To Julius Wess

Noncommutative QFT and Renormalization

Harald Grosse and Raimar Wulkenhaar

It was a great pleasure for me (Harald Grosse) to be invited to talk at the meeting celebrating the 70th birthday of Prof. Julius Wess. I remember various interactions with Julius during the last years: At the time of my studies at Vienna with Walter Thirring, Julius left already Vienna, I learned from his work on effective chiral Lagrangians. Next we met at various conferences and places like CERN (were I worked with Andre Martin, an old friend of Julius), and we all learned from Julius' and Bruno's creation of supersymmetry, next we realized our common interests in noncommutative quantum field theory and did have an intensive exchange. Julius influenced our perturbative approach to gauge field theories were we used the Seiberg-Witten map after his advice. And finally I lively remember the sad days when during my invitation to Vienna Julius did have the serious heart attack. So we are very happy, that you recovered so well, and we wish you all the best for the forthcoming years. Many happy recurrences.

1 Introduction

Four-dimensional quantum field theory suffers from infrared and ultraviolet divergences as well as from the divergence of the renormalized perturbation expansion. Despite the impressive agreement between theory and experiments and despite many attempts, these problems are not settled and remain a big challenge for theoretical physics. Furthermore, attempts to formulate a quantum theory of gravity have not yet been fully successful. It is astonishing that the two pillars of modern physics, quantum field theory and general relativity, seem incompatible. This convinced physicists to look for more general descriptions: After the formulation of supersymmetry (initiated by Bruno Zumino and Julius Wess) and supergravity, string theory was developed, and anomaly cancellation forced the introduction of six additional dimensions. On the other hand, loop gravity was formulated, and led to spin networks and space-time foams. Both approaches are not fully satisfactory. A third impulse came from noncommutative geometry developed by Alain Connes, providing a natural interpretation of the Higgs effect at the classical level. This finally led to noncommutative quantum field theory, which is the subject of this contribution. It allows to incorporate fluctuations of space into quantum field theory. There are of course relations among these three developments. In particular, the field theory limit of string theory leads to certain noncommutative field theory models, and some models defined over fuzzy spaces are related to spin networks.

The argument that space-time should be modified at very short distances goes back to Schrödinger and Heisenberg. Noncommutative coordinates appeared already in the work of Peierls for the magnetic field problem, and are obtained after projecting onto a particular Landau level. Pauli communicated this to Oppenheimer, whose student Snyder [27] wrote down the first deformed space-time algebra preserving Lorentz symmetry. After the development of noncommutative geometry by Connes [8], it was first applied in physics to the integer quantum Hall effect. Gauge models on the two-dimensional noncommutative tori were formulated, and the relevant projective modules over this space were classified.

Through interactions with John Madore I realized that such Fuzzy geometries allow to obtain natural cutoffs for quantum field theory [13]. This line of work was further developed together with Peter Prešnajder and Ctirad Klimčík [12]. At almost the same time, Filk [11] developed his Feynman rules for the canonically deformed four-dimensional field theory, and Doplicher, Fredenhagen and Roberts [9] published their work on deformed spaces. The subject experienced a major boost after one realized that string theory leads to noncommutative field theory under certain conditions [25, 26], and the subject developed very rapidly; see e.g. [19, 30, 10].

2 Noncommutative Quantum Field Theory

The formulation of Noncommutative Quantum Field Theory (NCFT) follows a dictionary worked out by mathematicians. Starting from some manifold $\mathcal{M}$ one obtains the commutative algebra of smooth functions over $\mathcal{M}$, which is then quantized along with additional structure. Space itself then looks locally like a phase space in quantum mechanics. Fields are elements of the algebra resp. a finitely generated projective module, and integration is replaced by a suitable trace operation.

Following these lines, one obtains field theory on quantized (or deformed) spaces, and Feynman rules for a perturbative expansion can be worked out. However some unexpected features such as IR/UV mixing arise upon quantization, which are described below. In 2000 Minwalla, van Raamsdonk and Seiberg realized [21] that perturbation theory for field theories defined on the Moyal plane faces a serious problem. The planar contributions show the standard singularities which can be handled by a renormalization procedure. The nonplanar one loop contributions are finite for generic momenta, however they become singular at exceptional momenta. The usual UV divergences are then reflected in new singularities in the infrared, which is called IR/UV mixing. This spoils the usual renormalization procedure: Inserting many such loops to a higher order diagram generates singularities of any inverse power. Without imposing a special structure such as supersymmetry, the renormalizability seems lost; see also [6, 7].

However, progress was made recently, when we were able to give a solution of this problem for the special case of a scalar four-dimensional theory defined on the Moyal-deformed space $\mathbb{R}^4_\theta$ [16]. The IR/UV mixing contributions were taken into account through a modification of the free Lagrangian by adding an oscillator term with parameter Ω, which modifies the spectrum of the free Hamiltonian. The harmonic oscillator term was obtained as a result of the renormalization proof. The model fulfills then the Langmann-Szabo duality [18] relating short distance and long distance behavior. Our proof followed ideas of Polchinski. There are indications that a constructive procedure might be possible and give a nontrivial ϕ^4 model, which is currently under investigation [24]. At $\Omega = 1$ the model becomes self-dual, and we are presently studying this model in greater details.

Nonperturbative aspects of NCFT have also been studied in recent years. The most significant and surprising result is that the IR/UV mixing can lead to a new phase denoted as "striped phase" [17], where translational symmetry is spontaneously broken. The existence of such a phase has indeed been confirmed in numerical studies [4, 20]. To understand better the properties of this phase and the phase transitions, further work and better analytical techniques are required, combining results from perturbative renormalization with nonperturbative techniques.

Here a particular feature of scalar NCFT is very suggestive: the field can be described as a hermitian matrix, and the quantization is defined nonperturbatively by integrating over all such matrices. This provides a natural starting point for nonperturbative studies. In particular, it suggests and allows to apply ideas and techniques from random matrix theory.

Remarkably, gauge theories on quantized spaces can also be formulated in a similar way [1, 5, 28, 2]. The action can be written as multi-matrix models, where the gauge fields are encoded in terms of matrices which can be interpreted as "covariant coordinates". The field strength can be written as commutator, which induces the usual kinetic terms in the commutative limit. Again, this allows a natural nonperturbative quantization in terms of matrix integrals.

Numerical studies for gauge theories have also been published including the 4-dimensional case [3], which again show a very intriguing picture of nontrivial phases and spontaneous symmetry breaking. These studies also strongly suggest the nonperturbative stability and renormalizability of NC gauge theory, adding to the need of further theoretical work.

3 Renormalization of ϕ^4-theory on the $4D$ Moyal plane

We briefly sketch the methods used by ourselves [16] in the proof of renormalizability for scalar field theory defined on the 4-dimensional quantum plane $\mathbb{R}^4_\theta$, with commutation relations $[x_\mu, x_\nu] = i\theta_{\mu\nu}$. The IR/UV mixing was taken into account through a modification of the free Lagrangian, by adding an oscillator term which modifies the spectrum of the free Hamiltonian:

$$S = \int d^4x \Big(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi) \star (\tilde{x}^\mu\phi) + \frac{\mu^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi\Big)(x) \ . \qquad (1)$$

Here, $\tilde{x}_\mu = 2(\theta^{-1})_{\mu\nu}x^\nu$ and $\star$ is the Moyal star product

$$(a \star b)(x) := \int d^4y \frac{d^4k}{(2\pi)^4} a(x+\tfrac{1}{2}\theta\cdot k) b(x+y)\, \mathrm{e}^{iky} \ , \qquad \theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R} \ . \qquad (2)$$

The harmonic oscillator term in (1) was found as a result of the renormalization proof. The model is covariant under the Langmann-Szabo duality relating short distance and long distance behavior. At $\Omega = 1$ the model becomes self-dual, and connected to integrable models.

This leads to the hope that a constructive procedure around this particular case allows the construction of a nontrivial interacting ϕ^4 model, which would be an extremely interesting and remarkable achievement.

The renormalization proof proceeds by using a matrix base, which leads to a dynamical matrix model of the type:

$$S[\phi] = (2\pi\theta)^2 \sum_{m,n,k,l\in\mathbb{N}^2} \Big(\frac{1}{2}\phi_{mn}\Delta_{mn;kl}\phi_{kl} + \frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\Big) \;, \tag{3}$$

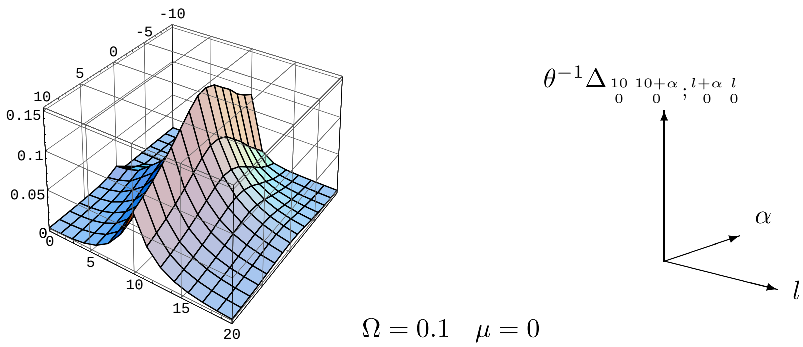
where

$$\begin{aligned}
\Delta_{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{k^1 \\ k^2}\substack{l^1 \\ l^2}} &= \big(\mu^2+\tfrac{2+2\Omega^2}{\theta}(m^1+n^1+m^2+n^2+2)\big)\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2} \\
&- \tfrac{2-2\Omega^2}{\theta}\big(\sqrt{k^1l^1}\,\delta_{n^1+1,k^1}\delta_{m^1+1,l^1} + \sqrt{m^1n^1}\,\delta_{n^1-1,k^1}\delta_{m^1-1,l^1}\big)\delta_{n^2k^2}\delta_{m^2l^2} \\
&- \tfrac{2-2\Omega^2}{\theta}\big(\sqrt{k^2l^2}\,\delta_{n^2+1,k^2}\delta_{m^2+1,l^2} + \sqrt{m^2n^2}\,\delta_{n^2-1,k^2}\delta_{m^2-1,l^2}\big)\delta_{n^1k^1}\delta_{m^1l^1} \;.
\end{aligned} \tag{4}$$

The interaction part becomes a trace of product of matrices, and no oscillations occur in this basis. The propagator obtained from the free part is quite complicated, in 4 dimensions it is:

$$\begin{aligned}
&G_{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{k^1 \\ k^2}\substack{l^1 \\ l^2}} \\
&= \frac{\theta}{2(1+\Omega)^2} \sum_{v^1=\frac{|m^1-l^1|}{2}}^{\frac{m^1+l^1}{2}} \sum_{v^2=\frac{|m^2-l^2|}{2}}^{\frac{m^2+l^2}{2}} B\big(1+\tfrac{\mu^2\theta}{8\Omega}+\tfrac{1}{2}(m^1+k^1+m^2+k^2)-v^1-v^2, 1+2v^1+2v^2\big) \\
&\times {}_2F_1\left(\begin{matrix}1+2v^1+2v^2\,,\ \frac{\mu^2\theta}{8\Omega}-\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2 \\ 2+\frac{\mu^2\theta}{8\Omega}+\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2\end{matrix}\,\middle|\,\frac{(1-\Omega)^2}{(1+\Omega)^2}\right)\Big(\frac{1-\Omega}{1+\Omega}\Big)^{2v^1+2v^2} \\
&\times \prod_{i=1}^{2}\delta_{m^i+k^i,n^i+l^i}\sqrt{\binom{n^i}{v^i+\frac{n^i-k^i}{2}}\binom{k^i}{v^i+\frac{k^i-n^i}{2}}\binom{m^i}{v^i+\frac{m^i-l^i}{2}}\binom{l^i}{v^i+\frac{l^i-m^i}{2}}}\;.
\end{aligned} \tag{5}$$

These propagators (in 2 and 4 dimensions) show asymmetric decay properties:



(6)

They decay exponentially on particular directions (in l-direction in the picture), but have power law decay in others (in α-direction in the picture). These decay properties are crucial for the perturbative renormalizability of the models.

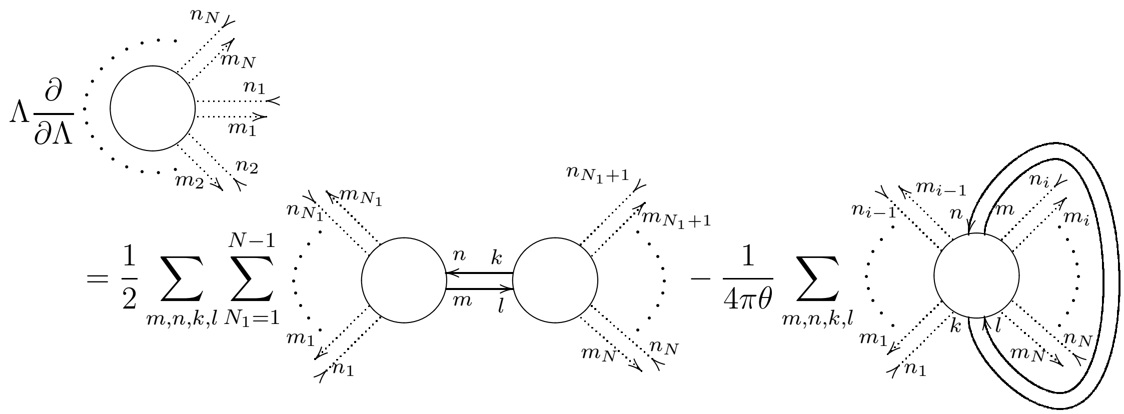
Our proof in [15, 16] then followed the ideas of Polchinski [22]. The quantum field theory corresponding to the action (3) is defined — as usual — by the partition function

$$Z[J] = \int \left(\prod_{m,n} d\phi_{mn} \right) \exp \left(-S[\phi] - \sum_{m,n} \phi_{mn} J_{nm} \right) . \tag{7}$$

The strategy due to Wilson [31] consists in integrating in the first step only those field modes ϕ_{mn} which have a matrix index bigger than some scale $\theta\Lambda^2$. The result is an effective action for the remaining field modes which depends on Λ. One can now adopt a smooth transition between integrated and not integrated field modes so that the Λ-dependence of the effective action is given by a certain differential equation, the Polchinski equation.

Now, renormalization amounts to prove that the Polchinski equation admits a regular solution for the effective action which depends on only a finite number of initial data. This requirement is hard to satisfy because the space of effective actions is infinite dimensional and as such develops an infinite dimensional space of singularities when starting from generic initial data.

The Polchinski equation can be iteratively solved in perturbation theory where it can be graphically written as

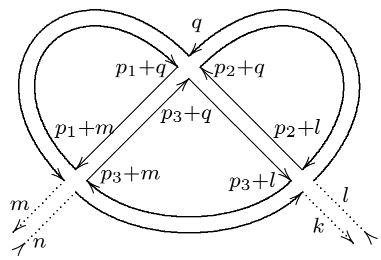


(8)

The graphs are graded by the number of vertices and the number of external legs. Then, to the Λ-variation of a graph on the lhs there only contribute graphs with a smaller number of vertices and a bigger number of legs. A general graph is thus obtained by iteratively adding a propagator to smaller building blocks, starting with the initial ϕ^4-vertex, and integrating over Λ. Here, these propagators are differentiated cut-off propagators $Q_{mn;kl}(\Lambda)$ which vanish (for an appropriate choice of the cut-off function) unless the maximal index is in the interval $[\theta\Lambda^2, 2\theta\Lambda^2]$. As the field carry two matrix indices and the propagator four of them, the graphs are ribbon graphs familiar from matrix models.

We have then proven that the cut-off propagator $Q(\Lambda)$ is bounded by $\frac{C}{\theta\Lambda^2}$. This was achieved numerically in [16] and later confirmed analytically in [24]. A nonvanishing frequency parameter Ω is required for such a decay behavior. As the volume of each two-component index $m \in \mathbb{N}^2$ is bounded by $C'\theta^2\Lambda^4$ in graphs of the above type, the power counting degree of divergence is (at first sight) $\omega = 4S - 2I$, where I is the number of propagators and S the number of summation indices.

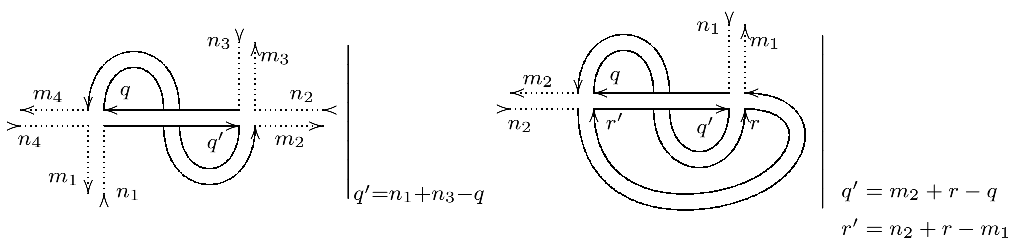
It is now important to take into account that if three indices of a propagator $Q_{mn;kl}(\Lambda)$ are given, the fourth one is determined by $m + k = n + l$, see (5). Then, for simple planar graphs one finds that $\omega = 4 - N$ where N is the number of external legs. But this conclusion is too early, there is a difficulty in presence of completely inner vertices, which require additional index summations. The graph



(9)

entails four independent summation indices p_1, p_2, p_3 and q, whereas for the powercounting degree $2 = 4 - N = 4S - 5 \cdot 2$ we should only have $S = 3$ of them. It turns out that due to the quasi-locality of the propagator (the exponential decay in l-direction in (6)), the sum over q for fixed m can be estimated without the need of the volume factor.

Remarkably, the quasi-locality of the propagator not only ensures the correct powercounting degree for planar graphs, it also renders all nonplanar graphs superficially convergent. For instance, in the nonplanar graphs



(10)

the summation over q and q, r, respectively, is of the same type as over q in (9) so that the graphs in (10) can be estimated without any volume factor.

After all, we have obtained the powercounting degree of divergence

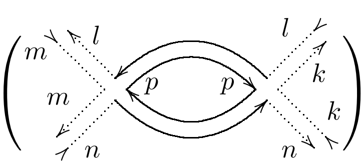
$$\omega = 4 - N - 4(2g + B - 1) \tag{11}$$

for a general ribbon graph, where g is the genus of the Riemann surface on which the graph is drawn and B the number of holes in the Riemann surface. Both are directly determined by the graph. It should be stressed, however, that although the number (11) follows from counting the required volume factors, its proof in our scheme is not so obvious: The procedure consists in adding a new cut-off propagator to a given graph, and in doing so the topology (B, g) has many possibilities to arise from the topologies of the smaller parts for which we have estimates by induction. The proof that in every situation of adding a new propagator one obtains (11) goes alone over 20 pages in [15]. Moreover, the boundary conditions for the integration have to be correctly chosen to confirm (11), see below.

The powercounting behavior (11) is good news because it implies that (in contrast to the situation without the oscillator potential) all nonplanar graphs are superficially convergent. However, this does not mean that all problems are solved: The remaining planar two- and four-leg graphs which are divergent carry matrix indices, and (11) suggests that these are divergent independent of the matrix indices. An infinite number of adjusted initial data would be necessary in order to remove these divergences.

Fortunately, a more careful analysis shows that the powercounting behavior is improved by the index jump along the trajectories of the graph. For example, the index jump for the graph (9) is defined as $J = \|k-n\|_1 + \|q-l\|_1 + \|m-q\|_1$. Then, the amplitude is suppressed by a factor of order $\left(\dfrac{\max(m,n\ldots)}{\theta\Lambda^2}\right)^{\frac{J}{2}}$ compared with the naive estimation. Thus, only planar four-leg graphs with $J=0$ and planar two-leg graphs with $J=0$ or $J=2$ are divergent (the total jumps is even). For these cases, we have invented a discrete Taylor expansion about the graphs with vanishing indices. Only the leading terms of the expansion, i.e. the reference graphs with vanishing indices, are divergent whereas the difference between original graph and reference graph is convergent. Accordingly, in our scheme only the reference graphs must be integrated in a way that involves initial conditions. For example, if the contribution to the rhs of the Polchinski equation (8) is given by the graph

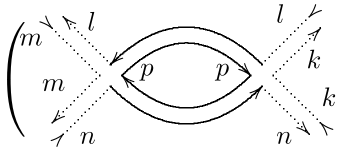
$$\Lambda\frac{\partial}{\partial\Lambda} A^{(2)\text{planar,1PI}}_{mn;nk;kl;lm}[\Lambda] = \sum_{p\in\mathbb{N}^2}$$



$$(\Lambda)\ , \qquad (12)$$

the Λ-integration is performed as follows:

$$A^{(2)\text{planar},1\text{PI}}_{mn;nk;kl;lm}[\Lambda] = -\int_\Lambda^\infty \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg($$



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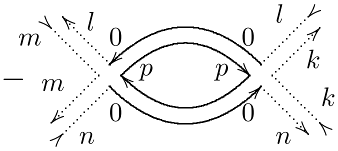
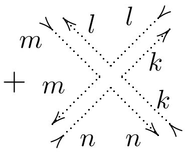
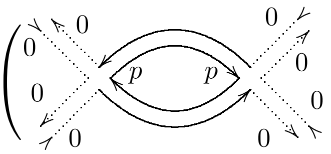


$$\Bigg)[\Lambda'] \;+$$



$$\left[\int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg(\right.$$



$$\left.\Bigg)[\Lambda'] + A^{(2,1,0)1\text{PI}}_{00;00;00;00}[\Lambda_R]\right] . \tag{13}$$

Only one initial condition, $A^{(2,1,0)1\text{PI}}_{00;00;00;00}[\Lambda_R]$, is required for an infinite number of planar four-leg graphs (distinguished by the matrix indices). We need one further initial condition for the two-leg graphs with $J = 2$ and two more initial condition for the two-leg graphs with $J = 0$ (for the leading quadratic and the subleading logarithmic divergence). This is one condition more than in a commutative ϕ^4-theory, and this additional condition justifies a posteriori our starting point of adding one new term to the action (1), the oscillator term Ω.

This being established, it was straightforward to derive beta functions for the coupling constant flow. To one-loop order we have found [14]

$$\beta_\lambda = \frac{\lambda^2_{\text{phys}}}{48\pi^2} \frac{(1-\Omega^2_{\text{phys}})}{(1+\Omega^2_{\text{phys}})^3} , \qquad \beta_\Omega = \frac{\lambda_{\text{phys}}\Omega_{\text{phys}}}{96\pi^2} \frac{(1-\Omega^2_{\text{phys}})}{(1+\Omega^2_{\text{phys}})^3} , \tag{14}$$

$$\beta_\mu = -\frac{\lambda_{\text{phys}}\Big(4\mathcal{N}\ln(2) + \frac{(8+\theta\mu^2_{\text{phys}})\Omega^2_{\text{phys}}}{(1+\Omega^2_{\text{phys}})^2}\Big)}{48\pi^2\theta\mu^2_{\text{phys}}(1+\Omega^2_{\text{phys}})} , \qquad \gamma = \frac{\lambda_{\text{phys}}}{96\pi^2} \frac{\Omega^2_{\text{phys}}}{(1+\Omega^2_{\text{phys}})^3} . \tag{15}$$

Together with the differential equation for the β-functions,

$$\lim_{\mathcal{N}\to\infty} \Big(\mathcal{N}\frac{\partial}{\partial\mathcal{N}} + N\gamma + \mu^2\beta_\mu \frac{\partial}{\partial\mu_0^2} + \beta_\lambda\frac{\partial}{\partial\lambda} + \beta_\Omega\frac{\partial}{\partial\Omega}\Big)\Gamma_{m_1n_1;\dots;m_Nn_N}[\mu,\lambda,\Omega,\mathcal{N}] = 0 , \tag{16}$$

(14) shows that the ratio of the coupling constants $\frac{\lambda}{\Omega^2}$ remains bounded along the renormalization group flow up to first order. Starting from given small values for Ω_R, λ_R at $\mathcal{N}_R$, the frequency grows in a small region around $\ln\frac{\mathcal{N}}{\mathcal{N}_R} = \frac{48\pi^2}{\lambda_R}$ to $\Omega \approx 1$. The coupling constant approaches $\lambda_\infty = \frac{\lambda_R}{\Omega_R^2}$, which can be made small for sufficiently small λ_R. This leaves the chance

of a nonperturbative construction [23] of the model.

In particular, the β-function vanishes at the self-dual point $\Omega = 1$, indicating special properties of the model.

4 Matrix-model techniques

Our recent interests turned towards dynamical matrix models, which are closely connected to integrable models. We briefly explain this method. Consider e.g. the scalar field theory defined by (3). Since ϕ is a hermitian matrix, it can be diagonalized as $\phi = U^{-1} diag(\phi_i) U$ where ϕ_i are the real eigenvalues. Hence the field theory can be reformulated in terms of the eigenvalues ϕ_i and the unitary matrix U. The main idea is now the following: consider the probability measure for the (suitably rescaled) eigenvalues ϕ_i induced the path integral by integrating out U:

$$\begin{aligned} Z &= \int \mathcal{D}\phi\ \exp(-S(\phi))) = \int d\phi_i \Delta^2(\phi_i) \int dU \exp(-S(U^{-1}(\phi_i)U)) \\ &= \int d\phi_i \exp(-\tilde{\mathcal{F}}(\phi) - (2\pi\theta)^{d/2} \sum_i V(\phi_i) + \sum_{i\neq j} \log|\phi_i - \phi_j|), \end{aligned} \tag{17}$$

where the analytic function

$$e^{-\tilde{\mathcal{F}}(\phi)} := \int dU \exp(-S_{kin}(U^{-1}(\phi)U)) \tag{18}$$

is introduced, which depends only on the eigenvalues of ϕ. The crucial point is that the logarithmic terms in the effective action above implies a repulsion of the eigenvalues ϕ_i, which therefore arrange themselves according to some distribution similar as in the standard matrix models of the form $\tilde{S} = \int d\phi \exp(\mathrm{Tr}\tilde{V}(\phi))$. This is related to the fact that nonplanar diagrams are suppressed. The presence of the unknown function $\tilde{\mathcal{F}}(\phi)$ in (17) cannot alter this conclusion qualitatively, since it is analytic. The function $\tilde{\mathcal{F}}(\phi)$ can be determined approximately by considering the weak coupling regime. For example, the effective action of the eigenvalue sector for the ϕ^4 model in the noncommutative regime $\frac{1}{\theta} \ll \Lambda^2$ becomes essentially

$$\tilde{S}(\phi) = f_0(m) + \frac{2N}{\alpha_0^2(m)} \mathrm{Tr}\phi^2 + g\phi^4, \tag{19}$$

where $\alpha_0^2(m)$ depends on the degree of divergence of a basic diagram [29].

This effective action (19) can now be studied using standard results from random matrix theory. For example, this allows to study the renormalization of the effective potential using matrix model techniques. The basic mechanism is the following: In the free case, the eigenvalue sector follows Wigner's semicircle law, where the size of the eigenvalue distribution depends on m via $\alpha_0(m)$. Turning on the coupling g alters that eigenvalue distribution. The effective or renormalized mass can be found by matching that distribution with the "closest" free distribution. To have a finite renormalized mass then requires a negative mass counterterm as usual.

This approach is particularly suitable to study the thermodynamical properties of the field theory. For the ϕ^4 model, the above effective action (19) implies a phase transition at strong coupling, to a phase which was identified with the striped or matrix phase in [29]. Based on the known universality properties of matrix models, these results on phase transitions are expected to be realistic, and should not depend on the details of the unknown function $\tilde{\mathcal{F}}(\phi)$. The method is applicable to 4 dimensions, where a critical line is found which terminates at a nontrivial point, with finite critical coupling. This can be seen as evidence for a new nontrivial fixed-point in the 4-dimensional NC ϕ^4 model. This is in accordance with results from the RG analysis of [14], which also point to the existence of nontrivial ϕ^4 model in 4 dimensions.

References

[1] J. Ambjørn, Y. M. Makeenko, J. Nishimura, and R. J. Szabo. Nonperturbative dynamics of noncommutative gauge theory. *Phys. Lett.*, B480:399–408, 2000.

[2] W. Behr, F. Meyer, and H. Steinacker. Gauge theory on fuzzy $S^2 \times S^2$ and regularization on noncommutative $\mathbb{R}^4$. *JHEP*, 07:040, 2005.

[3] W. Bietenholz et al. Numerical results for U(1) gauge theory on $2d$ and $4d$ noncommutative spaces. *Fortsch. Phys.*, 53:418–425, 2005.

[4] W. Bietenholz, F. Hofheinz, and J. Nishimura. Phase diagram and dispersion relation of the non-commutative $\lambda\phi^4$-model in $d = 3$. *JHEP*, 06:042, 2004.

[5] U. Carow-Watamura and S. Watamura. Noncommutative geometry and gauge theory on fuzzy sphere. *Commun. Math. Phys.*, 212:395–413, 2000.

[6] I. Chepelev and R. Roiban. Renormalization of quantum field theories on noncommutative $\mathbb{R}^d$. I: Scalars. *JHEP*, 05:037, 2000.

[7] I. Chepelev and R. Roiban. Convergence theorem for non-commutative Feynman graphs and renormalization. *JHEP*, 03:001, 2001.

[8] A. Connes. Noncommutative differential geometry. *Inst. Hautes Etudes Sci. Publ. Math.*, 62:257, 1986.

[9] S. Doplicher, K. Fredenhagen, and J. E. Roberts. The quantum structure of space-time at the Planck scale and quantum fields. *Commun. Math. Phys.*, 172:187–220, 1995.

[10] M. R. Douglas and N. A. Nekrasov. Noncommutative field theory. *Rev. Mod. Phys.*, 73:977–1029, 2001.

[11] T. Filk. Divergencies in a field theory on quantum space. *Phys. Lett.*, B376:53–58, 1996.

[12] H. Grosse, C. Klimcik, and P. Presnajder. Towards finite quantum field theory in noncommutative geometry. *Int. J. Theor. Phys.*, 35:231–244, 1996.

[13] H. Grosse and J. Madore. A noncommutative version of the Schwinger model. *Phys. Lett.*, B283:218–222, 1992.

[14] H. Grosse and R. Wulkenhaar. The beta-function in duality-covariant noncommutative ϕ^4 theory. *Eur. Phys. J.*, C35:277–282, 2004.

[15] H. Grosse and R. Wulkenhaar. Power-counting theorem for non-local matrix models and renormalisation. *Commun. Math. Phys.*, 254:91–127, 2005.

[16] H. Grosse and R. Wulkenhaar. Renormalisation of ϕ^4 theory on noncommutative $\mathbb{R}^4$ in the matrix base. *Commun. Math. Phys.*, 256:305–374, 2005.

[17] S. S. Gubser and S. L. Sondhi. Phase structure of non-commutative scalar field theories. *Nucl. Phys.*, B605:395–424, 2001.

[18] E. Langmann and R. J. Szabo. Duality in scalar field theory on noncommutative phase spaces. *Phys. Lett.*, B533:168–177, 2002.

[19] J. Madore, S. Schraml, P. Schupp, and J. Wess. Gauge theory on noncommutative spaces. *Eur. Phys. J.*, C16:161–167, 2000.

[20] X. Martin. A matrix phase for the ϕ^4 scalar field on the fuzzy sphere. *JHEP*, 04:077, 2004.

[21] S. Minwalla, M. Van Raamsdonk, and N. Seiberg. Noncommutative perturbative dynamics. *JHEP*, 02:020, 2000.

[22] J. Polchinski. Renormalization and effective Lagrangians. *Nucl. Phys.*, B231:269–295, 1984.

[23] V. Rivasseau. From perturbative to constructive renormalization. Princeton, USA: Univ. Pr. (1991) 336 p. (Princeton series in physics).

[24] V. Rivasseau, F. Vignes-Tourneret, and R. Wulkenhaar. Renormalization of noncommutative ϕ^4-theory by multi-scale analysis. *Commun. Math. Phys.*, 2005.

[25] V. Schomerus. D-branes and deformation quantization. *JHEP*, 06:030, 1999.

[26] N. Seiberg and E. Witten. String theory and noncommutative geometry. *JHEP*, 09:032, 1999.

[27] H. S. Snyder. Quantized space-time. *Phys. Rev.*, 71:38–41, 1947.

[28] H. Steinacker. Quantized gauge theory on the fuzzy sphere as random matrix model. *Nucl. Phys.*, B679:66–98, 2004.

[29] H. Steinacker. A non-perturbative approach to non-commutative scalar field theory. *JHEP*, 03:075, 2005.

[30] R. J. Szabo. Quantum field theory on noncommutative spaces. *Phys. Rept.*, 378:207–299, 2003.

[31] K. G. Wilson and John B. Kogut. The renormalization group and the epsilon expansion. *Phys. Rept.*, 12:75–200, 1974.