expanded around the vacuum depends then on four indices, so that diagonalization is very difficult (see Sect. 5.2). Inclusion of ghost terms into the action stemming from some further gauge fixing might well improve this situation as can be realized by inspection of the relevant expressions in the matrix basis. This would remain to be investigated. In any case, if the above conjecture was not verified, it is easy to obtain the coefficients a_m and a_{m_1,m_2} of all the solutions of the equation of motion in the matrix basis from a rather straightforward adaptation of the results of Sect. 5. Notice, however, that this produces a huge number of possible solutions. This will not be considered in the present paper.

Let us finally focus on another special case: $\Omega = 1$ and $\kappa < 0$. The equation of motion (31) simplifies into

$$2\mathcal{A}_\mu \star \mathcal{A}_\nu \star \mathcal{A}_\nu + 2\mathcal{A}_\nu \star \mathcal{A}_\nu \star \mathcal{A}_\mu + 2\kappa \mathcal{A}_\mu = 0, \quad (94)$$

which can be reexpressed in terms of the gauge invariant condensate $C(x) = (\mathcal{A}_\mu \star \mathcal{A}_\mu)(x)$ as

$$\{\mathcal{A}_\mu, 2C + \kappa\}_\star = 0. \quad (95)$$

It is obvious that the constant condensate

$$C(x) = -\frac{\kappa}{2} \quad (96)$$

satisfies this equation of motion. The solutions found in Sect. 5 by requiring the assumption $(\mathfrak{H})$ are

- $u_0 = 0$, $u_{2k} = -\frac{\kappa}{2}$ and $u_{2k+1} = 0$, in two dimensions,
- $v_0 = 0$, $v_{2k} = v_{2k+1} = -\frac{\kappa}{8k+4}$, in four dimensions,

and all these solutions are of constant condensate type (96).

Appendix

In this appendix, we will prove recurrently that the coefficients a_{m_1,m_2} depend only on $m = m_1 + m_2$ for all

$m_1, m_2 \in \mathbb{N}$. We have assumed that $a_{m_1,m_2} \neq 0$. Define $b_0 = a_{0,0}$. Now suppose that for a certain $m \in \mathbb{N}$, $\forall k_1, k_2 \in \mathbb{N}$, so that $k_1 + k_2 \leq m$, a_{k_1,k_2} depends only on $k_1 + k_2$, and we write $a_{k_1,k_2} = b_{k_1+k_2}$. Set also m_1, m_2 so that $m_1 + m_2 = m$. Let us prove that $a_{m_1+1,m_2} = a_{m_1,m_2+1}$.

As $a_{m_1,m_2} = a_{m_1+1,m_2-1} = b_m$ and $a_{m_1-1,m_2} = a_{m_1,m_2-1} = b_{m-1}$, (58) can then be reexpressed as

$$\big((3\Omega^2-1)m|b_{m-1}|^2 + (1+\Omega^2)(2m+3)|b_m|^2 + 2\kappa + ((3\Omega^2-1)(m_1+2) + (1+\Omega^2)(m_2+1))|a_{m_1+1,m_2}|^2 - 2(1-\Omega^2)(m_2+1)\overline{a}_{m_1+1,m_2}a_{m_1,m_2+1}\big)b_m = 0\,, \tag{A.1a}$$

$$\big((3\Omega^2-1)m|b_{m-1}|^2 + (1+\Omega^2)(2m+3)|b_m|^2 + 2\kappa + ((3\Omega^2-1)(m_2+2) + (1+\Omega^2)(m_1+1))|a_{m_1,m_2+1}|^2 - 2(1-\Omega^2)(m_1+1)\overline{a}_{m_1+1,m_2}a_{m_1,m_2+1}\big)\overline{b}_m = 0\,. \tag{A.1b}$$

If we transformat $m_1 \to m_1 + 1$ and $m_2 \to m_2 - 1$ in (A.1b), and $m_1 \to m_1 - 1$ and $m_2 \to m_2 + 1$ in (A.1b), and simplify by $b_m \neq 0$, we obtain

$$(3\Omega^2-1)m|b_{m-1}|^2 + (1+\Omega^2)(2m+3)|b_m|^2 + 2\kappa + \big((3\Omega^2-1)(m_2+1) + (1+\Omega^2)(m_1+2)\big)|a_{m_1+1,m_2}|^2 - 2(1-\Omega^2)(m_1+2)\overline{a}_{m_1+1,m_2}a_{m_1,m_2+1} = 0\,, \tag{A.2a}$$

$$(3\Omega^2-1)m|b_{m-1}|^2 + (1+\Omega^2)(2m+3)|b_m|^2 + 2\kappa + \big((3\Omega^2-1)(m_1+1) + (1+\Omega^2)(m_2+2)\big)|a_{m_1,m_2+1}|^2 - 2(1-\Omega^2)(m_2+2)\overline{a}_{m_1+1,m_2}a_{m_1,m_2+1} = 0\,. \tag{A.2b}$$

Then simplification by b_m in (A.1a) and addition by (A.2a) give rise to

$$2(3\Omega^2-1)m|b_{m-1}|^2 + 2(1+\Omega^2)(2m+3)|b_m|^2 + 4\kappa + 4\Omega^2(m+3)|a_{m_1+1,m_2}|^2 - 2(1-\Omega^2)(m+3)\overline{a}_{m_1+1,m_2}a_{m_1,m_2+1} = 0\,. \tag{A.3}$$

In the same way, with (A.1b) and (A.2b), we obtain

$$2(3\Omega^2-1)m|b_{m-1}|^2 + 2(1+\Omega^2)(2m+3)|b_m|^2 + 4\kappa + 4\Omega^2(m+3)|a_{m_1,m_2+1}|^2 - 2(1-\Omega^2)(m+3)\overline{a}_{m_1+1,m_2}a_{m_1,m_2+1} = 0\,. \tag{A.4}$$

The comparison of (A.3) and (A.4) gives

$$|a_{m_1+1,m_2}|^2 = |a_{m_1,m_2+1}|^2\,. \tag{A.5}$$

By substracting (A.1a) by (A.2b) and using (A.5), we find

$$-2(1-\Omega^2)|a_{m_1+1,m_2}|^2 + 2(1-\Omega^2)\overline{a}_{m_1+1,m_2}a_{m_1,m_2+1} = 0\,, \tag{A.6}$$

and this is the aim of the proof:

$$a_{m_1+1,m_2} = a_{m_1,m_2+1}\,. \tag{A.7}$$

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