On Feynman graphs as elements of a Hopf algebra

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Abstract

We review Kreimer's construction of a Hopf algebra associated to the Feynman graphs of a perturbative quantum field theory.

1 Introduction

This article is a companion to the contribution of Dirk Kreimer [1] to this volume. With his discovery that divergent Feynman graphs can be regarded as elements of a Hopf algebra whose antipode implements renormalization [2], Kreimer has initiated a dynamic development on the frontier of quantum field theory and noncommutative geometry. In particular since Alain Connes and Dirk Kreimer [3] established a structural link of this renormalization Hopf algebra to a Hopf algebra found in the study of an index problem in noncommutative geometry [4], this topic belongs to the most promising ones towards a unification of quantum field theory with gravity. Connes and Kreimer realized that a subalgebra of the Hopf algebra of renormalization (of a QFT given by a single divergent Feynman graph) is isomorphic to the dual of the diffeomorphism group of a manifold. The central quest is now the search for the counterpart of the entire renormalization Hopf algebra (of a realistic QFT) replacing the diffeomorphism group. There is no doubt that the latter object will deliver precious information on the short distance structure of spacetime. Renormalization is our most powerful microscope to see the smallest structures of the world!

Compared with these dreams, this article is extremely modest. We review how the Hopf algebra is derived from Feynman graphs. This review is intended to be pedagogical, we try to omit technical details as far as possible and prefer to illustrate the essential steps by typical examples from QED. The reader will find supplementary information in the original papers [2, 5, 3]. Our strategy is to focus first on Feynman graphs without overlapping divergences (section 2), because they yield a fairly simple Hopf algebra (section 3). In section 4 we include overlapping divergences and extend the Hopf algebra according to ideas developed in [5]. On that level, the antipode of the Hopf algebra recovers the combinatorics of renormalization. It reproduces the entire renormalization if we allow for a deformation of the Hopf algebra by a renormalization map R, see section 5. That map R spoils however the Hopf algebra axioms, and we hope to gain a deeper understanding of this new structure in the future.

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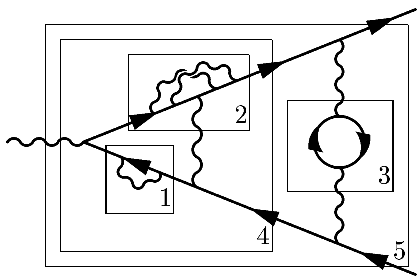
2 Feynman graphs, parenthesized words and rooted trees

Given a Feynman graph, let us draw boxes around all of its superficially (UV-) divergent sectors, for example

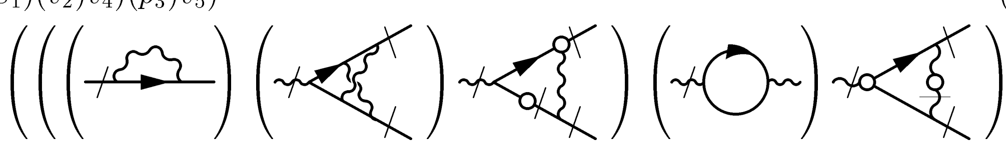


$$= (((s_1)(v_2)v_4)(p_3)v_5) \qquad (1)$$

(As usual, straight lines stand for fermions and wavy lines for bosons.) A criterion for superficial divergence of a region confined in a box is power counting. If a box has n_B bosonic and n_F fermionic outgoing legs, the power counting degree of divergence d is (in four dimensions) defined by $d := 4 - n_B - \frac{3}{2}n_F \geq 0$. Owing to symmetries the actual degree of divergence of one graph or a sum of graphs can be lower than d, see ref. [7].

Our example (1) is taken from a special class of Feynman graphs which contain no overlapping divergences. This means that the boxes can always be chosen non-intersecting. Any two boxes are, therefore, either disjoint from each other, or one box is nested in the other box. It is now convenient to rewrite such a Feynman graph in a form where the relative position of the subdivergences is more apparent. Starting with the set of innermost disjoint boxes we cut them out of the graph but leave them in the next-larger box. In our example, these are the boxes 1 and 2 which are cut out from the graph in box 4. In box 4 we are therefore left with a graph with two holes and two separate subgraphs flying around[1]. Now we pass to the next larger box (no. 5) and cut out the disjoint boxes from the graph. In the example we get a vertex graph with two holes and the two boxes 3 and 4 flying around, where box 4 itself contains boxes 1 and 2. Now we replace the boxes by pairs of opening-closing parentheses and order their contents such that the mother graph stands on the right of its children graphs. This gives us a "parenthesized word", PW for short. The PW of our example (1) looks as follows:

$$(((s_1)(v_2)v_4)(p_3)v_5) \qquad (2)$$

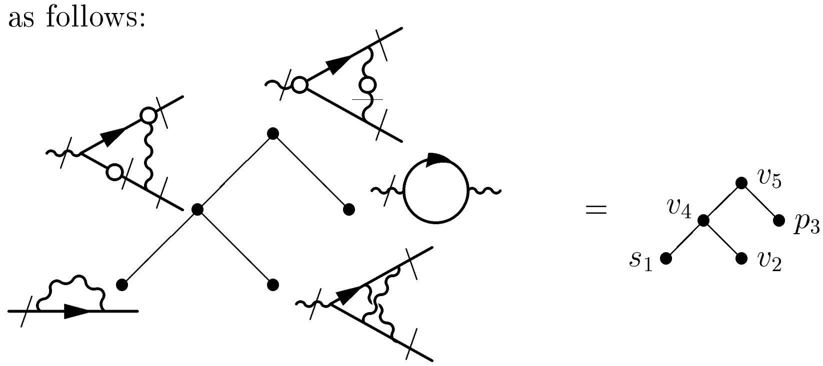
$=$ 

A slash through a propagator means amputation and a small circle symbolizes a hole. We see that our building blocks are the Feynman graphs with possible holes at any vertex and in any propagator. In an irreducible parenthesized word

[1]When we cut out a self-energy insertion, which means cutting a propagator into two, we attach one of these new propagators to the self-energy graph cut out. In this way we keep the number of holes in a graph finite.

(iPW for short) the leftmost opening parenthesis matches its rightmost closing parenthesis. A special type of iPWs are the primitive PWs which contain no inner parentheses.

There is a second way of writing the same, which turns out to be the adapted language in connection with noncommutative geometry. In an iPW we call the rightmost graph (mother graph) the root. We connect to this root the mother graphs of all its children boxes, and to these mother graphs the mother graphs of their children boxes, and so on. In this way we get a "rooted tree" whose vertices are labelled by Feynman graphs with holes. The rooted tree of our example (1) clearly looks as follows:



(3)

The tree is connected and simply connected for irreducible PWs and disconnected for reducible PWs. The tree of a primitive PWs consists solely of a labelled point.

3 The Hopf algebra

Dirk Kreimer has discovered [2] that the PWs or rooted trees of Feynman graphs form a Hopf algebra, whose antipode axiom reproduces the forest formula of renormalization [8]. Let us review these ideas in some detail. We refer e.g. to appendix 1 in [9] for a list of basic properties of Hopf algebras.

We consider the algebra $\mathcal{A}$ of polynomials over the rational numbers $\mathbf{Q}$ of irreducible PWs resp. connected rooted trees. The multiplication in $\mathcal{A}$ is given by writing the trees or PWs disjoint to each other. That multiplication is clearly associative and commutative. We also adjoin a formal unit e to $\mathcal{A}$.

We are going to equip that algebra with the structure of a Hopf algebra. The counit $\varepsilon : \mathcal{A} \to \mathbf{Q}$ is an operation which annihilates trees resp. PWs:

$$\varepsilon[qe] = q \ , \quad q \in \mathbf{Q}, \qquad\qquad \varepsilon[X] = 0 \ , \quad \mathcal{A} \ni X \neq e \ . \tag{4}$$

The product being an assignment of one element to sums of pairs of other elements, we expect the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ to be the splitting of a given PW or tree into a sum of pairs of PWs / trees. Before we give the details we have to define the notion of a subword or subtree. A parenthesized subword (PSW for short) of a PW is any of its iPW contained in it. In our example (1), the subwords of $(((s_1)(v_2)v_4)(p_3)v_5)$ are

$$(s_1) \ , \quad (v_2) \ , \quad (p_3) \ , \quad ((s_1)(v_2)v_4) \ , \quad (((s_1)(v_2)v_4)(p_3)v_5) \ . \tag{5}$$

The idea of the coproduct of a PW X is that it returns as the left factor of the tensor product any admissible product of PSWs X_i of X and as the right factor that what remains when we remove this left factor from X. A product $X_1 \cdots X_n$ is admissible if any two X_i, X_j contained in it do not intersect. For instance, $(s_1)(p_3)$ or $(s_1)(v_2)$ are admissible, but $(v_2)((s_1)(v_2)v_4)$ is not. We symbolize the removal of $X_1 \cdots X_n$ from X by $X/\prod_{i=1}^{n} X_i$ (replaced by 0 if $X_1 \cdots X_n$ is not admissible). A special case is $X/X = e$. Now, if the PSWs of the PW X are $X_1, \ldots, X_n$, we let U be the set of all (ordered) subsets of $\{1, \ldots, n\}$ and define

$$\Delta[e] := e \otimes e \;,$$
$$\Delta[X] := e \otimes X + \sum_U \Big\{ \prod_{i \in U} X_i \otimes X/\prod_{i \in U} X_i \Big\} \;. \tag{6}$$

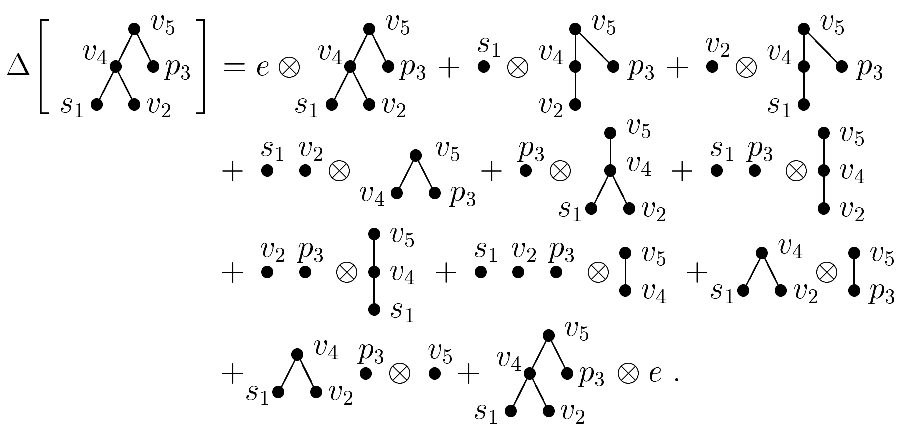
For our example we find

$$\begin{aligned}\Delta\big[(((s_1)(v_2)v_4)(p_3)v_5)\big] = {} & e \otimes (((s_1)(v_2)v_4)(p_3)v_5) + (s_1) \otimes ((v_2)v_4)(p_3)v_5) \\ & + (v_2) \otimes ((s_1)v_4)(p_3)v_5) + (s_1)(v_2) \otimes ((v_4)(p_3)v_5) \\ & + (p_3) \otimes (((s_1)(v_2)v_4)v_5) + (s_1)(p_3) \otimes (((v_2(v_4)v_5) \\ & + (v_2)(p_3) \otimes (((s_1(v_4)v_5) + (s_1)(v_2)(p_3) \otimes ((v_4)v_5) \\ & + ((s_1)(v_2)v_4) \otimes ((p_3)v_5) + ((s_1)(v_2)v_4)(p_3) \otimes (v_5) \\ & + (((s_1)(v_2)v_4)(p_3)v_5) \otimes e \;. \end{aligned} \tag{7}$$

In terms of rooted trees, the coproduct has an even more natural interpretation. A subtree T_i of a tree T is what falls down if we cut one edge of T, or it is the connected tree itself cut out of the plane. The notion of an admissible product of PSWs finds its counterpart in the set of admissible (multi-) cuts: the product $T_1 \cdots T_n$ is admissible iff on the path from any bottom vertex to the root of T we meet at most one of the n cuts that have produced the T_i. The rest $T/(T_1 \cdots T_n)$ is what remains attached to the root if we cut away the subtrees $T_1, \ldots, T_n$. The analogue of (6) is

$$\Delta[T] := e \otimes T + \sum_C \Big\{ \prod_{i \in U(C)} T_i \otimes T/\prod_{i \in U(C)} T_i \Big\} \;, \tag{8}$$

where each C is an admissible multi-cut of T which produces the subtrees $\{T_i\}_{i \in U(C)}$. For our example (1) , the coproduct reads in terms of trees



(9)

The coproduct is coassociative,

$$(\Delta \otimes \mathrm{id}) \circ \Delta[X] = X = (\mathrm{id} \otimes \Delta) \circ \Delta[X] \ . \tag{10}$$

If we split a PW into a sum of two, it is the same to split the right or the left factors further. The proof of (10) is non-trivial, it can be performed by induction [2, 3] or directly [5]. The coproduct is, however, not cocommutative, which means that in general

$$\tau \circ \Delta[X] \neq \Delta[X] \ .$$

Here, $\tau[X \otimes Y] = Y \otimes X$ is the flip operator.

From (4) and (6) the following relation between counit and coproduct is obvious:

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta[X] = (\mathrm{id} \otimes \varepsilon) \circ \Delta[X] = X \ . \tag{11}$$

Indeed, if we consider a monomial $X = \prod_i X_i$ of iPWs resp. connected trees, we get $\Delta[X] = e \otimes X + \sum Z \otimes Z' + X \otimes e$, where $\sum Z \otimes Z'$ represents all terms which do not contain the unit e and which are therefore annihilated by ε. The identity (11) means that our algebra $\mathcal{A}$ is also a coalgebra. It is even a bialgebra as the algebra and coalgebra operations are compatible. Denoting the multiplication in $\mathcal{A}$ by m we have

$$\Delta \circ m[X \otimes Y] = (m \otimes m) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)[X \otimes Y] \ , \tag{12}$$

due to the fact that the subwords of a product word XY are the subwords of X and the subwords of Y together.

The bialgebra $\mathcal{A}$ also has an antipode $S : \mathcal{A} \to \mathcal{A}$ which makes it to a Hopf algebra. It is on PWs and rooted trees recursively defined by

$$\begin{aligned} S[e] &= e \ , \\ S[XY] &= S[Y]S[X] \ , \qquad \forall X, Y \in \mathcal{A} \ , \\ S[X] &= -X - m \circ (S \otimes \mathrm{id}) \circ P_2 \Delta[X] \ , \qquad \forall \text{ iPW } X \in \mathcal{A} \ , \end{aligned} \tag{13}$$

where $P_2 : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is an operation annihilating all terms which contain the unit e. Note that $S[X] = -X$ if X is primitive. On rooted trees we can give a more natural definition of the antipode. In $P_2\Delta$ we take precisely the proper admissible cuts into account, proper in the sense that the cut of the entire tree out of the plane is not included. Since the recursive definition of S is translated into a recursive application of $P_2\Delta$ to the leftmost factor, it is not difficult to see [3] that (13) is equivalent to

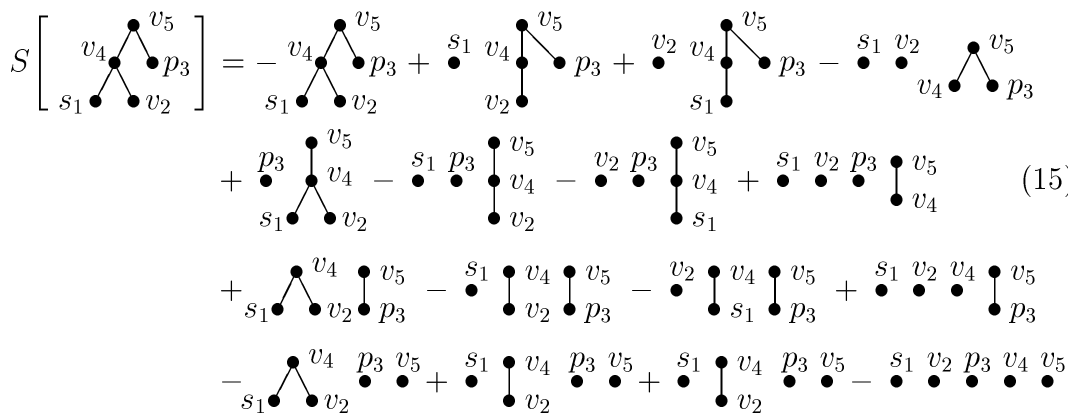
$$S[X] = -X - \sum_{C_a} \Big\{ (-1)^{\#(C_a)} \{ \prod_{i \in U(C_a)} X_i \} \, \{ X / \prod_{i \in U(C_a)} X_i \} \Big\} \ . \tag{14}$$

Now, the sum runs over all proper multi-cuts C_a consisting of $\#(C_a)$ single cuts, where each C_a cuts away the trees X_i, $i \in U(C_a)$ which need no longer to be subtrees of X in the previous sense. It remains the tree $X/\prod_{i\in U(C_a)} X_i$ which contains the root. The last line of (13) can equivalently be written as

$$S[X] = -X - m \circ (\mathrm{id} \otimes S) \circ P_2\Delta[X] \;, \tag{13'}$$

The easiest way to see this is to realize that both versions yield the same formula (14).

It is probably a good idea to illustrate the antipode for our example, although it is a bit lengthy:


(15)

We can now check the antipode axioms

$$\begin{aligned} m \circ (S \otimes \mathrm{id}) \circ \Delta[e] &= m \circ (\mathrm{id} \otimes S) \circ \Delta[e] = e \;, \\ m \circ (S \otimes \mathrm{id}) \circ \Delta[X] &= m \circ (\mathrm{id} \otimes S) \circ \Delta[X] = 0 \qquad \forall X \neq e \;. \end{aligned} \tag{16}$$

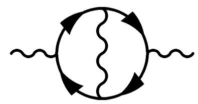
The second line is an immediate consequence of (13), (13′) and of the identity $\Delta[X] = e \otimes X + X \otimes e + P_2\Delta[X]$.

The reader may worry what all that has to do with Feynman graphs. What we have only used are parenthesized words or rooted trees whose building blocks are letters of some alphabet. Indeed, it was pointed out in [6] that the Hopf algebra structure is based on elementary set theoretical considerations. Feynman graphs are just an example. There is however an important observation, due to Dirk Kreimer [2], which makes the application of these set theoretic tools tremendously important for Feynman graphs. For X being the parenthesized word or rooted tree of a Feynman graph, the antipode S of (13) reproduces precisely the combinatorics of the forest formula which governs the renormalization of perturbative QFTs. We will return to this achievement in section 5.

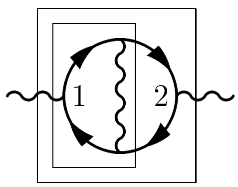
4 Overlapping divergences

Feynman graphs may contain overlapping divergences which at first sight do not fit into the language of parenthesized words or rooted trees. However, it turns out that an overlapping divergence can be represented by a linear combination of words or trees, where additional primitive elements arise. We present here a particular construction [5] of this linear combination. For this purpose we enlarge the class of PWs by including words with several lines – one line for each maximal forest. We extend the Hopf algebra operations to this larger class and show that they give rise to a linear combination of one-line PWs and new primitive elements.

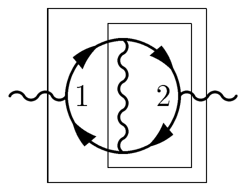
Let us consider the following graph borrowed from QED:



There is no possibility to draw non-intersecting boxes around all superficially divergent sectors. We can however try to draw non-intersecting boxes around a maximal number of superficial divergences, which will be possible in several ways. In our example we have two possibilities:



$$= ((v_1)p_2) \qquad \text{or}$$



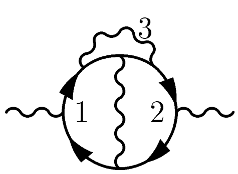
$$= ((v_2)p_1)\ . \tag{17}$$

We have resolved the overlapping divergence into (here two) maximal forests. A forest of a Feynman graph is a set of one-particle-irreducible (the graph remains connected after cutting an arbitrary line) divergent subgraphs which do not overlap. A maximal forest for a given Feynman graph is a forest which is not contained in any other forest of that graph.

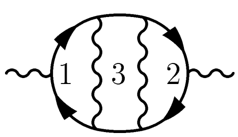
Our idea is now to bundle the parenthesized words of the n maximal forests of a given graph to an n-line PW. We write the single PWs of each maximal forest as different rows and connect by a tree of lines the closing parentheses of identical boxes occurring in different rows. The PW of our example is

$$\begin{array}{l} ((v_1)p_2) \\ ((v_2)p_1) \end{array}\Big] \ , \tag{18}$$

because the outermost parentheses of both rows represent the same large box in (17). Other examples are



$$\begin{array}{l} (((v_3)\,v_{13})p_2) \\ (((v_3)\,v_{23})p_1) \end{array} \tag{19}$$



$$\begin{array}{llll} (& (v_1)\,(v_2) & & p_3) \\ (((v_1) & & v_{13})p_2) & \\ ((& (v_2) & v_{23})p_1) & \end{array} \tag{20}$$

All we have to change now is the notion of the parenthesized subword (PSW) and of its removal. A PSW X_i of X is everything between a set of connected closing parentheses and its matching opening parentheses. Disconnected rows of X which are accidentally between connected rows are not part of the PSW X under consideration, and identical rows are condensed to one copy. Thus, apart from the total PW, the proper PSWs in the examples are

$$
\begin{array}{lll}
(18): & (v_1)\,,\ (v_2) & \\
(19): & (v_3)\,,\ ((v_3)v_{13})\,,\ ((v_3)v_{23})\,, & \qquad (21)\\
(20): & (v_1)\,,\ (v_2)\,,\ ((v_1)v_{13})\,,\ ((v_2)v_{23})\,. &
\end{array}
$$

The removal of a product $\prod_i X_i$ of PSWs of X from X is defined as follows: If $\prod_i X_i = X$ we define $X/X = e$. Otherwise we label the rows of X. We give to the X_i-rows the labels of the X-rows they are contained in. We delete from X and all X_i all but those rows whose labels occur in each of the chosen PSWs X_i. Let the results be X' and X'_i. If there remains no row at all or if $X'_i \cap X'_j \neq \emptyset$ for some pair $\{X'_i, X'_j\}$ then we put $X/\prod_i X_i = 0$. Otherwise $X/\prod X_i$ is given by removing all X'_i from X'.

With these modifications, the formulae (4), (6) and (13) define counit, coproduct and antipode of a Hopf algebra, and the properties (10), (11) and (16) remain unchanged, see [5]. The antipode reproduces now the combinatorics of the forest formula for any Feynman graph.

There is a way to return to the one-line PWs or rooted trees. It suffices to define a "primitivator" $\mathcal{P}$ which maps overlapping divergences to primitive elements. Let X be an iPW with proper PSWs $X_i \neq X$, $i = 1, \dots, n$, and $U \subset \{1, \dots, n\}$. Let us write the exterior parentheses of iPWs explicitly, i.e. (X) instead of X and (X_i) instead of X_i and $(\mathcal{P}[X/\prod_{i\in U} X_i])$ instead of $\mathcal{P}[X/\prod_{i\in U} X_i]$. With this convention we define

$$
\mathcal{P}[(X)] := (X) - \sum_U \Big(\prod_{i\in U} (X_i)\, \mathcal{P}\big[X/\prod_{i\in U} X_i\big]\Big)\ . \tag{22}
$$

We have proved in [5] that $\mathcal{P}[(X)]$ is primitive in the following sense:

$$
\Delta[\mathcal{P}[(X)]] = e \otimes \mathcal{P}[(X)] + \mathcal{P}[(X)] \otimes e\ . \tag{23}
$$

If (X) is primitive it contains no PSWs. Hence we have $U = \emptyset$ and $\mathcal{P}[(X)] = (X)$. For (X) and (Y) being primitive we compute $\mathcal{P}[((Y)X)] = ((Y)X) - ((Y)X) = 0$. By induction it is easy to show that $\mathcal{P}[Y] = 0$ for any non-primitive one-line iPW Y. On the other hand, $\mathcal{P}[(X)] \neq 0$ if (X) is an overlapping divergence, and we can replace X by the linear combination $\mathcal{P}[(X)] + \sum_U \Big(\prod_{i\in U} (X_i)\, \mathcal{P}\big[X/\prod_{i\in U} X_i\big]\Big)$. If (X) contains no overlapping subdivergences, all X_i are one-line PWs. Since the $\mathcal{P}[(X)]$ form additional primitive (i.e. one-line) elements of the Hopf algebra, we have written the multi-line overlapping divergence (X) as a linear combination of one-line PWs. In other words, our Hopf algebra of arbitrary Feynman graphs is isomorphic to a Hopf algebra of one-line PWs, and this is precisely Kreimer's original Hopf algebra [2]. The primitive elements of Kreimer's Hopf algebra are the graphically primitive elements and from each overlapping divergence a computational-primitive element. We have given an explicit construction of the latter. The same can be achieved, for instance, by Schwinger-Dyson techniques [2] or set theoretic considerations [6].

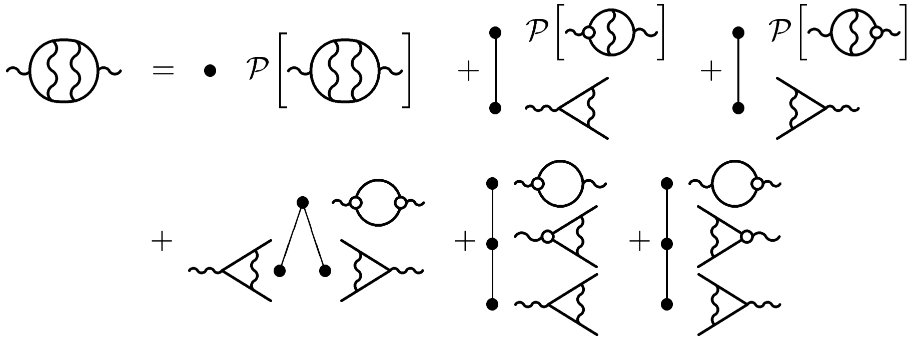
Let us evaluate the primitivators of our examples, thereby giving the decomposition into rooted trees. Using (21) we get

$$o_{1,2|2,1} := \mathcal{P}\left[\begin{matrix}((v_1)p_2)\\((v_2)p_1)\end{matrix}\right] = \begin{matrix}((v_1)p_2)\\((v_2)p_1)\end{matrix} - ((v_1)p_2) - ((v_2)p_1)\ ,$$

$$\mathcal{P}\left[\begin{matrix}(((v_3)v_{13})p_2)\\(((v_3)v_{23})p_1)\end{matrix}\right] = \begin{matrix}(((v_3)v_{13})p_2)\\(((v_3)v_{23})p_1)\end{matrix}$$
$$- ((v_3)o_{13,2|23,1}) - (((v_3)v_{13})p_2) - (((v_3)v_{23})p_1)\ ,$$

$$\mathcal{P}\left[\begin{matrix}(\ (v_1)(v_2)\ \ p_3)\\(((v_1)\ \ v_{13})p_2)\\((\ \ (v_2)v_{23})p_1)\end{matrix}\right] = \begin{matrix}(\ (v_1)(v_2)\ \ p_3)\\(((v_1)\ \ v_{13})p_2)\\((\ \ (v_2)v_{23})p_1)\end{matrix} - ((v_1)o_{2,3|13,2}) - ((v_2)o_{2,3|23,1})$$
$$- ((v_1)(v_2)p_3) - (((v_1)v_{13})p_2) - (((v_2)v_{23})p_1)\ .$$

The meaning of the index structure of $o_{i,j|k,l}$ is obvious from the first equation; we have to save this type of a primitive element for later use in the second and third equation. The third equation, for example, can be rewritten as the following decomposition of (20) into a sum of rooted trees:



5 Renormalization

We have already mentioned that the antipode recovers the combinatorics of the forest formula. In this section we will make this statement explicit. In fact the antipode reproduces precisely the forest formula if we allow for a deformation of our Hopf algebra by a renormalization map R. That map R should be considered as the projection onto the divergent part of the integral encoded in the Feynman graph. In terms of a regularization parameter ϵ, those integrals deliver a Laurent series, and R is any projection of the Laurent series which preserves the divergent part. Renormalization schemes differ in the way they handle the finite part in $\epsilon \to 0$. A special scheme is BPHZ renormalization where the projection is given by Taylor expansion of the integrals with respect to the external momenta.

We now enlarge our algebra $\mathcal{A}$ by copies $R[X]$ for each element $X \in \mathcal{A}$, subject to the convention $R[e] = e$. The Hopf algebra definitions are now modified as follows:

$$\Delta[X] := e \otimes X + \sum_U \Big\{ \prod_{i\in U} R[X_i] \otimes X / \prod_{i\in U} X_i \Big\}\ , \qquad (6_R)$$

$$\Delta[R[X]] = \Delta[X] \qquad \text{or} \qquad (6'_R)$$

$$\Delta[R[X]] = (\mathrm{id} \otimes R) \circ \Delta[X]\ , \qquad (6''_R)$$

$$S[X] = -X - m \circ (\mathrm{id} \otimes S) \circ P_2\Delta[X] \;, \qquad \forall \text{ iPW } X \in \mathcal{A} \;, \qquad (13_R)$$
$$S[R[X]] = -R[X - m \circ (S \otimes \mathrm{id}) \circ P_2\Delta[X]] \;, \qquad \forall \text{ iPW } X \in \mathcal{A} \;, \qquad (13'_R)$$

It turns out that a general R spoils several Hopf algebra axioms, somewhat depending on whether we prefer $(6'_R)$ or $(6''_R)$. If we choose $(6'_R)$ then counit, coassociativity and right antipode axiom are given up, for the choice $(6''_R)$ we keep coassociativity but violate the right counit and right antipode axioms.

Let us explore the left antipode axiom:

$$\begin{aligned} 0 \sim m \circ (S \otimes \mathrm{id}) \circ \Delta[X] &= m \circ (S \otimes \mathrm{id}) \circ (e \otimes X + R[X] \otimes e + P_2\Delta[X]) \\ &= (\mathrm{id} - R)[X + m \circ (S \otimes \mathrm{id}) \circ P_2\Delta[X]] \\ =: (\mathrm{id} - R)[\bar{X}] &= (\mathrm{id} - R)\Big[X + \sum_U \Big\{ \prod_{i \in U} \{-R[\bar{X}_i]\} \, \{X / \prod_{i \in U} X_i\} \Big\}\Big] \;, \\ R[\bar{X}_i] := -S[R[X_i]] &\equiv R[X_i + m \circ (S \otimes \mathrm{id}) \circ P_2\Delta[X_i]] \;. \end{aligned}$$

We see that we obtain a recursion formula for the determination of $\bar{X}$, and this is precisely Bogolyubov's recursion formula [10] of renormalization! That recursion formula has an explicit solution, the forest formula of Zimmermann [8]. In view of the general results of [6] it is not surprising that Feynman graphs can be regarded as elements of a Hopf algebra. It is however a deep message that the antipode of this (R-deformed) Hopf algebra implements renormalization. And the story continues with the discovery by Connes and Kreimer of a connection between the renormalization Hopf algebra and the diffeomorphism group of spacetime [3]. We can expect this subject to become one of the most promising activities in theoretical physics.

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