Renormalisation of noncommutative quantum field theories

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We sketch our proof that the real Euclidean ϕ^4-model on the four-dimensional Moyal plane is renormalisable to all orders. The bare action of relevant and marginal couplings of the model is parametrised by four (divergent) quantities which require normalisation to the experimental data. The corresponding physical parameters are the mass, the field amplitude (to be normalised to 1), the coupling constant and—in addition to the commutative version—the frequency of a harmonic oscillator potential.

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1 Introduction

In recent years there has been considerable interest in quantum field theories on the Moyal plane characterised by the $\star$-product

$$(a\star b)(x) := \int d^4y \frac{d^4k}{(2\pi)^4} a(x{+}\tfrac{1}{2}\theta{\cdot}k) b(x{+}y)\, \mathrm{e}^{\mathrm{i}ky} , \qquad \theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R} . \qquad (1)$$

The interest was to a large extent motivated by the observation that this kind of field theories arise in the zero-slope limit of open string theory in presence of a magnetic background field [1]. A few months later it was discovered [2] (first for scalar models) that these noncommutative field theories are not renormalisable beyond a certain loop order due to the mixing of ultraviolet and infrared divergences. A more rigorous explanation was given in [3] where the problem was traced back to divergences in some of the Hepp sectors which correspond to *disconnected* ribbon subgraphs wrapping the same handle of a Riemann surface.

We have proven in [4, 5] that the ϕ^4-model on the four-dimensional Moyal plane is renormalisable to all orders. Our proof rests on two concepts:

– the use of the harmonic oscillator base of the Moyal plane, which avoids the phase factors appearing in momentum space,
– the renormalisation by flow equations.

The renormalised ϕ^4-model corresponds to the classical action

$$S = \int d^4x \Big(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{\mu_0^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x) , \quad (2)$$

with $\tilde{x}_\mu := 2(\theta^{-1})_{\mu\nu}\, x^\nu$. The appearance of the harmonic oscillator term $\frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi)$ in the action (2) is a result of the renormalisation proof.

2 The ϕ^4-action in the matrix base

Expanding the fields $\phi(x) = \sum_{m,n\in\mathbb{N}^2} \phi_{mn} b_{mn}(x)$ in the matrix base (harmonic oscillator base) of the Moyal plane (see e.g. [6]), the action (2) takes the form

$$S = (2\pi\theta)^2 \sum_{m,n,k,l\in\mathbb{N}^2} \Big(\frac{1}{2}\phi_{mn}G_{mn;kl}\phi_{kl} + \frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\Big)\,, \tag{3}$$

$$\begin{aligned} G_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}} &= \big(\mu_0^2+\tfrac{2+2\Omega^2}{\theta}(m^1+n^1+m^2+n^2+2)\big)\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2} \\ &\quad - \tfrac{2-2\Omega^2}{\theta}\big(\sqrt{k^1l^1}\,\delta_{n^1+1,k^1}\delta_{m^1+1,l^1}+\sqrt{m^1n^1}\,\delta_{n^1-1,k^1}\delta_{m^1-1,l^1}\big)\delta_{n^2k^2}\delta_{m^2l^2} \\ &\quad - \tfrac{2-2\Omega^2}{\theta}\big(\sqrt{k^2l^2}\,\delta_{n^2+1,k^2}\delta_{m^2+1,l^2}+\sqrt{m^2n^2}\,\delta_{n^2-1,k^2}\delta_{m^2-1,l^2}\big)\delta_{n^1k^1}\delta_{m^1l^1}\;. \end{aligned} \tag{4}$$

We assume for simplicity that $\theta_{12} = -\theta_{21} = \theta_{34} = -\theta_{43}$ are the only non-vanishing components.

The quantum field theory is constructed as a perturbative expansion about the free theory, which is solved by the propagator $\Delta_{mn;kl}$, the inverse of $G_{mn;kl}$. After diagonalisation of $G_{mn;kl}$ (which leads to orthogonal Meixner polynomials) and the use of identities for hypergeometric functions one arrives at

$$\begin{aligned} &\Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}} \\ &= \frac{\theta}{2(1+\Omega)^2}\sum_{v^1=\frac{|m^1-l^1|}{2}}^{\frac{m^1+l^1}{2}}\;\sum_{v^2=\frac{|m^2-l^2|}{2}}^{\frac{m^2+l^2}{2}} B\big(1+\tfrac{\mu_0^2\theta}{8\Omega}+\tfrac{1}{2}(m^1+k^1+m^2+k^2)-v^1-v^2,\, 1+2v^1+2v^2\big) \\ &\times {}_2F_1\left(\begin{matrix}1+2v^1+2v^2\,,\ \frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2 \\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2\end{matrix}\,\middle|\,\frac{(1-\Omega)^2}{(1+\Omega)^2}\right)\Big(\frac{1-\Omega}{1+\Omega}\Big)^{2v^1+2v^2} \\ &\times \prod_{i=1}^{2}\delta_{m^i+k^i,n^i+l^i}\sqrt{\binom{n^i}{v^i+\frac{n^i-k^i}{2}}\binom{k^i}{v^i+\frac{k^i-n^i}{2}}\binom{m^i}{v^i+\frac{m^i-l^i}{2}}\binom{l^i}{v^i+\frac{l^i-m^i}{2}}}\;. \end{aligned} \tag{5}$$

It is important that the sums in (5) are finite.

3 Renormalisation group approach to dynamical matrix models

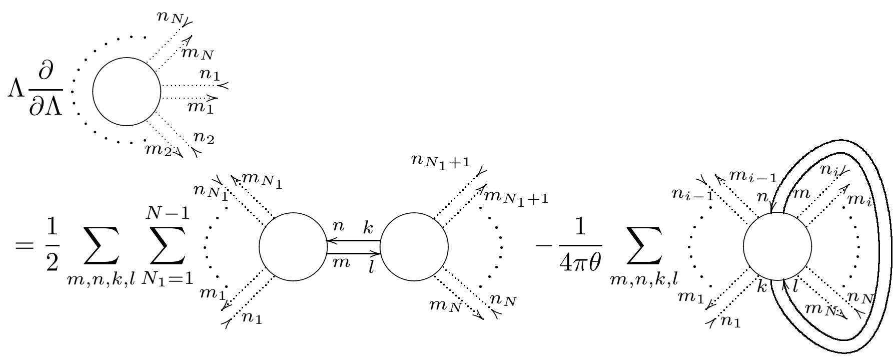
The (Euclidean) quantum field theory is defined by the partition function

$$Z[J] = \int \mathcal{D}[\phi]\,\exp\Big(-S[\phi]-(2\pi\theta)^2\sum_{m,n}\phi_{mn}J_{nm}\Big)\,. \tag{6}$$

The idea inspired by [7] is to change the weights of the matrix indices in the kinetic part of $S[\phi]$ as a smooth function of an energy scale Λ and to compensate this by a careful adaptation of the effective action $L[\phi,\Lambda]$ such that $Z[J]$ becomes independent of the scale Λ. If the modification of the weights of a matrix index $m \in \mathbb{N}$ is described by a function $K\big(\frac{m}{\theta\Lambda^2}\big)$, then the required Λ-dependence of the expansion coefficients of the effective action

$$L[\phi,\Lambda] = \sum_{V=1}^{\infty} \lambda^V \sum_{N=2}^{2V+2} \frac{(2\pi\theta)^{\frac{N}{2}-2}}{N!} \sum_{m_1,n_i \in \mathbb{N}^2} A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\phi_{m_1n_1}\cdots\phi_{m_Nn_N}\,. \quad (7)$$

is described by a differential equation for *ribbon graphs*:



(8)

An internal double line $\overset{n\quad k}{\underset{m\quad l}{\rightleftarrows}}$ symbolises the propagator

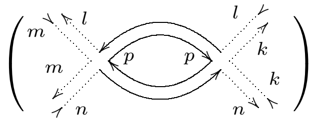
$$Q_{mn;kl}(\Lambda) := \Lambda\frac{\partial}{\partial\Lambda}\bigg(\prod_{i\in m^1,m^2,\dots,l^1,l^2} K\Big(\frac{i}{\theta\Lambda^2}\Big)\Delta_{mn;kl}(\Lambda)\bigg)\,. \quad (9)$$

In this way, very complicated ribbon graphs can be produced which cannot be drawn any more in a plane. Ribbon graphs define a Riemann surface on which they can be drawn. The Riemann surface is characterised by its genus g computable via the Euler characteristic of the graph, $g = 1 - \frac{1}{2}(L - I + V)$, and the number B of holes. Here, L is the number of single-line loops if we close the external lines of the graph, I is the number of double-line propagators and V the number of vertices. The number B of holes coincides with the number of single-line cycles which carry external legs. Accordingly, we also label the expansion coefficients in (7) by the topology, $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}$.

We have proven in [4] a power-counting estimation for these coefficients which relates the Λ-scaling of a ribbon graph to the topology of the graph and to two asymptotic scaling dimensions of the differentiated cut-off propagator $Q_{mn;kl}(\Lambda)$. As a result, if these scaling dimensions coincide with the classical momentum space dimensions, then all non-planar graphs are suppressed by the renormalisation flow. This is a necessary requirement for the renormalisability of a model. On the other hand, as the expansion coefficients $A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ carry an infinite number of matrix indices, the general power-counting estimation proven in [4] leaves, a priory, an infinite number of divergent planar graphs. These planar graphs require a separate analysis which has to be performed model by model.

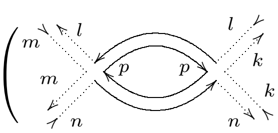
4 Renormalisation of the noncommutative ϕ^4-model

The key is the integration procedure of the Polchinski equation (8), which involves the entire magic of renormalisation. We consider the example of the planar one-particle irreducible four-point function with two vertices, $A^{(2,1,0)1\mathrm{PI}}_{m_1n_1;\dots;m_Nn_N}$. The Polchinski equation (8) provides the Λ-derivative of that function:

$$\Lambda\frac{\partial}{\partial\Lambda}A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda] = \sum_{p\in\mathbb{N}^2}\Bigg($$


$$\Bigg)(\Lambda) + \text{ permut's}\,. \qquad (10)$$

Performing the Λ-integration of (10) from some initial scale Λ_0 (sent to ∞ at the end) down to Λ, we obtain $A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda] \sim \ln\frac{\Lambda_0}{\Lambda}$, which diverges for $\Lambda_0\to\infty$. Renormalisation can be understood as the change of the boundary condition for the integration. Thus, integrating (10) from a renormalisation scale Λ_R up to Λ, we have $A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda] \sim \ln\frac{\Lambda}{\Lambda_R}$, and there would be no problem for $\Lambda_0\to\infty$. However, since there is an *infinite number* of matrix indices and there is no symmetry which could relate the amplitudes, that integration procedure entails an infinite number of initial conditions $A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda_R]$. To have a renormalisable model, we can only afford a finite number of integrations from Λ_R up to Λ. Thus, the correct choice is

$$A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda]$$
$$= -\int_{\Lambda}^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\sum_{p\in\mathbb{N}^2}\Bigg($$


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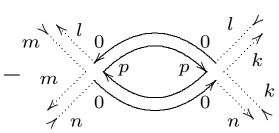
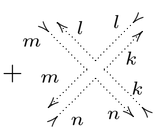
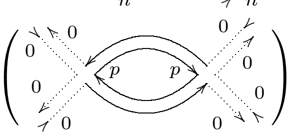


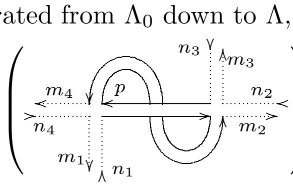
$$\Bigg)[\Lambda']$$
$$+$$


$$\left[\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\sum_{p\in\mathbb{N}^2}\Bigg(\right.$$


$$\left.\Bigg)[\Lambda'] + A^{(2,1,0)1\mathrm{PI}}_{00;00;00;00}[\Lambda_R]\right]. \qquad (11)$$

The second graph in the first line on the rhs and the graph in brackets in the last line are identical, because only the indices on the propagators determine the value of the graph. Moreover, the vertex in the last line in front of the bracket equals 1. Thus, differentiating (11) with respect to Λ we obtain indeed (10). As a further check one can consider (11) for $m=n=k=l=0$. Finally, the independence of $A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda_0]$ on the indices m,n,k,l is built-in. This property is, for $\Lambda_0\to\infty$, dynamically generated by the model.

There is a similar Λ_0-Λ_R-mixed integration procedure for the planar 1PI two-point functions $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$, $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$, $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1\\m^2+1}\substack{n^1\\n^2+1};\substack{n^1\\n^2}\substack{m^1\\m^2}}$ and all other $A^{(V,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}$. These involve in total four different sub-integrations from Λ_R up to Λ. We refer to [5] for details. All other graphs are integrated from Λ_0 down to Λ, e.g.

$$A^{(2,2,0)1\mathrm{PI}}_{m_1n_1;\dots;m_4n_4}[\Lambda] = -\int_{\Lambda}^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\sum_{p\in\mathbb{N}^2}\Bigg($$


$$\Bigg)[\Lambda']\,. \qquad (12)$$

Theorem 1 *The previous integration procedure yields*

$$\begin{aligned}&\big|A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \\ &\leq \big(\sqrt{\theta}\Lambda\big)^{(4-N)+4(1-B-2g)}\; P^{4V-N}\Big[\frac{\max(\|m_1\|,\|n_1\|,\dots\|n_N\|)}{\theta\Lambda^2}\Big]\; P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;,\end{aligned} \tag{13}$$

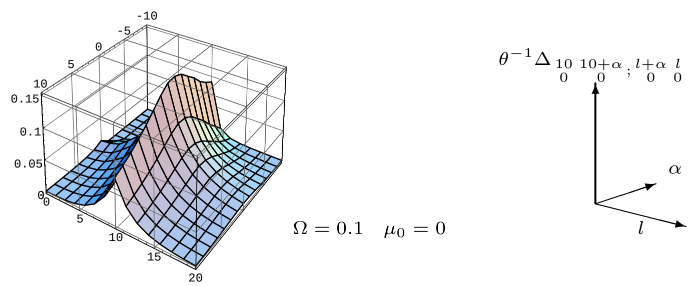
where $P^q[X]$ *stands for a polynomial of degree* q *in* X.

Idea of the proof. For the choice $K(x)=1$ for $0\leq x\leq 1$ and $K(x)=0$ for $x\geq 2$ of the cut-off function in (9) one has

$$|Q_{mn;kl}(\Lambda)| < \frac{C_0}{\Omega\theta\Lambda^2}\delta_{m+k,n+l}\;. \tag{14}$$

Thus, the propagator and the volume of a loop summation have the same power-counting dimensions as a commutative ϕ^4-model in momentum space, giving the total power-counting degree $4-N$ for an N-point function.

This is (more or less) correct for planar graphs. The scaling behaviour of non-planar graphs is considerably improved by the *quasi-locality* of the propagator:



(15)

As a consequence, for given index m of the propagator $Q_{mn;kl}(\Lambda) = \underset{m\quad l}{\overset{n\quad k}{\rightleftarrows}}$, the contribution to a graph is strongly suppressed unless the other index l on the trajectory through m is close to m. Thus, the sum over l for given m converges and does not alter (apart from a factor Ω^{-1}) the power-counting behaviour of (14):

$$\sum_{l\in\mathbb{N}^2}\Big(\max_{n,k}|Q_{mn;kl}(\Lambda)|\Big) < \frac{C_1}{\theta\Omega^2\Lambda^2}\;. \tag{16}$$

In a non-planar graph like the one in (12), the index n_3—fixed as an external index—localises the summation index $p\approx n_3$. Thus, we save one volume factor $\theta^2\Lambda^4$ compared with a true loop summation as in (11). In general, each hole in the Riemann surface saves one volume factor, and each handle even saves two.

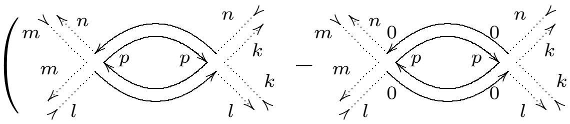
A more careful analysis of (5) shows that also planar graphs get suppressed with $\big|Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(\Lambda)\big| < \frac{C_2}{\Omega\theta\Lambda^2}\prod_{i=1}^2\big(\frac{\max(m^i,l^i)+1}{\theta\Lambda^2}\big)^{\frac{|m^i-l^i|}{2}}$, for $m^i\leq n^i$, if the index along a trajectory jumps. This leaves the functions $A^{(V,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}$, $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$,

$A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1+1\\ m^2}\substack{n^1+1\\ n^2};\substack{n^1\\ n^2}\substack{m^1\\ m^2}}$ and $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1\\ m^2+1}\substack{n^1\\ n^2+1};\substack{n^1\\ n^2}\substack{m^1\\ m^2}}$ as the only relevant or marginal ones. In these functions one has to use a discrete version of the Taylor expansion such as

$$\Big| Q_{\substack{m^1\\ m^2}\substack{n^1\\ n^2};\substack{n^1\\ n^2}\substack{m^1\\ m^2}}(\Lambda) - Q_{\substack{0\\ 0}\substack{n^1\\ n^2};\substack{n^1\\ n^2}\substack{0\\ 0}}(\Lambda)\Big| < \frac{C_3}{\Omega\theta\Lambda^2}\Big(\frac{\max(m^1,m^2)}{\theta\Lambda^2}\Big)\,, \qquad (17)$$

which can be traced back to the Meixner polynomials. The discrete Taylor subtractions are used in the integration from Λ_0 down to Λ in prescriptions like (11):

$$-\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\sum_{p\in\mathbb{N}^2}\Bigg($$



$$\Bigg)[\Lambda']$$

$$= \int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\int_{\Lambda'}^{\Lambda_0}\frac{d\Lambda''}{\Lambda''}\sum_{p\in\mathbb{N}^2}\Big((Q_{np;pn}-Q_{0p;p0})(\Lambda')Q_{lp;pl}(\Lambda'') + Q_{0p;p0}(\Lambda')(Q_{lp;pl}-Q_{0p;p0})(\Lambda'')\Big) \sim \frac{C(\|n\|+\|l\|)}{\theta\Omega^2\Lambda^2}\,. \qquad (18)$$

This explains the polynomial in fractions like $\frac{\|m\|}{\theta\Lambda^2}$ in (13). □

As the estimation (13) is achieved by a finite number of initial conditions at Λ_R (see (11)), the noncommutative ϕ^4-model with oscillator term is renormalisable to all orders in perturbation theory. These initial conditions correspond to normalisation experiments for the mass, the field amplitude, the coupling constant and the oscillator frequency in the bare action related to (2).

We have proven renormalisability of the two-dimensional case in [6], where the oscillator frequency required in intermediate steps can be switched off at the end.

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