International Journal of Modern Physics A
Vol. 27, No. 12 (2012) 1250067 (10 pages)

DOI: 10.1142/S0217751X12500674

RENORMALIZATION OF A NONCOMMUTATIVE FIELD THEORY*

HARALD GROSSE
Faculty of Physics, University of Vienna, Vienna, Austria
harald.grosse@univie.ac.at

RAIMAR WULKENHAAR
Mathematisches Institut der Westfälischen Wilhelms-Universität, Münster, Germany
raimar@math.uni-muenster.de

Received 23 March 2012
Accepted 28 March 2012
Published 5 May 2012

We discuss a special Euclidean ϕ^4_4-quantum field theory over quantized space–time as an example of a renormalizable field theory. Using a Ward identity, it was possible to prove the vanishing of the beta function for the coupling constant to all orders in perturbation theory. We extend this work and obtain from the Schwinger–Dyson equation a nonlinear integral equation for the renormalized two-point function alone. The nontrivial renormalized four-point function fulfills a linear integral equation with the inhomogeneity determined by the two-point function. These integral equations might be the starting point of a nonperturbative construction of a Euclidean quantum field theory on a noncommutative space. We expect to learn about renormalization from this almost solvable model.

Keywords: Noncommutative quantum fields; renormalization; Ward identity; Schwinger–Dyson equation.

PACS numbers: 11.10.Gh, 11.90.+t

1. Personal Remarks

My (H. G.) interactions with Julius Wess went through various periods: Being younger, I met Prof. Wess first at conferences in the early seventies and several times during my stay at CERN and admired the famous Austrian physicist. Next I was invited several times to Karlsruhe by Julius and learnt him knowing better.

*Based on a talk presented at a meeting devoted to the scientific and human legacy of Julius Wess, in Serbia, August 2011.

But the third period was the most enjoyable one: After he moved to Munich, he invited me quite often to visit him. His interests turned to field theories defined over noncommutative space–time, and therefore we met quite often.

He told several times to me: His group formulates models and I should try to select the better behaving ones, those being renormalizable. During the last years of Julius life, we came much closer. I was the first visiting him in the hospital after he got the serious heart attack. His first question to me was: "Harald, what is measured at LHC." In addition, he took a sheet of paper and discussed with me deformed gravity. All this I remember very lively. While walking around at various conferences, we came quite close.

> *We all lost a great humanitarian man and a great physicist. I lost an elder friend.*

2. Introduction

Our present fundamental physics rests on two pillars: Quantum Field Theory and General Relativity. One of the main questions in this area of physics concerns the question of matching these two concepts.

In addition we hope to improve quantum field theory models by adding gravity effects. Constructive methods led years ago to many beautiful ideas and results, but the main goal to construct a mathematical consistent model of a four-dimensional local quantum field theory has not been reached. Renormalized perturbation expansions allow one to get quantum corrections order by order in a coupling constant. The convergence of this expansion, for example as a Borel summable series, can be questioned.

In recent years, a modification of the space–time structure led to new models, which are nonlocal in a particular sense. But these models, in general suffer from an additional disease, which is called the Infrared–Ultraviolet mixing.[1] Additional infrared singularities show up. A possible way to cure this problem has been found by us in previous work.[2] It led to special models, which needed four (instead of three) relevant/marginal operators in the defining Lagrangian. We have been able to show that the resulting model is renormalizable up to all orders in perturbation theory. In addition a new fixed point appeared at a special value of the additional coupling constant. This way, we were able to tame the Landau ghost problem. Since the old problems of additional singularities due to partial summing up the perturbation expansion do not show up, we believe that the perturbation expansion will be Borel summable. That this new fixed point exists in perturbation theory to all orders has been shown in work by Rivasseau and collaborators.[3]

The main open question concerns the nonperturbative construction of a nontrivial noncommutative quantum field theory, with which we are dealing mainly here.

Classical field theories for fundamental interactions (electroweak, strong, gravitational) are of geometrical origin. We may remind, that the 4 Fermi interaction is nonrenormalizable, it needs a cutoff around 300 GeV, otherwise unitarity is violated! This is nicely resolved by adding new particles, the W^+, Z^0, W^-, and the confirmation of their existence was a great step towards consistence of quantum field theory models. On the other hand the quantum field theory for Standard Model (electroweak + strong) is renormalizable, while gravity is not! In the sense of renormalization theory one might argue, that space–time should not be a smooth manifold at tiny distances, gravity would be scaled away. Or stated otherwise, the weakness of gravity determines the Planck scale, and geometry at these tiny distances should be something different.

A promising approach concerns noncommutative geometry, which allows us to unify the Standard Model with gravity as a classical field theory.

In addition we hope that adding "Gravity" effects, or quantizing Space–Time, will improve field theory: This led to our program of merging general relativity ideas with quantum physics through noncommutative geometry.

Space–time structure

That one should limit localization in space–time follows from a very simple old argument due to Wheeler. Early ideas of modifying space–time were phrased already by Riemann, Schrödinger and Heisenberg, but Snyder in 1947 was the first to formulate a deformed space–time geometry. Such ideas became popular after 1986, when Alain Connes published his work on Noncommutative Geometry. One of us (H. G.) started in 1992 (in work with J. Madore) to use noncommutative manifolds (algebras) as a cutoff for quantized field theory models. A treatment of free fields and uncertainty relations on deformed Minkowski space–time was given by Doplicher, Fredenhagen and Roberts. Filk in 1995 was the first to elaborate on Feynman rules for models defined over deformed space–time, and finally they became popular due to the work by Schomerus, who observed, that such models may result from string theory after taking the zero slope limit.

Here we treat only the simplest deformed space–time: Starting from the algebra of smooth functions over D-dimensional Euclidean space, we define the $\star$-product as

$$(a \star b)(x) = \int d^D y \, d^D k a \left(x + \frac{1}{2} \Theta \cdot k \right) b(x+y) \, \mathrm{e}^{\mathrm{i}ky},$$

where $\Theta = -\Theta^T \in M_D(\mathbb{R})$.

Next question results: Can we make sense of renormalization in Noncommutative Geometric Models?

In local quantum field models we know, that the sign of the beta function determines the behavior of the RG flow. If the sign is positive, it indicates the appearance if the so-called Landau ghost, or phrased differently, triviality of this model results. A positive sign of β indicates asymptotic freedom. As for $D = 4$ dimensions we have to rely first on renormalized perturbation theory and follow the renormalization group flow.

A simple model

As a first step we intend to construct simple quantum field theory models on simple noncommutative geometries, e.g. the Moyal space. Of course, this way we obtain models with nonlocal interaction. The naïve application of this procedure to the ϕ^4-action (ϕ-real, Euclidean space) leads on Moyal plane to the action:

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{m^2}{2}\phi \star \phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi\right)(x)\,. \tag{1}$$

The Feynman rules can be obtained easily. Since we obtain only cyclic invariance at the Vertex, Graphs are best drawn as Ribbon Graphs on Riemann surfaces with a certain genus and a certain number of boundary components. We obtain planar regular contribution and nonplanar graphs. The planar graphs still reveal UV divergences, the nonplanar ones are finite for generic momenta. On the other hand for exceptional momenta (if sums of incoming or outgoing momenta vanish) the contributions develop an IR singularity, which spoils *Renormalizability*! In our previous work[2] we realized that the UV/IR-mixing problem can be solved by adding a fourth relevant/marginal operator to the Lagrangian.

Theorem 2.1. *The quantum field theory defined by the action*

$$S = \int d^4x \left(\frac{1}{2}\phi \star (-\Delta + \Omega^2\tilde{x}^2 + \mu^2)\phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi\right)(x) \tag{2}$$

(*with* $\tilde{x} = 2\Theta^{-1}\cdot x$, ϕ — *real, Euclidean metric*) *is perturbatively renormalizable to all orders in* λ.

The additional oscillator potential $\Omega^2\tilde{x}^2$ implements mixing between large and small distance scales and results from the renormalization proof. M. Buric and M. Wohlgenannt found an interesting interpretation of this additional term: It results as the coupling of the scalar field to the scalar curvature within the truncation procedure.

Here, $\star$ refers to the Moyal product parametrized by the antisymmetric 4×4-matrix Θ, and $\tilde{x} = 2\Theta^{-1}x$. The model is covariant under the Langmann–Szabo duality transformation[4] and becomes self-dual at $\Omega = 1$. Certain variants have also been treated, see Ref. 5 for a review. Evaluation of the β-functions for the coupling constants Ω, λ in first order of perturbation theory leads to a coupled dynamical system which indicates a fixed-point at $\Omega = 1$, while λ remains bounded.[6,7] The vanishing of the β-function at $\Omega = 1$ was next proven in Ref. 8 at three-loop order and finally in Ref. 3 to all orders of perturbation theory. It implies that there is no infinite renormalization of λ, and a nonperturbative construction seems possible.[9] The Landau ghost problem is solved. The vanishing of the β-function to all orders has been obtained using a Ward identity.[3] We extend this work and derive an integral equation for the two-point function alone by using the Ward identity and Schwinger–Dyson equations. Usually, Schwinger–Dyson equations couple the two-point function to the four-point function. In our model, we show that the

Ward identity allows us to express the four-point function in terms of the two-point function, resulting in an equation for the two-point function alone. This is achieved in the first step for the bare two-point function. We are able to perform the mass and wave function renormalization directly in the integral equation, giving a *self-consistent nonlinear equation for the renormalized two-point function alone.* Higher n-point functions fulfill a *linear* (inhomogeneous) Schwinger–Dyson equation, with the inhomogeneity given by m-point functions with $m < n$. This means that solving our equation for the two-point function leads to a full nonperturbative construction of this interacting quantum field theory in four dimensions. So far we treated our equation perturbatively up to third order in λ. The solution shows an interesting number-theoretic structure. We hope that a detailed analysis of our model will help for a nonperturbative treatment of more realistic Euclidean quantum field theories. We expect that we can learn much about nonperturbative renormalization of Euclidean quantum field theories in four dimensions from this almost solvable model.

3. Matrix Model

It is convenient to write the action (2) in the matrix base of the Moyal space, see Refs. 2, 5 and 10. It simplifies enormously at the self-duality point $\Omega = 1$. We write down the resulting action functionals for the *bare* quantities, which involves the bare mass μ_{bare} and the wave function renormalization $\phi \mapsto Z^{\frac{1}{2}}\phi$. For simplicity we fix the length scale to $\theta = 4$. This gives

$$S[\phi] = \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V[\phi]\,, \tag{3}$$

$$H_{mn} = Z(\mu^2_{\text{bare}} + |m| + |n|)\,, \quad V[\phi] = \frac{Z^2\lambda}{4} \sum_{m,n,k,l\in\mathbb{N}^2_\Lambda} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\,, \tag{4}$$

It is already used that this model has no renormalization of the coupling constant.[3] All summation indices $m, n, \ldots$ belong to $\mathbb{N}^2$, with $|m| := m_1 + m_2$. The symbol $\mathbb{N}^2_\Lambda$ refers to a cutoff in the matrix size. The scalar field is real, $\phi_{mn} = \overline{\phi_{nm}}$.

4. Ward Identity

The key step in the proof[3] that the β-function vanishes is the discovery of a Ward identity induced by inner automorphisms $\phi \mapsto U\phi U^\dagger$. Performing this transformation in the partition function $\mathcal{Z}[J] = \int \mathcal{D}[\phi] \exp(-S[\phi] + \text{tr}(\phi J))$ and expressing ϕ as functional derivative with respect to the source one obtains a *system of Ward identities*

$$0 = \sum_{n\in\mathbb{N}^2_\Lambda} \left((H_{pn} - H_{an}) \frac{\partial^2 \mathcal{Z}}{\partial J_{an}\partial J_{np}} + J_{na}\frac{\partial \mathcal{Z}}{\partial J_{np}} - J_{pn}\frac{\partial \mathcal{Z}}{\partial J_{an}} \right)\,. \tag{5}$$

We write Feynman graphs in the self-dual ϕ^4_4-model as ribbon graphs on a genus-g Riemann surface with B external faces. Adding for each external face an external vertex to get a closed surface, the matrix index is constant at every face. This means that the right index b of a source J_{ab} coincides with the left index of another source J_{bc}, or of the same source J_{bb}. Accordingly, there is a decomposition of the generating functional $\mathcal{W}[J] = \ln \mathcal{Z}[J]$ of connected functions into products of J-cycles $J_{p_1p_2}J_{p_1p_2}\cdots J_{p_{n-1}p_n}J_{p_np_1} =: J_{P^n}$, where $P^n = p_1\dots p_n$ stands for a collection of n indices. The decomposition according to the longest cycle reads

$$\mathcal{W}[J] = \mathcal{W}[0] + \sum_{N=1}^{\infty} \sum_{\substack{n_1,\dots,n_N=0\\ n_N\geq 1}}^{\infty} \left(\prod_{j=1}^{N} \frac{1}{n_j! j^{n_j}}\right) \times \sum_{P^j_{i_j}\in I^j} G_{|P^1_1|\dots|P^1_{n_1}|\dots|P^1_N|\dots|P^N_{n_N}|} \prod_{j=1}^{N}\prod_{i_j=1}^{n_j} J_{P^j_{i_j}} \,. \qquad (6)$$

One can then prove the following refinement of the Ward identity (5):

Theorem 4.1. *The partition function* $\mathcal{Z}[J]$ *for the action* (3) *satisfies*

$$\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} = \delta_{ap}(W^1_a[J] + W^2_a[J])\mathcal{Z} + \frac{1}{Z(|p|-|a|)} \sum_{n\in I}\left(J_{pn}\frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}[J]}{\partial J_{np}}\right) , \qquad (7)$$

$$W^2_a[J] := \sum_{N=1}^{\infty} \sum_{\substack{n_1,\dots,n_N=0\\ n_N\geq 1}}^{\infty} \left(\prod_{j=1}^{N} \frac{1}{n_j! j^{n_j}}\right) \sum_{P^j_{i_j}\in I^j} \left(\prod_{j=1}^{N}\prod_{i_j=1}^{n_j} J_{P^j_{i_j}}\right) \times \Bigg(G_{|a|a|P^1_1|\dots|P^N_{n_N}|} + \sum_{n\in I} G_{|P^1_1|\dots|P^1_{n_1}|an|P^2_1|\dots|P^N_{n_N}|} + \sum_{k=3}^{N} \times \sum_{n,q_1,\dots,q_{k-3}\in I} G_{|P^1_1|\dots|P^{k-1}_{n_{k-1}}|nanq_1\dots q_{k-3}|P^k_1|\dots|P^N_{n_N}|} J_{nq_1}J_{q_1q_2}\dots J_{q_{k-3}n}\Bigg),$$

$$W^1_a[J] := \sum_{N,M=1}^{\infty} \sum_{\substack{n_1,\dots,n_N,m_1,\dots,m_M=0\\ n_N,m_M\geq 1}}^{\infty} \left(\prod_{j=1}^{N} \frac{1}{n_j! j^{n_j}}\right) \left(\prod_{k=1}^{m} \frac{1}{m_k! k^{m_k}}\right) \sum_{P^j_{i_j}\in I^j} \sum_{Q^k_{l_k}\in I^k} \times \left(\prod_{j=1}^{N}\prod_{i_j=1}^{n_j} J_{P^j_{i_j}}\right) \left(\prod_{k=1}^{N}\prod_{l_k=1}^{m_k} J_{Q^k_{l_k}}\right) G_{|a|P^1_1|\dots|P^N_{n_N}|} G_{|a|Q^1_1|\dots|Q^M_{m_M}|} \Bigg\} .$$

Since $(H_{pn} - H_{an}) = Z(|p| - |a|)$ is independent of n, multiplication of (7) by $Z(|p|-|a|)$ gives the previous identity (5). The functions W^1_a and W^2_a are identified by careful discussion of the derivations of $W[J]$ as given in (6) with respect to J_{an} and J_{np}.

5. Schwinger–Dyson Equation

Schwinger–Dyson equations arise from functional integration of the two-point function in the partition function to

$$\mathcal{Z}[J] = e^{-V[\frac{\partial}{\partial J}]} e^{\frac{1}{2}\langle J,J\rangle_H}\,, \quad \langle J, J\rangle_H := \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{J_{mn}J_{nm}}{H_{mn}}\,. \tag{8}$$

The functional derivatives $\phi_{pq} = \frac{\partial}{\partial J_{qp}}$ applied to $\mathcal{Z}$ in order to produce the connected functions G in the expansion (6) combine with $V[\frac{\partial}{\partial J}]$ to certain identities called Schwinger–Dyson equations. It turns out that for each function G the Ward identity of Theorem 4.1 can be used at an intermediate step to generate new relations. As a result, we obtain for the regular ($B = 1$) two-point function $G_{|ab|}$, the irregular ($B = 2$) two-point function $G_{|a|b|}$ and the regular ($B = 1$) four-point function $G_{|abcd|}$ the identities (here B denotes the number of boundary components of the Riemann surface on which the graphs are drawn)

$$\begin{aligned} G_{|ab|} = \frac{1}{H_{ab}} + \frac{(-Z^2\lambda)}{H_{ab}}\Bigg\{&\Bigg(G_{|ab|}\Bigg(G_{|a|a|} + \sum_{n\in\mathbb{N}^2_\Lambda} G_{|an|}\Bigg) \\ &+ G_{|a|a|ab|} + \sum_{n\in I} G_{|an|ab|} + G_{|aaab|} + G_{|baba|}\Bigg) \\ &+ \sum_{p\in\mathbb{N}^2_\Lambda} \frac{G_{|ab|} - G_{|pb|}}{Z(|p|-|a|)} + \frac{G_{|a|b|} - G_{|b|b|}}{Z(|b|-|a|)}\Bigg\}, \end{aligned} \tag{9}$$

$$\begin{aligned} G_{|a|b|} = \frac{(-Z^2\lambda)}{H_{aa}}\Bigg\{&\Bigg(G_{|a|b|}\Bigg(G_{|a|a|} + \sum_{n\in\mathbb{N}^2_\Lambda} G_{|an|}\Bigg) + G_{|a|a|a|b|} \\ &+ \sum_{n\in\mathbb{N}^2_\Lambda} G_{|a|b|an|} + G_{|b|aaa|} + G_{|a|bab|} + 2G_{|a|a|}G_{|a|b|}\Bigg) \\ &+ \sum_{p\in\mathbb{N}^2_\Lambda} \frac{G_{|a|b|} - G_{|p|b|}}{Z(|p|-|a|)} + \frac{G_{|ab|} - G_{|bb|}}{Z(|b|-|a|)}\Bigg\}, \end{aligned} \tag{10}$$

$$\begin{aligned} G_{|abcd|} = \frac{(-Z^2\lambda)}{H_{ab}}\Bigg\{&\Bigg(G_{|abcd|}\Bigg(G_{|a|a|} + \sum_{n\in\mathbb{N}^2_\Lambda} G_{|an|}\Bigg) + G_{|a|a|abcd|} \\ &+ \sum_{n\in\mathbb{N}^2_\Lambda} G_{|an|abcd|} + G_{|aaabcd|} + G_{|babcda|} + G_{|cacdab|} + G_{|dadabc|}\Bigg) \end{aligned}$$

$$+ \sum_{p\in\mathbb{N}^2_\Lambda} \frac{G_{|abcd|}-G_{|pbcd|}}{Z(|p|-|a|)} + \frac{G_{|b|cda|}-G_{|b|cdb|}}{Z(|b|-|a|)} + \frac{G_{|bc|da|}-G_{|bc|dc|}}{Z(|c|-|a|)}$$

$$+ \frac{G_{|a|bcd|}-G_{|d|bcd|}}{Z(|d|-|a|)} + G_{bc}\frac{G_{|da|}-G_{|dc|}}{Z(|c|-|a|)}\Bigg\}. \qquad (11)$$

These connected functions involve all topologies. Decomposing them according to the genus, $G_{|P^1_1|\dots|P^N_{n_N}|} = \sum_{g=0}^\infty G^g_{|P^1_1|\dots|P^N_{n_N}|}$, we find for (9) after detailed discussion similar to Ref. 11:

Theorem 5.1. *The (unrenormalized) planar regular connected two-point function $G^0_{|ab|}$ satisfies the* <u>*closed equation*</u>*:*

$$G^0_{|ab|} = \frac{1}{Z(|a|+|b|+\mu^2_{\text{bare}})} - \frac{Z\lambda}{|a|+|b|+\mu^2_{\text{bare}}} G^0_{|ab|} \sum_{p\in\mathbb{N}^2_\Lambda} G^0_{|ap|}$$

$$+ \frac{\lambda}{|a|+|b|+\mu^2_{\text{bare}}} \sum_{p\in\mathbb{N}^2_\Lambda} \frac{G^0_{|pb|}-G^0_{|ab|}}{|p|-|a|}. \qquad (12)$$

6. Renormalization

Introducing the renormalized planar two-point function Γ^{ren}_{ab} by Taylor expansion $\Gamma^0_{|ab|} = Z\mu^2_{\text{bare}} - \mu^2 + (Z-1)(|a|+|b|) + \Gamma^{\text{ren}}_{ab}$, with $\Gamma^{\text{ren}}_{00} = 0$ and $(\partial\Gamma^{\text{ren}})_{00} = 0$, we obtain from (12) a coupled system of equations for Γ^{ren}_{ab}, Z and μ_{bare}. It leads to a closed equation for the renormalized function Γ^{ren}_{ab} alone, which is further analyzed in the integral representation. We replace the indices in $a, b, \dots \mathbb{N}$ by continuous variables in $\mathbb{R}_+$. Equation (12) depends only on the length $|a| = a_1 + a_2$ of indices. The Taylor expansion respects this feature, so that we replace $\sum_{p\in\mathbb{N}^2_\Lambda}$ by $\int_0^\Lambda |p|\,dp$. After a convenient change of variables $|a| =: \mu^2\frac{\alpha}{1-\alpha}$, $|p| =: \mu^2\frac{\rho}{1-\rho}$ and

$$\Gamma^{\text{ren}}_{ab} =: \mu^2 \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)}\left(1-\frac{1}{G_{\alpha\beta}}\right), \qquad (13)$$

and using an identity resulting from the symmetry $G_{0\alpha} = G_{\alpha 0}$, we arrive at:[12]

Theorem 6.1. *The renormalized planar connected two-point function $G_{\alpha\beta}$ of the self-dual noncommutative ϕ^4_4-theory satisfies, and is determined by, the integral equation*

$$G_{\alpha\beta} = 1 - \lambda\Bigg(\frac{1-\alpha}{1-\alpha\beta}(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}) + \frac{1-\beta}{1-\alpha\beta}(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y})$$

$$+ \frac{1-\beta}{1-\alpha\beta}\left(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1\right)(\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0})$$

$$- \frac{\alpha(1-\beta)}{1-\alpha\beta}(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}) + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta}-1)\mathcal{Y}\Bigg), \qquad (14)$$

where $\alpha,\ \beta \in [0,1)$,

$$\mathcal{L}_\alpha := \int_0^1 d\rho \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho}\,, \quad \mathcal{M}_\alpha := \int_0^1 d\rho \frac{\alpha G_{\alpha\rho}}{1-\alpha\rho}\,, \quad \mathcal{N}_{\alpha\beta} := \int_0^1 d\rho \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho-\alpha}\,,$$

and $\mathcal{Y} = \lim_{\alpha\to 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}$.

7. Perturbation Expansion

These integral equations are the starting point for a perturbative solution. In this way, the renormalized correlation functions are directly obtained, without Feynman graph computation and further renormalization steps. We obtain

$$\begin{aligned} G_{\alpha\beta} = 1 &+ \lambda\{A(I_\beta - \beta) + B(I_\alpha - \alpha)\} \\ &+ \lambda^2\{AB((I_{\substack{\alpha\\ \bullet}} - \alpha) + (I_{\substack{\beta\\ \bullet}} - \beta) + (I_\alpha - \alpha)(I_\beta - \beta) + \alpha\beta(\zeta(2)+1)) \\ &+ A(\beta I_{\substack{\beta\\ \bullet}} - \beta I_\beta) - \alpha AB((I_\beta)^2 - 2\beta I_\beta + I_\beta) \\ &+ B(\alpha I_{\substack{\alpha\\ \bullet}} - \alpha I_\alpha) - \beta BA((I_\alpha)^2 - 2\alpha I_\alpha + I_\alpha)\} + \mathcal{O}(\lambda^3)\,, \end{aligned} \tag{15}$$

where $A := \frac{1-\alpha}{1-\alpha\beta}$, $B := \frac{1-\beta}{1-\alpha\beta}$ and the following iterated integrals appear:

$$\begin{aligned} I_\alpha &:= \int_0^1 dx \frac{\alpha}{1-\alpha x} = -\ln(1-\alpha)\,, \\ I_{\substack{\alpha\\ \bullet}} &:= \int_0^1 dx \frac{\alpha I_x}{1-\alpha x} = \mathrm{Li}_2(\alpha) + \frac{1}{2}(\ln(1-\alpha))^2\,. \end{aligned} \tag{16}$$

We conjecture that $G_{\alpha\beta}$ is at any order a polynomial with rational coefficients in α, β, A, B and iterated integrals labeled by rooted trees.

8. Four-Point Schwinger–Dyson Equation

The knowledge of the two-point function allows a successive construction of the whole theory. As an example we treat the connected planar regular four-point function $G^0_{|abcd|}$ obtained as the genus-zero part of (11). It is convenient to express $\sum_p G^0_{|ap|}$ in that equation by (12). After amputation of the external two-point functions we obtain the Schwinger–Dyson equation for the *renormalized* 1PI four-point function definded by $G^0_{|abcd|} =: -G^0_{|ab|}G^0_{|bc|}G^0_{|cd|}G^0_{|da|}\Gamma^{\mathrm{ren}}_{abcd}$ as follows:

$$\Gamma^{\mathrm{ren}}_{abcd} = \frac{Z\lambda}{|a|-|c|}\left(\frac{1}{G^0_{|ad|}} - \frac{1}{G^0_{|cd|}}\right) + Z\lambda \sum_{p\in\mathbb{N}^2_\Lambda} \frac{1}{|a|-|p|} G^0_{|pb|}\left(\frac{G^0_{|dp|}}{G^0_{|ad|}}\Gamma^{\mathrm{ren}}_{pbcd} - \Gamma^{\mathrm{ren}}_{abcd}\right). \tag{17}$$

Inserting $(G^0_{|ab|})^{-1} = |a| + |b| + \mu^2 - \Gamma^{\rm ren}_{ab}$ and eliminating Z by the previously used equation resulting from the derivative of (12) at $a = b = 0$ we obtain in the integral representation for $\Gamma_{\alpha\beta\gamma\delta} := \Gamma^{\rm ren}_{abcd}$:

Theorem 8.1. *The renormalized planar regular 1PI four-point function $\Gamma_{\alpha\beta\gamma\delta}$ of self-dual noncommutative ϕ^4_4-theory (with continuous indices α, β, γ, $\delta \in [0,1)$) satisfies the integral equation*

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda \cdot \frac{\Big(1 - \frac{(1-\alpha)(1-\gamma\delta)(G_{\alpha\delta}-G_{\gamma\delta})}{G_{\gamma\delta}(1-\delta)(\alpha-\gamma)} + \int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}G_{\delta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{\Gamma_{\rho\beta\gamma\delta}-\Gamma_{\alpha\beta\gamma\delta}}{\rho-\alpha}\Big)}{G_{\alpha\delta} + \lambda\Big((\mathcal{M}_\beta - \mathcal{L}_\beta - \mathcal{Y})G_{\alpha\delta} + \int_0^1 d\rho \frac{G_{\alpha\delta}G_{\beta\rho}(1-\beta)}{(1-\delta\rho)(1-\beta\rho)} + \int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{(G_{\rho\delta}-G_{\alpha\delta})}{(\rho-\alpha)}\Big)} . \tag{18}$$

In lowest order we find

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda - \lambda^2\bigg(\frac{(1-\gamma)(I_\alpha - \alpha) - (1-\alpha)(I_\gamma - \gamma)}{\alpha - \gamma} + \frac{(1-\delta)(I_\beta - \beta) - (1-\beta)(I_\delta - \delta)}{\beta - \delta}\bigg) + \mathcal{O}(\lambda^3) . \tag{19}$$

Note that $\Gamma_{\alpha\beta\gamma\delta}$ is cyclic in the four indices, and that $\Gamma_{0000} = \lambda + \mathcal{O}(\lambda^3)$.

Acknowledgments

I (H. G.) would like to thank Goran Djordjevic for the kind invitation to the JW2011 Workshop and for the invitation to contribute to the Proceedings.

References

1. S. Minwalla, M. Van Raamsdonk and N. Seiberg, *J. High Energy Phys.* **0002**, 020 (2000).
2. H. Grosse and R. Wulkenhaar, *Commun. Math. Phys.* **256**, 305 (2005).
3. M. Disertori, R. Gurau, J. Magnen and V. Rivasseau, *Phys. Lett. B* **649**, 95 (2007).
4. E. Langmann and R. J. Szabo, *Phys. Lett. B* **533**, 168 (2002).
5. V. Rivasseau, Non-commutative renormalization, in *Quantum Spaces* (*Séminaire Poincaré X*), eds. B. Duplantier and V. Rivasseau (Birkhäuser Verlag, Basel, 2007), p. 19.
6. H. Grosse and R. Wulkenhaar, *Eur. Phys. J. C* **35**, 277 (2004).
7. H. Grosse and R. Wulkenhaar, *Lett. Math. Phys.* **71**, 13 (2005).
8. M. Disertori and V. Rivasseau, *Eur. Phys. J. C* **50**, 661 (2007).
9. V. Rivasseau, *J. High Energy Phys.* **0709**, 008 (2007).
10. H. Grosse and R. Wulkenhaar, *J. High Energy Phys.* **0312**, 019 (2003).
11. H. Grosse and R. Wulkenhaar, *Commun. Math. Phys.* **254**, 91 (2005).
12. H. Grosse and R. Wulkenhaar, Progress in solving a noncommutative quantum field theory in four dimensions, arXiv:0909.1389.