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The Energy-Momentum Tensor on Noncommutative Spaces - Some Pedagogical Comments

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Abstract. We present the discussion of the energy-momentum tensor of the scalar ϕ^4-theory on a noncommutative space. The Noether procedure is performed at the operator level. Additionally, the broken dilatation symmetry will be considered in a Moyal-Weyl deformed scalar field theory at the classical level.

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1 Introduction

The aim of this work is to investigate the translation and dilatation symmetry at least at the classical level for a noncommutative ϕ^4-theory. Not much work has yet been done in this direction, only in [1] and [2] one finds some scattered remarks concerning the energy-momentum tensor and its Noether procedure for Moyal-Weyl deformed scalar field theories. In this paper we extend the analysis of [2] and formulate the Noether procedure for translations already at the operator level. By the use of the Moyal-Weyl correspondence between operators and fields we are able to confirm the results of [2].

This work is organized as follows. Section 2 is devoted to some special features of the quantum space in connection with a ϕ^4-theory.

In Section 3 we study the construction of the energy momentum tensor at the operator level and in a Moyal-Weyl deformed ϕ^4-theory.

Finally, in the last section we investigate the broken dilatation symmetry of the noncommutative ϕ^4-theory.

2 The quantum phase space and the scalar field theory

We consider the scalar field theory which is described at the classical level by the following action[1]

$$S^{(0)}[\phi] = \int dx \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right), \tag{1}$$

where $\phi(x)$ is a real valued scalar field on the four dimensional Euclidean space time E_4. For fields in a Schwartz space of functions which decrease sufficiently fast at infinity we may define a Fourier transformation by

$$\begin{aligned} \phi(x) &= \int dk\, e^{ik_\mu x_\mu}\tilde{\phi}(k), \\ \tilde{\phi}(k) &= \int dx\, e^{-ik_\mu x_\mu}\phi(x), \end{aligned} \tag{2}$$

with $\tilde{\phi}(-k) = \tilde{\phi}^*(k)$. In order to generalize a field theory on an ordinary space-time to one on a noncommutative space-time we replace the local coordinates x_μ by hermitian operators $\hat{x}_\mu$ obeying the relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\sigma_{\mu\nu}, \quad [\hat{x}_\mu, \sigma_{\mu\nu}] = 0, \tag{3}$$

[1]We use the shorthand notations $dx := d^4x$ and $dk := \frac{d^4k}{(2\pi)^4}$.

where $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$ is a real invertible matrix. Consequently, fields on space-time are replaced by operators. Replacing x_μ by $\hat{x}_\mu$ in (2) we obtain

$$\phi(\hat{x}_\mu) = \int dk\, e^{ik\hat{x}} \tilde{\phi}(k). \tag{4}$$

With (2) one gets

$$\begin{aligned} \phi(\hat{x}) &= \int dk \int dx\, \phi(x) e^{ik\hat{x} - ikx} = \int dx \int dk\, \hat{T}(k) e^{-ikx} \phi(x) = \\ &= \int dx \hat{\Delta}(x) \phi(x). \end{aligned} \tag{5}$$

$\phi(\hat{x})$ is an element of an algebra $\mathcal{A}_x$ in the sense of [6].

In (5) we have introduced the operators $\hat{T}(k)$ and $\hat{\Delta}(k)$ which were originally defined by Balasz et al. [3]. More recently, these operators were also used by Filk [4]:

$$\hat{T}(k) = e^{ik\hat{x}}, \tag{6}$$

and by Ambjorn et al. [5]:

$$\hat{\Delta}(x) = \int dk\, e^{ik\hat{x} - ikx} = \int dk\, \hat{T}(k) e^{-ikx}. \tag{7}$$

$\hat{T}(k)$ and $\hat{\Delta}(x)$ have different useful properties for practical calculations. In order to list these properties let us define the trace operations for $\hat{T}(k)$ and $\hat{\Delta}(x)$.

For simplicity we choose the space time dimension $d = 2$ and consider in a first step the trace of $\hat{T}(k)$. The operator $\hat{T}(k)$ has the following properties [4]:

$$\begin{aligned} \hat{T}^\dagger(k) &= \hat{T}(-k) \\ \hat{T}(k)\hat{T}(k') &= e^{-ik\times k'} \hat{T}(k + k'), \end{aligned} \tag{8}$$

where $k \times k' := \frac{1}{2}\sigma_{\mu\nu} k_\mu k'_\nu$. For $d = 2$ we have

$$\sigma_{\mu\nu} = \sigma \varepsilon_{\mu\nu} = \sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{9}$$

and eq. (3) becomes

$$[\hat{x}_1, \hat{x}_2] = i\sigma. \tag{10}$$

The following remarks concerning the definitions of traces can be deduced with the methods of [3]. Eq. (10) looks like the usual commutation relation of ordinary quantum mechanics between $\hat{q}$ and $\hat{p}$ if one identifies $\hat{x}_1 = \hat{q}$ and $\hat{x}_2 = \hat{p}$. The corresponding eigenstates are defined by [3]:

$$\begin{aligned} \hat{x}_1 |x\rangle &= x|x\rangle, \\ \hat{x}_2 |p\rangle &= \sigma p |p\rangle \end{aligned} \tag{11}$$

with

$$\begin{aligned} \langle x|x'\rangle = \delta(x - x'), \quad & \int dx\, |x\rangle\langle x| = 1, \\ \langle p|p'\rangle = \delta(p - p'), \quad & \int dp\, |p\rangle\langle p| = 1, \end{aligned} \tag{12}$$

and

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipx}. \tag{13}$$

Now it is straightforward to calculate the matrix elements of $\hat{T}(k)$. The result is

$$\langle x'|\hat{T}(k)|x''\rangle = \delta(k_2\sigma + x' - x'') e^{ik_1(x'+x'')/2}, \tag{14}$$

implying that the trace is (with an appropriate normalization factor)

$$\begin{aligned} \mathrm{Tr}\, \hat{T}(k) &:= 2\pi\sigma \int dx\, \langle x|T(k)|x\rangle = \\ &= 2\pi\sigma\, \delta(k_2\sigma) \int dx\, e^{ik_1 x} = (2\pi)^2 \delta^{(2)}(k_\mu). \end{aligned} \tag{15}$$

With eq. (14) we are also able to calculate the matrix elements of $\hat{\Delta}(x)$. A short calculation gives

$$\begin{aligned} \langle x'|\hat{\Delta}(x)|x''\rangle &= \int dk\, \langle x'|\hat{T}(k)|x''\rangle e^{-ikx} = \\ &= \frac{1}{2\pi\sigma}\, \delta\left(x_1 - \frac{x' + x''}{2}\right) e^{i(x'-x'')x_2/\sigma} \end{aligned} \tag{16}$$

and the trace of $\hat{\Delta}(x)$ becomes

$$\mathrm{Tr}\, \hat{\Delta}(x) := 2\pi\sigma \int dx\, \langle x|\hat{\Delta}(x)|x\rangle = \int dx\, \delta(x_1 - x) = 1. \tag{17}$$

Eqs.(15) and (17) confirm the results of [4, 5]. Additionally, one can derive the following relations

$$\begin{aligned} \mathrm{Tr}\, [\hat{T}(k)\hat{T}(k')] &= (2\pi)^2 e^{-ik\times k'} \delta^{(2)}(k_\mu + k'_\mu) = (2\pi)^2\, \delta^{(2)}(k_\mu + k'_\mu), \\ \mathrm{Tr}\, [\hat{\Delta}(x)\hat{\Delta}(x')] &= \delta^{(2)}(x_\mu - x'_\mu). \end{aligned} \tag{18}$$

In order to be complete, we present an alternative way of calculating the trace of $\hat{T}(k)$:

$$\mathrm{Tr}\,\hat{T}(k) = 2\pi\sigma \int dx' \, \langle x'|\hat{T}(k)|x'\rangle, \tag{19}$$

where $|x'\rangle$ is now an appropriate representation of the algebra (3):

$$[\hat{x}_\mu, \hat{x}_\nu]|x'\rangle = i\sigma\varepsilon_{\mu\nu}|x'\rangle. \tag{20}$$

A possible solution for (20) is

$$\hat{x}_\mu|x'\rangle = \left(x_\mu + \frac{i}{2}\sigma_{\mu\rho}\partial_\rho\right)|x'\rangle. \tag{21}$$

However, in this case x' cannot be identified with x_μ due to the fact that x_μ and ∂_ρ represent $2 \times d$ degrees of freedom ($d = 2$). Therefore we need an irreducible representation which eliminates the redundant degrees of freedom.

An irreducible representation is given by (renaming $x' \to x$):

$$\begin{aligned} \hat{x}_1|x\rangle &= x|x\rangle \\ \hat{x}_2|x\rangle &= -i\sigma\frac{d}{dx}|x\rangle. \end{aligned} \tag{22}$$

Using the Baker-Campbell-Hausdorff-formula and the fact that

$$e^{\sigma k_2 \frac{d}{dx}}|x\rangle = |x - \sigma k_2\rangle, \tag{23}$$

one obtains again the result (15).

In order to define a scalar field theory at the operator level we need a derivation prescription [5, 6, 7]:

$$\hat{\partial}_\mu\phi(\hat{x}) = -i[\hat{x}'_\mu, \phi(\hat{x})] = \int dx\, \partial_\mu\phi(x)\hat{\Delta}(x), \tag{24}$$

where $\hat{x}'_\mu = \sigma^{-1}_{\mu\nu}\hat{x}_\nu$ and $\sigma_{\mu\rho}\sigma^{-1}_{\rho\nu} = \delta_{\mu\nu}$. This definition implies

$$[\hat{\partial}_\mu, \hat{x}_\nu] = \delta_{\mu\nu}, \quad [\hat{\partial}_\mu, \hat{\partial}_\nu] = 0. \tag{25}$$

Furthermore, we have the Leibniz rule

$$\hat{\partial}_\mu(f(\hat{x})g(\hat{x})) = \hat{\partial}_\mu f(\hat{x})g(\hat{x}) + f(\hat{x})\hat{\partial}_\mu g(\hat{x}). \tag{26}$$

Additionally, one can show that one has the following useful relation

$$[\hat{\partial}_\mu, \hat{\Delta}(x)] = -\partial_\mu\hat{\Delta}(x). \tag{27}$$

Eq. (27) implies

$$e^{-v_\mu \hat{\partial}_\mu} \hat{\Delta}(x) e^{v_\mu \hat{\partial}_\mu} = \hat{\Delta}(x+v). \tag{28}$$

The existence of such an operator implies that $\mathrm{Tr}\, \hat{\Delta}(x)$ is independent of x for any trace operation on the algebra of operators. (28) gives therefore

$$\mathrm{Tr}\, \hat{\Delta}(x) = \mathrm{Tr} \hat{\Delta}(x+v) \tag{29}$$

and thus one has in consistency with (17):

$$\mathrm{Tr}\, \phi(\hat{x}) = \int dx\, \phi(x) \mathrm{Tr}\, \hat{\Delta}(x) = \mathrm{Tr}\, \hat{\Delta}(x) \int dx\, \phi(x). \tag{30}$$

In normalizing $\mathrm{Tr}\, \hat{\Delta}(x)$ to one we get

$$\mathrm{Tr}\, \phi(\hat{x}) = \int dx\, \phi(x). \tag{31}$$

Now we are able to define the inverse map of (5). In Filk's [4] notation one obtains

$$\phi(x) = \int dk\, e^{ikx}\, \mathrm{Tr}\, [\phi(\hat{x}) T^\dagger(k)] \tag{32}$$

and corresponding to Ambjorn et al. [5] one has

$$\phi(x) = \mathrm{Tr}[\phi(\hat{x}) \hat{\Delta}(x)], \tag{33}$$

allowing now to define a Moyal-Weyl product [4, 5] in the following manner

$$\begin{aligned} (\phi_1 * \phi_2)(x) &:= \int dk\, e^{ikx}\, \mathrm{Tr}\, [\phi_1(\hat{x}) \phi_2(\hat{x}) T^\dagger(k)] = \mathrm{Tr}\, [\phi_1(\hat{x}) \phi_2(\hat{x}) \hat{\Delta}(x)] \\ &= \int dk_1 \int dk_2\, e^{i(k_1+k_2)x} e^{-ik_1 \times k_2} \tilde{\phi}_1(k_1) \tilde{\phi}_2(k_2). \end{aligned} \tag{34}$$

Eqs. (33) and (34) show that there is a one-to-one correspondence between fields (of sufficiently rapid decrease at infinity) and operators. From (34) follows also

$$\int dx\, (\phi_1 * \phi_2)(x) = \int dx\, \phi_1(x) \phi_2(x). \tag{35}$$

Furthermore one has

$$\mathrm{Tr}\, [\phi_1(\hat{x}) \phi_2(\hat{x})] = \int dx\, (\phi_1 * \phi_2)(x). \tag{36}$$

and

$$\mathrm{Tr}[\phi(\hat{x})^4] = \int dx\, (\phi(x))^4_*. \tag{37}$$

One can easily show that cyclic rotation is allowed:

$$\int dx\,(\phi_1 * \phi_2 * \ldots * \phi_n)(x) = \int dx\,(\phi_n * \phi_1 * \ldots * \phi_{n-1})(x). \tag{38}$$

Using now all these definitions one is able to define a scalar field theory on a noncommutative space-time at the "algebra" level as

$$\begin{aligned} \bar{S}^{(0)}[\phi] &= \mathrm{Tr}\left(\frac{1}{2}(\hat{\partial}_\mu \phi(\hat{x}))^2 + \frac{m^2}{2}\phi(\hat{x})^2 + \frac{\lambda}{4!}\phi(\hat{x})^4\right) = \\ &= \mathrm{Tr}\left(\bar{\mathcal{L}}^{(0)}(\phi(\hat{x}))\right). \end{aligned} \tag{39}$$

With help of (24) the latter expression may be rewritten as a "Moyal-Weyl deformed" action:

$$\begin{aligned} S^{(0)}[\phi] &= \int dx \left(\frac{1}{2}\partial_\mu\phi * \partial_\mu\phi + \frac{m^2}{2}\phi * \phi + \frac{\lambda}{4!}(\phi)^4_*\right) = \\ &= \int dx \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}(\phi)^4_*\right) = \\ &= \int dx\, \mathcal{L}^{(0)}_*(\phi(x)). \end{aligned} \tag{40}$$

We conclude this section with some remarks concerning the equation of motion at the algebra level. In order to see how this works it is sufficient to discuss the free kinetic part

$$\bar{S}^{(0)}_{free}[\phi] = -\frac{1}{2}\mathrm{Tr}\left([\hat{x}'_\mu, \phi(\hat{x})][\hat{x}'_\mu, \phi(\hat{x})]\right) = \frac{1}{2}\mathrm{Tr}\,(\hat{\partial}_\mu\phi(\hat{x}))^2. \tag{41}$$

The "classical" equation of motion, similar to the commutative case, is obtained by minimizing the action:

$$\frac{\delta \bar{S}^{(0)}_{free}[\phi]}{\delta\phi(\hat{x})} = 0. \tag{42}$$

We define the functional derivative as usual [2]:

$$\bar{S}^{(0)}_{free}[\phi + \delta\phi] - \bar{S}^{(0)}_{free}[\phi] =: \mathrm{Tr}\left(\frac{\delta \bar{S}^{(0)}_{free}[\phi]}{\delta\phi(\hat{x})}\delta\phi(\hat{x})\right). \tag{43}$$

Using cyclic rotation we obtain

$$\frac{\delta \bar{S}^{(0)}_{free}[\phi]}{\delta\phi(\hat{x})} = [\hat{x}'_\rho, [\hat{x}'_\rho, \phi(\hat{x})]] = -\hat{\partial}_\rho\hat{\partial}_\rho\phi(\hat{x}) = 0. \tag{44}$$

This is the massless free field equation of the theory. The inclusion of the mass term and the interaction gives the following equation of motion:

$$\frac{\delta \bar{S}^{(0)}[\phi]}{\delta \phi(\hat{x})} = -\hat{\partial}_\rho \hat{\partial}_\rho \phi(\hat{x}) + m^2 \phi(\hat{x}) + \frac{\lambda}{3!} \phi(\hat{x})^3 = 0. \tag{45}$$

Eq. (45) will be used for the construction of the energy momentum tensor in the next section. For the Moyal-Weyl deformed field theory one gets in a similar way the equation of motion [2]

$$\frac{\delta S^{(0)}[\phi(x)]}{\delta \phi(x)} = -\partial_\rho \partial_\rho \phi(x) + m^2 \phi(x) + \frac{\lambda}{3!} (\phi)^3_*(x) = 0. \tag{46}$$

3 Noether theorem for translation symmetry at the algebra level and its Moyal-deformed counterpart

In order to define infinitesimal translations at the operator level one generalizes the usual transformation law for a scalar field

$$\delta_\mu \phi(x) = \partial_\mu \phi(x) \tag{47}$$

into

$$\delta_\mu \phi(\hat{x}) = \hat{\partial}_\mu \phi(\hat{x}) = -i[\hat{x}'_\mu, \phi(\hat{x})] \tag{48}$$

in accordance with (24). Since the action

$$\bar{S}^{(0)}[\phi] = \mathrm{Tr}\left(\frac{1}{2}(\hat{\partial}_\mu \phi(\hat{x}))^2 + \frac{m^2}{2}\phi(\hat{x})^2 + \frac{\lambda}{4!}\phi(\hat{x})^4\right) \tag{49}$$

is invariant under translations we can try to derive a Noether current in the following way. One calculates the variation of $\bar{S}^{(0)}[\phi(\hat{x})]$ in two different ways, once using the equation of motion and alternatively without using the equation of motion [9].

First we note that with help of (24) and performing cyclic rotations under the trace one obtains the following formula for "partial integration"

$$\mathrm{Tr}\left(\phi_1(\hat{x})\hat{\partial}_\mu \phi_2(\hat{x})\right) = -\mathrm{Tr}\left(\hat{\partial}_\mu \phi_1(\hat{x})\phi_2(\hat{x})\right). \tag{50}$$

Then we have

$$
\begin{aligned}
&\delta_\mu \bar{S}^{(0)}[\phi]_1 = \\
&\quad = \mathrm{Tr}\Big(\hat{\partial}_\mu \hat{\partial}_\rho \phi(\hat{x}) \hat{\partial}_\rho \phi(\hat{x}) + m^2 \hat{\partial}_\mu \phi(\hat{x}) \phi(\hat{x}) + \frac{\lambda}{3!} \hat{\partial}_\mu \phi(\hat{x}) \phi^3(\hat{x})\Big) = \\
&\quad = \mathrm{Tr}\,\Big(\hat{\partial}_\mu \mathcal{L}(\phi(\hat{x}))\Big), \\
&\delta_\mu \bar{S}^{(0)}[\phi]_2 = \\
&\quad \mathrm{Tr}\Big(\hat{\partial}_\rho(\hat{\partial}_\mu \phi(\hat{x}) \hat{\partial}_\rho \phi(\hat{x})) + \hat{\partial}_\mu \phi(\hat{x}) \Big(- \hat{\partial}_\rho \hat{\partial}_\rho \phi(\hat{x}) + m^2 \phi(\hat{x}) + \frac{\lambda}{3!} \phi(\hat{x})^3\Big)\Big).
\end{aligned}
\tag{51}
$$

Clearly, one has for the difference

$$
\delta_\mu \bar{S}^{(0)}[\phi]_1 - \delta_\mu \bar{S}^{(0)}[\phi]_2 = 0. \tag{52}
$$

This leads to

$$
\mathrm{Tr}\,\Big(\hat{\partial}_\rho \Big[\frac{1}{2}(\hat{\partial}_\rho \phi(\hat{x}) \hat{\partial}_\mu \phi(\hat{x}) + \hat{\partial}_\mu \phi(\hat{x}) \hat{\partial}_\rho \phi(\hat{x})) - \delta_{\rho\mu} \bar{\mathcal{L}}(\phi(\hat{x}))\Big]\Big) = \mathrm{Tr}\,\Big(\hat{\partial}_\rho T_{\rho\mu}\Big) = 0. \tag{53}
$$

where we have defined the (symmetrized) energy-momentum tensor at the algebra level

$$
T_{\rho\mu}(\phi(\hat{x})) := \frac{1}{2}(\hat{\partial}_\rho \phi(\hat{x}) \hat{\partial}_\mu \phi(\hat{x}) + \hat{\partial}_\mu \phi(\hat{x}) \hat{\partial}_\rho \phi(\hat{x})) - \delta_{\rho\mu} \bar{\mathcal{L}}^{(0)}. \tag{54}
$$

It is important to note that eq. (53) does not imply $\hat{\partial}_\rho T_{\rho\mu} = 0$ locally.

For the further discussion we switch to Minkowski space M_4. Using the Moyal-Weyl prescription one can rewrite (54) as[2]

$$
T_{\rho\mu}(\phi(x)) = \frac{1}{2}(\partial_\rho \phi * \partial_\mu \phi + \partial_\mu \phi * \partial_\rho \phi) - \eta_{\rho\mu} \mathcal{L}^{(0)}_*. \tag{55}
$$

The construction (55) is symmetric - therefore no Belinfante-procedure is needed [8]. The result (55) is consistent with

$$
W_\mu S^{(0)}[\phi] = \int dx\, \partial_\mu \phi * \frac{\delta S^{(0)}[\phi]}{\delta \phi(x)} = \int dx\, \partial^\rho T_{\rho\mu} = 0. \tag{56}
$$

We add an improvement term in order to get an improved energy-momentum tensor which is traceless for $m = 0$ [8]:

$$
T^I_{\rho\mu} = T_{\rho\mu} + \frac{1}{6}(\eta_{\rho\mu}\Box - \partial_\rho \partial_\mu)(\phi * \phi). \tag{57}
$$

[2]In [13] one finds some further useful remarks concerning translation symmetry in deformed quantum field theories.

The improvement term does not contribute to the divergence of the energy-momentum tensor which is given by

$$\partial^\rho T_{\rho\mu} = \partial^\rho T^I_{\rho\mu} = \frac{\lambda}{4!}[[\phi, \partial_\mu \phi]_{_M}, \phi * \phi]_{_M} \neq 0, \tag{58}$$

where we have introduced the Moyal bracket

$$[\phi_1(x), \phi_2(x)]_{_M} := (\phi_1 * \phi_2)(x) - (\phi_2 * \phi_1)(x). \tag{59}$$

The result (58) is already given in [2]

For a physical interpretation one chooses $\sigma^{0i} = 0$ [1, 2]. Then one has[3]

$$\int d^3x \, (\phi_1 * \phi_2 * \ldots * \phi_n)(x) = \int d^3x \, (\phi_n * \phi_1 * \ldots * \phi_{n-1})(x). \tag{60}$$

Eq. (58) implies

$$\begin{aligned} \int d^3x \, \partial^\rho T_{\rho\mu} &= \partial^0 \int d^3x \, T_{0\mu} + \int d^3x \partial^i T_{i\mu} = \partial^0 \int d^3x \, T_{0\mu} = \\ &= \int d^3x \, \frac{\lambda}{4!}[[\phi, \partial_\mu \phi]_{_M}, \phi * \phi]_{_M} = 0 \end{aligned} \tag{61}$$

which means that in this case there exists a conserved four momentum:

$$\partial^0 P_\mu := \partial^0 \int d^3x \, T_{0\mu} = 0. \tag{62}$$

Additionally, $\sigma^{0i} = 0$ allows to establish unitarity [10].

As it is well known, in the commutative case the generators of the conformal group are given by moments of the energy-momentum tensor [8]. E.g. in the commutative case the conserved current for dilatation symmetry is given by

$$D_\mu = x^\rho T^I_{\rho\mu}. \tag{63}$$

However, in the noncommutative case one expects a breaking of the dilatation symmetry due to the fact that the energy-momentum tensor is not conserved. As a simple example we study in the last section the broken dilatation symmetry in a Moyal-Weyl deformed field theory.

[3] Note that the $\sigma_{\mu\nu}$-matrix is no longer invertible, and therefore we restrict our attention to the Moyal-deformed field theory.

4 The broken dilatation symmetry

In this section we express the dilatation transformation in terms of a functional differential operator, i.e. we consider

$$W_D = \int dx\, \delta_D\phi * \frac{\delta}{\delta\phi(x)} = \int dx\, (1 + x^\mu * \partial_\mu)\phi * \frac{\delta}{\delta\phi(x)} \tag{64}$$

acting on the Minkowskian action $S^{(0)}[\phi]$ for a massless field given by

$$S^{(0)} = \int dx \left(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \frac{\lambda}{4!}(\phi)^4_*\right). \tag{65}$$

Using

$$\begin{aligned} x^\mu &= (2\pi)^4 \int dp\, e^{ipx}\, i\frac{\partial}{\partial p_\mu}\delta^{(4)}(p), \\ \partial_\mu\phi(x) &= \int dp\, e^{ipx}\, ip_\mu\tilde{\phi}(p) \end{aligned} \tag{66}$$

one verifies with the definition of the Moyal product (34)

$$x^\mu * \partial_\mu\phi(x) = x^\mu\partial_\mu\phi(x). \tag{67}$$

Then one gets using the improved energy-momentum tensor (57)

$$\begin{aligned} W_D S^{(0)}[\phi] = -\int dx \Big[&\partial^\rho\left(x^\mu * T^I_{\rho\mu}\right) + \\ &+\frac{1}{2}(\phi * \Box\phi - \Box\phi * \phi) + \frac{1}{2}x^\mu * (\partial_\mu\phi * \Box\phi - \Box\phi * \partial_\mu\phi) + \\ &+\frac{\lambda}{4!}x^\mu * \left(4\partial_\mu\phi * (\phi)^3_* - \partial_\mu(\phi)^4_*\right)\Big]. \end{aligned} \tag{68}$$

It is straightforward to show that the terms in the second line of (68) vanish and thus one has

$$W_D S^{(0)}[\phi] = -\int dx \Big[\partial^\rho\left(x^\mu * T^I_{\rho\mu}\right) + \underbrace{\frac{\lambda}{4!}x^\mu * \left(4\partial_\mu\phi * (\phi)^3_* - \partial_\mu(\phi)^4_*\right)}_{=:B}\Big]. \tag{69}$$

A rather lenghty but straightforward calculation shows that the breaking B can be written as

$$B = -2\sigma^{\mu\nu}\frac{\partial S^{(0)}[\phi]}{\partial\sigma^{\mu\nu}} \tag{70}$$

which demonstrates that the breaking is determined by the deformation parameter $\sigma^{\mu\nu}$. The result (70) can be understood in the following way. An "infinitesimal" dilatation

$$\hat{x}'^{\mu} = (1+\varepsilon)\hat{x}^{\mu} \qquad (\varepsilon \ll 1) \tag{71}$$

yields the following modified algebra for the operators $\hat{x}'^{\mu}$:

$$[\hat{x}'^{\mu}, \hat{x}'^{\nu}] = i(1+2\varepsilon)\sigma^{\mu\nu} + \mathcal{O}(\varepsilon^2). \tag{72}$$

This means that the change in the deformation parameter induced by infinitesimal dilatations is given by $\delta\sigma^{\mu\nu} = 2\sigma^{\mu\nu}$. Therefore one expects the following relation:

$$\int dx\, \delta_D\phi \frac{\delta S^{(0)}}{\delta\phi} + \delta\sigma^{\mu\nu}\frac{\partial S^{(0)}}{\partial\sigma^{\mu\nu}} = 0. \tag{73}$$

This reproduces exactly the result (69), (70).

5 Conclusion and Outlook

In the previous sections we have shown that one is able to construct an energy-momentum tensor which allows to define a conserved four momentum if $\sigma^{0i} = 0$. We have also demonstrated that the Noether theorem for translations exists already at the operator level in terms of the operators $\phi(\hat{x})$. Using the Moyal-Weyl correspondence between operators $\phi(\hat{x})$ and fields $\phi(x)$ we have also derived the energy momentum tensor in the presence of a Moyal deformed interaction. Our result confirms the results of [1, 2].

In the last section we have also considered the dilatation symmetry directly in a deformed field theory. We found that the Ward-identity of dilatation symmetry picks up a breaking proportional to the deformation parameter $\sigma^{\mu\nu}$. All our considerations are classical, i.e. without inclusion of radiative corrections. Our investigations may be the basis to study the trace anomaly at least at the one loop level. In a further work [12] we will try to give an answer whether the well-known trace anomaly [11] is modified in a Moyal-Weyl deformed scalar quantum field theory.

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