Renormalisation of scalar quantum field theory on noncommutative $\mathbb{R}^4$

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Abstract: We give the main ideas our proof that the noncommutative ϕ^4-model is renormalisable to all orders. Compared with the commutative case, the bare action of relevant and marginal couplings contains necessarily an additional term: an harmonic oscillator potential in the free field action which solves the UV/IR-mixing problem.

1 Introduction

There is no doubt that at very short distance scales, space-time can no longer be described by a differentiable manifold. To the most advanced frameworks towards more realistic short-distance structures belong *string theory* and *noncommutative geometry*. Although these two approaches are very different in their strategy, a remarkable connection between them has been established [1]: Field theories on noncommutative spaces arise in the zero-slope theory limit of string theory in presence of D-branes with Neveu-Schwarz B-field. The simplest noncommutative geometry obtained in this way is the Moyal plane characterised by the $\star$-product (in 4 dimensions)

$$(a\star b)(x) := \int d^4y \frac{d^4k}{(2\pi)^4} a(x+\tfrac{1}{2}\theta\cdot k)b(x+y)\,\mathrm{e}^{\mathrm{i}ky}\;, \qquad \theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R}\;. \tag{1}$$

Although from string theory's point of view there is no reason that the limit is a well-defined quantum field theory, there has been an enormous activity aiming at renormalisation proofs for noncommutative quantum field theories. It turned out that the noncommutative analogues of typical field theoretical (in particular four-dimensional) models are not renormalisable due to the UV/IR-mixing problem [2]. The construction of dangerous non-planar graphs was made precise in [3] where the problem was traced back to divergences in some of the Hepp sectors which correspond to *disconnected* ribbon subgraphs wrapping the same handle of a Riemann surface.

We have proven in [4, 5] that the ϕ^4-model on the four-dimensional Euclidean Moyal plane is renormalisable to all orders. A summary is given in [6]. Our proof rests on two concepts:

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– the use of the harmonic oscillator base of the Moyal plane, which avoids the phase factors appearing in momentum space,
– the renormalisation by flow equations.

The renormalised ϕ^4-model corresponds to the classical action

$$S = \int d^4x \Big(\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{\Omega^2}{2} (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{\mu_0^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \Big)(x) \;, \tag{2}$$

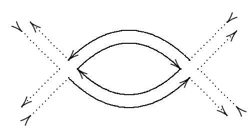
with $\tilde{x}_\mu := 2(\theta^{-1})_{\mu\nu}\, x^\nu$. The appearance of the harmonic oscillator term $\frac{\Omega^2}{2}(\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi)$ in the action (2) is a result of the renormalisation proof.

The harmonic oscillator term is the solution of the UV/IR-mixing problem. It can be motivated by the Langmann-Szabo duality [7]: The replacement

$$p_\mu \leftrightarrow \tilde{x}_\mu \;, \qquad \hat{\phi}(p) \leftrightarrow \pi^2 \sqrt{|\det \theta|}\; \phi(x) \tag{3}$$

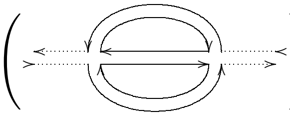
in a ϕ^{2n} mass or interaction term and application of the Fourier transformation $\hat{\phi}(p_a) = \int d^4x\; \mathrm{e}^{(-1)^a \mathrm{i} p_{a,\mu} x_a^\mu} \phi(x_a)$ for a being a cyclic label leaves $\int d^4x\, (\phi \star \cdots \star \phi)(x)$ invariant. This implies that with

$$\Big(\quad \Big)(p_1, p_2, p_3, p_4) \;, \tag{4}$$



also its dual is divergent. Then, as the same rules for propagators and vertices are used, it is plausible that also the dual of

$$\Big(\quad \Big)(p_1, p_2) \tag{5}$$



will be divergent. We need a counterterm in the initial action to absorb this divergence, which is precisely the harmonic oscillator term in (2).

2 The ϕ^4-action in the matrix base

Clearly, the action (2) is difficult to treat in momentum space. Therefore, we use the very convenient matrix base $\{b_{mn}\}_{m,n \in \mathbb{N}^2}$ of the Moyal plane which is distinguished by

$$(b_{mn} \star b_{kl})(x) = \delta_{nk} b_{ml}(x) \;, \qquad \int d^4x\, b_{mn}(x) = (2\pi\theta)^2 \delta_{mn} \;. \tag{6}$$

We assume for simplicity that $\theta_{12} = -\theta_{21} = \theta_{34} = -\theta_{43}$ are the only non-vanishing components. Then, expanding the fields $\phi(x) = \sum_{m,n \in \mathbb{N}^2} \phi_{mn} b_{mn}(x)$ in the matrix base, the interaction term in (2) becomes a simple matrix product, at the price of a rather complicated bilinear term:

$$S = (2\pi\theta)^2 \sum_{m,n,k,l \in \mathbb{N}^2} \Big(\frac{1}{2} \phi_{mn} G_{mn;kl} \phi_{kl} + \frac{\lambda}{4!} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} \Big) \;, \tag{7}$$

$$\begin{aligned} G_{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{k^1 \\ k^2}\substack{l^1 \\ l^2}} &= \big(\mu_0^2 + \tfrac{2+2\Omega^2}{\theta}(m^1+n^1+m^2+n^2+2)\big)\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2} \\ &\quad - \tfrac{2-2\Omega^2}{\theta}\big(\sqrt{k^1l^1}\,\delta_{n^1+1,k^1}\delta_{m^1+1,l^1} + \sqrt{m^1n^1}\,\delta_{n^1-1,k^1}\delta_{m^1-1,l^1}\big)\delta_{n^2k^2}\delta_{m^2l^2} \\ &\quad - \tfrac{2-2\Omega^2}{\theta}\big(\sqrt{k^2l^2}\,\delta_{n^2+1,k^2}\delta_{m^2+1,l^2} + \sqrt{m^2n^2}\,\delta_{n^2-1,k^2}\delta_{m^2-1,l^2}\big)\delta_{n^1k^1}\delta_{m^1l^1} \;. \end{aligned} \tag{8}$$

The quantum field theory is constructed as a perturbative expansion about the free theory, which is solved by the propagator $\Delta_{mn;kl}$, the inverse of $G_{mn;kl}$. After diagonalisation of $G_{mn;kl}$ (which leads to orthogonal Meixner polynomials) and the use of identities for hypergeometric functions one arrives at

$$\begin{aligned}
&\Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}} \\
&= \frac{\theta}{2(1+\Omega)^2} \sum_{v^1=\frac{|m^1-l^1|}{2}}^{\frac{m^1+l^1}{2}} \sum_{v^2=\frac{|m^2-l^2|}{2}}^{\frac{m^2+l^2}{2}} B\Big(1+\tfrac{\mu_0^2\theta}{8\Omega}+\tfrac{1}{2}(m^1+k^1+m^2+k^2)-v^1-v^2, 1+2v^1+2v^2\Big) \\
&\times {}_2F_1\left(\begin{matrix}1+2v^1+2v^2\,,\ \frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2 \\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2\end{matrix}\Bigg|\frac{(1-\Omega)^2}{(1+\Omega)^2}\right)\Big(\frac{1-\Omega}{1+\Omega}\Big)^{2v^1+2v^2} \\
&\times \prod_{i=1}^{2} \delta_{m^i+k^i,n^i+l^i} \sqrt{\binom{n^i}{v^i+\frac{n^i-k^i}{2}}\binom{k^i}{v^i+\frac{k^i-n^i}{2}}\binom{m^i}{v^i+\frac{m^i-l^i}{2}}\binom{l^i}{v^i+\frac{l^i-m^i}{2}}}\,. \qquad (9)
\end{aligned}$$

It is important that the sums in (9) are finite. This allows a fast numerical evaluation of the propagator as well as analytical estimations.

3 Renormalisation group approach to dynamical matrix models

The (Euclidean) quantum field theory is defined by the partition function

$$Z[J] = \int \mathcal{D}[\phi]\, \exp\Big(-S[\phi] - (2\pi\theta)^2 \sum_{m,n} \phi_{mn} J_{nm}\Big)\,. \qquad (10)$$

Instead of expanding the partition function into Feynman graphs, we use the Wilson-Polchinski approach [8, 9] adapted to our case of dynamical matrix models [4]. The idea is to integrate out the modes of the field with matrix indices larger than $\Lambda^2\theta$ (in a smooth way). This results in replacing the original ϕ^4-interaction by an effective action $L[\phi,\Lambda]$. Then, renormalisation of the model amounts to prove that the matrix Polchinski equation

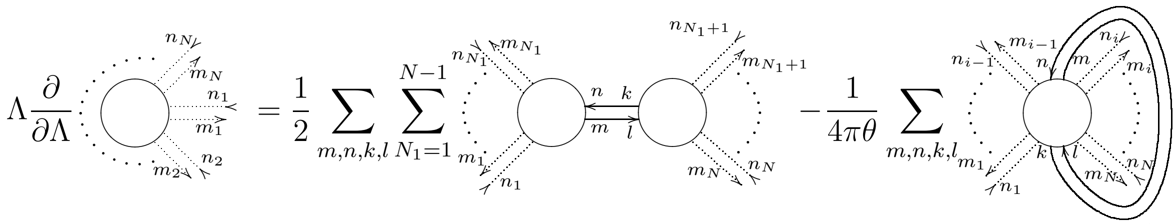
$$\Lambda\frac{\partial L[\phi,\Lambda]}{\partial\Lambda} = \sum_{m,n,k,l\in\mathbb{N}^2} \frac{1}{2} Q_{mn;kl}(\Lambda)\Big(\frac{\partial L[\phi,\Lambda]}{\partial\phi_{mn}}\frac{\partial L[\phi,\Lambda]}{\partial\phi_{kl}} - \frac{1}{(2\pi\theta)^2}\frac{\partial^2 L[\phi,\Lambda]}{\partial\phi_{mn}\,\partial\phi_{kl}}\Big)\,, \qquad (11)$$

admits a *regular solution* which depends on *finitely many initial data*. Here,

$$Q_{mn;kl}(\Lambda) := \Lambda\frac{\partial}{\partial\Lambda}\Big(\prod_{i\in m^1,m^2,\dots,l^1,l^2} K\Big[\frac{i}{\theta\Lambda^2}\Big]\Delta_{mn;kl}(\Lambda)\Big) \qquad (12)$$

is the differentiated cut-off propagator, where $K[x]$ is the smooth cut-off function with $K[x]=1$ for $x\le 1$ and $K[x]=0$ for $x\ge 2$.

Expanding the effective action $L[\phi,\Lambda]$ in a Taylor series with respect to ϕ_{mn}, we obtain a graphical interpretation of the Polchinski equation,



(13)

where $\overset{n\quad k}{\underset{m\quad l}{\rightleftarrows}} = Q_{mn;kl}$. The resulting *ribbon graphs* define a Riemann surface on which they can be drawn. In a perturbative expansion with respect to the number V of vertices, the Riemann surface is characterised by its genus g computable via the Euler characteristic of the graph, $g = 1 - \frac{1}{2}(L - I + V)$, and the number B of holes. Here, L is the number of single-line loops if we close the external lines of the graph, I is the number of double-line propagators and V the number of vertices. The number B of holes coincides with the number of single-line cycles which carry external legs. Accordingly, we label the expansion coefficients of $L[\phi,\Lambda]$ by the topology, $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}$.

4 Power-counting theorem for the noncommutative ϕ^4-model

The asymptotic behaviour of the propagator and the topology of the graph determine the power-counting estimation of the expansion coefficients of the effective action:

$$\begin{aligned}&\big|A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \\ &\leq \big(\sqrt{\theta}\Lambda\big)^{(4-N)+4(1-B-2g)}\; P^{4V-N}\Big[\frac{\max(\|m_1\|,\|n_1\|,\dots\|n_N\|)}{\theta\Lambda^2}\Big]\; P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\,, \end{aligned} \tag{14}$$

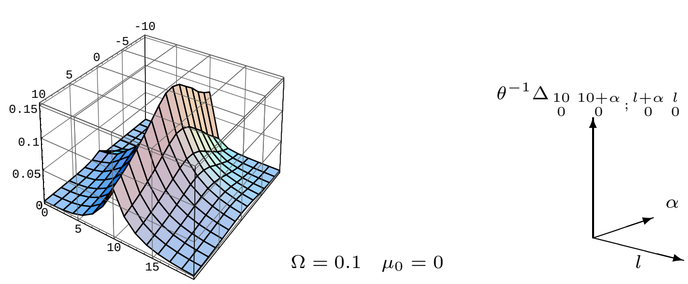
where $P^q[X]$ stands for a polynomial of degree q in X.

There are two ingredients to the proof. First, the cut-off propagator $Q_{mn;kl}(\Lambda)$ decays quadratically in Λ^{-1}:

$$|Q_{mn;kl}(\Lambda)| < \frac{C_0}{\Omega\theta\Lambda^2}\delta_{m+k,n+l}\ . \tag{15}$$

Thus, the propagator and the volume of a loop summation have the same power-counting dimensions as a commutative ϕ^4-model in momentum space, giving the total power-counting degree $4-N$ for an N-point function.

Second, the scaling behaviour of non-planar graphs is considerably improved by the *quasi-locality* of the propagator:

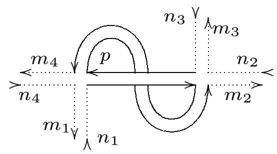


(16)

As a consequence, for given index m of the propagator $Q_{mn;kl}(\Lambda) = \overset{n\quad k}{\underset{m\quad l}{\rightleftarrows}}$, the contribution to a graph is strongly suppressed unless the other index l on the trajectory through m is close to m. Thus, the sum over l for given m converges and does not alter (apart from a factor Ω^{-1}) the power-counting behaviour of (15):

$$\sum_{l\in\mathbb{N}^2}\Big(\max_{n,k}|Q_{mn;kl}(\Lambda)|\Big) < \frac{C_1}{\theta\Omega^2\Lambda^2}\;. \qquad (17)$$

In a non-planar graph like



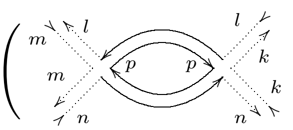
(18)

the index n_3—fixed as an external index—localises the summation index $p \approx n_3$. Thus, we save one volume factor $\theta^2\Lambda^4$ compared with a true loop summation. In general, each hole in the Riemann surface saves one volume factor, and each handle even saves two.

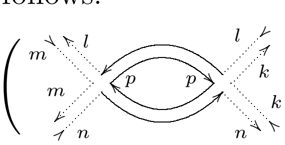
These observations capture the power-counting degree $(4-N)+4(1-B-2g)$ in (14). However, this is not the full story. First, one has also to prove that the power-counting degree is independent on the way one produces the graph[1]. Second, the previous arguments on the scaling (15) and (17) concern the rhs of the Polchinski equation (13). One has to prove that this is preserved after Λ-integration.

There is no problem with irrelevant interaction coefficients, i.e. for $(4-N)+4(1-B-2g) < 0$ because the integration can be performed from Λ_0 sent to ∞ in the end down to Λ. For relevant and marginal functions with $(4-N)+4(1-B-2g) \geq 0$, however, the integration requires initial conditions at some renormalisation scale Λ_R. The problem is that there are infinitely many relevant and marginal functions distinguished by different matrix indices, and a model with infinitely many initial data does not make any sense.

The solution of the problem consists in an introduction of reference graphs with vanishing external indices. Taking the example of the planar one-particle irreducible four-point function with two vertices, $A^{(2,1,0)1\mathrm{PI}}_{m_1n_1;\dots;m_Nn_N}$, the Polchinski equation (13) provides the Λ-derivative of that function:

$$\Lambda\frac{\partial}{\partial\Lambda}A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda] = \sum_{p\in\mathbb{N}^2}\Bigg($$

$$\Bigg)(\Lambda) + \text{ permutations}\;. \qquad (19)$$

The integration is then defined as follows:

$$A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda] = -\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\sum_{p\in\mathbb{N}^2}\Bigg($$

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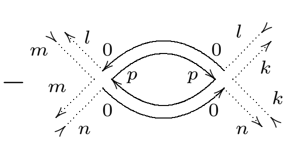
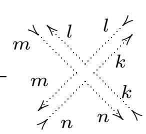
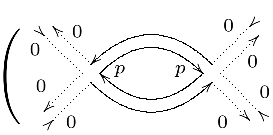
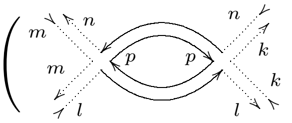

$$\Bigg)[\Lambda']$$
$$+$$

$$\Bigg[\int_{\Lambda_R}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\sum_{p\in\mathbb{N}^2}\Bigg($$

$$\Bigg)[\Lambda'] + A^{(2,1,0)1\mathrm{PI}}_{00;00;00;00}[\Lambda_R]\Bigg]\;. \qquad (20)$$

The second graph on the rhs and the graph in brackets in the last line are identical, because only the indices on the propagators determine the value of the graph. Moreover, the vertex in the last line in front of the bracket equals 1. Thus, differentiating (20) with respect to Λ we obtain indeed (19).

[1] This part of the proof goes over 20 pages in [4].

The difference to the reference graph in (20) is reduced to a difference of propagators. Then, a discrete Taylor expansion of the propagator shows that the integral exists in the limit $\Lambda_0 \to \infty$:

$$-\int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg($$



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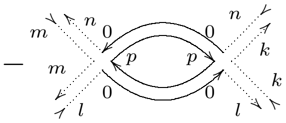

$$\Bigg)[\Lambda']$$

$$= \int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \int_{\Lambda'}^{\Lambda_0} \frac{d\Lambda''}{\Lambda''} \sum_{p\in\mathbb{N}^2} \Big((Q_{np;pn} - Q_{0p;p0})(\Lambda') Q_{lp;pl}(\Lambda'') + Q_{0p;p0}(\Lambda')(Q_{lp;pl} - Q_{0p;p0})(\Lambda'') \Big) \sim \frac{C(\|n\| + \|l\|)}{\theta\Omega^2\Lambda^2}\,. \qquad (21)$$

This explains the polynomial in fractions like $\frac{\|m\|}{\theta\Lambda^2}$ in (14).

In total, the following reference functions are required for the Λ-integration:

- $A^{(V,1,0)1\mathrm{PI}}_{\substack{0\,0;0\,0\\0\,0;0\,0}}$, which corresponds to the mass renormalisation,
- $A^{(V,1,0)1\mathrm{PI}}_{\substack{1\,0;0\,1\\0\,0;0\,0}} - A^{(V,1,0)1\mathrm{PI}}_{\substack{0\,0;0\,0\\0\,0;0\,0}}$, which corresponds to the wave function renormalisation,
- $A^{(V,1,0)1\mathrm{PI}}_{\substack{1\,1;0\,0\\0\,0;0\,0}}$, which has no commutative counterpart and corresponds to the renormalisation of the oscillator potential in (2),
- $A^{(V,1,0)1\mathrm{PI}}_{\substack{0\,0;0\,0;0\,0;0\,0\\0\,0;0\,0;0\,0;0\,0}}$, which corresponds to the renormalisation of the coupling constant.

We refer to [5] for details.

References

[1] N. Seiberg and E. Witten, “String theory and noncommutative geometry,” JHEP **09** (1999) 032 [arXiv:hep-th/9908142].

[2] S. Minwalla, M. Van Raamsdonk and N. Seiberg, “Noncommutative perturbative dynamics,” JHEP **02** (2000) 020 [arXiv:hep-th/9912072].

[3] I. Chepelev and R. Roiban, “Convergence theorem for non-commutative Feynman graphs and renormalization,” JHEP **03** (2001) 001 [arXiv:hep-th/0008090].

[4] H. Grosse and R. Wulkenhaar, “Power-counting theorem for non-local matrix models and renormalisation,” Commun. Math. Phys. **254** (2005) 91 [arXiv:hep-th/0305066].

[5] H. Grosse and R. Wulkenhaar, “Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base,” Commun. Math. Phys. (online first) DOI:10.1007/s00220-004-1285-2 [arXiv:hep-th/0401128].

[6] H. Grosse and R. Wulkenhaar, “Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ to all orders,” Lett. Math. Phys. **71** (2005) 13 [arXiv:hep-th/0403232].

[7] E. Langmann and R. J. Szabo, “Duality in scalar field theory on noncommutative phase spaces,” Phys. Lett. B **533** (2002) 168 [arXiv:hep-th/0202039].

[8] K. G. Wilson and J. B. Kogut, “The renormalization group and the epsilon expansion,” Phys. Rept. **12** (1974) 75.

[9] J. Polchinski, “Renormalization and effective Lagrangians,” Nucl. Phys. **B231** (1984) 269.