

Renormalizable noncommutative quantum field theory

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Abstract We discuss the Euclidean noncommutative ϕ_4^4 -quantum field theory as an example of a renormalizable field theory. Using a Ward identity, Disertori, Gurau, Magnen and Rivasseau were able to prove the vanishing of the beta function for the coupling constant to all orders in perturbation theory. We extend this work and obtain from the Schwinger–Dyson equation a non-linear integral equation for the renormalised two-point function alone. The non-trivial renormalised four-point function fulfils a linear integral equation with the inhomogeneity determined by the two-point function. These integral equations might be the starting point of a nonperturbative construction of a Euclidean quantum field theory on a noncommutative space. We expect to learn about renormalisation from this almost solvable model.

Keywords Noncommutative field theory · Ward identity · Schwinger–Dyson equation · Integral equations

1 Introduction

This report is based on our recent work [1].

In order to improve the problems of four-dimensional quantum field theory it was suggested to include “gravity effects” through deforming space–time. The canonical deformation is particularly simple, but the resulting models suffer from the

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UV/IR-mixing [2]. In our previous work [3] we found a way to handle this problem. We realized that the model defined by the action

$$S = \int d^4x \left(\frac{1}{2} \phi(-\Delta + \Omega^2 \tilde{x}^2 + \mu^2) \phi + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \right) (x) \quad (1)$$

is renormalisable to all orders of perturbation theory. Here, \star refers to the Moyal product parametrised by the antisymmetric 4×4 -matrix Θ , and $\tilde{x} = 2\Theta^{-1}x$. The model is covariant under the Langmann–Szabo duality transformation [4] and becomes self-dual at $\Omega = 1$. Certain variants have also been treated, see [5] for a review.

Evaluation of the β -functions for the coupling constants Ω , λ in first order of perturbation theory leads to a coupled dynamical system which indicates a fixed-point at $\Omega = 1$, while λ remains bounded [6, 7]. The vanishing of the β -function at $\Omega = 1$ was next proven in [8] at three-loop order and finally in [9] to all orders of perturbation theory. It implies that there is no infinite renormalisation of λ , and a non-perturbative construction seems possible [10]. The Landau ghost problem is solved.

The vanishing of the β -function to all orders has been obtained using a Ward identity [9]. We extend this work and derive an integral equation for the two-point function alone by using the Ward identity and Schwinger–Dyson equations. Usually, Schwinger–Dyson equations couple the two-point function to the four-point function. In our model, we show that the Ward identity allows to express the four-point function in terms of the two-point function, resulting in an equation for the two-point function alone. This is achieved in the first step for the bare two-point function. We are able to perform the mass and wave function renormalisation directly in the integral equation, giving a *self-consistent non-linear equation for the renormalised two-point function alone*.

Higher n -point functions fulfil a *linear* (inhomogeneous) Schwinger–Dyson equation, with the inhomogeneity given by m -point functions with $m < n$. This means that solving our equation for the two-point function leads to a full non-perturbative construction of this interacting quantum field theory in four dimensions.

So far we treated our equation perturbatively up to third order in λ . The solution shows an interesting number-theoretic structure.

We hope that a detailed analysis of our model will help for a non-perturbative treatment of more realistic Euclidean quantum field theories. We expect that we can learn much about non-perturbative renormalisation of Euclidean quantum field theories in four dimensions from this almost solvable model. In addition, the relation to a similar model over deformed Minkowski space–time might reveal interesting new perspectives. Such a model is dealt with by Andre Fischer and Richard Szabo in this volume.

2 Matrix model

It is convenient to write the action (1) in the matrix base of the Moyal space, see [3, 13]. It simplifies enormously at the self-duality point $\Omega = 1$. We write down the resulting action functionals for the *bare* quantities, which involves the bare mass μ_{bare} and the

wave function renormalisation $\phi \mapsto Z^{\frac{1}{2}}\phi$. For simplicity we fix the length scale to $\theta = 4$. This gives

$$S = \sum_{m,n \in \mathbb{N}_\Lambda^2} \frac{1}{2} \phi_{mn} H_{mn} \phi_{nm} + V(\phi), \tag{2}$$

$$H_{mn} = Z \left(\mu_{bare}^2 + |m| + |n| \right), \quad V(\phi) = \frac{Z^2 \lambda}{4} \sum_{m,n,k,l \in \mathbb{N}_\Lambda^2} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm}, \tag{3}$$

It is already used that this model has no renormalisation of the coupling constant [9]. All summation indices m, n, \dots belong to \mathbb{N}^2 , with $|m| := m_1 + m_2$. The symbol \mathbb{N}_Λ^2 refers to a cut-off in the matrix size. The scalar field is real, $\phi_{mn} = \overline{\phi_{nm}}$.

3 Ward identity

The key step in the proof [9] that the β -function vanishes is the discovery of a Ward identity induced by inner automorphisms $\phi \mapsto U\phi U^\dagger$. Inserting into the connected graphs the special insertion vertex

$$V_{ab}^{ins} := \sum_n (H_{an} - H_{nb}) \phi_{bn} \phi_{na} \tag{4}$$

is the same as the difference of graphs with external indices b and a , respectively, $Z(|a| - |b|)G_{[ab]...}^{ins} = G_{b...} - G_{a...}$:

We write Feynman graphs in the self-dual ϕ_4^4 -model as ribbon graphs on a genus- g Riemann surface with B external faces. Adding for each external face an external vertex to get a closed surface, the matrix index is constant at every face. Inserting the special vertex V_{ab}^{ins} leads, however, to an index jump from a to b in an external face which meets an external vertex. The corresponding external sources at the jumped face are thus J_{na} and J_{bm} for some other indices m, n . According to the Ward identity, this is the same as the difference between the graphs with face index b and a , respectively:

$$Z(|a| - |b|) G_{[ab]...}^{ins} = G_{b...} - G_{a...} \tag{5}$$

$$Z(|a| - |b|) G_{[ab]...}^{ins} = G_{b...} - G_{a...} \tag{6}$$

The dots in (6) stand for the remaining face indices. We have used $H_{an} - H_{nb} = Z(|a| - |b|)$.

4 Schwinger–Dyson equation

The Schwinger–Dyson equation for the one-particle irreducible two-point function Γ^{ab} reads

$$\Gamma_{ab} = \text{bubble} + \text{loop with insertion} + \text{bubble with internal line} \tag{7}$$

The sum of the last two graphs can be reexpressed in terms of the two-point function with insertion vertex,

$$\begin{aligned} \Gamma_{ab} &= Z^2 \lambda \sum_p \left(G_{ap} + G_{ab}^{-1} G_{[ap]b}^{ins} \right) = Z^2 \lambda \sum_p \left(G_{ap} - G_{ab}^{-1} \frac{G_{bp} - G_{ba}}{Z(|p| - |a|)} \right) \\ &= Z^2 \lambda \sum_p \left(\frac{1}{H_{ap} - \Gamma_{ap}} + \frac{1}{H_{bp} - \Gamma_{bp}} - \frac{1}{H_{bp} - \Gamma_{bp}} \frac{(\Gamma_{bp} - \Gamma_{ab})}{Z(|p| - |a|)} \right). \end{aligned} \tag{8}$$

This is a closed equation for the two-point function alone. It involves the divergent quantities Γ_{bp} and Z, μ_{bare} .

5 Renormalisation

Introducing the renormalised planar two-point function Γ_{ab}^{ren} by Taylor expansion $\Gamma_{ab} = Z\mu_{bare}^2 - \mu^2 + (Z-1)(|a|+|b|) + \Gamma_{ab}^{ren}$, with $\Gamma_{00}^{ren} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$, we obtain a coupled system of equations for Γ_{ab}^{ren}, Z and μ_{bare} . It leads to a closed equation for the renormalised function Γ_{ab}^{ren} alone, which is further analysed in the integral representation.

We replace the indices in $a, b, \dots \mathbb{N}$ by continuous variables in \mathbb{R}_+ . Equation (8) depends only on the length $|a| = a_1 + a_2$ of indices. The Taylor expansion respects this feature, so that we replace $\sum_{p \in \mathbb{N}_\Lambda^2}$ by $\int_0^\Lambda |p| dp$. After a convenient change of variables $|a| =: \mu^2 \frac{\alpha}{1-\alpha}, |p| =: \mu^2 \frac{\rho}{1-\rho}$ and

$$\Gamma_{ab}^{ren} =: \mu^2 \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)} \left(1 - \frac{1}{G_{\alpha\beta}} \right), \tag{9}$$

and using an identity resulting from the symmetry $G_{0\alpha} = G_{\alpha 0}$, we arrive at [1]:

Theorem 1 *The renormalised planar connected two-point function $G_{\alpha\beta}$ of the self-dual noncommutative ϕ_4^4 -theory satisfies the integral equation*

$$\begin{aligned}
 G_{\alpha\beta} = & 1 + \lambda \left(\frac{1 - \alpha}{1 - \alpha\beta} (\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}) + \frac{1 - \beta}{1 - \alpha\beta} (\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}) \right. \\
 & + \frac{1 - \beta}{1 - \alpha\beta} \left(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1 \right) (\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0}) \\
 & \left. - \frac{\alpha(1 - \beta)}{1 - \alpha\beta} (\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}) + \frac{(1 - \alpha)(1 - \beta)}{1 - \alpha\beta} (G_{\alpha\beta} - 1)\mathcal{Y} \right), \tag{10}
 \end{aligned}$$

where $\alpha, \beta \in [0, 1)$,

$$\mathcal{L}_\alpha := \int_0^1 d\rho \frac{G_{\alpha\rho} - G_{0\rho}}{1 - \rho}, \quad \mathcal{M}_\alpha := \int_0^1 d\rho \frac{\alpha G_{\alpha\rho}}{1 - \alpha\rho}, \quad \mathcal{N}_{\alpha\beta} := \int_0^1 d\rho \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho - \alpha},$$

and $\mathcal{Y} = \lim_{\alpha \rightarrow 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}$.

6 Perturbation expansion

These integral equations are the starting point for a perturbative solution. In this way, the renormalised correlation functions are directly obtained, without Feynman graph computation and further renormalisation steps. We obtain

$$\begin{aligned}
 G_{\alpha\beta} = & 1 + \lambda \{ A(I_\beta - \beta) + B(I_\alpha - \alpha) \} \\
 & + \lambda^2 \{ AB \left((I_\alpha - \alpha) + (I_\beta - \beta) + (I_\alpha - \alpha)(I_\beta - \beta) + \alpha\beta(\zeta(2) + 1) \right) \\
 & + A \left(\beta I_\beta - \beta I_\beta \right) - \alpha AB \left((I_\beta)^2 - 2\beta I_\beta + I_\beta \right) \\
 & + B \left(\alpha I_\alpha - \alpha I_\alpha \right) - \beta BA \left((I_\alpha)^2 - 2\alpha I_\alpha + I_\alpha \right) \} + \mathcal{O}(\lambda^3), \tag{11}
 \end{aligned}$$

where $A := \frac{1-\alpha}{1-\alpha\beta}$, $B := \frac{1-\beta}{1-\alpha\beta}$ and the following iterated integrals appear:

$$\begin{aligned}
 I_\alpha & := \int_0^1 dx \frac{\alpha}{1 - \alpha x} = -\ln(1 - \alpha), \\
 I_\alpha & := \int_0^1 dx \frac{\alpha I_x}{1 - \alpha x} = \text{Li}_2(\alpha) + \frac{1}{2} (\ln(1 - \alpha))^2.
 \end{aligned} \tag{12}$$

We conjecture that $G_{\alpha\beta}$ is at any order a polynomial with rational coefficients in α, β, A, B and iterated integrals labelled by rooted trees.

7 Four-point Schwinger–Dyson equation

The knowledge of the two-point function allows a successive construction of the whole theory. As an example we treat the planar connected four-point function G_{abcd} .

Following the a -face in direction of an arrow, there is a distinguished vertex at which the first ab -line starts. For this vertex there are two possibilities for the matrix index of the diagonally opposite corner to the a -face: either c or a summation vertex p :

$$G_{abcd} = Z^2 \lambda \dots + Z \lambda \dots \tag{13}$$

We write the first contribution as a product of the vertex $Z^2\lambda$, the left connected two-point function, the downward two-point function and an insertion, which is reexpressed by means of the Ward-identity. After amputation of the external two-point functions we obtain the Schwinger–Dyson equation for the *renormalised* 1PI four-point function $G_{abcd} = G_{ab}G_{bc}G_{cd}G_{da}\Gamma_{abcd}^{ren}$ as follows:

$$\Gamma_{abcd}^{ren} = Z\lambda \frac{1}{|a| - |c|} \left(\frac{1}{G_{ad}} - \frac{1}{G_{cd}} \right) + Z\lambda \sum_p \frac{1}{|a| - |p|} G_{pb} \left(\frac{G_{dp}}{G_{ad}} \Gamma_{pbcd}^{ren} - \Gamma_{abcd}^{ren} \right). \tag{14}$$

In terms of the 1PI function we have

$$\begin{aligned} Z^{-1}\Gamma_{abcd}^{ren} &= \lambda \left(1 - \frac{\Gamma_{ad}^{ren} - \Gamma_{cd}^{ren}}{|a| - |c|} \right) \\ &+ \lambda \sum_p \frac{|a| + |d| + \mu^2 - \Gamma_{ad}^{ren}}{|p| + |b| + \mu^2 - \Gamma_{pb}^{ren}} \frac{\frac{\Gamma_{pbcd}^{ren} - \Gamma_{abcd}^{ren}}{|p| - |a|}}{|p| + |d| + \mu^2 - \Gamma_{pd}^{ren}} \\ &+ \lambda \Gamma_{abcd}^{ren} \sum_p \frac{1 - \frac{\Gamma_{ad}^{ren} - \Gamma_{pd}^{ren}}{|a| - |p|}}{(|p| + |b| + \mu^2 - \Gamma_{pb}^{ren})(|p| + |d| + \mu^2 - \Gamma_{pd}^{ren})}. \end{aligned} \tag{15}$$

Passing to the integral representation and the variables α and β , we find for $\Gamma_{\alpha\beta\gamma\delta} := \Gamma_{abcd}^{ren}$ an integral equation, which manipulated appropriately allows again to take the limit $\xi \rightarrow 1$ after insertion of the expression for the wave function renormalisation constant.

Theorem 2 *The renormalised planar IPI four-point function $\Gamma_{\alpha\beta\gamma\delta}$ of self-dual non-commutative ϕ_4^4 -theory (with continuous indices $\alpha, \beta, \gamma, \delta \in [0, 1)$) satisfies the integral equation*

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda \cdot \frac{\left(1 - \frac{(1-\alpha)(1-\gamma\delta)(G_{\alpha\delta} - G_{\gamma\delta})}{G_{\gamma\delta}(1-\delta)(\alpha-\gamma)} + \int_0^1 \rho d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}G_{\delta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{\Gamma_{\rho\beta\gamma\delta} - \Gamma_{\alpha\beta\gamma\delta}}{\rho - \alpha} \right)}{G_{\alpha\delta} + \lambda \left((\mathcal{M}_\beta - \mathcal{L}_\beta - \mathcal{Y})G_{\alpha\delta} + \int_0^1 d\rho \frac{G_{\alpha\delta}G_{\beta\rho}(1-\beta)}{(1-\delta\rho)(1-\beta\rho)} + \int_0^1 \rho d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{(G_{\rho\delta} - G_{\alpha\delta})}{(\rho - \alpha)} \right)}. \tag{16}$$

In lowest order we find

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda - \lambda^2 \left(\frac{(1-\gamma)(I_\alpha - \alpha) - (1-\alpha)(I_\gamma - \gamma)}{\alpha - \gamma} + \frac{(1-\delta)(I_\beta - \beta) - (1-\beta)(I_\delta - \delta)}{\beta - \delta} \right) + \mathcal{O}(\lambda^3). \tag{17}$$

Note that $\Gamma_{\alpha\beta\gamma\delta}$ is cyclic in the four indices, and that $\Gamma_{0000} = \lambda + \mathcal{O}(\lambda^3)$.

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