On the Structure of a Differential Algebra used by Connes and Lott

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Abstract

We investigate the structure of a differential algebra associated with the simplest two–point K–cycle used by Connes and Lott for physical model building. We show that, in this case, the factorization of the universal differential algebra with respect to the canonical ideal can be performed explicitly. This enables us to present a complete analysis of the structure of the differential algebra under investigation, including explicit multiplication and differentiation rules. Moreover, we get explicit formulae for the scalar product, induced by the Dixmier trace, of elements of the differential algebra of arbitrary degree.

1 Introduction

Several aspects of non–commutative geometry have bee investigated during the last few years. In particular, A. Connes discovered ([7], [6]) that the "classical" Dirac K–cycle of a Riemannian manifold X contains all information about this manifold. Given the K–cycle, one can reconstruct the Riemannian manifold. This led to the abstract notion of a K–cycle over an – in general – non–commutative algebra, giving the possibility to discuss geometric structures, which – in general – do not possess an underlying "classical" manifold. But, interesting enough, already slight modifications of the "classical" K–cycle, namely such that the algebra remains commutative, give rise to interesting physical applications. This is due to the existence of non–commutative differential calculi even on "classical" manifolds. The most prominent example of the above type [7] is the K–cycle over the algebra $C^\infty(X) \otimes (\mathbb{C} \oplus \mathbb{C})$ leading to a unification of gauge and Higgs bosons. Taking the tensor product of this algebra with the vector space of fermions, one

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can derive the Salam–Weinberg model of electroweak interactions ([7], [6], [8]). The above algebra is the simplest example of the class of algebras $C^\infty(X) \otimes (\mathrm{M}_k\mathbb{C} \oplus \mathrm{M}_l\mathbb{C})$, which we call two–point algebras. To derive the full standard model, Connes and Lott [8] proposed to use a K–cycle over the algebra $C^\infty(X) \otimes (\mathbb{C} \oplus \mathbb{H})$, where $\mathbb{H}$ denotes the field of quaternions. For a detailed presentation of this construction we also refer to a series of papers by Kastler ([20], [21], [22], [24]). A comprehensive exposition of the mathematical background can be found in [27] and a physicist's review in [10].

The basic structure occurring in the construction of Connes and Lott is the differential algebra Ω_D^*, which is obtained from the universal differential algebra (associated with the algebra of the K–cycle) by factorizing with respect to a canonically given ideal. For formulating physical actions one only needs an explicit knowledge of the structure of the subspaces Ω_D^n, for $n = 0, 1, 2$, see [8]. However, if one wants to get deeper insight into the mathematical structure of Connes' approach, one should investigate the structure of the whole differential algebra Ω_D^*. This is done here for the simplest two–point algebra. At first sight, the motivation for this analysis seems to be on purely technical grounds. However, as we will show in our next paper [25], the results presented here can be taken as a starting point for constructing graded Lie algebras with derivation. This in turn makes it possible to establish a rigorous link between Connes' theory and the approach to model building proposed by Coquereaux et. al. ([13], [12], [9], [11], [14]). In this approach one postulates ad hoc a certain graded matrix algebra and considers a generalized connection with values in this algebra. The connection contains both differential one forms and zero forms, representing the classical gauge fields of the electroweak interaction and the scalar Higgs fields respectively. Adding by hand the gauge bosons of the strong interaction and choosing appropriate fermionic representations, one can derive the standard model in this way. As already mentioned, there are some deeper structural relations between the two approaches discussed here. We will prove in [25] that given the simplest two–point K–cycle together with a projective module, we are able to construct a graded Lie algebra with derivation. If one chooses the module appropriately, then one arrives at the graded Lie algebra used by Coquereaux and Scheck for the derivation of the standard model.

A technically different attempt to analyse the general two–point case was presented in [18]. However, an explicit exposition of the structure of this algebra (in the sense of explicit multiplication and derivation rules) has not been given. Following this line, it would be interesting to derive, for more general examples contained in the two–point class, the explicit multiplication and differentiation rules given in the present paper for the K–cycle over the algebra $C^\infty(X) \otimes (\mathbb{C} \oplus \mathbb{C})$. A similar analysis of Connes' differential algebra for the N–point case would be interesting, because this case seems to be relevant for the construction of grand unified theories, see [2], [3] and [5]. For the time being these authors circumvent the analysis of the algebra Ω_D^*. Instead of that they introduce – as in

the earlier papers of Connes [6] – additional auxiliary fields, which finally have to be eliminated. Finally, let us mention that there also exist attempts to describe gravity by methodes of non–commutative geometry ([4], [19], [23]).

The paper is organized as follows: In section 2 we review the K–cycle $(\mathcal{A}, h, D)$ of Connes and Lott given in [8] and the construction of the basic differential algebra Ω_D^* obtained from the universal differential algebra Ω^* by factorization with respect to a certain canonically given ideal. In section 3 we define an a priori different factorization procedure leading – for the simplest two–point case – to a differential algebra denoted by $\Lambda_{\mathcal{A}}^*$. We give explicit formulae for the multiplication and differentiation of elements of $\Lambda_{\mathcal{A}}^*$. In section 4 we prove that $\Lambda_{\mathcal{A}}^*$ and Ω_D^* coincide. Finally, in section 5 we show, using the Dixmier trace, that there is a natural embedding of the subspace $\Lambda_{\mathcal{A}}^k$ of elements of degree k in $\Lambda_{\mathcal{A}}^*$ into the representation $\pi(\Omega^k)$ of elements of degree k of the universal algebra acting on the Hilbert space h of the K–cycle. This provides us with a natural scalar product on $\Lambda_{\mathcal{A}}^*$. Since for the K–cycle under consideration we know the above embedding explicitly, we get explicit formulae for elements of arbitrary degree.

2 The K–cycle and the Differential Algebra of Connes and Lott

Let X be a compact even dimensional Riemannian spin manifold, $dim\, X =: N \geq 4$, and $L^2(X, S)$ be the Hilbert space of square integrable sections of the spinor bundle over X. Let F be a finite dimensional Hilbert space. The Hilbert space h of the K–cycle is chosen as

$$h := L^2(X, S) \otimes F \otimes \mathbb{C}^2 , \tag{1}$$

with natural scalar product $< , >_h :\ h \times h \to \mathbb{C}$.

Let C be the Clifford bundle over X, whose fibre at each point $x \in X$ is the complexified Clifford algebra $Cliff_{\mathbb{C}}(T_x^* X)$ of the cotangent space $T_x^* X$. We denote by $\Gamma^\infty(C)$ the set of smooth sections of C, and $C^k \subset \Gamma^\infty(C)$ is the set of those sections of C, whose values at each point $x \in X$ belong to the subspace spanned by products of less than or equal k elements of $T_x^* X$ of the same parity.

The algebra $\mathcal{A}$ of the K–cycle is the following subalgebra of the algebra $\Gamma^\infty(C) \otimes End\,(F) \otimes \mathrm{M}_2\mathbb{C} \subset B(h)$, where $B(h)$ is the algebra of bounded operators on h :

$$\mathcal{A} := \left\{ \ a = \begin{pmatrix} c_1 \otimes 1^F & 0 \\ 0 & c_4 \otimes 1^F \end{pmatrix} \ , \quad c_1, c_4 \in C^0 \ \right\} , \tag{2}$$

where 1^F is the identical endomorphism of F. The identity of $\mathcal{A}$ is $\mathrm{I} = 1^C \otimes 1^F \otimes \mathrm{I}_{2\times 2}$, where 1^C is the unit section of C and $\mathrm{I}_{2\times 2}$ is the 2×2 identity matrix. The involution $^*:\ \mathcal{A} \to \mathcal{A}$ is given by $< a^* \phi^1, \phi^2 >_h :=< \phi^1, a\phi^2 >_h$, $\quad \forall \phi^1, \phi^2 \in h$, $a \in \mathcal{A}$.

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We denote $\mathcal{M} = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}$, $M \in End\,(F)$, and demand $MM^* \neq c\,1^F$, for any $c \in \mathbb{C}$. Next we define the generalized Dirac operator D of the K–cycle as

$$D := D^{cl} \otimes 1^F \otimes I_{2 \times 2} + \gamma^{N+1} \otimes \mathcal{M} = \begin{pmatrix} D^{cl} \otimes 1^F & \gamma^{N+1} \otimes M \\ \gamma^{N+1} \otimes M^* & D^{cl} \otimes 1^F \end{pmatrix} . \tag{3}$$

Here D^{cl} is the classical Dirac operator on $L^2(X, S)$, which is locally given by $D^{cl} = i\gamma^\mu(\partial_\mu + \omega_\mu^g)$, where ω_μ^g is the spin connection associated to the Levi–Civita connection of the manifold X. The gamma matrices γ^μ, $\mu = 1, \ldots, N$, are local orthonormal sections of C^1, i.e.

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\,\delta^{\mu\nu}\,1^C . \tag{4}$$

We choose the γ^μ to be selfadjoint and denote

$$\gamma^{N+1} := i^{\frac{N}{2}}\,\gamma^1 \gamma^2 \cdots \gamma^{N-1} \gamma^N . \tag{5}$$

The matrix γ^{N+1} is a grading operator on $L^2(X, S)$, i.e. it commutes with C^0 and it anticommutes with the Dirac operator. Moreover, γ^{N+1} is selfadjoint and $(\gamma^{N+1})^2 = 1$.

Using (2) and (3) one finds

$$[D, a] = \begin{pmatrix} D^{cl}(c_1) \otimes 1^F & (c_4 - c_1)\gamma^{N+1} \otimes M \\ (c_1 - c_4)\gamma^{N+1} \otimes M^* & D^{cl}(c_4) \otimes 1^F \end{pmatrix} , \quad a \in \mathcal{A} . \tag{6}$$

It was proved in [27] that $[D, a]$ is bounded for all $a \in \mathcal{A}$. Obviously, D is selfadjoint because of the selfadjointness of D^{cl} and the special choice of $\mathcal{M}$. It can be shown [16] that $(I + D^2)^{-1}$ is compact.

The selfadjoint $\mathbb{R}_2$–grading operator Γ of the K–cycle is given by

$$\Gamma = \begin{pmatrix} -\gamma^{N+1} \otimes 1^F & 0 \\ 0 & \gamma^{N+1} \otimes 1^F \end{pmatrix} , \tag{7}$$

with $\Gamma^2 = I$, $\Gamma D + D\Gamma = 0$, and $\Gamma a = a\Gamma$, $\forall a \in \mathcal{A}$.

Hence, $(\mathcal{A}, h, D, \Gamma)$ is an even K–cycle according to Definition 1 in [8].

Let Ω^* be the universal differential algebra over $\mathcal{A}$ generated by symbols a and da, $a \in \mathcal{A}$:

$$\Omega^* = \bigoplus_{n=0}^\infty \Omega^n , \quad \Omega^0 = \mathcal{A} , \tag{8}$$

$$\Omega^n = \{ \sum_\alpha a_\alpha^0 da_\alpha^1 \ldots da_\alpha^n : a_\alpha^i \in \mathcal{A} , \ i = 0, \ldots, n , \ \text{finite sum} \} .$$

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The universal differential d is defined by

$$d(\sum_\alpha a_\alpha^0 da_\alpha^1 \ldots da_\alpha^n) := \sum_\alpha da_\alpha^0 da_\alpha^1 \ldots da_\alpha^n \equiv \sum_\alpha \mathrm{I} da_\alpha^0 \ldots da_\alpha^n \ , \qquad (9)$$

and has the properties

$$\begin{aligned} d(ab) &= (da)b + a\,db \ , \\ (da)^* &= da^* \ , \\ d\mathrm{I} &= 0 \ . \end{aligned} \qquad (10)$$

From this one gets

$$\begin{aligned} d(\omega_1\omega_2) &= (d\omega_1)\,\omega_2 + (-1)^n \omega_1 d\omega_2 \ , \quad \forall \omega_1 \in \Omega^n \ , \quad \omega_2 \in \Omega^* \ , \qquad (11) \\ d^2\omega &= 0 \ , \quad \forall \omega \in \Omega^* \ . \end{aligned}$$

It was proved in [27] that

$$\pi\left(\sum_\alpha a_\alpha^0 da_\alpha^1 \ldots da_\alpha^n\right) := \sum_\alpha (-i)^n\, a_\alpha^0 [D, a_\alpha^1] \cdots [D, a_\alpha^n] \qquad (12)$$

defines an involutive representation π of Ω^* on h . From (2) and (6) one obtains by induction the structure of the space $\pi(\Omega^k)$:

$$\begin{aligned} \pi(\Omega^k) = &\sum_{r=0}^{[k/2]} \left[\begin{array}{cc} C^{k-2r} \otimes \mathbb{C}\,(MM^*)^r & 0 \\ 0 & C^{k-2r} \otimes \mathbb{C}\,(M^*M)^r \end{array} \right] \qquad (13) \\ &\oplus \sum_{r=0}^{[(k-1)/2]} \left[\begin{array}{cc} 0 & C^{k-2r-1}\gamma^{N+1} \otimes \mathbb{C}\, M(M^*M)^r \\ C^{k-2r-1}\gamma^{N+1} \otimes \mathbb{C}\, M^*(MM^*)^r & 0 \end{array} \right] \ , \end{aligned}$$

where $[n/2] := n/2$ for n even and $[n/2] := (n-1)/2$ for n odd. Here we use the convention $C^n \equiv C^N$ for even $n > N$ and $C^n \equiv C^{N-1}$ for odd $n > N$.

Obviously, $\pi(\omega) = 0$ does not imply $\pi(d\omega) = 0$, for $\omega \in \Omega^*$. One defines $J = J_0 + dJ_0$, with $J_0 = \ker \pi$. It was shown in [27] that J is a two–sided graded ideal of Ω^* , which is invariant under d . Factorization with respect to this ideal leads to the differential algebra

$$\Omega_D^* := \bigoplus_{n=0}^\infty \Omega_D^n \ , \quad \Omega_D^n := \pi(\Omega^n)/\pi(J \cap \Omega^n) \ , \qquad (14)$$

of Connes and Lott.

3   The Differential Algebra $\Lambda_{\mathcal{A}}^*$

Here we are going to construct a differential algebra $\Lambda_{\mathcal{A}}^*$, which turns out to coincide with Ω_D^*. Since $\pi(\Omega^{n-2})$ is a vector subspace of $\pi(\Omega^n)$, for $n \geq 2$, we have the natural projection

$$\sigma_n : \pi(\Omega^n) \to \pi(\Omega^n) / \pi(\Omega^{n-2}) , \tag{15}$$

which we extend trivially, putting

$$\sigma_n := id_{\pi(\Omega^n)} \quad \text{for} \quad n = 0, 1 . \tag{16}$$

In order to perform this factorization explicitly, we decompose each vector space $\pi(\Omega^k)$ into a direct sum. We denote by V_q, $q = 1, \ldots, 4$, the vector spaces spanned by M_q^r, where $M_1^r := (MM^*)^r$, $M_2^r := M(M^*M)^r$, $M_3^r := M^*(MM^*)^r$, $M_4^r := (M^*M)^r$. Since F is finite dimensional, there is a maximal number $m + 1$ of linear independent elements in V_1. Then it is easy to show that $\{M_1^i\}_{i=0}^m$ is a basis in V_1. Moreover, one proves that $\{M_q^i\}_{i=0}^m$ are bases in V_q, for $q = 2, 3, 4$, i.e. these bases have equal dimensions. Therefore, we have unique decompositions

$$M_q^r = \sum_{i=0}^m v_{q,i}^r M_q^i , \quad q = 1, \ldots, 4 , \quad \text{with} \quad v_{q,i}^r = \delta_i^r \quad \text{for} \quad r \leq m . \tag{17}$$

Denoting the left upper corner of $\pi(\Omega^k)$, see (13), by P_1^k, we obtain

$$P_1^k = \sum_{r=0}^{[\frac{k}{2}]} C^{k-2r} \otimes \mathbb{C}M_1^r = \sum_{r=0}^{\min(m,[\frac{k}{2}])} C^{k-2r} \otimes \mathbb{C}\, M_1^r + \sum_{r=m+1}^{[\frac{k}{2}]} C^{k-2r} \otimes \mathbb{C}\, M_1^r$$

$$= \bigoplus_{r=0}^{\min(m,[\frac{k}{2}])} C^{k-2r} \otimes \mathbb{C}\, M_1^r . \tag{18}$$

The last step follows from (17) and the fact that $C^{n-2t} \subset C^n$. Putting $C^n = \{0\}$, $\forall n < 0$, we can formally extend this direct sum from $r = 0$ to m. By the same procedure one gets a direct sum decomposition of the other three blocks in the matrix (13). Thus, $\pi(\Omega^k)$ finally reads

$$\pi(\Omega^k) = \begin{bmatrix} \displaystyle\bigoplus_{r=0}^m C^{k-2r} \otimes \mathbb{C}\, M_1^r & ; & \displaystyle\bigoplus_{r=0}^m C^{k-2r-1}\gamma^{N+1} \otimes \mathbb{C}\, M_2^r \\ \\ \displaystyle\bigoplus_{r=0}^m C^{k-2r-1}\gamma^{N+1} \otimes \mathbb{C}\, M_3^r & ; & \displaystyle\bigoplus_{r=0}^m C^{k-2r} \otimes \mathbb{C}\, M_4^r \end{bmatrix} . \tag{19}$$

Elements of $\pi(\Omega^k)$ will be denoted by

$$
\tau^k = \begin{pmatrix} \displaystyle\sum_{r=0}^{m} c_1^{k-2r} \otimes M_1^r & ; & \displaystyle\sum_{r=0}^{m} c_2^{k-2r-1}\gamma^{N+1} \otimes M_2^r \\[4mm] \displaystyle\sum_{r=0}^{m} c_3^{k-2r-1}\gamma^{N+1} \otimes M_3^r & ; & \displaystyle\sum_{r=0}^{m} c_4^{k-2r} \otimes M_4^r \end{pmatrix} , \tag{20}
$$

where $c_q^n \in C^n$. We define $L^n := C^n/C^{n-2}$, in particular we have $L^0 \equiv C^0$ and $L^1 \equiv C^1$. Obviously, $L^n = \{0\}$ for $n > N$ and for $n < 0$. In the Appendix we show that there is a natural right action of γ^{N+1} on $L^* = \bigoplus_n L^n$ inherited from the multiplication in the Clifford algebra. This action commutes with the above factorization, i.e.

$$
L^n \gamma^{N+1} \cong (C^n \gamma^{N+1})/(C^{n-2}\gamma^{N+1}) . \tag{21}
$$

Lemma 1 *The space $\sigma_k \circ \pi(\Omega^k)$ has the structure*

$$
\sigma_k \circ \pi(\Omega^k) \cong \begin{bmatrix} \displaystyle\bigoplus_{r=0}^{m} L^{k-2r} \otimes \mathbb{C}\, M_1^r & ; & \displaystyle\bigoplus_{r=0}^{m} L^{k-2r-1}\gamma^{N+1} \otimes \mathbb{C}\, M_2^r \\[4mm] \displaystyle\bigoplus_{r=0}^{m} L^{k-2r-1}\gamma^{N+1} \otimes \mathbb{C}\, M_3^r & ; & \displaystyle\bigoplus_{r=0}^{m} L^{k-2r} \otimes \mathbb{C}\, M_4^r \end{bmatrix} . \tag{22}
$$

Proof: For $k < 2$ no factorization occurs. We prove the Lemma for the block P_1^k in the upper left corner of the matrix (19). Using the decomposition (66) for C^{k-2r} and C^{k-2r-2} we get

$$
\begin{aligned}
P_1^k &\cong (\bigoplus_{r=0}^{m} L^{k-2r} \otimes \mathbb{C}\, M_1^r) \oplus (\bigoplus_{r=0}^{m} C^{k-2r-2} \otimes \mathbb{C}\, M_1^r) \\
&= (\bigoplus_{r=0}^{m} L^{k-2r} \otimes \mathbb{C}\, M_1^r) \oplus P_1^{k-2} .
\end{aligned}
$$

The proof for the other diagonal block is identical. For the off–diagonal blocks one has to use (21). $\qquad\square$

For $\tau^k \in \pi(\Omega^k)$ the projection $\sigma_k(\tau^k)$ is calculated by decomposing τ^k according to (20) and by keeping only the part, which contributes to $\sigma_k \circ \pi(\Omega^k)$ according to Lemma 1. We denote the block in the left upper corner of $\sigma_k \circ \pi(\Omega^k)$ by S_1^k. Then from Lemma 1 we read off the explicit structure of the non–vanishing S_1^k (assuming $m \geq N/2$):

$$
S_1^0 \cong L^0 \otimes \mathbb{C}M_1^0 , \quad S_1^1 \cong L^1 \otimes \mathbb{C}M_1^0 ,
$$

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$$S_1^2 \cong L^2 \otimes \mathbb{C}M_1^0 \oplus L^0 \otimes \mathbb{C}M_1^1 \ , \ \ldots \ ,$$

$$S_1^{N-1} \cong \bigoplus_{r=0}^{(N/2)-1} L^{N-2r-1} \otimes \mathbb{C}M_1^r \ , \quad S_1^N \cong \bigoplus_{r=0}^{N/2} L^{N-2r} \otimes \mathbb{C}M_1^r \ ,$$

$$S_1^{N+2t-1} \cong \left(1^C \otimes \mathbb{C}(MM^*)^t\right) S_1^{N-1} \ , \quad \text{for} \ \ t = 1,\ldots,m-(N/2)+1 \ ,$$

$$S_1^{N+2t} \cong \left(1^C \otimes \mathbb{C}(MM^*)^t\right) S_1^N \ , \qquad \text{for} \ \ t = 1,\ldots,m-(N/2) \ , \tag{23}$$

$$S_1^{2m+2} \cong \bigoplus_{r=0}^{(N/2)-1} L^{N-2r} \otimes \mathbb{C}M_1^{r+m-(N/2)+1} \ ,$$

$$S_1^{2m+3} \cong \bigoplus_{r=0}^{(N/2)-2} L^{N-2r-1} \otimes \mathbb{C}M_1^{r+m-(N/2)+2} \ , \ \ldots \ ,$$

$$S_1^{N+2m-1} \cong L^{N-1} \otimes \mathbb{C}M_1^m \ , \quad S_1^{N+2m} \cong L^N \otimes \mathbb{C}M_1^m \ .$$

Similar formulae can easily be written down for the other three blocks of $\sigma^k \circ \pi(\Omega^k)$. Let us denote $\Lambda_{\mathcal{A}}^k := \sigma_k \circ \pi(\Omega^k)$ and observe that $\Lambda_{\mathcal{A}}^k = \{0\}$ for $k > N + 2m + 1$. Now we define a graded algebra $\Lambda_{\mathcal{A}}^*$ by

$$\Lambda_{\mathcal{A}}^* = \bigoplus_{k=0}^{\infty} \Lambda_{\mathcal{A}}^k \ , \tag{24}$$

with multiplication

$$\Lambda_{\mathcal{A}}^k \times \Lambda_{\mathcal{A}}^l \ni (\lambda^k, \tilde{\lambda}^l) \mapsto \lambda^k \bullet \tilde{\lambda}^l := \sigma_{k+l}(\tau^k \cdot \tilde{\tau}^l) \in \Lambda_{\mathcal{A}}^{k+l} \ , \tag{25}$$

where $\tau^k \in \pi(\Omega^k)$, $\tilde{\tau}^l \in \pi(\Omega^l)$, such that $\sigma_k(\tau^k) = \lambda^k$, $\sigma_l(\tilde{\tau}^l) = \tilde{\lambda}^l$. It is easy to check that the definition of this multiplication is independent of the choice of representatives. The general form of an element $\lambda^k \in \Lambda_{\mathcal{A}}^k$ is – according to Lemma 1 – given by

$$\lambda^k = \begin{pmatrix} \displaystyle\sum_{t=0}^m \alpha_1^{k-2t} \otimes M_1^t & ; & \displaystyle\sum_{t=0}^m \alpha_2^{k-2t-1} \gamma^{N+1} \otimes M_2^t \\[4mm] \displaystyle\sum_{t=0}^m \alpha_3^{k-2t-1} \gamma^{N+1} \otimes M_3^t & ; & \displaystyle\sum_{t=0}^m \alpha_4^{k-2t} \otimes M_4^t \end{pmatrix} \ , \quad \alpha_q^n \in L^n \ . \tag{26}$$

This shows that an element $\lambda^k \in \Lambda_{\mathcal{A}}^k$ is completely characterized by the following $4(m+1)$ elements of L^* (some of them may be identically equal to zero):

$$\alpha_1^{k-2t} \ , \quad \alpha_2^{k-2t-1} \ , \quad \alpha_3^{k-2t-1} \ , \quad \alpha_4^{k-2t} \ , \quad t = 0,\ldots,m \ . \tag{27}$$

Let ι_k be the classical vector space isomorphism

$$\iota_k : L^k \equiv C^k/C^{k-2} \to \Lambda^k(X) \ , \tag{28}$$

where $\Lambda^k(X)$ is the set of (complex–valued) k–forms on X . The exterior product $\wedge$ in $\Lambda^*(X) = \bigoplus_{k=0}^{N} \Lambda^k(X)$ transported by the isomorphism ι is denoted by the same symbol. We denote the factorization in C^k with respect to C^{k-2} by $\sigma_k^{c\ell} : C^k \to L^k \equiv C^k/C^{k-2}$, fulfilling

$$\sigma_{k+l}^{c\ell}(c^k \tilde{c}^l) = \sigma_k^{c\ell}(c^k) \wedge \sigma_l^{c\ell}(\tilde{c}^l) , \quad \forall c^k \in C^k , \quad \tilde{c}^l \in C^l . \tag{29}$$

Lemma 2 *Let $\alpha_q^n \in L^n$ and $\tilde{\alpha}_q^n \in L^n$ be the characterizing elements of $\lambda^k \in \Lambda_{\mathcal{A}}^k$ and $\tilde{\lambda}^l \in \Lambda_{\mathcal{A}}^l$ respectively. Then the characterizing elements β_q^n of $\lambda^k \bullet \tilde{\lambda}^l \in \Lambda_{\mathcal{A}}^{k+l}$ are given by*

$$\beta_1^{k+l-2t} = \sum_{r=0}^{t} (\alpha_1^{k-2r} \wedge \tilde{\alpha}_1^{l-2(t-r)} + (-1)^{l-1}\alpha_2^{k-2r-1} \wedge \tilde{\alpha}_3^{l-2(t-r)+1}) ,$$

$$\beta_2^{k+l-2t-1} = \sum_{r=0}^{t} (\alpha_1^{k-2r} \wedge \tilde{\alpha}_2^{l-2(t-r)-1} + (-1)^{l}\alpha_2^{k-2(t-r)-1} \wedge \tilde{\alpha}_4^{l-2r}) ,$$

$$\beta_3^{k+l-2t-1} = \sum_{r=0}^{t} (\alpha_4^{k-2r} \wedge \tilde{\alpha}_3^{l-2(t-r)-1} + (-1)^{l}\alpha_3^{k-2(t-r)-1} \wedge \tilde{\alpha}_1^{l-2r}) ,$$

$$\beta_4^{k+l-2t} = \sum_{r=0}^{t} (\alpha_4^{k-2r} \wedge \tilde{\alpha}_4^{l-2(t-r)} + (-1)^{l-1}\alpha_3^{k-2r-1} \wedge \tilde{\alpha}_2^{l-2(t-r)+1}) ,$$

where $t = 0, \ldots, m$ and elements α_q^n , $\tilde{\alpha}_q^n$, which do not occur in (27) are understood to be equal to zero.

Proof: For λ^k and $\tilde{\lambda}^l$ we take representatives $\tau^k \in \pi(\Omega^k)$ and $\tilde{\tau}^l \in \pi(\Omega^l)$, such that $\sigma_k(\tau^k) = \lambda^k$ and $\sigma_l(\tilde{\tau}^l) = \tilde{\lambda}^l$:

$$\tau^k = \left(\begin{array}{cc} \displaystyle\sum_{r=0}^{m} c_1^{k-2r} \otimes (MM^*)^r & ; \quad \displaystyle\sum_{r=0}^{m} c_2^{k-2r-1}\gamma^{N+1} \otimes M(M^*M)^r \\[4ex] \displaystyle\sum_{r=0}^{m} c_3^{k-2r-1}\gamma^{N+1} \otimes M^*(MM^*)^r & ; \quad \displaystyle\sum_{r=0}^{m} c_4^{k-2r} \otimes (M^*M)^r \end{array} \right) ,$$

$$\tilde{\tau}^l = \left(\begin{array}{cc} \displaystyle\sum_{s=0}^{m} \tilde{c}_1^{l-2s} \otimes (MM^*)^s & ; \quad \displaystyle\sum_{s=0}^{m} \tilde{c}_2^{l-2s-1}\gamma^{N+1} \otimes M(M^*M)^s \\[4ex] \displaystyle\sum_{s=0}^{m} \tilde{c}_3^{l-2s-1}\gamma^{N+1} \otimes M^*(MM^*)^s & ; \quad \displaystyle\sum_{s=0}^{m} \tilde{c}_4^{l-2s} \otimes (M^*M)^s \end{array} \right) ,$$

where $c_q^n \in C^n$, with $\sigma_n^{c\ell}(c_q^n) = \alpha_q^n$, and $\tilde{c}_q^n \in C^n$, with $\sigma_n^{c\ell}(\tilde{c}_q^n) = \tilde{\alpha}_q^n$. We prove the Lemma for the upper right block s_2^{k+l} of $(\lambda^k \bullet \tilde{\lambda}^l)$. Let p_2^{k+l} be the upper right

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block of the product $\tau^k \cdot \tilde{\tau}^l$ in $\pi(\Omega^*)$. Then we have

$$
\begin{aligned}
p_2^{k+l} &= \Big(\sum_{r=0}^{m} c_1^{k-2r} \otimes (MM^*)^r \Big) \cdot \Big(\sum_{s=0}^{m} \tilde{c}_2^{l-2s-1} \gamma^{N+1} \otimes M(M^*M)^s \Big) \\
&\quad + \Big(\sum_{r=0}^{m} c_2^{k-2r-1} \gamma^{N+1} \otimes M(M^*M)^r \Big) \cdot \Big(\sum_{s=0}^{m} \tilde{c}_4^{l-2s} \otimes (M^*M)^s \Big) \\
&= \sum_{r=0}^{m} \sum_{s=0}^{m} c_1^{k-2r} \tilde{c}_2^{l-2s-1} \gamma^{N+1} \otimes M(M^*M)^{r+s} \\
&\quad + \sum_{r=0}^{m} \sum_{s=0}^{m} (-1)^{l-2s} \, c_2^{k-2r-1} \tilde{c}_4^{l-2s} \gamma^{N+1} \otimes M(M^*M)^{r+s} \\
&= \sum_{t=0}^{m} \sum_{r=0}^{t} \big(c_1^{k-2r} \tilde{c}_2^{l-2(t-r)-1} + (-1)^l \, c_2^{k-2r-1} \tilde{c}_4^{l-2(t-r)} \big) \gamma^{N+1} \otimes M_2^t \\
&\quad + \sum_{t=m+1}^{2m} \sum_{r=t-m}^{m} \big(c_1^{k-2r} \tilde{c}_2^{l-2(t-r)-1} + (-1)^l \, c_2^{k-2r-1} \tilde{c}_4^{l-2(t-r)} \big) \gamma^{N+1} \otimes M_2^t \, .
\end{aligned}
$$

Observing that the second term of this sum lies in the kernel of σ_{k+l} and using (29) we get

$$
s_2^{k+l} = \sum_{t=0}^{m} \sum_{r=0}^{t} \big(\alpha_1^{k-2r} \wedge \tilde{\alpha}_2^{l-2(t-r)-1} + (-1)^l \, \alpha_2^{k-2r-1} \wedge \tilde{\alpha}_4^{l-2(t-r)} \big) \gamma^{N+1} \otimes M_2^t \, . \tag{30}
$$

The proof for the other blocks of $\lambda^k \bullet \tilde{\lambda}^l$ is similar. $\qquad\square$

Next, we introduce an involution on $\Lambda_{\mathcal{A}}^*$, putting

$$
\lambda^{k*} := \sigma_k(\tau^{k*}) \, , \quad \text{where} \quad \tau^k \in \pi(\Omega^k) \, , \quad \text{such that} \quad \sigma_k(\tau^k) = \lambda^k \in \Lambda_{\mathcal{A}}^k \, . \tag{31}
$$

This definition is independent of the choice of the representative τ^k. One gets

$$
(\lambda^k \bullet \tilde{\lambda}^l)^* = (\tilde{\lambda}^l)^* \bullet (\lambda^k)^* \, , \quad \forall \lambda^k \in \Lambda_{\mathcal{A}}^k \, , \quad \tilde{\lambda}^l \in \Lambda_{\mathcal{A}}^l \, . \tag{32}
$$

Using the explicit representation (26) we obtain

$$
\lambda^{k*} = \begin{pmatrix} \displaystyle\sum_{t=0}^{m} \alpha_1^{k-2t*} \otimes M_1^t \, ; & \displaystyle\sum_{t=0}^{m} (-1)^{k-1} \alpha_3^{k-2t-1*} \gamma^{N+1} \otimes M_2^t \\[2em] \displaystyle\sum_{t=0}^{m} (-1)^{k-1} \alpha_2^{k-2t-1*} \gamma^{N+1} \otimes M_3^t \, ; & \displaystyle\sum_{t=0}^{m} \alpha_4^{k-2t*} \otimes M_4^t \end{pmatrix} \, . \tag{33}
$$

Now we endow $\Lambda_{\mathcal{A}}^*$ with the structure of a differential algebra. First we transport the ordinary exterior differential d from $\Lambda^*(X)$ to L^* using the isomorphism ι, see (28),

$$
\mathbf{d}\alpha^k := \iota_{k+1}^{-1} \circ d \circ \iota_k(\alpha^k) \, , \quad \forall \alpha^k \in L^k \, , \tag{34}
$$

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and define the codifferential $\mathbf{d}^*$ on L^*, see Appendix, by

$$\mathbf{d}^* = \gamma^{N+1}\mathbf{d}\gamma^{N+1} \ . \tag{35}$$

Here γ^{N+1} means the left action on L^*, which is related to the right action by (70), see Appendix. Observe that

$$\left((\mathbf{d} - \mathbf{d}^*) \otimes 1^F \otimes \mathrm{I}_{2\times2} \right)(\lambda^k) \in \Lambda_{\mathcal{A}}^{k+1} \oplus \Lambda_{\mathcal{A}}^{k-1} \ , \quad \forall \lambda^k \in \Lambda_{\mathcal{A}}^k \ . \tag{36}$$

Denoting the projection onto the first component of this direct sum by pr_{k+1}, we define an operator $\mathbf{D} : \Lambda_{\mathcal{A}}^k \to \Lambda_{\mathcal{A}}^{k+1}$, putting

$$\mathbf{D}\lambda^k := pr_{k+1} \circ \left((\mathbf{d} - \mathbf{d}^*) \otimes 1^F \otimes \mathrm{I}_{2\times2} \right)(\lambda^k) \ , \quad \forall \lambda^k \in \Lambda_{\mathcal{A}}^k \ . \tag{37}$$

Explicitly, for $\lambda^k \in \Lambda_{\mathcal{A}}^k$ given in (26) we get

$$\mathbf{D}\lambda^k = \begin{pmatrix} \displaystyle\sum_{t=0}^{m}(\mathbf{d}\alpha_1^{k-2t}) \otimes M_1^t & ; & \displaystyle\sum_{t=0}^{m}(\mathbf{d}\alpha_2^{k-2t-1})\gamma^{N+1} \otimes M_2^t \\[2em] \displaystyle\sum_{t=0}^{m}(\mathbf{d}\alpha_3^{k-2t-1})\gamma^{N+1} \otimes M_3^t & ; & \displaystyle\sum_{t=0}^{m}(\mathbf{d}\alpha_4^{k-2t}) \otimes M_4^t \end{pmatrix} \ . \tag{38}$$

Indeed, in the diagonal blocks of λ^k there only survives the differential because the codifferential applied to them contributes only to Λ_A^{k-1}. In the off–diagonal blocks only the codifferential survives, giving

$$\begin{aligned} \mathbf{d}^*(\alpha_q^{k-2t-1}\gamma^{N+1}) &= \gamma^{N+1}\mathbf{d}(\gamma^{N+1}\alpha_q^{k-2t-1}\gamma^{N+1}) \\ &= (-1)^{k-2t-1}\gamma^{N+1}\mathbf{d}(\alpha_q^{k-2t-1}) = -(\mathbf{d}\alpha_q^{k-2t-1})\gamma^{N+1} \ . \end{aligned}$$

Lemma 3 $\mathbf{D}$ *is a graded differential on $\Lambda_{\mathcal{A}}^*$, i.e.*

$i)$ $\mathbf{D}\lambda^k \in \Lambda_{\mathcal{A}}^{k+1}$, $\lambda^k \in \Lambda_{\mathcal{A}}^k$,

$ii)$ $\mathbf{D}(\lambda^k \bullet \tilde{\lambda}^l) = (\mathbf{D}\lambda^k) \bullet \tilde{\lambda}^l + (-1)^k \lambda^k \bullet (\mathbf{D}\tilde{\lambda}^l)$, $\lambda^k \in \Lambda_{\mathcal{A}}^k$, $\tilde{\lambda}^l \in \Lambda_{\mathcal{A}}^l$,

$iii)$ $\mathbf{D}^2\lambda^k = 0$, $\lambda^k \in \Lambda_{\mathcal{A}}^k$.

Proof: $i)$ is clear by definition. $ii)$ can be checked by explicit calculation using (26), (38), and Lemma 2. $iii)$ is a consequence of (38) and the identity $\mathbf{d}^2 \equiv 0$.

$\square$

We observe that $\gamma^{N+1} \otimes \mathcal{M} \in \Lambda_{\mathcal{A}}^1$. Therefore, it makes sense to introduce the following graded commutator:

$$[\gamma^{N+1} \otimes \mathcal{M}, \lambda^k]_g := (\gamma^{N+1} \otimes \mathcal{M}) \bullet \lambda^k - (-1)^k \lambda^k \bullet (\gamma^{N+1} \otimes \mathcal{M}) \ . \tag{39}$$

We define

$$\hat{d} := \mathbf{D} - i[\gamma^{N+1} \otimes \mathcal{M}, \, . \,]_g \ . \tag{40}$$

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Lemma 4 $\hat{d}$ *is a graded differential on* $\Lambda_{\mathcal{A}}^*$, *i.e.*

$$i) \quad \hat{d}\lambda^k \in \Lambda_{\mathcal{A}}^{k+1} ,$$
$$ii) \quad \hat{d}(\lambda^k \bullet \tilde{\lambda}^l) = (\hat{d}\lambda^k) \bullet \tilde{\lambda}^l + (-1)^k \lambda^k \bullet (\hat{d}\tilde{\lambda}^l) , \tag{41}$$
$$iii) \quad \hat{d}^2 \lambda^k = 0 .$$

Proof: *i*) is clear by definition. From (39) we find

$$[\gamma^{N+1} \otimes \mathcal{M}, \lambda^k \bullet \tilde{\lambda}^l]_g = [\gamma^{N+1} \otimes \mathcal{M}, \lambda^k]_g \bullet \tilde{\lambda}^l + (-1)^k \lambda^k \bullet [\gamma^{N+1} \otimes \mathcal{M}, \tilde{\lambda}^l]_g .$$

Then *ii*) follows from Lemma 3. To see *iii*) we calculate

$$
\begin{aligned}
\hat{d}^2 \lambda^k &= \mathbf{D}\{\, \mathbf{D}(\lambda^k) - i[\gamma^{N+1} \otimes \mathcal{M}, \lambda^k]_g \,\} \\
&\quad -i[\gamma^{N+1} \otimes \mathcal{M}, \{\, \mathbf{D}(\lambda^k) - i[\gamma^{N+1} \otimes \mathcal{M}, \lambda^k]_g \,\}]_g \\
&= \mathbf{D}^2(\lambda^k) - i\mathbf{D}((\gamma^{N+1} \otimes \mathcal{M}) \bullet \lambda^k) + (-1)^k i \mathbf{D}(\lambda^k \bullet (\gamma^{N+1} \otimes \mathcal{M})) \\
&\quad -i(\gamma^{N+1} \otimes \mathcal{M}) \bullet \mathbf{D}(\lambda^k) + (-1)^{k+1} i \mathbf{D}(\lambda^k) \bullet (\gamma^{N+1} \otimes \mathcal{M}) \\
&\quad -(\gamma^{N+1} \otimes \mathcal{M}) \bullet (\gamma^{N+1} \otimes \mathcal{M}) \bullet \lambda^k + \lambda^k \bullet (\gamma^{N+1} \otimes \mathcal{M}) \bullet (\gamma^{N+1} \otimes \mathcal{M}) \\
&= -(\gamma^{N+1} \otimes \mathcal{M}) \bullet (\gamma^{N+1} \otimes \mathcal{M}) \bullet \lambda^k + \lambda^k \bullet (\gamma^{N+1} \otimes \mathcal{M}) \bullet (\gamma^{N+1} \otimes \mathcal{M}) \\
&= -\sigma_{k+2} \circ [1^C \otimes \mathcal{M}^2, \tau^k] ,
\end{aligned}
$$

for any $\tau^k \in \pi(\Omega^k)$, such that $\sigma_k(\tau^k) = \lambda^k$. We used *ii*) and *iii*) of Lemma 3 and $\mathbf{D}(\gamma^{N+1} \otimes \mathcal{M}) \equiv 0$. From (13) we see that $(1^C \otimes \mathcal{M}^2)$ commutes with any $\tau \in \pi(\Omega^*)$, and hence we conclude *iii*). $\qquad\square$

Lemma 5 *Let* α_q^n *be the characterizing elements of* $\lambda^k \in \Lambda_{\mathcal{A}}^k$. *Then the characterizing elements* β_q^n *of* $\hat{d}\lambda^k \in \Lambda_{\mathcal{A}}^{k+1}$ *are given by*

$$
\begin{aligned}
\beta_1^{k-2t+1} &= \mathbf{d}\alpha_1^{k-2t} + (-1)^k i\,(\alpha_2^{k-2t+1} + \alpha_3^{k-2t+1}) , \\
\beta_2^{k-2t} &= \mathbf{d}\alpha_2^{k-2t-1} + (-1)^k i\,(\alpha_1^{k-2t} - \alpha_4^{k-2t}) , \\
\beta_3^{k-2t} &= \mathbf{d}\alpha_3^{k-2t-1} + (-1)^k i\,(\alpha_4^{k-2t} - \alpha_1^{k-2t}) , \\
\beta_4^{k-2t+1} &= \mathbf{d}\alpha_4^{k-2t} + (-1)^k i\,(\alpha_3^{k-2t+1} + \alpha_2^{k-2t+1}) ,
\end{aligned}
$$

where $t = 0, \ldots, m$ *and elements* α_q^n, *which do not occur in (27) are understood to be equal to zero.*

Proof: The Lemma follows immediately from (40), (38), and Lemma 2. $\qquad\square$

Though, at first sight, the structure of the differential algebra $\Lambda_{\mathcal{A}}^*$ seems to berather complicated, it can be characterized, at least locally, in a nice way in terms of generators and relations.

Proposition 6 *Let* $(U, \{x^\mu\}_{\mu=1,\dots,N})$ *be a local coordinate system on* X *and*

$$e^\mu \;:=\; \hat{d}\left(\begin{array}{cc} x^\mu \otimes 1^F & 0 \\ 0 & x^\mu \otimes 1^F \end{array}\right) = \left(\begin{array}{cc} \mathbf{d}x^\mu \otimes 1^F & 0 \\ 0 & \mathbf{d}x^\mu \otimes 1^F \end{array}\right) , \qquad (42)$$

$$e^0 \;:=\; \hat{d}\left(\begin{array}{cc} \frac{1}{2} \otimes 1^F & 0 \\ 0 & -\frac{1}{2} \otimes 1^F \end{array}\right) = \left(\begin{array}{cc} 0 & i\gamma^{N+1} \otimes M \\ -i\gamma^{N+1} \otimes M^* & 0 \end{array}\right) .$$

Let ϵ *be the automorphism of* $\mathcal{A}$ *given by* $\epsilon : \left(\begin{array}{cc} f_1 & 0 \\ 0 & f_2 \end{array}\right) \longmapsto \left(\begin{array}{cc} f_2 & 0 \\ 0 & f_1 \end{array}\right)$. *Then* $\{e^0, e^1, \dots, e^N\}$ *is a basis of the* $\mathcal{A}$*–bimodule* $\Lambda^1_{\mathcal{A}}$ *and we have*

$$\begin{array}{rcl} a \bullet e^\mu & = & e^\mu \bullet a , \\ a \bullet e^0 & = & e^0 \bullet \epsilon(a) , \quad a \in \mathcal{A} , \end{array} \qquad (43)$$

$$\begin{array}{rcl} e^\mu \bullet e^\nu & = & -e^\nu \bullet e^\mu , \\ e^\mu \bullet e^0 & = & -e^0 \bullet e^\mu , \end{array} \qquad (44)$$

$$(e^0)^{2m+2} := \underbrace{e^0 \bullet \cdots \bullet e^0}_{2m+2} = 0 . \qquad (45)$$

Thus, every $\lambda^k \in \Lambda^k_{\mathcal{A}}$ *has a unique representation*

$$\lambda^k = \sum_{l=0,\dots,min(k,2m+1)} \;\; \sum_{1 \le \mu_1 < \cdots < \mu_{k-l} \le N} a_{l\mu_1\dots\mu_{k-l}} \bullet e^{\mu_1} \bullet \cdots \bullet e^{\mu_{k-l}} \bullet (e^0)^l , \qquad (46)$$

with $a_{l\mu_1\dots\mu_{k-l}} \in \mathcal{A}$.

Proof: Using the general form

$$\lambda^1 = \left(\begin{array}{cc} \alpha^1_1 \otimes 1^F & \alpha^0_2 \gamma^{N+1} \otimes M \\ \alpha^0_3 \gamma^{N+1} \otimes M^* & \alpha^1_4 \otimes 1^F \end{array}\right) , \;\; \alpha^1_1, \alpha^1_4 \in L^1 , \;\; \alpha^0_2, \alpha^0_3 \in L^0 ,$$

of an element of $\Lambda^1_{\mathcal{A}}$ we see that elements of the form (42) generate $\Lambda^1_{\mathcal{A}}$. Linear independence follows from the fact that $\{\mathbf{d}x^\mu\}_{\mu=1,\dots,N}$ is a local frame of L^*. Thus, $\{e^0, e^1, \dots, e^N\}$ is a local basis of the A–bimodule $\Lambda^1_{\mathcal{A}}$.

Formulae (43) and (44) follow immediately from the multiplication rules given in Lemma 2, and (45) is a direct consequence of (17). Since elements of $\pi(\Omega^*)$ can be represented as sums of products of elements of $\mathcal{A}$ and $\pi(\Omega^1)$, see (12), and since σ respects this property, see (25), we get that elements of $\Lambda^*_{\mathcal{A}}$ are sums of products of elements of $\mathcal{A}$ and $\Lambda^1_{\mathcal{A}}$. Thus, $\{e^{\mu_1} \bullet \cdots \bullet e^{\mu_{k-l}} \bullet (e^0)^l\}_{k,l \in \mathbb{N}_0}$ generate the algebra $\Lambda^*_{\mathcal{A}}$. Linear independence is again obvious. $\qquad\square$

4   Main Theorem

In this section we will show that the differential algebra $\Lambda^*_{\mathcal{A}}$ coincides with Ω^*_D. For this purpose we first prove a technical Lemma, which turns out to be a generalization of Lemma 5 in [8] to the non–commutative case.

First, note that on the space $C^0 \equiv L^0$ of functions on X we have (cf. (77) and [1], Proposition 3.38)

$$-i[D^{c\ell}, c_0] = (\mathbf{d} - \mathbf{d}^*)(c_0) \ , \quad \forall c_0 \in C^0 \ . \tag{47}$$

Using (47) we see that

$$\hat{d}\tau^0 = -i[D, \tau^0] \ , \quad \forall \tau^0 \in \mathcal{A} \equiv \pi(\Omega^0) \equiv \Lambda_{\mathcal{A}}^0 \ . \tag{48}$$

Lemma 7 *i)* *For any $\omega^k \in \Omega^k$ we have $\hat{d} \circ \sigma_k \circ \pi(\omega^k) = \sigma_{k+1} \circ \pi(d\omega^k)$.*
ii) *Let $\tau^k \in \pi(\Omega^k)$ and $\tilde{\tau}^{k+1} \in \pi(\Omega^{k+1})$ with $\hat{d} \circ \sigma_k(\tau^k) = \sigma_{k+1}(\tilde{\tau}^{k+1})$. Then there exists an $\omega^k \in \Omega^k$, such that $\pi(\omega^k) = \tau^k$ and $\pi(d\omega^k) = \tilde{\tau}^{k+1}$.*

Proof: *i)* For $\omega^k = \sum_\alpha a_\alpha^0 da_\alpha^1 \ldots da_\alpha^k \in \Omega^k$, $a_\alpha^i \in \mathcal{A}$, we have
$\pi(\omega^k) = \sum_\alpha (-i)^k a_\alpha^0 [D, a_\alpha^1] \cdots [D, a_\alpha^k]$ and
$\pi(d\omega^k) = \sum_\alpha (-i)^{k+1} [D, a_\alpha^0][D, a_\alpha^1] \cdots [D, a_\alpha^k]$.
Using (25) and (48), we get

$$\begin{aligned}
\sigma_k \circ \pi(\omega^k) &= \sum_\alpha a_\alpha^0 \bullet \hat{d}a_\alpha^1 \bullet \ldots \bullet \hat{d}a_\alpha^k \ , \\
\sigma_{k+1} \circ \pi(d\omega^k) &= \sum_\alpha \hat{d}a_\alpha^0 \bullet \hat{d}a_\alpha^1 \bullet \ldots \bullet \hat{d}a_\alpha^k \ .
\end{aligned} \tag{49}$$

Now *i)* follows from Lemma 4.

ii) For $k = 0$ we have $\pi(\Omega^0) = \mathcal{A}$, $\ker \sigma_0 = \{0\}$, and $\ker \sigma^1 = \{0\}$. Hence, for a given $\tau^0 \in \mathcal{A}$ we find with (48) $\tilde{\tau}^1 = \hat{d}\tau^0 = -i[D, \tau^0]$. Thus, we can take $\omega = \tau^0$.
    Now, let $k \geq 1$. From $\hat{d} \circ \sigma_k(\tau^k) = \sigma_{k+1}(\tilde{\tau}^{k+1})$ and the general form
$$\tau^k = \sum_\alpha (-i)^k a_\alpha^0 [D, a_\alpha^1] \cdots [D, a_\alpha^k] \ , \quad a_\alpha^i \in \mathcal{A} \ ,$$
of an element of $\pi(\Omega^k)$ we get – using (49) –
$$\tilde{\tau}^{k+1} = \sum_\alpha (-i)^{k+1} [D, a_\alpha^0][D, a_\alpha^1] \cdots [D, a_\alpha^k] + \kappa^{k+1} \ , \quad \text{with } \kappa^{k+1} \in \ker \sigma_{k+1} \ .$$
Making the ansatz $\omega^k = \sum_\alpha a_\alpha^0 da_\alpha^1 \ldots da_\alpha^k + j^k$, $j^k \in \Omega^k$, and assuming $\tau^k = \pi(\omega^k)$ and $\tilde{\tau}^{k+1} = \pi(d\omega^k)$, there follows

$$\pi(j^k) = 0 \quad \text{and} \quad \pi(dj^k) = \kappa^{k+1} \ . \tag{50}$$

Thus, it remains to show that for any given $\kappa^{k+1} \in \ker \sigma_{k+1} \equiv \pi(\Omega^{k-1})$ we can always find a j^k fulfilling (50).
    From *i)* we find $0 = \hat{d} \circ \sigma_k \circ \pi(\ker \pi \cap \Omega^k) = \sigma_{k+1} \circ \pi \circ d(\ker \pi \cap \Omega^k)$, which means that $\pi \circ d(\ker \pi \cap \Omega^k) \subset \ker \sigma_{k+1}$. Thus, any element $j^k = \sum_\beta b_\beta^0 db_\beta^1 \ldots db_\beta^k \in \ker \pi \cap \Omega^k$, $b_\beta^i \in \mathcal{A}$, fulfils $\pi(j^k) = 0$ and $\pi(dj^k) \in \ker \sigma_{k+1}$. To find the form of $\pi(dj^k)$, we calculate
$$-iD(\pi(j)\phi) = \sum_\beta (-i)^{k+1} [D, b_\beta^0][D, b_\beta^1] \cdots [D, b_\beta^k]\phi +$$

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$$\sum_{\beta} (-i)^{k+1} b_\beta^0 D(\, [D, b_\beta^1] \cdots [D, b_\beta^k] \phi \,) \, , \quad \text{where } \phi \in h \, .$$

Since $\pi(j) = 0$, we have $\pi(dj)\phi = -\sum_{\beta} (-i)^{k+1} b_\beta^0 D(\, [D, b_\beta^1] \cdots [D, b_\beta^k] \, \phi \,)$, and, therefore,

$$\pi(dj) \cdot = \sum_{\beta} b_\beta^0 K_\beta \, , \quad \text{with } K_\beta := -(-i)^{k+1} D([D, b_\beta^1] \cdots [D, b_\beta^k] \cdot) \, .$$

We underline that $\sum_{\beta} b_\beta^0 K_\beta \in \pi(\Omega^{k-1}) \equiv \ker \sigma_{k+1}$, which is not obvious from looking at K_β. Now we are left with solving the following equations:

$$\begin{aligned}
0 &= \sum_{\beta} (-i)^k b_\beta^0 [D, b_\beta^1] \cdots [D, b_\beta^k] \, , \\
\kappa^{k+1} &= \sum_{\beta} b_\beta^0 K_\beta \, ,
\end{aligned}$$

where $\kappa^{k+1} \in \pi(\Omega^{k-1})$ is given. We represent κ^{k+1}, $\pi(j^k)$, and $\pi(dj^k) = \sum_{\beta} b_\beta^0 K_\beta$ according to (20), which reduces the above equations to a system of equations for sections in the Clifford bundle C. We solve this system locally by choosing a local frame in C. Observing that we can fix all b_β^i with $i = 1, \ldots, k$, we are left with a system of linear algebraic equations for b_β^0, which has for a generic choice of b_β^i, $i = 1, \ldots, k$, (such that the determinant of the system of linear algebraic equations does not vanish) a solution. Using a partition of unity one constructs a global solution j^k for any given κ^{k+1}. $\qquad \square$

Using Lemma 7, the fact that π is an involutive representation, and the identity $d(\omega^{k*}) = (-1)^k (d\omega^k)^*$, $\forall \omega^k \in \Omega^k$, we have

$$\begin{aligned}
\hat{d}(\lambda^{k*}) &= \hat{d} \circ \sigma_k(\tau^{k*}) = \hat{d} \circ \sigma_k \circ \pi(\omega^{k*}) = \sigma_{k+1} \circ \pi(d(\omega^{k*})) \\
&= \sigma_{k+1} \circ \pi(\, (-1)^k (d\omega^k)^*) = (-1)^k (\sigma_{k+1} \circ \pi(d\omega^k))^* \, ,
\end{aligned}$$

where $\sigma_k(\tau^k) = \lambda^k \in \Lambda_A^k$ and $\pi(\omega^k) = \tau^k \in \pi(\Omega^k)$. Thus, we obtain

$$\hat{d}(\lambda^{k*}) = (-1)^k (\hat{d}\lambda^k)^* \, . \tag{51}$$

Theorem 8 Ω_D^k and Λ_A^k are identical as involutive graded differential algebras.

Proof: i) First we show that Ω_D^k and Λ_A^k are identical as vector spaces, i.e. we have to prove that $\pi(J \cap \Omega^k) = \ker \sigma_k$. We begin with the inclusion $\ker \sigma_k \subset \pi(J \cap \Omega^k)$. Let there be given $\tau \in \pi(\Omega^k)$ with $\sigma_k(\tau) = 0$. The pair $\tau^{k-1} \equiv 0 \in \pi(\Omega^{k-1})$, $\tilde{\tau}^k \equiv \tau \in \pi(\Omega^k)$ fulfils the conditions of ii) in Lemma 7. Hence, there exists an $\omega^{k-1} \in \Omega^{k-1}$, with $\pi(\omega^{k-1}) = 0$ and $\pi(d\omega^{k-1}) = \tau$. This means that $\omega^{k-1} \in J \cap \Omega^{k-1}$ and $d\omega^{k-1} \in J \cap \Omega^k$, and, therefore, $\tau \in \pi(J \cap \Omega^k)$.

To prove the inclusion $\pi(J \cap \Omega^k) \subset \ker \sigma_k$, observe that every $\omega^k \in J \cap \Omega^k$ can be represented as $\omega^k = \omega_1^k + d\omega_2^{k-1}$, with $\pi(\omega_1^k) = 0$ and $\pi(\omega_2^{k-1}) = 0$. Then from i) of Lemma 7 we find $\sigma_k \circ \pi(\omega^k) = \sigma_k \circ \pi(d\omega_2^{k-1}) = \hat{d} \circ \sigma_{k-1} \circ \pi(\omega_2^{k-1}) = 0$, and, therefore, $\pi(\omega^k) \in \ker \sigma_k$.

ii) The identity of the differentials is proved on the level of equivalence classes. Using ii) of Lemma 7 and i) of this proof, we get for $\lambda^k \in \Lambda_A^k$:

$$\lambda^k = \pi(\omega^k) + \ker \sigma_k = \pi(\omega^k) + \pi(J \cap \Omega^k) \,,$$
$$\hat{d}\lambda^k = \pi(d\omega^k) + \ker \sigma_{k+1} = \pi(d\omega^k) + \pi(J \cap \Omega^{k+1}) \,,$$
where ω^k belongs to Ω^k, as in ii) of Lemma 7. But this is just the definition of the differential in Ω_D^*.

iii) To prove that the involutions on $\pi(\Omega^k)$ and $\Lambda_{\mathcal{A}}^k$ are identical, observe that for $\lambda^k = \pi(\omega^k) + \ker \sigma_k \in \Lambda_{\mathcal{A}}^k$ we have with (31)
$$\lambda^{k*} = \pi(\omega^k)^* + \ker \sigma_k = \pi(\omega^k)^* + \pi(J \cap \Omega^k) = \pi(\omega^{k*}) + \pi(J \cap \Omega^k) \,,$$
where we have used that the ideals occurring in this factorization are invariant under the involution.

iv) The fact that multiplications coincide is again proved on the level of equivalence classes. Let $\lambda^k = \pi(\omega^k) + \ker \sigma_k \in \Lambda_{\mathcal{A}}^k$ and $\tilde{\lambda}^l = \pi(\tilde{\omega}^l) + \ker \sigma_l \in \Lambda_{\mathcal{A}}^l$, where $\omega^k \in \Omega^k$ and $\tilde{\omega}^l \in \Omega^l$ are as in ii). Since π is a representation and $\ker \sigma$ is an ideal, we have $\lambda^k \bullet \tilde{\lambda}^l = \pi(\omega^k \tilde{\omega}^l) + \ker \sigma_{k+l} = \pi(\omega^k \tilde{\omega}^l) + \pi(J \cap \Omega^{k+l})$, where the last identity follows from i). This is the multiplication rule in Ω_D^*. $\qquad\square$

This generalizes the result of Connes and Lott stating the isomorphism of Ω_D^* with the classical de Rham complex for the case $\mathcal{A} = C^\infty(X)$. We note that for $k = 1$ Lemma 5 reproduces the equations following Lemma 15 in [8].

5 The natural scalar product on $\Lambda_{\mathcal{A}}^*$ induced by the Dixmier trace

Here we will show that the Dixmier trace Tr_ω, see [7], [27], [6], [8], provides a scalar product $< \,,\, >_{\Lambda_{\mathcal{A}}^*}$ on $\Lambda_{\mathcal{A}}^*$. For $\tau^k \in \pi(\Omega^k)$, $\tilde{\tau}^l \in \pi(\Omega^l)$, a scalar product on $\pi(\Omega^*)$ is given by
$$< \tau^k, \tilde{\tau}^l > := Tr_\omega \left((\tau^k)^* \tilde{\tau}^l |D|^{-N} \right) \,. \tag{52}$$

For the K–cycle under consideration this scalar product can be expressed as a combination of the usual trace and integration over the manifold. Let $\tau^k \in \pi(\Omega^k)$ and $\tilde{\tau}^l \in \pi(\Omega^l)$ be represented as in (20). Let tr_C denote the trace in the Clifford algebra (normalized by $tr_C(1^C) = 2^{(N/2)}$), tr_F the trace in $End(F)$, and v_g the canonical volume formon X. Then one finds ([8], before Example 4, and [27], Theorem 5.3)

$$< \tau^k, \tilde{\tau}^l > = \frac{1}{(\frac{N}{2})!(4\pi)^{\frac{N}{2}}} \times$$
$$\sum_{q=1}^4 \sum_{r,s=0}^m (-1)^{(k+l)\zeta_q} \int_X v_g \, tr_C\{(c_q^{k-2r-\zeta_q})^* \tilde{c}_q^{l-2s-\zeta_q}\} \, tr_F\{(M_q^r)^* M_q^s\}, \tag{53}$$

where $\zeta_q = 0$ for $q = 1, 4$ and $\zeta_q = 1$ for $q = 2, 3$.

For $k \geq 2$, let $\tilde{\sigma}_k$ be the orthogonal projection from $\pi(\Omega^k)$ onto the orthogonal complement (with respect to (52)) of the subspace $\pi(\Omega^{k-2})$. For $k = 0$ and $k = 1$

let $\tilde{\sigma}_k$ be the identity on $\pi(\Omega^0)$ and $\pi(\Omega^1)$. Here and in what follows a completion of $\pi(\Omega^n)$ in the sense of the above defined scalar product (52) is meant. We have a natural embedding $i_k : \Lambda^k_{\mathcal{A}} \to \pi(\Omega^k)$ given by

$$i_k(\lambda^k) := \tilde{\sigma}_k(\tau^k) \in \pi(\Omega^k) , \qquad (54)$$

for $\sigma_k(\tau^k) = \lambda^k$. By construction, i_k is well defined. In particular,

$$\sigma_k \circ i_k = id_{\Lambda^k_{\mathcal{A}}} , \qquad (55)$$

because with $(id - \tilde{\sigma}_k)(\tau^k) \in \ker \sigma_k$ we have
$$\sigma_k \circ i_k(\lambda^k) = \sigma_k \circ \tilde{\sigma}_k(\tau_k) = \sigma_k(\tilde{\sigma}_k(\tau^k) + (id - \tilde{\sigma}_k)(\tau^k)) = \sigma_k(\tau^k) = \lambda^k .$$
Next we define a scalar product $< , >_{\Lambda^*_{\mathcal{A}}} : \Lambda^*_{\mathcal{A}} \times \Lambda^*_{\mathcal{A}} \to \mathbb{C}$, putting

$$< \lambda^k , \tilde{\lambda}^l >_{\Lambda^*_{\mathcal{A}}} := < i_k(\lambda^k) , i_l(\tilde{\lambda}^l) > . \qquad (56)$$

Observe that for $k \neq l$ this product gives zero. First, for $k + l$ being odd, the trace in the Clifford algebra, see (53), is taken over an odd number of γ–matrices and, therefore, vanishes:
$$tr_C(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}}) = tr_C(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}}(\gamma^{N+1})^2)$$
$$= tr_C(\gamma^{N+1}\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}} \gamma^{N+1}) = (-1)^{2n+1} tr_C(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}}) .$$
For $k + l$ being even, assume $k > l$. Then $i_l(\lambda^l) \in \pi(\Omega^{k-2})$, and hence $i_k(\lambda^k)$ is by definition orthogonal to $i_l(\lambda^l)$.

Let us calculate $i_k(\lambda^k)$ for $\lambda^k \in \Lambda^k_{\mathcal{A}}$. For this purpose it is convenient to perform a linear transformation of bases $M^i_q \mapsto N^i_q$ in the vector spaces V_q spanned by $\{M^i_q\}^m_{i=0}$. Inserting this transformation into (13), we see that the only transformations leaving $\pi(\Omega^k)$ invariant are

$$N^i_q = \sum_{t=0}^{i} f^i_{q,t} M^t_q , \quad i = 0, \ldots, m , \quad q = 1, \ldots, 4 , \quad f^i_{q,t} \in \mathbb{C} , \quad \det(f_q) \neq 0 . \quad (57)$$

In particular, $N^0_1 = f^0_{1,0} 1^F$, $N^0_2 = f^0_{2,0} M$, $N^0_3 = f^0_{3,0} M^*$, $N^0_4 = f^0_{4,0} 1^F$. By the Schmidt orthogonalization procedure we get a unique (up to normalization) orthogonal (with respect to tr_F) basis $\{N^i_q\}^m_{i=0}$, which is in accordance with (57):

$$N^0_q = M^0_q , \quad N^i_q = M^i_q - \sum_{t=0}^{i-1} f^i_{q,t} N^t_q , \quad f^i_{q,t} = \frac{tr_F (N^{t*}_q M^i_q)}{tr_F (N^{t*}_q N^t_q)} , \qquad (58)$$

for $i = 1, \ldots, m$. Since in model building the norms of N^i_q have to be fitted to physical parameters, we cannot normalize the basis $\{N^i_q\}^m_{i=0}$. We have chosen $f^i_{q,i} = 1$, $i = 0, \ldots, m$, which is especially convenient, because in that case the factorization procedure leading from $\pi(\Omega^k)$ to $\Lambda^k_{\mathcal{A}}$ done in terms of $\{N^i_q\}^m_{i=0}$ yields

the same result as given in (26). In this sense the above choice of the orthogonal basis can be considered as being canonical. For this basis we get

$$\lambda^k = \begin{pmatrix} \sum_{t=0}^{m} \alpha_1^{k-2t} \otimes N_1^t & ; & \sum_{t=0}^{m} \alpha_2^{k-2t-1} \gamma^{N+1} \otimes N_2^t \\ \sum_{t=0}^{m} \alpha_3^{k-2t-1} \gamma^{N+1} \otimes N_3^t & ; & \sum_{t=0}^{m} \alpha_4^{k-2t} \otimes N_4^t \end{pmatrix} . \tag{59}$$

Therefore, the corresponding formulae for products and the differential, see Lemma 2 and Lemma 5, remain unchanged. Hence, $i_k(\lambda^k)$ has the form

$$i_k(\lambda^k) = \begin{pmatrix} \sum_{t=0}^{m} b_1^{k-2t} \otimes N_1^t & ; & \sum_{t=0}^{m} b_2^{k-2t-1} \gamma^{N+1} \otimes N_2^t \\ \sum_{t=0}^{m} b_3^{k-2t-1} \gamma^{N+1} \otimes N_3^t & ; & \sum_{t=0}^{m} b_4^{k-2t} \otimes N_4^t \end{pmatrix} \in \pi(\Omega^k) ,$$

with

$$b_q^n \in C^n , \quad \sigma_n^{c\ell}(b_q^n) = \alpha_q^n . \tag{60}$$

The element $i_k(\lambda^k)$ has to be orthogonal to any $\tilde{\tau}^{k-2} \in \pi(\Omega^{k-2})$,

$$\tilde{\tau}^{k-2} = \begin{pmatrix} \sum_{t=0}^{m} \tilde{b}_1^{k-2t-2} \otimes N_1^t & ; & \sum_{t=0}^{m} \tilde{b}_2^{k-2t-3} \gamma^{N+1} \otimes N_2^t \\ \sum_{t=0}^{m} \tilde{b}_3^{k-2t-3} \gamma^{N+1} \otimes N_3^t & ; & \sum_{t=0}^{m} \tilde{b}_4^{k-2t-2} \otimes N_4^t \end{pmatrix} , \quad \tilde{b}_q^n \in C^n .$$

Then, using the orthogonality of the basis $\{N_q^i\}_{i=0}^m$ we find

$$0 = < i_k(\lambda^k), \tilde{\tau}^{k-2} >$$

$$= \frac{1}{(\frac{N}{2})!(4\pi)^{\frac{N}{2}}} \sum_{q=1}^{4} \sum_{r,s=0}^{m} \int_X v_g \, tr_C \left\{ (b_q^{k-2r-\zeta_q})^* \, \tilde{b}_q^{k-2s-\zeta_q-2} \right\} tr_F \{ (N_q^r)^* N_q^s \}$$

$$= \frac{1}{(\frac{N}{2})!(4\pi)^{\frac{N}{2}}} \sum_{q=1}^{4} \sum_{r=0}^{m} \int_X v_g \, tr_C \left\{ (b_q^{k-2r-\zeta_q})^* \, \tilde{b}_q^{k-2r-\zeta_q-2} \right\} tr_F \{ (N_q^r)^* N_q^r \} .$$

This must be true for any $\tilde{b}_q^{k-2r-\zeta_q-2} \in C^{k-2r-\zeta_q-2}$, which means that $b_q^{k-2r-\zeta_q}$ is a maximal homogeneous element of $C^{k-2r-\zeta_q}$, treated in the sense of (66), fulfilling (60).

In more detail, if we write both $i_k(\lambda^k)$ and λ^k in terms of the canonical orthogonal bases $\{N_q^i\}_{i=0}^m$ as above, then i_k splits blockwise into $i_k = i_k^{c\ell} \otimes id$, where $i_k^{c\ell} : L^k \to C^k$ denotes the classical embedding

$$i_0^{c\ell}([1^C]) = 1^C , \quad i_k^{c\ell}([\gamma^{\mu_1}] \wedge [\gamma^{\mu_2}] \wedge \ldots \wedge [\gamma^{\mu_k}]) = \gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_k} , \tag{61}$$

with $1 \leq \mu_1 < \mu_2 < \ldots < \mu_k \leq N$ and $[\gamma^\mu] := \sigma_1^{c\ell}(\gamma^\mu)$. This means

$$
\mathrm{i}_k(\lambda^k) = \begin{pmatrix} \displaystyle\sum_{t=0}^{m} \mathrm{i}_{k-2t}^{c\ell}(\alpha_1^{k-2t}) \otimes N_1^t \; ; & \displaystyle\sum_{t=0}^{m} \mathrm{i}_{k-2t-1}^{c\ell}(\alpha_2^{k-2t-1})\gamma^{N+1} \otimes N_2^t \\ \displaystyle\sum_{t=0}^{m} \mathrm{i}_{k-2t-1}^{c\ell}(\alpha_3^{k-2t-1})\gamma^{N+1} \otimes N_3^t \; ; & \displaystyle\sum_{t=0}^{m} \mathrm{i}_{k-2t}^{c\ell}(\alpha_4^{k-2t}) \otimes N_4^t \end{pmatrix}. \quad (62)
$$

Observe that the use of the orthogonal basis $\{N_q^i\}_{i=0}^m$ leads to an extremely simple representation of i_k, namely one has to apply the classical embedding $\mathrm{i}_k^{c\ell}$ on the first component of every block of λ^k. Now we can write down the explicit structure of the product (56) for $\lambda^k, \tilde{\lambda}^k \in \Lambda_{\mathcal{A}}^k$ represented as in (59) with α_q^n and $\tilde{\alpha}_q^n$ denoting their characterizing elements:

$$
< \lambda^k, \tilde{\lambda}^k >_{\Lambda_{\mathcal{A}}^*} \;\; = \;\; \frac{1}{(\frac{N}{2})!(4\pi)^{\frac{N}{2}}} \times
$$

$$
\sum_{q=1}^{4}\sum_{r=0}^{m} \int_X v_g \, tr_C\{\mathrm{i}_{k-2r-\zeta_q}^{c\ell}(\alpha_q^{k-2r-\zeta_q})^* \, \mathrm{i}_{k-2r-\zeta_q}^{c\ell}(\tilde{\alpha}_q^{k-2r-\zeta_q})\} tr_F\{(N_q^r)^* N_q^r\}. \quad (63)
$$

Thus, $< \; , \; >_{\Lambda_{\mathcal{A}}^*}$ is a positive definite scalar product on $\Lambda_{\mathcal{A}}^k$ for each k.

For the classical embedding $\mathrm{i}^{c\ell}$ there holds the following equality between the scalar product for sections of the Clifford bundle und the scalar product for differential forms:

$$
\int_X v_g \, tr_C \{(\mathrm{i}_k^{c\ell}(\alpha^k))^* \, \mathrm{i}_k^{c\ell}(\beta^k)\} = (-1)^{\frac{k(k-1)}{2}} 2^{\frac{N}{2}} \int_X \iota_k((\alpha^k)^*) \wedge *\iota_k(\beta^k) \,, \quad (64)
$$

where $\alpha^k, \beta^k \in L^k$, $\iota_k : L^k \to \Lambda^k(X)$ is the classical isomorphism (28), and $*$ denotes the Hodge star operator, see (81). Using this, we can write (63) as

$$
< \lambda^k, \tilde{\lambda}^k >_{\Lambda_{\mathcal{A}}^*} = \frac{1}{(\frac{N}{2})!(2\pi)^{\frac{N}{2}}} \sum_{q=1}^{4}\sum_{r=0}^{m} (-1)^{(k-2r-\zeta_q)(k-2r-\zeta_q-1)/2} \times
$$

$$
\int_X \{\iota_{k-2r-\zeta_q}(\alpha_q^{k-2r-\zeta_q})^* \wedge *\iota_{k-2r-\zeta_q}(\tilde{\alpha}_q^{k-2r-\zeta_q})\} tr_F\{(N_q^r)^* N_q^r\}. \quad (65)
$$

We see that – due to the explicit knowledge of the canonical ideal – we get an explicit embedding of $\Lambda_{\mathcal{A}}^k \equiv \Omega_D^k$ into $\pi(\Omega^k)$ and, therefore, explicit formulae for scalar products of elements of arbitrary degree.

A   Appendix: Some Remarks on γ^{N+1}

With $L^k := C^k / C^{k-2}$ we have the canonical isomorphism of vector spaces [1]

$$
C^k \cong L^k \oplus L^{k-2} \oplus \ldots \oplus L^{\zeta_k} \,, \quad (66)
$$

where $\zeta_k := 0$ for k even and $\zeta_k = 1$ for k odd. Using the local orthonormal basis γ^μ, $\mu = 1, \ldots, N$, of C^1, see (4), we can consider L^k, $k \geq 1$, as spanned (locally) by completely antisymmetrized products of k elements γ^μ.

On C^k we have natural right and left multiplications by γ^{N+1}. One finds that right multiplication of a completely antisymmetrized product of k elements γ^μ $(1 \leq k \leq N)$ by γ^{N+1} gives a completely antisymmetrized product of the $N-k$ complementary elements γ^μ. This means that we have

$$L^k \gamma^{N+1} = L^{N-k} . \tag{67}$$

We denote $B^{N-k} \equiv C^k \gamma^{N+1}$. Then from (66) and (67) we find

$$B^{N-k} \cong L^{N-k} \oplus L^{N-(k-2)} \oplus \ldots \oplus L^{N-\zeta_k} . \tag{68}$$

This gives

$$
\begin{aligned}
(C^k \gamma^{N+1})/(C^{k-2} \gamma^{N+1}) &\equiv B^{N-k}/B^{N-(k-2)} \\
&\cong L^{N-k} = L^k \gamma^{N+1} \equiv (C^k/C^{k-2})\gamma^{N+1} .
\end{aligned} \tag{69}
$$

For $c^n \in C^n$ we have $c^n \gamma^{N+1} = (-1)^n \gamma^{N+1} c^n$. This enables us to interchange left and right multiplication with γ^{N+1} on L^*. Putting $\alpha^k = c^k + C^{k-2} \in L^k$, where $c^k \in C^k$, we have

$$\gamma^{N+1} \alpha^k = \gamma^{N+1}(c^k + C^{k-2}) = (-1)^k(c^k + C^{k-2})\gamma^{N+1} = (-1)^k \alpha^k \gamma^{N+1} \in L^{N-k} . \tag{70}$$

Because of $(\gamma^{N+1})^2 = 1$ this gives

$$\gamma^{N+1} \alpha^k \gamma^{N+1} = (-1)^k \alpha^k \in L^k , \quad \forall \alpha^k \in L^k . \tag{71}$$

The isomorphism $\iota_k : L^k \to \Lambda^k(X)$, $\iota_k(\gamma^\mu) = dx^\mu$, relates left multiplication with γ^{N+1} to the Hodge star $\star$ on $\Lambda^*(X)$ as defined in [1] via

$$\star \iota_k(\alpha^k) := \iota_{N-k}(\gamma^{N+1} \alpha^k) , \quad \forall \alpha^k \in L^k . \tag{72}$$

Note, however, that – comparing with [1] – we use a different sign convention in the defining relations of the Clifford algebra, see (4). Because of (70) we obviously have

$$\iota_{N-k}(\alpha^k \gamma^{N+1}) = (-1)^k \star \iota_k(\alpha^k) , \quad \forall \alpha^k \in L^k . \tag{73}$$

Now we prove that $\mathbf{d}^* = \gamma^{N+1} \mathbf{d} \gamma^{N+1}$ (left multiplication by γ^{N+1}) is the codifferential on L^*, see (35). On L^* we have the natural scalar product

$$< \alpha^k, \beta^k > := \int_X v_g \, tr_C \left\{ (\mathrm{i}_k^{c\ell}(\alpha^k))^* \, \mathrm{i}_k^{c\ell}(\beta^k) \right\} , \quad \forall \alpha^k, \beta^k \in L^k , \tag{74}$$

where $\mathrm{i}_k^{c\ell}$ was given in (61). An equivalent representation of this product is

$$< \alpha^k, \beta^k > := \int_X v_g \, tr_C \left\{ \gamma^{N+1} \mathrm{i}_N^{c\ell}((\gamma^{N+1} \alpha^{k*}) \wedge \beta^k) \right\} . \tag{75}$$

We remark that we have a natural involution on the exterior differential algebra $\Lambda^*(X)$ inherited from the involution in the Clifford algebra, which gives

$$(a^k \wedge \tilde{a}^l)^* = (\tilde{a}^l)^* \wedge (a^k)^* \;, \quad \forall a^k \in \Lambda^k(X) \;, \quad \tilde{a}^l \in \Lambda^l(X) \;. \tag{76}$$

Therefore,

$$(da^k)^* = (-1)^k d(a^{k^*}) \;, \quad \forall a^k \in \Lambda_{\mathcal{A}}^k \;. \tag{77}$$

Now we calculate

$$
\begin{aligned}
< \alpha^k, \mathbf{d}\beta^{k-1} > &= \int_X v_g \, tr_C \left\{ \gamma^{N+1} \, \mathrm{i}_N^{c\ell}((\gamma^{N+1}\alpha^{k^*}) \wedge \mathbf{d}\beta^{k-1}) \right\} \\
&= \int_X v_g \, tr_C \left\{ \gamma^{N+1} \, \mathrm{i}_N^{c\ell}((-1)^{N-k}\mathbf{d}((\gamma^{N+1}\alpha^{k^*}) \wedge \beta^{k-1})) \right\} \\
&\quad - \int_X v_g \, tr_C \left\{ \gamma^{N+1} \, \mathrm{i}_N^{c\ell}((-1)^{N-k}(\mathbf{d}(\gamma^{N+1}\alpha^{k^*})) \wedge \beta^{k-1}) \right\} \\
&= -(-1)^{N-k} \int_X v_g \, tr_C \left\{ \gamma^{N+1} \, \mathrm{i}_N^{c\ell}(((-1)^k\mathbf{d}((\gamma^{N+1}\alpha^k)^*)) \wedge \beta^{k-1}) \right\} \\
&= -(-1)^k \int_X v_g \, tr_C \left\{ \gamma^{N+1} \, \mathrm{i}_N^{c\ell}(\gamma^{N+1}\gamma^{N+1}(\mathbf{d}(\gamma^{N+1}\alpha^k))^* \wedge \beta^{k-1}) \right\} \\
&= -(-1)^k \int_X v_g \, tr_C \left\{ \gamma^{N+1} \, \mathrm{i}_N^{c\ell}(\gamma^{N+1}(-1)^{N-k+1}(\gamma^{N+1}\mathbf{d}(\gamma^{N+1}\alpha^k))^* \wedge \beta^{k-1}) \right\} \\
&= < \mathbf{d}^*\alpha^k, \beta^{k-1} > \;,
\end{aligned}
$$

where $\int_X v_g \, tr_C \left\{ \gamma^{N+1} \, \mathrm{i}_N^{c\ell}(\mathbf{d}\alpha^{N-1}) \right\} = 0$, $\alpha^{N-1} \in L^{N-1}$, and equations (70) and (77) were used. The codifferential $\mathbf{d}^* = \gamma^{N+1}\mathbf{d}\gamma^{N+1}$ on L^* is related to the codifferential $d^* = \star d \star$ on $\Lambda^*(X)$ by

$$\iota_{k-1}(\mathbf{d}^*(\alpha^k)) = d^*(\iota_k(\alpha^k)) \;, \quad \forall \alpha^k \in L^k \;. \tag{78}$$

With $(\gamma^{N+1})^2 = 1$ and $\star^2 = 1$ this yields

$$\gamma^{N+1}(\mathbf{d} - \mathbf{d}^*) + (\mathbf{d} - \mathbf{d}^*)\gamma^{N+1} = 0 \quad \text{and} \quad \star(d - d^*) + (d - d^*)\star = 0 \;. \tag{79}$$

These equations make it, in principle, possible to replace in this paper the calculus based upon L^* and γ^{N+1} by a calculus based upon $\Lambda^*(X)$ and $\star$. For example, we could write – omitting the isomorphism ι – the element λ^k given in (26) as:

$$
\lambda^k = \begin{pmatrix} \displaystyle\sum_{t=0}^m a_1^{k-2t} \otimes M_1^t \;; & \displaystyle\sum_{t=0}^m (-1)^{k-1} \star a_2^{k-2t-1} \otimes M_2^t \\[2em] \displaystyle\sum_{t=0}^m (-1)^{k-1} \star a_3^{k-2t-1} \otimes M_3^t \;; & \displaystyle\sum_{t=0}^m a_4^{k-2t} \otimes M_4^t \end{pmatrix} \;, \quad a_q^n \in \Lambda^n(X) \;. \tag{80}
$$

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We underline, however, that for calculating the product $\bullet$ one has either to return to representatives in the Clifford algebra, or one has to use (after blockwise multiplication of matrices of the form (80)) the following multiplication rules:

$$
\begin{aligned}
a^k \dot{\wedge} \tilde{a}^l &= a^k \wedge \tilde{a}^l \ , \\
\star a^k \dot{\wedge} \tilde{a}^l &= \star(a^k \wedge \tilde{a}^l) \ , \\
a^k \dot{\wedge} \star \tilde{a}^l &= (-1)^k \star (a^k \wedge \tilde{a}^l) \ , \\
\star a^k \dot{\wedge} \star \tilde{a}^l &= (-1)^k a^k \wedge \tilde{a}^l \ .
\end{aligned}
$$

We have omitted the lower index q of differential forms occurring in (80), because the multipication rules do not depend on q .

Finally, we relate the definition of the Hodge star $\star$ in [1] to the Hodge star $*$ used in classical differential geometry. One has

$$
*a^k := \Lambda^k g^{-1}(a^k) \, \lrcorner \, v_g \ , \quad a^k \in \Lambda^k(X) \ , \tag{81}
$$

where $\Lambda^k g^{-1}$ denotes the natural isomorphism between k–differential forms and k–vector fields provided by the metric $g : \Gamma^\infty(TX) \to \Gamma^\infty(T^*X)$, $\lrcorner$ is the contraction of a k–vector field with an l–differential form $(l \geq k)$ giving an $(l-k)$–differential form, and v_g is the canonical volume form associated to the metric g on X . Then one can show that for an even dimensional Riemannian manifold the following relations hold:

$$
\begin{aligned}
* * a^k &= (-1)^k a^k \ , \\
\star a^k &= (-i)^{\frac{N}{2}} (-1)^{\frac{k(k+1)}{2}} * a^k \ .
\end{aligned} \tag{82}
$$

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