Towards a construction of a quantum field theory in four dimensions

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Abstract

We summarise our recent construction of the $\lambda\phi^4_4$-model on four-dimensional Moyal space. In the limit of infinite noncommutativity, this model is exactly solvable in terms of the solution of a non-linear integral equation. Surprisingly, this limit describes Schwinger functions of a Euclidean quantum field theory on standard $\mathbb{R}^4$ which satisfy the easy Osterwalder-Schrader axioms boundedness, invariance and symmetry. We prove that the decisive reflection positivity axiom is, for the 2-point function, equivalent to the question whether or not the solution of the integral equation is a Stieltjes function. A numerical investigation leaves no doubt that this is true for coupling constants $\lambda_c < \lambda \leq 0$ with $\lambda_c \approx -0.39$.

1 Introduction

1.1 Rigorous quantum field theories

Perturbatively renormalised quantum field theory is an enormous phenomenological success, a success which lacks a mathematical understanding. The perturbation series is at best an asymptotic expansion which cannot converge at physical coupling constants. Some physical effects such as confinement are out of reach for perturbation theory. In two, and partly three dimensions, methods of constructive physics [GJ87, Riv91], often combined with the Euclidean approach [Sch59, OS73, OS75], were used to rigorously establish existence and certain properties of quantum field theory models.

In four dimensions there has been little success so far. It is generally believed that due to asymptotic freedom [GW73, Pol73], non-Abelian gauge theory (i.e. Yang-Mills theory) has the chance of a rigorous construction. But this is a hard problem [JW00]. What makes it so difficult is the fact that any simpler model

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such as quantum electrodynamics or the $\lambda\phi^4$-model cannot be constructed in four dimensions (Landau ghost problem [LAK54] or triviality [Aiz81, Frö82]).

One of the main difficulties is the *non-linearity* of the models under consideration. Fixed point methods provide a standard approach to non-linear problems, but they are rarely used in quantum field theory. In this contribution we review a sequence of papers [GW14a, GW13b, GW14b, GW14c] in which we successfully used symmetry and fixed point methods to exactly solve a toy model for a quantum field theory in four dimensions.

1.2 Regularisation and renormalisation

We follow the Euclidean approach, starting from a partition function with source term $\mathcal{Z}[J]$, which itself involves the action functional of the model. It is absolutely crucial that, as in any rigorous construction, this action functional cannot be taken as the naïve action of the underlying classical field theory. Instead, we have to *regularise* the action in a twofold manner:

1. We have to place the model into finite volume V. As known from statistical physics, the logarithm of the partition function is, in leading approximation, proportional to V. Conversely, $\mathcal{Z}[J] \propto C^V$, which shows the necessity of the finite volume (infrared) regularisation.
2. The laws of quantum physics imply that confining a system to a small length scale $\delta x = \frac{1}{\Lambda}$ leads to large momentum fluctuations $\delta p \propto \frac{1}{\delta x}$. In order to have, at least temporarily, a finite energy density we must discard all information which is sharper localised than an ultraviolet cutoff $\frac{1}{\Lambda}$.

It is perfectly possible that Nature itself provides a finite global volume $V < \infty$ and a fundamental minimal length scale $\frac{1}{\Lambda} > 0$. But there are two important (and related) reasons to take the limits $V \to \infty$ and $\frac{1}{\Lambda} \to 0$: symmetry and simplicity. Restricting the Euclidean space $\mathbb{R}^D$ e.g. to a finite lattice of volume V and spacing $a = \frac{1}{\Lambda}$, the Euclidean group $\mathbb{R}^D \rtimes O(D)$, a continuous Lie group, is restricted to a finite crystallographic subgroup. By Noether's theorem, symmetries imply conservation laws, i.e. first integrals of motion, so that a model with more symmetry is simpler to describe. This is already visible in simple spin systems. Although defined on a discrete lattice, one typically has to go to a critical point of divergent correlation length, which simulates the continuum, to compute physical quantities.

But how to deal with the limits $V \to \infty$ and $\frac{1}{\Lambda} \to 0$ which are forbidden by the above remarks?

1. For $V \to \infty$ it suffices to study the free energy density $\frac{1}{V}\log(\mathcal{Z}[J])$ and functions derived from that. Their limits $V \to \infty$ exist and, as discussed before, are simpler to characterise than in finite volume.
2. The limit $\frac{1}{\Lambda} \to 0$ is achieved by the renormalisation philosophy. There is a family of maps $R(\Lambda', \Lambda)$, forming the *renormalisation group* [WK74], with

the following property: the physics computed from

- a partition function resulting from action $S(\Lambda)$ taken at cut-off Λ,
- and from another partition function resulting from action $S'(\Lambda') = R(\Lambda',\Lambda)S(\Lambda)$ taken at cut-off Λ',

coincides. This 'physics' can again be interpreted as an effective action $S_{phys}(\Lambda_{phys})$ at some scale Λ_{phys} (which in models with mass can be put to zero). We thus have $S_{phys}(\Lambda_{phys}) = R(\Lambda_{phys},\Lambda)S(\Lambda) = R(\Lambda_{phys},\Lambda')S(\Lambda')$ and consequently the semigroup law $R(\Lambda_{phys},\Lambda')R(\Lambda',\Lambda) = R(\Lambda_{phys},\Lambda)$.

As noted above, given $S(\Lambda)$, the effective action $S_{phys}(\Lambda_{phys})$ diverges for $\Lambda \to \infty$. In the class of *renormalisable theories*, these divergences result from a finite-dimensional subspace $\Sigma(\Lambda)$ of actions, termed relevant or marginal interactions. Identifying a corresponding subspace $S^{\Sigma}_{phys}(\Lambda_{phys})$ such that $R_{\Sigma}(\Lambda_{phys},\Lambda) := R(\Lambda_{phys},\Lambda)\big|_{\Sigma} : \Sigma(\Lambda) \to S^{\Sigma}_{phys}(\Lambda_{phys})$ is injective, one can re-normalise the model as follows: Take any point $\mathcal{C} \in S^{\Sigma}_{phys}(\Lambda_{phys})$ and consider

$$S^{\mathcal{C}}_{phys}(\Lambda_{phys}) := R(\Lambda_{phys},\Lambda) \circ R^{-1}_{\Sigma}(\Lambda_{phys},\Lambda)(\mathcal{C}) \;. \tag{1}$$

If properly done, the limit $\lim_{\Lambda\to\infty} S^{\mathcal{C}}_{phys}(\Lambda_{phys})$ exists. But note that the inversion $R_{\Sigma}(\Lambda_{phys},\Lambda) \mapsto R^{-1}_{\Sigma}(\Lambda_{phys},\Lambda)$ is a difficult problem. Except for a few superrenormalisable models in low dimensions, this can only be handled in perturbation theory.

Of course it remains to prove that these sophisticated limits $V \to \infty$ and $\frac{1}{\Lambda} \to 0$ restore the original symmetry.

In summary, it is important *that* one regularises, but there is freedom *how* to do it. As symmetry is so important, one will use the freedom to preserve the symmetry as much as possible. It is at this point where our approach sets in. We had the luck to identify a regularisation which in a certain sense has *more symmetry* than the original problem. This symmetry turned out to be so strong that it completely solves our Euclidean quantum field theory model in four dimensions [GW14a]. The limits $V \to \infty$ and $\frac{1}{\Lambda} \to 0$ exist, and to our great surprise they restore the *Euclidean symmetry group* [GW13b], i.e. the Osterwalder-Schrader [OS75] axiom *(OS1)*. In addition the other easy axioms *(OS3) permutation symmetry* (for trivial reasons) and *(OS0) factorial growth* (after some work) are realised. We thus turned towards the most decisive axiom, *(OS2) reflection positivity*, and so far everything looks promising [GW14c]. We give more details in the final sections of this contribution.

1.3 The Moyal space as symmetry-enhancing regulator

For concreteness, let us look at the $\lambda\phi^4_4$-model defined by the action functional

$$S[\phi] = \int_{\mathbb{R}^4} dx\,\Big\{\frac{1}{2}\phi(x)\cdot(-\Delta+\mu^2)(\phi(x)) + \frac{\lambda}{4}\big(\phi(x)\big)^4\Big\}\;, \tag{2}$$

where $\Delta = \delta^{\mu\nu}\partial_\mu\partial_\nu$ is the (negatively definite) 4D Laplacian. As described in sec. 1.2, in order to give rise to a meaningful partition function, we have to put it into finite volume $V < \infty$ and restrict its energy density via $\Lambda < \infty$. Finite volume is actually a compactification of the underlying geometry. A more sophisticated method than putting the model into a box is to add in (2) a harmonic oscillator potential $-\Delta \mapsto H := -\Delta + \omega^2\|x\|^2$. The result is the same, the resolvent $(H + \mathrm{i})^{-1}$ is a compact operator (with discrete spectrum), and the D-dimensional spectral volume is proportional to $\omega^{-\frac{D}{2}}$.

A well-known restriction of the energy density consists in introducing a lattice of spacing $a = \frac{1}{\Lambda}$. This is the heart of the lattice gauge theory approach [Wil74] to quantum chromodynamics. As action one takes, for example, Wilson's plaquette action, which is *non-local* but converges for $a \to 0$ to the local Yang-Mills action. This key example tells us that non-local actions are a decisive part of regularisation.

A particularly convenient type of non-locality for (rapidly decaying) functions on $\mathbb{R}^D$, with D even, is given by the Moyal product

$$(f \star g)(x) = \int_{\mathbb{R}^D \times \mathbb{R}^D} \frac{dy\, dk}{(2\pi)^D}\, f(x+\tfrac{1}{2}\Theta k)\, g(x+y)\, e^{\mathrm{i}\langle k,y\rangle}\;, \quad \Theta = -\Theta^t \in M_D(\mathbb{R})\,. \tag{3}$$

It is an associative but noncommutative product which is thoroughly investigated [GV88]. Together with the standard $\mathbb{R}^D$ Dirac operator it gives rise to a spectral noncommutative geometry [GGISV03].

Some 15 years ago the Moyal space $(\mathbb{R}^4, \star)$ became popular in quantum field theory. The motivation first came from gravitational considerations [DFR95] and from string theory [SW99], later from the UV/IR-mixing phenomenon [MVS00] observed in these models. Common to these early works is the point of view that the non-locality due to the Moyal product is a true physical effect. We also shared this view throughout a decade. During the last years we noticed that *a degradation of the Moyal product to a temporary regularisation, to be removed in the end, has much larger potential.*

This regularisation consists in replacing in (2) the pointwise product by the Moyal product, $\big(\phi(x)\big)^4 \mapsto (\phi \star \phi \star \phi \star \phi)(x)$. Optionally also the bilinear terms could be replaced, but due to $\int dx\; (f \star g)(x) = \int dx\; f(x)\, g(x)$ this leads to the same result as without $\star$.

The explicit occurrence of a skew-symmetric matrix $\Theta = -\Theta^t \in M_4(\mathbb{R})$ in the action introduces two preferred orthogonal planes in $\mathbb{R}^4$. They distinguish a reference frame in which Θ has block-diagonal form $\Theta = \mathrm{diag}\Big(\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}\Big)$. The original $SO(4)$ symmetry group is thus restricted to the stabiliser subgroup $SO(2) \times SO(2) \times \mathbb{Z}_2$ (there would be no $\mathbb{Z}_2$ if Θ had different complex eigenvalues $\pm\mathrm{i}\theta_1 \neq \pm\mathrm{i}\theta_2$). With these conventions, the redefinition $\omega =: \frac{2\Omega}{\theta}$ of the frequency and with additional multiplicative prefactors needed for renormalisation, the action (2) is turned into (note that $\|\Theta^{-1}x\|^2 = \theta^{-2}\|x\|^2$)

$$S[\phi] = \frac{1}{64\pi^2}\int d^4x \Big(\frac{Z}{2}\phi\star\big(-\Delta+\Omega^2_{bare}\|2\Theta^{-1}x\|^2 + \mu^2_{bare}\big)\phi + \frac{\lambda_{bare}Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)\ . \tag{4}$$

The action (4) is not yet properly regularised because so far there is no Λ. The most convenient regularisation is a matrix cut-off. The two-dimensional Moyal space $(\mathbb{R}^2,\star)$ possesses distinguished (complex-valued) functions [GV88] which under the identification $(x_1,x_2) \equiv z = x_1 + \mathrm{i}x_2$ read

$$f_{mn}(z) = 2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\frac{2}{\theta}}z\Big)^{n-m} L_m^{n-m}\Big(\frac{2|z|^2}{\theta}\Big)e^{-\frac{|z|^2}{\theta}}\ , \qquad m,n\in\mathbb{N}\ , \tag{5}$$

where $L^a_n(r)$ are the associated Laguerre polynomials. These functions satisfy $(f_{mn}\star f_{kl})(z) = \delta_{nk}f_{ml}(z)$ and $\int dz\ f_{mn}(z) = 2\pi\theta\delta_{mn}$. Consequently, expanding ϕ in (4) in the f_{mn}-bases of the two orthogonal planes, $\phi(x_1,\dots,x_4) = \sum_{\underline{m},\underline{n}\in\mathbb{N}^2}\Phi_{\underline{mn}}f_{m_1n_1}(x_1+\mathrm{i}x_2)f_{m_2n_2}(x_3+\mathrm{i}x_4)$, with $\underline{m} = (m_1,m_2)$, we arrive at

$$S[\phi] = \Big(\frac{\theta}{4}\Big)^2\sum_{\underline{k},\underline{l},\underline{m},\underline{n}\in\mathbb{N}^2}\Big(\frac{Z}{2}\Phi_{\underline{kl}}(\Delta_{\underline{kl};\underline{mn}} + \mu^2_{bare}\delta_{\underline{kn}}\delta_{\underline{lm}})\Phi_{\underline{mn}} + \frac{Z^2\lambda_{bare}}{4}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\Big)\ . \tag{6}$$

In this way (and after appropriate enumeration of $\mathbb{N}^2$) we turn the non-local $\int dx\ \phi^{\star4}(x)$-interaction into an ordinary matrix product $(2\pi\theta)^2\mathrm{tr}(\Phi^4)$. The kinetic term $-\Delta + \|2\Theta^{-1}x\|^2$ is turned into the matrix kernel [GW05b]

$$\begin{aligned}\Delta_{\underline{kl};\underline{mn}} &= \frac{2}{\theta}(1+\Omega^2)(|\underline{m}|+|\underline{n}|+2)\delta_{n_1k_1}\delta_{m_1l_1}\delta_{n_2k_2}\delta_{m_2l_2}\\ &\quad - \frac{2}{\theta}(1-\Omega^2)\Big(\big(\sqrt{k_1l_1}\,\delta_{n_1+1,k_1}\delta_{m_1+1,l_1} + \sqrt{m_1n_1}\,\delta_{n_1-1,k_1}\delta_{m_1-1,l_1}\big)\delta_{n_2k_2}\delta_{m_2l_2}\\ &\qquad + \big(\sqrt{k_2l_2}\,\delta_{n_2+1,k_2}\delta_{m_2+1,l_2} + \sqrt{m_2n_2}\,\delta_{n_2-1,k_2}\delta_{m_2-1,l_2}\big)\delta_{n_1k_1}\delta_{m_1l_1}\Big)\ ,\end{aligned} \tag{7}$$

where $|\underline{m}| := m_1 + m_2$. The regularisation now consists in restricting the matrix sizes to $\mathcal{N} = \frac{\theta}{4}\mu^2\Lambda^2$ for a renormalised mass scale μ introduced later.

With these preparations we have proved in [GW05b] that for given θ one can choose $Z[\Omega,\lambda,\frac{\Lambda}{\mu}]$, $\frac{\mu_{bare}}{\mu}[\Omega,\lambda,\frac{\Lambda}{\mu}]$, $\Omega_{bare}[\Omega,\lambda,\frac{\Lambda}{\mu}]$, $\lambda_{bare}[\Omega,\lambda,\frac{\Lambda}{\mu}]$ in such a way that the limit $\lim_{\Lambda\to\infty}R(0,\Lambda)\circ R^{-1}_{\Sigma}(0,\Lambda)(\mathcal{C})$ described in (1) exists as formal power series, where $\mathcal{C}$ is given by restricting the effective action to (4) but with renormalised values $Z\mapsto 1$, $\mu_{bare}\mapsto\mu$, $\Omega_{bare}\mapsto\Omega$ and $\lambda_{bare}\mapsto\lambda$. This result was reestablished by various methods, see [Riv07] for a review.

In this way a perturbatively renormalised $\lambda\phi^{\star4}_4$-Euclidean quantum field theory on Moyal space with harmonic propagation (which is a spectral noncommutative geometry [GW13a]) is obtained. To extract the ordinary $\lambda\phi^{\star4}_4$-model one has to take final limits $\Omega \to 0$ and $\theta \to 0$, *but these limits do not exist.* That $\theta \to 0$ is not continuous follows from the partition of the effective action (for $\theta \neq 0$) into topological sectors (B, g). This is clear in matrix formulation as we discuss below, but can already be seen in position or momentum space [CR00]. The amplitude of (B, g)-correlation functions is proportional to $\theta^{(2-2B-4g)}$ [GW05b] so that only the so-called planar regular sector ($B{\leq}1, g{=}0$) has a commutative limit $\theta \to 0$ (which differs from the computation at $\theta = 0$!). The limit $\Omega \to 0$ is prevented by UV/IR-mixing [MVS00]. In summary, one can take $|\Omega|, |\theta|$ as small as one wants, but not exactly zero.

For some time one was happy with the $\lambda\phi_4^{\star 4}$-model at finite θ, Ω. The reason is the observation [GW04] that in one-loop approximation one has

$$\lim_{\Lambda\to\infty} \Omega_{bare}[\Omega, \lambda, \tfrac{\Lambda}{\mu}] = 1 \;, \qquad \lim_{\Lambda\to\infty} \lambda_{bare}[\Omega, \lambda, \tfrac{\Lambda}{\mu}] < \infty \;. \tag{8}$$

The second limit is in sharp contrast to the ordinary $\lambda\phi_4^4$-model where $\lambda_{bare}[\Omega, \lambda, \frac{\Lambda}{\mu}]$ *diverges already at finite* $\Lambda = \Lambda_L$, *the Landau pole* [LAK54]. There was thus considerable hope that the $\lambda\phi_4^{\star 4}$-model (4) could permit a rigorous construction along the methods described in [GJ87, Riv91], circumventing triviality [Aiz81, Frö82]).

During the last two years this hope became reality. In fact something beyond any dream was achieved:

1. The $\lambda\phi_4^{\star 4}$-model is, at a special point in parameter space, exactly solvable [GW14a].
2. The symmetry group $SO(2) \times SO(2) \times \mathbb{Z}_2$ of the $\lambda\phi_4^{\star 4}$-model is enlarged to the full Euclidean group $\mathbb{R}^4 \rtimes SO(4)$. See [GW13b].
3. There are good chances [GW14c] that all Osterwalder-Schrader axioms [OS75] are fulfilled. There remains much work to complete the proof, but in any case there is well-founded hope that in the end this can lead to a Wightman quantum field theory [Wig56, SW64] in four dimensions.

A key step for these achievements was the first limit in (8). Namely, the limit $\Omega = 1$ is an exact renormalisation group fixed point $\Omega_{bare}[\Omega = 1, \lambda, \frac{\Lambda}{\mu}] = 1$ for any value of the other parameters. It makes the $\lambda\phi_4^{\star 4}$-model self-dual under the Langmann-Szabo transformation [LS02]. But most importantly, the running coupling constant $\lambda_{bare}[\Omega = 1, \lambda, \frac{\Lambda}{\mu}]$ remains finite for all Λ! This was first proved in [DR07] at three-loop order, then in [DGMR07] to all orders in perturbation theory, and eventually in [GW14a] exactly.

At first sight, putting $\Omega = 1$ in (4) cuts all ties to translation invariance. The only chance to kill the oscillator potential for $\Omega = 1$ is to let $\theta \to \infty$. This is a highly singular limit in (3), although already mentioned in [MVS00] as 'stringy'. We were able to make sense of this limit first for matrices [GW14a] but later in position space [GW13b], and surprisingly it not only restores translation invariance but also full rotation invariance.

If asked to locate the core of the whole construction, the answer would be: *symmetry.* The matrix cut-off $\mathcal{N} = \frac{\theta}{4}\mu^2\Lambda^2$ of the Moyal-oscillator regularisation

(4) still carries the action of a continuous symmetry group $U(\mathcal{N})$. This group action simplifies the Schwinger-Dyson equations enormously. Starting from a closed (non-linear) equation [GW09] for the 2-point function, there is a hierarchy of correlation functions such that all other functions are either algebraically determined or solve linear equations with inductively known parameters.

In the sequel we give a few details of this construction and review the main results.

2 Exact solution of the quartic matrix model

For $\Omega = 1$ in (7) and with cut-off $\mathbb{N}^2 \mapsto \mathbb{N}^2_{\mathcal{N}}$ in the matrix indices, the action (4) takes the form

$$S[\Phi] = V\Bigg(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}} E_{\underline{m}}\, \Phi_{\underline{m}\underline{n}}\Phi_{\underline{n}\underline{m}} + \frac{Z^2\lambda}{4} \sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}} \Phi_{\underline{m}\underline{n}}\Phi_{\underline{n}\underline{k}}\Phi_{\underline{k}\underline{l}}\Phi_{\underline{l}\underline{m}} \Bigg) , \tag{9}$$

$$E_{\underline{m}} = Z\Big(\frac{|\underline{m}|}{\sqrt{V}} + \frac{\mu^2_{bare}}{2}\Big) , \qquad |\underline{m}| := m_1 + m_2 \leq \mathcal{N} , \quad V = \Big(\frac{\theta}{4}\Big)^2 .$$

So it gives rise to a matrix model $S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + P[\Phi])$, where $P[\Phi]$ is a polynomial in Hilbert-Schmidt operators $\Phi = (\Phi_{mn})_{m,n\in I}$ on Hilbert space $L^2(I)$ with scalar coefficients, and E is an unbounded selfadjoint positive operator with compact resolvent. Adding a source term to the action, we define the partition function as

$$\mathcal{Z}[J] = \int \mathcal{D}[\Phi]\ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J)) , \tag{10}$$

where $\mathcal{D}[\Phi]$ is the extension of the Lebesgue measure from finite-rank operators to the Hilbert-Schmidt class and J a test function matrix.

2.1 Ward identity and topological expansion

There is a subgroup of unitary operators U on Hilbert space such that the transformed operator $\tilde{\Phi} = U\Phi U^*$ belongs to the same class as Φ. This implies

$$\int \mathcal{D}[\Phi]\ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J)) = \int \mathcal{D}[\tilde{\Phi}]\ \exp(-S[\tilde{\Phi}] + V\,\mathrm{tr}(\tilde{\Phi} J)) .$$

Unitary invariance $\mathcal{D}[\tilde{\Phi}] = \mathcal{D}[\Phi]$ of the Lebesgue measure leads to

$$0 = \int \mathcal{D}[\Phi] \Big\{ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J)) - \exp(-S[\tilde{\Phi}] + V\,\mathrm{tr}(\tilde{\Phi} J)) \Big\} .$$

Note that the integrand $\{\dots\}$ itself does not vanish because $\mathrm{tr}(E\Phi^2)$ and $\mathrm{tr}(\Phi J)$ are not unitarily invariant; we only have $\mathrm{tr}(P[\Phi]) = \mathrm{tr}(P[\tilde{\Phi}])$ due to $UU^* =$

$U^*U = \mathrm{id}$ together with the trace property. Linearisation of U about the identity operator leads to the *Ward identity*

$$0 = \int \mathcal{D}[\Phi] \Big\{ E\Phi\Phi - \Phi\Phi E - J\Phi + \Phi J \Big\} \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J)) \,. \qquad (11)$$

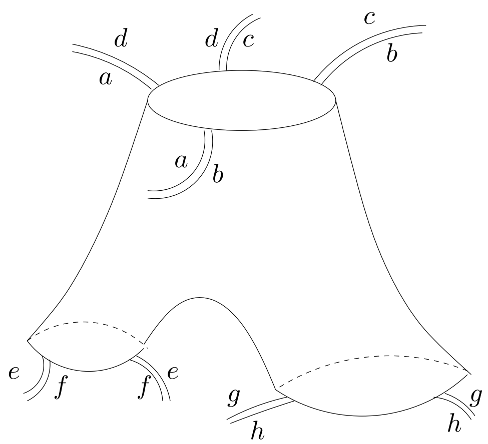
We can always choose an orthonormal basis where E is diagonal (but J is not). Since E is of compact resolvent, E has eigenvalues $E_a > 0$ of finite multiplicity μ_a. We thus label the matrices by an enumeration of the (necessarily discrete) eigenvalues of E and an enumeration of the basis vectors of the finite-dimensional eigenspaces. Writing Φ in $\{\dots\}$ of (11) as functional derivative $\Phi_{ab} = \frac{\partial}{V\partial J_{ba}}$, we have thus proved (first obtained in [DGMR07]):

Proposition 1 *The partition function $\mathcal{Z}[J]$ of the matrix model defined by the external matrix E satisfies the $|I| \times |I|$ Ward identities*

$$0 = \sum_{n\in I} \Big(\frac{(E_a - E_p)}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{an}\partial J_{np}} + J_{pn}\frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}}{\partial J_{np}} \Big) \,. \qquad (12)$$

The compactness of the resolvent of E implies that at the expense of adding a measure $\mu_{[m]} = \dim\ker(E - E_m\mathrm{id})$, we can assume that $m \mapsto E_m$ is injective.

In a perturbative expansion, Feynman graphs in matrix models are *ribbon graphs*. Viewed as simplicial complexes, they encode the topology (B, g) of a genus-g Riemann surface with B boundary components. The k^{th} boundary face is characterised by $N_k \geq 1$ external double lines to which we attach the source matrices J. See e.g. [GW05a]. Since E is diagonal, the matrix index is conserved along each strand of the ribbon graph. Therefore, the right index of J_{ab} coincides with the left index of another J_{bc}, or of the same J_{bb}. Accordingly, the k^{th} boundary component carries a cycle $J^{N_k}_{p_1\dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, with $N_k + 1 \equiv 1$. Here is a drawing for a $(B = 3, g = 0)$ Riemann surface with cycles of lengths $N_1 = 4$, $N_2 = 2$, $N_3 = 2$:



$$(J_{ab}J_{bc}J_{cd}J_{da})\,(J_{ef}J_{fe})\,(J_{gh}J_{hg}) \qquad (13)$$

We know from sec. 1.2 that only $\frac{1}{V}\log\mathcal{Z}[J]$, not $\mathcal{Z}[J]$ itself, can have an infinite volume limit. Consequently we *define* $\log\mathcal{Z}[J]$ as an expansion according to the cycle structure:

$$\log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \sum_{B=1}^{\infty} \sum_{1\leq N_1 \leq \dots \leq N_B}^{\infty} \sum_{p_1^\beta,\dots,p_{N_\beta}^\beta \in I} \frac{V^{2-B}}{S_{N_1\dots N_B}} G_{|p_1^1\dots p_{N_1}^1|\dots|p_1^B\dots p_{N_B}^B|} \prod_{\beta=1}^{B} \Big(\frac{J^{N_\beta}_{p_1^\beta\dots p_{N_\beta}^\beta}}{N_\beta}\Big) . \tag{14}$$

The symmetry factor $S_{N_1\dots N_B}$ is obtained as follows: If ν_i of the B numbers N_β in a given tuple $(N_1,\dots,N_B)$ are equal to i, then $S_{N_1\dots N_B} = \prod_{i=1}^{N_B} \nu_i!$.

Now comes the crucial step. We turn the Ward identity (12) into a formula for the second derivative $\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}$ of the partition function. This amounts to identify the kernel of multiplication by $(E_p - E_a)$. For injective $m \mapsto E_m$ this kernel is given by $W_a[J]\delta_{ap}$ for some function $W_a[J]$. This function is identified by inserting (14) into $\sum_{n\in I} \frac{\partial^2 \exp(\log \mathcal{Z}[J])}{\partial J_{an}\partial J_{np}}$ and carefully registering the possibilities which give rise to a factor δ_{ap}. We find [GW14a]:

Theorem 2

$$\begin{aligned}\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} &= \delta_{ap}\Big\{V^2 \sum_{(K)} \frac{J_{P_1}\cdots J_{P_K}}{S_{(K)}} \Big(\sum_{n\in I} \frac{G_{|an|P_1|\dots|P_K|}}{V^{|K|+1}} + \frac{G_{|a|a|P_1|\dots|P_K|}}{V^{|K|+2}} \\ &\qquad + \sum_{r\geq 1} \sum_{q_1,\dots,q_r\in I} \frac{G_{|q_1 a q_1\dots q_r|P_1|\dots|P_K|} J^r_{q_1\dots q_r}}{V^{|K|+1}} \Big) \\ &\quad + V^4 \sum_{(K),(K')} \frac{J_{P_1}\cdots J_{P_K} J_{Q_1}\cdots J_{Q_{K'}}}{S_{(K)}S_{(K')}} \frac{G_{|a|P_1|\dots|P_K|}}{V^{|K|+1}} \frac{G_{|a|Q_1|\dots|Q_{K'}|}}{V^{|K'|+1}} \Big\} \mathcal{Z}[J] \\ &\quad + \frac{V}{E_p - E_a} \sum_{n\in I} \Big(J_{pn} \frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}[J]}{\partial J_{np}} \Big) . \end{aligned} \tag{15}$$

2.2 Schwinger-Dyson equations

Formula (15) is the core of our approach. It is a consequence of the unitary group action and the cycle structure of the partition function. The importance lies in the fact that the formula allows to kill two J-derivatives in the partition function. As we describe below, this is the key step in breaking up the tower of Schwinger-Dyson equations.

We can write the action as $S = \frac{V}{2}\sum_{a,b}(E_a + E_b)\Phi_{ab}\Phi_{ba} + VS_{int}[\Phi]$, where E_a are the eigenvalues of E. Functional integration yields, up to an irrelevant constant,

$$\mathcal{Z}[J] = e^{-VS_{int}[\frac{\partial}{V\partial J}]} e^{\frac{V}{2}\langle J,J\rangle_E} , \qquad \langle J,J\rangle_E := \sum_{m,n\in I} \frac{J_{mn}J_{nm}}{E_m + E_n} . \tag{16}$$

Instead of a perturbative expansion of $e^{-VS_{int}[\frac{\partial}{V\partial J}]}$ we apply those J-derivatives to (16) which give rise to a correlation function $G_{\dots}$ on the lhs. On the rhs of (16), these external derivatives combine with internal derivatives from $S_{int}[\frac{\partial}{V\partial J}]$ to certain identities for $G_{\dots}$. These Schwinger-Dyson equations are often of little use because they express an N-point function in terms of $(N{+}2)$-point functions. But thanks to (15) we can express the $(N{+}2)$-point function on the rhs in terms of N'-point functions with $N' \leq N$.

Let us look at this mechanism for the 2-point function $G_{|ab|}$ for $a \neq b$. According to (14), $G_{|ab|}$ is obtained by deriving (16) with respect to J_{ba} and J_{ab}:

$$
\begin{aligned}
G_{|ab|} &= \frac{1}{V\mathcal{Z}[0]} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{ba}\partial J_{ab}}\Big|_{J=0} \qquad \text{(disconnected part of } \mathcal{Z} \text{ does not contribute for } a \neq b\text{)} \\
&= \frac{1}{V\mathcal{Z}[0]} \Big\{ \frac{\partial}{\partial J_{ba}} e^{-VS_{int}\big[\frac{\partial}{V\partial J}\big]} \frac{\partial}{\partial J_{ab}} e^{\frac{V}{2}\langle J,J\rangle_E} \Big\}_{J=0} \\
&= \frac{1}{(E_a+E_b)\mathcal{Z}[0]} \Big\{ \frac{\partial}{\partial J_{ba}} e^{-VS_{int}\big[\frac{\partial}{V\partial J}\big]} J_{ba} e^{\frac{V}{2}\langle J,J\rangle_E} \Big\}_{J=0} \\
&= \frac{1}{E_a+E_b} + \frac{1}{(E_a+E_b)\mathcal{Z}[0]} \Big\{ \Big(\Phi_{ab} \frac{\partial(-VS_{int})}{\partial \Phi_{ab}} \Big) \Big[\frac{\partial}{V\partial J} \Big] \Big\} \mathcal{Z}[J] \Big|_{J=0} .
\end{aligned}
\tag{17}
$$

Now observe that $\frac{\partial(-VS_{int})}{\partial \Phi_{ab}}$ contains, for any $P[\Phi]$, the derivative $\sum_n \frac{\partial^2}{\partial J_{an}\partial J_{np}}$ which we know from (15). In case of the quartic matrix model $P[\Phi] = \frac{\lambda}{4}\Phi^4$ we have $\frac{\partial(-VS_{int})}{\partial \Phi_{ab}} = -\lambda V \sum_{n,p\in I} \Phi_{bp}\Phi_{pn}\Phi_{na}$, hence

$$
\Big(\Phi_{ab} \frac{\partial(-VS_{int})}{\partial \Phi_{ab}} \Big) \Big[\frac{\partial}{V\partial J} \Big] = -\frac{\lambda}{V^3} \sum_{p,n\in I} \frac{\partial^2}{\partial J_{pb}\partial J_{ba}} \frac{\partial^2}{\partial J_{an}\partial J_{np}} ,
$$

and the Schwinger-Dyson equation (17) for $G_{|ab|}$ becomes with (15)

$$
\begin{aligned}
G_{|ab|} &= \frac{1}{E_a+E_b} - \frac{\lambda}{V^3(E_a+E_b)\mathcal{Z}[0]} \sum_{p\in I} \frac{\partial^2}{\partial J_{pb}\partial J_{ba}} \sum_{n\in I} \frac{\partial^2 \mathcal{Z}}{\partial J_{an}\partial J_{np}} \Big|_{J=0} \\
&= \frac{1}{E_a+E_b} - \frac{\lambda}{V(E_a+E_b)\mathcal{Z}[0]} \frac{\partial^2}{\partial J_{ab}\partial J_{ba}} \Big\{ \\
&\quad \Big(\sum_{n\in I} \frac{G_{|an|}}{V} + \sum_{n,q,r\in I} \frac{G_{|an|qr|}}{V^2} \frac{J_{qr}J_{rq}}{2} + \sum_{n,q,r\in I} \frac{G_{|an|q|r|}}{V^3} \frac{J_{qq}}{1} \frac{J_{rr}}{1} \\
&\quad + \frac{G_{|a|a|}}{V^2} + \sum_{q,r\in I} \frac{G_{|a|a|qr|}}{V^3} \frac{J_{qr}J_{rq}}{2} + \sum_{q,r\in I} \frac{G_{|a|a|q|r|}}{V^4} \frac{J_{qq}}{1} \frac{J_{rr}}{1} \\
&\quad + \sum_{q,r\in I} \frac{G_{|qaqr|}}{V} J_{qr}J_{rq} + V^2 \frac{G_{|a|q|}}{V^2} \frac{J_{qq}}{1} \frac{G_{|a|r|}}{V^2} \frac{J_{rr}}{1} \Big) \mathcal{Z}[J] \Big\} \Big|_{J=0} \\
&\quad - \frac{\lambda}{V^2(E_a+E_b)\mathcal{Z}[0]} \sum_{p\in I} \frac{\Big(\frac{\partial^2 \mathcal{Z}[J]}{\partial J_{ab}\partial J_{ba}} + \delta_{pb} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{aa}\partial J_{bb}} - \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{pb}\partial J_{bp}} \Big)}{E_p - E_a} \Bigg|_{J=0} .
\end{aligned}
\tag{18}
$$

Taking $\frac{\partial^2 \mathcal{Z}[J]}{\partial J_{pb}\partial J_{bp}} = (VG_{|pb|} + \delta_{pb}G_{|p|b|})\mathcal{Z}[0] + \mathcal{O}(J)$ and $\frac{\partial J_{rr}}{\partial J_{ab}} = 0$ for $a \neq b$ into account, we have proved:

Proposition 3 *The 2-point function of a quartic matrix model with action $S = V\,\mathrm{tr}(E\Phi^2 + \frac{\lambda}{4}\Phi^4)$ satisfies for injective $m \mapsto E_m$ the Schwinger-Dyson equation*

$$G_{|ab|} = \frac{1}{E_a+E_b} - \frac{\lambda}{E_a+E_b}\frac{1}{V}\sum_{p\in I}\Big(G_{|ab|}G_{|ap|} - \frac{G_{|pb|}-G_{|ab|}}{E_p-E_a}\Big) \qquad \Big\} \tag{19a}$$

$$-\frac{\lambda}{V^2(E_a+E_b)}\Big(G_{|a|a|}G_{|ab|} + \frac{1}{V}\sum_{n\in I}G_{|an|ab|} + G_{|aaab|} + G_{|baba|} - \frac{G_{|b|b|}-G_{|a|b|}}{E_b-E_a}\Big) \qquad \Big\} \tag{19b}$$

$$-\frac{\lambda}{V^4(E_a+E_b)}G_{|a|a|ab|}\;. \qquad \Big\} \tag{19c}$$

It can be checked [GW14a] that in a genus expansion $G_{\dots} = \sum_{g=0}^{\infty} V^{-2g}G^{(g)}_{\dots}$ (which is probably not convergent but Borel summable), precisely the line (19a) preserves the genus, the lines (19b) increase $g \mapsto g+1$ and the line (19c) increases $g \mapsto g+2$.

We will not rely on a genus expansion. Instead we consider a scaling limit $V \to \infty$ such that the densitised index summation $\frac{1}{V}\sum_{p\in I}$ remains finite. Then the exact Schwinger-Dyson equation for $G_{|ab|}$ coincides with its restriction (19a) to the planar sector $g = 0$, a closed non-linear equation for $G^{(0)}_{|ab|}$ alone.

By similar calculation we derive the Schwinger-Dyson equation for higher N-point functions. This expresses the N-point function $G_{|ab_1\dots b_{N-1}|}$ in terms of its summation $\displaystyle\frac{\lambda}{E_a+E_{b_1}}\frac{1}{V}\sum_{p\in I}\Big(G_{|ap|}G_{|ab_1\dots b_{N-1}|} - \frac{G_{|pb_1\dots b_{N-1}|}-G_{|ab_1\dots b_{N-1}|}}{E_p-E_a}\Big)$ and several other functions [GW14a]. It turns out that a real theory with $\Phi = \Phi^*$ admits a short-cut which directly gives the higher N-point functions without any index summation. Since the equations for $G_{\dots}$ are real and $\overline{J_{ab}} = J_{ba}$, the reality $\mathcal{Z} = \overline{\mathcal{Z}}$ implies (in addition to invariance under cyclic permutations) invariance under orientation reversal

$$G_{|p_1^1p_2^1\dots p_{N_1}^1|\dots|p_1^Bp_2^B\dots p_{N_B}^B|} = G_{|p_1^1p_{N_1}^1\dots p_2^1|\dots|p_1^Bp_{N_B}^B\dots p_2^B|}\;. \tag{20}$$

Whereas empty for $G_{|ab|}$, in $(E_a+E_{b_1})G_{ab_1b_2\dots b_{N-1}} - (E_a+E_{b_{N-1}})G_{ab_{N-1}\dots b_2b_1}$ the identities (20) lead to many cancellations which result in a universal algebraic recursion formula [GW14a]:

Proposition 4 *Given a quartic matrix model $S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + \frac{\lambda}{4}\Phi^4)$ with E of compact resolvent. Then in a scaling limit $V \to \infty$ with $\frac{1}{V}\sum_{i\in I}$ finite, the $(B = 1)$-sector of $\log\mathcal{Z}$ is given by*

$$G_{|ab|} = \frac{1}{E_a+E_b} - \frac{\lambda}{E_a+E_b}\frac{1}{V}\sum_{p\in I}\Big(G_{|ab|}G_{|ap|} - \frac{G_{|pb|}-G_{|ab|}}{E_p-E_a}\Big)\;, \tag{21}$$

$$G_{|b_0b_1\dots b_{N-1}|} = (-\lambda)\sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0b_1\dots b_{2l-1}|}G_{|b_{2l}b_{2l+1}\dots b_{N-1}|} - G_{|b_{2l}b_1\dots b_{2l-1}|}G_{|b_0b_{2l+1}\dots b_{N-1}|}}{(E_{b_0}-E_{b_{2l}})(E_{b_1}-E_{b_{N-1}})}\;. \tag{22}$$

The self-consistency equation (21) was first obtained in [GW09] for the Moyal model by the graphical method proposed by [DGMR07]. There we also solved the renormalisation problem resulting from the divergent summation $\sum_{p\in I}$. The non-linearity of (21) was a considerable challenge which we successfully addressed in [GW14a, GW14c].

The other topological sectors $B \geq 2$ made of $(N_1+\dots+N_B)$-point functions $G_{|b_1^1\dots b_{N_1}^1|\dots|b_1^1\dots b_{N_B}^B|}$ are similar in the following sense [GW14a]: The basic functions with all $N_i \leq 2$ satisfy an equation with index summation as (21), but in contrast to the 2-point function these equations are linear. The other functions with one $N_i \geq 3$ are purely algebraic.

We make the following key observation: An affine transformation $E \mapsto ZE+C$ together with a corresponding rescaling $\lambda \mapsto Z^2\lambda$ leaves the algebraic equations invariant:

Theorem 5 *Given a real quartic matrix model with $S = V\,\mathrm{tr}(E\Phi^2 + \frac{\lambda}{4}\Phi^4)$ and $m \mapsto E_m$ injective, which determines the set $G_{|p_1^1\dots p_{N_1}^1|\dots|p_1^B\dots p_{N_B}^B|}$ of $(N_1+\dots+N_B)$-point functions. Assume that the basic functions with all $N_i \leq 2$ are turned finite by $E_a \mapsto Z(E_a + \frac{\mu^2}{2} - \frac{\mu_{bare}^2}{2})$ and $\lambda \mapsto Z^2\lambda$. Then all functions with one $N_i \geq 3$*

1. *are finite without further need of a renormalisation of λ, i.e. all renormalisable quartic matrix models have vanishing β-function,*
2. *are given by universal algebraic recursion formulae in terms of renormalised basic functions with $N_i \leq 2$.*

The theorem tells us that vanishing of the β-function for the self-dual Φ^4_4-model on Moyal space (proved in [DGMR07] to all orders in perturbation theory) is generic to all quartic matrix models, and the result even holds non-perturbatively!

3 Renormalisation and integral representation

We return to the Moyal-space regularisation (9) of the $\lambda\phi^4_4$-model. We have proved in sec. 2.2 that the *unrenormalised* 2-point function $G_{|\underline{a}\underline{b}|}$ satisfies the self-consistency equation (21) for $E_{\underline{m}} = Z\Big(\frac{|\underline{m}|}{\sqrt{V}} + \frac{\mu_{bare}^2}{2}\Big)$ and $\lambda \mapsto Z^2\lambda$. Because of the vanishing β-function (Theorem 5), there is no need to introduce a bare coupling λ_{bare}. The matrix indices have ranges $\underline{a},\dots \in I := \mathbb{N}^2_{\mathcal{N}}$, i.e. pairs of natural numbers with certain cut-off. The index sum diverges for $\mathbb{N}^2_{\mathcal{N}} \mapsto \mathbb{N}^2$. As usual, the renormalisation strategy consists in adjusting Z, μ_{bare} in such a way that the limit $\mathbb{N}^2_{\mathcal{N}} \mapsto \mathbb{N}^2$ exists. This will be achieved by normalisation conditions for the

1PI function $\Gamma_{\underline{a}\underline{b}}$ defined by $G_{|\underline{a}\underline{b}|} =: (H_{\underline{a}\underline{b}} - \Gamma_{\underline{a}\underline{b}})^{-1}$, where $H_{\underline{a}\underline{b}} := E_{\underline{a}} + E_{\underline{b}}$. We express (21) in terms of $\Gamma_{\underline{a}\underline{b}}$,

$$\Gamma_{\underline{a}\underline{b}} = -\frac{\lambda Z^2}{V}\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}} \Big(\frac{1}{H_{\underline{a}\underline{p}} - \Gamma_{\underline{a}\underline{p}}} + \frac{1}{H_{\underline{p}\underline{b}} - \Gamma_{\underline{p}\underline{b}}} - \frac{1}{(H_{\underline{p}\underline{b}} - \Gamma_{\underline{p}\underline{b}})} \frac{\Gamma_{\underline{p}\underline{b}} - \Gamma_{\underline{a}\underline{b}}}{\frac{Z}{\sqrt{V}}(|\underline{p}| - |\underline{a}|)}\Big) , \qquad (23)$$

and write $\Gamma_{\underline{a}\underline{b}}$ as first-order Taylor formula with remainder $\Gamma^{ren}_{\underline{a}\underline{b}}$,

$$\Gamma_{\underline{a}\underline{b}} = Z\mu^2_{bare} - \mu^2 + \tfrac{(Z-1)}{\sqrt{V}}(|\underline{a}| + |\underline{b}|) + \Gamma^{ren}_{\underline{a}\underline{b}} , \quad \Gamma^{ren}_{\underline{0}\underline{0}} = 0 , \quad (\partial\Gamma^{ren})_{\underline{0}\underline{0}} = 0 .$$

Equation (23) for $\Gamma_{\underline{a}\underline{b}}\big[\Gamma^{ren}_{\underline{a}\underline{b}}, \mu^2_{bare}, Z\big]$ together with $\Gamma^{ren}_{\underline{0}\underline{0}} = 0$ and $(\partial\Gamma^{ren})_{\underline{0}\underline{0}}$ constitute three equations to determine the three functions $\Gamma^{ren}_{\underline{a}\underline{b}}, \mu^2_{bare}, Z$. Eliminating μ^2_{bare}, Z thus gives rise to a *closed equation for the renormalised function* $\Gamma^{ren}_{\underline{a}\underline{b}}$ *alone*. For this elimination it is important to note that the equations for $\Gamma^{ren}_{\underline{a}\underline{b}}, \mu^2_{bare}, Z$ depend on $\underline{a}, \underline{b}$ only via the norms $|\underline{a}|, |\underline{b}|$ which parametrise the spectrum of E. Therefore, $\Gamma_{\underline{a}\underline{b}}$ is actually a function only of $|\underline{a}|, |\underline{b}|$, and consequently the index sum reduces to $\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}} f(|\underline{p}|) = \sum_{|\underline{p}|=0}^{\mathcal{N}}(|\underline{p}|+1)f(|\underline{p}|)$.

The equations (21) and hence (23) result from (19) in a scaling limit $V \to \infty$ and $\frac{1}{V}\sum_{|\underline{p}|=0}^{\mathcal{N}}(|\underline{p}|+1)f(|\underline{p}|)$ finite for all $f(|\underline{p}|)$. The most natural way to achieve this is to keep the ratio $\frac{\mathcal{N}}{\sqrt{V\mu^4}} = \Lambda^2(1+\mathcal{Y})$ fixed. Note that $V = \big(\frac{\theta}{4}\big)^2 \to \infty$ is a limit of extreme noncommutativity! The new parameter $(1+\mathcal{Y})$ corresponds to a finite wavefunction renormalisation, identified later to decouple our equations. The parameter Λ^2 represents an ultraviolet cut-off which is sent to $\Lambda \to \infty$ in the very end (continuum limit). In the scaling limit, functions of $\frac{|\underline{p}|}{\sqrt{V}} =: \mu^2(1+\mathcal{Y})p$ converge to functions of 'continuous matrix indices' $p \in [0, \Lambda^2]$. In the same way, $\Gamma^{ren}_{\underline{a}\underline{b}}$ converges to a function $\mu^2\Gamma_{ab}$ with $a, b \in [0, \Lambda^2]$, and the discrete sum converges to a Riemann integral

$$\frac{1}{V}\sum_{|\underline{p}|=0}^{\mathcal{N}}(|\underline{p}|+1)f\big(\tfrac{|\underline{p}|}{\sqrt{V}}\big) \longrightarrow \mu^4(1+\mathcal{Y})^2\int_0^{\Lambda^2} p\,dp\; f\big(\mu^2(1+\mathcal{Y})p\big) .$$

Eliminating μ_{bare} and Z as described above we thus obtain a highly non-linear equation for Γ_{ab}. We found it convenient to express this equation in terms of $G_{ab} := \big((a+b)(1+\mathcal{Y}) + 1 - \Gamma_{ab}\big)^{-1}$. Its non-linearity was for many years a tremendous obstacle. Then a breakthrough resulted from a simple observation: The difference of this equation for G_{ab} to the equation for G_{a0}, for appropriate choice of $\mathcal{Y}$, is linear in the function $a \mapsto D_{ab} := \frac{a}{b}(G_{ab} - G_{a0})$. After elemination of μ_{bare}, before elimination of Z, this difference equation reads

$$\frac{Z^{-1}}{(1+\mathcal{Y})}\Big(\frac{1}{G_{ab}} - \frac{1}{G_{a0}}\Big) = b - \lambda\int_0^{\Lambda^2} p\,dp\,\frac{\frac{G_{pb}}{G_{ab}} - \frac{G_{p0}}{G_{a0}}}{p-a} . \qquad (24)$$

Differentiation $\frac{d}{db}\Big|_{a=b=0}$ of (24) yields Z^{-1} in terms of G_{ab} and its derivative. The resulting derivative G' can be avoided by adjusting $\mathcal{Y} := -\lambda \lim_{b\to 0} \int_0^{\Lambda^2} dp\ \dfrac{G_{pb}-G_{p0}}{b}$. This choice leads to $\dfrac{Z^{-1}}{(1+\mathcal{Y})} = 1 - \lambda \displaystyle\int_0^{\Lambda^2} dp\ G_{p0}$, which is a perturbatively divergent integral for $\Lambda \to \infty$. Inserting Z^{-1} and $\mathcal{Y}$ back into (24) we end up in a *linear* integral equation for the difference function $D_{ab} := \frac{a}{b}(G_{ab} - G_{a0})$ to the boundary. The non-linearity restricts to the boundary function G_{a0} where the second index is put to zero. Assuming $a \mapsto G_{ab}$ Hölder-continuous, we can pass to Cauchy principal values. In terms of the *finite Hilbert transform*

$$\mathcal{H}^\Lambda_a[f(\bullet)] := \frac{1}{\pi} \lim_{\epsilon\to 0} \Big(\int_0^{a-\epsilon} + \int_{a+\epsilon}^{\Lambda^2} \Big) \frac{f(q)\,dq}{q-a}\ , \tag{25}$$

the integral equation becomes

$$\Big(\frac{b}{a} + \frac{1+\lambda\pi a\mathcal{H}^\Lambda_a\big[G_{\bullet 0}\big]}{aG_{a0}}\Big) D_{ab} - \lambda\pi\mathcal{H}^\Lambda_a\big[D_{\bullet b}\big] = -G_{a0}\ . \tag{26}$$

Equation (26) is a well-known singular integral equation of Carleman type [Car22, Tri57]:

Theorem 6 ([Tri57], transformed from $[-1,1]$ to $[0,\Lambda^2]$) *The singular linear integral equation*

$$h(a)y(a) - \lambda\pi\mathcal{H}^\Lambda_a[y] = f(a)\ , \qquad a \in\ ]0,\Lambda^2[\ ,$$

is for $h(a)$ continuous on $]0,\Lambda^2[$, Hölder-continuous near $0,\Lambda^2$, and $f \in L^p$ for some $p > 1$ (determined by $\vartheta(0)$ and $\vartheta(\Lambda^2)$) solved by

$$\begin{aligned} y(a) &= \frac{\sin(\vartheta(a))e^{-\mathcal{H}^\Lambda_a[\pi-\vartheta]}}{\lambda\pi a} \Big(a\,f(a)e^{\mathcal{H}^\Lambda_a[\pi-\vartheta]}\cos(\vartheta(a)) \\ &\qquad\qquad + \mathcal{H}^\Lambda_a\Big[e^{\mathcal{H}^\Lambda_\bullet[\pi-\vartheta]}\bullet f(\bullet)\sin(\vartheta(\bullet))\Big] + C\Big) \qquad (27a) \\ &\overset{*}{=} \frac{\sin(\vartheta(a))e^{\mathcal{H}^\Lambda_a[\vartheta]}}{\lambda\pi} \Big(f(a)e^{-\mathcal{H}^\Lambda_a[\vartheta]}\cos(\vartheta(a)) \\ &\qquad\qquad + \mathcal{H}^\Lambda_a\Big[e^{-\mathcal{H}^\Lambda_\bullet[\vartheta]}f(\bullet)\sin(\vartheta(\bullet))\Big] + \frac{C'}{\Lambda^2-a}\Big)\ , \qquad (27b) \end{aligned}$$

where $\vartheta(a) = \underset{[0,\pi]}{\arctan}\Big(\dfrac{\lambda\pi}{h(a)}\Big)$, $\sin(\vartheta(a)) = \frac{|\lambda\pi|}{\sqrt{(h(a))^2+(\lambda\pi)^2}} \geq 0$ and C, C' are arbitrary constants.

The possibility of $C, C' \neq 0$ is due to the fact that the finite Hilbert transform has a kernel, in contrast to the infinite Hilbert transform with integration over

$\mathbb{R}$. The two formulae (27a) and (27b) are formally equivalent, but the solutions belong to different function classes and normalisation conditions may (and will) make a choice.

A lengthy discussion [GW14c] shows that such a constant C, C' arises for $\lambda > 0$ but not for $\lambda < 0$. The key step in this analysis is to regard the defining equation for ϑ as a Carleman type singular integral equation for G_{a0}. This allows to express G_{a0} in terms of ϑ, and various identities in [Tri57] and trigonometric addition theorems give the result:

Theorem 7 ([GW14c]) *The matrix* 2*-point function G_{ab} of the $\lambda\phi_4^{\star 4}$-model is in infinite volume limit given in terms of the boundary* 2*-point function G_{0a} by the equation*

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} \mathrm{e}^{\mathrm{sign}(\lambda)(\mathcal{H}_0^\Lambda[\tau_0(\bullet)] - \mathcal{H}_a^\Lambda[\tau_b(\bullet)])} \begin{cases} 1 & \text{for } \lambda < 0\,, \\ \big(1+\frac{Ca+bF(b)}{\Lambda^2-a}\big) & \text{for } \lambda > 0\,, \end{cases} \tag{28}$$

$$\tau_b(a) := \operatorname*{arctan}_{[0,\pi]} \left(\frac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a \mathcal{H}_a^\Lambda[G_{0\bullet}]}{G_{0a}}} \right), \tag{29}$$

where C is a undetermined constant and $b\,F(b)$ an undetermined function of b vanishing at $b=0$.

Some remarks:

- We proved this theorem in 2012 for $\lambda > 0$ under the assumption $C' = 0$ in (27b), but knew that non-trivial solutions of the homogeneous Carleman equation parametrised by $C' \neq 0$ are possible. That no such term arises for $\lambda < 0$ (if angles are redefined $\vartheta \mapsto \tau$) is a recent result [GW14b, GW14c].
- Strictly speaking, we have $G_{\bullet 0}, G_{a0}$ and not $G_{0\bullet}, G_{0a}$ on the rhs of (29), i.e. G_{ab} is expressed in terms of G_{a0} and not the other boundary G_{0b}. The modification is justified by the required symmetry $G_{ab} = G_{ba}$ of the 2-point function.
- We expect C, F to be Λ-dependent so that $\big(1+\frac{Ca+bF(b)}{\Lambda^2-a}\big) \stackrel{\Lambda\to\infty}{\longrightarrow} 1+\tilde{C}a+b\tilde{F}(b)$.
- An important observation is $G_{ab} \geq 0$, at least for $\lambda < 0$. This is a truly non-perturbative result; individual Feynman graphs show no positivity at all!
- As in [GW09], the equation for G_{ab} can be solved perturbatively. This reproduces exactly [GW14a] the Feynman graph calculation! Matching at $\lambda = 0$ requires C, F to be flat functions of λ (all derivatives vanish at zero). Because of $\mathcal{H}_a^\Lambda[G_{\bullet 0}] \stackrel{a\to\Lambda^2}{\longrightarrow} -\infty$, the naïve arctan series is dangerous for $\lambda > 0$. Unless there are cancellations, we expect zero radius of convergence!

- From (28) we deduce the finite wavefunction renormalisation

$$\mathcal{Y} := -1 - \frac{dG_{ab}}{db}\Big|_{a=b=0} = \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \big(\frac{1+\lambda\pi p\mathcal{H}_p^\Lambda[G_{\bullet 0}]}{G_{p0}}\big)^2} - \begin{cases} 0 & \text{for } \lambda < 0\,, \\ F(0) & \text{for } \lambda > 0\,. \end{cases} \tag{30}$$

- The partition function $\mathcal{Z}$ is undefined for $\lambda < 0$. But the Schwinger-Dyson equations for G_{ab} and for higher functions, and with them $\log \mathcal{Z}$, extend to $\lambda < 0$. These extensions are unique but probably not analytic in a neighbourhood of $\lambda = 0$.

It remains to identify the boundary function, which strictly speaking is G_{a0}. It is determined from the equation we had subtracted in order to get (24). The equation involves subtle cancellations which so far we did not succeed to control. As substitute we use a symmetry argument, already prepared in the replacement $G_{\bullet 0} \mapsto G_{0\bullet}$ in (29). Given the boundary function G_{0b}, the Carleman theory computes the full 2-point function G_{ab} via (28). In particular, we get G_{a0} as function of G_{0b}. But the 2-point function is symmetric, $G_{ab} = G_{ba}$, and the special case $a = 0$ leads to the following self-consistency equation:

Proposition 8 *The limit $\theta \to \infty$ of $\lambda\phi_4^4$-theory on Moyal space is determined by the solution of the fixed point equation $G = TG$,*

$$G_{b0} = \frac{\begin{Bmatrix} 1 & for\ \lambda{<}0 \\ 1{+}bF(b) & for\ \lambda{>}0 \end{Bmatrix}}{1{+}b} \exp\left(-\lambda\int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \big(t{+}\frac{1+\lambda\pi p\mathcal{H}_p^\Lambda[G_{\bullet 0}]}{G_{p0}}\big)^2}\right)\,. \tag{31}$$

At this point we can eventually send $\Lambda \to \infty$. Any solution of (31) is automatically smooth and (for $\lambda > 0$ but $F = 0$) monotonously decreasing. Any solution of the true equation (23) (without the difference to $b = 0$) also solves the master equation (31), but not necessarily conversely. In case of a unique solution of (31), it is enough to check one candidate.

Existence of a solution of (31) is established (for $\lambda > 0$ but $F(b) = 0$) by the Schauder fixed point theorem [GW14a]. For $\lambda < 0$ we know that $G_{0b} = 1$ is an exact solution for $\Lambda \to \infty$ [GW14d].

This solution provides G_{ab} via (28) and all higher correlation functions via the universal algebraic recursion formulae (22), or via the linear equations for the basic $(N_1{+}\ldots{+}N_B)$-point functions [GW14a]. The recursion formula (22) becomes after transition to continuous matrix indices

$$G_{b_0\ldots b_{N-1}} = \frac{(-\lambda)}{(1+\mathcal{Y})^2} \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{b_0 b_1\ldots b_{2l-1}} G_{b_{2l} b_{2l+1}\ldots b_{N-1}} - G_{b_{2l} b_1\ldots b_{2l-1}} G_{b_0 b_{2l+1}\ldots b_{N-1}}}{(b_0 - b_{2l})(b_1 - b_{N-1})}\,. \tag{32}$$

It involves the finite wavefunction renormalisation $1 + \mathcal{Y} = -\frac{dG_{ab}}{db}\big|_{a=b=0}$ given by (30).

4 Schwinger functions and reflection positivity

4.1 Schwinger functions

In the previous section we have constructed the connected matrix correlation functions $G_{|\underline{q}_1^1\dots\underline{q}_{N_1}^1|\dots|\underline{q}_1^B\dots\underline{q}_{N_B}^B|}$ of the ($\theta\to\infty$)-limit of $\lambda\phi_4^4$-theory on Moyal space. These functions arise from the topological expansion (14) of the free energy

$$\log\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}=\sum_{B=1}^{\infty}\sum_{1\leq N_1\leq\dots\leq N_B}^{\infty}\frac{(V\mu^4)^{2-B}}{S_{N_1\dots N_B}}\sum_{\underline{q}_i^\beta\in\mathbb{N}^2}G_{|\underline{q}_1^1\dots\underline{q}_{N_1}^1|\dots|\underline{q}_1^B\dots\underline{q}_{N_B}^B|}\prod_{\beta=1}^{B}\frac{1}{N_\beta}\Big(\frac{J_{\underline{q}_1^\beta\underline{q}_2^\beta}}{\mu^3}\cdots\frac{J_{\underline{q}_{N_\beta}^\beta\underline{q}_1^\beta}}{\mu^3}\Big). \tag{33}$$

Our goal is to revert the introduction of the matrix basis (5) to obtain Schwinger functions [Sch59] in position space,

$$\big\langle\phi(\mu x_1)\dots\phi(\mu x_N)\big\rangle\equiv\sum_{\underline{m}_1,\underline{n}_1,\dots,\underline{m}_N,\underline{n}_N\in\mathbb{N}^2}f_{\underline{m}_1\underline{m}_2}(\mu x_1)\cdots f_{\underline{m}_N\underline{m}_N}(\mu x_N)\big\langle\Phi_{\underline{m}_1\underline{n}_1}\dots\Phi_{\underline{m}_N\underline{n}_N}\big\rangle.$$

Here the matrix correlation functions $\big\langle\Phi_{\underline{m}_1\underline{n}_1}\dots\Phi_{\underline{m}_N\underline{n}_N}\big\rangle$ are obtained by derivatives of $\frac{1}{\text{volume}}$ times (33) with respect to $J_{\underline{m}_1\underline{n}_1},\dots,J_{\underline{m}_N\underline{n}_N}$, and we absorbed all mass dimensions into μ to work with densities which admit the limit $V\mu^4\to\infty$.

Definition 9 *The connected Schwinger functions associated with the regularised action (4) are*

$$\begin{aligned}&\mu^N S_c(\mu x_1,\dots,\mu x_N)\\&:=\lim_{V\mu^4\to\infty}\sum_{\underline{m}_1,\underline{n}_1,\dots,\underline{m}_N,\underline{n}_N\in\mathbb{N}^2_{\mathcal{N}}}f_{\underline{m}_1\underline{n}_1}(\mu x_1)\cdots f_{\underline{m}_N\underline{n}_N}(\mu x_N)\frac{\mu^{4N}\partial^N\mathcal{F}[J]}{\partial J_{\underline{m}_1\underline{n}_1}\dots\partial J_{\underline{m}_N\underline{n}_N}}\bigg|_{J=0}, \qquad (34)\\&\mathcal{F}[J]:=\frac{1}{64\pi^2V^2\mu^8}\log\left(\frac{\int\mathcal{D}[\Phi]\;e^{-S[\Phi]+V\sum_{\underline{a},\underline{b}\in\mathbb{N}^2_{\mathcal{N}}}\Phi_{\underline{a}\underline{b}}J_{\underline{b}\underline{a}}}}{\int\mathcal{D}[\Phi]\;e^{-S[\Phi]}}\right)_{\substack{Z\mu^2_{bare}\mapsto\mu^2\\Z\mapsto(1+\mathcal{Y})}},\end{aligned}$$

where $S[\Phi]$ is given by (9) and $f_{\underline{m}\underline{n}}$ by (5). By $(\quad)_{\substack{Z\mu^2_{bare}\mapsto\mu^2\\Z\mapsto(1+\mathcal{Y})}}$ we symbolise the renormalisation of sec. 3 including the matrix cut-off $\mathcal{N}=\sqrt{V\mu^4}(1+\mathcal{Y})\Lambda^2$ coupled to the volume. A final limit $\Lambda\to\infty$ is understood.

The (at first sight surprising) squared volume factor $\frac{1}{(V\mu^4)^2}$ has its origin in the spectral geometry of the Moyal plane with harmonic propagation [GW13a, GW12]. Note that the J-derivatives, and hence the Schwinger functions, are fully symmetric in $\mu x_1,\dots,\mu x_N$. Applying the J-derivatives to the topological expansion (33) into J-cycles produces an $f_{\underline{m}\underline{n}}$-cycle for each of the B boundary components:

$$S_c(\mu x_1, \ldots, \mu x_N) = \lim_{V\mu^4\to\infty} \frac{1}{64\pi^2} \sum_{N_1+\cdots+N_B=N} \sum_{\underline{q}_i^\beta \in \mathbb{N}^2} G_{|\underline{q}_1^1 \ldots \underline{q}_{N_1}^1|\ldots|\underline{q}_1^B \ldots \underline{q}_{N_B}^B|}$$
$$\times \sum_{\sigma\in\mathcal{S}_N} \prod_{\beta=1}^{B} \frac{f_{\underline{q}_1\underline{q}_2}(\mu x_{\sigma(N_1+\ldots+N_{\beta-1}+1)}) \cdots f_{\underline{q}_{N_\beta}\underline{q}_1}(\mu x_{\sigma(N_1+\ldots+N_\beta)})}{V\mu^4 N_\beta} .$$

The summation over $\underline{q}_i^\beta \in \mathbb{N}^2$ is performed by Laplace-Fourier transform of G and use of the following summation of Laguerre polynomials arising via (5):

$$\sum_{q_1,\ldots,q_{N'}=0}^{\infty} \prod_{j=1}^{N'} z_j^{q_j} L_{q_j}^{q_{j+1}-q_j}(r_j) = \frac{\exp\left(-\dfrac{\sum_{j,k=1}^{N'} r_j z_{k+j}\cdots z_{N'+j}}{1-z_1\cdots z_{N'}}\right)}{1-z_1\cdots z_{N'}} , \quad |z_i|<1 . \tag{35}$$

We refer to [GW13b, GW14b] for details. The final result is:

Theorem 10 *The connected N-point Schwinger functions of the $\lambda\phi_4^4$-model on extreme Moyal space $\theta\to\infty$ are given by*

$$\begin{aligned} &S_c(\mu x_1, \ldots, \mu x_N) \\ &= \frac{1}{64\pi^2} \sum_{\substack{N_1+\ldots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma\in\mathcal{S}_N} \left(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{dp_\beta}{4\pi^2\mu^4}\, e^{\mathrm{i}\left\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta} (-1)^{i-1} \mu x_{\sigma(N_1+\ldots+N_{\beta-1}+i)} \right\rangle} \right) \\ &\qquad \times G_{\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}, \ldots, \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1} \Big| \ldots \Big| \underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}, \ldots, \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}} . \end{aligned} \tag{36}$$

Some comments:

- Only a restricted sector of the underlying matrix model contributes to position space: All strands of the same boundary component carry the same matrix index.
- Schwinger functions are symmetric and invariant under the full Euclidean group. This comes truly surprising since $\theta \neq 0$ breaks both translation invariance and manifest rotation invariance. The limit $\theta\to\infty$ was expected to make this symmetry violation even worse!
- The most interesting sector is the case where every boundary component has $N_\beta = 2$ indices. It is described by the $(2+\ldots+2)$-point functions $G_{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}\big|\ldots\big|\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}$. The corresponding matrix functions $G_{a_1a_1|\ldots|a_Ba_B}$ satisfy more complicated singular (but linear!) integral equations. The solution techniques of the Carleman problem can be used in a first step to regularise these equations to linear integral equations of Fredholm type. These have always a unique solution for $|\lambda|$ small enough.

- This (2+ … +2)-sector describes the propagation and interaction of B (at the moment Euclidean) particles without any momentum exchange. In two dimensions, this is familiar from quantum field theories with factorising S-matrix [Iag78, ZZ79]. Ideally, our Schwinger functions describe a four-dimensional analogue of this mechanism. This would be far away from a realistic physical model, but also far outside the scope of any other four-dimensional quantum field theory we know of.
- We are aware of the problem that the absence of momentum transfer in four dimensions is a sign of *triviality*. Typical triviality proofs rely on clustering, analyticity in Mandelstam representation or absence of bound states. All this needs verification.
- It is already clear that clustering is maximally violated. There is always a permutation σ in (36) for which the contribution $S_c^{(\sigma)}(\mu x_1, \dots, \mu x_N)$ to the Schwinger function has individual Euclidean invariance in the arguments $x_1, \dots, \mu x_{2l}$ and $x_{2l+1}, \dots, \mu x_N$.
- That the $\theta \to \infty$ limit is so close to an ordinary field theory expected for $\theta \to 0$ can be seen from the following observation: The interaction term in momentum space

$$\frac{\lambda}{4} \int_{(\mathbb{R}^4)^4} \Big(\prod_{i=1}^{4} \frac{dp_i}{(2\pi)^4} \Big) \, \delta(p_1 + \dots + p_4) \, \exp\Big(\mathrm{i} \sum_{i<j} \langle p_i, \Theta p_j \rangle \Big) \, \prod_{i=1}^{4} \hat{\phi}(p_i)$$

  leads to the Feynman rule $\lambda \exp\big(\mathrm{i} \sum_{i<j} \langle p_i, \Theta p_j \rangle\big)$, plus momentum conservation. For $\theta \to \infty$, this converges to zero almost everywhere by the Riemann-Lebesgue lemma, *unless p_i, p_j are linearly dependent.* This case of linearly dependent momenta might be protected for topological reasons, and these are precisely the boundary components $B > 1$ which guarantee full Lebesgue measure!

4.2 Osterwalder-Schrader axioms

Under conditions identified by Osterwalder-Schrader [OS73, OS75], Schwinger functions [Sch59] of a Eulidean quantum field theory permit an analytical continuation to Wightman functions [Wig56, SW64] of a true relativistic quantum field theory. In simplified terms, the reconstruction theorem of Osterwalder-Schrader for a field theory on $\mathbb{R}^D$ says:

Theorem 11 ([OS73, OS75]) *Assume the Schwinger functions $S(x_1, \dots, x_N)$ satisfy*

(OS0) growth conditions,

(OS1) Euclidean invariance,

(OS2) reflection positivity[1]*,*

(OS3) permutation symmetry.

Then the $S(\xi_1, \dots \xi_{N-1})\big|_{\xi_i^0>0}$*, with* $\xi_i = x_i - x_{i+1}$*, are Laplace-Fourier transforms of Wightman functions in a relativistic quantum field theory. If in addition the* $S(x_1, \dots, x_N)$ *satisfy*

(OS4) clustering

then the Wightman functions satisfy clustering, too.

The Schwinger functions (36) clearly satisfy (OS1)+(OS3). Clustering (OS4) is not realised. Bounds on $S_c(\mu x_1, \dots, \mu x_N)$ for large N follow from bounds on (32) at coinciding indices, i.e. from bounds on derivatives of the 2-point function. In [GW14c] we have derived integral formulae for $\frac{\partial^{n+\ell} G_{ab}}{\partial a^n\, \partial b^\ell}$. These are lengthy so that we will not reproduce them in full generality; only the special case $n = 0$ should be mentioned:

$$\begin{aligned}\frac{\partial^\ell \log G_{ab}}{\partial b^\ell}\Big|_{\ell\geq 1} &= (-1)^\ell(\ell-1)!\,\mathrm{sign}(\lambda)\,\mathcal{H}^\Lambda_a\Big[\sin\big(\ell\tau_b(\bullet)\big)\Big(\frac{\sin\tau_b(\bullet)}{|\lambda|\pi\bullet}\Big)^\ell\Big] \\ &\quad + (-1)^\ell(\ell-1)!\cos\big(\ell\tau_b(a)\big)\Big(\frac{\sin\tau_b(a)}{|\lambda|\pi a}\Big)^\ell\,. \end{aligned} \tag{37}$$

These formulae give $|\frac{\partial^{n+\ell} G_{ab}}{\partial a^n\, \partial b^\ell}| \leq n!\ell! C_{n\ell}$ where $C_{n\ell}$ involves combinatorial numbers and bounds of Hilbert transforms. This makes it plausible, although still to prove, that $C_{n\ell}$ is bounded by a polynomial in $n!, \ell!$, which would suffice to establish (OS0).

Thus the remaining problem is (OS2) reflection positivity. Representation as Laplace transform in ξ^0 requires analyticity in $\mathrm{Re}(\xi^0) > 0$. For the Schwinger 2-point function (36), such analyticity in ξ^0 is a corollary of analyticity of the function $a \mapsto G_{aa}$ in $\mathbb{C} \setminus\,]-\infty, 0]$. We will show that analyticity and reflection positivity boil down to *Stieltjes functions*, i.e. functions $f : \mathbb{R}_+ \to \mathbb{R}$ which have a representation as a Stieltjes transform (see [Wid38])

$$f(x) = c + \int_0^\infty \frac{d(\rho(t))}{x+t}\;, \qquad c = f(\infty) \geq 0\;, \tag{38}$$

where ρ is non-negative and non-decreasing. We prove:

Proposition 12 *The Schwinger function* $S_c(\mu\xi) = \displaystyle\int_{\mathbb{R}^4} \frac{dp}{(2\pi\mu)^4} e^{\mathrm{i}p\xi} G_{\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}}$ *identified in (36) is the analytic continuation of a Wightman* 2*-point function if and only if* $a \mapsto G_{aa}$ *is Stieltjes.*

[1]For each assignment $N \mapsto f_N \in \mathcal{S}^N$ of test functions, one has

$$\sum_{M,N} \int dx\, dy\; S(x_1, \dots, x_N, y_1, \dots, y_M)\overline{f_N(x_1^r, \dots, x_N^r)} f_M(y_1, \dots, y_M) \geq 0\;,$$

where $(x^0, x^1, \dots x^{D-1})^r := (-x^0, x^1, \dots x^{D-1})$

Proof. This is verified by explicit calculation. If $a \mapsto G_{aa}$ is Stieltjes, we have in terms of $\omega_{\vec{p}}(t) := \sqrt{\vec{p}^2 + 2\mu^2(1+\mathcal{Y})t}$

$$
\begin{aligned}
S_c(\mu\xi)\big|_{\xi^0>0} &= 2\mu(1{+}\mathcal{Y})\int_{\mathbb{R}^3}\frac{d\vec{p}\, e^{\mathrm{i}\vec{p}\cdot\vec{\xi}}}{(2\pi\mu)^3}\int_0^\infty\frac{d\rho(t)}{2\omega_{\vec{p}}(t)}\int_{-\infty}^\infty\frac{dp^0}{2\pi\mathrm{i}}\Big(\frac{e^{\mathrm{i}p^0\xi^0}}{p^0{-}\mathrm{i}\omega_{\vec{p}}(t)}-\frac{e^{\mathrm{i}p^0\xi^0}}{p^0{+}\mathrm{i}\omega_{\vec{p}}(t)}\Big)\\
&= 2\mu(1{+}\mathcal{Y})\int_0^\infty\frac{d\rho(t)}{2\omega_{\vec{p}}(t)}\int_{\mathbb{R}^3}\frac{d\vec{p}\, e^{-\xi^0\omega_{\vec{p}}(t)+\mathrm{i}\vec{p}\cdot\vec{\xi}}}{(2\pi\mu)^3}\\
&= \int_0^\infty\frac{2(1+\mathcal{Y})\,d\rho(t)}{\mu^4}\int_0^\infty dq^0\int_{\mathbb{R}^3}d\vec{q}\,\hat{W}_t(q)e^{-q^0\xi^0+\mathrm{i}\vec{q}\cdot\vec{\xi}}\,, \qquad (39)\\
\hat{W}_t(q) &:= \frac{\theta(q^0)}{(2\pi)^3}\delta\Big(\frac{(q^0)^2-\vec{q}^2-2\mu^2(1{+}\mathcal{Y})t}{\mu^2}\Big)\,.
\end{aligned}
$$

The step from the first to second line is the residue theorem. We observe that $\hat{W}_t(q)$ is precisely the Källén-Lehmann spectral representation [Käl52, Leh54] of a Wightman 2-point function. □

Stieltjes functions form an important subclass of the class $\mathcal{C}$ of *completely monotonic functions*. We refer to [Ber08] for an overview about completely monotonic functions and their relations to other important classes of functions. The class $\mathcal{C}$ characterises the positive definite functions on $\mathbb{R}_+$, i.e. for any $x_1,\dots,x_n \geq 0$ the matrix $a_{ij} = f(x_i+x_j)$, with $f \in \mathcal{C}$, is positive (semi-)definite. A function $f : \mathbb{R}_+ \to \mathbb{R}$ is positive definite, bounded and continuous if and only if it is the Laplace transform of a positive finite measure, $f(x) = \displaystyle\int_0^\infty e^{-xt}d\mu(t)$. This representation provides a unique analytic continuation of such functions to the half space $\mathrm{Re}(z) > 0$. Remarkably, such analyticity is a consequence of the purely real conditions $(-1)^n f^{(n)}(x) \geq 0$ for all $n \in \mathbb{N}$ and $x > 0$.

The Stieltjes integral (38) provides a unique analytic continuation of a Stieltjes function to the cut plane $\mathbb{C}\setminus\,]{-\infty},0[$. Remarkably again, this analyticity can be tested by purely real conditions identified by Widder [Wid38]: A smooth non-negative function f on $\mathbb{R}_+$ is Stieltjes iff $L_{n,t}[f(\bullet)] \geq 0$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$, where $L_{0,t}[f(\bullet)] = f(t)$, $L_{1,t}[f(\bullet)] = \frac{d}{dt}\big(tf(t)\big)$ and

$$
L_{n,t}[f(\bullet)] := \frac{(-t)^{n-1}}{n!(n-2)!}\frac{d^{2n-1}}{dt^{2n-1}}\big(t^n f(t)\big)\,, \qquad n \geq 2\,. \qquad (40)
$$

If Widder's criterion is satisfied, the sequence $\{L_{n,t}[f(\bullet)]\}$ converges for $n \to \infty$ in distributional sense and almost everywhere to the measure function of the Stieltjes transform, $\displaystyle\int_0^T dt\,\rho'(t) = \lim_{n\to\infty}\int_0^T dt\; L_{n,t}[f(\bullet)]$.

5 Computer simulations [GW14c]

A first hint about reflection positivity can be obtained from a numerical solution of the fixed point equation (31), assuming $F(b) \equiv 0$ (which is the case for $\lambda < 0$).

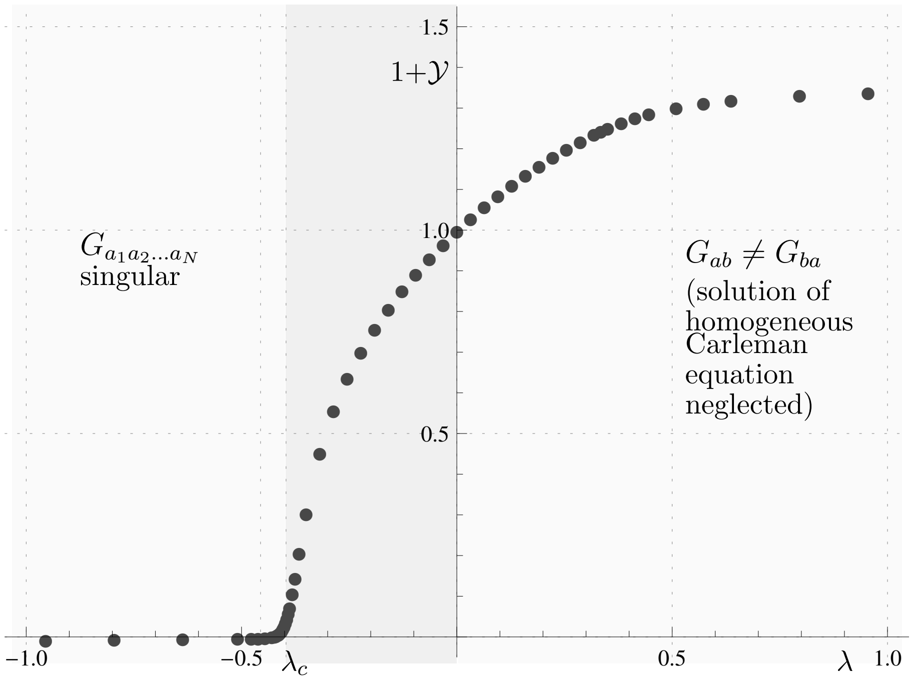


Figure 1: $1+\mathcal{Y} := -\frac{dG_{0b}}{db}\big|_{b=0}$ as function of λ, based on G_{0b} computed for $\Lambda^2{=}10^7$ with $L = 2000$ sample points.

This was done in [GW14c] using *Mathematica*^{TM}. The idea is to approximate G_{0b} as a piecewise linear function on $[0, \Lambda^2]$ sampled according to a geometric progression and view (31) as iteration $G^{i+1}_{0b} = (TG^i)_{0b}$ for some initial function G^0. We confirmed the convergence of this iteration in Lipschitz norm for any $\lambda \in \mathbb{R}$. It turned out that the required symmetry $G_{ab} = G_{ba}$ does not hold for $\lambda > 0$, which is a clear hint that $F(b) \neq 0$ for $\lambda > 0$. For $\lambda < 0$ everything is consistent within small numerical errors. This allows us to compute for $\lambda < 0$ all quantities of the model with sufficient precision.

We find clear evidence for a second-order phase transition at $\lambda_c \approx -0.39$, which is a common critical value in several independent problems. The first one is the derivative $1 + \mathcal{Y} := -\frac{dG_{0b}}{db}\big|_{b=0}$, viewed as function of λ (fig. 1). In good approximation we find a critical behaviour $1 + \mathcal{Y} = \begin{cases} A(\lambda - \lambda_c)^\alpha & \text{for } \lambda \geq \lambda_c \\ 0 & \text{for } \lambda < \lambda_c \end{cases}$ for some $A, \alpha > 0$. Of course, there cannot be a discontinuity in $(1 + \mathcal{Y})'(\lambda)$ for finite Λ, but we have numerical evidence for a critical behaviour in the limit $\Lambda^2 \to \infty$. More precisely, for $\lambda < \lambda_c$ we have $G_{0b} = 1$ in a whole neighbourhood of $b = 0$, i.e. the exact but unstable solution $G_{0b} = 1$ for $\Lambda \to \infty$ [GW14d] becomes locally stable. The end point $b_\lambda := \sup\{b \,:\, G_{0b} = 1\}$ serves as an order parameter: $b_\lambda = 0$ in the phase $\lambda > \lambda_c$ and $b_\lambda > 0$ in the phase $\lambda < \lambda_c$. Because of $1 + \mathcal{Y} = 0$ for $\lambda < \lambda_c$, all higher correlation functions (32) become singular in that phase. Another phase transition occurs at $\lambda = 0$. It is not visible in G_{0b} but

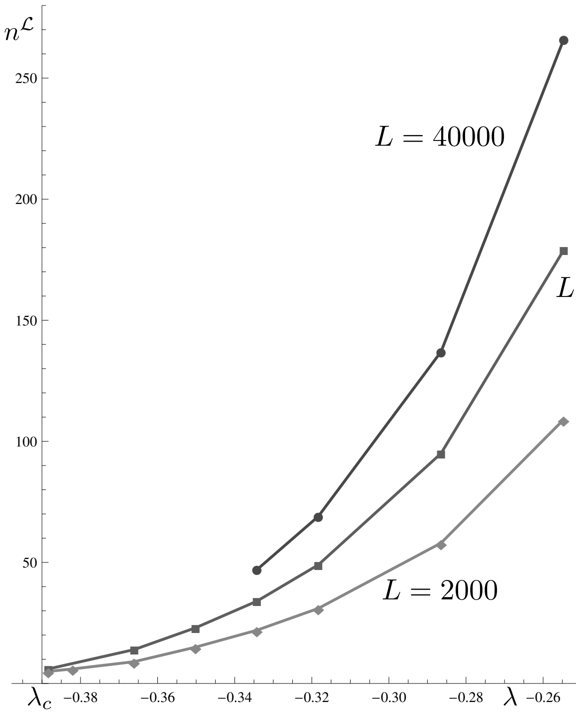


Figure 2: The critical indices $n^{\mathcal{L}} := \min\{n : (-1)^n \frac{d^n(\log G_{0b})}{db^n} < 0\}$ where logarithmically complete monotonicity of the discretised function G_{0b} fails, as function of λ and of the number L of sample points. The cut-off is $\Lambda^2 = 10^7$.

in the full 2-point function G_{ab}. As mentioned before, the symmetry $G_{ab} = G_{ba}$ is violated for $\lambda > 0$. This means that the 'good' phase is $\lambda_c < \lambda \leq 0$.

Of paramount importance is the question whether or not $a \mapsto G_{aa}$ is a Stieltjes function. We cannot expect a definite answer from a numerical simulation because a discrete approximation, here a piecewise linear function, cannot be analytic. This means that the criteria $(-1)^n \frac{d^n G_{aa}}{da^n} \geq 0$ of complete monotonicity and $L_{n,a}[G_{\bullet\bullet}] \geq 0$ of Stieltjes property must fail for some n. But refining the approximation, i.e. increasing the number L of sample points, the failure should occur at larger n, with no failure in the limit. This is precisely what we observe.

The cleanest results are obtained for the boundary function G_{0b}. Fig. 2 shows the failure $n^{\mathcal{L}}$ of complete monotonicity of $\log G_{0b}$ (see also [Ber08]) as function of λ for several resolutions L. We notice that $n^{\mathcal{L}}$ increases with L, but this increase slows down for larger $|\lambda|$ and stops at precisely the same value $\lambda_c \approx -0.39$ that located the discontinuity in fig. 1!

We are mainly interested in the diagonal 2-point function G_{aa}. Here the results are less clean because multiple a-differentiation of (28) produces many terms and thus amplified numerical errors. But the tendency is exactly the same: We prove in [GW14c] that Widder's criterion $L_{n,a}[G_{\bullet\bullet}] \geq 0$ for Stieltjes functions

- is clearly violated for $\lambda < \lambda_c \approx -0.39$; the failure is at $n = 2$ for all $\lambda < -0.42$ and increases to $n = 5$ slightly below λ_c,
- fails in the phase $\lambda_c < \lambda \leq 0$ for the discretised function G_{aa}, but the order $n^{\mathcal{S}}$ of failure shows qualitatively the same dependence on λ and the discretisation L as in fig. 2,
- is manifestly violated for any $\lambda > 0$ with already $L_{1,a}[G_{\bullet\bullet}] < 0$ for a large enough.

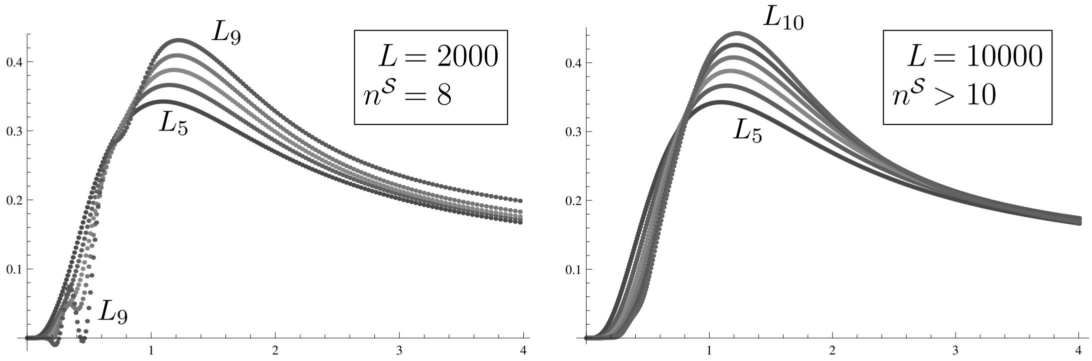


Figure 3: Widder's operations $L_{n,t}[G_{\bullet\bullet}]$ for the discretised function G_{aa} at $\lambda = -0.366$, for resolutions $L = 2000$ (left) and $L = 10000$ (right). There is a significant increase of the failure index $n^{\mathcal{S}}$ with L.

Fig. 3 demonstrates the improvement of $L_{n,t}[G_{\bullet\bullet}]$ for $\lambda = -0.366$ for larger resolution L. There is clear evidence for

- positivity of $L_{n,a}[G_{\bullet\bullet}]$,
- convergence $\lim_{n\to\infty} L_{n,t}[G_{\bullet\bullet}]$ to the derivative $\rho'(t)$ of the Stieltjes measure,
- a mass gap $\rho(t) = 0$ for $0 \leq t < \frac{m_0^2}{\mu^2}$,
- absence of a further mass gap, i.e. scattering right away from m_0^2 and not only from the two-particle threshold on.

In summary we have clear evidence, albeit no proof, of reflection positivity of the Schwinger 2-point function $S_c(\mu x_1, \mu x_2)$ precisely in the phase $\lambda_c < \lambda \leq 0$.

6 Summary and outlook

We have introduced a Moyal-space regularisation of the Euclidean $\lambda\phi_4^4$-model and showed that an unconventional limit $\theta \to \infty$, which does not give back the original $\lambda\phi_4^4$, leads to complete integrability in four dimensions. To our great surprise, the resulting Schwinger functions on $\mathbb{R}^4$ satisfy the easy Osterwalder-Schrader axioms boundedness, invariance and symmetry. A numerical investigation discovered an intriguing phase structure with second-order phase transition at $\lambda_c \approx -0.39$ and perfectly nice behaviour in the phase $\lambda_c < \lambda \leq 0$. In particular, we have overwhelming evidence for reflection positivity of the Schwinger 2-point function precisely in the good phase.

Suppose these miracles continue and all Osterwalder-Schrader axioms (except for clustering) hold for the family (36) of Schwinger functions. Then the Osterwalder-Schrader theorem [OS75] reconstructs Wightman functions of a relativistic quantum field theory [Wig56, SW64] in four dimensions. This theory is somewhat strange as 'particles' keep their momenta in interaction processes. Such behaviour is close to triviality. In two dimensions there is the possiblity of factorised S-matrix [Iag78, ZZ79] which is related to integrability [Kul76]. It would be highly interesting to investigate whether our exotic limit $\theta \to \infty$ violates the assumptions of those theorems which forbid 4D factorised S-matrices. The possibility of a non-trivial quantum field theory in four dimensions is enough motivation to proceed our work.

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