Non–Associative Geometry — Unified Models Based on L–Cycles

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Referat:

Ausgehend von einem Hilbertraum und der Darstellung einer schiefadjungierten Lie–Algebra sowie der Wirkung eines verallgemeinerten Dirac–Operators auf diesem Hilbertraum wird ein mathematisches Konzept für eine Konstruktion von Eichtheorien entwickelt. Dieses Konzept besitzt gewisse Analogien zur Nichtkommutativen Geometrie à la Connes / Lott, unterscheidet sich von dieser jedoch durch die Verwendung schiefadjungierter Lie–Algebren statt assoziativer $*$–Algebren. Wählt man als Lie–Algebra das Tensorprodukt aus der Funktionenalgebra und einer geeigneten Matrix–Lie–Algebra, zusammen mit einer physikalisch motivierten Darstellung auf dem Hilbertraum der Fermionen, sowie eine Kombination aus dem klassischen Dirac–Operator und den fermionischen Massenmatrizen als verallgemeinerten Dirac–Operator, dann führt der entwickelte mathematische Kalkül auf interessante Modelle der Teilchenphysik, erweitert diese jedoch um Vorhersagen für die Massen der Higgs– und Eich–Bosonen in klassischer Näherung.

English version of the Abstract:

Starting with a Hilbert space endowed with a representation of a skew–adjoint Lie algebra and an action of a generalized Dirac operator, we develop a mathematical concept towards gauge field theories. This concept shares common features with the non–commutative geometry à la Connes / Lott, differs from that, however, by the implementation of skew–adjoint Lie algebras instead of associative $*$–algebras. Taking as the Lie algebra the tensor product of the functions algebra and an appropriate matrix Lie algebra, together with physically motivated representations on the fermionic Hilbert space and a combination of the classical Dirac operator and fermionic mass matrices as the generalized Dirac operator, we recover prominent models of particle physics, extended by tree–level predictions for the masses of Higgs and gauge bosons.

Contents

1 Introduction

As the thesis is very long and contains numerous technical formulae, there exists the danger that the reader worries about local problems and loses the overview over the whole picture. To minimize this danger I decided to include this section on the physical motivation, the main ideas and the physical results. Moreover, at some places in the main part there occur comments in *italics*, which should inform the reader what is going on.

This Introduction is a brief tour through the thesis, without technical subtleties and proofs for the statements. The reader who wishes to know why certain claims are true and how the construction works in detail will (hopefully) find the answer in the main part. To ease a selected search I give at the end of this Introduction a description of the contents.

1.1 Physical Motivation

We would like to construct (the classical action of) gauge field theories on a space–time manifold X with trivial topology out of the following input data:

1) The unitary matrix Lie group G and its associated group $\mathscr{G} = C^\infty(X) \otimes G$ of local gauge transformations. Here, $C^\infty(X)$ denotes the algebra of real–valued smooth functions on X.
2) Chiral fermions ψ transforming under a representation $\tilde{\pi}_0$ of G. The induced representation of the gauge group $\mathscr{G}$ is $\tilde{\pi} = \mathrm{id} \otimes \tilde{\pi}_0$.
3) The fermionic mass matrix $\widetilde{\mathcal{M}}$, i.e. fermion masses plus generalized Kobayashi–Maskawa matrices.
4) Possibly the spontaneous symmetry breaking pattern of G.

Let us comment on these data. It is common knowledge that the free Dirac action for fermions,

$$S_F^{\text{free}} = \int_X dx\, \psi^*(\mathsf{D} + \widetilde{\mathcal{M}})\psi \,, \tag{1.1}$$

is not gauge invariant. In this equation, D is the free Dirac operator and dx the volume form on X. First, the kinetic term $\psi^*\mathsf{D}\psi$ of the Dirac Lagrangian is not gauge invariant, because $\tilde{\pi}(\mathscr{G})$ does not commute with D. Usually, one restores gauge invariance by adding gauge fields A minimally coupled to the fermions. The gauge field A and its action on ψ are determined by the condition that there exist transformations of A under $\mathscr{G}$ that compensate the disturbing part of the transformation of $\psi^*\mathsf{D}\psi$. Second, if the action of only a subgroup $\mathscr{G}_0$ of $\mathscr{G}$ commutes with $\widetilde{\mathcal{M}}$, then the mass term $\psi^*\widetilde{\mathcal{M}}\psi$ of the Dirac Lagrangian is not gauge invariant. In this case, one restores gauge invariance by extending the fermionic mass matrix to Higgs fields $\widetilde{\mathcal{M}} + \Phi$ with appropriate transformation behavior. Thus, the gauge invariant fermionic action can be written symbolically (i.e. up to signs and constants of the order one) as

$$S_F^{\text{inv}} = \int_X dx\, \psi^*(\mathsf{D} + \widetilde{\mathcal{M}} + \mathrm{A} + \Phi)\psi \,. \tag{1.2}$$

Moreover, one wishes to have a dynamics for the fields A and Φ. This is achieved by adding the free bosonic action

$$S_B^{\text{free}} = \int_X dx \left(\langle \mathbf{d}\mathrm{A}, \mathbf{d}\mathrm{A} \rangle_2 + \langle \mathbf{d}(\Phi + \widetilde{\mathcal{M}}), \mathbf{d}(\Phi + \widetilde{\mathcal{M}}) \rangle_1 \right) , \tag{1.3}$$

where $\langle\ ,\ \rangle_2$ and $\langle\ ,\ \rangle_1$ are appropriate scalar products. However, the action S_B^{free} is not gauge invariant, one has to add interaction terms for A and Φ. Moreover, the vacuum expectation value of $\Phi + \widetilde{\mathcal{M}}$ must be just the mass matrix $\widetilde{\mathcal{M}}$ in order to reproduce the correct fermionic sector. This is achieved by adding quartic interaction terms $V(\Phi + \widetilde{\mathcal{M}})$ such that $\Phi + \widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}$ is a local minimum of $V(\Phi + \widetilde{\mathcal{M}})$. Here, one has to implement the desired spontaneous symmetry breaking scheme 4), which in some gauge theories is already determined by the fermionic mass matrix $\widetilde{\mathcal{M}}$. However, in extended theories, one may need supplementary information on the spontaneous symmetry breaking scheme that is not contained in $\widetilde{\mathcal{M}}$. In summary, the invariant bosonic action has the symbolic form

$$\begin{aligned} S_B^{\text{inv}} = \int_X dx \, \big(& \langle \mathbf{d}\mathrm{A} + \mathrm{A}^2, \mathbf{d}\mathrm{A} + \mathrm{A}^2 \rangle_2 \\ & + \langle (\mathbf{d} + \mathrm{A})(\Phi + \widetilde{\mathcal{M}}), (\mathbf{d} + \mathrm{A})(\Phi + \widetilde{\mathcal{M}}) \rangle_1 + V(\Phi + \widetilde{\mathcal{M}}) \big) . \end{aligned} \tag{1.4}$$

We see that our input data 1) to 4) should suffice to reconstruct a complete classical gauge field theory. In particular, the fermionic sector determines candidates for the bosonic configuration space. Of course, the actions (1.2) and (1.4) are not unique, but we can fix much of the ambiguity by a minimal choice of A and Φ.

Usually, the above construction scheme is carried out more or less by hand. This is not difficult, for example, in the case of the standard model. However, in Grand Unified Theories with very large Higgs multiplets this is a highly non–trivial puzzle. One may wish to have a machinery at disposal which is able to do this work. This machinery should consist of an algorithm which has to be fed with the data 1) to 4) as the input and which returns the desired action, in particular, the Higgs multiplets and the Higgs potential. My PhD thesis is one possible description of such a machinery, which even does much more: It also returns tree–level predictions for the masses of the gauge fields and the Higgs fields.

An idea how to find this machinery is caused by the following observation [48]: The gauge field A is a vector field and the Higgs field Φ a scalar field. From that point of view, both are completely different objects. However, in the above sketch they play precisely the same rôle. Both A and Φ occur via minimal coupling in the fermionic action (1.2) and restore in this way the gauge invariance. Both have the same type of kinetic Lagrangians (1.3). Both occur as fourth order polynomials in the bosonic action (1.4). Moreover, also D and $\widetilde{\mathcal{M}}$ play the same rôle. All that may be an accident. But accidents have often inspired new theories. It might be promising to search for a new type of mathematics that deals with vector and scalar fields in the same way. Such mathematics does already exist in form of Alain Connes' non–commutative geometry [17]!

1.2 Non–Commutative Geometry

1.2.1 General Remarks

The evolution of non–commutative topology started with Gel'fands discovery that the unital C^*–algebra $C(X)$ of continuous functions over a compact manifold X contains all information about that manifold: Given $C(X)$ one can reconstruct the manifold X (up to homeomorphisms) as the set of characters. In the other direction, each commutative unital C^*–algebra is isomorphic to $C(X)$ for a certain compact manifold X. This language was transcribed to the case that the C^*–algebra is not commutative, and one considers general C^*–algebras as function algebras over "non–commutative manifolds". This programme, to dualize geometric or topological objects and to deform them within the dual picture, has been very successful. It led for instance to algebraic K–theory [5] and quantum groups [25, 63].

1.2.2 The Connes–Lott Prescription

Gel'fands theorem establishes the duality between the function algebra $C(X)$ and the topology of X. The discovery of Connes [17, 20] was that, taking in addition the Dirac operator acting on the spinor Hilbert space, one can also recover the metric properties of X. It is possible to reconstruct the distance between two points and the de Rham complex. Formalizing this method, Connes introduced the basic object of non–commutative geometry, the K–cycle or[1] spectral triple:

Definition 1. *A K–cycle $(\mathcal{A}, h, D, \pi, \Gamma)$ over a unital associative $*$–algebra $\mathcal{A}$ is given by*

i) *an involutive representation π of $\mathcal{A}$ in the algebra $\mathscr{B}(h)$ of bounded operators on a Hilbert space h,*

ii) *a (possibly unbounded) selfadjoint operator D on h such that $(\mathbb{1}_{\mathscr{B}(h)} + D^2)^{-1}$ is compact and for all $a \in \mathcal{A}$ there is $[D, \pi(a)] \in \mathscr{B}(h)$.*

The K–cycle is called even iff in addition there is a selfadjoint operator Γ on h, fulfilling $\Gamma^2 = \mathbb{1}_{\mathscr{B}(h)}$, $\Gamma D + D\Gamma = 0$ and $\Gamma\pi(a) - \pi(a)\Gamma = 0$, for all $a \in \mathcal{A}$.

Non–commutative geometry (NCG) as sketched above seems to be perfectly adapted to the setting 1) to 4): For technical reasons one first has to pass from the space–time manifold to a compact Euclidian spin manifold X. Then, the fermions ψ constitute the Hilbert space h. Next, one chooses the selfadjoint operator D of Definition 1 to be equal to $\mathsf{D} + \widetilde{\mathcal{M}}$ on physical fermions ψ. A matrix algebra $\mathcal{A}_M$ is chosen in such a way that the gauge group $\mathscr{G} = C^\infty(X) \otimes G$ is isomorphic to the group of unitary elements of the algebra $\mathcal{A} = C^\infty(X) \otimes \mathcal{A}_M$. The action $\pi = \mathrm{id} \otimes \pi_0$ of $\mathcal{A} = C^\infty(X) \otimes \mathcal{A}_M$ on h is the extension[2] of the group representation $\tilde{\pi} = \mathrm{id} \otimes \tilde{\pi}_0$ of $\mathscr{G} = C^\infty(X) \otimes G$ on the fermions ψ. At the very end, one returns to an indefinite metric by a Wick rotation. Chiral fermions are obtained by means of a chirality condition via the operator Γ.

[1] We prefer the ancient notation 'K–cycle'.

[2] Provided that this is possible!

To any K–cycle $(\mathcal{A}, h, D, \pi, \Gamma)$ there is canonically associated a differential algebra $\Omega_D^*\mathcal{A}$: One considers the universal graded differential algebra $\Omega^*\mathcal{A}$ over the algebra $\mathcal{A}$ of the K–cycle,

$$\Omega^*\mathcal{A} = \bigoplus_{n=0}^{\infty} \Omega^n\mathcal{A} \;, \qquad \Omega^n\mathcal{A} = \big\{ \textstyle\sum_\alpha a_\alpha^0\, da_\alpha^1\, da_\alpha^2 \dots da_\alpha^n \big\} \;, \tag{1.5}$$

where d is the universal differential and $a_\alpha^i \in \mathcal{A}$. In particular, $\Omega^0\mathcal{A} \cong \mathcal{A}$. One defines a linear representation π of $\Omega^*\mathcal{A}$ on the Hilbert space h by [62]

$$\pi(a_0\, da_1\, da_2 \dots da_n) := \pi(a_0) \cdot [-\mathrm{i}D, \pi(a_1)] \cdot [-\mathrm{i}D, \pi(a_2)] \cdots [-\mathrm{i}D, \pi(a_n)] \;. \tag{1.6}$$

One remarks the defect that $\pi(\Omega^*\mathcal{A})$ is not a differential algebra. Fortunately, this defect can be repaired, and the canonical graded differential algebra is

$$\Omega_D^*\mathcal{A} = \bigoplus_{n=0}^{\infty} \Omega_D^n\mathcal{A} \;, \qquad \begin{aligned} \Omega_D^n\mathcal{A} &:= \Omega^n\mathcal{A} \,/\, ((\ker\pi + d\ker\pi) \cap \Omega^n\mathcal{A}) \\ &\cong \pi(\Omega^n\mathcal{A}) \,/\, \pi(d\ker\pi \cap \Omega^n\mathcal{A}) \;. \end{aligned} \tag{1.7}$$

For the physically interesting case of even K–cycles over a subalgebra of $C^\infty(X)_{\mathbb{C}} \otimes \mathrm{M}_F\mathbb{C}$ and generalized Dirac operators of the form $D = \mathsf{D} \otimes \mathbb{1}_F + \gamma^5 \otimes \mathcal{M}$, a generally applicable construction of $\Omega_D^*\mathcal{A}$ has been given in [37]. The non–commutative gauge potential is an element of $\Omega_D^1\mathcal{A}$ and the field strength an element of $\Omega_D^2\mathcal{A}$. Using invariant scalar products one defines bosonic and fermionic actions [17, 20]. A further improvement is a new spectral action principle [1,2,10–12,36,61] that gives a coupling of the Yang–Mills (–Higgs) action to Einstein plus Weyl gravity. For an NCG–treatment of gravity see [15, 18, 38, 41].

1.2.3 Application to the Standard Model

This NCG–prescription has proved very successful in reformulating the standard model. There exists an "old scheme" initiated by Connes and Lott in [20], see also [17, 33, 34, 39, 40, 42–44, 59, 62], and a "new scheme" based upon real structures introduced by Connes in [19], see also [9–12, 35, 36, 48] for the application to model building. The algebra $\mathcal{A}$ and its group of unitary elements $\mathcal{U}(\mathcal{A})$ are given by

$$\begin{aligned} \mathcal{A}_{\mathrm{old}} &= C^\infty(X) \otimes \big((\mathbb{H} \oplus \mathbb{C}) \oplus (\mathrm{M}_3\mathbb{C} \oplus \mathbb{C})\big) \;, \\ \mathcal{U}(\mathcal{A}_{\mathrm{old}}) &= C^\infty(X) \otimes (\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(3) \times \mathrm{U}(1)) \;, \\ \mathcal{A}_{\mathrm{new}} &= C^\infty(X) \otimes (\mathbb{H} \oplus \mathbb{C} \oplus \mathrm{M}_3\mathbb{C}) \;, \\ \mathcal{U}(\mathcal{A}_{\mathrm{new}}) &= C^\infty(X) \otimes (\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(3)) \;. \end{aligned} \tag{1.8}$$

The additional U(1)–groups are eliminated by unimodularity conditions. The most important improvement compared with the usual formulation of the standard model is that the non–commutative gauge potential contains both the $\mathrm{su}(3) \oplus \mathrm{su}(2) \oplus \mathrm{u}(1)$

Yang–Mills fields and the complex Higgs doublet. Moreover, the bosonic action contains the Yang–Mills Lagrangian, the covariant derivatives of the Higgs fields and the Higgs potential in a unified form. The fermionic action unifies the gauge field couplings with the Yukawa–couplings. Numerically, one gets a very promising "fuzzy" relation between the mass of the W boson and the mass of the top quark, and the prediction for the mass of the Higgs field is compatible with LEP precision experiments, see [9, 17, 19, 33–35, 40, 42–44, 48, 59, 62].

1.2.4 The Mainz–Marseille Model

There exists a different non–commutative geometric formulation of the standard model [21–23, 29–31, 57] elaborated by groups in Mainz and Marseille. This formulation leads to the same unification of the Yang–Mills and Higgs sectors in the bosonic and fermionic actions. The essential mathematical difference is the use of the graded Lie algebra $\Lambda^* \otimes \mathrm{su}(2|1)$ of differential form–valued matrices as the starting point instead of K–cycles and differential algebras constructed thereof in the Connes–Lott prescription. The essential physical difference is that the purely bosonic sector of the standard model can be formulated. This is in contrast to the Connes–Lott model, where the bosonic sector can only be reproduced if at least two generations of fermions occur in nature (which is the case, of course). The Mainz–Marseille model yields no relations between fermion and boson masses, but an interesting relation between the Cabibbo angle and quark masses can be obtained [21, 31].

The inseparable tie between bosons and fermions in the Connes–Lott model, which is responsible for relations between fermion and boson masses obtained in that model, has been criticized by the Mainz–Marseille group [54], mainly for two reasons: First, purely bosonic theories are mathematically interesting as well. Second, relations between fermion and boson masses do not survive the usual quantization procedure. However, there exist examples where parameter relations that are not stemming from a symmetry of the theory are respected on quantum level, see [52, 53, 60, 69]. Thus, our point of view is to consider the interpretation of the mass relations in the Connes–Lott model as a challenge for the future.

1.2.5 An Attempt to Bridge between Connes–Lott and Mainz–Marseille

One may ask whether there exists a link between the Connes–Lott prescription and the Mainz–Marseille approach. A construction of such a link was proposed in [50]. The idea is to consider finite projective modules $\mathcal{E}^n := \mathcal{E} \otimes_{\mathcal{A}} \Omega^n_D \mathcal{A}$ over the algebra $\mathcal{A}$ of the K–cycle, where $\mathcal{E}$ is a finite projective $\mathcal{A}$–module itself and $\Omega^*_D \mathcal{A}$ the differential algebra associated to the K–cycle. The algebra $\mathcal{H}^* := \bigoplus_{n=0}^{\infty} \mathcal{H}^n$, $\mathcal{H}^n := \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}^n)$, of homomorphisms of projective modules can be considered as a graded Lie algebra with derivation. One takes a K–cycle over the algebra $\mathcal{A} = C^\infty(X)_{\mathbb{C}} \otimes (\mathbb{C} \oplus \mathbb{C})$, which can be used to obtain the Salam–Weinberg model within the concept of finite projective modules. The construction of the differential algebra $\Omega^*_D \mathcal{A}$ proposed in [37] becomes so trivial in this case that a primitive matrix calculation [49] is successful as well. For an appropriate module $\mathcal{E}$ one can find a projection of the geometric objects of the module formulation of the Connes–Lott scheme to geometric objects used in the Mainz–Marseille model, see [50, 51]. It is, however, not possible to obtain the graded Lie algebra $\Lambda^* \otimes \mathrm{su}(2|1)$ as a homomorphic image of $\mathcal{H}^*$.

There has been an attempt [64] to construct the standard model using the graded Lie algebra $\mathcal{H}^*$ and ideas of both the Mainz–Marseille approach and the module formulation of the Connes–Lott scheme. However, one has to make certain assumptions which can not be explained within the scheme. The failure of that module construction of the standard model was a strong motivation to develop the calculus presented in this thesis.

1.2.6 Non–Commutative Geometry and Grand Unification

The overwhelming success of non–commutative geometry leads to the expectation that its application to other gauge field theories should be not difficult. However, if one follows the Connes–Lott prescription one runs into certain problems. It was shown in [46] that, besides the standard model, there are only two more or less realistic models which can be constructed within the above understanding of non–commutative geometry: the $\mathrm{SU}(4)_{PS}\times\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$–model and the $\mathrm{SU}(3)_C\times\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\times\mathrm{U}(1)_{B-L}$–model. However, if one additionally demands a real structure [19] for the K–cycle, then also these two models are ruled out. The only more or less realistic physical model that is compatible with the most elegant prescription of non–commutative geometry is the standard model! It is certainly to early to judge from experimental results whether the standard model is correct or not. At least there exist good reasons [45] why one could be interested in Grand Unified Theories (GUT's): Unified theories explain the quantization of electric charge, yield a fairly well prediction for the Weinberg angle, explain the convergence of running coupling constants at high energies, include massive neutrinos to solve the solar neutrino problem, produce the observed baryon asymmetry of the universe, etc. Unfortunately, the results of [46] imply that one needs additional structures or different methods for a formulation of these models within non–commutative geometry.

The perhaps most successful NCG–approach towards Grand Unification was proposed by Chamseddine, Felder and Fröhlich. In the SU(5)–model [13, 14], the authors start to construct an auxiliary K–cycle. Within this framework they construct the bosonic sector. Then they interpret some of these bosonic quantities as Lie algebra valued and consider Lie algebra representations on the physical Hilbert space to obtain the fermionic sector. This procedure is a systematic realization of the gauge theory construction programme set up at the beginning. However, an aesthetic shortcoming of that approach is the auxiliary character of the K–cycle, which of course is inevitable in view of [46]. The SO(10)–model [16] by Chamseddine and Fröhlich fits well[3] into

[3]Nevertheless, the use of Lie algebras instead of algebras could probably justify certain assumptions made in [16].

the NCG–scheme. The reason why this model was excluded in [46] is that only models possessing complex fundamental irreducible representations were admitted in that paper.

It turns out that only a slight modification of the Connes–Lott prescription enables the formulation of a large class of physical models without additional structures. The development of that formulation and its application to interesting physical models is the concern of this thesis. As far as I know, a construction of a Grand Unification model following the Mainz–Marseille approach does not exist. But we will see that our formulation uses similar algebraic objects (graded Lie algebras) as the Mainz–Marseille model. In this sense, it extends ideas of the Mainz–Marseille approach to Grand Unification. Though, our formulation has more or less the same physical behavior as the Connes–Lott model, in the sense that the bosonic sector is derived from the fermionic sector and mass relations between fermion and boson masses are obtained.

1.3 Non–Associative Geometry

Let us investigate why the most elegant prescription of non–commutative geometry is so restrictive to admissible models. The obstruction is the extension of the representations of the gauge group $\mathcal{G} = C^\infty(X) \otimes G$ to representations of the unital associative $*$–algebra $\mathcal{A} = C^\infty(X) \otimes \mathcal{A}_M$ containing $\mathcal{G}$ as the set of unitary elements. That $\tilde{\pi} = \mathrm{id} \otimes \tilde{\pi}_0$ is a representation of $\mathcal{G}$ on the Hilbert space h means that

$$\tilde{\pi}_0(g_1)\,\tilde{\pi}_0(g_2) = \tilde{\pi}_0(g_1 g_2)\,, \quad \forall g_1, g_2 \in G\,. \tag{1.9}$$

The representation $\tilde{\pi}_0$ of the matrix group G should coincide with the representation π_0 of the matrix algebra $\mathcal{A}_M$ on the subset $G \subset \mathcal{A}_M$,

$$\pi_0(g_1)\,\pi_0(g_2) = \pi_0(g_1 g_2)\,, \quad \forall g_1, g_2 \in G \subset \mathcal{A}_M\,. \tag{1.10}$$

It is perhaps not the problem to extend the multiplication rule (1.10) to the entire matrix algebra $\mathcal{A}_M$. The essential problem is that this extension must be compatible with linear operations,

$$\lambda_1 \pi_0(a_1) + \lambda_2 \pi_0(a_2) = \pi_0(\lambda_1 a_1 + \lambda_2 a_2)\,, \quad \forall a_1, a_2 \in \mathcal{A}_M\,,\ \forall \lambda_1, \lambda_2 \in \mathbb{R}\,. \tag{1.11}$$

Addition and multiplication by scalars are not defined on G, and the representation $\tilde{\pi}_0$ does not care whether it is linear or not. A priory, there are two types of irreducible representations that fulfil (1.11): the identity and – in the case of real algebras – the complex conjugation. In general, this is all what is possible. We see: The reason why the most elegant prescription of non–commutative geometry is so restrictive is that it is compatible only with linear representations of the matrix group. Most of the Grand Unified Theories are not of that type.

Fortunately, our observation also shows the way how to overcome the restriction: We have to linearize the matrix group! But linearization of a Lie group means to work within the tangent space at a fixed group element, for instance the unit element. The tangent space at the unit element is isomorphic to the Lie algebra $\mathfrak{a}$ of G. Thus, *the Lie algebra* $\mathfrak{g} = C^\infty(X) \otimes \mathfrak{a}$ *of the gauge group* $\mathscr{G} = C^\infty(X) \otimes G$ *is the correct object to use, not an algebra extending* $\mathscr{G}$. The linearized group multiplication is described by the commutator of Lie algebra elements. It is clear that the representation of a Lie group induces a representation of its Lie algebra. The point is that this Lie algebra representation is always linear.

In analogy to the procedure in non–commutative geometry we formalize our observation. We simply replace in Definition 1 the unital associative $*$–algebra $\mathcal{A}$ by a skew–adjoint Lie algebra $\mathfrak{g}$. The outcome can no longer be called a K–cycle; I propose the name "L–cycle", where the letter L stands for Lie (and it is the next letter in the alphabet). We also cannot keep the name non–commutative geometry, because a Lie bracket is always (anti–)commutative. I suggest the name "non–associative geometry", because – in general – the Lie bracket is not associative. However, it must be stressed that our approach can not be applied to general non–associative algebras. Thus, the title could be misleading, but any title carries the risk of wrong associations.

The point of departure in our approach is the following definition:

Definition 2. *An L–cycle* $(\mathfrak{g}, h, D, \pi, \Gamma)$ *over a skew–adjoint Lie algebra* $\mathfrak{g}$ *is given by*

i) *an involutive representation* π *of* $\mathfrak{g}$ *in the Lie algebra* $\mathscr{B}(h)$ *of bounded operators on a Hilbert space* h*, i.e.* $(\pi(a))^* = \pi(a^*) \equiv -\pi(a)$*, for any* $a \in \mathfrak{g}$,

ii) *a (possibly unbounded) selfadjoint operator* D *on* h *such that* $(\mathrm{id}_h + D^2)^{-1}$ *is compact and for all* $a \in \mathfrak{g}$ *there is* $[D, \pi(a)] \in \mathscr{B}(h)$*, where* id_h *denotes the identity on* h.

iii) *a selfadjoint operator* Γ *on* h*, fulfilling* $\Gamma^2 = \mathrm{id}_h$, $\Gamma D + D\Gamma = 0$ *and* $\Gamma\pi(a) - \pi(a)\Gamma = 0$*, for all* $a \in \mathfrak{g}$.

It seems obvious that the concept of non–associative geometry is perfectly adapted to the setting 1) to 4) at the beginning[4]: As in non–commutative geometry we start with the construction of the Euclidian gauge field theory. Again, the Euclidian fermions ψ constitute our Hilbert space h. For technical reasons it may sometimes be necessary to work with several copies of the fermions. The Lie algebra $\mathfrak{g} = C^\infty(X) \otimes \mathfrak{a}$ is simply the Lie algebra of the gauge group $\mathscr{G}$, up to possible modifications if U(1)–groups occur[4]. We assume that X has a trivial topology in order to avoid discussions of transition functions between different charts of the manifold. The Lie algebra representation $\pi = \mathrm{id} \otimes \hat{\pi}$ is just the differential $\tilde{\pi}_*$ of the group representation $\tilde{\pi} = \mathrm{id} \otimes \tilde{\pi}_0$. The selfadjoint operator D is chosen in such a way that on physical fermions it equals $\mathsf{D} + \widetilde{\mathcal{M}}$. The operator Γ represents the chirality properties of the fermions. Finally, one returns to Minkowski space by a Wick rotation and imposes a chirality condition for the fermions ψ by means of Γ.

[4]There can occur obstructions and modifications if Abelian Lie groups are present. In particular, a purely Abelian gauge field theory can be constructed only with partial success, see Section 4.1. In some cases, Abelian Lie algebras are automatically generated. If such a Lie algebra is desired, one can omit this part when deriving the Lie algebra $\mathfrak{g}$ out of $\mathscr{G}$.

The programme of non–associative geometry is clear: We "simply" have to transcribe the Connes–Lott prescription of non–commutative geometry to our case. However, this is not as easy as one probably expects. The associativity of the algebra and the existence of a unit element are very powerful tools. Without them we are forced to go long detours where non–commutative geometry uses short cuts.

1.4 The General Scheme

Now for the sketch of the construction in the general context, without relation to physical models. In analogy to the first step in non–commutative geometry we enlarge our Lie algebra $\mathfrak{g}$ to a universal graded differential Lie algebra $\Omega^*\mathfrak{g}$. One can imagine $\Omega^*\mathfrak{g}$ as the set of repeated graded commutators of $\mathfrak{g}$ and $d\mathfrak{g}$, where $d\mathfrak{g}$ is a second copy of $\mathfrak{g}$. Thus, elements $\omega \in \Omega^*\mathfrak{g}$ have the form

$$\omega = \sum_{\alpha, z \geq 0} [v_\alpha^z, [v_\alpha^{z-1}, [\dots, [v_\alpha^1, v_\alpha^0] \dots]]] \,, \quad \text{finite sum} \,, \tag{1.12}$$

where v_α^i either belongs to $\mathfrak{g}$ or $d\mathfrak{g}$. The vector space $\Omega^*\mathfrak{g}$ is $\mathbb{N}$–graded. The homogeneous element $[v^z, [v^{z-1}, [\dots, [v^1, v^0] \dots]]]$ belongs to $\Omega^n\mathfrak{g}$ iff n elements of $\{v^0, \dots, v^z\}$ belong to $d\mathfrak{g}$. The graded commutator $[\ ,\ ]$ is compatible with that grading structure; one has $[\Omega^k\mathfrak{g}, \Omega^l\mathfrak{g}] \subset \Omega^{k+l}\mathfrak{g}$. Moreover, $[\ ,\ ]$ respects the usual graded antisymmetry and the graded Jacobi identity. The symbol d is extended to a graded differential on $\Omega^*\mathfrak{g}$, it is nilpotent and obeys the graded Leibniz rule. The graded Lie algebra $\Omega^*\mathfrak{g}$ is universal in the following sense: Each graded differential Lie algebra generated by $\pi(\mathfrak{g})$ and $\mathrm{d}\pi(\mathfrak{g})$ can be obtained by factorization of $\Omega^*\mathfrak{g}$ with respect to a differential ideal. For instance, the information contained in an L–cycle determines uniquely such a differential ideal. Thus, there is a canonical graded differential Lie algebra $\Omega_D^*\mathfrak{g}$ associated to an L–cycle.

To find this differential Lie algebra, we represent $\Omega^*\mathfrak{g}$ on the Hilbert space h, using the data specified in the L–cycle. This representation extends the representation π of the L–cycle and is defined by

$$\begin{aligned} \pi(da) &= [-\mathrm{i}D, \pi(a)] \,, \\ \pi([\omega^k, \tilde{\omega}^l]) &= [\pi(\omega^k), \pi(\tilde{\omega}^l)]_g := \pi(\omega^k)\pi(\tilde{\omega}^l) - (-1)^{kl}\pi(\tilde{\omega}^l)\pi(\omega^k) \,, \end{aligned} \tag{1.13}$$

for $a \in \mathfrak{g}$, $\omega^k \in \Omega^k\mathfrak{g}$ and $\tilde{\omega}^l \in \Omega^l\mathfrak{g}$. Here, it is essential to have the grading operator Γ, which detects the correct sign for $(-1)^{kl}$.

As one expects from non–commutative geometry, the representation π does not transport the differential d on $\Omega^*\mathfrak{g}$ to a differential on $\pi(\Omega^*\mathfrak{g})$. To cure this, we use the same trick as in non–commutative geometry. One shows that

$$\mathcal{J}^*\mathfrak{g} = \ker\pi + d\ker\pi \subset \Omega^*\mathfrak{g} \tag{1.14}$$

is a graded differential ideal of $\Omega^*\mathfrak{g}$. Factorizing out the "junk" $\mathcal{J}^*\mathfrak{g}$ we obtain the graded differential Lie algebra $\Omega^*_D\mathfrak{g}$,

$$\Omega^*_D\mathfrak{g} = \bigoplus_{n=0}^{\infty} \Omega^n_D\mathfrak{g} \;, \qquad \Omega^n_D\mathfrak{g} = \frac{\Omega^n\mathfrak{g}}{\mathcal{J}^n\mathfrak{g}} \cong \frac{\pi(\Omega^n\mathfrak{g})}{\pi(\mathcal{J}^n\mathfrak{g})} \;. \tag{1.15}$$

The differential and the commutator are defined as usual for equivalence classes.

It is extremely useful to introduce a linear map σ from $\Omega^*\mathfrak{g}$ to (possibly unbounded) operators on h. The operator σ is odd with respect to the $\mathbb{Z}_2$–grading and is within the same notations as before defined by

$$\begin{aligned} \sigma(a) = 0 \;, \qquad \sigma(da) = [D^2, \pi(a)] \;, \\ \sigma([\omega^k, \tilde{\omega}^l]) = [\sigma(\omega^k), \pi(\tilde{\omega}^l)]_g + (-1)^k [\pi(\omega^k), \sigma(\tilde{\omega}^l)]_g \;. \end{aligned} \tag{1.16}$$

The importance of the map σ is that it measures the defect if one represents the universal differential d by graded commutators with $-\mathrm{i}D$,

$$\pi(d\omega^k) = [-\mathrm{i}D, \pi(\omega^k)]_g + \sigma(\omega^k) \;, \quad \omega^k \in \Omega^k\mathfrak{g} \;. \tag{1.17}$$

In particular, taking $\omega^k \in \ker\pi$, we get

$$\pi(\mathcal{J}^{k+1}\mathfrak{g}) = \{\; \sigma(\omega^k) \;, \;\; \omega^k \in \Omega^k\mathfrak{g} \cap \ker\pi \;\} \;. \tag{1.18}$$

This characterization of $\pi(\mathcal{J}^*\mathfrak{g})$ is especially convenient, because $\sigma(\omega^k)$ is derived successively from lower degrees, see (1.16). Indeed, this is the way how we can eventually compute $\pi(\mathcal{J}^*\mathfrak{g})$: The real problem is to find $\sigma(\Omega^1\mathfrak{g})$. Then we derive for $k \geq 2$ by induction a formula for $\sigma(\omega^k)$ for given $\pi(\omega^k)$. Clearly, $\sigma(\omega^k)$ is not uniquely defined by $\pi(\omega^k)$, and this ambiguity is nothing but $\pi(\mathcal{J}^{k+1}\mathfrak{g})$. However, the explicit realization of this line is not done within a couple of pages. We also point out that, once knowing $\sigma(\omega^k)$, formula (1.17) provides the explicit differentiation rule for elements of $\Omega^*_D\mathfrak{g}$.

In non–commutative geometry, all work is done at this point. There, the connection form is simply an element of $\Omega^1_D\mathcal{A}$ and the curvature an element of $\Omega^2_D\mathcal{A}$. It is straightforward to write down the fermionic and bosonic actions. In non–associative geometry, the situation is different. If one tries to find a reasonable definition for the connection (the covariant derivative), one encounters more freedom than one expects. Moreover, it is not possible to describe gauge field theories containing U(1)–groups if one takes $\Omega^1_D\mathfrak{g}$–valued connection forms. Therefore, an additional structure is necessary: Not the graded differential Lie algebra $\Omega^*_D\mathfrak{g}$ is the correct space where the connection form and the curvature live, but the space of certain graded Lie endomorphisms of $\Omega^*_D\mathfrak{g}$. This is not completely unreasonable. For instance, connections within the framework of finite projective modules [50] are of a similar type. Formally, we introduce the space $\mathcal{H}^*\mathfrak{g} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}^n\mathfrak{g}$ of certain graded Lie homomorphisms of $\pi(\Omega^*\mathfrak{g})$. The space $\mathcal{H}^n\mathfrak{g}$ consists of linear (possibly unbounded) operators on h of $\mathbb{Z}_2$–degree $n \mod 2$, which raise the $\mathbb{N}$–degree of $\pi(\Omega^*\mathfrak{g})$ and $\pi(\mathcal{J}^*\mathfrak{g})$ by n,

$$[\mathcal{H}^n\mathfrak{g}, \pi(\Omega^k\mathfrak{g})]_g \subset \pi(\Omega^{k+n}\mathfrak{g}) \;, \qquad [\mathcal{H}^n\mathfrak{g}, \pi(\mathcal{J}^k\mathfrak{g})]_g \subset \pi(\mathcal{J}^{k+n}\mathfrak{g}) \;. \tag{1.19}$$

Factorizing $\mathcal{H}^*\mathfrak{g}$ with respect to the graded centratizer $\tilde{\mathfrak{c}}^*\mathfrak{g}$ of $\pi(\Omega^*\mathfrak{g})$ and the ideal $\pi(\mathcal{J}^*\mathfrak{g})$, we obtain the graded Lie algebra

$$\hat{\mathcal{H}}^*\mathfrak{g} := \bigoplus_{n\in\mathbb{N}} \hat{\mathcal{H}}^n\mathfrak{g} \;, \qquad \hat{\mathcal{H}}^n\mathfrak{g} := \mathcal{H}^n\mathfrak{g} \,/\, (\pi(\mathcal{J}^n\mathfrak{g}) + \tilde{\mathfrak{c}}^n\mathfrak{g}) \;. \tag{1.20}$$

The differential and the commutator on $\hat{\mathcal{H}}^*\mathfrak{g}$ are defined as usual for dual spaces: via the graded Leibniz rule and the graded Jacobi identity. From our definitions it is clear that

$$\pi(\Omega^n\mathfrak{g}) \subset \mathcal{H}^n\mathfrak{g} \;, \qquad \Omega^n_D\mathfrak{g} \subset \hat{\mathcal{H}}^n\mathfrak{g} \;. \tag{1.21}$$

In some sense, this framework is an extension of the primary spaces $\pi(\Omega^*\mathfrak{g})$ and $\Omega^*_D\mathfrak{g}$.

The formal definition of a connection on L–cycles will be given in the main part. Here, we shall only quote the result: A connection ∇ acting on $\Omega^*_D\mathfrak{g}$ is closely related to the covariant derivative ∇_h acting on the Hilbert space h. The general form of these two objects is

$$\nabla_h = -\mathrm{i}D + \rho \;, \quad \nabla = d + [\hat{\rho}\,,\,.\,]_g \;, \quad \rho \in \mathcal{H}^1\mathfrak{g} \;, \;\; \hat{\rho} := \rho + \tilde{\mathfrak{c}}^1\mathfrak{g} \in \hat{\mathcal{H}}^1\mathfrak{g} \;. \tag{1.22}$$

The Lie homomorphism ρ is called the connection form (gauge potential). The curvature (field strength) of the connection ∇ is

$$\nabla^2 = [\theta,\,.\,] \;, \qquad \theta = d\hat{\rho} + \tfrac{1}{2}\{\hat{\rho},\hat{\rho}\} \in \hat{\mathcal{H}}^2\mathfrak{g} \;. \tag{1.23}$$

We see that our formulae look very similar to what one knows from non–commutative geometry or classical gauge field theory. However, we have no control over the space of connections in that general context. All what we know is that elements of $\Omega^1_D\mathfrak{g}$ are possible connection forms, but it is completely unclear what else. Also the operations $d\hat{\rho}$ and $\{\hat{\rho},\hat{\rho}\}$ are difficult to perform, because they are only indirectly defined. It is a visible complication compared with non–commutative geometry to find not only $\Omega^*_D\mathfrak{g}$ but also $\hat{\mathcal{H}}^*\mathfrak{g}$ (up to second degree).

The group $\mathcal{U}(\mathfrak{g})$ obtained via the exponential mapping of a neighbourhood of the zero element of $\mathcal{H}^0\mathfrak{g}$ plays the rôle of a gauge group in our approach. Comparing for a physical model this group with the original gauge group $\mathscr{G}$ we had started with, we see that the global topology of $\mathscr{G}$ cannot always be reconstructed. But for most physical applications it suffices to know the gauge group locally. One can define an adjoint representation Ad of $\mathcal{U}(\mathfrak{g})$ on $\Omega^*_D\mathfrak{g}$. Local gauge transformations are given by

$$\begin{aligned} &\nabla \mapsto \mathrm{Ad}_u\,\nabla\,\mathrm{Ad}_{u^*} \;, && \nabla_h \mapsto u\nabla_h u^* \;, && \\ &\rho \mapsto u du^{-1} + u\rho u^* \;, && \theta \mapsto \mathrm{Ad}_u(\theta) \;, && \psi \mapsto u\psi \;, \end{aligned} \tag{1.24}$$

where $u \in \mathcal{U}(\mathfrak{g})$ and $\psi \in h$. The bosonic and fermionic actions are defined in the same way as in non–commutative geometry. Using the Dixmier trace Tr_ω we define the bosonic action

$$S_B(\nabla) := \min_{j^2 \in \tilde{\mathfrak{c}}^2\mathfrak{g} + \pi(\mathcal{J}^2\mathfrak{g})} \mathrm{Tr}_\omega((\theta_0 + j^2)^2\,|D|^{-\mathrm{d}}) \;, \tag{1.25}$$

where $\theta_0 \in \mathcal{H}^2\mathfrak{g}$ is any representative of θ. For the fermionic action we use the scalar product on the Hilbert space:

$$S_F(\psi, \nabla_h) := \langle \psi, \mathrm{i}\nabla_h \psi \rangle_h \,, \quad \psi \in h \,. \tag{1.26}$$

Both S_B and S_F are invariant under gauge transformations (1.24).

1.5 Functions $\otimes$ Matrices

In physical applications one is especially interested in the case that the Lie algebra $\mathfrak{g}$ is the tensor product of the algebra of functions on the space–time manifold X and a matrix Lie algebra $\mathfrak{a}$. We are able to handle this situation. However, it turns out that we must impose restrictions on the matrix Lie algebra. If $\mathfrak{a}$ is semisimple then there are no problems at all. The situation that $\mathfrak{a}$ is Abelian can not be satisfactorily treated. We are able to deal with L–cycles over the Lie algebra

$$\mathfrak{g} = C^\infty(X) \otimes (\mathfrak{a}' \oplus \mathfrak{a}'') \,, \tag{1.27}$$

where $C^\infty(X)$ is the algebra of real smooth functions over the space–time manifold, $\mathfrak{a}'$ is a semisimple Lie algebra and $\mathfrak{a}''$ an optional Abelian Lie algebra. For $\mathfrak{a}''$ we have to impose constraints on the representations. Remarkably, for the models I considered so far, the u(1)–representations realized in nature are admissible. The Hilbert space is

$$h = L^2(X,S) \otimes \mathbb{C}^F \,, \tag{1.28}$$

where $L^2(X,S)$ is the Hilbert space of square integrable sections on the spinor bundle over X. The representation π of $\mathfrak{g}$ on h is given by

$$\pi = \mathrm{id} \otimes \hat{\pi} \,, \tag{1.29}$$

where $\hat{\pi}$ is a representation of $\mathfrak{a}' \oplus \mathfrak{a}''$ on $\mathbb{C}^F$. The selfadjoint operator D of the L–cycle is

$$D = \mathsf{D} \otimes \mathbb{1}_F + \gamma \otimes \mathcal{M} \,, \tag{1.30}$$

where D and γ are the Dirac operator of the spin connection and the chirality operator on $L^2(X,S)$. We have $\gamma = \gamma^5$ in four dimensions. Moreover, $\mathcal{M}$ is a symmetrical complex $F \times F$–matrix such that there exists a symmetrical $F \times F$–matrix $\hat{\Gamma}$, fulfilling $\hat{\Gamma}^2 = \mathbb{1}_F$, $\mathcal{M}\hat{\Gamma} = -\hat{\Gamma}\mathcal{M}$ and $\hat{\pi}(a)\hat{\Gamma} = \hat{\Gamma}\hat{\pi}(a)$, for all $a \in \mathfrak{a}$. Then, the chirality operator is

$$\Gamma = \gamma \otimes \hat{\Gamma} \,. \tag{1.31}$$

As mentioned before, the representation $\hat{\pi}(\mathfrak{a}'')$ is not arbitrary, we have a constraint relation between $\mathcal{M}$ and $\hat{\pi}(\mathfrak{a}'')$. Observe that the tuple $(\mathfrak{a}, \mathbb{C}^F, \mathcal{M}, \hat{\pi}, \hat{\Gamma})$ itself forms an L–cycle. In some sense, the L–cycle $(\mathfrak{g}, h, D, \pi, \Gamma)$ is the product of the Dirac K–cycle $(C^\infty(X), L^2(X,S), \mathsf{D}, \gamma)$ with the matrix L–cycle $(\mathfrak{a}, \mathbb{C}^F, \mathcal{M}, \hat{\pi}, \hat{\Gamma})$.

One may ask how the spaces $\pi(\Omega^*\mathfrak{g}), \pi(\jmath^*\mathfrak{g})$ and $\Omega^*_D\mathfrak{g}$ depend on the geometric objects of the underlying Dirac K–cycle and the matrix L–cycle[5]. It turns out that $\pi(\Omega^*\mathfrak{g}), \pi(\jmath^*\mathfrak{g})$ and $\Omega^*_D\mathfrak{g}$ can be universally written as a sum of tensor products of differential forms of homogeneous degree (partly coboundaries only) with certain commutators and anticommutators of homogeneous subspaces of $\hat{\pi}(\Omega^*\mathfrak{a})$ and $\hat{\pi}(\jmath^*\mathfrak{a})$. Thus, if one has complete knowledge of $\hat{\pi}(\Omega^*\mathfrak{a})$ and $\hat{\pi}(\jmath^*\mathfrak{a})$, then also $\pi(\Omega^*\mathfrak{g}), \pi(\jmath^*\mathfrak{g})$ and $\Omega^*_D\mathfrak{g}$ are known. The formulae of lowest degree read:

$$\begin{aligned}
\pi(\Omega^0\mathfrak{g}) &= \Lambda^0 \otimes (\hat{\pi}(\mathfrak{a}') \oplus \hat{\pi}(\mathfrak{a}''))\,, \\
\pi(\Omega^1\mathfrak{g}) &= (\Lambda^1 \otimes \hat{\pi}(\mathfrak{a}')) \oplus (B^1 \otimes \hat{\pi}(\mathfrak{a}'')) \oplus (\Lambda^0\gamma \otimes \hat{\pi}(\Omega^1\mathfrak{a}))\,, \qquad (1.32)\\
\pi(\Omega^2\mathfrak{g}) &= (\Lambda^2 \otimes \hat{\pi}(\mathfrak{a}')) \oplus (\Lambda^1\gamma \otimes \hat{\pi}(\Omega^1\mathfrak{a})) \oplus (\Lambda^0 \otimes (\hat{\pi}(\Omega^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}))\,,
\end{aligned}$$

$$\pi(\jmath^0\mathfrak{g}) = 0\,, \qquad \pi(\jmath^1\mathfrak{g}) = 0\,, \qquad \pi(\jmath^2\mathfrak{g}) = \Lambda^0 \otimes (\hat{\pi}(\jmath^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\})\,, \qquad (1.33)$$

$$\begin{aligned}
\Omega^0_D\mathfrak{g} &= \pi(\Omega^0\mathfrak{g})\,, \qquad \Omega^1_D\mathfrak{g} = \pi(\Omega^1\mathfrak{g})\,, \\
\Omega^2_D\mathfrak{g} &= (\Lambda^2 \otimes \hat{\pi}(\mathfrak{a}')) \oplus (\Lambda^1\gamma \otimes \hat{\pi}(\Omega^1\mathfrak{a})) \qquad (1.34)\\
&\quad \oplus (\Lambda^0 \otimes \big((\hat{\pi}(\Omega^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}) \mod (\hat{\pi}(\jmath^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\})\big))\,.
\end{aligned}$$

Here, Λ^k is the space of k–differential forms, $B^1 = \mathbf{d}\Lambda^0 \subset \Lambda^1$ the space of 1–coboundaries and

$$\{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} = \{ \textstyle\sum_\alpha \{\hat{\pi}(a_\alpha), \hat{\pi}(\tilde{a}_\alpha)\}\,,\ a_\alpha, \tilde{a}_\alpha \in \mathfrak{a}\,,\ \text{finite sum} \}\,. \qquad (1.35)$$

For higher degrees, the formulae for the matrix part belonging to a fixed space of k–differential forms become more and more complicated. Corresponding formulae in non–commutative geometry are less difficult, because an associative algebra does not care, at which sites in the product $\omega_1 \circ \omega_2 \circ \ldots \circ \omega_n$ one inserts brackets distinguishing commutators and anticommutators. As it can be seen, the Abelian Lie algebra $\mathfrak{a}''$ plays a special rôle. For instance, if the connection form ρ belongs to $\Omega^1_D\mathfrak{g}$, then the field strength of a u(1)–gauge field is always zero. That u(1)–gauge fields can have a non–vanishing field strength in our theory is due to the extension of $\Omega^1_D\mathfrak{g}$ to $\hat{\mathcal{H}}^1\mathfrak{g}$.

An additional feature of L–cycles over functions ⊗ matrix Lie algebra is the possibility to consider local connections. For local connections, the connection form ρ commutes with functions. Therefore, it has the decomposition

$$\rho \in (\Lambda^1 \otimes \mathbb{r}^0\mathfrak{a}) \oplus (\Lambda^0\gamma \otimes \mathbb{r}^1\mathfrak{a})\,, \qquad (1.36)$$

where $\mathbb{r}^0\mathfrak{a}$ and $\mathbb{r}^1\mathfrak{a}$ are certain subspaces of $M_F\mathbb{C}$. The defining equations (1.19), decomposed according to their differential form degree, yield certain equations for commutators and anticommutators of $\mathbb{r}^0\mathfrak{a}$ and $\mathbb{r}^1\mathfrak{a}$ with $\hat{\pi}(\Omega^*\mathfrak{a})$ and $\hat{\pi}(\jmath^*\mathfrak{a})$. These equations and $\mathbb{Z}_2$–grading properties and involution identities make it possible to find the space of gauge potentials (1.36). Moreover, one also gets a decomposition for the

[5] The corresponding analysis for the product of the Dirac K–cycle with a matrix K–cycle was performed in [37].

ideal $\mathbb{J}^2\mathfrak{g} := \tilde{\mathbb{c}}^2\mathfrak{g} + \pi(\jmath^2\mathfrak{g})$ commuting with functions, which we need to write down the bosonic action (1.25):

$$\mathbb{J}^2\mathfrak{g} = (\Lambda^0 \otimes \mathbb{c}^2\mathfrak{a}) \oplus (\Lambda^1\gamma \otimes \mathbb{c}^1\mathfrak{a}) \oplus (\Lambda^2 \otimes (\mathbb{c}^0\mathfrak{a} + \hat{\pi}(\jmath^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\})) . \qquad (1.37)$$

Again, one finds certain equations between $\mathbb{c}^i\mathfrak{a}$ and $\hat{\pi}(\Omega^*\mathfrak{a})$ that make it possible to determine $\mathbb{J}^2\mathfrak{g}$. For the computation of the bosonic action one makes use of the fact that in the present situation one can express the Dixmier trace by a combination of the usual trace over the matrix structures (including gamma matrices) and integration over the space–time manifold.

1.6 Electrodynamics and Standard Model

One can try to formulate the chiral spinor electrodynamics within our approach. However, since the Lie algebra to use is purely Abelian, there occur certain problems. It is no problem to get the correct fermionic action. In particular, the photon has the usual properties and a non–vanishing classical curvature. Nevertheless, in our approach we get a vanishing curvature and, therefore, no bosonic action.

The reformulation of the standard model is more successful. The L–cycle is the direct transcription of the physical situation. Clearly, the Lie algebra to use is $C^\infty(X) \otimes (\mathrm{su}(3) \oplus \mathrm{su}(2) \oplus \mathrm{u}(1))$. We can formulate the standard model with or without right neutrinos. For a generic mass matrix, the space of gauge potentials is just the usual one. The formalism generates one complex Higgs doublet and a quartic potential for it. Three of its components are absorbed by the Higgs mechanism and give mass to the $W^\pm$ and Z bosons. One massive scalar Higgs field survives. In the same way as in non–commutative geometry we obtain tree–level predictions for all bosonic masses. For the simplest scalar product we find in the case that right neutrinos are included

$$m_W = \tfrac{1}{2} m_t \,, \qquad m_Z = m_W / \cos\theta_W \,, \qquad \sin^2\theta_W = \tfrac{3}{8} \,, \qquad m_H = \tfrac{3}{2} m_t \,. \qquad (1.38)$$

Without right neutrinos, the only modification is $m_H = \sqrt{\frac{43}{20}}m_t$. Here, m_t, m_W, m_Z, m_H are the masses of the top quark, the W bosons, the Z boson and the Higgs boson. The photon and the gluons remain massless. The Weinberg angle θ_W coincides with the SU(5)–GUT prediction. Moreover, we get the same coupling constants for the weak and strong interactions. In the Connes–Lott formulation of non–commutative geometry one uses the algebra $\mathcal{A} = C^\infty(X) \otimes (\mathrm{M}_3\mathbb{C} \oplus \mathbb{H} \oplus \mathbb{C})$ to derive the standard model, together with a rather complicated representation of $\mathcal{A}$. For the simplest scalar product, the numerical results are [44]

$$m_W = \tfrac{1}{2} m_t \,, \quad m_Z = m_W / \cos\theta_W \,, \quad \sin^2\theta_W = \tfrac{12}{29} \,, \quad m_H = \sqrt{\tfrac{69}{28}} m_t \approx 1.57\, m_t \,. \quad (1.39)$$

Thus, we see that the predictions from non–associative and non–commutative geometry do not differ very much.

1.7 The Flipped SU(5)×U(1)–Grand Unification Model

1.7.1 The Matrix L–Cycle

Whereas the construction of the standard model within our prescription is very short, the same work for the flipped SU(5)×U(1)–Grand Unification model consumes more than 70 pages. For the classical treatment of that model see [3,4,6,24] and for a Kaluza–Klein approach [56].

The matrix L–cycle is given by the following data: The matrix Lie algebra is $\mathfrak{a} = \mathrm{su}(5)$. Nevertheless, we will obtain an additional u(1)–gauge field and U(1)–gauge transformations due to the extension of $\pi(\Omega^1\mathfrak{g})$ to $\mathcal{H}^1\mathfrak{g}$. Remarkably, the representation of that u(1)–gauge field on the fermionic Hilbert space is unique and realized in nature[6]! The internal Hilbert space is $\mathbb{C}^{192}$. This means that we must deal with huge matrices, a problem which should not be underestimated. The strange number $192 = 4\cdot 48$ arises because there are 48 fermions in nature (including right neutrinos), and we need four copies of them: Two copies because we need particles and antiparticles in one representation (the SU(5) exchanges particles and antiparticles – proton decay!), and an additional doubling to include the essential grading operator. The 48 fermions occur in three generations, each generation contains 16 fermions. These 16 fermions are assigned to the su(5)–representations $\underline{10}, \underline{5}^*, \underline{1}$. Now, for $a \in \mathrm{su}(5)$ we define the representation $\hat{\pi}$ of the Lie algebra su(5) of our matrix L–cycle in terms of 48×48–block matrices

$$\hat{\pi}(a) = \left(\begin{array}{cc|cc} \hat{A} & 0 & 0 & 0 \\ 0 & \hat{A} & 0 & 0 \\ \hline 0 & 0 & \overline{\hat{A}} & 0 \\ 0 & 0 & 0 & \overline{\hat{A}} \end{array}\right) . \tag{1.40}$$

In terms of the decomposition $\mathbb{C}^{48} = (\underline{10} \oplus \underline{5}^* \oplus \underline{1}) \otimes \mathbb{C}^3$ we have

$$\hat{A} = \mathrm{diag}\left(\pi_{10}(a) \otimes \mathbb{1}_3\,,\ \overline{\pi_5(a)} \otimes \mathbb{1}_3\,,\ 0_3\right) . \tag{1.41}$$

Here, $\pi_5(a) = a$ is the adjoint representation $\underline{24}$ of su(5) and $\pi_{10}(a)$ the embedding of $\underline{24}$ into $\mathrm{End}(\underline{10}) = \underline{1} \oplus \underline{24} \oplus \underline{75}$. The fact that the su(5) representations are tensorized by $\mathbb{1}_3$ means that the gauge group does not distinguish between the three generations of fermions.

The mass matrix $\mathcal{M}$ of the L–cycle consists of two different contributions. The first one is diagonal and the other one off–diagonal in the sense of the indicated decompo-

[6] This is a purely algebraic result, for which I have no geometric interpretation. I suppose that this has something to do with anomaly–freedom of the model.

sition into two by two blocks in (1.40):

$$\mathcal{M} = \left(\begin{array}{cc|cc} 0 & \mathcal{M}_i & \mathcal{M}_f & 0 \\ \mathcal{M}_i^* & 0 & 0 & \mathcal{M}_f \\ \hline \mathcal{M}_f^* & 0 & 0 & \overline{\mathcal{M}}_i \\ 0 & \mathcal{M}_f^* & \mathcal{M}_i^T & 0 \end{array}\right) . \tag{1.42}$$

The 48×48–matrix $\mathcal{M}_f = \mathcal{M}_f^T$ is the fermionic mass matrix. A convenient picture is to imagine the two–two structure as the left–right decomposition. Since mass terms exchange left and right fermions, they must stand in the off–diagonal blocks. With this picture in mind it is not difficult to assign the 3×3–fermion mass matrices M_u, M_d, M_e, M_n, M_N to the 16×16–block matrix $\mathcal{M}_f$. Here, M_u is the mass matrix for the (u,c,t)–quark sector, M_d the mass matrix for the (d,s,b)–quark sector and M_e the mass matrix for the (e,μ,τ)–lepton sector. Moreover, M_n and M_N are Dirac and Majorana mass matrices for the (ν_e,ν_μ,ν_τ)–neutrino sector. These mass matrices include the fermion masses and generalized Kobayashi–Maskawa mixing angles. Mathematically, the sites where these generation matrices occur in $\mathcal{M}_f$ coincide with a combination of the representations $\underline{5}$, $\underline{45}$ and $\underline{50}$ of $\mathrm{su}(5)$. The relevant decomposition rules of tensor products are

$$\mathrm{Hom}(\underline{10}^*, \underline{10}) = \underline{5}^* \oplus \underline{45} \oplus \underline{50} , \quad \mathrm{Hom}(\underline{5}, \underline{10}) = \underline{5} \oplus \underline{45}^* , \quad \mathrm{Hom}(\underline{1}, \underline{5}^*) = \underline{5}^* . \tag{1.43}$$

Let n, n', m' be appropriate elements of $\underline{5}, \underline{45}^*, \underline{50}$, in this order. Then one has

$$\mathcal{M}_f := \begin{pmatrix} \mathrm{i}\pi_{10,10}(n)\otimes M_d + \mathrm{i}m'\otimes M_N & \mathrm{i}\pi_{10,5}(n)\otimes M_{\tilde{u}} + \mathrm{i}n'\otimes M_{\tilde{n}} & 0 \\ \mathrm{i}\pi_{10,5}(n)^T\otimes M_{\tilde{u}}^T + \mathrm{i}n'^T\otimes M_{\tilde{n}}^T & 0 & \mathrm{i}\pi_{5,1}(n)\otimes M_e \\ 0 & \mathrm{i}\pi_{5,1}(n)^T\otimes M_e^T & 0 \end{pmatrix} , \tag{1.44}$$

where $\pi_{10,10}(n)$ is the embedding of $n \in \underline{5}$ into $\mathrm{Hom}(\underline{10}^*, \underline{10})$, $\pi_{10,5}(n)$ the embedding of n into $\mathrm{Hom}(\underline{5}, \underline{10})$ and $\pi_{5,1}(n)$ the embedding of n into $\mathrm{Hom}(\underline{1}, \underline{5}^*)$. Moreover,

$$M_{\tilde{u}} = \tfrac{1}{4}(3M_u + M_n) , \qquad M_{\tilde{n}} = \tfrac{1}{4}(M_u - M_n) . \tag{1.45}$$

The block diagonal part $\mathcal{M}_i$ of $\mathcal{M}$ couples left–left and right–right sectors. Thus, it has no interpretation in terms of fermion masses. It is responsible for the desired spontaneous symmetry breaking pattern from $\mathrm{su}(5)\oplus\mathrm{u}(1)$ to $\mathrm{su}(3)\oplus\mathrm{su}(2)\oplus\mathrm{u}(1)\oplus\mathrm{u}(1)$, see item 4) at the very beginning. The non–Abelian part of $\mathrm{su}(5)\oplus\mathrm{u}(1)$ commuting with $\mathcal{M}_i$ must coincide with the non–Abelian part of the standard model Lie algebra. In terms of the decomposition

$$\mathrm{su}(5) = \left(\begin{array}{c|c} \mathrm{su}(3) & \cdot \\ \hline \cdot & \mathrm{su}(2) \end{array}\right) \tag{1.46}$$

we put

$$m = \mathrm{i}\ \mathrm{diag}(-\tfrac{2}{5}, -\tfrac{2}{5}, -\tfrac{2}{5}, \tfrac{3}{5}, \tfrac{3}{5}) \in \mathrm{su}(5)\ . \tag{1.47}$$

With this notation, the desired symmetry breaking pattern is achieved for

$$\mathcal{M}_i := \mathrm{diag}(\mathrm{i}\pi_{10}(m) \otimes M_{10}\ ,\ \overline{-\mathrm{i}\pi_5(m) \otimes M_5}\ ,\ 0_3)\ , \tag{1.48}$$

where M_{10} and M_5 are arbitrary 3×3–matrices. In contrast to the parameters entering M_f we have no experimental hints how to choose M_{10} and M_5 except that their norm must be very large. Namely, in the flipped SU(5)×U(1)–Grand Unification model there occur interactions which lead to proton decay. It turns out that the lifetime predicted for the proton depends on $\mathrm{tr}(M_{10}M_{10}^* + M_5M_5^*)$. The larger the trace (in units of m_t), the larger is the lifetime of the proton. It is essential that the matrices $M_{u,d,e,n,N}$ and $M_{10,5}$ are generically chosen, because otherwise there would be unwanted contributions from the extension (1.21). Finally, the grading operator is

$$\hat{\Gamma} = \begin{pmatrix} -\mathbb{1}_{48} & 0 & 0 & 0 \\ 0 & \mathbb{1}_{48} & 0 & 0 \\ 0 & 0 & \mathbb{1}_{48} & 0 \\ 0 & 0 & 0 & -\mathbb{1}_{48} \end{pmatrix}\ . \tag{1.49}$$

1.7.2 Remarks on the Construction

To this L–cycle we apply our formalism, which performs the following job: First, it extends the matrix $a \in \mathrm{su}(5)$ to a su(5)–gauge field A. This step is obvious, because we have $A \in \pi(\Omega^1\mathfrak{g}) = \Omega^1_D\mathfrak{g}$. Second, a rather long calculation reveals that those local elements of $\mathcal{H}^1\mathfrak{g}$ that are not already contained in $\pi(\Omega^1\mathfrak{g})$ are u(1)–gauge fields A''. The representations π of A and A'' on the fermionic Hilbert space are fixed by the formalism. In the notation of (1.40) they are given by

$$\begin{aligned} \pi(A) &= \mathrm{diag}\big(\tilde{A}, \tilde{A}, \gamma_C\overline{\tilde{A}}\gamma_C, \gamma_C\overline{\tilde{A}}\gamma_C\big)\ , \\ \tilde{A} &= \mathrm{diag}\big(\pi_{10}(A) \otimes \mathbb{1}_3\ ,\ \gamma_C\overline{\pi_5(A)}\gamma_C \otimes \mathbb{1}_3\ ,\ 0_3\ \big)\ , \end{aligned} \tag{1.50a}$$

$$\begin{aligned} \pi(A'') &= \mathrm{diag}\big(\tilde{A}'', \tilde{A}'', \gamma_C\overline{\tilde{A}''}\gamma_C, \gamma_C\overline{\tilde{A}''}\gamma_C\big)\ , \\ \tilde{A}'' &= \mathrm{diag}\big(-\tfrac{1}{2}A''\mathbb{1}_{10} \otimes \mathbb{1}_3\ ,\ -\tfrac{3}{2}\gamma_C\overline{A''}\gamma_C\mathbb{1}_5 \otimes \mathbb{1}_3\ ,\ -\tfrac{5}{2}A'' \otimes \mathbb{1}_3\big)\ , \end{aligned} \tag{1.50b}$$

where γ_C is the complex conjugation matrix: $\gamma^\mu = \gamma_C\overline{\gamma^\mu}\gamma_C$, $(\gamma_C)^2 = \pm\mathbb{1}_4$. Third, the formalism extends the matrices

m to a $\underline{24}$–Higgs multiplet $\tilde{\Psi} = \Psi + m$,
n to a complex $\underline{5}$–Higgs multiplet $\tilde{\Phi} = \Phi + n$,
n' to a complex $\underline{45}^*$–Higgs multiplet $\tilde{\Upsilon} = \Upsilon + n'$,
m' to a complex $\underline{50}$–Higgs multiplet $\tilde{\Xi} = \Xi + m'$.

This is an immediate consequence of the fact that m, n, n', m' belong to irreducible representations. Thus, the formalism generates the complete bosonic configuration space of the flipped SU(5)×U(1)–model out of the given L–cycle. Totally, there are 224 Higgs fields and 25 gauge bosons. The connection form has the structure

$$\rho = \tag{1.51}$$
$$\begin{pmatrix} \tilde\pi(A+A'') & \gamma^5\tilde\pi(\tilde\Psi) & \tilde\pi(\tilde\Phi+\tilde\Xi+\tilde\Upsilon) & 0 \\ -(\gamma^5\tilde\pi(\tilde\Psi))^* & \tilde\pi(A+A'') & 0 & \gamma^5\tilde\pi(\tilde\Phi+\tilde\Xi+\tilde\Upsilon) \\ -(\gamma^5\tilde\pi(\tilde\Phi+\tilde\Xi+\tilde\Upsilon))^* & 0 & -\gamma_C\overline{(\tilde\pi(A+A''))}\gamma_C & \overline{\gamma^5\tilde\pi(\tilde\Psi)} \\ 0 & -(\gamma^5\tilde\pi(\tilde\Phi+\tilde\Xi+\tilde\Upsilon))^* & -(\overline{\gamma^5\tilde\pi(\tilde\Psi)})^* & -\gamma_C\overline{(\tilde\pi(A+A''))}\gamma_C \end{pmatrix}.$$

Here, we have denoted by $\tilde\pi$ the embeddings (1.50a) and (1.50b) of the gauge fields A and A'', the embedding (1.44) of the Higgs multiplets $\tilde\Phi$, $\tilde\Upsilon$ and $\tilde\Xi$ and the embedding (1.48) of the Higgs multiplet $\tilde\Psi$ into $\mathrm{M}_{48}\mathbb{C}$ each. Thus, gauge and Higgs fields are treated in a unified way. Since the embeddings (1.44) and (1.48) include the matrices $M_{u,d,e,n,N}$ and $M_{10,5}$, the bosonic masses will depend on the fermion masses and the parameters of $M_{10,5}$.

The bosonic Lagrangian contains the usual Yang–Mills Lagrangian, the covariant derivatives of the Higgs fields and the Higgs potential. The Higgs potential is very complicated as a fourth order polynomial in 224 variables. All gauge invariant combinations of

$$\pi_{10}(\tilde\Psi),\ \pi_5(\tilde\Psi),\ \pi_{10,10}(\tilde\Phi),\ \pi_{10,5}(\tilde\Phi),\ \pi_{5,1}(\tilde\Phi),\ \tilde\Upsilon,\ \pi_{10,10}(\tilde\Upsilon),\ \tilde\Xi \tag{1.52}$$

really do occur. A computation of the minimum of such a monster seems hopeless. However, we do not have to work. The minimum is simply given by

$$\tilde\Psi = m\,, \qquad \tilde\Phi = n\,, \qquad \tilde\Upsilon = n' \qquad \tilde\Xi = m'\,. \tag{1.53}$$

This is a general feature of both non–commutative and non–associative geometry; the Higgs fields occur already in the broken phase. Just to give an impression of the power of our approach we list few examples of occurring contributions to the Higgs potential. Let

$$\begin{aligned} &V_1 = \tilde\Psi^2 - \tfrac{1}{5}\,\mathrm{tr}(\tilde\Psi^2)\mathbb{1}_5 - \tfrac{1}{5}\mathrm{i}\tilde\Psi\,, \qquad V_2 = (\tilde\Upsilon\tilde\Upsilon^*)' + \tfrac{8}{3}\mathrm{i}\tilde\Psi - \tilde\Phi^*\tilde\Phi + \tfrac{1}{5}\,\mathrm{tr}(\tilde\Phi^*\tilde\Phi)\mathbb{1}_5\,, \\ &V_3 = \tilde\Upsilon^*\Upsilon - \tfrac{1}{5}\,\mathrm{tr}(\tilde\Upsilon^*\Upsilon)\mathbb{1}_5 + 8\mathrm{i}\tilde\Psi + 9\tilde\Phi^*\Phi - \tfrac{9}{5}\,\mathrm{tr}(\Phi^*\Phi)\mathbb{1}_5\,, \\ &V_4 = \Upsilon^*\pi_{10,5}(\tilde\Phi) + \pi_{10,5}(\tilde\Phi)^*\Upsilon - 8\mathrm{i}\tilde\Psi - 6\Phi^*\Phi + \tfrac{6}{5}\,\mathrm{tr}(\Phi^*\Phi)\mathbb{1}_5\,, \\ &V_5 = \Upsilon^*\pi_{10,5}(\Phi) - \pi_{10,5}(\Phi)^*\Upsilon\,, \quad V_6 = (\tilde\Xi\tilde\Xi^*)' + \tfrac{1}{3}\mathrm{i}\tilde\Psi\,. \end{aligned} \tag{1.54}$$

Here, $\mathrm{i}Y'$ denotes the $\underline{24}$–component of the 10×10–matrix $\mathrm{i}Y$. Then,

$$\textstyle\sum_{i,j=1}^6 \mu_{ij}\,\mathrm{tr}(V_iV_j) \tag{1.55}$$

is a typical contribution to the Higgs potential. If one came to the idea to change the relative coefficients a bit, say, to omit the linear terms in V_i, then (1.53) is no longer the minimum and one has to deal with the monster. At this point at the latest one realizes the advantage that non–associative geometry brings to gauge field theory. The linear terms in (1.54) arise from the part $\sigma(\omega^1)$ in equation (1.17) for the differential. They lead to cubic terms in the Higgs potential, which must not be omitted! Principally, we have the freedom to choose the global parameters in the Higgs potential such as μ_{ij} in (1.55) arbitrarily (but such that the Higgs potential remains positive definite). In the classical construction this freedom exists indeed, and that is the reason why one obtains no predictions for the masses of the Higgs fields. In our approach, also these global parameters are fixed. They are given by traces over certain combinations of the matrices $M_{u,d,e,n,N}$ and $M_{10,5}$. Thus, if we fix the mass matrix $\mathcal{M}$ then all Higgs masses are determined on tree–level.

In the flipped SU(5)×U(1)–model, the Lie subalgebra which leaves the vacuum (1.53) invariant is $C^\infty(X)\otimes(\mathrm{su}(3)_C\oplus\mathrm{u}(1)_{EM})$. The $\mathrm{su}(3)_C$ corresponds to the colour symmetry and the $\mathrm{u}(1)_{EM}$ to the symmetry generated by the electric charge of the particles. The remaining 16 gauge degrees of freedom, corresponding to

$$C^\infty(X)\otimes((\mathrm{su}(5)\oplus\mathrm{u}(1))/(\mathrm{su}(3)_C\oplus\mathrm{u}(1)_{EM}))\,, \tag{1.56}$$

are used to gauge away 16 Higgs fields, twelve of the $\underline{24}$–representation, three of the $\underline{5}$–representation and one of the $\underline{50}$–representation. This in turn gives a mass to the 16 former gauge bosons corresponding to (1.56). These are the $W^\pm$ and Z bosons, an additional neutral heavy gauge boson Z' and the twelve leptoquarks X and Y (six each). Thus, there remain 208 Higgs fields

$$\psi_1',\psi_2',\psi_3',\psi_0,\psi_1,\dots,\psi_8,\phi_0',\phi_1,\dots,\phi_6,\upsilon_0',\upsilon_1,\dots,\upsilon_{89},\xi_0,\xi_1,\dots,\xi_{98}\,, \tag{1.57}$$

whose masses are obtained by diagonalization of the bilinear terms of the Higgs potential. These bilinear terms to select is still a tedious procedure (without computer algebra it is almost impossible to avoid errors).

1.7.3 The SU(5)–Grand Unification Model

If we omit ad hoc the u(1)–gauge field A'' and put M_N equal to zero, we can "derive" the SU(5)–Grand Unification Model out of the flipped SU(5)×U(1)–Grand Unification Model. This derivation violates the principles introduced in the thesis. However, if we do not perform the extension (1.21), then the SU(5)–model is obtained from the same L–cycle introduced above, after renaming $M_u \leftrightarrow M_d$, $M_n \mapsto M_e$ and $M_e \mapsto M_\nu$. If one omits the $\underline{5}$–representations and the matrix M_ν then one gets a model without right neutrinos. For the classical treatment of the SU(5)–GUT see [8, 26, 45].

1.7.4 Physical Results from the Grand Unification Model

We present the final results (on tree–level) for the flipped SU(5)×U(1)–Grand Unification model in Table 1. In this table, we denote by m_t and m_b the masses of the top quark

Particle	Mass	Particle	Mass
1. The completely neutral Higgs fields:			
ϕ_0'	$(0\ldots 1.45)\,m_t$	ξ_0	$(\sqrt{\frac{1}{60}}\ldots\sqrt{\frac{7}{4}})m_N$
υ_0'	λm_t	υ_{45}	$\frac{1}{2}\sqrt{3}\lambda m_t$
ψ_0	$\sqrt{\frac{2}{5}}m_N$	ψ_3'	$(0\ldots\frac{1}{12}\sqrt{\frac{11}{3}})\frac{m_N^2}{M}$
2. The colour–neutral Higgs fields of charge ∓ 1:			
$\frac{1}{\sqrt{2}}(\upsilon_{18}\pm \mathrm{i}\upsilon_{63})$	$\frac{1}{2}\sqrt{3}\lambda m_t$	$\frac{1}{\sqrt{2}}(\psi_1\pm \mathrm{i}\psi_2)$	$(0\ldots\frac{1}{12}\sqrt{\frac{11}{3}})\frac{m_N^2}{M}$
3. The neutral Higgs fields, for $i=0,\ldots,7$:			
ψ_{1+i}	$(0\ldots\frac{1}{12}\sqrt{\frac{11}{3}})\frac{m_N^2}{M}$		
υ_{1+i}	$(\lambda\ldots\lambda+\check{\lambda})m_n$	υ_{46+i}	$(\lambda\ldots\lambda+\check{\lambda})m_n$
ξ_{33+i}	$3M$	ξ_{82+i}	$3M$
4. The Higgs fields of charge ∓ 1, for $i=0\ldots 7$:			
$\frac{1}{\sqrt{2}}(\upsilon_{19+i}\pm \mathrm{i}\upsilon_{64+i})$	$(\lambda\ldots\lambda+\check{\lambda})m_n$	$\frac{1}{\sqrt{2}}(\xi_{25+i}\pm \mathrm{i}\xi_{74+i})$	$3M$
5. The Higgs fields of charge $\mp\frac{1}{3}$, for $i=0,1,2$ and $j=0,\ldots,5$:			
$\frac{1}{\sqrt{2}}(\phi_{1+i}\pm \mathrm{i}\phi_{4+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{9+i}\pm \mathrm{i}\upsilon_{54+i})$	M
$\frac{1}{\sqrt{2}}(\upsilon_{12+i}\pm \mathrm{i}\upsilon_{57+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{39+i}\pm \mathrm{i}\upsilon_{84+i})$	$2M$
$\frac{1}{\sqrt{2}}(\xi_{44+i}\pm \mathrm{i}\xi_{93+i})$	M	$\frac{1}{\sqrt{2}}(\xi_{47+i}\pm \mathrm{i}\xi_{96+i})$	$2M$
$\frac{1}{\sqrt{2}}(\xi_{19+j}\pm \mathrm{i}\xi_{68+j})$	$2M$	$\frac{1}{\sqrt{2}}(\upsilon_{30+j}\pm \mathrm{i}\upsilon_{75+j})$	M
6. The Higgs fields of charge $\pm\frac{2}{3}$, for $i=0,1,2$ and $j=0,\ldots,5$:			
$\frac{1}{\sqrt{2}}(\upsilon_{15+i}\pm \mathrm{i}\upsilon_{60+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{36+i}\pm \mathrm{i}\upsilon_{81+i})$	$2M$
$\frac{1}{\sqrt{2}}(\upsilon_{42+i}\pm \mathrm{i}\upsilon_{87+i})$	M	$\frac{1}{\sqrt{2}}(\xi_{41+i}\pm \mathrm{i}\xi_{90+i})$	M
$\frac{1}{\sqrt{2}}(\xi_{7+j}\pm \mathrm{i}\xi_{56+j})$	$2M$	$\frac{1}{\sqrt{2}}(\xi_{13+j}\pm \mathrm{i}\xi_{62+j})$	$4M$
7. The Higgs fields of charge $\mp\frac{4}{3}$, for $i=0,1,2$ and $j=0,\ldots,5$:			
$\frac{1}{\sqrt{2}}(\upsilon_{27+i}\pm \mathrm{i}\upsilon_{72+i})$	M	$\frac{1}{\sqrt{2}}(\xi_{1+j}\pm \mathrm{i}\xi_{50+j})$	$2M$
8. The neutral massive gauge fields:			
Z	$\sqrt{\frac{2}{5}}\,m_t$	Z'	$\frac{1}{2}\sqrt{\frac{5}{3}}m_N$
9. The massive gauge fields of charge ± 1:			
$\frac{1}{\sqrt{2}}(W_1\mp \mathrm{i}W_2)$	$\frac{1}{2}m_t$	Weinberg angle: $\sin^2\theta_W=\frac{3}{8}$	
10. The leptoquarks leading to proton decay, for $i=0,1,2$:			
$\frac{1}{\sqrt{2}}(X_{1+i}\mp \mathrm{i}X_{4+i})$	M	charge: $\mp\frac{1}{3}$	
$\frac{1}{\sqrt{2}}(Y_{1+i}\mp \mathrm{i}Y_{4+i})$	M	charge: $\pm\frac{2}{3}$	

Table 1: The particle masses for the SU(5)×U(1)–model

and by m_n and m_N the mass scales of the Dirac and Majorana masses for the neutrinos, respectively. The masses in Table 1 are correct for

$$m_n, m_b < m_t \ll \lambda m_t, (\lambda + \check{\lambda}) m_n < M, m_N \,, \tag{1.58}$$

which is physically plausible. The parameter $M \gg m_t$ is the Grand Unification scale. Moreover, we assume that the Majorana mass of the right neutrinos is of the same order of magnitude as M. The parameters $M, \lambda, \check{\lambda}$ are certain combinations of the unknown parameters of the matrices M_{10} and M_5. For generic matrices M_{10} and M_5, the masses λm_t and $(\lambda \ldots \lambda + \check{\lambda}) m_n$ are not significantly smaller than M and m_N. Let us comment on some observations:

1) There occur three mass scales in the models: The mass scale of the fermions determined by m_t, the Grand Unification scale M and an intermediate scale determined by λm_t and $(\lambda \ldots \check{\lambda}) m_n$. All particles with fractional–valued electric charge, which therefore lead to proton decay, have a mass of the order M.

2) In the flipped SU(5)×U(1)–model there exists precisely one light Higgs field ϕ_0', whose upper bound for the mass is independent of the Grand Unification matrices M_{10} and M_5. The reason that only an upper bound can be given is the incomplete knowledge of the input parameters. The Higgs field ϕ_0' is a certain linear combination of neutral Higgs fields of the $\underline{5}$–representation and the $\underline{45}^*$–representation[7]. It has precisely the same properties as the standard model Higgs field.

3) The predictions for the SU(5)–model are qualitatively the same, except that the gauge field Z' and all Higgs fields ξ_i are absent. Moreover, the electric charges of certain Higgs fields are modified.

4) The standard model is in perfect agreement with experiment. However, our results show that the low energy sector of both the SU(5)×U(1) and SU(5) Grand Unification models is identical with the standard model. This means that it is not possible to decide by means of present energy experiments which of the three models is correct. One essential advantage of the Grand Unification models is that they explain why proton and electron have up to the sign the same electric charge. On the other hand, the proton is not a stable particle in Grand Unification models. Concerning this question, the SU(5)×U(1)–model is favoured over the SU(5)–model, because it yields a larger lifetime for the proton [24].

We see that non–associative geometry has the flexibility to describe Grand Unification models.

[7] This shows impressively that the $\underline{45}$–representation, which is absent in the formulations [13, 14] of the SU(5)–GUT by non–commutative geometry, is an essential part of our model.

1.8 Description of the Contents

The general scheme of non–associative geometry is presented in Section 2. We start in Section 2.1 with basic definitions concerning L–cycles. In Section 2.2 we construct the universal graded differential Lie algebra $\Omega^*\mathfrak{g}$ and derive properties of its elements. Using the data specified in the L–cycle we define in Section 2.3 a Lie algebra representation π of $\Omega^*\mathfrak{g}$ in $\mathscr{B}(h)$. Factorization of $\pi(\Omega^*\mathfrak{g})$ with respect to the ideal $\pi(\mathcal{J}^*\mathfrak{g})$ yields the graded differential Lie algebra $\Omega^*_D\mathfrak{g}$. In Section 2.4 we introduce the important map σ, which enables us to give a convenient form to the ideal $\pi(\mathcal{J}^*\mathfrak{g})$. Using the language of graded Lie homomorphisms introduced in Section 2.5 we define in Section 2.6 the fundamental objects of gauge field theories: connections, curvatures, gauge transformations, bosonic and fermionic actions.

In Section 3 we apply this formalism to L–cycles over functions $\otimes$ matrices. Such L–cycles are generally defined in Section 3.1. For the space–time part it is convenient to redefine the exterior differential algebra Λ^*, see Section 3.2. This enables us to work with exterior and interior multiplication, exterior differential and codifferential on Λ^*. Using induction techniques we decompose in Section 3.3 the graded Lie algebra $\pi(\Omega^*\mathfrak{g})$ into space–time part and matrix part. This calculation is more or less understandable. However, the decomposition of $\Omega^*_D\mathfrak{g}$ in Section 3.5, again by induction, is perfectly designed to produce headache. The first step, the decomposition for $\Omega^2_D\mathfrak{g}$, uses some interesting ideas. But the very boring induction requires a big number of properties, which we derive in Section 3.4 and which at first sight have nothing to do with our goal. The decomposition of the formulae for the differential and the commutator into space–time and matrix part is given in Section 3.6. The decomposition also enables us to consider local connections, see Section 3.7.

In Section 4 we consider two simple test models for our approach: The chiral spinor electrodynamics is investigated in Section 4.1. Section 4.2 is devoted to the derivation of the standard model within our scheme.

In Section 5 we start with the construction of the matrix L–cycle of the flipped SU(5)$\times$U(1)–Grand Unification model. In Section 5.1 we consider relevant representations of the matrix Lie algebra su(5). The mass matrix $\mathcal{M}$ is defined in Section 5.2. Then it is not difficult to obtain the spaces $\pi(\Omega^1\mathfrak{a})$ and $\pi(\Omega^2\mathfrak{a})$, see Section 5.3. The structure of the connection form is derived in Section 5.4. Finally, we perform in Section 5.6 the factorization of the curvature with respect to the ideal $\mathbb{J}^2\mathfrak{g}$, which we construct before in Section 5.5.

In Section 6 we include the space–time part and derive the action for our model. Out of the curvature obtained in Section 6.1 we built in Section 6.2 the bosonic action. To compare it with usual formulae of gauge field theory we write down this action (except the Higgs potential) in terms of local coordinates, see Section 6.3. The fermionic action is derived in Section 6.4. Comparing it with measurements we can identify certain parameters of the generalized Dirac operator $\mathcal{M}$ with fermion masses and Kobayashi–Maskawa mixing angles.

This information plays an essential rôle in deriving the masses of the Higgs bosons in Section 7. First we determine in Section 7.1 the quadratic terms and in Section 7.2 the global parameters of the Higgs potential. Then we specify to the SU(5)$\times$U(1)–model in Section 7.3 and to the SU(5)–model in Section 7.4. The tree–level predictions for the Higgs masses are discussed in Section 8.

Preprints

There exist a preprint and three publications [65–68] yet[rw], which deal with the physics developed in this thesis. The paper [65] is essentially the Introduction of this thesis. The article [66] contains the mathematical scheme developed in Sections 2 and 3. The article [67] contains[8] the construction of the standard model given in Section 4.2. Finally, the preprint [68] is an abbreviated version of the construction of the Grand Unification models of Sections 5, 6 and 7.

[rw]Note added: After completion of the thesis, there has appeared the proceeding contribution [70] and a preprint [71] that improves the general framework of Section 2.

[8]In the meantime I have found a way to formulate the standard model without right neutrinos, which in [67] was claimed to be impossible.

2 L–Cycles and Graded Differential Lie Algebras

We are going to transcribe step by step the Connes–Lott procedure of non–commutative geometry to the case where the unital associative $*$-algebra is replaced by a skew–adjoint Lie algebra. The first modification concerns the definition of the K–cycle, which becomes an L–cycle in Section 2.1. But this is only a matter of completeness. The real story starts with Section 2.2, where we construct a "universal graded differential Lie algebra", modifying the "universal graded differential algebra" used in non–commutative geometry.

2.1 The L–Cycle

The basic geometric object in non–associative geometry is the L–cycle, which differs from the K–cycle (Definition 1) used in non–commutative geometry by the implementation of skew–adjoint Lie algebras instead of unital associative $*$-algebras. Let us recall how an L–cycle was defined.

Definition 2. *An L–cycle* $(\mathfrak{g}, h, D, \pi, \Gamma)$ *over a skew–adjoint Lie algebra* $\mathfrak{g}$ *is given by*

i) *an involutive representation* π *of* $\mathfrak{g}$ *in the Lie algebra* $\mathscr{B}(h)$ *of bounded operators on a Hilbert space* h, *i.e.* $(\pi(a))^* = \pi(a^*) \equiv -\pi(a)$, *for any* $a \in \mathfrak{g}$,

ii) *a (possibly unbounded) selfadjoint operator* D *on* h *such that* $(\mathrm{id}_h + D^2)^{-1}$ *is compact and for all* $a \in \mathfrak{g}$ *there is* $[D, \pi(a)] \in \mathscr{B}(h)$, *where* id_h *denotes the identity on* h.

iii) *a selfadjoint operator* Γ *on* h, *fulfilling* $\Gamma^2 = \mathrm{id}_h$, $\Gamma D + D\Gamma = 0$ *and* $\Gamma\pi(a) - \pi(a)\Gamma = 0$, *for all* $a \in \mathfrak{g}$.

Any Lie algebra $\mathfrak{g}$ can be embedded into its universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$, and the representation $\pi : \mathfrak{g} \to \mathscr{B}(h)$ extends to a representation $\pi : \mathfrak{U}(\mathfrak{g}) \to \mathscr{B}(h)$ (Poincaré–Birkhoff–Witt theorem, see [32]). In this sense, any L–cycle can be embedded into its "enveloping K–cycle". However, the gauge field theory obtained by the Connes–Lott prescription [17, 20] from this enveloping K–cycle differs from the gauge field theory we are going to develop for the L–cycle. Our construction follows the ideas of Connes and Lott, but the methods and results are different.

Although we do not need it, let us translate properties of a K–cycle into definitions for the L–cycle. We use the definition of the distance on a K–cycle [17, 20] to define the distance between linear functionals $\mathrm{x}_1, \mathrm{x}_2 : \mathfrak{g} \to \mathbb{C}$ of the Lie algebra:

Definition 3. *Let* X *be the space of linear functionals of* $\mathfrak{g}$. *The distance* $\mathrm{dist}(\mathrm{x}_1, \mathrm{x}_2)$ *between* $\mathrm{x}_1, \mathrm{x}_2 \in \mathrm{X}$ *is given by*

$$\mathrm{dist}(\mathrm{x}_1, \mathrm{x}_2) := \sup_{a \in \mathfrak{g}} \{\, |\mathrm{x}_1(a) - \mathrm{x}_2(a)| \;:\; \|\,[D, \pi(a)]\,\| \leq 1 \,\}\,.$$

This definition makes $(\mathrm{X}, \mathrm{dist})$ to a metric space, and there is no need for π being an algebra homomorphism.

Next, we can take the definition of integration on a K–cycle [17, 20] to define the notion of integration on an L–cycle:

Definition 4. *Let* $\mathrm{d} \in [1,\infty)$ *be a real number. An L–cycle* $(\mathfrak{g}, h, D, \pi, \Gamma)$ *is called* d^+–*summable if the eigenvalues* E_n *of* D *– arranged in increasing order – satisfy*

$$\textstyle\sum_{n=1}^N E_n^{-1} = O(\sum_{n=1}^N n^{-1/\mathrm{d}}) \,.$$

We define the integration

$$\int_{\mathrm{X}} |a|^2 \, d\mu := const.(\mathrm{d}) \; \mathrm{Tr}_\omega((\pi(a))^2 \, |D|^{-\mathrm{d}}) \,, \qquad a \in \mathfrak{g} \,,$$

where Tr_ω *is the Dixmier trace,* $d\mu$ *is the "volume measure" on* X *and* $const.(\mathrm{d})$ *refers to a constant depending on* d.

2.2 The Universal Graded Differential Lie Algebra $\Omega^*\mathfrak{g}$

To construct differential algebras over a K–cycle $(\mathcal{A}, h, D, \pi)$, *one starts from the universal differential algebra* $\Omega^*\mathcal{A}$ *over* $\mathcal{A}$ *and factorizes this differential algebra with respect to a differential ideal determined by the representation* π *of* $\Omega^*\mathcal{A}$ *in* $\mathscr{B}(h)$. *In analogy to this procedure we first define a universal differential Lie algebra* $\Omega^*\mathfrak{g}$ *over the Lie algebra* $\mathfrak{g}$ *of the L–cycle. The method*[9,rw] *is to start with the tensor algebra of the free vector spaces generated by* $\mathfrak{g}$ *and* $d\mathfrak{g}$. *There, we can define a canonical grading structure, a canonical graded commutator, a canonical differential and a canonical involution. Then we restrict ourselves to a certain graded Lie subalgebra* $\Omega^*\mathfrak{g}$. *Now, the natural algebraic structures on the tensor algebra induce corresponding algebraic structures on the Lie subalgebra. This is the concern of Sections 2.2.1 and 2.2.2. In Section 2.2.3 we prove a universality property for the differential Lie subalgebra* $\Omega^*\mathfrak{g}$, *which provides a link to the construction of the graded differential Lie algebra* $\Omega^*_D\mathfrak{g}$ *relevant for physics in Section 2.3.*

2.2.1 The Tensor Algebra $T(\mathfrak{g})$

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ with involution fulfilling $a^* = -a$, for $a \in \mathfrak{g}$. The construction of the universal graded differential Lie algebra $\Omega^*\mathfrak{g}$ over the Lie algebra $\mathfrak{g}$ goes as follows: First, let $d\mathfrak{g}$ be another copy of $\mathfrak{g}$. Let $V(\mathfrak{g})$ be the free vector space generated by $\mathfrak{g}$ and let $V(d\mathfrak{g})$ be the free vector space generated by $d\mathfrak{g}$,

$$\begin{aligned} V(\mathfrak{g}) &:= \bigoplus_{a\in\mathfrak{g}} V_a \,, &\qquad V_a &= \mathbb{R} \;\forall a \in \mathfrak{g} \,, \\ V(d\mathfrak{g}) &:= \bigoplus_{a\in\mathfrak{g}} V_{da} \,, &\qquad V_{da} &= \mathbb{R} \;\forall a \in \mathfrak{g} \,. \end{aligned} \tag{2.1}$$

[9] I am grateful to Rainer Matthes for a substantial improvement of a prior construction of $\Omega^*\mathfrak{g}$.

[rw] Note added: After completion of the thesis, I have found a further simplification of that construction, see [71].

For a vector space $\mathfrak{X}$ we denote by δ_x the function on $\mathfrak{X}$, which takes the value 1 at the point $x \in \mathfrak{X}$ and the value 0 at all points $y \neq x$. Then,

$$\begin{aligned} V(\mathfrak{g}) &= \{ \textstyle\sum_\alpha \lambda_\alpha \delta_{a_\alpha} \ , \quad a_\alpha \in \mathfrak{g} \ , \ \ \lambda_\alpha \in \mathbb{R} \ \} \ , \\ V(d\mathfrak{g}) &= \{ \textstyle\sum_\alpha \lambda_\alpha \delta_{da_\alpha} \ , \ a_\alpha \in \mathfrak{g} \ , \ \ \lambda_\alpha \in \mathbb{R} \ \} \ , \end{aligned} \tag{2.2}$$

where the sums are finite. Let $T(\mathfrak{g})$ be the tensor algebra of $V(\mathfrak{g}) \oplus V(d\mathfrak{g})$, which carries a natural $\mathbb{N}$–grading structure. We define $\deg(v) = 0$ for $v \in V(\mathfrak{g})$ and $\deg(v) = 1$ for $v \in V(d\mathfrak{g})$. For tensor products $v_1 \otimes v_2 \otimes \ldots \otimes v_n \in T(\mathfrak{g})$, where each v_i, $i = 1, \ldots, n$, belongs either to $V(\mathfrak{g})$ or to $V(d\mathfrak{g})$, we define

$$\deg(v_1 \otimes v_2 \otimes \ldots \otimes v_n) := \textstyle\sum_{i=1}^n \deg(v_i) \ . \tag{2.3}$$

Now we have

$$T(\mathfrak{g}) = \bigoplus_{n \in \mathbb{N}} T^n(\mathfrak{g}) \ , \qquad T^n(\mathfrak{g}) := \{ \ t \in T(\mathfrak{g}) \ , \ \ \deg(t) = n \ \} \ . \tag{2.4}$$

In particular, we have $T^k(\mathfrak{g}) \otimes T^l(\mathfrak{g}) \subset T^{k+l}(\mathfrak{g})$.

Next, we regard $T(\mathfrak{g})$ as a graded Lie algebra with graded commutator given by

$$[t^k, \tilde{t}^l] := t^k \otimes \tilde{t}^l - (-1)^{kl} \, \tilde{t}^l \otimes t^k \ , \quad t^k \in T^k(\mathfrak{g}) \ , \ \ \tilde{t}^l \in T^l(\mathfrak{g}) \ . \tag{2.5}$$

Obviously, one has

$$\begin{array}{ll} 1) & [t^k, \tilde{t}^l] = -(-1)^{kl}[\tilde{t}^l, t^k] \ , \\ 2) & [t^k, \lambda\tilde{t}^l + \tilde{\lambda}\tilde{\tilde{t}}^l] = \lambda[t^k, \tilde{t}^l] + \tilde{\lambda}[t^k, \tilde{\tilde{t}}^l] \ , \\ 3) & (-1)^{km}[t^k, [\tilde{t}^l, \tilde{\tilde{t}}^m]] + (-1)^{lk}[\tilde{t}^l, [\tilde{\tilde{t}}^m, t^k]] + (-1)^{ml}[\tilde{\tilde{t}}^m, [t^k, \tilde{t}^l]] = 0 \ , \end{array} \tag{2.6}$$

for $t^k \in T^k(\mathfrak{g})$, $\tilde{t}^l, \tilde{\tilde{t}}^l \in T^l(\mathfrak{g})$, $\tilde{\tilde{t}}^m \in T^m(\mathfrak{g})$ and $\lambda, \tilde{\lambda} \in \mathbb{R}$.

2.2.2 Definition of and Structures on $\Omega^*\mathfrak{g}$

Let $\tilde{\Omega}^*\mathfrak{g} = \bigoplus_{n\in\mathbb{N}} \tilde{\Omega}^n\mathfrak{g}$ be the $\mathbb{N}$–graded Lie subalgebra[10] of $T(\mathfrak{g})$ given by the set of all repeated commutators (in the sense of (2.5)) of elements of $V(\mathfrak{g})$ and $V(d\mathfrak{g})$. Let $I'(\mathfrak{g})$ be the vector subspace of $\tilde{\Omega}^*\mathfrak{g}$ of elements of the following type:

$$\begin{array}{ll} \lambda\delta_a - \delta_{\lambda a} \ , & \lambda\delta_{da} - \delta_{d(\lambda a)} \ , \\ \delta_a + \delta_{\tilde{a}} - \delta_{a+\tilde{a}} \ , & \delta_{da} + \delta_{d\tilde{a}} - \delta_{d(a+\tilde{a})} \ , \\ [\delta_a, \delta_{\tilde{a}}] - \delta_{[a,\tilde{a}]} \ , & [\delta_{da}, \delta_{\tilde{a}}] + [\delta_a, \delta_{d\tilde{a}}] - \delta_{d[a,\tilde{a}]} \ , \end{array} \tag{2.7}$$

[10] This is due to the graded Jacobi identity $[[s^k, \tilde{s}^l], [t^m, \tilde{t}^n]] = [s^k, [\tilde{s}^l, [t^m, \tilde{t}^n]]] - (-1)^{kl}[\tilde{s}^l, [s^k, [t^m, \tilde{t}^n]]]$, for $s^k \in \tilde{\Omega}^k\mathfrak{g}$, $\tilde{s}^l \in \tilde{\Omega}^l\mathfrak{g}$, $t^m \in \tilde{\Omega}^m\mathfrak{g}$ and $\tilde{t}^n \in \tilde{\Omega}^n\mathfrak{g}$.

for $a, \tilde{a} \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$. Obviously,

$$I(\mathfrak{g}) := I'(\mathfrak{g}) + [V(\mathfrak{g}) \oplus V(d\mathfrak{g}), I'(\mathfrak{g})] + [V(\mathfrak{g}) \oplus V(d\mathfrak{g}), [V(\mathfrak{g}) \oplus V(d\mathfrak{g}), I'(\mathfrak{g})]] + \dots \tag{2.8}$$

is an $\mathbb{N}$–graded ideal of $\tilde{\Omega}^*\mathfrak{g}$, $I(\mathfrak{g}) = \bigoplus_{n\in\mathbb{N}} I^n(\mathfrak{g})$. Then,

$$\Omega^*\mathfrak{g} := \bigoplus_{n\in\mathbb{N}} \Omega^n\mathfrak{g} \,, \qquad \Omega^n\mathfrak{g} := \tilde{\Omega}^n\mathfrak{g} \,/\, I^n(\mathfrak{g}) \,, \tag{2.9}$$

is an $\mathbb{N}$–graded Lie algebra, with commutator given by

$$[\varpi + I(\mathfrak{g}), \tilde{\varpi} + I(\mathfrak{g})] := [\varpi, \tilde{\varpi}] + I(\mathfrak{g}) \,, \quad \varpi, \tilde{\varpi} \in \tilde{\Omega}^*\mathfrak{g} \,. \tag{2.10}$$

On $T(\mathfrak{g})$ we define recursively a graded differential as an $\mathbb{R}$–linear map $d : T^n(\mathfrak{g}) \to T^{n+1}(\mathfrak{g})$ by

$$\begin{aligned} d(\lambda\delta_a) &:= \lambda\delta_{da} \,, & d(\lambda\delta_{da}) &:= 0 \,, \\ d(\lambda\delta_a \otimes t) &:= \lambda\delta_{da} \otimes t + \lambda\delta_a \otimes dt \,, & d(\lambda\delta_{da} \otimes t) &:= -\lambda\delta_{da} \otimes dt \,, \end{aligned} \tag{2.11}$$

for $a \in \mathfrak{g}$, $t \in T(\mathfrak{g})$ and $\lambda \in \mathbb{R}$. From this definition we get

$$\begin{aligned} d^2(\lambda\delta_a) &= d(\lambda\delta_{da}) = 0 \,, \qquad d^2(\lambda\delta_{da}) = 0 \,, \\ d^2(\lambda\delta_a \otimes t) &= d(\lambda\delta_{da} \otimes t) + d(\lambda\delta_a \otimes dt) \\ &= -\lambda\delta_{da} \otimes dt + \lambda\delta_{da} \otimes dt + \lambda\delta_a \otimes d^2t = \lambda\delta_a \otimes d^2t \,, \\ d^2(\lambda\delta_{da} \otimes t) &= \lambda\delta_{da} \otimes d^2t \,, \end{aligned} \tag{2.12}$$

therefore, by induction, $d^2 \equiv 0$ on $T(\mathfrak{g})$. In order to show that d is a graded differential we use the following equivalent characterization of (2.11):

$$d(v_1 \otimes \dots \otimes v_n) = \sum_{i=1}^{n} (-1)^{\sum_{j=1}^{i-1} \deg(v_j)} v_1 \otimes \dots \otimes v_{i-1} \otimes dv_i \otimes v_{i+1} \otimes \dots \otimes v_n \,. \tag{2.13}$$

For $t^k = v_1 \otimes \dots \otimes v_n \in T^k(\mathfrak{g})$, $k = \sum_{i=1}^n \deg(v_i)$, and $\tilde{t}^l \in T^l(\mathfrak{g})$ we get from (2.13)

$$d(t^k \otimes \tilde{t}^l) = d(t^k) \otimes \tilde{t}^l + (-1)^k t^k \otimes d\tilde{t}^l \,. \tag{2.14}$$

Thus, d defined by (2.11) is a graded differential of the tensor algebra $T(\mathfrak{g})$. Moreover, d is also a graded differential of the graded Lie algebra $T(\mathfrak{g})$:

$$\begin{aligned} d[t^k, \tilde{t}^l] &= d(t^k \otimes \tilde{t}^l - (-1)^{kl} \tilde{t}^l \otimes t^k) \\ &= (d(t^k) \otimes \tilde{t}^l - (-1)^{(k+1)l} \tilde{t}^l \otimes dt^k) + (-1)^k (t^k \otimes d\tilde{t}^l - (-1)^{k(l+1)} d(\tilde{t}^l) \otimes t^k) \\ &= [dt^k, \tilde{t}^l] + (-1)^k [t^k, d\tilde{t}^l] \,. \end{aligned}$$

Now, from $d(V(\mathfrak{g})\oplus V(d\mathfrak{g}))\subset V(\mathfrak{g})\oplus V(d\mathfrak{g})$ we conclude that d is also a graded differential of the graded Lie subalgebra $\tilde{\Omega}^*\mathfrak{g}\subset T(\mathfrak{g})$.

Next, we show that $dI'(\mathfrak{g})\subset I'(\mathfrak{g})$:

$$\begin{gathered} d(\lambda\delta_a-\delta_{\lambda a})=\lambda\delta_{da}-\delta_{d(\lambda a)}\ , \qquad d(\lambda\delta_{da}-\delta_{d(\lambda a)})=0\ , \\ d(\delta_a+\delta_{\tilde a}-\delta_{a+\tilde a})=\delta_{da}+\delta_{d\tilde a}-\delta_{d(a+\tilde a)}\ , \qquad d(\delta_{da}+\delta_{d\tilde a}-\delta_{d(a+\tilde a)})=0\ , \\ d([\delta_a,\delta_{\tilde a}]-\delta_{[a,\tilde a]})=[\delta_{da},\delta_{\tilde a}]+[\delta_a,\delta_{d\tilde a}]-\delta_{d[a,\tilde a]}\ , \\ d([\delta_{da},\delta_{\tilde a}]+[\delta_a,\delta_{d\tilde a}]-\delta_{d[a,\tilde a]})=-[\delta_{da},\delta_{d\tilde a}]+[\delta_{da},\delta_{d\tilde a}]=0\ . \end{gathered} \tag{2.15}$$

Since $d(V(\mathfrak{g})\oplus V(d\mathfrak{g}))\subset V(\mathfrak{g})\oplus V(d\mathfrak{g})$, we get from (2.8)

$$dI(\mathfrak{g})\subset I(\mathfrak{g})\ . \tag{2.16}$$

Therefore, the graded differential d on $\tilde{\Omega}^*\mathfrak{g}$ induces a graded differential on $\Omega^*\mathfrak{g}$ denoted by the same symbol:

$$d(\varpi+I(\mathfrak{g})):=d\varpi+I(\mathfrak{g})\ , \quad \varpi\in\tilde{\Omega}^*\mathfrak{g}\ . \tag{2.17}$$

Hence, $(\Omega^*\mathfrak{g}, [\ ,\ ], d)$ is a graded differential Lie algebra.

We extend the involution $*: a\mapsto -a$ on $\mathfrak{g}$ to an involution of the free vector spaces $V(\mathfrak{g})$ and $V(d\mathfrak{g})$ by

$$(\lambda\delta_a)^*:=-\lambda\delta_a\ , \qquad (\lambda\delta_{da})^*:=-\lambda\delta_{da}\ . \tag{2.18}$$

We obtain an involution of $T(\mathfrak{g})$ by

$$(v_1\otimes v_2\otimes\ldots\otimes v_n)^*:=v_n^*\otimes\ldots\otimes v_2^*\otimes v_1^*\ , \tag{2.19}$$

fulfilling

$$(t\otimes\tilde t)^*=\tilde t^*\otimes t^*\ . \tag{2.20}$$

Formula (2.20) induces the following property of the Lie bracket (2.5):

$$[t^k,\tilde t^l]^*=-(-1)^{kl}[t^{k^*},\tilde t^{l*}]\ . \tag{2.21}$$

Because of $(V(\mathfrak{g})\oplus V(d\mathfrak{g}))^*=V(\mathfrak{g})\oplus V(d\mathfrak{g})$ we get an involution on $\tilde{\Omega}^*\mathfrak{g}$ by restricting the involution on $T(V)$ to its graded Lie subalgebra $\tilde{\Omega}^*\mathfrak{g}$. Obviously, we have $I'(\mathfrak{g})^*=I'(\mathfrak{g})$, giving $I(\mathfrak{g})^*=I(\mathfrak{g})$. Therefore, we obtain an involution on $\Omega^*\mathfrak{g}$ by

$$(\varpi+I(\mathfrak{g}))^*:=\varpi^*+I(\mathfrak{g})\ , \quad \varpi\in\tilde{\Omega}^*\mathfrak{g}\ . \tag{2.22}$$

2.2.3 The Universality Property of $\Omega^*\mathfrak{g}$

The graded differential Lie algebra $\Omega^*\mathfrak{g}$ is universal in the following sense:

Proposition 5. *Let $\Lambda^*\mathfrak{g} = \bigoplus_{n\in\mathbb{N}} \Lambda^n\mathfrak{g}$ be an $\mathbb{N}$–graded Lie algebra with graded differential* $\mathrm{d} : \Lambda^n\mathfrak{g} \to \Lambda^{n+1}\mathfrak{g}$ *such that*
i) $\Lambda^0\mathfrak{g} = \pi(\mathfrak{g})$, *for a surjective homomorphism $\pi : \mathfrak{g} \to \pi(\mathfrak{g})$ of Lie algebras,*
ii) $\Lambda^*\mathfrak{g}$ *is generated by $\pi(\mathfrak{g})$ and $\mathrm{d}\pi(\mathfrak{g})$ as the set of repeated commutators.*
Then there exists a differential ideal $I_\Lambda \subset \Omega^*\mathfrak{g}$ such that $\Lambda^*\mathfrak{g} \cong \Omega^*\mathfrak{g}/I_\Lambda$.

Proof. We define a surjective mapping $\tilde{p} : \tilde{\Omega}^*\mathfrak{g} \to \Lambda^*\mathfrak{g}$ by

$$\tilde{p}(\lambda\delta_a) := \pi(\lambda a) ,$$
$$\tilde{p}(d\varpi) := \mathrm{d}(\tilde{p}(\varpi)) ,$$
$$\tilde{p}([\varpi, \tilde{\varpi}]) := [\tilde{p}(\varpi), \tilde{p}(\tilde{\varpi})] ,$$

for $a \in \mathfrak{g}$, $\varpi, \tilde{\varpi} \in \tilde{\Omega}^*\mathfrak{g}$ and $\lambda \in \mathbb{R}$. Obviously, $\tilde{p}(I(\mathfrak{g})) = 0$. Therefore, by factorization with respect to $I(\mathfrak{g})$ we get a surjection $p : \Omega^*\mathfrak{g} \to \Lambda^*\mathfrak{g}$ by $p(\varpi + I(\mathfrak{g})) := \tilde{p}(\varpi)$, for $\varpi \in \tilde{\Omega}^*\mathfrak{g}$. We have $p(d \ker p) = 0$, therefore, $I_\Lambda = \ker p$ is the desired differential ideal of $\Omega^*\mathfrak{g}$:

$$\Lambda^*\mathfrak{g} \cong \Omega^*\mathfrak{g} / \ker p . \qquad \square$$

Proposition 5 tells us that each graded differential Lie algebra generated by $\pi(\mathfrak{g})$ and its differential is obtained by factorizing $\Omega^*\mathfrak{g}$ with respect to a differential ideal. For the setting described by an L–cycle, such a differential ideal is canonically given. This leads to a canonical graded differential Lie algebra, see Section 2.3.

2.2.4 Summary

To summarize: We have defined a universal graded differential Lie algebra $\Omega^*\mathfrak{g} = \bigoplus_{n=0}^{\infty} \Omega^n\mathfrak{g}$ over a Lie algebra $\mathfrak{g}$, with:
– graded commutator $[\ ,\ ] : \Omega^k\mathfrak{g} \times \Omega^l\mathfrak{g} \to \Omega^{k+l}\mathfrak{g}$,
– universal differential $d : \Omega^k\mathfrak{g} \to \Omega^{k+1}\mathfrak{g}$, which is linear, nilpotent and obeys the graded Leibniz rule.
– involution $* : \Omega^k\mathfrak{g} \to \Omega^k\mathfrak{g}$.
Explicitly, we have the following properties:

1) $[\omega^k, \tilde{\omega}^l] = -(-1)^{kl}[\tilde{\omega}^l, \omega^k]$, (2.23a)
2) $[\omega^k, \lambda\tilde{\omega}^l + \tilde{\lambda}\tilde{\tilde{\omega}}^l] = \lambda[\omega^k, \tilde{\omega}^l] + \tilde{\lambda}[\omega^k, \tilde{\tilde{\omega}}^l]$, (2.23b)
3) $(-1)^{km}[\omega^k, [\tilde{\omega}^l, \tilde{\tilde{\omega}}^m]] + (-1)^{lk}[\tilde{\omega}^l, [\tilde{\tilde{\omega}}^m, \omega^k]] + (-1)^{ml}[\tilde{\tilde{\omega}}^m, [\omega^k, \tilde{\omega}^l]] = 0$, (2.23c)
4) $d[\omega^k, \tilde{\omega}^l] = [d\omega^k, \tilde{\omega}^l] + (-1)^k[\omega^k, d\tilde{\omega}^l]$, (2.23d)
5) $d^2\omega^k = 0$, (2.23e)
6) $[\omega^k, \tilde{\omega}^l]^* = -(-1)^{kl}[\omega^{k*}, \tilde{\omega}^{l*}]$, (2.23f)

for $\omega^k \in \Omega^k\mathfrak{g}$, $\tilde{\omega}^l, \tilde{\tilde{\omega}}^l \in \Omega^l\mathfrak{g}$, $\tilde{\tilde{\omega}}^m \in \Omega^m\mathfrak{g}$ and $\lambda, \tilde{\lambda} \in \mathbb{R}$.

2.2.5 A Canonical Representation of Elements of $\Omega^*\mathfrak{g}$

For technical reasons it is convenient to fix a canonical ordering in elements of $\Omega^k\mathfrak{g}$, $k \geq 1$. The main result is that each element of $\Omega^k\mathfrak{g}$, $k > 1$, can be represented as a sum of graded commutators of elements of $\Omega^1\mathfrak{g}$ with elements of $\Omega^{k-1}\mathfrak{g}$. This is very important for the proofs of almost all propositions, because we shall mainly use induction techniques.

First, let

$$\imath(a) := \delta_a + I(\mathfrak{g}) , \qquad \imath(da) := \delta_{da} + I(\mathfrak{g}) , \tag{2.24}$$

for $a \in \mathfrak{g}$. The first equation establishes an isomorphism $\Omega^0\mathfrak{g} \cong \mathfrak{g}$. We shall represent elements $\omega^1 \in \Omega^1\mathfrak{g}$ as

$$\begin{aligned} \omega^1 &= \imath(d\tilde{a}) + \textstyle\sum_{\alpha,z\geq 1}[\imath(a^z_\alpha),[\ldots[\imath(a^2_\alpha),[\imath(a^1_\alpha),\imath(da^0_\alpha)]]\ldots]] \\ &\equiv \textstyle\sum_{\alpha,z\geq 0}[\imath(a^z_\alpha),[\ldots[\imath(a^2_\alpha),[\imath(a^1_\alpha),\imath(da^0_\alpha)]]\ldots]] , \end{aligned} \tag{2.25}$$

where $\tilde{a}, a^i_\alpha \in \mathfrak{g}$ and the sums are finite. To avoid possible misunderstandings concerning this notation we fix throughout this thesis the following convention: Beginning with $z = 1$, the index α first runs from 1 to $\alpha_1 > 0$ and labels the terms

$[\imath(a^1_1),\imath(da^0_1)],\ldots,[\imath(a^1_{\alpha_1}),\imath(da^0_{\alpha_1})]$

in (2.25). Then, for $z = 2$, the index α runs from $\alpha_1 + 1$ to $\alpha_2 > \alpha_1$ and labels the commutators

$[\imath(a^2_{\alpha_1+1}),[\imath(a^1_{\alpha_1+1}),\imath(da^0_{\alpha_1+1})]],\ldots,[\imath(a^2_{\alpha_2}),[\imath(a^1_{\alpha_2}),\imath(da^0_{\alpha_2})]]$

in (2.25), and so on. Therefore, the pair (i,β) of indices labelling an element $a^i_\beta \in \mathfrak{g}$ does never occur more than once in the sum (2.25). Moreover, we identify the term belonging to the pair $(\alpha = 0, z = 0)$ of indices with $\imath(d\tilde{a})$, as already indicated in (2.25).

Now, we write down elements $\omega^k \in \Omega^k\mathfrak{g}$, $k \geq 2$, recursively as

$$\omega^k = \textstyle\sum_\alpha[\omega^1_\alpha, \tilde{\omega}^{k-1}_\alpha] , \quad \omega^1_\alpha \in \Omega^1\mathfrak{g} , \;\; \tilde{\omega}^{k-1}_\alpha \in \Omega^{k-1}\mathfrak{g} , \;\; \text{finite sum} . \tag{2.26}$$

There are two things to check concerning (2.26). First, for $\tilde{\omega}^n \equiv \sum_\alpha[\tilde{\omega}^1_\alpha, \tilde{\tilde{\omega}}^{n-1}_\alpha] \in \Omega^n\mathfrak{g}$, with $\tilde{\omega}^1_\alpha \in \Omega^1\mathfrak{g}$ and $\tilde{\tilde{\omega}}^{n-1}_\alpha \in \Omega^{n-1}\mathfrak{g}$, we must show that also $[\omega^0, \tilde{\omega}^n] \in \Omega^n\mathfrak{g}$ can be represented in the standard form (2.26), for any $\omega^0 \in \Omega^0\mathfrak{g}$. But this follows from the graded Jacobi identity (2.23c):

$$\begin{aligned} [\omega^0,\tilde{\omega}^n] &= [\omega^0, \textstyle\sum_\alpha[\tilde{\omega}^1_\alpha,\tilde{\tilde{\omega}}^{n-1}_\alpha]] = -\textstyle\sum_\alpha[\tilde{\omega}^1_\alpha,[\tilde{\tilde{\omega}}^{n-1}_\alpha,\omega^0]] - (-1)^{n-1}\textstyle\sum_\alpha[\tilde{\tilde{\omega}}^{n-1}_\alpha,[\omega^0,\tilde{\omega}^1_\alpha]] \\ &= \textstyle\sum_\alpha\left([\tilde{\omega}^1_\alpha,[\omega^0,\tilde{\tilde{\omega}}^{n-1}_\alpha]] + [[\omega^0,\tilde{\omega}^1_\alpha],\tilde{\tilde{\omega}}^{n-1}_\alpha]\right) . \end{aligned}$$

Second, we must show that the commutator $[\omega^k, \tilde{\omega}^l] \in \Omega^{k+l}\mathfrak{g}$, for $2 \leq k \leq l$, can be represented in the standard form (2.26) of an element of $\Omega^{k+l}\mathfrak{g}$, provided that both $\omega^k \in$

$\Omega^k\mathfrak{g}$ and $\tilde{\omega}^l \in \Omega^l\mathfrak{g}$ are written down recursively in the form (2.26). Using again (2.23b) and (2.23c) we get for $\omega^k = \sum_\alpha [\omega^1_\alpha, \tilde{\tilde{\omega}}^{k-1}_\alpha]$

$$\begin{aligned}[\omega^k, \tilde{\omega}^l] &= -(-1)^{lk} \sum_\alpha [\tilde{\omega}^l, [\omega^1_\alpha, \tilde{\tilde{\omega}}^{k-1}_\alpha]] \\ &= \sum_\alpha \big([\omega^1_\alpha, [\tilde{\tilde{\omega}}^{k-1}_\alpha, \tilde{\omega}^l]] + (-1)^k [\tilde{\tilde{\omega}}^{k-1}_\alpha, [\omega^1_\alpha, \tilde{\omega}^l]]\big)\ .\end{aligned}$$

Repeating this calculation for the commutators $[\tilde{\tilde{\omega}}^{k-1}_\alpha, \tilde{\omega}^l]$ and $[\tilde{\tilde{\omega}}^{k-1}_\alpha, [\omega^1_\alpha, \tilde{\omega}^l]]$, we can recursively decrease the degree k until we arrive at degree 1.

Now we can easily prove

$$(\omega^k)^* = -(-1)^{k(k-1)/2}\omega^k\ , \quad \omega^k \in \Omega^k\mathfrak{g}\ . \tag{2.27}$$

By definition, (2.27) holds for $k = 0$. From (2.25) and (2.23f) we get for $\omega^1 \in \Omega^1\mathfrak{g}$

$$\begin{aligned}{\omega^1}^* &= \sum_{\alpha,z\geq 0} [\imath(a^z_\alpha), [\dots, [\imath(a^1_\alpha), \imath(da^0_\alpha)] \dots]]^* \\ &= \sum_{\alpha,z\geq 0} [\imath(a^z_\alpha), ([\dots, [\imath(a^1_\alpha), \imath(da^0_\alpha)] \dots])^*] \\ &= \dots = \sum_{\alpha,z\geq 0} [\imath(a^z_\alpha), [\dots, [\imath(a^1_\alpha), (\imath(da^0_\alpha))^*] \dots]] = -\omega^1\ .\end{aligned}$$

In the same way we get from (2.26) and (2.23f) for $\omega^k \in \Omega^k\mathfrak{g}$

$$\begin{aligned}{\omega^k}^* &= \sum_\alpha [\omega^1_\alpha, \tilde{\omega}^{k-1}_\alpha]^* = (-1)^{k-1} \sum_\alpha [\omega^1_\alpha, (\tilde{\omega}^{k-1}_\alpha)^*] = (-1)^{(\sum_{i=2}^{k-1} i)}\omega^k \\ &= -(-1)^{k(k-1)/2}\omega^k\ .\end{aligned}$$

2.3 The Graded Differential Lie Algebra $\Omega^*_D\mathfrak{g}$

In non–commutative geometry, the next step after having defined the universal graded differential algebra $\Omega^*\mathcal{A}$ would be to represent $\Omega^*\mathcal{A}$ on the Hilbert space, using the data specified in the K–cycle. That representation of $\Omega^*\mathcal{A}$ is not a differential algebra, but the defect gives rise to a canonical differential ideal. Analogously, we represent our universal graded differential Lie algebra $\Omega^*\mathfrak{g}$ on the Hilbert space, using the data specified in the L–cycle. Again, we get a canonical differential ideal and, therefore, a canonical graded differential Lie algebra $\Omega^*_D\mathfrak{g}$.

2.3.1 Definition of $\pi(\Omega^*\mathfrak{g})$

Following the procedure for K–cycles we define an involutive representation[11] π of the universal differential Lie algebra $\Omega^*\mathfrak{g}$ introduced in Section 2.2 in the graded Lie algebra $\mathscr{B}(h)$ of bounded operators on h, where h is the Hilbert space of the L–cycle given in Definition 2. We underline that π will not be a representation of graded Lie algebras with differential. The definition of π uses almost the whole input contained in the L–cycle. First, using the grading operator Γ, we define a $\mathbb{Z}_2$–grading structure on

[11] The symbol π is already used but there is no danger of confusion.

the vector space $\mathscr{O}(h)$ of linear operators on the Hilbert space h, $\mathscr{O}(h) = \mathscr{O}_0(h) \oplus \mathscr{O}_1(h)$, by

$$\mathscr{O}_0(h)\Gamma = \Gamma\mathscr{O}_0(h) , \qquad \mathscr{O}_1(h)\Gamma = -\Gamma\mathscr{O}_1(h) . \tag{2.28}$$

This enables us to introduce the graded commutator for $\mathbb{Z}_2$–graded linear operators on h : For $A_i \in \mathscr{O}_i(h)$ and $B_j \in \mathscr{O}_j(h) \cap \mathscr{B}(h)$, where both A_i, B_j are selfadjoint or skew–adjoint on h, we define

$$[A_i, B_j]_g := A_i \circ B_j - (-1)^{ij} B_j \circ A_i \equiv -(-1)^{ij}[B_j, A_i]_g , \tag{2.29}$$

on the subset $h' = \text{domain}(A_i) \cap \{\psi \in h\ ,\ \ B_j\psi \in \text{domain}(A_i)\}$ of h. *We remark that we present our formalism in a very general context where we cannot guarantee that all occurring graded commutators make sense. In those cases, our formulae hold only formally. There are deep questions of functional analysis behind which we are not going to touch. We are interested in applications to physics. In those class of examples introduced in Section 3, all graded commutators that occur do have a strict meaning. Nevertheless, even in this case we shall apply formal calculations: We can guarantee that h' is dense in h. We will only work on that dense subspace*[rw]. *Extension problems are not studied.*

Let us define a linear mapping $\tilde{\pi} : \tilde{\Omega}^*\mathfrak{g} \to \mathscr{B}(h)$ by

$$\tilde{\pi}(\lambda\delta_a) := \pi(\lambda a) , \tag{2.30a}$$

$$\tilde{\pi}(\lambda\delta_{da}) := [-\mathrm{i}D, \pi(\lambda a)]_g \equiv [-\mathrm{i}D, \pi(\lambda a)] , \tag{2.30b}$$

$$\tilde{\pi}([\varpi^k, \tilde{\varpi}^l]) := [\tilde{\pi}(\varpi^k), \tilde{\pi}(\tilde{\varpi}^l)]_g , \tag{2.30c}$$

for $a \in \mathfrak{g}$, $\varpi^k \in \tilde{\Omega}^k\mathfrak{g}$, $\tilde{\varpi}^l \in \tilde{\Omega}^l\mathfrak{g}$ and $\lambda \in \mathbb{R}$. Note that $\pi(a)$ and $[D, \pi(a)]$ are bounded due to Definition 2 so that the r.h.s. of equations (2.30a) and (2.30b) belong to $\mathscr{B}(h)$. Now, due to $\pi(\mathfrak{g}) \subset \mathscr{O}_0(h)$ and $D \in \mathscr{O}_1(h)$, we get from (2.30)

$$\tilde{\pi}(\tilde{\Omega}^{2k}\mathfrak{g}) \subset \mathscr{O}_0(h) , \qquad \tilde{\pi}(\tilde{\Omega}^{2k+1}\mathfrak{g}) \subset \mathscr{O}_1(h) . \tag{2.31}$$

Next, we show that $\tilde{\pi} : \tilde{\Omega}^*\mathfrak{g} \to \mathscr{B}(h)$ is an involutive representation, where we recall that the involution in $\mathscr{B}(h)$ is defined as usual by means of the scalar product $\langle\ ,\ \rangle_h$ on h :

$$\langle\psi, \tau^*\tilde{\psi}\rangle_h := \langle\tau\psi, \tilde{\psi}\rangle_h , \quad \forall\psi, \tilde{\psi} \in h\ ,\ \ \tau \in \mathscr{B}(h) . \tag{2.32}$$

First, from (2.18), (2.30a) and the fact that $\pi : \mathfrak{g} \to \mathscr{B}(h)$ is an involutive representation we get

$$\tilde{\pi}((\lambda\delta_a)^*) = -\tilde{\pi}(\lambda\delta_a) = -\pi(\lambda a) = (\pi(\lambda a))^* = (\tilde{\pi}(\lambda\delta_a))^* .$$

[rw] Note added after completion of the thesis: In [71] we have reformulated our construction in terms of linear bounded operators only. In that paper, all the formal considerations of the thesis have found a strict explanation.

Second, from (2.18), (2.30b) and the selfadjointness of D we obtain

$$\begin{aligned}\tilde{\pi}((\lambda\delta_{da})^*) &= -\tilde{\pi}(\lambda\delta_{da}) = \mathrm{i}(D\circ\pi(\lambda a) - \pi(\lambda a)\circ D) \\ &= -(-\mathrm{i})^*(D^*\circ(\pi(\lambda a))^* - (\pi(\lambda a))^*\circ D^*) \\ &= -\{-\mathrm{i}(\pi(\lambda a)\circ D - D\circ\pi(\lambda a))\}^* = (\tilde{\pi}(\lambda\delta_{da}))^* .\end{aligned}$$

Now we get by induction that $\tilde{\pi}$ is an involutive representation on $\tilde{\Omega}^*\mathfrak{g}$.

Observe that

$$\tilde{\pi}(I(\mathfrak{g})) \equiv 0 . \tag{2.33}$$

Therefore, the involutive representation $\tilde{\pi} : \tilde{\Omega}^*\mathfrak{g} \to \mathscr{B}(h)$ induces an involutive representation $\pi : \Omega^*\mathfrak{g} \to \mathscr{B}(h)$ by

$$\pi(\varpi + I(\mathfrak{g})) := \tilde{\pi}(\varpi) , \quad \varpi \in \tilde{\Omega}^*\mathfrak{g} . \tag{2.34}$$

2.3.2 Definition of $\Omega_D^*\mathfrak{g}$

In the same way as for K–cycles, there may exist $\omega \in \Omega^*\mathfrak{g}$ fulfilling $\pi(\omega) = 0$ but not $\pi(d\omega) = 0$. Therefore, $\pi(\Omega^*\mathfrak{g})$ is not a differential Lie algebra. But there is a canonical construction towards such an object. Let us define

$$\mathcal{J}^*\mathfrak{g} = \ker\pi + d\ker\pi = \bigoplus_{k=0}^{\infty} \mathcal{J}^k\mathfrak{g} , \qquad \mathcal{J}^k\mathfrak{g} = \mathcal{J}^*\mathfrak{g} \cap \Omega^k\mathfrak{g} . \tag{2.35}$$

To obtain a differential Lie algebra we first prove:

Lemma 6. *$\mathcal{J}^*\mathfrak{g}$ is a graded differential ideal of the graded Lie algebra $\Omega^*\mathfrak{g}$.*

Proof. It is clear that $\ker\pi$ is an ideal of $\Omega^*\mathfrak{g}$. Then, for $j^k \in \ker\pi \cap \Omega^k\mathfrak{g}$ and $\omega \in \Omega^*\mathfrak{g}$ we have, see (2.23d),

$$[d j^k, \omega] = d([j^k, \omega]) - (-1)^k [j^k, d\omega] .$$

Because of $[j^k, d\omega] \in \ker\pi$ and $d([j^k, \omega]) \in d\ker\pi$, $\mathcal{J}^*\mathfrak{g}$ is an ideal of $\Omega^*\mathfrak{g}$. Moreover, it is obviously a differential ideal: $d\mathcal{J}^*\mathfrak{g} \subset \mathcal{J}^*\mathfrak{g}$, due to $d^2 = 0$. □

By virtue of Proposition 5, the canonical differential ideal (2.35) gives rise to a graded differential Lie algebra $\Omega_D^*\mathfrak{g}$:

$$\Omega_D^*\mathfrak{g} = \bigoplus_{k=0}^{\infty} \Omega_D^k\mathfrak{g} , \qquad \Omega_D^k\mathfrak{g} := \Omega^k\mathfrak{g}/\mathcal{J}^k\mathfrak{g} . \tag{2.36a}$$

There is a canonical isomorphism

$$\frac{\Omega^k\mathfrak{g}}{\jmath^k\mathfrak{g}} \cong \frac{\Omega^k\mathfrak{g}/(\ker\pi\cap\Omega^k\mathfrak{g})}{\jmath^k\mathfrak{g}/(\ker\pi\cap\Omega^k\mathfrak{g})}\,, \tag{2.36b}$$

establishing the isomorphism

$$\Omega^k_D\mathfrak{g} \cong \pi(\Omega^k\mathfrak{g})/\pi(\jmath^k\mathfrak{g})\,. \tag{2.36c}$$

In particular, one has

$$\Omega^0_D\mathfrak{g} \cong \pi(\Omega^0\mathfrak{g}) \equiv \pi(\mathfrak{g})\,, \qquad\qquad \Omega^1_D\mathfrak{g} \cong \pi(\Omega^1\mathfrak{g})\,. \tag{2.36d}$$

Let ς denote the projection onto equivalence classes, $\varsigma : \pi(\Omega^k\mathfrak{g}) \to \Omega^k_D\mathfrak{g}$. In this notation, the commutator and the differential on $\Omega^*_D\mathfrak{g}$ are defined as

$$[\varsigma\circ\pi(\omega^k),\varsigma\circ\pi(\tilde{\omega}^l)]_g := \varsigma([\pi(\omega^k),\pi(\tilde{\omega}^l)]_g)\,, \tag{2.37a}$$

$$d(\varsigma\circ\pi(\omega^k)) := \varsigma\circ\pi(d\omega^k)\,, \tag{2.37b}$$

for $\omega^k \in \Omega^k\mathfrak{g}$ and $\tilde{\omega}^l \in \Omega^l\mathfrak{g}$. From (2.37a) there follows that $\Omega^*_D\mathfrak{g}$ is a graded Lie algebra, and the bracket $[\ ,\ ]_g : \Omega^*_D\mathfrak{g}\times\Omega^*_D\mathfrak{g} \to \Omega^*_D\mathfrak{g}$ has properties analogous to (2.23). For $\varrho^k = \varsigma\circ\pi(\omega^k)$ and $\tilde{\varrho}^l = \varsigma\circ\pi(\tilde{\omega}^l)$ we have with (2.37a) and (2.37b)

$$\begin{aligned} d[\varrho^k,\tilde{\varrho}^l]_g &= \varsigma\circ\pi(d[\omega^k,\tilde{\omega}^l]) = \varsigma\circ\pi([d\omega^k,\tilde{\omega}^l]+(-1)^k[\omega^k,d\tilde{\omega}^l]) \\ &= [d\varrho^k,\tilde{\varrho}^l]_g + (-1)^k[\varrho^k,d\tilde{\varrho}^l]_g\,. \end{aligned} \tag{2.37c}$$

Obviously, $d^2 \equiv 0$ on $\Omega^*_D\mathfrak{g}$. This means that d is a graded differential on $\Omega_D\mathfrak{g}$. Moreover, we have

$$(\varsigma\circ\pi(\omega^k))^* = \varsigma\circ\pi((\omega^k)^*)\,, \quad \omega^k \in \Omega^k_D\mathfrak{g}\,, \tag{2.38}$$

because π is an involutive representation and $\pi(\jmath^*\mathfrak{g})$ is invariant under the involution. From (2.27) we get

$$\varrho^{n*} = -(-1)^{n(n-1)/2}\varrho^n\,, \quad \varrho^n \in \Omega^n_D\mathfrak{g}\,. \tag{2.39}$$

2.4 Towards the Analysis of the Differential Ideal

*This subsection is extremely important. We define formally a linear representation σ of $\Omega^*\mathfrak{g}$ on the Hilbert space. It turns out that σ measures the defect if we represent the universal differential d by graded commutators with –i*D*, see Proposition 7. This result is the key to compute both the differential on $\Omega^*_D\mathfrak{g}$ and the differential ideal $\pi(\jmath^*\mathfrak{g})$ itself. Here, it is essential that, once knowing $\sigma(\Omega^1\mathfrak{g})$, we can easily iteratively compute $\sigma(\Omega^k\mathfrak{g})$ for higher degrees using formula* (2.42)*. One can define an analogue of σ in non–commutative geometry as well. But for even matrix K–cycles, the result* [37] *is* *simply that $\sigma(\Omega^k\mathscr{A})$ coincides with the $(k+1)^{th}$ degree of the differential ideal, where only the second degree of that ideal is non–trivial to compute. In our approach, we can have $(\sigma(\Omega^*\mathfrak{g}) \mod \pi(\mathscr{J}^*\mathfrak{g})) \neq 0$. This fact has important physical consequences: In Grand Unification models, the occurrence of $\sigma(\Omega^1\mathfrak{g}) \neq 0$ (modulo the ideal) leads to cubic terms in the Higgs potential, which guarantee the desired spontaneous symmetry breaking pattern.*

Our goal is the analysis of the ideal $\pi(\mathscr{J}^*\mathfrak{g})$. For this purpose we define formally

$$\begin{aligned}\sigma\Big(\sum_{\alpha,z\geq 0}[\iota(a^z_\alpha),[\dots[\iota(a^1_\alpha),\iota(da^0_\alpha)]\dots]]\Big)\\ := \sum_{\alpha,z\geq 0}[\pi(a^z_\alpha),[\dots[\pi(a^1_\alpha),[D^2,\pi(a^0_\alpha)]]\dots]]\;, \qquad (2.40)\end{aligned}$$

where $a^i_\alpha \in \mathfrak{g}$. In particular, from (2.40) we get

$$\sigma(\iota(da)) = [D^2,\pi(a)]\;, \qquad \sigma([\iota(a),\omega^1]) = [\pi(a),\sigma(\omega^1)]\;, \qquad (2.41)$$

for $a \in \mathfrak{g}$ and $\omega^1 \in \Omega^1\mathfrak{g}$. *We recall that D is not necessarily bounded so that it is unclear how to treat*[rw] *the commutators with D^2. However, in our only class of examples, the dangerous part of D is the Dirac operator of the spin connection, whose square is the Laplacian. This case can be treated. A restriction to that class of examples would not be a good idea, because certain formulae become so complicated that nobody realizes the structure behind. In the general context, the structure is apparent, subject to the price*[rw] *that we can only give a formal meaning to certain equations.*

We extend σ formally to $\Omega^*\mathfrak{g}$, putting $\sigma(\Omega^0\mathfrak{g}) \equiv 0$ and

$$\sigma(\sum_\alpha[\omega^k_\alpha,\tilde{\omega}^l_\alpha]) := \sum_\alpha\big([\sigma(\omega^k_\alpha),\pi(\tilde{\omega}^l_\alpha)]_g + (-1)^k[\pi(\omega^k_\alpha),\sigma(\tilde{\omega}^l_\alpha)]_g\big)\;, \qquad (2.42)$$

for $\omega^k_\alpha \in \Omega^k\mathfrak{g}$ and $\tilde{\omega}^l_\alpha \in \Omega^l\mathfrak{g}$. Note that $\sigma(\omega^k) \in \mathscr{O}_{z_{k+1}}(h)$ if $\pi(\omega^k) \in \mathscr{O}_{z_k}(h)$, where $z_n = n \mod 2$. We do not necessarily have $\sigma(\omega^k) \in \mathscr{B}(h)$. Now we prove:

Proposition 7. *We have $\pi(d\omega^k) = [-\mathrm{i}D,\pi(\omega^k)]_g + \sigma(\omega^k)$, for $\omega^k \in \Omega^k\mathfrak{g}$.*

Proof. The Proposition is clearly true for $k=0$. To prove the Proposition for $k=1$ we first consider the case $\omega^1 = \iota(da) \in \Omega^1\mathfrak{g}$. Then we have

$$[-\mathrm{i}D,\pi(\omega^1)]_g = [-\mathrm{i}D,[-\mathrm{i}D,\pi(a)]_g]_g = [(-\mathrm{i}D)^2,\pi(a)] = -\sigma(\iota(da)),$$

so that $\pi(d\omega^1) = 0$. But this is consistent with $d\omega^1 = d^2(\iota(a)) = 0$. Now we prove the Proposition for $k=1$ by induction. Because of (2.41), the linearity of π and the structure of elements of $\Omega^1\mathfrak{g}$, see (2.25), it suffices to assume that the Proposition is true for all $\omega^1 \in \Omega^1\mathfrak{g}$ and to show that from this assumption there follows

$$\pi(d[\iota(a),\omega^1]) = [-\mathrm{i}D,\pi([\iota(a),\omega^1])]_g + \sigma([\iota(a),\omega^1]),$$

[rw] Note added after completion of the thesis: This problem has been resolved in [71].

for all $a \in \mathfrak{g}$. We calculate

$$\begin{aligned}\pi(d[\imath(a),\omega^1]) &= [\pi(\imath(da)),\pi(\omega^1)]_g + [\pi(\imath(a)),\pi(d\omega^1)]_g \\ &= [[-\mathrm{i}D,\pi(a)]_g,\pi(\omega^1)]_g + [\pi(a),[-\mathrm{i}D,\pi(\omega^1)]_g + \sigma(\omega^1)]_g \\ &= [-\mathrm{i}D,[\pi(a),\pi(\omega^1)]_g]_g + \sigma([\imath(a),\omega^1]) \\ &= [-\mathrm{i}D,\pi([\imath(a),\omega^1])]_g + \sigma([\imath(a),\omega^1])\,.\end{aligned}$$

Finally, we extend the proof to any k by induction. For that purpose let us assume that the Proposition holds for $k-1$. Due to linearity we can restrict ourselves to elements $\omega^k = [\omega^1,\tilde{\omega}^{k-1}] \in \Omega^k\mathfrak{g}$. Using (2.42) and the graded Jacobi identity we calculate

$$\begin{aligned}\pi(d[\omega^1,\tilde{\omega}^{k-1}]) &= [\pi(d\omega^1),\pi(\tilde{\omega}^{k-1})]_g - [\pi(\omega^1),\pi(d\tilde{\omega}^{k-1})]_g \\ &= [[-\mathrm{i}D,\pi(\omega^1)]_g + \sigma(\omega^1),\pi(\tilde{\omega}^{k-1})]_g - [\pi(\omega^1),[-\mathrm{i}D,\pi(\tilde{\omega}^{k-1})]_g + \sigma(\tilde{\omega}^{k-1})]_g \\ &= -[\pi(\tilde{\omega}^{k-1}),[-\mathrm{i}D,\pi(\omega^1)]_g]_g - (-1)^k[\pi(\omega^1),[\pi(\tilde{\omega}^{k-1}),-\mathrm{i}D]_g]_g + \sigma([\omega^1,\tilde{\omega}^{k-1}]) \\ &= [-\mathrm{i}D,[\pi(\omega^1),\pi(\tilde{\omega}^{k-1})]_g]_g + \sigma([\omega^1,\tilde{\omega}^{k-1}])\,.\end{aligned}$$ □

Looking at Proposition 7 one remarks that we have splitted the bounded operator $\pi(d\omega^k)$ *into the two possibly unbounded operators* $[-\mathrm{i}D,\pi(\omega^k)]_g$ *and* $\sigma(\omega^k)$. *However, the relevant problem is to compute the differential* $\pi(d\omega^k)$ *for given* $\pi(\omega^k)$. *The element* $\omega^k \in \Omega^k\mathfrak{g}$ *is not known itself so that it is completely unclear how to compute the differential. The computation of* $[-\mathrm{i}D,\pi(\omega^k)]_g$ *is easy on a formal level. Moreover, it turns out that – for the class of examples we are interested in – also* $\sigma(\omega^k)$ *can be computed if* $\pi(\omega^k)$ *is given, see Section 3.5. Thus, the splitting of Proposition 7 is indeed a useful tool.*

There is another important application of Proposition 7. We recall that

$$\pi(\mathcal{J}^k\mathfrak{g}) = \{\pi(d\omega^{k-1})\,,\ \ \omega^{k-1} \in \Omega^{k-1}\mathfrak{g} \cap \ker\pi\ \}\,. \tag{2.43a}$$

From Proposition 7 we get the following equivalent characterization:

$$\pi(\mathcal{J}^k\mathfrak{g}) = \{\sigma(\omega^{k-1})\,,\ \ \omega^{k-1} \in \Omega^{k-1}\mathfrak{g} \cap \ker\pi\ \}\,. \tag{2.43b}$$

Obviously, $\sigma(\omega^{k-1})$ is bounded if $\pi(\omega^{k-1}) = 0$. Of course, (2.43b) is only a rewriting of (2.43a), but it is a convenient starting point for the analysis of $\pi(\mathcal{J}^*\mathfrak{g})$.

2.5 Graded Lie Homomorphisms

In this subsection we provide the framework for the definition of connections and gauge transformations. In non–commutative geometry, the connection form (gauge potential) is an element of $\Omega^1_D\mathcal{A}$ *and the curvature (field strength) an element of* $\Omega^2_D\mathcal{A}$. *One could expect that our goal is to find a calculus where the connection form is an element of* $\Omega^1_D\mathfrak{g}$ *and the curvature an element of* $\Omega^2_D\mathfrak{g}$. *However, there are good reasons to proceed differently. The most important reason is that in the physically relevant case of L–cycles* *over functions ⊗ matrices treated in Section 3 it would not be possible to describe gauge field theories where Abelian Lie algebras occur (see Section 3.3 for details). In particular, there would be no way to reproduce the standard model. Moreover, it would even be mathematically difficult to give a good definition for a connection where the connection form belongs to $\Omega^1_D\mathfrak{g}$. Therefore, we must invent a new framework solving the problems. Our proposal is to consider certain graded Lie homomorphisms as defined and discussed below.*

2.5.1 Definition of $\mathcal{H}^*\mathfrak{g}$ and $\hat{\mathcal{H}}^*\mathfrak{g}$

We consider a set $[\mathcal{H}^n\mathfrak{g}, \,.\,]_g$ of certain graded Lie homomorphisms of $\pi(\Omega^*\mathfrak{g})$ of n^{th} degree, which is formally defined by

$$\begin{aligned}\mathcal{H}^n\mathfrak{g} := \{\, \eta^n \in \mathcal{O}_{z_n}(h)\,,\ z_n = n \mod 2\,,\ \eta^{n*} = -(-1)^{n(n-1)/2}\,\eta^n\,,\\ [\eta^n,\pi(\Omega^k\mathfrak{g})]_g \subset \pi(\Omega^{k+n}\mathfrak{g})\,,\ [\eta^n,\pi(\jmath^k\mathfrak{g})]_g \subset \pi(\jmath^{k+n}\mathfrak{g})\,\}\,. \qquad (2.44)\end{aligned}$$

This definition requires some explanation. It is obvious that $\pi(\Omega^n\mathfrak{g}) \subset \mathcal{H}^n\mathfrak{g}$. But it is unclear what other linear operators meet the definition.[rw] *If there is any chance that the commutator $[\eta^n,\pi(\Omega^k\mathfrak{g})]_g$ belongs to $\pi(\Omega^{k+n}\mathfrak{g}) \subset \mathcal{B}(h)$ then the domain*

$$h' = \text{domain}(\eta^n) \cap \{\psi \in h\,,\ \pi(\Omega^*\mathfrak{g})\psi \subset \text{domain}(\eta^n)\}$$

where the commutator is defined must be dense in h. This is necessary to ensure that the sequence $\{\, [\eta^n,\pi(\omega)]_g\psi_k\,\}_k$ of elements of h, for $\psi_k \in h'$ and any $\omega \in \Omega^*\mathfrak{g}$, converges to $\pi(\tilde\omega)\psi$ if ψ_k tends to $\psi \in h$, where $\pi(\tilde\omega) \in \pi(\Omega^*\mathfrak{g})$ is independent of ψ_k. But this condition does not necessarily exclude unbounded linear operators on h. Nevertheless, we are forced to play with the spaces $\mathcal{H}^n\mathfrak{g}$. Therefore, it must be stressed that the following formulae have only a formal meaning. Again, we apply our physical–example–argument and assure that no bad things happen in the cases we are interested in.

Let

$$\tilde{\mathfrak{C}}^n\mathfrak{g} := \{\, j^n \in \mathcal{H}^n\mathfrak{g}\,,\ [j^n,\pi(\Omega^*\mathfrak{g})]_g = 0\,\} \qquad (2.45)$$

be the graded centralizer of $\pi(\Omega^*\mathfrak{g})$ of n^{th} degree. Then, the factor space

$$\tilde{\mathcal{H}}^*\mathfrak{g} := \bigoplus_{n\in\mathbb{N}} \tilde{\mathcal{H}}^n\mathfrak{g}\,, \qquad \tilde{\mathcal{H}}^n\mathfrak{g} := \mathcal{H}^n\mathfrak{g}\,/\,\tilde{\mathfrak{C}}^n\mathfrak{g}\,, \qquad (2.46a)$$

is a graded Lie algebra, with the graded commutator given by

$$\begin{aligned}&[[\eta^k + \tilde{\mathfrak{C}}^k\mathfrak{g}, \tilde\eta^l + \tilde{\mathfrak{C}}^l\mathfrak{g}]_g, \pi(\omega^n)]_g\\ &\qquad := [\eta^k,[\tilde\eta^l,\pi(\omega^n)]_g]_g - (-1)^{kl}[\tilde\eta^l,[\eta^k,\pi(\omega^n)]_g]_g\,, \qquad (2.46b)\end{aligned}$$

[rw] Note added after completion of the thesis: An explicit characterization of $\mathcal{H}^*\mathfrak{g}$ has been given in [71].

for $\eta^k \in \mathcal{H}^k\mathfrak{g}$, $\tilde{\eta}^l \in \mathcal{H}^l\mathfrak{g}$ and $\omega^n \in \Omega^n\mathfrak{g}$. It is clear that this equation is well–defined. Obviously, $\pi(\Omega^*\mathfrak{g})$ is a graded Lie subalgebra of $\tilde{\mathcal{H}}^*\mathfrak{g}$.

It is clear that the graded ideal $\pi(\jmath^*\mathfrak{g})$ of $\pi(\Omega^*\mathfrak{g})$ yields a graded ideal $\pi(\jmath^*\mathfrak{g}) + \tilde{\mathbb{c}}^*\mathfrak{g}$ of $\mathcal{H}^*\mathfrak{g}$, see (2.44). Therefore,

$$\hat{\mathcal{H}}^*\mathfrak{g} := \bigoplus_{n\in\mathbb{N}} \hat{\mathcal{H}}^n\mathfrak{g}\,, \qquad \hat{\mathcal{H}}^n\mathfrak{g} = \mathcal{H}^n\mathfrak{g}\,/\,\mathbb{J}^n\mathfrak{g}\,, \qquad \mathbb{J}^n\mathfrak{g} = \tilde{\mathbb{c}}^n\mathfrak{g} + \pi(\jmath^n\mathfrak{g})\,, \tag{2.47a}$$

is a graded Lie algebra. Moreover, it is a graded differential Lie algebra, too, where the graded differential is defined by

$$\begin{aligned}&[d(\eta^k + \pi(\jmath^k\mathfrak{g}) + \tilde{\mathbb{c}}^k\mathfrak{g}), \pi(\omega^n) + \pi(\jmath^n\mathfrak{g})]_g \\ &\qquad := \pi\circ d\circ\pi^{-1}([\eta^k, \pi(\omega^n)]_g) - (-1)^k[\eta^k, \pi(d\omega^n)]_g + \pi(\jmath^{k+n+1}\mathfrak{g})\,,\end{aligned} \tag{2.47b}$$

for $\eta^k \in \mathcal{H}^k\mathfrak{g}$ and $\omega^n \in \Omega^n\mathfrak{g}$. It is obvious that this equation is well–defined and that $\Omega^*_D\mathfrak{g}$ is a graded Lie subalgebra of $\hat{\mathcal{H}}^*\mathfrak{g}$.

2.5.2 The Exponential Mapping

Let

$$\begin{aligned}\mathbb{u}(\mathfrak{g}) := \{\, &\eta^0 \in \mathcal{H}^0\mathfrak{g} \cap \mathscr{B}(h)\,, \\ &\sigma\circ\pi^{-1}([\eta^0, \pi(\omega^k)]_g) - [\eta^0, \sigma(\omega^k)]_g \in \pi(\jmath^{k+1}\mathfrak{g})\,,\ \ \forall\omega^k \in \Omega^k\mathfrak{g}\,\}\,.\end{aligned} \tag{2.48}$$

Obviously, $\pi(\mathfrak{g}) \subset \mathbb{u}(\mathfrak{g})$. Let $\mathcal{O}_0 \subset \mathbb{u}(\mathfrak{g})$ be an open neighbourhood of the zero element of $\mathbb{u}(\mathfrak{g})$ and $\mathcal{O}_1 \subset \mathscr{B}(h)$ be an open neighbourhood of $\mathbb{1}_{\mathscr{B}(h)}$. For an appropriate choice of $\mathcal{O}_0$ and $\mathcal{O}_1$ we define the exponential mapping

$$\exp : \mathcal{O}_0 \to \mathcal{O}_1\,, \qquad \exp(\eta) := \mathbb{1}_{\mathscr{B}(h)} + \sum_{k=1}^{\infty}\frac{1}{k!}(\eta)^k\,,\ \ \eta \in \mathcal{O}_0\,. \tag{2.49}$$

The Baker–Campbell–Hausdorff formula for $\eta_\alpha, \eta_\beta \in \mathcal{O}_0$,

$$\begin{aligned}&\exp(\eta_\alpha)\exp(\eta_\beta) = \exp(\eta_\gamma)\,, \\ &\eta_\gamma = \eta_\alpha + \eta_\beta + \tfrac{1}{2}[\eta_\alpha, \eta_\beta] + \tfrac{1}{12}([\eta_\alpha,[\eta_\alpha,\eta_\beta]] - [\eta_\beta,[\eta_\alpha,\eta_\beta]]) + \cdots \in \mathbb{u}(\mathfrak{g})\,,\end{aligned} \tag{2.50}$$

implies that we have a multiplication in $\exp(\mathcal{O}_0)$. In particular, for η_β proportional to η_α we get

$$\exp(\lambda_1\eta)\exp(\lambda_2\eta) = \exp((\lambda_1+\lambda_2)\eta) = \exp(\lambda_2\eta)\exp(\lambda_1\eta)\,, \tag{2.51}$$

for $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\eta \in \mathcal{O}_0$. Thus, $\exp(\eta)$ is invertible in $\mathscr{B}(h)$ for each $\eta \in \mathcal{O}_0$, and the inverse is given by

$$(\exp(\eta))^{-1} = \exp(-\eta) = \exp(\eta^*) = (\exp(\eta))^*\,. \tag{2.52}$$

Therefore, all elements $\exp(\eta)$ are unitary. Since $\mathscr{B}(h)$ is a C^*–algebra we conclude that for all $\eta \in \mathbb{u}(\mathfrak{g})$ we have

$$\|\exp(\eta)\| = \|\exp(\eta)^* \exp(\eta)\|^{1/2} = \|\mathbb{1}_{\mathscr{B}(h)}\|^{1/2} = 1 \,. \tag{2.53}$$

Hence, our construction leads to the subgroup

$$\exp(\mathbb{u}(\mathfrak{g})) := \{ \textstyle\prod_{\alpha=1}^{N} \exp(\eta_\alpha) \,, \quad \eta_\alpha \in \mathcal{O}_0 \,, \quad N \text{ finite } \} \tag{2.54}$$

of the group of unitary elements of $\mathscr{B}(h)$.

For A being a linear operator on h and $\eta \in \mathcal{O}_0$ we have

$$\exp(\eta) A \exp(-\eta) = A + \sum_{k=1}^{\infty} \frac{1}{k!} [\underbrace{\eta, [\eta, \dots, [\eta,}_{k} A] \dots]] \,. \tag{2.55}$$

For $A = \pi(a) \in \pi(\mathfrak{g})$ and $\exp(\eta) = u \in \mathcal{O}_1$ we get $u\pi(a)u^* \in \pi(\mathfrak{g})$. For $A = -\mathrm{i}D$ we get $u[-\mathrm{i}D, u^*] = -\mathrm{i}(uDu^* - D) \equiv ud(u^*) \in \hat{\mathcal{H}}^1\mathfrak{g}$, because with (2.48) and (2.47b) we have

$$\begin{aligned}
&[[-\mathrm{i}D, \eta], \pi(\omega^k)]_g + \pi(\jmath^{\,k+1}\mathfrak{g}) = [-\mathrm{i}D, [\eta, \pi(\omega^k)]]_g - [\eta, [-\mathrm{i}D, \pi(\omega^k)]_g] + \pi(\jmath^{\,k+1}\mathfrak{g}) \\
&\quad = \pi \circ d \circ \pi^{-1}([\eta, \pi(\omega^k)]) - \sigma \circ \pi^{-1}([\eta, \pi(\omega^k)]) - [\eta, \pi(d\omega^k)] + [\eta, \sigma(\omega^k)] + \pi(\jmath^{\,k+1}\mathfrak{g}) \\
&\quad = [d\eta, \pi(\omega^k)]_g + \pi(\jmath^{\,k+1}\mathfrak{g}) \,.
\end{aligned}$$

If $\pi(\omega^k) \in \pi(\jmath^{\,k}\mathfrak{g})$ then $[[-\mathrm{i}D, \eta], \pi(\omega^k)]_g \in \pi(\jmath^{\,k+1}\mathfrak{g})$. Therefore, there is a natural degree–preserving representation Ad of $\exp(\mathbb{u}(\mathfrak{g}))$ in $\Omega^*_D\mathfrak{g}$ defined by

$$\begin{aligned}
\mathrm{Ad}_u \,\pi(a) &:= u\pi(a)u^* \,, \\
\mathrm{Ad}_u \,[-\mathrm{i}D, \pi(a)] &:= [-\mathrm{i}D, \mathrm{Ad}_u \,\pi(a)] + [u[-\mathrm{i}D, u^*], \mathrm{Ad}_u \,\pi(a)] \,, \\
\mathrm{Ad}_u \,(\pi(\omega^k) + \pi(\jmath^{\,k}\mathfrak{g})) &:= (\mathrm{Ad}_u \,\pi(\omega^k)) + \pi(\jmath^{\,k}\mathfrak{g}) \,, \\
\mathrm{Ad}_u \,[\varrho, \tilde{\varrho}]_g &:= [\mathrm{Ad}_u \,\varrho, \mathrm{Ad}_u \,\tilde{\varrho}]_g \,,
\end{aligned} \tag{2.56}$$

for $u \in \exp(\mathbb{u}(\mathfrak{g}))$, $a \in \mathfrak{g}$, $\omega^k \in \Omega^k\mathfrak{g}$ and $\varrho, \tilde{\varrho} \in \Omega^*_D\mathfrak{g}$. Note that due to (2.55) we have $\mathrm{Ad}_u \,\pi(\jmath^{\,k}\mathfrak{g}) \subset \pi(\jmath^{\,k}\mathfrak{g})$.

2.6 Connections and Gauge Transformations

Now we have the means at disposal to define the notion of a connection, of its curvature, of gauge transformations and of bosonic and fermionic actions. Many formulae look very similar as in non–commutative geometry and classical gauge field theory. However, our geometric objects live in complicated spaces, which enables a unified description of physical models.

2.6.1 Connection and Curvature

Definition 8. *A connection on an L–cycle is a pair* (∇, ∇_h), *where*

i) $\nabla_h : h \to h$ *is linear, odd and skew–adjoint,*
$\nabla_h \in \mathscr{O}_1(h)$, $\langle \psi, \nabla_h \tilde{\psi}\rangle_h = -\langle \nabla_h \psi, \tilde{\psi}\rangle_h$, $\forall \psi, \tilde{\psi} \in h$,

ii) $\nabla : \Omega^n_D \mathfrak{g} \to \Omega^{n+1}_D \mathfrak{g}$ *is linear,*

iii) $\nabla(\pi(\omega^n) + \pi(\jmath^n \mathfrak{g})) = [\nabla_h, \pi(\omega^n)]_g + \sigma(\omega^n) + \pi(\jmath^{n+1}\mathfrak{g})$, $\omega^n \in \Omega^n \mathfrak{g}$.

The operator $\nabla^2 : \Omega^n_D \mathfrak{g} \to \Omega^{n+2}_D \mathfrak{g}$ *is called the curvature of the connection.*

As a consequence of iii) we get with (2.42)

$$\nabla([\varrho^k, \tilde{\varrho}^l]_g) = [\nabla(\varrho^k), \tilde{\varrho}^l]_g + (-1)^k [\varrho^k, \nabla(\tilde{\varrho}^l)]_g , \quad \varrho^k \in \Omega^k_D \mathfrak{g} , \ \tilde{\varrho}^l \in \Omega^l_D \mathfrak{g} . \tag{2.57}$$

Proposition 9. *Any connection has the form* $\big(\nabla = d + [\hat{\rho}, \,.\,]_g , \nabla_h = -\mathrm{i}D + \rho\big)$, *for* $\rho \in \mathcal{H}^1 \mathfrak{g}$ *and* $\hat{\rho} := \rho + \tilde{\mathfrak{c}}^1 \mathfrak{g} \in \hat{\mathcal{H}}^1 \mathfrak{g}$. *Its curvature is* $\nabla^2 = [\theta, \,.\,]$, *with* $\theta = d\hat{\rho} + \frac{1}{2}[\hat{\rho}, \hat{\rho}]_g \in \hat{\mathcal{H}}^2 \mathfrak{g}$.

Proof. There is a canonical connection given by $(\nabla = d, \nabla_h = -\mathrm{i}D)$. Items i) and ii) of Definition 8 are obvious. For iii) we find with Proposition 7

$$[-\mathrm{i}D, \pi(\omega^k)]_g + \sigma(\omega^k) = \pi(d\omega^k) . \tag{2.58}$$

Taking $\omega \in \ker \pi$ we see that iii) is well–defined. Let $(\nabla^{(1)}, \nabla^{(1)}_h)$ and $(\nabla^{(2)}, \nabla^{(2)}_h)$ be two connections. Then we get from iii) of Definition 8

$$(\nabla^{(1)} - \nabla^{(2)})(\pi(\omega^k) + \pi(\jmath^k \mathfrak{g})) = [\nabla^{(1)}_h - \nabla^{(2)}_h, \pi(\omega^k)]_g + \pi(\jmath^{k+1}\mathfrak{g}) , \tag{2.59}$$

for $\omega^k \in \Omega^k \mathfrak{g}$. Now, item ii) yields $\rho := \nabla^{(1)}_h - \nabla^{(2)}_h \in \mathcal{H}^1 \mathfrak{g}$. Since a modification of ρ by an element of $\tilde{\mathfrak{c}}^1 \mathfrak{g} \equiv \mathbb{J}^1 \mathfrak{g}$ does not change formula (2.59), we get $\nabla^{(1)} - \nabla^{(2)} = [\hat{\rho}, \,.\,]$, where $\hat{\rho} := \rho + \tilde{\mathfrak{c}}^1 \mathfrak{g} \in \hat{\mathcal{H}}^1 \mathfrak{g}$. Taking $(\nabla^{(2)}, \nabla^{(2)}_h) = (d, -\mathrm{i}D)$ we obtain $(\nabla^{(1)}, \nabla^{(1)}_h) = (d + [\hat{\rho}, \,.\,]_g , -\mathrm{i}D + \rho)$.

Note that if $\sigma(\omega^k) \in \pi(\jmath^{k+1}\mathfrak{g})$ for all $\omega^k \in \pi(\Omega^k \mathfrak{g})$ then there is $-\mathrm{i}D \in \mathcal{H}^1 \mathfrak{g}$. Thus, the assertion remains true although the connection $(\nabla = d, \nabla_h = -\mathrm{i}D)$ is not distinguished in this case.

Finally, we compute the curvature ∇^2 . For $\omega^k \in \Omega^k \mathfrak{g}$ we have with (2.46)

$$\begin{aligned}
\nabla^2(\pi(\omega^k) + \pi(\jmath^k \mathfrak{g})) &= \nabla(\pi(d\omega^k) + [\hat{\rho}, \pi(\omega^k)]_g + \pi(\jmath^{k+1}\mathfrak{g})) \\
&= [\hat{\rho}, \pi(d\omega^k)]_g + \pi \circ d \circ \pi^{-1}([\hat{\rho}, \pi(\omega^k)]_g) + [\hat{\rho}, [\hat{\rho}, \pi(\omega^k)]_g]_g + \pi(\jmath^{k+2}\mathfrak{g}) \\
&\equiv [d\hat{\rho} + \tfrac{1}{2}[\hat{\rho}, \hat{\rho}]_g, \pi(\omega^k) + \pi(\jmath^k \mathfrak{g})]_g =: [\theta, \pi(\omega^k) + \pi(\jmath^k \mathfrak{g})] . \qquad \square
\end{aligned}$$

Proposition 9 tells us that the connection form ρ *is an element of* $\mathcal{H}^1 \mathfrak{g}$. *However, we do not learn very much, because we have no control over that space. This situation* *changes drastically in the case of functions $\otimes$ matrices studied in Section 3. There, we consider local connection forms which commute with functions. Then, the defining formula* (2.44) *for $\mathcal{H}^1\mathfrak{g}$ can be evaluated.*

Note that the relation between $\rho \in \mathcal{H}^1\mathfrak{g}$ and $\rho' \in \mathcal{H}^1\mathfrak{g}$ in (2.59),

$$[\rho, \pi(\omega^k)]_g + \pi(\jmath^{k+1}\mathfrak{g}) = [\rho', \pi(\omega^k)]_g + \pi(\jmath^{k+1}\mathfrak{g}) \;,$$

may have more solutions than $\rho' = \rho + \tilde{\mathfrak{c}}^1\mathfrak{g}$. However, we shall regard ρ and ρ' as different connection forms if $\rho - \rho' \notin \tilde{\mathfrak{c}}^1\mathfrak{g}$. Analogously, the determining equation for $\theta' \in \hat{\mathcal{H}}^2\mathfrak{g}$,

$$[\theta', \varrho]_g = [\theta, \varrho]_g \quad \text{for all } \varrho \in \Omega_D^*\mathfrak{g} \;,$$

may have more solutions than $\theta' = \theta$. However, we shall select always the canonical representative $\theta = d\hat{\rho} + \frac{1}{2}[\hat{\rho}, \hat{\rho}]_g$ in the curvature form of the connection ∇^2. Often we shall denote $\theta \in \hat{\mathcal{H}}^2\mathfrak{g}$ itself instead of ∇^2 the curvature of the connection (∇, ∇_h).

One may ask whether a different definition of the ideal $\mathbb{J}^*\mathfrak{g}$ given in (2.47a) *could eliminate the above ambiguity. It turns out that this is the case. However, if we enlarge $\mathbb{J}^*\mathfrak{g}$ then there is a risk that the factorization removes interesting components of the curvature. Indeed, this happens in the standard model. A definition of $\mathbb{J}^*\mathfrak{g}$ removing the ambiguity also removes the Higgs potential.*

2.6.2 The Gauge Group

Definition 10. *The gauge group of the L–cycle is the group $\mathcal{U}(\mathfrak{g}) := \exp(\mathbb{u}(\mathfrak{g}))$ defined in* (2.54)*. Gauge transformations of the connection are given by*

$$(\nabla, \nabla_h) \mapsto (\nabla', \nabla_h') := (\mathrm{Ad}_u \nabla \mathrm{Ad}_{u^*} \,, u\nabla_h u^*) \,, \quad u \in \mathcal{U}(\mathfrak{g}) \,.$$

We must check that the definition of gauge transformations of a connection is compatible with Definition 8:

$$\begin{aligned}
[\nabla_h', \pi(\omega^n)]_g + \pi(\jmath^{n+1}\mathfrak{g}) &= u[\nabla_h, u^*\pi(\omega^n)u]_g u^* + \pi(\jmath^{n+1}\mathfrak{g}) \\
&= \mathrm{Ad}_u \big(\nabla(\mathrm{Ad}_{u^*}(\pi(\omega^n) + \pi(\jmath^n\mathfrak{g}))) - \sigma(\pi^{-1} \circ \mathrm{Ad}_{u^*} \circ \pi(\omega^n)) + \pi(\jmath^{n+1}\mathfrak{g})\big) \\
&= \nabla'(\pi(\omega^n) + \pi(\jmath^n\mathfrak{g})) - \mathrm{Ad}_u(\sigma(\pi^{-1} \circ \mathrm{Ad}_{u^*} \circ \pi(\omega^n))) + \pi(\jmath^{n+1}\mathfrak{g}) \;.
\end{aligned}$$

Thus, the definition is consistent iff $\sigma(\pi^{-1} \circ \mathrm{Ad}_u \circ \pi(\omega^n)) + \pi(\jmath^{n+1}\mathfrak{g}) = \mathrm{Ad}_u(\sigma(\omega^n)) + \pi(\jmath^{n+1}\mathfrak{g})$. But this equation is satisfied due to (2.48).

The gauge transformation of the connection form ρ occurring in the connection $\nabla_h = -\mathrm{i}D + \rho$ is defined by

$$\nabla_h' =: -\mathrm{i}D + \gamma_u(\rho) \;. \tag{2.60}$$

From $\nabla_h'\psi = u(-\mathrm{i}D + \rho)u^*\psi = (-\mathrm{i}D + u[-\mathrm{i}D, u^*] + u\rho u^*)\psi$ one finds

$$\gamma_u(\rho) = udu^* + u\rho u^* \;. \tag{2.61}$$

The gauge transformation of the curvature is due to

$$(\mathrm{Ad}_u \nabla \mathrm{Ad}_{u^*})^2(\varrho^k) = \mathrm{Ad}_u \nabla^2 \mathrm{Ad}_{u^*} \varrho^k = u[\theta, u^* \varrho^k u]u^*$$

given by

$$\gamma_u(\theta) = \mathrm{Ad}_u \theta = u\theta u^* \,. \tag{2.62}$$

2.6.3 Bosonic and Fermionic Actions

The Dixmier trace provides a canonical scalar product $\langle\, , \rangle$ on $\mathscr{B}(h)$, see [17]. If the L–cycle is d^+–summable (see Definition 4) we define for $\tau, \tilde{\tau} \in \mathscr{B}(h)$

$$\langle \tau, \tilde{\tau} \rangle := \mathrm{Tr}_\omega(\tau^* \tilde{\tau}\, |D|^{-\mathrm{d}}) \,. \tag{2.63}$$

We assume that in some sense there exists an extension of this formula to linear operators on h belonging to $\mathcal{H}^2\mathfrak{g}$ (recall that $\mathcal{H}^2\mathfrak{g}$ is bounded on a dense subset of h).

Definition 11. *The bosonic action S_B and the fermionic action S_F of the connection (∇, ∇_h) are given by*

$$S_B(\nabla) = \langle \theta, \theta \rangle_{\hat{\mathcal{H}}^2\mathfrak{g}} := \min_{j^2 \in \mathbb{J}^2\mathfrak{g}} \mathrm{Tr}_\omega((\theta_0 + j^2)^2\, |D|^{-\mathrm{d}}) \,, \tag{2.64a}$$

$$S_F(\psi, \nabla_h) := \langle \psi, \mathrm{i}\nabla_h \psi \rangle_h \,, \quad \psi \in h \,, \tag{2.64b}$$

where Tr_ω is the Dixmier trace, $\langle\, , \rangle_h$ the scalar product on h and $\theta_0 \in \mathcal{H}^2\mathfrak{g}$ any representative of the curvature of ∇.

Since both $\langle\, , \rangle_{\hat{\mathcal{H}}^2\mathfrak{g}}$ and $\langle\, , \rangle_h$ are invariant under unitary transformations [17] we get from (2.62) and Definition 10 that the action (2.64) is invariant under gauge transformations

$$(\nabla, \nabla_h) \mapsto (\mathrm{Ad}_u \nabla \mathrm{Ad}_{u^*}, u\nabla_h u^*) \,, \quad \psi \mapsto u\psi \,, \;\; u \in \mathcal{U}(\mathfrak{g}) \,. \tag{2.65}$$

There is an equivalent formulation of (2.64a). Let $\mathfrak{e}(\theta_0 + j^2) \in \mathcal{H}^2\mathfrak{g}$ be those representative of $\theta \in \hat{\mathcal{H}}^2\mathfrak{g}$, for which the minimum in (2.64a) is attained. Let $j^2 = \sum_\alpha \lambda_\alpha j^2_\alpha$, for $\lambda_\alpha \in \mathbb{R}$, be a parametrization of $j^2 \in \mathbb{J}^2\mathfrak{g}$. Then,

$$0 = \frac{d}{d\lambda_\alpha} \mathrm{Tr}_\omega((\theta_0 + j^2)^2\, |D|^{-\mathrm{d}}) = 2\, \mathrm{Tr}_\omega((\theta_0 + j^2) j^2_\alpha)\, |D|^{-\mathrm{d}}) \,.$$

Thus, $\mathfrak{e}(\theta_0 + j^2) \equiv \mathfrak{e}(\theta)$ is those representative of θ, which is orthogonal to the ideal $\mathbb{J}^2\mathfrak{g}$ with respect to $\langle\, , \rangle_{\hat{\mathcal{H}}^2\mathfrak{g}}$:

$$S_B = \mathrm{Tr}_\omega((\mathfrak{e}(\theta))^2\, |D|^{-\mathrm{d}}) \,, \qquad\qquad \mathrm{Tr}_\omega(\mathfrak{e}(\theta)\, \mathbb{J}^2\mathfrak{g}\, |D|^{-\mathrm{d}}) \equiv 0 \,. \tag{2.66}$$

The representative $\mathfrak{e}(\theta)$ is unique, because $\mathrm{Tr}_\omega(\,.\, |D|^{-\mathrm{d}})$ is positive definite [17]:

$$\mathrm{Tr}_\omega((\mathfrak{e}(\theta) + j^2)^2\, |D|^{-\mathrm{d}}) = \mathrm{Tr}_\omega((\mathfrak{e}(\theta))^2\, |D|^{-\mathrm{d}}) + \mathrm{Tr}_\omega((j^2)^2 |D|^{-\mathrm{d}}) > \mathrm{Tr}_\omega((\mathfrak{e}(\theta))^2\, |D|^{-\mathrm{d}}) \,,$$

for all $j^2 \neq 0$.

3 L–Cycles over Functions $\otimes$ Matrix Lie Algebra

So far, we have developed our approach in a very general context. This is not the most convenient picture for physical calculations. The construction of classical gauge field theories is related to the special situation that all data of an L–cycle split into space–time part and matrix part. Then, it is natural to ask how the algebraic and geometric objects of our theory split into space–time and matrix parts. Section 3 is devoted to the investigation of that question. Here, we treat the general case. This means that we study the decomposition of our graded Lie algebras and their graded ideals up to arbitrary degree. This leads to a plenty of confusing formulae. Fortunately, for any case of physical relevance we will only need the splitting of the lowest degrees. The corresponding formulae are rather simple. Thus, the reader who is interested in physical applications should focus her or his attention to the relevant degrees.

3.1 A Class of L–Cycles Relevant to Physics

Let $(\mathfrak{a}, \mathbb{C}^F, \mathcal{M}, \hat{\pi}, \hat{\Gamma})$ be an L–cycle over a matrix Lie algebra $\mathfrak{a}$. In particular, we have a representation $\hat{\pi}$ of $\mathfrak{a}$ in the Lie algebra $\mathrm{M}_F\mathbb{C}$ of endomorphisms of the Hilbert space $\mathbb{C}^F$. Moreover, the grading operator $\hat{\Gamma}$ anticommutes with the generalized Dirac operator $\mathcal{M}$ and commutes with $\hat{\pi}(\mathfrak{a})$. Both $\mathcal{M}$ and $\hat{\Gamma}$ belong to $\mathrm{M}_F\mathbb{C}$.

Let X be a compact even dimensional Riemannian spin manifold, $\dim(X) = N \geq 4$, and let $C^\infty(X)$ be the algebra of real–valued smooth functions on X. Since $C^\infty(X)$ is a commutative algebra, the tensor product

$$\mathfrak{g} := C^\infty(X) \otimes \mathfrak{a} \tag{3.1a}$$

over $\mathbb{R}$ is in a natural way a Lie algebra, where the commutator is given by

$$[f_1 \otimes a_1, f_2 \otimes a_2] \equiv f_1 f_2 \otimes [a_1, a_2]\,, \quad f_1, f_2 \in C^\infty(X)\,, \;\; a_1, a_2 \in \mathfrak{a}\,. \tag{3.1b}$$

We introduce the Hilbert space

$$h := L^2(X, S) \otimes \mathbb{C}^F\,, \tag{3.2}$$

where $L^2(X,S)$ denotes the Hilbert space of square integrable sections of the spinor bundle over X. The representation $\hat{\pi} : \mathfrak{a} \to \mathrm{End}(\mathbb{C}^F)$ and the $C^\infty(X)$–module structure of $L^2(X,S)$ induce a natural representation π of $\mathfrak{g}$ in $\mathscr{B}(h)$:

$$\pi(f \otimes a)(s \otimes \varphi) := fs \otimes \hat{\pi}(a)\varphi\,, \tag{3.3}$$

for $f \in C^\infty(X)$, $a \in \mathfrak{a}$, $s \in L^2(X,S)$ and $\varphi \in \mathbb{C}^F$. We denote by γ the grading operator and by D the Dirac operator associated to the spin connection on the Hilbert space $L^2(X,S)$, see Section 3.2 for more details. Then we put

$$D := \mathsf{D} \otimes \mathbb{1}_F + \gamma \otimes \mathcal{M}\,, \tag{3.4a}$$

$$\Gamma := \gamma \otimes \hat{\Gamma}\,. \tag{3.4b}$$

The operator $[D, \pi(f \otimes a)]$ is bounded on h for all $f \otimes a \in \mathfrak{g}$. Moreover, D is selfadjoint on h, because D and γ are selfadjoint on $L^2(X, S)$ and $\mathcal{M}$ is symmetrical. Next, Γ commutes with $\pi(\mathfrak{g})$ and anticommutes with D. Finally, $(\mathrm{id}_h + D^2)^{-1}$ is compact, see [27]: The operator $(\mathrm{id}_h + D^2)^{-1}$ is a pseudo–differential operator of order -2 with compact support and has, therefore, an extension to a continuous operator from H_s to H_{s+2} on the Sobolev scale $\{H_s\}$. Due to Rellich's lemma, the embedding $e : H_t \hookrightarrow H_s$ is compact for $t > s$. Thus, $(\mathrm{id}_h + D^2)^{-1}$ considered as

$$e \circ (\mathrm{id}_h + D^2)^{-1} : H_s \to H_s$$

is compact, and $(\mathfrak{g}, h, D, \pi, \Gamma)$ forms an L–cycle.

Finally, we briefly recall how the physical data specified in the Introduction fit into this scheme. First, one constructs a Euclidian version of the gauge field theory. This means that X is the one–point compactification of the Euclidian space–time manifold. The Euclidian fermions ψ constitute the Hilbert space h of the L–cycle. In some cases, it may be necessary to work with several copies of the fermions. Given the group $\mathcal{G} = C^\infty(X) \otimes G$ of local gauge transformations, we take $\mathfrak{g} = C^\infty(X) \otimes \mathfrak{a}$ as the Lie algebra of $\mathcal{G}$. The representation $\pi = \mathrm{id} \otimes \hat{\pi}$ of $\mathfrak{g}$ in $\mathscr{B}(h)$ is just the differential $\tilde{\pi}_*$ of the representation $\tilde{\pi} = \mathrm{id} \otimes \tilde{\pi}_0$ of $\mathcal{G}$. The matrix $\mathcal{M}$ occurring in the generalized Dirac operator (3.4a) *contains the fermionic mass parameters and possibly contributions required by the desired symmetry breaking scheme. The grading operator Γ represents the chirality properties of the fermions. We have $\gamma = \gamma^5$ in four dimensions. After the Wick rotation to Minkowski space we use Γ to impose a chirality condition on h. Then, $\gamma \otimes \mathcal{M}$ must coincide with the fermionic mass matrix $\widetilde{\mathcal{M}}$ on chiral fermions.*

3.2 Notations and Techniques

This subsection is devoted to definitions and techniques related to sections of the Clifford bundle. The essential idea is to represent the exterior differential algebra by gamma matrices. This language is perfectly adapted to our situation, because there is no rupture between exterior differential forms and "matrix differential forms". In particular, we can easily define the scalar product of differential forms, the interior product and the codifferential without introducing the Hodge star operator with its ugly sign conventions. We refer to Appendix A for longer proofs.

3.2.1 Exterior and Interior Products

We denote by $\Gamma^\infty(C)$ the set of smooth sections of the Clifford bundle C over X and by $C^k \subset \Gamma^\infty(C)$ the set of those sections of C, whose values at each point $x \in X$ belong to the subspace spanned by products of less than or equal k elements of $T^*_x X$ of the same parity. In particular, we identify $C^\infty(X) \equiv C^0$.

We recall [7] that there is an isomorphism of vector spaces

$$c : \Lambda^*(\Gamma^\infty(T^*X)) \to \Gamma^\infty(C) \tag{3.5}$$

between $\Gamma^\infty(C)$ and the exterior differential algebra $\Lambda^*(\Gamma^\infty(T^*X))$ of antisymmetrized tensor products of the vector space of smooth sections of the cotangent bundle over X. In particular, the restriction to the first degree yields a vector space isomorphism $c : \Gamma^\infty(T^*X) \to C^1$. Therefore, elements $c^1 \in C^1$ have the form $c^1 = c(\boldsymbol{\omega}^1)$, for $\boldsymbol{\omega}^1 \in \Gamma^\infty(T^*X)$. We use the following sign convention for the defining relation of the Clifford action:

$$\tfrac{1}{2}(c(\boldsymbol{\omega}^1)c(\tilde{\boldsymbol{\omega}}^1)+c(\tilde{\boldsymbol{\omega}}^1)c(\boldsymbol{\omega}^1)) \equiv \tfrac{1}{2}\{c(\boldsymbol{\omega}^1), c(\tilde{\boldsymbol{\omega}}^1)\} = g^{-1}(\boldsymbol{\omega}^1, \tilde{\boldsymbol{\omega}}^1)1 \in C^0 , \tag{3.6}$$

where $g^{-1} : \Gamma^\infty(T^*X)\times\Gamma^\infty(T^*X) \to C^\infty(X)$ is the inverse of the metric $g : \Gamma^\infty(T_*X)\times \Gamma^\infty(T_*X) \to C^\infty(X)$.

Let us define the notion of the exterior product $\wedge$:

$$c_1^1 \wedge c_2^1 \wedge \ldots \wedge c_n^1 := \frac{1}{n!}\sum_{\pi\in P^n}(-1)^{\mathrm{sign}(\pi)} c^1_{\pi(1)}c^1_{\pi(2)}\ldots c^1_{\pi(n)} , \quad c_i^1 \in C^1 , \tag{3.7}$$

where the sum runs over all permutations of the numbers $1,\ldots,n$ and the product on the r.h.s. is pointwise the product in the Clifford algebra. Observe that $\wedge$ is associative and that the antisymmetrization (3.7) yields zero for $n > N = \dim(X)$.

Definition 12. *$\Lambda^n \subset C^n$ is the vector subspace generated by elements of the form* (3.7)*, with $\Lambda^0 \equiv C^0$, $\Lambda^1 \equiv C^1$ and $\Lambda^n \equiv \{0\}$ for $n < 0$ and $n > \dim(X)$.*

We define the interior product $\lrcorner : \Lambda^1 \times \Lambda^n \to \Lambda^{n-1}$ by

$$c_0^1 \lrcorner (c_1^1 \wedge c_2^1 \wedge \ldots \wedge c_n^1) := \sum_{j=1}^{n}(-1)^{j+1}\tfrac{1}{2}\{c_0^1, c_j^1\}(c_1^1 \wedge \overset{\overset{j}{\vee}}{\ldots} \wedge c_n^1) , \tag{3.8a}$$

$$c_1^1 \wedge \overset{\overset{j}{\vee}}{\ldots} \wedge c_n^1 := c_1^1 \wedge c_2^1 \wedge \ldots \wedge c_{j-1}^1 \wedge c_{j+1}^1 \wedge \ldots \wedge c_n^1 . \tag{3.8b}$$

The interior product (3.8a) is extended to $\lrcorner : \Lambda^k \times \Lambda^n \to \Lambda^{n-k}$ by

$$\begin{aligned}(\tilde{c}_1^1 \wedge \tilde{c}_2^1 \wedge \ldots \wedge \tilde{c}_k^1) &\lrcorner (c_1^1 \wedge c_2^1 \wedge \ldots \wedge c_n^1) \\ &:= \tilde{c}_1^1 \lrcorner (\ldots \lrcorner (\tilde{c}_{k-1}^1 \lrcorner (\tilde{c}_k^1 \lrcorner (c_1^1 \wedge c_2^1 \wedge \ldots \wedge c_n^1)))\ldots).\end{aligned} \tag{3.9}$$

Lemma 13. *For $c_i^1 \in C^1$ we have*

$$\tfrac{1}{2}(c_0^1(c_1^1 \wedge \ldots \wedge c_n^1) + (-1)^n(c_1^1 \wedge \ldots \wedge c_n^1)c_0^1) = c_0^1 \wedge c_1^1 \wedge c_2^1 \wedge \ldots \wedge c_n^1 , \tag{3.10a}$$

$$\tfrac{1}{2}(c_0^1(c_1^1 \wedge \ldots \wedge c_n^1) - (-1)^n(c_1^1 \wedge \ldots \wedge c_n^1)c_0^1) = c_0^1 \lrcorner (c_1^1 \wedge c_2^1 \wedge \ldots \wedge c_n^1) . \tag{3.10b}$$

Proof. See Appendix A. □

We will make extensive use of Lemma 13.

3.2.2 Exterior Differential and Codifferential

Let $\{e^j\}_{j=1}^N$ be an arbitrary selfadjoint basis of $\Gamma^\infty(T^*X)$ and $\{e_j\}_{j=1}^N$ its dual basis of $\Gamma^\infty(T_*X)$. Duality of $\{e_j\}_{j=1}^N$ and $\{e^j\}_{j=1}^N$ is understood in the sense

$$e^j(e_i) \equiv \langle e^j, e_i\rangle = \delta_i^j \tag{3.11}$$

and selfadjointness means $c(e^j) = c(e^j)^*$. Let ∇_v be the Levi–Civita covariant derivative with respect to the vector field $v \in \Gamma^\infty(T_*X)$. Then we define the exterior differential $\mathbf{d} : \Lambda^k \to \Lambda^{k+1}$ on Λ^* by

$$\mathbf{d}c^k := \sum_{j=1}^N c(e^j) \wedge \nabla_{e_j}(c^k)\,, \quad c^k \in \Lambda^k\,. \tag{3.12}$$

We check in Appendix A that $\mathbf{d}$ is indeed a graded differential. There is a natural scalar product $\langle\ ,\ \rangle_{\Lambda^*}$ on Λ^*:

$$\langle c^k, \tilde{c}^l\rangle_{\Lambda^*} := \int_X \mathrm{v}_g \ \mathrm{tr}_c(c^{k^*}\tilde{c}^l)\,, \quad c^k \in \Lambda^k\,,\ \ \tilde{c}^l \in \Lambda^l\,, \tag{3.13}$$

where $\mathrm{tr}_c : \Gamma^\infty(C) \to C^\infty(X)$ is pointwise the trace in the Clifford algebra and v_g the canonical volume form on X. The scalar product (3.13) vanishes for $k \neq l$. Via this scalar product we define the codifferential $\mathbf{d}^* : \Lambda^k \to \Lambda^{k-1}$ on Λ^* as the operator dual to the exterior differential $\mathbf{d}$:

$$\langle \mathbf{d}c^k, \tilde{c}^{k+1}\rangle_{\Lambda^*} =: \langle c^k, \mathbf{d}^*\tilde{c}^{k+1}\rangle_{\Lambda^*}\,, \quad \forall c^k \in \Lambda^k\,,\ \ c^{k+1} \in \Lambda^{k+1}\,. \tag{3.14}$$

Lemma 14. *Within our conventions one has the representation*

$$\mathbf{d}^*c^k = -\sum_{j=1}^N c(e^j) \lrcorner \nabla_{e_j}(c^k)\,. \tag{3.15}$$

Proof. See Appendix A. □

Using (3.12) and (3.15) one easily derives for $c_i^1 \in C^1 \equiv \Lambda^1$ the formulae

$$\mathbf{d}^*(c_1^1 \wedge c_2^1 \wedge \ldots \wedge c_n^1) \tag{3.16a}$$
$$= \textstyle\sum_{k=1}^n \left(-(-1)^{k+1}\nabla_{g^{-1}(c^{-1}(c_k^1))}(c_1^1 \wedge \overset{k}{\overset{\vee}{\ldots}} \wedge c_n^1) + (-1)^{k+1}\mathbf{d}^*(c_k^1)(c_1^1 \wedge \overset{k}{\overset{\vee}{\ldots}} \wedge c_n^1)\right)\,,$$

$$\mathbf{d}(c_0^1 \lrcorner c_1^1) = \nabla_{g^{-1}(c^{-1}(c_0^1))}(c_1^1) + \nabla_{g^{-1}(c^{-1}(c_1^1))}(c_0^1) - c_0^1 \lrcorner \mathbf{d}c_1^1 - c_1^1 \lrcorner \mathbf{d}c_0^1\,, \tag{3.16b}$$

where g^{-1} is treated as an isomorphism from $\Gamma^\infty(T^*X)$ to $\Gamma^\infty(T_*X)$. Thus, $\mathbf{d}^*$ is not a derivation, in contrast to what its name suggests.

3.2.3 Identities for the Dirac Operator

In terms of the above introduced selfadjoint bases $\{e^j\}_{j=1}^N$ of $\Gamma^\infty(T^*X)$ and $\{e_j\}_{j=1}^N$ of $\Gamma^\infty(T_*X)$, the classical Dirac operator is given by [7]

$$\mathsf{D} = \sum_{j=1}^{N} \mathrm{i}c(e^j)\nabla^S_{e_j} \ . \tag{3.17}$$

Here, ∇^S_v is the spin covariant derivative on $L^2(X,S)$ with respect to the vector field v. It has the property

$$[\nabla^S_v, c(\boldsymbol{\omega})] = c(\nabla_v\boldsymbol{\omega}) \equiv \nabla_v c(\boldsymbol{\omega}) \ , \tag{3.18}$$

for any differential form $\boldsymbol{\omega}$. With (3.12) this gives immediately

$$[\mathsf{D}, f] = \textstyle\sum_{j=1}^N \mathrm{i}c(e^j)[\nabla^S_{e_j}, f] \equiv \mathrm{i}\mathbf{d}f \equiv \mathrm{i}c(\mathsf{d}f) \ , \quad f\in C^\infty(X) \ , \tag{3.19}$$

where d is the usual exterior differential on the exterior differential algebra. The grading operator on $L^2(X,S)$ is $\boldsymbol{\gamma} = -\mathrm{i}^{N/2}c(\mathrm{v}_g)$, fulfilling

$$\begin{aligned}\mathsf{D}\boldsymbol{\gamma} + \boldsymbol{\gamma}\mathsf{D} &= \mathrm{i}^{-1+N/2}\textstyle\sum_{j=1}^N (c(e^j)[\nabla^S_{e_j}, c(\mathrm{v}_g)] + (c(e^j)c(\mathrm{v}_g) + c(\mathrm{v}_g)c(e^j))\nabla^S_{e_j}) \\ &= \mathrm{i}^{-1+N/2}\textstyle\sum_{j=1}^N (c(e^j)c(\nabla_{e^j}(\mathrm{v}_g)) + 2c(e^j)\wedge c(\mathrm{v}_g)\nabla^S_{e_j}) \equiv 0 \ ,\end{aligned} \tag{3.20}$$

because of the properties $\nabla_v(\mathrm{v}_g)\equiv 0$ and $c(e^j)\wedge c(\mathrm{v}_g) \in \Lambda^{N+1}\equiv 0$. Therefore, the Dirac operator D is an odd first order differential operator. One has $\boldsymbol{\gamma}^2 = (-1)^{N/2}c(\mathrm{v}_g)c(\mathrm{v}_g) = \det g^{-1}$. If we restrict ourselves to an orthogonal metric, which we do for the rest of this thesis, then we have $\boldsymbol{\gamma}^2 = 1$.

Next, using (3.12), (3.15) and Lemma 13 we have for $c^k\in\Lambda^k$

$$\begin{aligned}(-\mathrm{i}\mathsf{D})c^k - (-1)^k c^k(-\mathrm{i}\mathsf{D}) &= \textstyle\sum_{j=1}^N (c(e^j)[\nabla^S_{e_j}, c^k] + (c(e^j)c^k - (-1)^k c^k c(e^j))\nabla^S_{e_j}) \\ &= \mathbf{d}c^k - \mathbf{d}^* c^k + 2\textstyle\sum_{j=1}^N c(e^j)\,\lrcorner\, c^k\,\nabla^S_{e_j} \\ &= \mathbf{d}c^k - \mathbf{d}^* c^k + 2\textstyle\sum_{i=1}^k (-1)^{i+1} c_1^1\wedge \overset{\overset{i}{\vee}}{\ldots}\wedge c_k^1\,\nabla^S_{g^{-1}(c^{-1}(c_i^1))} \ ,\end{aligned} \tag{3.21}$$

if $c^k = c_1^1\wedge c_2^1\wedge\ldots c_k^1$, $c_i^1\in\Lambda^1$. The last identity in (3.21) is due to

$$\begin{aligned}2\textstyle\sum_{j=1}^N c(e^j)\,\lrcorner\, c^k\,\nabla^S_{e^j} &= \textstyle\sum_{j=1}^N\sum_{i=1}^k (-1)^{i+1}\{c(e^j), c_i^1\}\, c_1^1\wedge \overset{\overset{i}{\vee}}{\ldots}\wedge c_k^1\,\nabla^S_{e^j} \\ &= 2\textstyle\sum_{j=1}^N\sum_{i=1}^k (-1)^{i+1} g^{-1}(e^j, c^{-1}(c_i^1))\, c_1^1\wedge \overset{\overset{i}{\vee}}{\ldots}\wedge c_k^1\,\nabla^S_{e_j} \\ &= 2\textstyle\sum_{i=1}^k (-1)^{i+1} c_1^1\wedge \overset{\overset{i}{\vee}}{\ldots}\wedge c_k^1\,\nabla^S_{g^{-1}(c^{-1}(c_i^1))} \ .\end{aligned}$$

In particular,

$$[\mathsf{D}^2, f] = \Delta f - 2\nabla^S_{\mathrm{grad}\,f} \ , \quad f\in C^\infty(X) \ , \tag{3.22}$$

where $\mathrm{grad}\,f := g^{-1}(\mathsf{d}f)$ is the vector field dual to $\mathsf{d}f$ and Δ the scalar Laplacian,

$$\Delta f \equiv \mathbf{d}^*\mathbf{d}f = -\textstyle\sum_{i,j=1}^N g^{-1}(e^i, e^j)(\nabla_{e_i}\nabla_{e_j} - \nabla_{\nabla_{e_i}e_j})(f) \ . \tag{3.23}$$

The proof is given in Appendix A.

3.3 The Representation of $\Omega^*\mathfrak{g}$ on the Hilbert Space

Now we begin with the decomposition of the first algebraic object, of the graded Lie algebra $\pi(\Omega^*\mathfrak{g})$, into space–time and matrix parts. It turns out that there are obstructions if Abelian Lie algebras occur, see Section 3.3.1 for the result and Section 3.3.2 for the motivation. The decomposition of $\pi(\Omega^1\mathfrak{g})$ is straightforward, see Section 3.3.2, and the result is given in formula (3.30). *There is a special behaviour of the Abelian part of the Lie algebra, which forced us to consider the graded Lie homomorphisms in Section 2.5. We have learned in Section 2.2.5 that elements of $\pi(\Omega^n\mathfrak{g})$, $n > 1$, are sums of repeated commutators of elements of $\pi(\Omega^1\mathfrak{g})$. Therefore, we can derive the decomposition of $\pi(\Omega^n\mathfrak{g})$ by induction, see Proposition 15. Again, the Abelian part plays a special rôle. In that decomposition there occur certain types of anticommutators of elements of $\pi(\Omega^*\mathfrak{g})$ of smaller degree. We define these anticommutators in Section 3.3.3.*

3.3.1 Decomposition of the Matrix Lie Algebra

For physical applications we are interested in the case that the matrix Lie algebra $\mathfrak{a}$ decomposes into

$$\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}'' , \tag{3.24}$$

Here, $\mathfrak{a}'$ is unitary and semisimple, i.e. a direct sum of simple unitary Lie algebras, and $\mathfrak{a}''$ is a direct sum of copies of the Abelian Lie algebra $\mathrm{u}(1)$, each of them represented in the form $\mathrm{u}(1)_{(i)} = \mathbb{R}\mathrm{b}_{(i)}$. In particular, direct sum means that elements of different direct sum subspaces always commute. For each copy of $\mathrm{u}(1)$, the representation $\hat\pi(\mathrm{b})$ shall have the following property: There exist $\lambda^z \in \mathbb{R}$ such that

$$[\hat\pi(\mathrm{b}), \mathcal{M}\,] = \textstyle\sum_{z\geq 2} \lambda^z [\underbrace{\hat\pi(\mathrm{b}), [\ldots[\hat\pi(\mathrm{b}), [\hat\pi(\mathrm{b}),}_{z} \mathcal{M}\,]]\ldots]] \,. \tag{3.25}$$

For simplicity, we restrict ourselves to the case $\mathfrak{a}'' = \mathrm{u}(1)$, where (3.25) is given by

$$\begin{aligned} [\hat\pi(\mathrm{b}), [\hat\pi(\mathrm{b}), [\hat\pi(\mathrm{b}), \mathcal{M}\,]]] &= [\hat\pi(\mathrm{b}), \mathcal{M}\,] \qquad \text{or} \\ [\hat\pi(\mathrm{b}), [\hat\pi(\mathrm{b}), [\hat\pi(\mathrm{b}), \mathcal{M}\,]]] &= -[\hat\pi(\mathrm{b}), \mathcal{M}\,] \,. \end{aligned} \tag{3.26}$$

The extension to the general case is obvious.

3.3.2 The Construction of $\pi(\Omega^1\mathfrak{g})$

Our goal is to construct the graded differential Lie algebra $\Omega^*_D\mathfrak{g}$ associated to the L–cycle $(\mathfrak{g}, h, D, \pi, \Gamma)$, see Section 2.3. For this purpose we first have to construct the graded Lie algebra $\pi(\Omega^*\mathfrak{g})$ associated to this L–cycle. We denote by $\hat\pi(\Omega^*\mathfrak{a})$ the corresponding graded Lie algebra associated to the L–cycle $(\mathfrak{a}, \mathbb{C}^F, \mathcal{M}\,, \hat\pi, \hat\Gamma)$. From (3.19) we get

$$[D, \pi(f\otimes a)] = \mathrm{i}\mathbf{d}f \otimes \hat\pi(a) + f\boldsymbol{\gamma} \otimes [\mathcal{M}\,, \hat\pi(a)] \,, \quad a \in \mathfrak{a} \,, \quad f \in C^\infty(X) \,, \tag{3.27}$$

where **d** is the exterior differential (3.12). Using that C^0 is an Abelian algebra, that elements of C^0 commute with elements of C^1 and that $\hat{\pi}$ is a representation we obtain for elements of $\pi(\Omega^1\mathfrak{g})$, see (2.25) and (2.30),

$$\begin{aligned}
&\pi\big(\textstyle\sum_{\alpha,z\geq 0}[\iota(f^z_\alpha\otimes a^z_\alpha),[\dots[\iota(f^1_\alpha\otimes a^1_\alpha),\iota(d(f^0_\alpha\otimes a^0_\alpha))]\dots]]\big)\\
&\qquad = \textstyle\sum_{\alpha,z\geq 0}[\pi(f^z_\alpha\otimes a^z_\alpha),[\dots[\pi(f^1_\alpha\otimes a^1_\alpha),[-\mathrm{i}D,\pi(f^0_\alpha\otimes a^0_\alpha)]]\dots]]\\
&\qquad = \textstyle\sum_{\alpha,z\geq 0} f^z_\alpha\cdots f^1_\alpha\mathbf{d}f^0_\alpha\otimes\hat{\pi}([a^z_\alpha,[\dots[a^1_\alpha,a^0_\alpha]\dots]]) \qquad (3.28a)\\
&\qquad + \textstyle\sum_{\alpha,z\geq 0} f^z_\alpha\cdots f^1_\alpha f^0_\alpha\gamma\otimes\hat{\pi}([\iota(a^z_\alpha),[\dots[\iota(a^1_\alpha),\iota(da^0_\alpha)]\dots]])\,. \qquad (3.28b)
\end{aligned}$$

Here, we have $f^j_\alpha\in C^0$, $a^j_\alpha\in\mathfrak{a}$, and d denotes the universal differential on both the universal differential Lie algebras over $\mathfrak{g}$ and $\mathfrak{a}$; it is clear from the context on which of them. The same notational simplification was used for the factorization mappings ι. There are two different contributions in this formula, (3.28a) belongs to $C^1\otimes\hat{\pi}(\Omega^0\mathfrak{a})$ and (3.28b) to $C^0\gamma\otimes\hat{\pi}(\Omega^1\mathfrak{a})$. If it was possible to put all f^0_α equal to constants without changing the range of (3.28b) then the lines (3.28a) and (3.28b) would be independent. This is possible iff for all $f^0_0\in C^\infty(X)$ and $a^0_0\in\mathfrak{a}$ there exists a solution of the equation

$$f^0_0\otimes\hat{\pi}(\iota(da^0_0)) = \textstyle\sum_{\alpha,z\geq 1} f^z_\alpha\cdots f^1_\alpha f^0_\alpha\gamma\otimes\hat{\pi}([\iota(a^z_\alpha),[\dots[\iota(a^1_\alpha),\iota(da^0_\alpha)]\dots]])\,.$$

But this is indeed the case, due to (3.25) for $a^0_0\in\mathfrak{a}''$ and the fact that $\mathfrak{a}'$ is semisimple. Namely, for a semisimple Lie algebra $\mathfrak{a}'$ we have $[\mathfrak{a}',\mathfrak{a}']=\mathfrak{a}'$, see [32]. This means that

$$\forall\, a'\in\mathfrak{a}\ \ \exists\, a'_\alpha,\tilde{a}'_\alpha\in\mathfrak{a}'\ :\ \ a'=\textstyle\sum_\alpha[a'_\alpha,\tilde{a}'_\alpha]\,. \qquad (3.29)$$

Then, $\iota(da')=\sum_\alpha\big([\iota(a'_\alpha),\iota(d\tilde{a}'_\alpha)]-[\iota(\tilde{a}'_\alpha),\iota(da'_\alpha)]\big)$. *Here we see the importance of the restrictions imposed to $\mathfrak{a}$, we will meet further examples in the sequel.*

Now, from the definition (2.25) of $\Omega^1\mathfrak{a}$ there follows that (3.28b) can attain any element of $C^0\gamma\otimes\hat{\pi}(\Omega^1\mathfrak{a})$. We split elements $a^j_\alpha\in\mathfrak{a}$ according to (3.24). Since commutators containing elements of the Abelian part vanish, there is a non–vanishing contribution of elements of $\mathfrak{a}''$ to (3.28a) only from the term $\mathbf{d}\tilde{f}^0_0\otimes\hat{\pi}(a^0_0)$, for $a^0_0\in\mathfrak{a}''$. Therefore, the coefficient of elements of $\hat{\pi}(\mathfrak{a}'')$ is the Clifford action of a total differential. We denote the space $\mathbf{d}C^0\subset C^1$ by B^1 ("[co]boundary"). In the case of the semisimple Lie algebra $\mathfrak{a}'$, the line (3.28a) attains any element of $C^1\otimes\hat{\pi}(\mathfrak{a}')$, due to (3.29). Thus, we get the final result

$$\pi(\Omega^1\mathfrak{g}) = (\Lambda^1\otimes\hat{\pi}(\mathfrak{a}'))\oplus(B^1\otimes\hat{\pi}(\mathfrak{a}''))\oplus(\Lambda^0\gamma\otimes\hat{\pi}(\Omega^1\mathfrak{a}))\,. \qquad (3.30)$$

This means that elements $\tau^1\in\pi(\Omega^1\mathfrak{g})$ are of the form

$$\tau^1=\sum_\alpha\big(c^1_\alpha\otimes\hat{\pi}(a'_\alpha)+b^1_\alpha\otimes\hat{\pi}(a''_\alpha)+f_\alpha\gamma\otimes\hat{\pi}(\omega^1_\alpha)\big)\,, \qquad (3.31)$$

where $c^1_\alpha\in C^1$, $b^1_\alpha\in B^1$, $f_\alpha\in C^0$, $a'_\alpha\in\mathfrak{a}'$, $a''_\alpha\in\mathfrak{a}''$ and $\omega^1_\alpha\in\Omega^1\mathfrak{a}$.

3.3.3 Definition of $T^j_n\mathfrak{a}$

Let us define the following vector subspaces $T^j_n\mathfrak{a}$ of the tensor algebra $T(\mathfrak{a})$ introduced in Section 2.2. The vector space $T^j_n\mathfrak{a}$ is zero for $j<0$, $n<j+2$ or $n>N+j+2$. For $j\geq 0$ and $j+2\leq n\leq N+j+2$ it is recursively defined by

$$
\begin{aligned}
T^0_2\mathfrak{a} &:= \{\iota(\mathfrak{a}),\iota(\mathfrak{a})\}\,, \qquad T^0_{N+2}\mathfrak{a} := [\iota(\mathfrak{a}),\{\iota(\mathfrak{a}),\iota(\mathfrak{a}')\}]\,,\\
T^0_n\mathfrak{a} &:= \{\iota(\mathfrak{a}),\iota(\mathfrak{a}')\}\,, \qquad 3\leq n\leq N+1\,,
\end{aligned}
\tag{3.32a}
$$

$$
\begin{aligned}
T^j_n\mathfrak{a} &:= \{\iota(\mathfrak{a}),\Omega^j\mathfrak{a}+T^{j-2}_{j+1}\mathfrak{a}\}+[\Omega^1\mathfrak{a},T^{j-1}_{j+1}\mathfrak{a}]\,,\\
&\qquad\qquad 2+j\leq n\leq N+j+1\,,\ j>0\,,
\end{aligned}
\tag{3.32b}
$$

$$
T^j_{N+j+2}\mathfrak{a} := [\iota(\mathfrak{a}),T^j_{j+2}\mathfrak{a}]+[\Omega^1\mathfrak{a},T^{j-1}_{N+j+1}\mathfrak{a}]\,, \qquad j>0\,.
$$

The importance of this definition will become clear in the following proposition about the structure of the space $\pi(\Omega^n\mathfrak{g})$. The reader who is mainly interested in physical applications can skip Step 2 of the proof; the construction of $\pi(\Omega^2\mathfrak{g})$ (Step 1) is sufficient for physics.

3.3.4 Proposition 15.

$$
\pi(\Omega^n\mathfrak{g}) = (\Lambda^n\otimes\hat{\pi}(\mathfrak{a}'))\oplus\Big(\bigoplus_{j=1}^{n}\Lambda^{n-j}\gamma^j\otimes(\hat{\pi}(\Omega^j\mathfrak{a})+\hat{\pi}(T^{j-2}_n\mathfrak{a}))\Big)\,, \quad n\geq 2\,,
\tag{3.33}
$$

where $\hat{\pi}: T^j_n\mathfrak{a}\to\hat{\pi}(T^j_n\mathfrak{a})$ is the obvious extension of $\hat{\pi}$ defined according to (2.30) *to anticommutors occurring in* (3.32).

Proof. The proposition is proved by induction. We need the following two identities:

$$
\begin{aligned}
&(\tilde{c}^1\otimes\hat{\pi}(\tilde{a}))(c^{n-j}\gamma^j\otimes A^j)-(-1)^n(c^{n-j}\gamma^j\otimes A^j)(\tilde{c}^1\otimes\hat{\pi}(\tilde{a}))\\
&\quad=\tfrac{1}{2}(\tilde{c}^1c^{n-j}+(-1)^{n-j}c^{n-j}\tilde{c}^1)\gamma^j\otimes(\hat{\pi}(\tilde{a})A^j-A^j\hat{\pi}(\tilde{a}))\\
&\quad+\tfrac{1}{2}(\tilde{c}^1c^{n-j}-(-1)^{n-j}c^{n-j}\tilde{c}^1)\gamma^j\otimes(\hat{\pi}(\tilde{a})A^j+A^j\hat{\pi}(\tilde{a}))\,,
\end{aligned}
\tag{3.34a}
$$

$$
\begin{aligned}
&(\tilde{f}\gamma\otimes\hat{\pi}(\tilde{\omega}^1))(c^{n-j}\gamma^j\otimes A^j)-(-1)^n(c^{n-j}\gamma^j\otimes A^j)(\tilde{f}\gamma\otimes\hat{\pi}(\tilde{\omega}^1))\\
&\quad=(-1)^{n-j}\tilde{f}c^{n-j}\gamma^{j+1}\otimes(\hat{\pi}(\tilde{\omega}^1)A^j-(-1)^jA^j\hat{\pi}(\tilde{\omega}^1))\,,
\end{aligned}
\tag{3.34b}
$$

for $\tilde{c}^1\in\Lambda^1$, $c^n\in\Lambda^n$, $\tilde{f}\in\Lambda^0$, $\tilde{a}\in\mathfrak{a}$, $\tilde{\omega}^1\in\Omega^1\mathfrak{a}$ and any $A^j\in\mathrm{M}_F\mathbb{C}$. We shall write (3.31) in the form

$$
\tau^1=\textstyle\sum_\alpha\big(c^1_\alpha\otimes\hat{\pi}(a_\alpha)+f_\alpha\gamma\otimes\hat{\pi}(\omega_\alpha)\big)\,,
$$

where $\sum_\alpha c^1_\alpha\otimes\hat{\pi}(a_\alpha)\equiv\sum_\alpha\big(c^{1\prime}_\alpha\otimes\hat{\pi}(a'_\alpha)+c^{1\prime\prime}_\alpha\otimes\hat{\pi}(a''_\alpha)\big)$.

Step 1: The Construction of $\pi(\Omega^2\mathfrak{g})$

Using (3.34a), (3.34b) and Lemma 13 we obtain from (2.26) the following form of elements $\tau^2 \in \pi(\Omega^2\mathfrak{g})$:

$$\begin{aligned}
\tau^2 &= \textstyle\sum_\alpha (\tau^1_\alpha \tilde{\tau}^1_\alpha + \tilde{\tau}^1_\alpha \tau^1_\alpha) \\
&= \textstyle\sum_{\alpha,\beta,\gamma} \big(c^1_{\alpha\beta} \wedge \tilde{c}^1_{\alpha\gamma} \otimes [\hat{\pi}(a_{\alpha\beta}), \hat{\pi}(\tilde{a}_{\alpha\gamma})] + f_{\alpha\beta} \tilde{f}_{\alpha\gamma} \otimes [\hat{\pi}(\omega^1_{\alpha\beta}), \hat{\pi}(\tilde{\omega}^1_{\alpha\gamma})]_g \qquad (3.35a) \\
&\quad + \tilde{f}_{\alpha\gamma} c^1_{\alpha\beta} \boldsymbol{\gamma} \otimes [\hat{\pi}(a_{\alpha\beta}), \hat{\pi}(\tilde{\omega}^1_{\alpha\gamma})] + f_{\alpha\beta} \tilde{c}^1_{\alpha\gamma} \boldsymbol{\gamma} \otimes [\hat{\pi}(\tilde{a}_{\alpha\gamma}), \hat{\pi}(\omega^1_{\alpha\beta})] \big) + \kappa^0 \;, \\
\kappa^0 &= \textstyle\sum_{\alpha,\beta,\gamma} c^1_{\alpha\beta} \,\lrcorner\, \tilde{c}^1_{\alpha\gamma} \otimes \{\hat{\pi}(a_{\alpha\beta}), \hat{\pi}(\tilde{a}_{\alpha\gamma})\} \;. \qquad (3.35b)
\end{aligned}$$

All five occurring different types of tensor products are independent. This is due to the fact that for non–vanishing $\tilde{c}^1 \in \Lambda^1$ and $c^n \in \Lambda^n$ the equality $\tilde{c}^1 \wedge c^n = 0$ implies $\tilde{c}^1 \lrcorner\, c^n \neq 0$ and $\tilde{c}^1 \lrcorner\, c^n = 0$ implies $\tilde{c}^1 \wedge c^n \neq 0$, see Lemma 13. First, κ^0 attains each element of $\Lambda^0 \otimes \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}$. Moreover, $\sum_\alpha f \tilde{f} \otimes [\hat{\pi}(\omega^1_\alpha), \hat{\pi}(\tilde{\omega}^1_\alpha)]_g$ gives an arbitrary element of $\Lambda^0 \otimes \hat{\pi}(\Omega^2\mathfrak{a})$ and each term in (3.35a) containing $\boldsymbol{\gamma}$ an arbitrary element of $\Lambda^1\boldsymbol{\gamma} \otimes \hat{\pi}(\Omega^1\mathfrak{a})$. The only not obvious elements are those of the form $[\mathcal{M}\ , \hat{\pi}(a)]$. However, they can be represented by (3.26) for $a = a''$ and for $a = a'$ due to (3.29) by

$$[\mathcal{M}\ , \hat{\pi}(\textstyle\sum_\alpha [a'_\alpha, \tilde{a}'_\alpha])] = \sum_\alpha \big([[\mathcal{M}\ , \hat{\pi}(a'_\alpha)], \hat{\pi}(\tilde{a}'_\alpha)] + [\hat{\pi}(a'_\alpha), [\mathcal{M}\ , \hat{\pi}(\tilde{a}'_\alpha)]] \big) \;. \qquad (3.36)$$

Finally, $\sum_{\alpha,\beta,\gamma} c^1_{\alpha\beta} \wedge \tilde{c}^1_{\alpha\gamma} \otimes [\hat{\pi}(a'_{\alpha\beta}), \hat{\pi}(\tilde{a}'_{\alpha\gamma})]$ represents an arbitrary element of $\Lambda^2 \otimes \hat{\pi}(\mathfrak{a}')$, because possible contributions from $\mathfrak{a}''$ are cancelled by the commutator. Collecting these results, we arrive at (3.33), for $n = 2$.

Step 2: The Construction of $\pi(\Omega^n\mathfrak{g})$ by Induction

We assume that (3.33) holds for $n = k$ and show that this implies (3.33) for $n = k+1$. Let $\tau^k \in \pi(\Omega^k\mathfrak{g})$ be represented as

$$\tau^k = \sum_\alpha \big(c^k_\alpha \otimes \hat{\pi}(a'_\alpha) + \sum_{j=1}^{k} (c^{k-j}_\alpha \boldsymbol{\gamma}^j \otimes \hat{\pi}(\omega^j_\alpha) + \hat{c}^{k-j}_\alpha \boldsymbol{\gamma}^j \otimes \hat{\pi}(\hat{\kappa}^{j-2}_\alpha)) \big) \;,$$

where $c^i_\alpha, \hat{c}^i_\alpha \in \Lambda^i$, $a'_\alpha \in \mathfrak{a}'$, $\omega^i_\alpha \in \Omega^i\mathfrak{a}$ and $\hat{\kappa}^i_\alpha \in T^i_k\mathfrak{a}$. This gives with (3.34)

$$\begin{aligned}
\tau^{k+1} &= \textstyle\sum_\alpha (\tilde{\tau}^1_\alpha \tau^k_\alpha - (-1)^k \tau^k_\alpha \tilde{\tau}^1_\alpha) \qquad (3.37a) \\
&= \sum_{\alpha,\beta,\gamma} \big(\tilde{c}^1_{\alpha\beta} \wedge c^k_{\alpha\gamma} \otimes [\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(a'_{\alpha\gamma})] \\
&\quad + \sum_{j=1}^{k} \tilde{c}^1_{\alpha\beta} \wedge c^{k-j}_{\alpha\gamma} \boldsymbol{\gamma}^j \otimes [\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(\omega^j_{\alpha\gamma})] + (-1)^k \tilde{f}_{\alpha\beta} c^k_{\alpha\gamma} \boldsymbol{\gamma} \otimes [\hat{\pi}(\tilde{\omega}^1_{\alpha\beta}), \hat{\pi}(a'_{\alpha\gamma})] \\
&\quad + \sum_{j=1}^{k} (-1)^{k-j} \tilde{f}_{\alpha\beta} c^{k-j}_{\alpha\gamma} \boldsymbol{\gamma}^{j+1} \otimes [\hat{\pi}(\tilde{\omega}^1_{\alpha\beta}), \hat{\pi}(\omega^j_{\alpha\gamma})]_g \big) + \textstyle\sum_{j=0}^{k-1} \kappa^j_{k+1} \;, \\
\kappa^0_{k+1} &= \sum_{\alpha,\beta,\gamma} \big(\tilde{c}^1_{\alpha\beta} \,\lrcorner\, c^k_{\alpha\gamma} \otimes \{\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(a'_{\alpha\gamma})\} + \tilde{c}^1_{\alpha\beta} \wedge \hat{c}^{k-2}_{\alpha\gamma} \otimes [\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(\hat{\kappa}^0_{\alpha\gamma})] \big) \;, \qquad (3.37b)
\end{aligned}$$

$$\kappa^j_{k+1} = \sum_{\alpha,\beta,\gamma} \big(\tilde{c}^1_{\alpha\beta} \lrcorner c^{k-j}_{\alpha\gamma}\gamma^j \otimes \{\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(\omega^j_{\alpha\gamma})\} + \tilde{c}^1_{\alpha\beta} \lrcorner \hat{c}^{k-j}_{\alpha\gamma}\gamma^j \otimes \{\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(\hat{\kappa}^{j-2}_{\alpha\gamma})\}$$
$$+ \tilde{c}^1_{\alpha\beta}\wedge \hat{c}^{k-j-2}_{\alpha\gamma}\gamma^j \otimes [\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(\hat{\kappa}^j_{\alpha\gamma})] - (-1)^{k-j}\,\tilde{f}_{\alpha\beta}\hat{c}^{k-j-1}_{\alpha\gamma}\gamma^j \otimes [\hat{\pi}(\tilde{\omega}^1_{\alpha\beta}), \hat{\pi}(\hat{\kappa}^{j-1}_{\alpha\gamma})]_g\big).$$

Again, all occurring tensor products are independent. It is clear that

$\sum_{\alpha,\beta,\gamma} \tilde{c}^1_{\alpha\beta} \wedge c^k_{\alpha\gamma} \otimes [\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(a'_{\alpha\gamma})]$

attains precisely each element of $\Lambda^{k+1} \otimes \hat{\pi}(\mathfrak{a}')$. For the same reasons as before,

$\sum_{\alpha,\beta,\gamma} \big(\tilde{c}^1_{\alpha\beta} \wedge c^{k-1}_{\alpha\gamma}\gamma \otimes [\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(\omega^1_{\alpha\gamma})] + (-1)^k \tilde{f}_{\alpha\beta} c^k_{\alpha\gamma}\gamma \otimes [\hat{\pi}(\tilde{\omega}^1_{\alpha\beta}), \hat{\pi}(a'_{\alpha\gamma})]\big)$

attains precisely each element of $\Lambda^k\gamma \otimes \hat{\pi}(\Omega^1\mathfrak{a})$, see formulae (3.26) and (3.36). Next, it is obvious that

$\sum_{\alpha,\beta,\gamma}(-1)^{k-j}\,\tilde{f}_{\alpha\beta} c^{k-j}_{\alpha\gamma}\gamma^{j+1} \otimes [\hat{\pi}(\tilde{\omega}^1_{\alpha\beta}), \hat{\pi}(\omega^j_{\alpha\gamma})]_g$

attains each element of $\Lambda^{k-j}\gamma^{j+1} \otimes \hat{\pi}(\Omega^{j+1}\mathfrak{a})$, for $j = 1, \dots k$. Moreover, we have

$\sum_{\alpha,\beta,\gamma} \tilde{c}^1_{\alpha\beta} \wedge c^{k-(j+1)}_{\alpha\gamma}\gamma^{j+1} \otimes [\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(\omega^{j+1}_{\alpha\gamma})] \in \Lambda^{k-j}\gamma^{j+1} \otimes \hat{\pi}(\Omega^{j+1}\mathfrak{a})$,

for $j = 1, \dots k-1$. Thus, we are left with the task to show that κ^j_{k+1} attains each element of $\Lambda^{k-j-1}\gamma^j \otimes \hat{\pi}(T^j_{k+1}\mathfrak{a})$.

Let us begin with the discussion of κ^0_{k+1}. For $k = 2$ it is clear that

$\sum_{\alpha,\beta,\gamma} \tilde{c}^1_{\alpha\beta} \lrcorner c^k_{\alpha\gamma} \otimes \{\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(a'_{\alpha\gamma})\}$

attains each element of $\Lambda^1 \otimes \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a}')\}$. Moreover, we have

$\sum_{\alpha,\beta,\gamma} \tilde{c}^1_{\alpha\beta} \wedge \hat{c}^{k-2}_{\alpha\gamma} \otimes [\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(\hat{\kappa}^0_{\alpha\gamma})] \in \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a}')\}$.

Hence, κ^0_3 attains each element of $\Lambda^1 \otimes \hat{\pi}(T^0_3\mathfrak{a})$. Since $[\pi(\mathfrak{a}), \hat{\pi}(T^0_3\mathfrak{a})] \subset \hat{\pi}(T^0_3\mathfrak{a})$, we get from (3.37b) for $3 < k \le N$ that κ^0_{k+1} attains each element of $\Lambda^{k-1} \otimes \hat{\pi}(T^0_3\mathfrak{a})$. This is no longer true for $k = N+1$. In this case we have

$\sum_{\alpha,\beta,\gamma} \tilde{c}^1_{\alpha\beta} \lrcorner c^k_{\alpha\gamma} \otimes \{\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(a'_{\alpha\gamma})\} = 0$

so that κ^0_{N+2} attains each element of $\Lambda^N \otimes [\hat{\pi}(\mathfrak{a}), \hat{\pi}(T^0_3\mathfrak{a})] \equiv \Lambda^N \otimes \hat{\pi}(T^0_{N+2}\mathfrak{a})$.

For $j > 0$ we proceed by induction in j, where the argumentation is roughly the same as before. For $k \le j$ we have $\kappa^j_{k+1} = 0$. For $k = j+1$ we have

$\sum_{\alpha,\beta,\gamma} \tilde{c}^1_{\alpha\beta} \wedge \hat{c}^{k-j-2}_{\alpha\gamma}\gamma^j \otimes [\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(\hat{\kappa}^j_{\alpha\gamma})] \equiv 0$.

Thus, κ^j_{j+2} attains each element of

$\Lambda^0\gamma^j \otimes (\{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\Omega^j\mathfrak{a}) + \hat{\pi}(T^{j-2}_{j+1}\mathfrak{a})\} + [\hat{\pi}(\Omega^1\mathfrak{a}), \hat{\pi}(T^{j-1}_{j+1}\mathfrak{a})]_g) = \Lambda^0\gamma^j \otimes \hat{\pi}(T^j_{j+2}\mathfrak{a})$.

From $[\hat{\pi}(\mathfrak{a}), \hat{\pi}(T^0_k\mathfrak{a})] \subset \hat{\pi}(T^0_k\mathfrak{a})$ we get by induction $[\hat{\pi}(\mathfrak{a}), \hat{\pi}(T^j_{j+2}\mathfrak{a})] \subset \hat{\pi}(T^j_{j+2}\mathfrak{a})$. Then, we see for $j+2 \le k \le N+j$ that κ^j_{k+1} attains each element of $\Lambda^{k-j-1}\gamma^j \otimes \hat{\pi}(T^j_{j+2}\mathfrak{a})$, i.e. we have $\hat{\pi}(T^j_{k+1}\mathfrak{a}) = \hat{\pi}(T^j_{j+2}\mathfrak{a})$. For $k = N+j+1$ we have

$\sum_{\alpha,\beta,\gamma} \big(\tilde{c}^1_{\alpha\beta} \lrcorner c^{k-j}_{\alpha\gamma}\gamma^j \otimes \{\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(\omega^j_{\alpha\gamma})\} + \tilde{c}^1_{\alpha\beta} \lrcorner \hat{c}^{k-j}_{\alpha\gamma}\gamma^j \otimes \{\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(\hat{\kappa}^{j-2}_{\alpha\gamma})\}\big) = 0$.

Therefore, the remaining space of which κ^j_{N+j+2} attains each element is

$\Lambda^N\gamma^j \otimes ([\hat{\pi}(\mathfrak{a}), \hat{\pi}(T^j_{j+2}\mathfrak{a})] + [\hat{\pi}(\Omega^1\mathfrak{a}), \hat{\pi}(T^{j-1}_{N+j+1}\mathfrak{a})]_g) \equiv \Lambda^N\gamma^j \otimes \hat{\pi}(T^j_{N+j+2}\mathfrak{a})$. $\square$

Thus, the computation of $\pi(\Omega^n\mathfrak{g})$ is reduced to an iterative multiplication of matrices only.

3.4 Towards the Structure of $\Omega^*_D\mathfrak{g}$

In Section 3.5 we will derive the structure of $\Omega^n_D\mathfrak{g}$. Before, we provide certain additional structures and related properties which enter the main theorem. Strictly speaking, these structures are relevant (almost) exclusively for $n>2$ and not for the physically interesting case $n=2$. The formulae are very technical and it is not easy to keep the overview. Therefore, the reader who is mainly interested in physics is advised to read only Section 3.4.1 (where the map $\hat\sigma_\mathfrak{g}$ is defined) and to pass then immediately to Section 3.5. There remains only a very small gap, namely Lemma 19 on page 59 for $k=1$, which reduces to $\hat\sigma_\mathfrak{g}(\Omega^1\mathfrak{g}\cap\ker\pi)=\Lambda^0\otimes\hat\sigma(\Omega^1\mathfrak{a}\cap\ker\hat\pi)$. But this identity, which is the only one needed in the case $n=2$, is more or less plausible if one looks at the structure of $\pi(\Omega^1\mathfrak{g})$ given in formula (3.30). The reader who is not satisfied can optionally regard the lowest two degrees in the technical formulae to follow.

For the mainly mathematically interested reader we give a short description of the "censored" part. Our goal is to prove Lemma 19. For that purpose we define in Section 3.4.2 a certain graded differential algebra $\Omega^*X\hat\otimes\hat T(\mathfrak{a})$, which is of course a graded differential Lie algebra as well. Next, we define a homomorphism $\mathbf{i}:\Omega^*\mathfrak{g}\to\Omega^*X\hat\otimes\hat T^*(\mathfrak{a})$ of graded differential Lie algebras. The homomorphism $\mathbf{i}$ enables us to rewrite elements of $\pi(\Omega^*\mathfrak{g})$ and of $\hat\sigma_\mathfrak{g}(\Omega^*\mathfrak{g})$, see Sections 3.4.3 and 3.4.5. This yields two ingredients for the proof of Lemma 19 in Section 3.4.6. The third ingredient is a formula for the structure of $\mathbf{i}(\Omega^*\mathfrak{g})$ derived in Section 3.4.4. Finally, we provide in Section 3.4.7 some identities on the differential of $T^j_k\mathfrak{a}$.

3.4.1 Definition of $\hat\sigma$ and $\hat\sigma_\mathfrak{g}$

First, we define in analogy to (2.40)

$$\hat\sigma\Big(\sum_{\alpha,z\geq 0}[\imath(a^z_\alpha),[\dots[\imath(a^1_\alpha),\imath(da^0_\alpha)]\dots]]\Big):=\sum_{\alpha,z\geq 0}[\hat\pi(a^z_\alpha),[\dots[\hat\pi(a^1_\alpha),[\mathcal{M}^{\,2},\hat\pi(a^0_\alpha)]]\dots]]\,, \tag{3.38}$$

for $a^i_\alpha\in\mathfrak{a}$. We extend $\hat\sigma$ to a linear map $\hat\sigma_\mathfrak{g}:\Omega^*\mathfrak{g}\to\Gamma^\infty(C)\otimes\mathrm{M}_F\mathbb{C}$ by

$$\begin{aligned}&\hat\sigma_\mathfrak{g}(\imath(f\otimes a)):=0\,,\qquad \hat\sigma_\mathfrak{g}(\imath(d(f\otimes a))):=f\otimes\hat\sigma(\imath(da))\,,\\&\hat\sigma_\mathfrak{g}([\omega^k,\tilde\omega^l]):=[\hat\sigma_\mathfrak{g}(\omega^k),\pi(\tilde\omega^l)]_g+(-1)^k[\pi(\omega^k),\hat\sigma_\mathfrak{g}(\tilde\omega^l)]_g\,,\end{aligned} \tag{3.39}$$

for $f\in C^\infty(X)$, $a\in\mathfrak{a}$, $\omega^k\in\Omega^k\mathfrak{g}$ and $\tilde\omega^l\in\Omega^l\mathfrak{g}$. In the usual way we extend $\hat\sigma$ to a linear map on $T^j_n\mathfrak{a}$, see (3.32):

$$\begin{aligned}&\hat\sigma(\{\imath(a),\omega^j+\hat\kappa^{j-2}_j\}+[\tilde\omega^1,\hat\kappa^{j-1}_{j+1}]_g)\\&\quad:=\{\hat\pi(a),\hat\sigma(\omega^j)+\hat\sigma(\hat\kappa^{j-2}_j)\}+[\hat\sigma(\tilde\omega^1),\hat\pi(\hat\kappa^{j-1}_{j+1})]_g-[\hat\pi(\tilde\omega^1),\hat\sigma(\hat\kappa^{j-1}_{j+1})]_g\,,\end{aligned} \tag{3.40}$$

for $a\in\mathfrak{a}$, $\omega^j\in\Omega^j\mathfrak{g}$, $\tilde\omega^1\in\Omega^1\mathfrak{g}$ and $\hat\kappa^n_{n+2}\in T^n_{n+2}\mathfrak{a}$. In particular, $\hat\sigma(\{\imath(a),\imath(\tilde a)\})\equiv 0$.

3.4.2 The Graded Differential Algebra $\Omega^* X \hat{\otimes} \hat{T}(\mathfrak{a})$

Now, we use an idea of [37] for the derivation of formula 3.60 some pages ahead. For this purpose we consider the graded differential algebra $\Omega^* X \hat{\otimes} \hat{T}(\mathfrak{a})$, where $\Omega^* X$ denotes the universal differential algebra over $C^\infty(X)$, $\hat{T}(\mathfrak{a})$ the universal differential algebra over $\mathfrak{a}$ and $\hat{\otimes}$ the skew–tensor product. The universal differential algebra $\Omega^* X = \bigoplus_{k=0}^\infty \Omega^k X$ is the factor algebra of the tensor algebra of the free vector spaces generated by $C^\infty(X)$ and $\mathfrak{d} C^\infty(X)$ with respect to an ideal $\hat{I}^*(X)$. The graded ideal $\hat{I}^*(X)$ ensures linearity and contains tensor products of $C^\infty(X)$ and $\mathfrak{d} C^\infty(X)$ with elements of the type

$$\begin{aligned} &\{ f_1 \otimes f_2 - f_1 f_2 \} \cup \{ f_3 \otimes \mathfrak{d} f_4 - \mathfrak{d} f_4 \otimes f_3 \} \\ &\qquad \cup \{ \mathfrak{d} f_5 \otimes f_6 + f_5 \otimes \mathfrak{d} f_6 - 1 \otimes \mathfrak{d}(f_5 f_6) \} , \quad f_i \in C^\infty(X) . \end{aligned} \tag{3.41}$$

The degree of homogeneous elements of $\Omega^* X$ is given by the number of elements of $\mathfrak{d} C^\infty(X)$ in the tensor product. Therefore, one has $\Omega^0 X \cong C^\infty(X)$ and for the vector subspaces $\Omega^k X$ the representation

$$\Omega^k X := \{ \textstyle\sum_\alpha f_\alpha^0 \otimes \mathfrak{d} f_\alpha^1 \otimes \ldots \otimes \mathfrak{d} f_\alpha^k , \ f_\alpha^i \in C^\infty(X) , \ \text{finite sum} \} . \tag{3.42}$$

The differential $\mathfrak{d}$ and product $\otimes$ in $\Omega^* X$ are defined by

$$\mathfrak{d}(\textstyle\sum_\alpha f_\alpha^0 \otimes \mathfrak{d} f_\alpha^1 \otimes \ldots \otimes \mathfrak{d} f_\alpha^n) := \sum_\alpha 1 \otimes \mathfrak{d} f_\alpha^0 \otimes \mathfrak{d} f_\alpha^1 \otimes \ldots \otimes \mathfrak{d} f_\alpha^n , \qquad \mathfrak{d} 1 = 0 , \tag{3.43a}$$

$$\begin{aligned} &(\textstyle\sum_\alpha f_\alpha^0 \otimes \mathfrak{d} f_\alpha^1 \otimes \ldots \otimes \mathfrak{d} f_\alpha^k) \otimes (\sum_\beta \tilde{f}_\beta^0 \otimes \mathfrak{d} \tilde{f}_\beta^1 \otimes \ldots \otimes_a \mathfrak{d} \tilde{f}_\beta^l) \\ &\qquad := \textstyle\sum_{\alpha\beta} f_\alpha^0 \tilde{f}_\beta^0 \otimes \mathfrak{d} f_\alpha^1 \otimes \ldots \otimes \mathfrak{d} f_\alpha^k \otimes \mathfrak{d} \tilde{f}_\beta^1 \otimes \ldots \otimes \mathfrak{d} \tilde{f}_\beta^l . \end{aligned} \tag{3.43b}$$

For the universal differential algebra $\hat{T}(\mathfrak{a})$ we have the identity

$$\begin{aligned} \hat{T}(\mathfrak{a}) = \bigoplus_{k=0}^\infty \hat{T}^k(\mathfrak{a}) , \qquad & \hat{T}^k(\mathfrak{a}) := T^k(\mathfrak{a}) / (\hat{I}(\mathfrak{a}) \cap T^k(\mathfrak{a})) , \\ & \hat{I}(\mathfrak{a}) := T(\mathfrak{a}) \otimes I'(\mathfrak{a}) \otimes T(\mathfrak{a}) , \end{aligned} \tag{3.44}$$

where $T(\mathfrak{a})$ is the tensor algebra (2.4) over $\mathfrak{a} \oplus d\mathfrak{a}$ and and $I'(\mathfrak{a})$ was defined in (2.7), see Section 2.2. Note that

$$I(\mathfrak{a}) \subset \hat{I}(\mathfrak{a}) , \qquad\qquad d\hat{I}(\mathfrak{a}) \subset \hat{I}(\mathfrak{a}) . \tag{3.45}$$

The skew–tensor product

$$\Omega^* X \hat{\otimes} \hat{T}(\mathfrak{a}) = \bigoplus_{k=0}^\infty (\Omega^* X \hat{\otimes} \hat{T}(\mathfrak{a}))^k , \qquad (\Omega^* X \hat{\otimes} \hat{T}(\mathfrak{a}))^k := \bigoplus_{l=0}^k \Omega^{k-l} X \hat{\otimes} \hat{T}^l(\mathfrak{a}) ,$$

is a graded differential algebra. Elements of degree k have the form $\sum_{l=0}^k \sum_\alpha \xi_\alpha^{k-l} \otimes t_\alpha^l$, where $\xi_\alpha^n \in \Omega^n X$ and $t_\alpha^l \in \hat{T}^l(\mathfrak{a})$. The product in $\Omega^* X \hat{\otimes} \hat{T}(\mathfrak{a})$ is the skew–tensor product defined on homogeneous elements by

$$(\textstyle\sum_\alpha \xi_\alpha^k \otimes t_\alpha^l) \hat{\otimes} (\sum_\beta \tilde{\xi}_\beta^m \otimes \tilde{t}_\beta^n) := \sum_{\alpha,\beta} (-1)^{lm} (\xi_\alpha^k \otimes \tilde{\xi}_\beta^m) \otimes (t_\alpha^l \otimes \tilde{t}_\beta^n) , \tag{3.46}$$

where $\xi_\alpha^k \in \Omega^k X$, $\tilde{\xi}_\beta^m \in \Omega^m X$, $t_\alpha^l \in \hat{T}^l(\mathfrak{a})$ and $\tilde{t}_\beta^n \in \hat{T}^n(\mathfrak{a})$. The differential $\hat{d}$ on $\Omega^* X \hat{\otimes} \hat{T}(\mathfrak{a})$ is defined on homogeneous elements by

$$\hat{d}(\textstyle\sum_\alpha \xi_\alpha^k \otimes t_\alpha^l) := \sum_\alpha \big((\mathfrak{d}\xi_\alpha^k) \otimes t_\alpha^l + (-1)^k \xi_\alpha^k \otimes (dt_\alpha^l)\big) \,. \tag{3.47}$$

The graded differential algebra $\Omega^* X \hat{\otimes} \hat{T}(\mathfrak{a})$ is in a natural way a graded differential Lie algebra with commutator defined by

$$[(\xi \otimes t)^k, (\tilde{\xi} \otimes \tilde{t})^l] := (\xi \otimes t)^k \hat{\otimes} (\tilde{\xi} \otimes \tilde{t})^l - (-1)^{kl} (\tilde{\xi} \otimes \tilde{t})^l \hat{\otimes} (\xi \otimes t)^k \,, \tag{3.48}$$

where $(\xi \otimes t)^k \in (\Omega^* X \hat{\otimes} \hat{T}(\mathfrak{a}))^k$ and $(\tilde{\xi} \otimes \tilde{t})^l \in (\Omega^* X \hat{\otimes} \hat{T}(\mathfrak{a}))^l$. From (3.47) we obtain the graded Leibniz rule for $\hat{d}$,

$$\hat{d}([(\xi \otimes t)^k, (\tilde{\xi} \otimes \tilde{t})^l]) := [\hat{d}(\xi \otimes t)^k, (\tilde{\xi} \otimes \tilde{t})^l] + (-1)^k [(\xi \otimes t)^k, \hat{d}(\tilde{\xi} \otimes \tilde{t})^l] \,.$$

The purpose of this construction was to provide a frame for a useful look at $\Omega^*\mathfrak{g}$. Namely, we have $\Omega^0\mathfrak{g} \subset (\Omega^* X \hat{\otimes} \hat{T}^*(\mathfrak{a}))^0$ as a vector subspace. Identifying

$$\Omega^1\mathfrak{g} \ni \imath(d(\textstyle\sum_\alpha f_\alpha \otimes a_\alpha)) \equiv \hat{d}(\sum_\alpha f_\alpha \otimes a_\alpha) \in (\Omega^* X \hat{\otimes} \hat{T}^*(\mathfrak{a}))^1 \,,$$

we can regard $\Omega^*\mathfrak{g}$ in a natural way as a graded Lie subalgebra of $\Omega^* X \hat{\otimes} \hat{T}^*(\mathfrak{a})$. Here we have omitted the factorization mappings in $\Omega^* X$ and $\hat{T}^*(\mathfrak{a})$, the complete notation would be $(\sum_\alpha (f_\alpha + \hat{I}^0(X)) \otimes (a_\alpha + \hat{I}^0(\mathfrak{a}))$ instead of $(\sum_\alpha f_\alpha \otimes a_\alpha)$. We shall adopt this simplifying notation in the sequel. The corresponding homomorphism of graded differential Lie algebras

$$\begin{aligned} &\mathbf{i} : \Omega^*\mathfrak{g} \to \Omega^* X \hat{\otimes} \hat{T}^*(\mathfrak{a}) \,, \\ &\mathbf{i}(\imath(f \otimes a)) := f \otimes a \,, \qquad \mathbf{i}(\imath(d(f \otimes a))) := \hat{d}(\mathbf{i}(\imath(f \otimes a))) \,, \\ &\mathbf{i}([\omega^k, \tilde{\omega}^l]) := [\mathbf{i}(\omega^k), \mathbf{i}(\tilde{\omega}^l)] \,, \end{aligned} \tag{3.49}$$

for $f \in C^\infty(X)$, $a \in \mathfrak{a}$, $\omega^k \in \Omega^k\mathfrak{a}$ and $\tilde{\omega}^l \in \Omega^l\mathfrak{a}$, is essentially a regrouping of tensor products.

3.4.3 Rewriting $\pi(\Omega^*\mathfrak{g})$

Lemma 16. *For all $\omega \in \Omega^*\mathfrak{g}$ we have*

$$\begin{aligned} &\pi(\omega) \equiv (c \circ \gamma^{\deg\restriction_{\hat{T}(\mathfrak{a})}} \otimes \hat{\pi}) \circ \mathbf{i}(\omega) \,, \qquad \textit{in the sense of} \\ &(c \circ \gamma^{\deg\restriction_{\hat{T}(\mathfrak{a})}} \otimes \hat{\pi})(\xi^k \otimes t^l) \equiv c(\xi^k)\gamma^l \otimes \hat{\pi}(t^l) \,, \quad \xi^k \in \Omega^k X \,,\ t^l \in \hat{T}^l(\mathfrak{a}) \,, \end{aligned} \tag{3.50}$$

where $c(\sum_\alpha f_\alpha^0 \otimes \mathfrak{d} f_\alpha^1 \otimes \ldots \otimes \mathfrak{d} f_\alpha^k) := \sum_\alpha f_\alpha^0 \mathbf{d}(f_\alpha^1) \cdots \mathbf{d}(f_\alpha^k)$.

Proof. For $\omega = \sum_\alpha \imath(f_\alpha \otimes a_\alpha) \in \Omega^0\mathfrak{g}$ we have

$$\pi(\textstyle\sum_\alpha \imath(f_\alpha \otimes a_\alpha)) = \sum_\alpha f_\alpha \otimes \hat{\pi}(a_\alpha) = (c \otimes \hat{\pi}) \circ (\sum_\alpha f_\alpha \otimes a_\alpha)\,. \tag{3.51}$$

For $\omega = \sum_\alpha \imath(d(f_\alpha \otimes a_\alpha)) \in \Omega^0\mathfrak{g}$ we have

$$\begin{aligned}
\pi(\textstyle\sum_\alpha \imath(d(f_\alpha \otimes a_\alpha))) &\equiv \textstyle\sum_\alpha [-\mathrm{iD} \otimes \mathbb{1}_F + \gamma \otimes -\mathrm{i}\mathcal{M}\;,\, f_\alpha \otimes \hat{\pi}(a_\alpha)] \\
&= \textstyle\sum_\alpha \big([-\mathrm{iD}, f_\alpha] \otimes \hat{\pi}(a_\alpha) + f_\alpha \gamma \otimes [-\mathrm{i}\mathcal{M}\;, \hat{\pi}(a_\alpha)]\big) \\
&= \textstyle\sum_\alpha \big(\mathbf{d}(f_\alpha) \otimes \hat{\pi}(a_\alpha) + f_\alpha \gamma \otimes \hat{\pi}(da_\alpha)\big) \\
&= (c \circ \gamma^{\deg\restriction_{\hat{T}(\mathfrak{a})}} \otimes \hat{\pi}) \circ \textstyle\sum_\alpha \big(\mathfrak{d} f_\alpha \otimes a_\alpha + f_\alpha \otimes da_\alpha\big) \\
&= (c \circ \gamma^{\deg\restriction_{\hat{T}(\mathfrak{a})}} \otimes \hat{\pi}) \circ \hat{d}(\textstyle\sum_\alpha f_\alpha \otimes a_\alpha)\,.
\end{aligned} \tag{3.52}$$

Since $\Omega^1\mathfrak{g}$ is generated by repeated commutators of (3.52) with (3.51), we must show for any $\omega^k \in \Omega^k\mathfrak{g}$ and $\tilde{\omega}^l \in \Omega^l\mathfrak{g}$, $k=1, l=0$, that

$$\begin{aligned}
&[(c \circ \gamma^{\deg\restriction_{\hat{T}(\mathfrak{a})}} \otimes \hat{\pi})(\mathbf{i}(\tilde{\omega}^l)), (c \circ \gamma^{\deg\restriction_{\hat{T}(\mathfrak{a})}} \otimes \hat{\pi})(\mathbf{i}(\omega^k))]_g \\
&\qquad = (c \circ \gamma^{\deg\restriction_{\hat{T}(\mathfrak{a})}} \otimes \hat{\pi})(\mathbf{i}([\tilde{\omega}^l, \omega^k]))\,.
\end{aligned} \tag{3.53}$$

We use the representation

$$\mathbf{i}(\tilde{\omega}^0) = \textstyle\sum_\alpha \tilde{f}_\alpha \otimes \tilde{a}_\alpha\;, \qquad \mathbf{i}(\omega^1) = \sum_\beta ((f^0_\beta \otimes \mathfrak{d} f^1_\beta) \otimes a_\beta + f^2_\beta \otimes \hat{\omega}^1_\beta)\;,$$

where $\tilde{f}_\alpha, f^i_\beta \in C^\infty(X)$, $\tilde{a}_\alpha, a_\beta \in \mathfrak{a} + \hat{I}^0(\mathfrak{a})$ and $\hat{\omega}^1_\beta \in \Omega^1\mathfrak{a} + \hat{I}^1(\mathfrak{a})$; if $a_\beta \in \mathfrak{a}'' + \hat{I}^0(\mathfrak{a})$ then $f^0_\beta = 1$. Since $\mathbf{i}$ is a homomorphism we have

$$\begin{aligned}
\mathbf{i}([\tilde{\omega}^0, \omega^1]) &= \mathbf{i}(\tilde{\omega}^0) \hat{\otimes} \mathbf{i}(\omega^1) - \mathbf{i}(\omega^1) \hat{\otimes} \mathbf{i}(\tilde{\omega}^0) \\
&= \textstyle\sum_{\alpha,\beta} ((f^0_\beta \tilde{f}_\alpha \otimes \mathfrak{d} f^1_\beta) \otimes [\tilde{a}_\alpha, a_\beta] + f^2_\beta \tilde{f}_\alpha \otimes [\tilde{a}_\alpha, \hat{\omega}^1_\beta])\;.
\end{aligned}$$

Applying $(c \circ \gamma^{\deg\restriction_{\hat{T}(\mathfrak{a})}} \otimes \hat{\pi})$ we get

$$\begin{aligned}
&(c \circ \gamma^{\deg\restriction_{\hat{T}(\mathfrak{a})}} \otimes \hat{\pi}) \circ \mathbf{i}([\tilde{\omega}^0, \omega^1]) \\
&\qquad = \textstyle\sum_{\alpha,\beta} \big(f^0_\beta \tilde{f}_\alpha \mathbf{d}(f^1_\beta) \otimes [\hat{\pi}(\tilde{a}_\alpha), \hat{\pi}(a_\beta)] + f^2_\beta \tilde{f}_\alpha \gamma \otimes [\hat{\pi}(\tilde{a}_\alpha), \hat{\pi}(\hat{\omega}^1_\beta)]\big) \\
&\qquad \equiv [\pi(\tilde{\omega}^0), \pi(\omega^1)]\;.
\end{aligned}$$

Now, it suffices to check (3.53) for $l=1$ and any k, because then the graded Jacobi identity yields (3.53) for arbitrary k, l. We use the representation

$$\mathbf{i}(\tilde{\omega}^1) = \textstyle\sum_\alpha (\tilde{\xi}^1_\alpha \otimes \tilde{a}_\alpha + \tilde{f}_\alpha \otimes \hat{\tilde{\omega}}^1_\alpha)\;, \qquad \mathbf{i}(\omega^k) = \sum_{j=0}^k \sum_\beta \xi^{k-j}_\beta \otimes t^j_\beta$$

where $\xi^{k-j} \in \Omega^{k-j}X$ and $t^j \in \hat{T}^j(\mathfrak{a})$ are appropriate combinations of symmetrized and anti–symmetrized tensor products such that $\mathbf{i}(\omega^k)$ is indeed the image of an $\omega^k \in \Omega^k\mathfrak{g}$

under $\mathbf{i}$. This yields

$$\begin{aligned}\mathbf{i}([\tilde\omega^1,\omega^k]) &= \mathbf{i}(\tilde\omega^1)\hat\otimes\mathbf{i}(\omega^k)-(-1)^k\mathbf{i}(\omega^k)\hat\otimes\mathbf{i}(\tilde\omega^1) \\ &= \textstyle\sum_{j=0}^k\sum_{\alpha,\beta}\Big(\tfrac12(\tilde\xi^1_\alpha\otimes\xi^{k-j}_\beta+(-1)^{k-j}\xi^{k-j}_\beta\otimes\tilde\xi^1_\alpha)\otimes(\tilde a_\alpha\otimes t^j_\beta-t^j_\beta\otimes\tilde a_\alpha) \\ &\qquad +\tfrac12(\tilde\xi^1_\alpha\otimes\xi^{k-j}_\beta-(-1)^{k-j}\xi^{k-j}_\beta\otimes\tilde\xi^1_\alpha)\otimes(\tilde a_\alpha\otimes t^j_\beta+t^j_\beta\otimes\tilde a_\alpha) \\ &\qquad +(-1)^{k-j}\tilde f_\alpha\xi^{k-j}_\beta\otimes(\hat{\tilde\omega}^1_\alpha\otimes t^j_\beta-(-1)^j t^j_\beta\otimes\hat{\tilde\omega}^1_\alpha)\Big)\;.\end{aligned} \tag{3.54}$$

Applying $(c\circ\gamma^{\deg\restriction_{\hat T(\mathfrak{a})}}\otimes\hat\pi)$ we get from Lemma 13

$$\begin{aligned}(c\circ\gamma^{\deg\restriction_{\hat T(\mathfrak{a})}}\otimes\hat\pi)\circ\mathbf{i}([\tilde\omega^1,\omega^l]) &= \textstyle\sum_{j=0}^k\sum_{\alpha,\beta}\Big(c(\tilde\xi^1_\alpha)\wedge c(\xi^{k-j}_\beta)\gamma^j\otimes[\hat\pi(\tilde a_\alpha),\hat\pi(t^j_\beta)] \\ &\qquad +c(\tilde\xi^1_\alpha)\lrcorner\, c(\xi^{k-j}_\beta)\gamma^j\otimes\{\hat\pi(\tilde a_\alpha),\hat\pi(t^j_\beta)\} \\ &\qquad +(-1)^{k-j}c(\tilde f_\alpha\xi^{k-j}_\beta)\gamma^{j+1}\otimes[\hat\pi(\hat{\tilde\omega}^1_\alpha),\hat\pi(t^j_\beta)]_g\Big) \\ &\equiv[\pi(\tilde\omega^1),\pi(\omega^k)]_g\;.\end{aligned} \tag{3.55}$$

We have used $\gamma c(\xi^n)=(-1)^n\gamma c(\xi^n)$, for $\xi^n\in\Omega^nX$. □

3.4.4 The Analysis of $\mathbf{i}(\Omega^*\mathfrak{g})$

It is worth studying $\mathbf{i}(\Omega^*\mathfrak{g})$ by means of formula (3.54). Here, it is very important to keep track of the occurring spaces. We introduce the following convenient notation: Let $\mathrm{t}=\mathrm{t}_k\mathrm{t}_{k-1}\ldots\mathrm{t}_1\mathrm{t}_0$, $\mathrm{t}_i\in\{1,2,3\}$, be a ternary number and $\#_i(\mathrm{t})$, $i=1,2,3$, be the number of digits i occurring in t. To each ternary number $\mathrm{t}>1$ we associate subspaces $\Omega_{\mathrm{t}}X\subset\Omega^*X$ and $T_{\mathrm{t}}(\mathfrak{a})\subset\hat T^*(\mathfrak{a})$ according to the following rule:

1. $\Omega_1X:=0$, $\quad T_1(\mathfrak{a}):=0$,
   $\Omega_2X:=\Omega^1X$, $\quad T_2(\mathfrak{a}):=\mathfrak{a}+\hat I^0(\mathfrak{a})$,
   $\Omega_3X:=\Omega^0X$, $\quad T_3(\mathfrak{a}):=\Omega^1\mathfrak{a}+\hat I^1(\mathfrak{a})$,
2. $$\left.\begin{aligned}\Omega_{1\mathrm{t}}X &:= \big\{\textstyle\sum_\alpha\tfrac12(\tilde\xi_{2,\alpha}\otimes\xi_{\mathrm{t},\alpha}-(-1)^{\#_1(\mathrm{t})+\#_2(\mathrm{t})}\xi_{\mathrm{t},\alpha}\otimes\tilde\xi_{2,\alpha})\big\}\;,\\ T_{1\mathrm{t}}(\mathfrak{a}) &:= \big\{\textstyle\sum_\alpha(\tilde\omega_{2,\alpha}\otimes\omega_{\mathrm{t},\alpha}+\omega_{\mathrm{t},\alpha}\otimes\tilde\omega_{2,\alpha})\big\}\;,\\ \Omega_{2\mathrm{t}}X &:= \big\{\textstyle\sum_\alpha\tfrac12(\tilde\xi_{2,\alpha}\otimes\xi_{\mathrm{t},\alpha}+(-1)^{\#_1(\mathrm{t})+\#_2(\mathrm{t})}\xi_{\mathrm{t},\alpha}\otimes\tilde\xi_{2,\alpha})\big\}\;,\\ T_{2\mathrm{t}}(\mathfrak{a}) &:= \big\{\textstyle\sum_\alpha(\tilde\omega_{2,\alpha}\otimes\omega_{\mathrm{t},\alpha}-\omega_{\mathrm{t},\alpha}\otimes\tilde\omega_{2,\alpha})\big\}\;,\\ \Omega_{3\mathrm{t}}X &:= \big\{\textstyle\sum_\alpha(-1)^{\#_1(\mathrm{t})-\#_2(\mathrm{t})}\tilde\xi_{3,\alpha}\otimes\xi_{\mathrm{t},\alpha}\big\}\;,\\ T_{3\mathrm{t}}(\mathfrak{a}) &:= \big\{\textstyle\sum_\alpha(\tilde\omega_{3,\alpha}\otimes\omega_{\mathrm{t},\alpha}-(-1)^{\#_3(\mathrm{t})}\omega_{\mathrm{t},\alpha}\otimes\tilde\omega_{3,\alpha})\big\}\;,\end{aligned}\right\}\quad\begin{aligned}&\xi_{\mathrm{t},\alpha},\tilde\xi_{\mathrm{t},\alpha}\in\Omega_{\mathrm{t}}X\;,\\ &\omega_{\mathrm{t},\alpha},\tilde\omega_{\mathrm{t},\alpha}\in T_{\mathrm{t}}(\mathfrak{a})\;.\end{aligned}$$

Here, it is extremely important that the symbols $\xi_{\mathrm{t},\alpha},\tilde\xi_{\mathrm{t},\alpha}$ and $\omega_{\mathrm{t},\alpha},\tilde\omega_{\mathrm{t},\alpha}$ are *the same* in each subformula of 2. Then, we obtain from (3.54) the interesting formula

$$\mathbf{i}(\Omega^k\mathfrak{g})=\sum_{\mathrm{t}=(3^k-1)/2}^{3(3^k-1)/2}\Omega_{\mathrm{t}}X\hat\otimes T_{\mathrm{t}}(\mathfrak{a})\;,\quad k\geq 2\;. \tag{3.56}$$

For $k=1$, formula (3.56) must be slightly modified, taking into account that the coefficient of $\mathfrak{a}''$ has to be a total differential.

As a demonstration of the power of this method we compute $(c\circ\gamma^{\deg\restriction_{\hat{T}(\mathfrak{a})}}\otimes\hat{\pi})\circ\mathbf{i}(\Omega^k\mathfrak{g})$. First, it is easy to see that $c(\Omega_{\mathrm{t}}X)=\Lambda^{\#_2(\mathrm{t})-\#_1(\mathrm{t})}$ if for all subsuccessions $\mathrm{t}'=\mathrm{t}_j\mathrm{t}_{j-1}\dots\mathrm{t}_1\mathrm{t}_0$ of $\mathrm{t}=\mathrm{t}_k\mathrm{t}_{k-1}\dots\mathrm{t}_1\mathrm{t}_0$, $j\le k$, there is always $0\le\#_2(\mathrm{t}')-\#_1(\mathrm{t}')\le N$. Otherwise we have $c(\Omega_{\mathrm{t}}X)=0$. In particular, $c(\Omega_{1\mathrm{t}}X)$ and $c(\Omega_{2\mathrm{t}}X)$ are independent. Next, we have $\hat{\pi}(T_{\mathrm{t}}(\mathfrak{a}))\subset\hat{\pi}(\hat{T}^{\#_3(\mathrm{t})}(\mathfrak{a}))$, in particular $\hat{\pi}(T_{\mathrm{t}}(\mathfrak{a}))\equiv\hat{\pi}(\Omega^{\#_3(\mathrm{t})}\mathfrak{a})$ for $\#_1(\mathrm{t})=0$. With every additional digit 1 in t we get one additional anticommutator with $\hat{\pi}(\mathfrak{a})$ in $\hat{\pi}(T_{\mathrm{t}}(\mathfrak{a}))$, which means that for $\#_1(\mathrm{t})>0$ and $\#_2(\mathrm{t})-\#_1(\mathrm{t})<N$ we have $\hat{\pi}(T_{\mathrm{t}}(\mathfrak{a}))\subset\hat{\pi}(T^{2\#_1(\mathrm{t})+\#_3(\mathrm{t})-2}_{\#_1(\mathrm{t})+\#_2(\mathrm{t})+\#_3(\mathrm{t})}\mathfrak{a})$. Actually, for $\#_2(\mathrm{t})-\#_1(\mathrm{t})<N$ each element of $\hat{\pi}(T^{2\#_1(\mathrm{t})+\#_3(\mathrm{t})-2}_{\#_1(\mathrm{t})+\#_2(\mathrm{t})+\#_3(\mathrm{t})}\mathfrak{a})$ is attained. For $\#_2(\mathrm{t})-\#_1(\mathrm{t})=N$ we must distinguish three cases. First, for $\mathrm{t}=2\mathrm{t}'$ we have $\hat{\pi}(T_{2\mathrm{t}'}(\mathfrak{a}))=[\hat{\pi}(\mathfrak{a}),\hat{\pi}(T_{\mathrm{t}'}(\mathfrak{a}))]$, second, for $\mathrm{t}=3\mathrm{t}'$ we have $\hat{\pi}(T_{3\mathrm{t}'}(\mathfrak{a}))=[\hat{\pi}(\Omega^1\mathfrak{a}),\hat{\pi}(T_{\mathrm{t}'}(\mathfrak{a}))]_g$, but third, for $\mathrm{t}=1\mathrm{t}'$ the coefficient of $\hat{\pi}(T_{1\mathrm{t}'}(\mathfrak{a}))=\{\hat{\pi}(\mathfrak{a}),\hat{\pi}(T_{\mathrm{t}'}(\mathfrak{a}))\}$ is zero due to $\#_2(\mathrm{t}')-\#_1(\mathrm{t}')=N+1$. Collecting these results, we get

$$
\begin{aligned}
(c\circ\gamma^{\deg\restriction_{\hat{T}(\mathfrak{a})}}\otimes\hat{\pi})\circ\mathbf{i}(\Omega^k\mathfrak{g}) &= \sum_{\substack{\mathrm{t}=(3^k-1)/2\\ 0\le\#_2(\mathrm{t}')-\#_1(\mathrm{t}')\le N}}^{3(3^k-1)/2}\Lambda^{\#_2(\mathrm{t})-\#_1(\mathrm{t})}\gamma^{\#_3(\mathrm{t})}\otimes\hat{\pi}(T_{\mathrm{t}}(\mathfrak{a})) \qquad (3.57)\\
&= \bigoplus_{j=0}^{k}\sum_{\substack{\mathrm{t}=(3^k-1)/2\\ 2\#_1(\mathrm{t})+\#_3(\mathrm{t})=j\\ 0\le\#_2(\mathrm{t}')-\#_1(\mathrm{t}')\le N}}^{3(3^k-1)/2}\Lambda^{k-j}\gamma^j\otimes\hat{\pi}(T_{\mathrm{t}}(\mathfrak{a})) = \bigoplus_{j=0}^{k}\Lambda^{k-j}\gamma^j\otimes(\hat{\pi}(\Omega^j\mathfrak{a})+\hat{\pi}(T_k^{j-2}\mathfrak{a}))\,.
\end{aligned}
$$

Surjectivity is due to analogous reasons as in the proof of Proposition 15.

3.4.5 Rewriting $\hat{\sigma}_{\mathfrak{g}}(\Omega^*\mathfrak{g})$

Next, there exists a relation between $\hat{\sigma}_{\mathfrak{g}}$ and $\hat{\sigma}$ analogous to the relation between π and $\hat{\pi}$ of Lemma 16:

Lemma 17. *For all $\omega\in\Omega^*\mathfrak{g}$ we have*

$$
\begin{aligned}
&\hat{\sigma}_{\mathfrak{g}}(\omega)\equiv(c\circ\gamma^{1+\deg\restriction_{\hat{T}(\mathfrak{a})}}\otimes\hat{\sigma})\circ\mathbf{i}(\omega)\,, \qquad \textit{in the sense of}\\
&(c\circ\gamma^{1+\deg\restriction_{\hat{T}(\mathfrak{a})}}\otimes\hat{\sigma})(\xi^k\otimes t^l)\equiv c(\xi^k)\gamma^{l+1}\otimes\hat{\sigma}(t^l)\,, \quad \xi^k\in\Omega^kX\,,\ t^l\in\hat{T}^l(\mathfrak{a})\,.
\end{aligned} \qquad (3.58)
$$

Proof. For $\omega=\sum_\alpha\iota(f_\alpha\otimes a_\alpha)\in\Omega^0\mathfrak{g}$ we have $0=\hat{\sigma}_{\mathfrak{g}}(\omega)=\sum_\alpha f_\alpha\gamma\otimes\hat{\sigma}(a_\alpha)$. For $\omega=\sum_\alpha\iota(d(f_\alpha\otimes a_\alpha))\in\Omega^0\mathfrak{g}$ we have

$$
\begin{aligned}
\hat{\sigma}_{\mathfrak{g}}(\textstyle\sum_\alpha\iota(d(f_\alpha\otimes a_\alpha)))&\equiv\textstyle\sum_\alpha f_\alpha\otimes\hat{\sigma}(da_\alpha)\\
&=(c\circ\gamma^{1+\deg\restriction_{\hat{T}(\mathfrak{a})}}\otimes\hat{\sigma})\circ\hat{d}(\textstyle\sum_\alpha f_\alpha\otimes a_\alpha)\,.
\end{aligned}
$$

Now, an analogous consideration as in the proof of Lemma 16 yields the assertion. □

3.4.6 An Identity for $\hat{\sigma}_{\mathfrak{g}}(\ker\pi\cap\Omega^k\mathfrak{g})$

The purpose of this analysis was the following. During the calculation of $\Omega^*_D\mathfrak{g}$, to which Section 3.5 is devoted, it is necessary to know the space $\hat{\sigma}_{\mathfrak{g}}(\ker\pi\cap\Omega^k\mathfrak{g})$. Since $\hat{\sigma}_{\mathfrak{g}}(\ker\mathbf{i}\cap\Omega^k\mathfrak{g})=0$ due to Lemma 17, we have

$$\hat{\sigma}_{\mathfrak{g}}(\ker\pi\cap\Omega^k\mathfrak{g})=\hat{\sigma}_{\mathfrak{g}}(\ker(c\circ\gamma^{\deg\restriction_{\hat{T}(\mathfrak{a})}}\otimes\hat{\pi})\cap\mathbf{i}(\Omega^k\mathfrak{g}))\,. \tag{3.59}$$

Now we need the following general result of multilinear algebra, which was proved in [55]:

Lemma 18. *Let V_1,V_2 be vector spaces and let there be given linear maps $\pi_i:V_i\to\pi_i(V_i)$, $i=1,2$. Then, the linear map*

$$\tilde{\pi}:V_1\otimes V_2\to\pi_1(V_1)\otimes\pi_2(V_2)\,,\qquad \tilde{\pi}(v_1\otimes v_2)=\pi_1(v_1)\otimes\pi_2(v_2)\,,$$

for $v_1\in V_1$ and $v_2\in V_2$, has the property

$$\ker\tilde{\pi}=V_1\otimes\ker\pi_2+\ker\pi_1\otimes V_2\,.\qquad\square$$

Lemma 19.

$$\hat{\sigma}_{\mathfrak{g}}(\Omega^k\mathfrak{g}\cap\ker\pi)=\bigoplus_{j=1}^{k}\Lambda^{k-j}\gamma^{j+1}\otimes\hat{\sigma}\big((\Omega^j\mathfrak{a}+T_k^{j-2}\mathfrak{a})\cap\ker\hat{\pi}\big)\,. \tag{3.60}$$

Proof. We specify Lemma 18 to our case, putting

$$V_1=\Omega^*X\,,\qquad \pi_1=c\circ\gamma^{k-\deg\restriction_{\Omega^*X}}\,,\qquad V_2=\hat{T}(\mathfrak{a})\,,\qquad \pi_2=\hat{\pi}\,,$$
$$\tilde{\pi}\equiv(c\circ\gamma^{k-\deg\restriction_{\Omega^*X}}\otimes\hat{\pi})\restriction_{\mathbf{i}(\Omega^k\mathfrak{a})}\,.$$

Now, the application of $c\circ\gamma^{k+1-\deg\restriction_{\Omega^*X}}\otimes\hat{\sigma}$ to $\ker(c\circ\gamma^{k-\deg\restriction_{\Omega^*X}}\otimes\hat{\pi})\cap\mathbf{i}(\Omega^k\mathfrak{g})$ according to (3.56) yields after the same procedure as the one leading to (3.57) our assertion (3.60). We have used that there is no contribution for $j=0$. A priory, (3.60) is only true for $k\geq 2$, see (3.56). However, putting $k=1$ in (3.56) then only the part $\Omega_2X\hat{\otimes}T_2(\mathfrak{a})$ is not reproduced correctly, but on terms of those type the mapping $\hat{\sigma}_{\mathfrak{g}}$ is zero anyway. Thus, (3.60) holds for any k. $\square$

3.4.7 Extending the Universal Differential to $T_k^j\mathfrak{a}$

Next, the natural differential on $T_k^j\mathfrak{a}$ compatible with the differential on $\hat{T}^*(\mathfrak{a})$ is

$$\begin{aligned}&d(\{\imath(a),\omega^j+\kappa_{k-1}^{j-2}\}+[\tilde{\omega}^1,\tilde{\kappa}_{k-1}^{j-1}]) \\ &\quad:=\{\imath(da),\omega^j+\kappa_{k-1}^{j-2}\}+\{\imath(a),d\omega^j+d\kappa_{k-1}^{j-2}\}+[d\tilde{\omega}^1,\tilde{\kappa}_{k-1}^{j-1}]-[\tilde{\omega}^1,d\tilde{\kappa}_{k-1}^{j-1}]\,,\end{aligned} \tag{3.61}$$

where $a\in\mathfrak{a}$, $\omega^j\in\Omega^j\mathfrak{a}$, $\tilde{\omega}^1\in\Omega^1\mathfrak{a}$, $\kappa_{k-1}^{j-2}\in T_{k-1}^{j-2}\mathfrak{a}$ and $\tilde{\kappa}_{k-1}^{j-1}\in T_{k-1}^{j-1}\mathfrak{a}$. Here,

$$\{\imath(da),\omega^j+\kappa_{k-1}^{j-2}\}:=\imath(da)\otimes(\omega^j+\kappa_{k-1}^{j-2})+(-1)^j(\omega^j+\kappa_{k-1}^{j-2})\otimes\imath(da)$$

is the graded anticommutator.

Lemma 20. *We have* $\hat\pi(dT^j_k\mathfrak{a}) \subset \hat\pi(T^{j+1}_{k+1}\mathfrak{a})$.

Remark: In general, we do not have $dT^j_k\mathfrak{a} \subset T^{j+1}_{k+1}\mathfrak{a}$ itself.

Proof. Due to (3.61) and induction it suffices to show that for $k \leq N+j+1$ one has

$$\hat\pi(\{da, \omega^j + \kappa^{j-2}_{k-1}\}) \equiv \{\hat\pi(da), \hat\pi(\omega^j) + \hat\pi(\kappa^{j-2}_{k-1})\}_g \in \hat\pi(T^{j+1}_{k+1}\mathfrak{a}) , \tag{3.62a}$$
$$\hat\pi([d\tilde\omega^1, \tilde\kappa^{j-1}_{k-1}]) \equiv [\hat\pi(d\tilde\omega^1), \hat\pi(\tilde\kappa^{j-1}_{k-1})]_g \in \hat\pi(T^{j+1}_{k+1}\mathfrak{a}) , \tag{3.62b}$$

where

$$\begin{aligned}&\{\hat\pi(\tilde\omega^n), \hat\pi(\omega^j) + \hat\pi(\kappa^{j-2}_{k-1})\}_g \\ &\quad := \hat\pi(\tilde\omega^n)(\hat\pi(\omega^j) + \hat\pi(\kappa^{j-2}_{k-1})) + (-1)^{nj}(\hat\pi(\omega^j) + \hat\pi(\kappa^{j-2}_{k-1}))\hat\pi(\tilde\omega^n)\end{aligned} \tag{3.63}$$

is the graded anticommutator, for $\tilde\omega^n \in \Omega^n\mathfrak{a}$, $\omega^j \in \Omega^j\mathfrak{a}$ and $\kappa^{j-2}_{k-1} \in T^{j-2}_{k-1}\mathfrak{a}$. Due to (3.26) and (3.29) we can replace $\hat\pi(da)$ in (3.62a) by $\sum_\alpha[\hat\pi(a_\alpha), \hat\pi(\omega^1_\alpha)]$, for certain $a_\alpha \in \mathfrak{a}$ and $\omega^1_\alpha \in \Omega^1\mathfrak{a}$. Then,

$$\begin{aligned}&\textstyle\sum_\alpha\{[\hat\pi(a_\alpha), \hat\pi(\omega^1_\alpha)], \hat\pi(\omega^j) + \hat\pi(\kappa^{j-2}_{k-1})\}_g \\ &\equiv \textstyle\sum_\alpha(\{\hat\pi(a_\alpha), [\hat\pi(\omega^1_\alpha), \hat\pi(\omega^j)+\hat\pi(\kappa^{j-2}_{k-1})]_g\} - [\hat\pi(\omega^1_\alpha), \{\hat\pi(a_\alpha), \hat\pi(\omega^j)+\hat\pi(\kappa^{j-2}_{k-1})\}]_g)\end{aligned}$$

belongs to $\hat\pi(T^{j+1}_{k+1}\mathfrak{a})$. In (3.62b) we have $\hat\pi(d\tilde\omega^1) = \sum_\alpha[\hat\pi(\omega^1_\alpha), \hat\pi(\tilde\omega^1_\alpha)]_g$, for certain elements $\omega^1_\alpha, \tilde\omega^1_\alpha \in \Omega^1\mathfrak{a}$. Then,

$$\begin{aligned}&\textstyle\sum_\alpha\ [[\hat\pi(\omega^1_\alpha), \hat\pi(\tilde\omega^1_\alpha)]_g, \hat\pi(\kappa^{j-1}_{k-1})]_g \\ &\quad \equiv \textstyle\sum_\alpha([\hat\pi(\omega^1_\alpha), [\hat\pi(\tilde\omega^1_\alpha), \hat\pi(\kappa^{j-1}_{k-1})]_g]_g + [\hat\pi(\tilde\omega^1_\alpha), [\hat\pi(\omega^1_\alpha), \hat\pi(\kappa^{j-1}_{k-1})]_g]_g) ,\end{aligned}$$

which obviously belongs to $\hat\pi(T^{j+1}_{k+1}\mathfrak{a})$. For $k = N+j+2$ we must additionally show that

$$\hat\pi([\imath(da), \{\imath(\tilde a), \omega^j + \kappa^{j-2}_{k-1}\}]) \equiv [\hat\pi(da), \{\hat\pi(\tilde a), \hat\pi(\omega^j) + \hat\pi(\kappa^{j-2}_{k-1})\}_g]_g \tag{3.64}$$

belongs to $\hat\pi(T^{j+1}_{N+j+3}\mathfrak{a})$. Again, we put $\hat\pi(da) = \sum_\alpha[\hat\pi(a_\alpha), \hat\pi(\omega^1_\alpha)]$, giving

$$\begin{aligned}(3.64) &= \textstyle\sum_\alpha[\hat\pi(a_\alpha), [\hat\pi(\omega^1_\alpha), \{\hat\pi(\tilde a), \hat\pi(\omega^j) + \hat\pi(\kappa^{j-2}_{k-1})\}_g]_g] \\ &\quad - \textstyle\sum_\alpha[\hat\pi(\omega^1_\alpha), [\hat\pi(a_\alpha), \{\hat\pi(\tilde a), \hat\pi(\omega^j) + \hat\pi(\kappa^{j-2}_{k-1})\}_g]]_g ,\end{aligned}$$

which belongs to $[\hat\pi(\mathfrak{a}), \hat\pi(T^{j+1}_{j+3}\mathfrak{a})] + [\hat\pi(\Omega^1\mathfrak{a}), \hat\pi(T^j_{N+j+2}\mathfrak{a})]_g \subset \hat\pi(T^{j+1}_{j+N+3}\mathfrak{a})$. □

Now, we have an analogous property as Lemma 7:

Lemma 21. $\hat\pi(d\kappa^j_k) = \hat\sigma(\kappa^j_k) + [-\mathrm{i}\mathcal{M}\ , \hat\pi(\kappa^j_k)]_g$, $\quad \kappa^j_k \in T^j_k\mathfrak{a}$.

Proof. First, for $j=0$, $\kappa_k^0=\{\imath(a),\imath(\tilde{a})\}$, we have $\hat{\sigma}(\kappa_k^0)=0$ and

$$\begin{aligned}\hat{\pi}(d\{\imath(a),\imath(\tilde{a})\}) &= \{\hat{\pi}(da),\hat{\pi}(\tilde{a})\}+\{\hat{\pi}(a),\hat{\pi}(d\tilde{a})\} \\ &= \{[-\mathrm{i}\mathcal{M}\,,\hat{\pi}(a)],\hat{\pi}(\tilde{a})\}+\{\hat{\pi}(a),[-\mathrm{i}\mathcal{M}\,,\hat{\pi}(\tilde{a})]\}=[-\mathrm{i}\mathcal{M}\,,\{\hat{\pi}(a),\hat{\pi}(\tilde{a})\}]\,.\end{aligned}$$

Next, for $d\kappa_k^j$ given by (3.61), $k\le j+N+1$, we have by induction and Lemma 7

$$\begin{aligned}&\hat{\pi}(d(\{\imath(a),\omega^j+\kappa_{k-1}^{j-2}\}+[\tilde{\omega}^1,\tilde{\kappa}_{k-1}^{j-1}])) \\ &\quad=\{[-\mathrm{i}\mathcal{M}\,,\hat{\pi}(a)],\hat{\pi}(\omega^j)+\hat{\pi}(\kappa_{k-1}^{j-2})\}_g \\ &\qquad+\{\hat{\pi}(a),[-\mathrm{i}\mathcal{M}\,,\hat{\pi}(\omega^j)+\hat{\pi}(\kappa_{k-1}^{j-2})]_g\}+\{\hat{\pi}(a),\hat{\sigma}(\omega^j)+\hat{\sigma}(\kappa_{k-1}^{j-2})\} \\ &\qquad+[[-\mathrm{i}\mathcal{M}\,,\hat{\pi}(\tilde{\omega}^1)]_g,\hat{\pi}(\tilde{\kappa}_{k-1}^{j-1})]+[\hat{\sigma}(\tilde{\omega}^1),\hat{\pi}(\tilde{\kappa}_{k-1}^{j-1})] \\ &\qquad-[\hat{\pi}(\tilde{\omega}^1),[-\mathrm{i}\mathcal{M}\,,\hat{\pi}(\tilde{\kappa}_{k-1}^{j-1})]_g]_g-[\hat{\pi}(\tilde{\omega}^1),\hat{\sigma}(\tilde{\kappa}_{k-1}^{j-1})]_g \\ &\quad=\hat{\sigma}(\{\imath(a),\omega^j+\kappa_{k-1}^{j-2}\}+[\tilde{\omega}^1,\tilde{\kappa}_{k-1}^{j-1}])+[-\mathrm{i}\mathcal{M}\,,\hat{\pi}(\{\imath(a),\omega^j+\kappa_{k-1}^{j-2}\}+[\tilde{\omega}^1,\tilde{\kappa}_{k-1}^{j-1}])]_g\,.\end{aligned} \tag{3.65}$$

The calculation for $k=N+j+2$ is similar. □

The results Lemma 19, Lemma 20 and Lemma 21 will play an essential rôle in the proof of our main theorem, however, a lot of work is still necessary.

3.5 Main Theorem

Here we will prove by induction the main theorem on the structure of the differential ideal $\pi(\mathcal{J}^n\mathfrak{g})$. Induction proofs sometimes have the tendency to be ugly, and our Theorem is a perfect example of an ugly proof. The induction starts with $n=2$. This step uses some interesting properties. But the proof for the higher degrees is really boring. It is a long and stupid calculation without a single new idea. Fortunately, for physical applications we need only the case $n=2$. Thus, the reader is – again – advised to pay attention only to the case $n=2$. This means that she or he ought to read the Introduction and Section 3.5.2, to skip the rest starting with Section 3.5.3 and to pass immediately to Section 3.6.

3.5.1 Theorem 22.

For $\{\hat{\pi}(\mathfrak{a}''),\hat{\pi}(\mathfrak{a}'')\}\cap\hat{\pi}(\Omega^2\mathfrak{a})=0$ we have

$$\begin{aligned}\pi(\mathcal{J}^n\mathfrak{g}) &= \bigoplus_{j=2}^{n}\Lambda^{n-j}\gamma^j\otimes(\hat{\pi}(\mathcal{J}^j\mathfrak{a})+\tilde{K}_n^{j-2}\mathfrak{a}) \\ &\quad+B^N\gamma^n\otimes(\{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\Omega^{n-N-2}\mathfrak{a})+\hat{\pi}(T_{n-2}^{n-N-4}\mathfrak{a})\}\cap\hat{\pi}(\Omega^{n-N}\mathfrak{a}))\,,\end{aligned} \tag{3.66}$$

where $B^N=\mathbf{d}\Lambda^{N-1}$, $\tilde{K}_n^0\mathfrak{a}\equiv\hat{\pi}(T_n^0\mathfrak{a})$ and

$$\begin{aligned}\tilde{K}_n^j\mathfrak{a} &= \{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\Omega^j\mathfrak{a})+\tilde{K}_{n-1}^{j-2}\mathfrak{a}\}+[\hat{\pi}(\Omega^1\mathfrak{a}),\tilde{K}_{n-1}^{j-1}\mathfrak{a}]_g \\ &\quad+\hat{\sigma}(\hat{\pi}^{-1}(\hat{\pi}(T_{j+1}^{j-1}\mathfrak{a})\cap\hat{\pi}(\Omega^{j+1}\mathfrak{a})))\,,\quad 2+j\le n\le N+j+1\,,\quad j>0\,,\end{aligned} \tag{3.67a}$$

$$\begin{aligned}\tilde{K}^j_{N+j+2}\mathfrak{a} &= [\hat{\pi}(\mathfrak{a}), \tilde{K}^j_{N+j+1}\mathfrak{a}] + [\hat{\pi}(\Omega^1\mathfrak{a}), \tilde{K}^{j-1}_{N+j+1}\mathfrak{a}]_g \\ &\quad + \hat{\sigma}(\hat{\pi}^{-1}(\hat{\pi}(T^{j-1}_{N+j+1}\mathfrak{a}) \cap \hat{\pi}(\Omega^{j+1}\mathfrak{a}))) , \quad j > 0 .\end{aligned} \tag{3.67b}$$

If $\{\hat{\pi}(\mathfrak{a}''), \hat{\pi}(\mathfrak{a}'')\} \cap \hat{\pi}(\Omega^2\mathfrak{a}) \neq 0$ *then* $\pi(\jmath^3\mathfrak{g})$ *must be replaced by*

$$\pi(\jmath^3\mathfrak{g}) = \pi(\jmath^3\mathfrak{g}) \restriction_{(3.66)} + B^1 \otimes (\{\hat{\pi}(\mathfrak{a}''), \hat{\pi}(\mathfrak{a}'')\} \cap \hat{\pi}(\Omega^2\mathfrak{a})) .$$

Proof. *The proof consists in deriving a formula for* $\sigma(\omega^k)$ *for a given* $\omega^k \in \Omega^k\mathfrak{g}$. *Taking* $\omega^k \in \Omega^k\mathfrak{g} \cap \ker\pi$, *we can derive the structure of* $\pi(\jmath^{k+1}\mathfrak{g})$, *see* (2.40). *We start with* $k = 1$ *and* $k = 2$ *and proceed for higher degrees by induction.*

Introduction

We consider the splitting

$$\hat{\omega}^1 = d(\imath(a') + \imath(a'')) + \textstyle\sum_{\alpha,z\geq 1}[\imath(a^z_\alpha), [\ldots, [\imath(a^2_\alpha), [\imath(a^1_\alpha), \imath(da^0_\alpha)]]\ldots]] \in \Omega^1\mathfrak{a} ,$$

for $a' = \sum_\beta [a'_\beta, \tilde{a}'_\beta] \in \mathfrak{a}'$ and $a'' \in \mathfrak{a}''$. Due to (3.26) and (3.29) we can replace $\omega^1_0 := \imath(d(a' + a''))$ by

$$\begin{aligned}\hat{\omega}^1_0 &= \pm\tfrac{5}{4}[\imath(\mathrm{b}), [\imath(\mathrm{b}), \imath(da'')]] - \tfrac{1}{4}[\imath(\mathrm{b}), [\imath(\mathrm{b}), [\imath(\mathrm{b}), [\imath(\mathrm{b}), \imath(da'')]]]] \\ &\quad + \textstyle\sum_\beta \left([\imath(a'_\beta), \imath(d\tilde{a}'_\beta)] - [\imath(\tilde{a}'_\beta), \imath(da'_\beta)]\right) .\end{aligned}$$

Here, in the first term the plus sign (minus sign) stands if in (3.26) the equation with the plus sign (minus sign) is realized. Indeed, we have

$$\hat{\pi}(\hat{\omega}^1_0) \equiv \hat{\pi}(\omega^1_0) , \qquad \hat{\sigma}(\hat{\omega}^1_0) \equiv \hat{\sigma}(\omega^1_0) . \tag{3.68}$$

The first formula is due to (3.26) for a'' and due to the Jacobi identity for a'. The $\mathfrak{a}'$–part of the second formula in (3.68) follows immediately from the Jacobi identity. The proof for the $\mathfrak{a}''$–part consists of algebraic manipulations of (3.26), which are not difficult but rather lengthy so that they are not listed in this work. *The importance of the identities* (3.68) *is that already elements of* $\Omega^1\mathfrak{a}$, *which do not contain terms labelled by* $z = 0$, *are sufficient for the construction of* $\hat{\pi}(\Omega^1\mathfrak{a})$ *and* $\hat{\sigma}(\Omega^1\mathfrak{a})$.

Next, we introduce the following very useful functions $\mathrm{f}_{\alpha A}, \tilde{\mathrm{f}}_{\alpha A}$, $A = 1,2,3$:

$$\begin{array}{lll}\tilde{\mathrm{f}}_{\alpha 1} = f_\alpha , & \tilde{\mathrm{f}}_{\alpha 2} = -\tfrac{1}{2} , & \tilde{\mathrm{f}}_{\alpha 3} = -\tfrac{1}{2}(f_\alpha)^2 , \\ \mathrm{f}_{\alpha 1} = f_\alpha \tilde{f}_\alpha , & \mathrm{f}_{\alpha 2} = (f_\alpha)^2 \tilde{f}_\alpha , & \mathrm{f}_{\alpha 3} = \tilde{f}_\alpha ,\end{array} \tag{3.69}$$

where $f_\alpha, \tilde{f}_\alpha \in C^\infty(X)$. These functions have the nice properties (the sum over A runs always from 1 to 3)

$$\begin{array}{lll}\sum_A \tilde{\mathrm{f}}_{\alpha A}\mathrm{f}_{\alpha A} = 0 , & \sum_A \tilde{\mathrm{f}}_{\alpha A}\mathbf{d}(\mathrm{f}_{\alpha A}) = 0 , & \sum_A \mathbf{d}(\tilde{\mathrm{f}}_{\alpha A})\mathrm{f}_{\alpha A} = 0 , \\ \multicolumn{3}{l}{\sum_A \nabla_{\mathrm{grad}(\tilde{\mathrm{f}}_{\alpha A})}(\mathrm{f}_{\alpha A}) = \tilde{f}_\alpha \nabla_{\mathrm{grad}(f_\alpha)}(f_\alpha) = \tilde{f}_\alpha g^{-1}(\mathsf{d}f_\alpha, \mathsf{d}f_\alpha) .}\end{array} \tag{3.70}$$

For an appropriate choice of $f_\alpha, \tilde{f}_\alpha$, the last sum attains in any chart $\mathcal{O}_i$ of the manifold X each given function on $\mathcal{O}_i$. Via a partition of unity we can even represent each function on X in this way.

3.5.2 The Proof for $n = 2$

We take an arbitrary element $\omega^1 \in \Omega^1\mathfrak{g}$ and compute $\sigma(\omega^1)$. Then we apply identities of the Dirac operator, of the differential and the codifferential and of covariant derivatives. This leads us to formula (3.75)*. Next, we discuss to which amount $\sigma(\omega^1)$ is determined by $\pi(\omega^1)$. The answer is given in formula* (3.81)*. That's all.*

Using (3.28) we can represent elements $\omega^1 \in \Omega^1\mathfrak{g}$ as

$$\begin{aligned}
&\omega^1 = \textstyle\sum_{\alpha,z\geq 0}[\imath(f^z_\alpha \otimes a^z_\alpha),[\ldots[\imath(f^1_\alpha \otimes a^1_\alpha),\imath(d(f^0_\alpha \otimes a^0_\alpha))]\ldots]]\,, && (3.71a)\\
&\Rightarrow \qquad \pi(\omega^1) = \textstyle\sum_{\alpha,z\geq 0}\big(\hat c^{1,z}_\alpha \otimes \hat\pi(\hat a^z_\alpha) + \hat f^z_\alpha \gamma \otimes \hat\pi(\hat\omega^{1,z}_\alpha)\big)\,, &&\\
&\hat f^z_\alpha = f^z_\alpha \cdots f^1_\alpha f^0_\alpha \in \Lambda^0\,, \qquad \hat c^{1,z}_\alpha = f^z_\alpha \cdots f^1_\alpha \mathbf{d} f^0_\alpha \in \Lambda^1\,, && (3.71b)\\
&\hat a^z_\alpha = [a^z_\alpha,[\ldots[a^1_\alpha,a^0_\alpha]\ldots]] \in \mathfrak{a}\,, \quad \hat\omega^{1,z}_\alpha = [\imath(a^z_\alpha),[\ldots[\imath(a^1_\alpha),\imath(da^0_\alpha)]\ldots]] \in \Omega^1\mathfrak{a}\,, &&
\end{aligned}$$

where $a^i_\alpha \in \mathfrak{a}$ and $f^i_\alpha \in \Lambda^0$. Applying the map σ to ω^1 in (3.71a) we get – using (3.22) and $D^2 \equiv \mathsf{D}^2 \otimes \mathbb{1}_F + 1 \otimes \mathcal{M}^{\,2}$, see (3.4a) –

$$\begin{aligned}
&\sigma(\omega^1) = \textstyle\sum_{\alpha,z\geq 0}[f^z_\alpha \otimes \hat\pi(a^z_\alpha),[\ldots[f^1_\alpha \otimes \hat\pi(a^1_\alpha),[D^2, f^0_\alpha \otimes \hat\pi(a^0_\alpha)]]\ldots]] \equiv \sum_{j=0}^3 s_j\,, &&\\
&s_0 = \hat\sigma_{\mathfrak{g}}(\omega^1) = \textstyle\sum_{\alpha,z\geq 0} f^z_\alpha \cdots f^1_\alpha f^0_\alpha \otimes [\hat\pi(a^z_\alpha),[\ldots[\hat\pi(a^1_\alpha),[\mathcal{M}^{\,2},\hat\pi(a^0_\alpha)]]\ldots]]\,, && (3.72a)\\
&s_1 = \textstyle\sum_{\alpha,z\geq 0} f^z_\alpha \cdots f^1_\alpha(\Delta f^0_\alpha) \otimes \hat\pi([a^z_\alpha,[\ldots[a^1_\alpha,a^0_\alpha]\ldots]])\,, && (3.72b)\\
&s_2 = -2\textstyle\sum_{\alpha,z\geq 0} f^z_\alpha \cdots f^1_\alpha \nabla^S_{\mathrm{grad}\, f^0_\alpha} \otimes \hat\pi([a^z_\alpha,[\ldots[a^1_\alpha,a^0_\alpha]\ldots]])\,, && (3.72c)\\
&s_3 = 2\textstyle\sum_{\alpha,z\geq 1}\big(f^z_\alpha \cdots f^2_\alpha \nabla_{\mathrm{grad}\, f^0_\alpha}(f^1_\alpha) \otimes [\hat\pi(a^z_\alpha),[\ldots[\hat\pi(a^2_\alpha),\hat\pi(a^0_\alpha)\hat\pi(a^1_\alpha)]\ldots]] &&\\
&\quad + f^z_\alpha \cdots f^3_\alpha \nabla_{\mathrm{grad}\, f^0_\alpha}(f^2_\alpha) f^1_\alpha \otimes [\hat\pi(a^z_\alpha),[\ldots[\hat\pi(a^3_\alpha),\hat\pi([a^1_\alpha,a^0_\alpha])\hat\pi(a^2_\alpha)]\ldots]] + \ldots &&\\
&\quad + \nabla_{\mathrm{grad}\, f^0_\alpha}(f^z_\alpha) f^{z-1}_\alpha \cdots f^1_\alpha \otimes \hat\pi([a^{z-1}_\alpha,[\ldots[a^1_\alpha,a^0_\alpha]\ldots]])\hat\pi(a^z_\alpha)\big)\,. && (3.72d)
\end{aligned}$$

From properties of covariant derivatives we find

$$f^z_\alpha \cdots f^1_\alpha \nabla^S_{\mathrm{grad}\, f^0_\alpha} = \nabla^S_{f^z_\alpha \cdots f^1_\alpha g^{-1}(\mathsf{d} f^0_\alpha)} = \nabla^S_{g^{-1}(f^z_\alpha \cdots f^1_\alpha \mathsf{d} f^0_\alpha)}\,.$$

Next, using (3.12) and (3.15) one easily shows

$$f^z_\alpha \cdots f^1_\alpha(\Delta f^0_\alpha) = \mathbf{d}^*(f^z_\alpha \cdots f^1_\alpha \mathbf{d} f^0_\alpha) + \nabla_{\mathrm{grad}\, f^0_\alpha}(f^z_\alpha \cdots f^1_\alpha)\,. \qquad (3.73)$$

Then, the sum of s_3 and the part of s_1 corresponding to the second term on the r.h.s. of (3.73) will be denoted by $\hat s(\omega^1)$:

$$\begin{aligned}
\hat s(\omega^1) &= s_3 + \textstyle\sum_{\alpha,z\geq 1}\nabla_{\mathrm{grad}\, f^0_\alpha}(f^z_\alpha \cdots f^1_\alpha) \otimes \hat\pi(\hat a^z_\alpha) \qquad (3.74)\\
&= \textstyle\sum_{\alpha,z\geq 1}\big(f^z_\alpha \cdots f^2_\alpha \nabla_{\mathrm{grad}\, f^0_\alpha}(f^1_\alpha) \otimes [\hat\pi(a^z_\alpha),[\ldots[\hat\pi(a^2_\alpha),\{\hat\pi(a^0_\alpha),\hat\pi(a^1_\alpha)\}]\ldots]]\\
&\quad + f^z_\alpha \cdots f^3_\alpha \nabla_{\mathrm{grad}\, f^0_\alpha}(f^2_\alpha) f^1_\alpha \otimes [\hat\pi(a^z_\alpha),[\ldots[\hat\pi(a^3_\alpha),\{\hat\pi([a^1_\alpha,a^0_\alpha]),\hat\pi(a^2_\alpha)\}]\ldots]] + \ldots\\
&\quad + \nabla_{\mathrm{grad}\, f^0_\alpha}(f^z_\alpha) f^{z-1}_\alpha \cdots f^1_\alpha \otimes \{\hat\pi([a^{z-1}_\alpha,[\ldots[a^1_\alpha,a^0_\alpha]\ldots]]),\hat\pi(a^z_\alpha)\}\big)\\
&\in \Lambda^0 \otimes \{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\}\,.
\end{aligned}$$

Observe that the terms labelled by $z=0$ do not occur in (3.74). Collecting the results we find

$$\sigma(\omega^1) = \hat{s}(\omega^1) + \hat{\sigma}_{\mathfrak{g}}(\omega^1) + \sum_{\alpha, z\geq 0} \big(-2\nabla^S_{g^{-1}\circ c^{-1}(\hat{c}^{1,z}_\alpha)} \otimes \hat{\pi}(\hat{a}^z_\alpha) + \mathbf{d}^*(\hat{c}^{1,z}_\alpha) \otimes \hat{\pi}(\hat{a}^z_\alpha)\big) \;. \tag{3.75}$$

The Relation between $\pi(\omega^1)$ and $\sigma(\omega^1)$

It is clear that $\hat{s}(\omega^1) \in \Lambda^0 \otimes \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}$ and $\hat{\sigma}_{\mathfrak{g}}(\omega^1) \in \Lambda^0 \otimes \hat{\sigma}(\Omega^1\mathfrak{a})$, the question is to which amount they are determined by $\pi(\omega^1)$. To answer this question we first consider

$$\omega^1 = \textstyle\sum_\alpha \sum_A [\imath(\tilde{\mathrm{f}}_{\alpha A} \otimes \tilde{a}_\alpha), \imath(d(\mathrm{f}_{\alpha A} \otimes a_\alpha))] \;, \quad a_\alpha, \tilde{a}_\alpha \in \mathfrak{a} \;. \tag{3.76}$$

Due to (3.70) we have $\pi(\omega^1) = 0$ and $\hat{\sigma}_{\mathfrak{g}}(\omega^1) = 0$, but for (3.74) we get

$$\begin{aligned} \hat{s}(\omega^1) &\equiv \textstyle\sum_\alpha \sum_A \nabla_{\mathrm{grad}\,\tilde{\mathrm{f}}_{\alpha A}}(\mathrm{f}_{\alpha A}) \otimes \{\hat{\pi}(\tilde{a}_\alpha), \hat{\pi}(a_\alpha)\} \\ &= \textstyle\sum_\alpha \tilde{f}_\alpha \nabla_{\mathrm{grad}\, f_\alpha}(f_\alpha) \otimes \{\hat{\pi}(\tilde{a}_\alpha), \hat{\pi}(a_\alpha)\} \;. \end{aligned}$$

Thus, $\hat{s}(\omega^1)$ is independent of $\pi(\omega^1)$ and attains each element of $\Lambda^0 \otimes \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} \equiv \Lambda^0 \otimes \hat{\pi}(T^0_2\mathfrak{a})$. We know from Lemma 19 that

$$\hat{\sigma}_{\mathfrak{g}}(\ker \pi \cap \Omega^1 \mathfrak{g}) = \Lambda^0 \otimes \hat{\sigma}(\ker \hat{\pi} \cap \Omega^1 \mathfrak{a}) \equiv \Lambda^0 \otimes \hat{\pi}(\jmath^2 \mathfrak{a}) \;. \tag{3.77}$$

It is interesting to check the inclusion $\supset$ directly. By virtue of (3.68) it suffices to take

$$\omega^1 = \textstyle\sum_\alpha \sum_{\beta, z\geq 1} [\imath(1 \otimes a^z_{\alpha\beta}), [\dots, [\imath(1 \otimes a^2_{\alpha\beta}), [\imath(f_\alpha \otimes a^1_{\alpha\beta}), \imath(d(1 \otimes a^0_{\alpha\beta}))]]\dots]] \;,$$

with

$$\hat{\omega}^1_\alpha := \textstyle\sum_{\beta, z\geq 1} [\imath(a^z_{\alpha\beta}), [\dots, [\imath(a^2_{\alpha\beta}), [\imath(a^1_{\alpha\beta}), \imath(d a^0_{\alpha\beta})]]\dots]] \in \ker \hat{\pi} \cap \Omega^1 \mathfrak{a} \;, \quad \forall \alpha \,,$$

where $f_\alpha \in \Lambda^0$ and $a^i_{\alpha\beta} \in \mathfrak{a}$. It is obvious that $\pi(\omega^1) \equiv 0$ and that $\sigma(\omega^1) = \hat{\sigma}_{\mathfrak{g}}(\omega^1) = \sum_\alpha f_\alpha \otimes \hat{\sigma}(\hat{\omega}^1_\alpha)$ attains each element of $\Lambda^0 \otimes \hat{\pi}(\jmath^2\mathfrak{a})$. But the proof of the inclusion $\subset$ requires considerations equivalent to those used in the proof of Lemma 19.

Some Definitions

We use the natural convention

$$\mathbf{d}^*(c^{n-j} \gamma^j \otimes \hat{\pi}(\hat{\omega}^j + \hat{\kappa}^{j-2}_n)) := \mathbf{d}^*(c^{n-j})\gamma^j \otimes \hat{\pi}(\hat{\omega}^j + \hat{\kappa}^{j-2}_n) \;, \tag{3.78a}$$

$$\mathbf{d}(c^{n-j} \gamma^j \otimes \hat{\pi}(\hat{\omega}^j + \hat{\kappa}^{j-2}_n)) := \mathbf{d}(c^{n-j})\gamma^j \otimes \hat{\pi}(\hat{\omega}^j + \hat{\kappa}^{j-2}_n) \;, \tag{3.78b}$$

for $c^{n-j} \in \Lambda^{n-j}$, $\hat{\omega}^j \in \Omega^j\mathfrak{a}$ and $\hat{\kappa}^{j-2}_n \in T^{j-2}_n\mathfrak{a}$. Moreover, we define a linear map ∇_Ω from $\pi(\Omega^*\mathfrak{g})$ to (unbounded) operators on h,

$$\begin{aligned} \nabla_\Omega(c^{n-j} \gamma^j \otimes \hat{\pi}(\hat{\omega}^j + \hat{\kappa}^{j-2}_n)) &:= \nabla^S_{c^{n-j}} \gamma^j \otimes \hat{\pi}(\hat{\omega}^j + \hat{\kappa}^{j-2}_n) \;, \quad n-j > 0 \;, \\ \nabla_\Omega(f \gamma^n \otimes \hat{\pi}(\hat{\omega}^n + \hat{\kappa}^{n-2}_n)) &:= 0 \;, \quad f \in C^\infty(X) \;. \end{aligned} \tag{3.79}$$

Here and in the sequel a covariant derivative with respect to elements of Λ^n is understood in the sense

$$\nabla_{c_1^1\wedge c_2^1\wedge\ldots\wedge c_n^1} := \sum_{l=1}^n (-1)^{l+1} c_1^1 \wedge \overset{l}{\cdot\vee\cdot} \wedge c_n^1 \nabla_{g^{-1}\circ c^{-1}(c_l^1)} \,, \quad c_i^1 \in \Lambda^1 \,, \tag{3.80}$$

where $c^{-1}: \Lambda^1 \to \Gamma^\infty(T^*X)$ and $g^{-1}: \Gamma^\infty(T^*X) \to \Gamma^\infty(T_*X)$ are isomorphisms.

The Final Formula for $\sigma(\Omega^1\mathfrak{g})$

Now, we can express (3.75) in terms of $\pi(\omega^1)$. For given $\tau^1 \in \pi(\Omega^1\mathfrak{g})$ let $\pi^{-1}(\tau^1) \in \Omega^1\mathfrak{g}$ be an arbitrary but fixed representative and $\omega^1 \in \Omega^1\mathfrak{g}$ be any representative. Then, the set $\{\sigma(\omega^1)\}$ of all elements $\sigma(\omega^1)$ fulfilling the just introduced conditions is

$$\{\sigma(\omega^1)\} = \Lambda^0 \otimes (\hat\pi(T_2^0\mathfrak{a}) + \hat\pi(\jmath^2\mathfrak{a})) + \hat\sigma_{\mathfrak{g}}(\pi^{-1}(\tau^1)) - 2\nabla_\Omega(\tau^1) + \mathbf{d}^*\tau^1 \,. \tag{3.81}$$

Putting $\tau^1 = 0$, i.e. $\omega^1 \in \ker\pi \cap \Omega^1\mathfrak{g}$, we obtain immediately the assertion of the theorem for $n = 2$.

3.5.3 The Proof for $n = 3$

One could think that we can perform the induction now to prove the theorem for arbitrary degree. However, the case $n = 3$ is a special one which must be separately treated. The reason is that certain "boundary terms" occur if the Abelian Lie algebra $\mathfrak{a}''$ is present. Therefore, we must take $n = 3$ as the starting point of the induction and not $n = 2$. The proof for $n = 3$ is based upon the same boring calculation as the general case.

We use formula (3.81) as the starting point for the construction of $\sigma(\Omega^n\mathfrak{g})$ for $n > 1$. For $\omega^2 = \sum_\alpha [\omega_\alpha^1, \tilde\omega_\alpha^1] \in \Omega^2\mathfrak{g}$ and

$$\begin{aligned}
\tau_\alpha^1 &= \pi(\omega_\alpha^1) = \textstyle\sum_\beta (c_{\alpha\beta}^1 \otimes \hat\pi(a_{\alpha\beta}) + f_{\alpha\beta}\gamma \otimes \hat\pi(\hat\omega_{\alpha\beta}^1)) \,, \\
\tilde\tau_\alpha^1 &= \pi(\tilde\omega_\alpha^1) = \textstyle\sum_\gamma (\tilde c_{\alpha\gamma}^1 \otimes \hat\pi(\tilde a_{\alpha\gamma}) + \tilde f_{\alpha\gamma}\gamma \otimes \hat\pi(\tilde{\hat\omega}_{\alpha\gamma}^1))
\end{aligned}$$

we have according to (2.42) $\sigma(\omega^2) = \sum_\alpha \big([\sigma(\omega_\alpha^1), \tilde\tau_\alpha^1] + [\sigma(\tilde\omega_\alpha^1), \tau_\alpha^1]\big)$. Inserting (3.81) we get

$$\begin{aligned}
\sigma(\omega^2) = {} & [\Lambda^0 \otimes (\hat\pi(T_2^0\mathfrak{a}) + \hat\pi(\jmath^2\mathfrak{a})), \pi(\Omega^1\mathfrak{g})] \\
& + \textstyle\sum_\alpha \big([\hat\sigma_{\mathfrak{g}}(\pi^{-1}(\tau_\alpha^1)), \tilde\tau_\alpha^1] + [\hat\sigma_{\mathfrak{g}}(\pi^{-1}(\tilde\tau_\alpha^1)), \tau_\alpha^1] \\
& + [-2\nabla_\Omega(\tau_\alpha^1) + \mathbf{d}^*(\tau_\alpha^1), \tilde\tau_\alpha^1] + [-2\nabla_\Omega(\tilde\tau_\alpha^1) + \mathbf{d}^*(\tilde\tau_\alpha^1), \tau_\alpha^1]\big) \,.
\end{aligned} \tag{3.82}$$

The Last Line in Formula (3.82)

First, let us analyse

$$
\begin{aligned}
&[-2\nabla_\Omega\,(\tau^1_\alpha)+\mathbf{d}^*(\tau^1_\alpha),\tilde\tau^1_\alpha]\\
&\quad=\textstyle\sum_{\beta,\gamma}[(-2\nabla^S_{c^1_{\alpha\beta}}+\mathbf{d}^*c^1_{\alpha\beta})\otimes\hat\pi(a_{\alpha\beta}),\tilde c^1_{\alpha\gamma}\otimes\hat\pi(\tilde a_{\alpha\gamma})+\tilde f_{\alpha\gamma}\gamma\otimes\hat\pi(\hat{\tilde\omega}^1_{\alpha\gamma})]\\
&\quad=\textstyle\sum_{\beta,\gamma}\big(-2\tilde c^1_{\alpha\gamma}\nabla^S_{c^1_{\alpha\beta}}\otimes\hat\pi([a_{\alpha\beta},\tilde a_{\alpha\gamma}])-2\nabla^S_{\tilde f_{\alpha\gamma}c^1_{\alpha\beta}}\gamma\otimes[\hat\pi(a_{\alpha\beta}),\hat\pi(\hat{\tilde\omega}^1_{\alpha\gamma})]\\
&\quad-2\nabla_{c^1_{\alpha\beta}}(\tilde c^1_{\alpha\gamma})\otimes\hat\pi(a_{\alpha\beta})\hat\pi(\tilde a_{\alpha\gamma})-2\nabla_{c^1_{\alpha\beta}}(\tilde f_{\alpha\gamma})\gamma\otimes\hat\pi(a_{\alpha\beta})\hat\pi(\hat{\tilde\omega}^1_{\alpha\gamma})\\
&\quad+\mathbf{d}^*(c^1_{\alpha\beta})\tilde c^1_{\alpha\gamma}\otimes\hat\pi([a_{\alpha\beta},\tilde a_{\alpha\gamma}])+\mathbf{d}^*(c^1_{\alpha\beta})\tilde f_{\alpha\gamma}\gamma\otimes[\hat\pi(a_{\alpha\beta}),\hat\pi(\hat{\tilde\omega}^1_{\alpha\gamma})]\big)\ .
\end{aligned}
$$

Using (3.16a), (3.16b), (3.78a) and (3.79) we get

$$
\begin{aligned}
&\textstyle\sum_\alpha([-\,2\nabla_\Omega(\tau^1_\alpha)+\mathbf{d}^*(\tau^1_\alpha),\tilde\tau^1_\alpha]+[-2\nabla_\Omega(\tilde\tau^1_\alpha)+\mathbf{d}^*(\tilde\tau^1_\alpha),\tau^1_\alpha])\\
&\quad=\textstyle\sum_{\alpha,\beta,\gamma}\big((-c^1_{\alpha\beta}\lrcorner\,\mathbf{d}(\tilde c^1_{\alpha\gamma})-\tilde c^1_{\alpha\gamma}\lrcorner\,\mathbf{d}(c^1_{\alpha\beta}))\otimes\{\hat\pi(a_{\alpha\beta}),\hat\pi(\tilde a_{\alpha\gamma})\} && (3.83a)\\
&\quad-\nabla_{c^1_{\alpha\beta}}(\tilde f_{\alpha\gamma})\gamma\otimes\{\hat\pi(a_{\alpha\beta}),\hat\pi(\hat{\tilde\omega}^1_{\alpha\gamma})\}-\nabla_{\tilde c^1_{\alpha\gamma}}(f_{\alpha\beta})\gamma\otimes\{\hat\pi(\tilde a_{\alpha\gamma}),\hat\pi(\hat\omega^1_{\alpha\beta})\}\big) && (3.83b)\\
&\quad+\mathbf{d}^*(\tau^2)-2\nabla_\Omega(\tau^2)-\textstyle\sum_{\alpha,\beta,\gamma}\mathbf{d}(c^1_{\alpha\beta}\lrcorner\,\tilde c^1_{\alpha\gamma})\otimes\{\hat\pi(a_{\alpha\beta}),\hat\pi(\tilde a_{\alpha\gamma})\}\ . && (3.83c)
\end{aligned}
$$

Observe that $\sum_{\alpha\beta\gamma}\mathbf{d}(c^1_{\alpha\beta}\lrcorner\,\tilde c^1_{\alpha\gamma})\otimes\{\hat\pi(a_{\alpha\beta}),\hat\pi(\tilde a_{\alpha\gamma})\}$ is the application of the exterior differential to the $\Lambda^0\otimes\hat\pi(T^0_2\mathfrak{a})$–component of $\tau^2=\hat\pi(\omega^2)$. We prove that the line (3.83a) is independent of the line (3.83c) for all pairs $(a_{\alpha\beta},\tilde a_{\alpha\gamma})\notin\mathfrak{a}''\times\mathfrak{a}''$. In this case, not both $c^1_{\alpha\beta},\tilde c^1_{\alpha\gamma}$ are total differentials. Therefore, we may choose $\tilde\tau^1_{\alpha A}=\mathbf{d}\tilde{\mathrm{f}}_{\alpha A}\otimes\hat\pi(\tilde a'_\alpha)$ and $\tau^1_{\alpha A}=\mathrm{f}_{\alpha A}c^1_\alpha\otimes\hat\pi(a_\alpha)$, giving $\sum_\alpha\sum_A\{\tau^1_{\alpha A},\tilde\tau^1_{\alpha A}\}=0$ and

$$
\begin{aligned}
&\textstyle\sum_\alpha\sum_A([-2\nabla_\Omega(\tau^1_{\alpha A})+\mathbf{d}^*(\tau^1_{\alpha A}),\tilde\tau^1_{\alpha A}]+[-2\nabla_\Omega(\tilde\tau^1_{\alpha A})+\mathbf{d}^*(\tilde\tau^1_{\alpha A}),\tau^1_{\alpha A}])\\
&\quad=-\textstyle\sum_\alpha\tilde f_\alpha g^{-1}(\mathrm{d}f_\alpha,\mathrm{d}f_\alpha)(c^1_\alpha-\frac{g^{-1}(c^{-1}(c^1_\alpha),\mathrm{d}f_\alpha)}{g^{-1}(\mathrm{d}f_\alpha,\mathrm{d}f_\alpha)}\mathbf{d}f_\alpha)\otimes\{\hat\pi(a_\alpha),\hat\pi(\tilde a'_\alpha)\}\ . && (3.84)
\end{aligned}
$$

Observe that for given f_α the first component of the tensor product (3.84) attains each element of the distribution defined as the orthogonal complement of $\mathbf{d}f_\alpha$ in Λ^1. Thus, if we vary f_α and c^1_α we see that (3.84) attains each element of

$$\Lambda^1\otimes\{\hat\pi(\mathfrak{a}'),\hat\pi(\mathfrak{a})\}\equiv\Lambda^1\otimes\hat\pi(T^0_3\mathfrak{a})\subset\sigma(\ker\pi\cap\Omega^2\mathfrak{g})\ . \tag{3.85}$$

In order to analyse the line (3.83b) we put

$$
\begin{aligned}
&\tilde\omega^1_{\alpha A}=\imath(d(\tilde{\mathrm{f}}_{\alpha A}\otimes\tilde a_\alpha))\ ,\\
&\omega^1_{\alpha A}=\textstyle\sum_{\beta,z\ge1}[\imath(1\otimes a^z_{\alpha\beta}),[\ldots[\imath(1\otimes a^2_{\alpha\beta}),[\imath(\mathrm{f}_{\alpha A}\otimes a^1_{\alpha\beta}),\imath(d(1\otimes a^0_{\alpha\beta}))]]\ldots]]\ .
\end{aligned}
$$

This gives $\pi(\sum_\alpha\sum_A[\tilde\omega^1_{\alpha A},\omega^1_{\alpha A}]_g)\equiv0$ and $\hat\sigma_{\mathfrak{g}}(\sum_\alpha\sum_A[\tilde\omega^1_{\alpha A},\omega^1_{\alpha A}]_g)\equiv0$, but

$$
\begin{aligned}
(3.83\mathrm{b})&=-\textstyle\sum_\alpha\sum_A\nabla_{\mathrm{grad}(\tilde{\mathrm{f}}_{\alpha A})}(\mathrm{f}_{\alpha A})\gamma\otimes\{\hat\pi(\tilde a_a),\hat\pi(\hat\omega^1_\alpha)\}\\
&=-\textstyle\sum_\alpha\tilde f_\alpha\nabla_{\mathrm{grad}(f_\alpha)}(f_\alpha)\gamma\otimes\{\hat\pi(\tilde a_a),\hat\pi(\hat\omega^1_\alpha)\} && (3.86)
\end{aligned}
$$

yields an arbitrary element of $\Lambda^0\gamma\otimes\{\hat\pi(\mathfrak a),\hat\pi(\Omega^1\mathfrak a)\}$, which is due to the fact that the matrix $\hat\pi(\hat\omega^1_\alpha):=\hat\pi(\sum_{\beta,z\geq 1}[\iota(a^z_{\alpha\beta}),[\dots[\iota(a^1_{\alpha\beta}),\iota(da^0_{\alpha\beta})]\dots]])$ is an arbitrary element of $\hat\pi(\Omega^1\mathfrak a)$.

The Remaining Terms in Formula (3.82)

From (3.39), Lemma 19, Lemma 20 and Lemma 21 we get

$$\begin{aligned}\big\{[\hat\sigma_{\mathfrak g}(\pi^{-1}(\tau^1)),\tilde\tau^1]+[\hat\sigma_{\mathfrak g}(\pi^{-1}(\tilde\tau^1)),\tau^1]\big\} &= \hat\sigma_{\mathfrak g}(\pi^{-1}(\tau^2))+\hat\sigma_{\mathfrak g}(\ker\pi\cap\Omega^2\mathfrak g)\\ &= \hat\sigma_{\mathfrak g}(\pi^{-1}(\tau^2))+\Lambda^1\otimes\hat\pi(\jmath^2\mathfrak a)+\Lambda^0\gamma\otimes\big(\hat\pi(\jmath^3\mathfrak a)+\hat\sigma(\hat\pi^{-1}(\hat\pi(T^0_2\mathfrak a)\cap\hat\pi(\Omega^2\mathfrak a)))\big)\\ &\quad+\text{a subspace of }\Lambda^0\otimes\hat\pi(T^1_3\mathfrak a)\,,\end{aligned}$$

where $\tau^2=\hat\pi(\omega^2)$. Moreover,

$$\begin{aligned}[\Lambda^0\otimes(\hat\pi(T^0_2\mathfrak a)+\hat\pi(\jmath^2\mathfrak a)),\pi(\Omega^1\mathfrak g)] &= \Lambda^1\otimes\big([\hat\pi(\mathfrak a'),\hat\pi(T^0_2\mathfrak a)]+[\hat\pi(\mathfrak a),\hat\pi(\jmath^2\mathfrak a)]\big)\\ &\quad+\Lambda^0\gamma\otimes\big([\pi(\Omega^1\mathfrak g),\hat\pi(T^0_2\mathfrak a)]+[\pi(\Omega^1\mathfrak g),\hat\pi(\jmath^2\mathfrak a)]\big)\,.\end{aligned}$$

The Final Formula for $\sigma(\Omega^2\mathfrak g)$

Collecting the results we get the final formula for $\sigma(\Omega^2\mathfrak g)$. For given $\tau^2\in\pi(\Omega^2\mathfrak g)$ let $\pi^{-1}(\tau^2)\in\Omega^2\mathfrak g$ be an arbitrary but fixed representative and $\omega^2\in\Omega^2\mathfrak g$ be any representative. Then we have

$$\begin{aligned}\{\sigma(\omega^2)\} &= \Lambda^1\otimes(\hat\pi(T^0_3\mathfrak a)+\hat\pi(\jmath^2\mathfrak a))+\Lambda^0\gamma\otimes(\tilde K^1_3\mathfrak a+\hat\pi(\jmath^3\mathfrak a))\\ &\quad+\hat\sigma_{\mathfrak g}(\pi^{-1}(\tau^2))-2\nabla_\Omega(\tau^2)+\mathbf d^*\tau^2-\mathbf d\big(\tau^2\restriction_{\Lambda^0\otimes\{\hat\pi(\mathfrak a''),\hat\pi(\mathfrak a'')\}}\big)\,.\end{aligned}\tag{3.87}$$

This gives the assertion of the theorem for $n=3$. Here, one has to take into account that for $\{\hat\pi(\mathfrak a''),\hat\pi(\mathfrak a'')\}\cap\hat\pi(\Omega^2\mathfrak a)=0$ and $\tau^2=0$ we have $\mathbf d\big(\tau^2\restriction_{\Lambda^0\otimes\{\hat\pi(\mathfrak a''),\hat\pi(\mathfrak a'')\}}\big)=0$. If $\{\hat\pi(\mathfrak a''),\hat\pi(\mathfrak a'')\}\cap\hat\pi(\Omega^2\mathfrak a)\neq 0$ then a non–vanishing $\Lambda^0\otimes\{\hat\pi(\mathfrak a''),\hat\pi(\mathfrak a'')\}$–part of $\tau^2=0$ can be compensated by $\Lambda^0\otimes\hat\pi(\Omega^2\mathfrak a)$, giving the contribution $B^1\otimes(\{\hat\pi(\mathfrak a''),\hat\pi(\mathfrak a'')\}\cap\hat\pi(\Omega^2\mathfrak a))$ to the ideal $\pi(\jmath^3\mathfrak g)$.

3.5.4 The Proof for $n>3$

Here comes the most boring part: the induction. The calculation is identical as in the proof for $n=3$, but the formulae are longer and more complicated. The method is to prove the following Lemma, which – with a lot of imagination – can be "guessed" from formula (3.87)*:*

Lemma 23. *For given $\tau^n\in\pi(\Omega^n\mathfrak g)$ let $\pi^{-1}(\tau^n)\in\Omega^n\mathfrak g$ be an arbitrary but fixed representative and ω^n be any representative. Then we have for $n\geq 3$*

$$\begin{aligned}\{\sigma(\omega^n)\} &= \hat\sigma_{\mathfrak g}(\pi^{-1}(\tau^n))-2\nabla_\Omega(\tau^n)+\mathbf d^*\tau^n+\sum_{j=2}^{n+1}\Lambda^{n+1-j}\gamma^j\otimes(\tilde K^{j-2}_{n+1}\mathfrak a+\hat\pi(\jmath^j\mathfrak a))\\ &\quad-\mathbf d\big(\tau^n\restriction_{\Lambda^{N-1}\gamma^{n+1}\otimes\{\hat\pi(\mathfrak a),\hat\pi(\Omega^{n-N-1}\mathfrak a)+\hat\pi(T^{n-N-1}_{n-1}\mathfrak a)\}}\big)\,.\end{aligned}\tag{3.88}$$

Proof. It is clear that if we prove Lemma 23 then will have finished the proof of Theorem 22. We assume that Lemma 23 is true for $n=k$. For the time being we neglect in the induction the terms in (3.87) and (3.88) containing total differentials. The justification will be given at the end of this proof. Then, for $\omega^{k+1}=\sum_\alpha[\tilde\omega^1_\alpha,\omega^k_\alpha]\in\Omega^{k+1}\mathfrak{g}$ and

$$\begin{aligned}&\tilde\tau^1_\alpha=\pi(\tilde\omega^1_\alpha)=\textstyle\sum_\beta(\tilde c^1_{\alpha\beta}\otimes\hat\pi(\tilde a_{\alpha\beta})+\tilde f_{\alpha\beta}\gamma\otimes\hat\pi(\hat{\tilde\omega}^1_{\alpha\beta}))\,, &&(3.89)\\ &\tau^k_\alpha=\pi(\omega^k_\alpha)=\textstyle\sum_\gamma\sum_{j=0}^k c^{k-j}_{\alpha\gamma}\gamma^j\otimes\hat\pi(\hat\omega^j_{\alpha\gamma}+\hat\kappa^{j-2}_{k,\alpha\gamma})\,, \qquad \hat\pi(\hat\omega^0_{\alpha\gamma}):=\hat\pi(a'_{\alpha\gamma})\,,\end{aligned}$$

where $c^n_{\alpha\gamma}\in\Lambda^n$, $\hat\omega^j_{\alpha\gamma}\in\Omega^j\mathfrak{a}$ and $\hat\kappa^j_{k,\alpha\gamma}\in T^j_k\mathfrak{a}$, we have according to (2.42)

$$\begin{aligned}\sigma(\omega^{k+1})&=\textstyle\sum_\alpha(-(-1)^k[\sigma(\omega^k_\alpha),\tilde\tau^1_\alpha]_g+[\sigma(\tilde\omega^1_\alpha),\tau^k_\alpha]_g)\\ &\left.\begin{aligned}&=[\textstyle\sum_{j=2}^{k+1}\Lambda^{k+1-j}\gamma^j\otimes(\tilde K^{j-2}_{k+1}\mathfrak{a}+\hat\pi(\jmath^j\mathfrak{a})),\pi(\Omega^1\mathfrak{g})]_g\\ &\qquad+[\Lambda^0\otimes(\tilde K^0_2\mathfrak{a}+\hat\pi(\jmath^2\mathfrak{a})),\pi(\Omega^k\mathfrak{g})]_g\end{aligned}\right\}\quad(3.90a)\\ &+\textstyle\sum_\alpha\left(-(-1)^k[\hat\sigma_{\mathfrak{g}}(\pi^{-1}(\tau^k_\alpha)),\tilde\tau^1_\alpha]_g+[\hat\sigma_{\mathfrak{g}}(\pi^{-1}(\tilde\tau^1_\alpha)),\tau^k_\alpha]_g\right) \qquad (3.90b)\\ &+\textstyle\sum_\alpha\left(-(-1)^k[-2\nabla_\Omega(\tau^k_\alpha)+\mathbf{d}^*(\tau^k_\alpha),\tilde\tau^1_\alpha]_g+[-2\nabla_\Omega(\tilde\tau^1_\alpha)+\mathbf{d}^*(\tilde\tau^1_\alpha),\tau^k_\alpha]_g\right). \qquad (3.90c)\end{aligned}$$

The Explicit Form of the Line (3.90c)

$$\begin{aligned}&-(-1)^k\textstyle\sum_\alpha[-2\nabla_\Omega(\tau^k_\alpha)+\mathbf{d}^*(\tau^k_\alpha),\tilde\tau^1_\alpha]_g \qquad (3.91a)\\ &=\textstyle\sum_{j=0}^{k-1}\sum_{\alpha,\beta,\gamma}(-1)^{k-1}[(-2\nabla^S_{c^{k-j}_{\alpha\gamma}}+\mathbf{d}^*c^{k-j}_{\alpha\gamma})\gamma^j\otimes\hat\pi(\hat\omega^j_{\alpha\gamma}+\hat\kappa^{j-2}_{k,\alpha\gamma}),\\ &\qquad\qquad\tilde c^1_{\alpha\beta}\otimes\hat\pi(\tilde a_{\alpha\beta})+\tilde f_{\alpha\beta}\gamma\otimes\hat\pi(\hat{\tilde\omega}^1_{\alpha\beta})]_g\\ &=\textstyle\sum_{j=0}^{k-1}\sum_{\alpha,\beta,\gamma}\big(2\tilde c^1_{\alpha\beta}\wedge\nabla^S_{c^{k-j}_{\alpha\gamma}}\gamma^j\otimes[\hat\pi(\tilde a_{\alpha\beta}),\hat\pi(\hat\omega^j_{\alpha\gamma}+\hat\kappa^{j-2}_{k,\alpha\gamma})]\\ &+2\tilde c^1_{\alpha\beta}\lrcorner\nabla^S_{c^{k-j}_{\alpha\gamma}}\gamma^j\otimes\{\hat\pi(\tilde a_{\alpha\beta}),\hat\pi(\hat\omega^j_{\alpha\gamma}+\hat\kappa^{j-2}_{k,\alpha\gamma})\}\\ &-2(-1)^{k-j}\nabla^S_{\tilde f_{\alpha\beta}c^{k-j}_{\alpha\gamma}}\gamma^{j+1}\otimes[\hat\pi(\hat{\tilde\omega}^1_{\alpha\beta}),\hat\pi(\hat\omega^j_{\alpha\gamma}+\hat\kappa^{j-2}_{k,\alpha\gamma})]_g\\ &+2(-1)^{k-j}\nabla_{c^{k-j}_{\alpha\gamma}}(\tilde c^1_{\alpha\beta})\gamma^j\otimes\hat\pi(\hat\omega^j_{\alpha\gamma}+\hat\kappa^{j-2}_{k,\alpha\gamma})\hat\pi(\tilde a_{\alpha\beta})\\ &+2(-1)^k\nabla_{c^{k-j}_{\alpha\gamma}}(\tilde f_{\alpha\beta})\gamma^{j+1}\otimes\hat\pi(\hat\omega^j_{\alpha\gamma}+\hat\kappa^{j-2}_{k,\alpha\gamma})\hat\pi(\hat{\tilde\omega}^1_{\alpha\beta})\\ &-\tilde c^1_{\alpha\beta}\wedge(\mathbf{d}^*c^{k-j}_{\alpha\gamma})\gamma^j\otimes[\hat\pi(\tilde a_{\alpha\beta}),\hat\pi(\hat\omega^j_{\alpha\gamma}+\hat\kappa^{j-2}_{k,\alpha\gamma})]\\ &-\tilde c^1_{\alpha\beta}\lrcorner(\mathbf{d}^*c^{k-j}_{\alpha\gamma})\gamma^j\otimes\{\hat\pi(\tilde a_{\alpha\beta}),\hat\pi(\hat\omega^j_{\alpha\gamma}+\hat\kappa^{j-2}_{k,\alpha\gamma})\}\\ &+(-1)^{k-j}\tilde f_{\alpha\beta}(\mathbf{d}^*c^{k-j}_{\alpha\gamma})\gamma^{j+1}\otimes[\hat\pi(\hat{\tilde\omega}^1_{\alpha\beta}),\hat\pi(\hat\omega^j_{\alpha\gamma}+\hat\kappa^{j-2}_{k,\alpha\gamma})]_g\big)\,,\end{aligned}$$

$$\begin{aligned}&\textstyle\sum_a[-2\nabla_\Omega(\tilde\tau^1_\alpha)+\mathbf{d}^*(\tilde\tau^k_\alpha),\tau^k_\alpha]_g \qquad (3.91b)\\ &=\textstyle\sum_{j=0}^k\sum_{\alpha,\beta,\gamma}[(-2\nabla^S_{\tilde c^1_{\alpha\beta}}+\mathbf{d}^*c^1_{\alpha\beta})\otimes\hat\pi(\tilde a_{\alpha\beta}),c^{k-j}_{\alpha\gamma}\gamma^j\otimes\hat\pi(\hat\omega^j_{\alpha\gamma}+\hat\kappa^{j-2}_{k,\alpha\gamma})]\\ &=\textstyle\sum_{j=0}^k\sum_{\alpha,\beta,\gamma}\big(-2c^{k-j}_{\alpha\gamma}\nabla^S_{\tilde c^1_{\alpha\beta}}\gamma^j\otimes[\hat\pi(\tilde a_{\alpha\beta}),\hat\pi(\hat\omega^j_{\alpha\gamma}+\hat\kappa^{j-2}_{k,\alpha\gamma})]\\ &-2\nabla_{\tilde c^1_{\alpha\beta}}(c^{k-j}_{\alpha\gamma})\gamma^j\otimes\hat\pi(\tilde a_{\alpha\beta})\hat\pi(\hat\omega_{\alpha\gamma}+\hat\kappa^{j-2}_{k,\alpha\gamma})\\ &+(\mathbf{d}^*\tilde c^1_{\alpha\beta})c^{k-j}_{\alpha\gamma}\gamma^j\otimes[\hat\pi(\tilde a_{\alpha\beta}),\hat\pi(\hat\omega_{\alpha\gamma}+\hat\kappa^{j-2}_{k,\alpha\gamma})]\big)\,.\end{aligned}$$

Collection of the Results Just Obtained

First, from (3.37) we get

$$
\begin{aligned}
&\textstyle\sum_{\alpha,\beta,\gamma}\big(\sum_{j=0}^{k-1}(2\tilde{c}^1_{\alpha\beta}\wedge\nabla^S_{c^{k-j}_{\alpha\gamma}}\gamma^j\otimes[\hat{\pi}(\tilde{a}_{\alpha\beta}),\hat{\pi}(\hat{\omega}^j_{\alpha\gamma}+\hat{\kappa}^{j-2}_{k,\alpha\gamma})]\\
&+2\tilde{c}^1_{\alpha\beta}\,\lrcorner\,\nabla^S_{c^{k-j}_{\alpha\gamma}}\gamma^j\otimes\{\hat{\pi}(\tilde{a}_{\alpha\beta}),\hat{\pi}(\hat{\omega}_{\alpha\gamma}+\hat{\kappa}^{j-2}_{k,\alpha\gamma})\}\\
&-2(-1)^{k-j}\nabla^S_{\tilde{f}_{\alpha\beta}c^{k-j}_{\alpha\gamma}}\gamma^{j+1}\otimes[\hat{\pi}(\hat{\tilde{\omega}}^1_{\alpha\beta}),\hat{\pi}(\hat{\omega}^j_{\alpha\gamma}+\hat{\kappa}^{j-2}_{k,\alpha\gamma})]_g)\\
&+\textstyle\sum_{j=0}^{k}(-2c^{k-j}_{\alpha\gamma}\nabla^S_{\tilde{c}^1_{\alpha\beta}}\gamma^j\otimes[\hat{\pi}(\tilde{a}_{\alpha\beta}),\hat{\pi}(\hat{\omega}^j_{\alpha\gamma}+\hat{\kappa}^{j-2}_{k,\alpha\gamma})])\big)\\
&=-2\nabla_\Omega(\textstyle\sum_\alpha[\tilde{\tau}^1_\alpha,\tau^k_\alpha]_g)\equiv-2\nabla_\Omega(\tau^{k+1})\,.
\end{aligned}
$$

Second, from (3.16a) we get for $c^{k-j}_{\alpha\gamma}\equiv c^1_{1,\alpha\gamma}\wedge c^1_{2,\alpha\gamma}\wedge\ldots\wedge c^1_{k-j,\alpha\gamma}$ the identity

$$
\begin{aligned}
&\textstyle\sum_{\alpha,\beta,\gamma}\big(\sum_{j=0}^{k-1}(-\tilde{c}^1_{\alpha\beta}\wedge(\mathbf{d}^*c^{k-j}_{\alpha\gamma})\gamma^j\otimes[\hat{\pi}(\tilde{a}_{\alpha\beta}),\hat{\pi}(\hat{\omega}^j_{\alpha\gamma}+\hat{\kappa}^{j-2}_{k,\alpha\gamma})]\\
&-\tilde{c}^1_{\alpha\beta}\,\lrcorner\,(\mathbf{d}^*c^{k-j}_{\alpha\gamma})\gamma^j\otimes\{\hat{\pi}(\tilde{a}_{\alpha\beta}),\hat{\pi}(\hat{\omega}^j_{\alpha\gamma}+\hat{\kappa}^{j-2}_{k,\alpha\gamma})\}\\
&+(-1)^{k-j}\,\tilde{f}_{\alpha\beta}(\mathbf{d}^*c^{k-j}_{\alpha\gamma})\gamma^{j+1}\otimes[\hat{\pi}(\hat{\tilde{\omega}}^1_{\alpha\beta}),\hat{\pi}(\hat{\omega}^j_{\alpha\gamma}+\hat{\kappa}^{j-2}_{k,\alpha\gamma})]_g)\\
&+\textstyle\sum_{j=0}^{k}(\mathbf{d}^*\tilde{c}^1_{\alpha\beta})c^{k-j}_{\alpha\gamma}\gamma^j\otimes[\hat{\pi}(\tilde{a}_{\alpha\beta}),\hat{\pi}(\hat{\omega}_{\alpha\gamma}+\hat{\kappa}^{j-2}_{k,\alpha\gamma})]\big)\\
&=\mathbf{d}^*(\textstyle\sum_\alpha[\tilde{\tau}^1_\alpha,\tau^k_\alpha]_g)\\
&+\textstyle\sum_{\alpha,\beta,\gamma}\big(\big(\sum_{j=0}^{k-1}(\sum_{n=1}^{k-j}(-1)^n(\nabla_{c^1_{n,\alpha\gamma}}(\tilde{c}^1_{\alpha\beta}))\wedge c^1_{1,\alpha\gamma}\wedge\overset{n}{\overset{\vee}{\ldots}}\wedge c^1_{k-j,\alpha\gamma})+\\
&\qquad\qquad+\textstyle\sum_{j=0}^{k}\nabla_{\tilde{c}^1_{\alpha\beta}}(c^{k-j}_{\alpha\gamma})\big)\gamma^j\otimes[\hat{\pi}(\tilde{a}_{\alpha\beta}),\hat{\pi}(\hat{\omega}^j_{\alpha\gamma}+\hat{\kappa}^{j-2}_{k,\alpha\gamma})]\\
&+\textstyle\sum_{j=0}^{k-1}(-1)^{k-j}\nabla_{c^{k-j}_{\alpha\gamma}}(\tilde{f}_{\alpha\beta})\gamma^{j+1}\otimes[\hat{\pi}(\hat{\tilde{\omega}}^1_{\alpha\beta}),\hat{\pi}(\hat{\omega}^j_{\alpha\gamma}+\hat{\kappa}^{j-2}_{k,\alpha\gamma})]_g\\
&+(\mathbf{d}\tilde{c}^1_{\alpha\beta})\,\lrcorner\,c^{k-j}_{\alpha\gamma}\otimes\{\hat{\pi}(\tilde{a}_{\alpha\beta}),\hat{\pi}(\hat{\omega}^j_{\alpha\gamma}+\hat{\kappa}^{j-2}_{k,\alpha\gamma})\}\big)\,.
\end{aligned}
$$

Third, using Lemma 13 we have

$$
\begin{aligned}
(-1)^{k-j}\nabla_{c^{k-j}_{\alpha\gamma}}(\tilde{c}^1_{\alpha\beta})&\equiv(-1)^{k-j}\textstyle\sum_{n=1}^{k-j}(-1)^{n+1}(c^1_{1,\alpha\gamma}\wedge\overset{n}{\overset{\vee}{\ldots}}\wedge c^1_{k-j,\alpha\gamma})\nabla_{c^1_{n,\alpha\gamma}}(\tilde{c}^1_{\alpha\beta})\\
&=\textstyle\sum_{n=1}^{k-j}(-1)^n\nabla_{c^1_{n,\alpha\gamma}}(\tilde{c}^1_{\alpha\beta})\wedge(c^1_{1,\alpha\gamma}\wedge\overset{n}{\overset{\vee}{\ldots}}\wedge c^1_{k-j,\alpha\gamma})\\
&-\textstyle\sum_{n=1}^{k-j}(-1)^n\nabla_{c^1_{n,\alpha\gamma}}(\tilde{c}^1_{\alpha\beta})\,\lrcorner\,(c^1_{1,\alpha\gamma}\wedge\overset{n}{\overset{\vee}{\ldots}}\wedge c^1_{k-j,\alpha\gamma})\\
&=\textstyle\sum_{n=1}^{k-j}(-1)^n\nabla_{c^1_{n,\alpha\gamma}}(\tilde{c}^1_{\alpha\beta})\wedge(c^1_{1,\alpha\gamma}\wedge\overset{n}{\overset{\vee}{\ldots}}\wedge c^1_{k-j,\alpha\gamma})-(\mathbf{d}\tilde{c}^1_{\alpha\beta})\,\lrcorner\,c^{k-j}_{\alpha\gamma}\,.
\end{aligned}
$$

Collecting the results we obtain

$$
\begin{aligned}
&\textstyle\sum_a([-2\nabla_\Omega(\tilde{\tau}^1_\alpha)+\mathbf{d}^*(\tilde{\tau}^k_\alpha),\tau^k_\alpha]-(-1)^k[-2\nabla_\Omega(\tau^k_\alpha)+\mathbf{d}^*(\tau^k_\alpha),\tilde{\tau}^1_\alpha])\\
&\quad=\mathbf{d}^*(\textstyle\sum_\alpha[\tilde{\tau}^1_\alpha,\tau^k_\alpha]_g)-2\nabla_\Omega(\sum_\alpha[\tilde{\tau}^1_\alpha,\tau^k_\alpha]_g) \qquad (3.92a)\\
&\quad+\textstyle\sum_{\alpha,\beta,\gamma}\big(-\nabla_{\tilde{c}^1_{\alpha\beta}}(c^0_{\alpha\gamma})\gamma^k\otimes\{\hat{\pi}(\tilde{a}_{\alpha\beta}),\hat{\pi}(\hat{\omega}^k_{\alpha\gamma}+\hat{\kappa}^{k-2}_{k,\alpha\gamma})\} \qquad (3.92b)
\end{aligned}
$$

$$\left.\begin{aligned} &+\textstyle\sum_{j=0}^{k-1}\sum_{n=1}^{k-j}(-1)^n(\nabla_{c^1_{n,\alpha\gamma}}(\tilde{c}^1_{\alpha\beta})+\nabla_{\tilde{c}^1_{\alpha\beta}}(c^1_{n,\alpha\gamma}))\wedge c^1_{1,\alpha\gamma}\wedge\overset{n}{\overset{\vee}{\ldots}}\wedge c^1_{k-j,\alpha\gamma}\gamma^j\otimes \\ &\qquad\qquad\qquad\otimes\{\hat{\pi}(\tilde{a}_{\alpha\beta}),\hat{\pi}(\hat{\omega}^j_{\alpha\gamma}+\hat{\kappa}^{j-2}_{k,\alpha\gamma})\}\end{aligned}\right\} \tag{3.92c}$$

$$+\textstyle\sum_{j=0}^{k-1}(-1)^{k-j}\nabla_{c^{k-j}_{\alpha\gamma}}(\tilde{f}_{\alpha\beta})\gamma^{j+1}\otimes\{\hat{\pi}(\hat{\tilde{\omega}}^1_{\alpha\beta}),\hat{\pi}(\hat{\omega}^j_{\alpha\gamma}+\hat{\kappa}^{j-2}_{k,\alpha\gamma})\}_g \tag{3.92d}$$

$$+\textstyle\sum_{j=0}^{k-2}(\mathbf{d}\tilde{c}^1_{\alpha\beta})\,\lrcorner\,(c^1_{1,\alpha\gamma}\wedge\ldots\wedge c^1_{k-j,\alpha\gamma})\gamma^j\otimes[\hat{\pi}(\tilde{a}_{\alpha\beta}),\hat{\pi}(\hat{\omega}^j_{\alpha\gamma}+\hat{\kappa}^{j-2}_{k,\alpha\gamma})]\big)\ . \tag{3.92e}$$

The Analysis of the Lines (3.92b) **and** (3.92c)

Taking $\tilde{\tau}^1_{\alpha A}=\mathbf{d}\tilde{\mathrm{f}}_{\alpha A}\otimes\hat{\pi}(\tilde{a}_\alpha)$ and $\tau^k_{\alpha A}=\mathrm{f}_{\alpha A}\gamma^k\otimes\hat{\pi}(\hat{\omega}^k_\alpha+\hat{\kappa}^{k-2}_{k,\alpha})$, we see that (3.92b) attains each element of

$$\Lambda^0\gamma^k\otimes\{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\Omega^k\mathfrak{a})+\tilde{K}^{k-2}_k\mathfrak{a}\} \tag{3.93}$$

independently of $\sum_\alpha\sum_A[\tilde{\tau}^1_{\alpha A},\tau^k_{\alpha A}]_g$. Next, putting

$$\tilde{\tau}^1_{\alpha A}=\mathbf{d}\tilde{\mathrm{f}}_{\alpha A}\otimes\hat{\pi}(\tilde{a}_\alpha)\ ,\qquad\qquad \tau^k_{\alpha A}=\mathrm{f}_{\alpha A}c^{k-j}_\alpha\gamma^j\otimes\hat{\pi}(\hat{\omega}^j_\alpha+\hat{\kappa}^{j-2}_{k,\alpha})\ ,$$

we get for (3.92c) after summing over A

$$\begin{aligned}&\textstyle\sum_\alpha\big(-\tilde{f}_\alpha\nabla_{\mathbf{d}f_\alpha}(f_\alpha)c^{k-j}_\alpha-\sum_{n=1}^{k-j}(-1)^n\tilde{f}_\alpha(\nabla_{c^1_{n,\alpha}}(f_\alpha))\mathbf{d}(f_\alpha)\wedge c^1_{1,\alpha}\wedge\overset{n}{\overset{\vee}{\ldots}}\wedge c^1_{k-j,\alpha}\big)\gamma^j\otimes\\ &\qquad\qquad\qquad\qquad\otimes\{\hat{\pi}(\tilde{a}_\alpha),\hat{\pi}(\hat{\omega}^j_\alpha+\hat{\kappa}^{j-2}_{k,\alpha})\}\ .\end{aligned}$$

For given f_α we decompose all $c^1_{n,\alpha}$ pointwise into its components parallel and perpendicular to $\mathbf{d}f_\alpha$. It is obvious that none of the components parallel to $\mathbf{d}f_\alpha$ gives a non–vanishing contribution to (3.92c). On the other hand,

$$-\tilde{f}_\alpha\nabla_{\mathbf{d}f_\alpha}(f_\alpha)c^{k-j}_\alpha-\textstyle\sum_{n=1}^{k-j}(-1)^n\tilde{f}_\alpha(\nabla_{c^1_{n,\alpha}}(f_\alpha))\mathbf{d}(f_\alpha)\wedge c^1_{1,\alpha}\wedge\overset{n}{\overset{\vee}{\ldots}}\wedge c^1_{k-j,\alpha} \tag{3.94}$$

attains each element of the space $\underbrace{\Lambda^1_{\mathbf{d}f^\perp_\alpha}\wedge\ldots\wedge\Lambda^1_{\mathbf{d}f^\perp_\alpha}}_{k-j}$, where $\Lambda^1_{\mathbf{d}f^\perp_\alpha}\subset\Lambda^1$ is the distribution perpendicular to $\mathbf{d}f_\alpha$. If we vary f_α then (3.94) attains for $k{-}j\le N{-}1$ each element of Λ^{k-j} and, therefore, (3.92c) attains each element of

$$\textstyle\sum_{j=0}^{k-1}\Lambda^{k-j}\gamma^j\otimes\{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\Omega^j\mathfrak{a})+\hat{\pi}(T^{j-2}_k\mathfrak{a})\}\ ,\quad k{-}j<N\ ,\ \ \hat{\pi}(\Omega^0\mathfrak{a})\equiv\hat{\pi}(\mathfrak{a}')\ , \tag{3.95}$$

independently of $\sum_\alpha\sum_A[\tilde{\tau}^1_{\alpha A},\tau^k_{\alpha A}]_g$. For $k-j=N$ there is no contribution to (3.95), because Λ^N is not completely orthogonal to $\mathbf{d}f_\alpha$. Therefore, in this case a different analysis is necessary. Using (3.16b) we find for $k{-}j=N$

$$\begin{aligned}&\textstyle\sum_{n=1}^{N}(-1)^n(\nabla_{c^1_{n,\alpha\gamma}}(\tilde{c}^1_{\alpha\beta})+\nabla_{\tilde{c}^1_{\alpha\beta}}(c^1_{n,\alpha\gamma}))\wedge c^1_{1,\alpha\gamma}\wedge\overset{n}{\overset{\vee}{\ldots}}\wedge c^1_{N,\alpha\gamma}\\ &=\textstyle\sum_{n=1}^{N}(-1)^n\big(\mathbf{d}(c^1_{n,\alpha\gamma}\,\lrcorner\,\tilde{c}^1_{\alpha\beta})+c^1_{n,\alpha\gamma}\,\lrcorner\,\mathbf{d}(\tilde{c}^1_{\alpha\beta})+\tilde{c}^1_{\alpha\beta}\,\lrcorner\,\mathbf{d}(c^1_{n,\alpha\gamma})\big)\wedge c^1_{1,\alpha\gamma}\wedge\overset{n}{\overset{\vee}{\ldots}}\wedge c^1_{N,\alpha\gamma}\\ &=-\mathbf{d}(\tilde{c}^1_{\alpha\beta}\,\lrcorner\,(c^1_{1,\alpha\gamma}\wedge\ldots\wedge c^1_{N,\alpha\gamma}))-\textstyle\sum_{n=1}^{N}(-1)^n(c^1_{n,\alpha\gamma}\,\lrcorner\,\tilde{c}^1_{\alpha\beta})\mathbf{d}(c^1_{1,\alpha\gamma}\wedge\overset{n}{\overset{\vee}{\ldots}}\wedge c^1_{N,\alpha\gamma})\\ &+\textstyle\sum_{n=1}^{N}(-1)^n(c^1_{n,\alpha\gamma}\,\lrcorner\,\mathbf{d}(\tilde{c}^1_{\alpha\beta})+\tilde{c}^1_{\alpha\beta}\,\lrcorner\,\mathbf{d}(c^1_{n,\alpha\gamma}))\wedge c^1_{1,\alpha\gamma}\wedge\overset{n}{\overset{\vee}{\ldots}}\wedge c^1_{N,\alpha\gamma}\ .\end{aligned}$$

Using orthogonal bases or a decomposition of all $c^1_{n,\alpha\gamma}$ into its components parallel and perpendicular to $\tilde{c}^1_{\alpha\beta}$ it is straightforward to check that

$$
\begin{aligned}
&(c^1_{n,\alpha\gamma} \lrcorner\, \mathbf{d}(\tilde{c}^1_{\alpha\beta})) \wedge c^1_{1,\alpha\gamma} \wedge \overset{n}{\overset{\vee}{\cdots}} \wedge c^1_{N,\alpha\gamma} = 0\;, \\
&\textstyle\sum_{n=1}^N (-1)^n \big((\tilde{c}^1_{\alpha\beta} \lrcorner\, c^1_{n,\alpha\gamma}) \mathbf{d}(c^1_{1,\alpha\gamma} \wedge \overset{n}{\overset{\vee}{\cdots}} \wedge c^1_{N,\alpha\gamma}) \\
&\qquad\qquad - (\tilde{c}^1_{\alpha\beta} \lrcorner\, \mathbf{d}(c^1_{n,\alpha\gamma})) \wedge c^1_{1,\alpha\gamma} \wedge \overset{n}{\overset{\vee}{\cdots}} \wedge c^1_{N,\alpha\gamma} \big) = 0\;.
\end{aligned}
$$

Therefore, the $k-j=N$–component of (3.92c) takes the form

$$
\begin{aligned}
(3.92\text{c}) \restriction_{k=N+j} &= -\textstyle\sum_{\alpha,\beta,\gamma} \mathbf{d}(\tilde{c}^1_{\alpha\beta} \lrcorner\, c^N_{\alpha\gamma}) \gamma^j \otimes \{\hat{\pi}(\tilde{a}_{\alpha\beta}), \hat{\pi}(\hat{\omega}^j_{\alpha\gamma} + \hat{\kappa}^{j-2}_{k,\alpha\gamma})\} \\
&= -\mathbf{d}\big(\tau^{N+j+1} \restriction_{\Lambda^{N-1}\gamma^j \otimes \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\Omega^j \mathfrak{a}) + \hat{\pi}(T^{j-2}_{N+j}\mathfrak{a})\}} \big)\;,
\end{aligned}
\tag{3.96}
$$

which gives the second line in (3.88).

Discussion of the Lines (3.90a) **and** (3.90b)

Before discussing the lines (3.92d) and (3.92e) we remark that the lines (3.90a) yield

$$
\begin{aligned}
&\textstyle\sum_{j=2}^{k+1} \Lambda^{k+2-j}\gamma^j \otimes [\hat{\pi}(\mathfrak{a}), \tilde{K}^{j-2}_{k+1}\mathfrak{a} + \hat{\pi}(\jmath^{\,j}\mathfrak{a})] \\
&+ \textstyle\sum_{j=2}^{k} \Lambda^{k-j}\gamma^j \otimes \{\hat{\pi}(\mathfrak{a}), \tilde{K}^{j-2}_{k+1}\mathfrak{a} + \hat{\pi}(\jmath^{\,j}\mathfrak{a})\}\Theta(N-(k-j)) \\
&+ \textstyle\sum_{j=2}^{k+1} \Lambda^{k-j+1}\gamma^{j+1} \otimes [\hat{\pi}(\Omega^1\mathfrak{a}), \tilde{K}^{j-2}_{k+1}\mathfrak{a} + \hat{\pi}(\jmath^{\,j}\mathfrak{a})]_g + \Lambda^k \otimes [\hat{\pi}(\mathfrak{a}'), \hat{\pi}(T^0_2\mathfrak{a}) + \hat{\pi}(\jmath^{\,2}\mathfrak{a})] \\
&+ \textstyle\sum_{j=1}^{k} \Lambda^{k-j}\gamma^j \otimes [\hat{\pi}(T^{j-2}_k\mathfrak{a}) + \hat{\pi}(\Omega^j\mathfrak{a}), \hat{\pi}(T^0_2\mathfrak{a}) + \hat{\pi}(\jmath^{\,2}\mathfrak{a})] \\
&= \Lambda^k \otimes [\hat{\pi}(\mathfrak{a}), \hat{\pi}(T^0_{k+1}\mathfrak{a}) + \hat{\pi}(\jmath^{\,2}\mathfrak{a})] \\
&+ \Lambda^{k-1}\gamma \otimes ([\hat{\pi}(\Omega^1\mathfrak{a}), \hat{\pi}(T^0_2\mathfrak{a}) + \hat{\pi}(\jmath^{\,2}\mathfrak{a})] + [\hat{\pi}(\mathfrak{a}), \tilde{K}^1_{k+1}\mathfrak{a} + \hat{\pi}(\jmath^{\,3}\mathfrak{a})]) \\
&+ \textstyle\sum_{j=2}^{k} \Lambda^{k-j}\gamma^j \otimes \big([\hat{\pi}(\mathfrak{a}), \tilde{K}^j_{k+1}\mathfrak{a} + \hat{\pi}(\jmath^{\,j+2}\mathfrak{a})]\Theta(k-j) \\
&+ \{\hat{\pi}(\mathfrak{a}), \tilde{K}^{j-2}_{k+1}\mathfrak{a} + \hat{\pi}(\jmath^{\,j}\mathfrak{a})\}\Theta(N-(k-j)) + [\hat{\pi}(\Omega^1\mathfrak{a}), \tilde{K}^{j-1}_{k+1}\mathfrak{a} + \hat{\pi}(\jmath^{\,j+1}\mathfrak{a})]_g \\
&+ [\hat{\pi}(T^{j-2}_k\mathfrak{a}) + \hat{\pi}(\Omega^j\mathfrak{a}), \hat{\pi}(T^0_2\mathfrak{a}) + \hat{\pi}(\jmath^{\,2}\mathfrak{a})]_g\big)\;.
\end{aligned}
\tag{3.97}
$$

The term $[\hat{\pi}(\mathfrak{a}), \tilde{K}^j_{k+1}\mathfrak{a} + \hat{\pi}(\jmath^{\,j+2}\mathfrak{a})]$ is not present for $k=j$. For $k-j=N$ the terms containing anticommutators with $\hat{\pi}(\mathfrak{a})$ do not occur. This is indicated by the step function $\Theta(n)=0$ for $n \le 0$ and $\Theta(n)=1$ for $n>0$. Thanks to Lemma 19, the line (3.90b) can be rewritten as

$$
\begin{aligned}
(3.90\text{b}) &= \hat{\sigma}_{\mathfrak{g}}(\pi^{-1}(\tau^{k+1})) + \hat{\sigma}_{\mathfrak{g}}(\ker\pi \cap \Omega^{k+1}\mathfrak{g}) \qquad\qquad (3.98\text{a})\\
&= \hat{\sigma}_{\mathfrak{g}}(\pi^{-1}(\tau^{k+1})) + \textstyle\sum_{j=0}^{k} \Lambda^{k-j}\gamma^j \otimes \big(\hat{\pi}(\jmath^{\,j+2}\mathfrak{a}) + \hat{\sigma}(T^{j-1}_{k+1}\mathfrak{a} \cap \ker\hat{\pi}) \\
&\qquad + \hat{\sigma} \circ \hat{\pi}^{-1}(\hat{\pi}(\Omega^{j+1}\mathfrak{a}) \cap \hat{\pi}(T^{j-1}_{k+1}\mathfrak{a}))\big)\;.
\end{aligned}
$$

Due to Lemma 21 and Lemma 20 we have

$$
\hat{\sigma}(T^{j-1}_{k+1} \cap \ker\hat{\pi}) \subset \hat{\pi}(T^j_{k+2}\mathfrak{a})\;.
\tag{3.98b}
$$

The Coefficient of Λ^k

As the next step we analyse the coefficients of Λ^n in the sum of (3.93), (3.95), (3.97) and (3.98a). First, we provide a useful property:

$$\begin{aligned}&[\hat{\pi}(\mathfrak{a}),\, \hat{\sigma}\circ\hat{\pi}^{-1}(\hat{\pi}(\Omega^j\mathfrak{a})\cap\hat{\pi}(T_k^{j-2}\mathfrak{a}))] \\ &\qquad\equiv \hat{\sigma}\circ\hat{\pi}^{-1}([\hat{\pi}(\mathfrak{a}),\hat{\pi}(\Omega^j\mathfrak{a})\cap\hat{\pi}(T_k^{j-2}\mathfrak{a})]) \subset \hat{\sigma}\circ\hat{\pi}^{-1}(\hat{\pi}(\Omega^j\mathfrak{a})\cap\hat{\pi}(T_{k+1}^{j-2}\mathfrak{a}))\,,\end{aligned} \tag{3.99}$$

for $k<N+j$. For the coefficient of Λ^k we get for $k\le N$

$$\{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\mathfrak{a}')\}\Theta(N-k)+[\hat{\pi}(\mathfrak{a}),\hat{\pi}(T_{k+1}^0\mathfrak{a})+\hat{\pi}(\jmath^2\mathfrak{a})]+\hat{\pi}(\jmath^2\mathfrak{a})\equiv\hat{\pi}(\jmath^2\mathfrak{a})+\tilde{K}_{k+2}^0\mathfrak{a}\,. \tag{3.100a}$$

This equation should be considered as the definition of $\tilde{K}_{k+2}^0\mathfrak{a}$. For $k<N$ all terms in (3.100a) occur and we find $\tilde{K}_{k+2}^0\mathfrak{a}=\hat{\pi}(T_3^0\mathfrak{a})=\{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\mathfrak{a}')\}$ independently of k. For $k=N$ the first term on the l.h.s. of (3.100a) does not occur, therefore, $\tilde{K}_{N+2}^0\mathfrak{a}=\hat{\pi}(T_{N+2}^0\mathfrak{a})$.

The Coefficient of $\Lambda^{k-1}\gamma$

For the coefficient of $\Lambda^{k-1}\gamma$ we get

$$\begin{aligned}&\{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\Omega^1\mathfrak{a})\}\Theta(N-(k-1))+[\hat{\pi}(\Omega^1\mathfrak{a}),\hat{\pi}(T_2^0\mathfrak{a})+\hat{\pi}(\jmath^2\mathfrak{a})]\\ &+[\hat{\pi}(\mathfrak{a}),\tilde{K}_{k+1}^1\mathfrak{a}+\hat{\pi}(\jmath^3\mathfrak{a})]\Theta(k-1)+\hat{\pi}(\jmath^3\mathfrak{a})+\hat{\sigma}\circ\hat{\pi}^{-1}(\hat{\pi}(\Omega^2\mathfrak{a})\cap\hat{\pi}(T_{k+1}^0\mathfrak{a}))\\ &\equiv\hat{\pi}(\jmath^3\mathfrak{a})+\tilde{K}_{k+2}^1\mathfrak{a}\,.\end{aligned} \tag{3.100b}$$

Again, this is the definition of $\tilde{K}_{k+2}^1\mathfrak{a}$. For $0<k-1<N$ all terms occur, however, for the time being we exclude the term $[\hat{\pi}(\mathfrak{a}),\tilde{K}_{k+1}^1\mathfrak{a}]$, which does not occur for $k=1$. This gives

$$\tilde{K}_{k+2}^1\mathfrak{a}=\tilde{K}_3^1\mathfrak{a}=\{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\Omega^1\mathfrak{a})\}+[\hat{\pi}(\Omega^1\mathfrak{a}),\hat{\pi}(T_2^0\mathfrak{a})]+\hat{\sigma}\circ\hat{\pi}^{-1}(\hat{\pi}(\Omega^2\mathfrak{a})\cap\hat{\pi}(T_3^0\mathfrak{a}))\,,$$

which, in particular, is correct for $k=1$. Now, using (3.99) we find by induction that $[\hat{\pi}(\mathfrak{a}),\tilde{K}_{k+1}^1\mathfrak{a}]\subset\tilde{K}_{k+2}^1\mathfrak{a}$, therefore, our formula for $\tilde{K}_{k+2}^1\mathfrak{a}$ remains true for $1<k\le N$ if the term $[\hat{\pi}(\mathfrak{a}),\tilde{K}_{k+1}^1\mathfrak{a}]$ is included. For $k=N+1$ the first term on the l.h.s. of (3.100b) does not occur, therefore,

$$\tilde{K}_{N+3}^1\mathfrak{a}=[\hat{\pi}(\Omega^1\mathfrak{a}),\hat{\pi}(T_2^0\mathfrak{a})]+[\hat{\pi}(\mathfrak{a}),\tilde{K}_3^1\mathfrak{a}]+\hat{\sigma}\circ\hat{\pi}^{-1}(\hat{\pi}(\Omega^2\mathfrak{a})\cap\hat{\pi}(T_{N+2}^0\mathfrak{a}))\,.$$

The Coefficient of $\Lambda^{k-j}\gamma^j$

For the coefficient of $\Lambda^{k-j}\gamma^j$, $j\ge 2$, we gain

$$\begin{aligned}&\{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\Omega^j\mathfrak{a})+\hat{\pi}(T_{k+1}^{j-2}\mathfrak{a})+\tilde{K}_{k+1}^{j-2}\mathfrak{a}+\hat{\pi}(\jmath^j\mathfrak{a})\}\Theta(N-(k-j))\\ &+[\hat{\pi}(\mathfrak{a}),\tilde{K}_{k+1}^j\mathfrak{a}+\hat{\pi}(\jmath^{j+2}\mathfrak{a})]\Theta(k-j)+[\hat{\pi}(\Omega^1\mathfrak{a}),\tilde{K}_{k+1}^{j-1}\mathfrak{a}+\hat{\pi}(\jmath^{j+1}\mathfrak{a})]_g\\ &+[\hat{\pi}(T_k^{j-2}\mathfrak{a})+\hat{\pi}(\Omega^j\mathfrak{a}),\hat{\pi}(T_2^0\mathfrak{a})+\hat{\pi}(\jmath^2\mathfrak{a})]_g+\hat{\pi}(\jmath^{j+2}\mathfrak{a})\\ &+\hat{\sigma}(T_{k+1}^{j-1}\mathfrak{a}\cap\ker\hat{\pi})+\hat{\sigma}\circ\hat{\pi}^{-1}(\hat{\pi}(\Omega^{j+1}\mathfrak{a})\cap\hat{\pi}(T_{k+1}^{j-1}\mathfrak{a}))\\ &\equiv\hat{\pi}(\jmath^{j+2}\mathfrak{a})+\tilde{K}_{k+2}^j\mathfrak{a}\,.\end{aligned} \tag{3.100c}$$

By induction we get $\hat\pi(T_k^j\mathfrak{a}) \subset \tilde K_k^j\mathfrak{a}$. For $0<k-j<N$ all terms occur, but for the moment we exclude the term $[\hat\pi(\mathfrak{a}), \tilde K_{k+1}^j\mathfrak{a}]$, which is not present for $k=j$. Using (3.98b) and

$$[\hat\pi(T_k^{j-2}\mathfrak{a}), \hat\pi(\jmath^2\mathfrak{a})]_g \subset [\hat\pi(T_k^{j-2}\mathfrak{a}), \hat\pi(\Omega^2\mathfrak{a})]_g \subset [\hat\pi(\Omega^1\mathfrak{a}), [\hat\pi(\Omega^1\mathfrak{a}), \hat\pi(T_k^{j-2}\mathfrak{a})]_g]_g$$

we find the formula

$$\begin{aligned}\tilde K_{k+2}^j\mathfrak{a} &= \{\hat\pi(\mathfrak{a}), \hat\pi(\Omega^j\mathfrak{a}) + \tilde K_{k+1}^{j-2}\mathfrak{a}\} + [\hat\pi(\Omega^1\mathfrak{a}), \tilde K_{k+1}^{j-1}\mathfrak{a}]_g \\ &\quad + \hat\sigma\circ\hat\pi^{-1}(\hat\pi(\Omega^{j+1}\mathfrak{a})\cap\hat\pi(T_{j+1}^{j-1}\mathfrak{a}))\,,\end{aligned}$$

which, in particular, is correct for $k=j$. Now, using (3.99) we find by induction that $[\hat\pi(\mathfrak{a}), \tilde K_{k+1}^j\mathfrak{a}] \subset \tilde K_{k+2}^j\mathfrak{a}$, therefore, our formula for $\tilde K_{k+2}^j\mathfrak{a}$ remains true for $0<k-j<N$ if the term $[\hat\pi(\mathfrak{a}), \tilde K_{k+1}^j\mathfrak{a}]$ is included. Note that $\tilde K_{k+2}^j\mathfrak{a} \subset \tilde K_{k+1}^j\mathfrak{a}$, but in general we do not have $\tilde K_{k+2}^j\mathfrak{a} = \tilde K_{k+1}^j\mathfrak{a}$. For $k-j=N$ the terms on the l.h.s. of (3.100c) containing anticommutators with $\hat\pi(\mathfrak{a})$ do not occur, therefore, we are left with

$$\begin{aligned}\tilde K_{N+j+2}^j\mathfrak{a} &= [\hat\pi(\mathfrak{a}), \tilde K_{N+j+1}^j\mathfrak{a}] + [\hat\pi(\Omega^1\mathfrak{a}), \tilde K_{N+j+1}^{j-1}\mathfrak{a}]_g \\ &\quad + \hat\sigma\circ\hat\pi^{-1}(\hat\pi(\Omega^{j+1}\mathfrak{a})\cap\hat\pi(T_{N+j+1}^{j-1}\mathfrak{a}))\,.\end{aligned}$$

Here, one has to use the identity

$$[A,\{a,\tilde a\}] = [\{A,a\},\tilde a] + [\{A,\tilde a\},a]\,,$$

giving $[\hat\pi(T_{N+j}^{j-2}\mathfrak{a}) + \hat\pi(\Omega^j\mathfrak{a}), \hat\pi(T_2^0\mathfrak{a})]_g \subset [\hat\pi(\mathfrak{a}), \hat\pi(T_{j+2}^j\mathfrak{a})]$.

Therefore, from (3.92a), (3.98), (3.96) and the structure of the spaces $\tilde K_k^j\mathfrak{a}$ just obtained we find precisely the assertions (3.88) and (3.67) of Lemma 23. It remains to show that the lines (3.92d) and (3.92e) give no additional contribution and that the terms containing the total differentials in (3.87) and (3.88), which we have neglected so far, do not propagate into higher degrees.

The Discussion of the Lines (3.92d) and (3.92e)

Due to (3.62a), with da replaced by $\hat{\tilde\omega}^1_{\alpha\beta}$, we have for $k-j\le N$

$$(3.92\text{d}) \in \textstyle\sum_{j=0}^{k-1}\Lambda^{k-j-1}\gamma^{j+1}\otimes\hat\pi(T_{k+2}^{j+1}\mathfrak{a})\,.$$

For $k-j>N$ the line (3.92d) is identical zero. In (3.92e) only the $\mathfrak{a}'$–part of $\tilde a_{\alpha\beta}$ survives, because the coefficient $\mathbf{d}\tilde c^1_{\alpha\beta}$ belonging to its $\mathfrak{a}''$–part is zero. Hence, we have $\tilde a_{\alpha\beta} = \sum_\delta[a_{\alpha\beta\delta}, \tilde a_{\alpha\beta\delta}]$, giving

$$\begin{aligned}[\hat\pi(\tilde a_{\alpha\beta}), \hat\pi(\hat\omega^j_{\alpha\gamma} + \hat\kappa^{j-2}_{k,\alpha\gamma})] &= \textstyle\sum_\delta \big(\{\hat\pi(a_{\alpha\beta\delta}), \{\hat\pi(\tilde a_{\alpha\beta\delta}), \hat\pi(\hat\omega^j_{\alpha\gamma} + \hat\kappa^{j-2}_{k,\alpha\gamma})\}\} \\ &\quad - \{\hat\pi(\tilde a_{\alpha\beta\delta}), \{\hat\pi(a_{\alpha\beta\delta}), \hat\pi(\hat\omega^j_{\alpha\gamma} + \hat\kappa^{j-2}_{k,\alpha\gamma})\}\}\big)\,,\end{aligned}$$

which belongs to $\hat\pi(T_{k+2}^{j+2}\mathfrak{a})$. Therefore, we have for $k-j\le N$

$$(3.92\text{e}) \in \textstyle\sum_{j=0}^{k-2}\Lambda^{k-j-2}\gamma^j\otimes\hat\pi(T_{k+2}^{j+2}\mathfrak{a})\,.$$

For $k-j>N$ the line (3.92e) is identical zero. Thus, there is no additional contribution from (3.92d) and (3.92e).

The Propagation of the Total Differentials

Finally, we analyse the propagation of the terms in (3.87) and (3.88) containing total differentials into higher degrees. We begin with the step from $n=2$ to $n=3$ in Lemma 23, where we have the additional term

$$-\mathbf{d}\big(\tau^2 \restriction_{\Lambda^0\otimes\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\}}\big) \ni -\mathbf{d} f_\alpha \otimes \{\hat\pi(\mathrm{b}),\hat\pi(\mathrm{b})\}\ .$$

The relevant contribution from this term to $\{\sigma(\omega^3)\}$ is in the notations of (3.89)

$$\begin{aligned}
&-[-\,\mathbf{d}(\tau^2_\alpha \restriction_{\Lambda^0\otimes\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\}}),\tilde\tau^1_\alpha]_g\\
&\quad=\textstyle\sum_{\alpha,\beta}[\mathbf{d} f_\alpha\otimes\{\hat\pi(\mathrm{b}),\hat\pi(\mathrm{b})\},\tilde c^1_{\alpha\beta}\otimes\hat\pi(\tilde a_{\alpha\beta})+\tilde f_{\alpha\beta}\gamma\otimes\hat\pi(\hat{\tilde\omega}^1_{\alpha\beta})]\\
&\quad=\textstyle\sum_{\alpha,\beta}\big(\mathbf{d}(f_\alpha)\,\lrcorner\,\tilde c^1_{\alpha\beta}\otimes\{\{\hat\pi(\mathrm{b}),\hat\pi(\mathrm{b})\},\hat\pi(\tilde a_{\alpha\beta})\}\\
&\qquad+\tilde f_{\alpha\beta}\mathbf{d}(f_\alpha)\gamma\otimes[\{\hat\pi(\mathrm{b}),\hat\pi(\mathrm{b})\},\hat\pi(\hat{\tilde\omega}^1_{\alpha\beta})]\big)\\
&\quad=\textstyle\sum_{\alpha,\beta}\big(\mathbf{d}(f_\alpha)\,\lrcorner\,\tilde c^1_{\alpha\beta}\otimes\{\hat\pi(\mathrm{b}),\{\hat\pi(\tilde a_{\alpha\beta}),\hat\pi(\mathrm{b})\}\}\\
&\qquad+2\tilde f_{\alpha\beta}\mathbf{d}(f_\alpha)\gamma\otimes\{\hat\pi(\mathrm{b}),[\hat\pi(\mathrm{b}),\hat\pi(\hat{\tilde\omega}^1_{\alpha\beta})]\}\big)\\
&\quad\subset\Lambda^0\otimes\hat\pi(T^2_4\mathfrak{a})+\Lambda^1\gamma\otimes\hat\pi(T^1_4\mathfrak{a})\ .
\end{aligned}\tag{3.101}$$

Therefore, there is no additional contribution of (3.101) to $\{\sigma(\omega^3)\}$, which means that it was correct to take (3.87) as the starting point for the induction. Next, we examine the propagation of the term

$$-\mathbf{d}\big(\tau^k \restriction_{\Lambda^{N-1}\gamma^{k+1}\otimes\{\hat\pi(\mathfrak{a}),\hat\pi(\Omega^{k-1-N}\mathfrak{a})+\hat\pi(T^{k-3-N}_{k-1}\mathfrak{a})\}}\big)\ni-\mathbf{d}c^{N-1}_{\alpha\gamma}\gamma^{k+1}\otimes\hat\pi(\hat\kappa^{k-1-N}_{k+1-N,\alpha\gamma})$$

into degree $k{+}1$, where $\hat\pi(\kappa^{k-1-N}_{k+1-N,\alpha\gamma})\equiv\{\hat\pi(a_{\alpha\gamma}),\hat\pi(\hat\omega^{k-1-N}_{\alpha\gamma}+\hat\kappa^{k-3-N}_{k-1,\alpha\gamma})\}\in\hat\pi(T^{k-1-N}_{k+1-N}\mathfrak{a})$. The relevant contribution from this term to $\{\sigma(\omega^{k+1})\}$ is in the notations of (3.89)

$$\begin{aligned}
&(-1)^k[\mathbf{d}\ \big(\tau^k \restriction_{\Lambda^{N-1}\gamma^{k+1}\otimes\{\hat\pi(\mathfrak{a}),\hat\pi(\Omega^{k-1-N}\mathfrak{a})+\hat\pi(T^{k-3-N}_{k-1}\mathfrak{a})\}}\big),\tilde\tau^1_\alpha]_g\\
&\quad=(-1)^k\textstyle\sum_{\alpha,\beta,\gamma}[\mathbf{d}c^{N-1}_{\alpha\gamma}\gamma^{k+1}\otimes\hat\pi(\hat\kappa^{k-1-N}_{k+1-N,\alpha\gamma}),\tilde c^1_{\alpha\beta}\otimes\hat\pi(\tilde a_{\alpha\beta})+\tilde f_{\alpha\beta}\gamma\otimes\hat\pi(\hat{\tilde\omega}^1_{\alpha\beta})]_g\\
&\quad=\textstyle\sum_{\alpha,\beta,\gamma}\tilde c^1_{\alpha\beta}\,\lrcorner\,\mathbf{d}c^{N-1}_{\alpha\gamma}\gamma^{k+1}\otimes\{\hat\pi(\tilde a_{\alpha\beta}),\hat\pi(\hat\kappa^{k-1-N}_{k+1-N,\alpha\gamma})\} &&(3.102\text{a})\\
&\quad+\textstyle\sum_{\alpha,\beta,\gamma}\tilde f_{\alpha\beta}\mathbf{d}c^{N-1}_{\alpha\gamma}\gamma^{k}\otimes[\hat\pi(\hat{\tilde\omega}^1_{\alpha\beta}),\hat\pi(\hat\kappa^{k-1-N}_{k+1-N,\alpha\gamma})]_g\ . &&(3.102\text{b})
\end{aligned}$$

The line (3.102a) belongs to $\Lambda^{N-1}\gamma^{k+1}\otimes\hat\pi(T^{k+1-N}_{k+3-N}\mathfrak{a})$, which is already contained in (3.100). In the line (3.102b) we make use of the possibility to represent $\hat\pi(\hat{\tilde\omega}^1_{\alpha\beta})=\sum_\delta[\hat\pi(a_{\alpha\beta\delta}),\hat\pi(\hat\omega^1_{\alpha\beta\delta})]_g$, with $a_{\alpha\beta\delta}\in\mathfrak{a}$ and $\hat\omega^1_{\alpha\beta\delta}\in\Omega^1\mathfrak{a}$. This gives

$$\begin{aligned}
(3.102\text{b})=\textstyle\sum_{\alpha,\beta,\gamma,\delta}\tilde f_{\alpha\beta}\mathbf{d}c^{N-1}_{\alpha\gamma}\gamma^k\otimes\big(&[\hat\pi(a_{\alpha\beta\delta}),[\hat\pi(\hat\omega^1_{\alpha\beta\delta}),\hat\pi(\hat\kappa^{k-1-N}_{k+1-N,\alpha\gamma})]_g]\\
&-[\hat\pi(\hat\omega^1_{\alpha\beta\delta}),[\hat\pi(a_{\alpha\beta\delta}),\hat\pi(\hat\kappa^{k-1-N}_{k+1-N,\alpha\gamma})]]_g\big)\\
\subset\Lambda^N\gamma^k\otimes([\hat\pi(\mathfrak{a}),\hat\pi(T^{k-N}_{k+2-N}&\mathfrak{a})]+[\hat\pi(\Omega^1\mathfrak{a}),\hat\pi(T^{k-1-N}_{k+1}\mathfrak{a})]_g)\ ,
\end{aligned}$$

which is already contained in (3.100). Thus, Lemma 23 and Theorem 22 are proved. □

3.6 The Structure of $\Omega^*_D\mathfrak{g}$, Commutator and Differential

Now we can harvest the fruits of our efforts. We can immediately or with very little work write down the structure of $\Omega^*_D\mathfrak{g}$ and of the commutator and differential of elements of $\Omega^*_D\mathfrak{g}$.

3.6.1 The Structure of $\Omega^*_D\mathfrak{g}$

As an immediate consequence of Theorem 22 and Proposition 15 we find:

Corollary 24. *If $\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\}\cap\hat\pi(\Omega^2\mathfrak{a})=0$ we have for $n\geq 2$*

$$\begin{aligned}\Omega^n_D\mathfrak{g} = &\big(\Lambda^n\otimes\hat\pi(\mathfrak{a}')\big)\oplus\big(\Lambda^{n-1}\gamma\otimes\hat\pi(\Omega^1\mathfrak{a})\big)\oplus\\ &\oplus\bigoplus_{j=2}^{n}\big(\Lambda^{n-j}\gamma^j\otimes\big((\hat\pi(\Omega^j\mathfrak{a})+\hat\pi(T_n^{j-2}\mathfrak{a}))\mod(\hat\pi(\jmath^j\mathfrak{a})+\tilde K_n^{j-2}\mathfrak{a})\big)\\ &\qquad \mod \delta^j_{n-N}B^N\gamma^n\otimes(\{\hat\pi(\mathfrak{a}),\hat\pi(\Omega^{n-N-2}\mathfrak{a})+\hat\pi(T^{n-N-4}_{n-1}\mathfrak{a})\}\cap\hat\pi(\Omega^{n-N}\mathfrak{a}))\big)\;.\end{aligned}\tag{3.103}$$

If $\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\}\cap\hat\pi(\Omega^2\mathfrak{a})\neq 0$ then $\Omega^3_D\mathfrak{g}$ must be replaced by

$$\Omega^3_D\mathfrak{g}=\Omega^3_D\mathfrak{g}\restriction_{(3.103)}\quad\mod B^1\otimes(\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\}\cap\hat\pi(\Omega^2\mathfrak{a}))\;.\qquad\square$$

Therefore, the construction of $\Omega^n_D\mathfrak{g}$ is reduced to the problem of finding the factor space $(\hat\pi(\Omega^j\mathfrak{a})+\hat\pi(T_n^{j-2}\mathfrak{a}))\,/\,(\hat\pi(\jmath^j\mathfrak{a})+\tilde K_n^{j-2}\mathfrak{a})$. Here, only the matrix Lie algebra $\mathfrak{a}$ plays a rôle. The influence of the Λ^*–part to $\Omega^n_D\mathfrak{g}$ is almost trivial.

3.6.2 The Commutator of Elements of $\Omega^*_D\mathfrak{g}$

For the sake of an easier notation we restrict ourselves to the case $\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\}\cap\hat\pi(\Omega^2\mathfrak{a})=0$ and $(\{\hat\pi(\mathfrak{a}),\hat\pi(\Omega^{n-N-2}\mathfrak{a})+\hat\pi(T^{n-N-4}_{n-1}\mathfrak{a})\}\cap\hat\pi(\Omega^{n-N}\mathfrak{a}))=0$. If these conditions are not fulfilled then there are obvious modifications to $\Omega^3_D\mathfrak{g}$ and $\Omega^n_D\mathfrak{g}$, $n\geq N+2$, see Corollary 24.

Due to (3.31), (3.33) and Corollary 24 we represent elements $\varrho^n\in\Omega^n_D\mathfrak{g}$ as

$$\varrho^n=\sum_\alpha\sum_{j=0}^n c_\alpha^{n-j}\gamma^j\otimes(\hat\pi(\omega^j_\alpha)+\tilde\jmath^j_n\mathfrak{a})\;,\tag{3.104a}$$

$$\tilde\jmath^j_n\mathfrak{a}:=\hat\pi(\jmath^j\mathfrak{a})+\tilde K_n^{j-2}\mathfrak{a}\;,\qquad \tilde\jmath^0_n\mathfrak{a}\equiv 0\;,\qquad \tilde\jmath^1_n\mathfrak{a}\equiv 0\;,\tag{3.104b}$$

$$\begin{aligned} n\geq 2\;:\;& c_\alpha^{n-j}\in\Lambda^{n-j}\;,\qquad \hat\pi(\omega^0_\alpha)\in\hat\pi(\mathfrak{a}')\;,\qquad \hat\pi(\omega^j_\alpha)\in\hat\pi(\Omega^j\mathfrak{a})\text{ for }j>0\;,\\ n=1\;:\;& c^1_\alpha\in\Lambda^1\text{ if }\hat\pi(\omega^0_\alpha)\in\hat\pi(\mathfrak{a}')\;,\qquad c^1_\alpha\in B^1\text{ if }\hat\pi(\omega^0_\alpha)\in\hat\pi(\mathfrak{a}'')\;,\\ & c^0_\alpha\in\Lambda^0\;,\qquad \hat\pi(\omega^1_\alpha)\in\hat\pi(\Omega^1\mathfrak{a})\;,\\ n=0\;:\;& c^0_\alpha\in\Lambda^0\;,\qquad \hat\pi(\omega^0_\alpha)\in\hat\pi(\mathfrak{a})\;.\end{aligned}\tag{3.104c}$$

The formula for the graded commutator of elements of $\Omega^*_D\mathfrak{g}$ is very simple,

$$
\begin{aligned}
\big[\sum_\alpha \sum_{i=0}^k c_\alpha^{k-i}\gamma^i \otimes (\hat\pi(\omega_\alpha^i) + \tilde{\jmath}^{\,i}_k\mathfrak{a}), \sum_\beta\sum_{j=0}^l \tilde c_\beta^{\,l-j}\gamma^j \otimes (\hat\pi(\tilde\omega^j_\beta) + \tilde{\jmath}^{\,j}_l\mathfrak{a})\big]_g \\
= \sum_{\alpha,\beta}\sum_{i=0}^k\sum_{j=0}^l (-1)^{i(l-j)} c_\alpha^{k-i}\wedge \tilde c_\beta^{\,l-j}\gamma^{i+j}\otimes([\hat\pi(\omega^i_\alpha),\hat\pi(\tilde\omega^j_\alpha)]_g + \tilde{\jmath}^{\,i+j}_{k+l}\mathfrak{a})\,,
\end{aligned}
\tag{3.105}
$$

because if the product between c_α^{k-i} and $\tilde c_\beta^{\,l-j}$ is not completely antisymmetrized then we get a combination of graded anticommutators of elements of $\hat\pi(\Omega^*\mathfrak{a})$ in the second component of the tensor product, which contributes to the ideal $\pi(\jmath^*\mathfrak{g})$. Thus, the graded commutator of elements of $\Omega^*_D\mathfrak{g}$ is given by the combination of the exterior product of the Λ^*–parts and the graded commutator of the $\hat\pi(\Omega^*\mathfrak{a})$–parts modulo $\pi(\jmath^*\mathfrak{g})$, where a graded sign due to the exchange with γ must be added.

3.6.3 The Differential of Elements of $\Omega^*_D\mathfrak{g}$

Due to (3.21) and (3.80) we have for $c^k \in \Lambda^k$

$$
(-\mathrm{i}\mathsf{D})c^k - (-1)^k c^k(-\mathrm{i}\mathsf{D}) = \mathbf{d}c^k - \mathbf{d}^* c^k + 2\nabla^S_{c^k}\,. \tag{3.106}
$$

We apply Proposition 7 and Lemma 23 to (3.104a), introducing $\tau^n := \sum_\alpha\sum_{j=0}^n c_\alpha^{n-j}\gamma^j \otimes \hat\pi(\omega^j_\alpha) \in \pi(\Omega^n\mathfrak{g})$ and using (3.78a), (3.79) and Lemma 17. This gives

$$
\begin{aligned}
d\varrho^n &\equiv \pi(d\pi^{-1}(\tau^n)) + \pi(\jmath^{\,n+1}\mathfrak{g}) \\
&= \textstyle\sum_\alpha\sum_{j=0}^n \big(((-\mathrm{i}\mathsf{D})c_\alpha^{n-j} - (-1)^{n-j}c_\alpha^{n-j}(-\mathrm{i}\mathsf{D}))\gamma^j\otimes\hat\pi(\omega^j_\alpha) \\
&\quad + (-1)^{n-j}c_\alpha^{n-j}\gamma^{j+1}\otimes((-\mathrm{i}\mathcal{M})\hat\pi(\omega^j_\alpha) - (-1)^j\hat\pi(\omega^j_\alpha)(-\mathrm{i}\mathcal{M}))\big) \\
&\quad + \mathbf{d}^*\tau^n - 2\nabla_\Omega(\tau^n) + \hat\sigma_{\mathfrak{g}}(\pi^{-1}(\tau^n)) + \pi(\jmath^{\,n+1}\mathfrak{g}) \\
&= \textstyle\sum_\alpha\sum_{j=0}^n \big(\mathbf{d}c_\alpha^{n-j}\gamma^j\otimes(\hat\pi(\omega^j_\alpha) + \tilde{\jmath}^{\,j}_{n+1}\mathfrak{a}) \\
&\quad + c_\alpha^{n-j}\gamma^{j+1}\otimes((-1)^{n-j}[-\mathrm{i}\mathcal{M},\hat\pi(\omega^j_\alpha)]_g + \hat\sigma(\omega^j_\alpha) + \tilde{\jmath}^{\,j+1}_{n+1}\mathfrak{a})\big)\,.
\end{aligned}
\tag{3.107}
$$

Let us say some words on the terms in (3.87) and (3.88) containing total differentials. In general, for

$$
\tau^k := c^{k-j}\gamma^j\otimes\hat\pi(\hat\kappa_k^{j-2}) \in \Lambda^{k-j}\gamma^j\otimes\hat\pi(T_k^{j-2}\mathfrak{a}) \subset \pi(\jmath^{\,k}\mathfrak{g})
$$

we have $\mathbf{d}\tau^k \in \pi(\jmath^{\,k+1}\mathfrak{g})$. This is no longer true for $k=2$ and $\hat\pi(\hat\kappa^0_2) = \{\hat\pi(a''),\hat\pi(\tilde a'')\}$, with $a'',\tilde a'' \in \mathfrak{a}''$. However, in this case the differential $\mathbf{d}\tau^2$ is eliminated by the counterterm $-\mathbf{d}\big(\tau^2\restriction_{\Lambda^0\otimes\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\}}\big)$ in (3.87). An analogous property holds for $k-j = N-1$, where the terms $\mathbf{d}\tau^k$ are cancelled by the differentials in (3.88). Therefore, in the following formula for the differentiation rule on $\Omega^*_D\mathfrak{g}$ one must omit these boundary terms. Then we obtain a simple formula:

$$
\begin{aligned}
d\varrho^n = \big(&(\mathbf{d}\otimes\mathbb{1}_F)(\tau^n) + [\gamma\otimes -\mathrm{i}\mathcal{M},\tau^n]_g \\
&+ (1\otimes\hat\sigma\circ\hat\pi^{-1})\circ\tau^n\circ(\gamma\otimes\mathbb{1}_F)\big) \quad \mathrm{mod}\ \pi(\jmath^{\,n+1}\mathfrak{g})\,,
\end{aligned}
\tag{3.108}
$$

where $\tau^n \in \pi(\Omega^n\mathfrak{g})$ is an arbitrary representative of $\varrho^n \in \Omega^n_D\mathfrak{g}$. Here, the differential $\mathbf{d}$ ignores the grading operator $\boldsymbol{\gamma}$, i.e. $\mathbf{d}(c^k\boldsymbol{\gamma}) := (\mathbf{d}c^k)\boldsymbol{\gamma}$. The non–trivial part in this formula is to find the spaces $\tilde{\jmath}^{\,j}_{n+1}\mathfrak{a}$ constituting the ideal $\pi(\jmath^{\,n+1}\mathfrak{g})$. The differential $\mathbf{d}\tau^n$, the graded commutator with $\boldsymbol{\gamma} \otimes -\mathrm{i}\mathcal{M}$ and even the computation of $(1 \otimes \hat{\sigma} \circ \hat{\pi}^{-1})(\tau^n)$ are not difficult for a concrete example.

3.7 Local Connections

Our definition of a connection given in Section 2.6 is too general for the desired application to gauge field theories. It is possible to convince oneself that Definition 8 does include non–local connections. To define local connections we need an additional structure: We must introduce something like a Λ^*–module structure on $\Omega^*_D\mathfrak{g}$, see Section 3.7.1. However, there are certain subtle points due to the existence of disturbing "boundary spaces". Therefore, only a certain generic subspace of $\Omega^*_D\mathfrak{g}$ has a module structure, but this is sufficient. Then we define local connection forms and local curvatures by the requirement that these objects commute with functions. Moreover, local gauge transformations commute with functions and preserve the space of local connection forms. Finally, we comment on the bosonic and fermionic actions in the case of local connections.

3.7.1 Making $\Omega^*_D\mathfrak{g}$ to a Λ^*–Module

In the case under consideration, an L–cycle over the tensor product of the algebra of functions and a matrix Lie algebra, there exists the notion of locality. Our goal is to define a multiplication

$$
\begin{aligned}
&\tilde{\wedge} : \Lambda^k \times \Omega^n_D\mathfrak{g} \to \Omega^{k+n}_D\mathfrak{g}\,, \qquad (3.109)\\
&\tilde{c}^k \tilde{\wedge} (\sum_\alpha \sum_{j=0}^n c^{n-j}_\alpha \boldsymbol{\gamma}^j \otimes (\hat{\pi}(\omega^j_\alpha) + \tilde{\jmath}^{\,j}_n\mathfrak{a})) := \sum_\alpha \sum_{j=0}^n (\tilde{c}^k \wedge c^{n-j}_\alpha)\boldsymbol{\gamma}^j \otimes (\hat{\pi}(\omega^j_\alpha) + \tilde{\jmath}^{\,j}_{k+n}\mathfrak{a})\,,
\end{aligned}
$$

see (3.104). However, we clearly have problems to do this on the whole differential Lie algebra $\Omega^*_D\mathfrak{g}$ due to the existence of the boundary spaces $\Lambda^0 \otimes \hat{\pi}(\mathfrak{a}'')$ in $\Omega^0_D\mathfrak{g} \equiv \pi(\mathfrak{g})$ and $B^1 \otimes \hat{\pi}(\mathfrak{a}'')$ in $\Omega^1_D\mathfrak{g} \equiv \pi(\Omega^1\mathfrak{g})$. These boundary spaces in general do not yield elements of $\Omega^*_D\mathfrak{g}$ when we multiply them by elements of Λ^*. Moreover, there are problems if the boundary terms $\delta^j_{n-N} B^N \boldsymbol{\gamma}^n \otimes (\{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\Omega^{n-N-2}\mathfrak{a}) + \hat{\pi}(T^{n-N-4}_{n-1}\mathfrak{a})\} \cap \hat{\pi}(\Omega^{n-N}\mathfrak{a}))$ and $B^1 \otimes (\{\hat{\pi}(\mathfrak{a}''), \hat{\pi}(\mathfrak{a}'')\} \cap \hat{\pi}(\Omega^2\mathfrak{a}))$ in Corollary 24 are present. Therefore, formula (3.109) is understood to hold on subspaces of $\Omega^*_D\mathfrak{g}$, where no collision with boundary terms occurs. Then, the multiplication (3.109) is associative,

$$
(c^k \wedge \tilde{c}^l)\tilde{\wedge}\varrho^n = c^k \tilde{\wedge}(\tilde{c}^l \tilde{\wedge} \varrho^n)\,, \qquad (3.110)
$$

for $\tilde{c}^k \in \Lambda^k$, $\tilde{\tilde{c}}^l \in \Lambda^l$ and $\varrho^n \in \Omega^n_D\mathfrak{g}$ (different from boundary spaces). In particular, $\Omega^n_D\mathfrak{g}$ carries a natural $C^\infty(X)$–module structure, where we omit the multiplication symbol $\tilde{\wedge}$

for simplicity:

$$f(\sum_\alpha \sum_{j=0}^{n} c_\alpha^{n-j}\gamma^j \otimes (\hat{\pi}(\omega_\alpha^j) + \tilde{\jmath}_n^j\mathfrak{a})) := \sum_\alpha \sum_{j=0}^{n} (fc_\alpha^{n-j})\gamma^j \otimes (\hat{\pi}(\omega_\alpha^j) + \tilde{\jmath}_n^j\mathfrak{a}) , \tag{3.111}$$

for $f \in C^\infty(X)$. Moreover, the Hilbert space $h = L^2(X,S) \otimes \mathbb{C}^F$ carries a natural $\Gamma^\infty(C)$–module structure induced by the $\Gamma^\infty(C)$–module structure of $L^2(X,S)$:

$$s^c(\sum_\alpha s_\alpha \otimes \varphi_\alpha) := \sum_\alpha s^c s_\alpha \otimes \varphi_\alpha , \quad s^c \in \Gamma^\infty(C) , \; s_\alpha \in L^2(X,S) , \; \varphi_\alpha \in \mathbb{C}^F . \tag{3.112}$$

The structures just introduced enable us to restrict the set of connections according to Definition 8 to the subset of local connections relevant for physical applications.

Definition 25. *A connection* (∇, ∇_h) *is called local connection iff for all* $f \in C^\infty(X)$, $\psi \in h$ *and* $\varrho^n \in \Omega_D^n\mathfrak{g}$ *different from boundary spaces one has*

$$\nabla_h(f\psi) = f\nabla_h(\psi) + (\mathbf{d}f)(\psi) , \tag{3.113a}$$

$$\nabla(f\varrho^n) = f\nabla(\varrho^n) + (\mathbf{d}f)\hat{\wedge}\varrho^n . \tag{3.113b}$$

The group of local gauge transformations is the group

$$\begin{aligned} \mathcal{U}_0(\mathfrak{g}) := \big\{ & u \in \mathcal{U}(\mathfrak{g}) \subset \mathscr{B}(h) , \; fu\psi = uf\psi , \; \forall f \in C^\infty(X) , \; \forall \psi \in h , \\ & (\mathrm{Ad}_u \nabla \mathrm{Ad}_{u^*} , u\nabla_h u^*) \text{ is a local connection if } (\nabla, \nabla_h) \text{ is } \big\} . \end{aligned} \tag{3.113c}$$

3.7.2 Local Connection Forms

We recall that a connection has the form $(\nabla = d + [\hat{\rho}, \, . \,]_g , \nabla_h = -\mathrm{i}D + \rho)$, where $\rho \in \mathcal{H}^1\mathfrak{g}$ and $\hat{\rho} := \rho + \tilde{\mathfrak{c}}^1\mathfrak{g} \in \hat{\mathcal{H}}^1\mathfrak{g}$, see Proposition 9. The insertion into Definition 25 yields

$$\rho \circ f = f \circ \rho , \quad \forall f \in C^\infty(X) . \tag{3.114}$$

Therefore, $\rho \in \Gamma(C) \otimes \mathrm{M}_F\mathbb{C}$. Since $\rho \in \mathcal{H}^1\mathfrak{g}$, there can only occur classical smooth differential forms up to first degree in the $\Gamma(C)$–component of ρ. This means that

$$\begin{aligned} & \rho \in (\Lambda^1 \otimes \mathbb{r}^0\mathfrak{a}) \oplus (\Lambda^0\gamma \otimes \mathbb{r}^1\mathfrak{a}) , \\ & \mathbb{r}^0\mathfrak{a} = -(\mathbb{r}^0\mathfrak{a})^* = \hat{\Gamma}(\mathbb{r}^0\mathfrak{a})\hat{\Gamma} \subset \mathrm{M}_F\mathbb{C} , \qquad \mathbb{r}^1\mathfrak{a} = -(\mathbb{r}^1\mathfrak{a})^* = -\hat{\Gamma}(\mathbb{r}^1\mathfrak{a})\hat{\Gamma} \subset \mathrm{M}_F\mathbb{C} . \end{aligned} \tag{3.115}$$

If we compute graded commutators with $\pi(\Omega^*\mathfrak{g})$ we get for $\Omega_D^2\mathfrak{g} \neq 0$

$$[\mathbb{r}^0\mathfrak{a}, \hat{\pi}(\mathfrak{a})] \subset \hat{\pi}(\mathfrak{a}') , \tag{3.116a}$$

$$[\mathbb{r}^0\mathfrak{a}, \hat{\pi}(\Omega^1\mathfrak{a})] \subset \hat{\pi}(\Omega^1\mathfrak{a}) , \tag{3.116b}$$

$$\{\mathbb{r}^0\mathfrak{a}, \hat{\pi}(\mathfrak{a})\} \subset \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} + \hat{\pi}(\Omega^2\mathfrak{a}) , \tag{3.116c}$$

$$
\begin{aligned}
\{\mathbb{r}^0\mathfrak{a}, \hat{\pi}(\Omega^1\mathfrak{a})\} &\subset \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\Omega^1\mathfrak{a})\} + \hat{\pi}(\Omega^3\mathfrak{a})\,, && (3.116\text{d})\\
[\mathbb{r}^1\mathfrak{a}, \hat{\pi}(\mathfrak{a})] &\subset \hat{\pi}(\Omega^1\mathfrak{a})\,, && (3.116\text{e})\\
\{\mathbb{r}^1\mathfrak{a}, \hat{\pi}(\Omega^1\mathfrak{a})\} &\subset \hat{\pi}(\Omega^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}\,. && (3.116\text{f})
\end{aligned}
$$

Moreover, one has to check that $[\rho, \pi(\jmath^n\mathfrak{g})]_g \subset \pi(\jmath^{n+1}\mathfrak{g})$. Comparing this formula with Theorem 22, we must demand

$$
\begin{aligned}
[\mathbb{r}^0\mathfrak{a}, \hat{\pi}(\jmath^k\mathfrak{a})] &\subset \tilde{\jmath}^k_{N+k}\mathfrak{a}\,, && (3.117\text{a})\\
\{\mathbb{r}^0\mathfrak{a}, \hat{\pi}(\jmath^k\mathfrak{a})\} &\subset \tilde{\jmath}^{k+2}_{N+k+2}\mathfrak{a}\,, && (3.117\text{b})\\
[\mathbb{r}^1\mathfrak{a}, \hat{\pi}(\jmath^k\mathfrak{a})]_g &\subset \tilde{\jmath}^{k+1}_{N+k+1}\mathfrak{a}\,, && (3.117\text{c})\\
[\mathbb{r}^0\mathfrak{a}, \hat{\sigma}\circ\hat{\pi}^{-1}(\hat{\pi}(T^k_n\mathfrak{a})\cap\hat{\pi}(\Omega^{k+2}\mathfrak{a}))] &\subset \tilde{\jmath}^{k+1}_{N+k+1}\mathfrak{a}\,, && (3.117\text{d})\\
\{\mathbb{r}^0\mathfrak{a}, \hat{\sigma}\circ\hat{\pi}^{-1}(\hat{\pi}(T^k_n\mathfrak{a})\cap\hat{\pi}(\Omega^{k+2}\mathfrak{a}))\} &\subset \tilde{\jmath}^{k+3}_{N+k+3}\mathfrak{a}\,, && (3.117\text{e})\\
[\mathbb{r}^1\mathfrak{a}, \hat{\sigma}\circ\hat{\pi}^{-1}(\hat{\pi}(T^k_n\mathfrak{a})\cap\hat{\pi}(\Omega^{k+2}\mathfrak{a}))] &\subset \tilde{\jmath}^{k+2}_{N+k+2}\mathfrak{a}\,, && (3.117\text{f})
\end{aligned}
$$

for all $k, n \in \mathbb{N}$. The remaining commutators and anticommutators

$[\mathbb{r}^0\mathfrak{a}, \hat{\pi}(T^k_n\mathfrak{a})]$, $\{\mathbb{r}^0\mathfrak{a}, \hat{\pi}(T^k_n\mathfrak{a})\}$ and $[\mathbb{r}^1\mathfrak{a}, \hat{\pi}(T^k_n\mathfrak{a})]_g$

can always be transformed into

$[\hat{\pi}(\mathfrak{a}), \hat{\pi}(T^k_n\mathfrak{a})]$, $\{\hat{\pi}(\mathfrak{a}), \hat{\pi}(T^k_n\mathfrak{a})\}$ and $[\hat{\pi}(\Omega^1\mathfrak{a}), \hat{\pi}(T^k_n\mathfrak{a})]_g$

by means of (3.116).

3.7.3 Local Curvatures

From (3.113b) one easily finds for the curvature of a local connection $\nabla^2 f = f\nabla^2$, for $f \in C^\infty(X)$. Thus,

$$
\begin{aligned}
f\theta = \theta f &= f(\pi\circ d\circ\pi^{-1}(\rho) + \tfrac{1}{2}[\rho,\rho]_g + \pi(\jmath^2\mathfrak{g}) + \tilde{\mathbb{c}}^2\mathfrak{g})\\
&= (\pi\circ d\circ\pi^{-1}(\rho) + \tfrac{1}{2}[\rho,\rho]_g + \pi(\jmath^2\mathfrak{g}) + \tilde{\mathbb{c}}^2\mathfrak{g})f\,. \qquad (3.118)
\end{aligned}
$$

Here, $\pi\circ d\circ\pi^{-1}(\rho) + \pi(\jmath^2\mathfrak{g}) + \tilde{\mathbb{c}}^2\mathfrak{g}$ is understood in the sense (2.47b). Hence, we must search for the subspace of $\tilde{\mathbb{c}}^2\mathfrak{g}$ commuting with functions. This space has the structure

$$
\tilde{\mathbb{c}}^2\mathfrak{g} = (\Lambda^2\otimes\mathbb{c}^0\mathfrak{a}) \oplus (\Lambda^1\gamma\otimes\mathbb{c}^1\mathfrak{a}) \oplus (\Lambda^0\otimes\mathbb{c}^2\mathfrak{a})\,, \quad \mathbb{c}^i\mathfrak{a} \subset \mathrm{M}_F\mathbb{C}\,, \qquad (3.119)
$$

because possible Λ^*–contributions of higher degree are already orthogonal to any representative of θ, see (2.66). The spaces $\mathbb{c}^i\mathfrak{a}$ have elementwise the following involution and $\mathbb{Z}_2$–grading properties:

$$
\begin{aligned}
&\mathbb{c}^0\mathfrak{a} = -(\mathbb{c}^0\mathfrak{a})^* = \hat{\Gamma}(\mathbb{c}^0\mathfrak{a})\hat{\Gamma}\,, \qquad \mathbb{c}^1\mathfrak{a} = -(\mathbb{c}^1\mathfrak{a})^* = -\hat{\Gamma}(\mathbb{c}^1\mathfrak{a})\hat{\Gamma}\,,\\
&\mathbb{c}^2\mathfrak{a} = \ \ (\mathbb{c}^2\mathfrak{a})^* = \hat{\Gamma}(\mathbb{c}^2\mathfrak{a})\hat{\Gamma}\,.
\end{aligned}
\qquad (3.120)
$$

From (2.45) one finds after a decomposition into Λ^*–components the equations

$$\begin{aligned} \mathbb{c}^0\mathfrak{a}\cdot\hat\pi(\mathfrak{a}') &= 0\;, & \mathbb{c}^0\mathfrak{a}\cdot\hat\pi(\Omega^1\mathfrak{a}) &= 0\;,\\ \mathbb{c}^1\mathfrak{a}\cdot\hat\pi(\mathfrak{a}') &= 0\;, & \mathbb{c}^1\mathfrak{a}\cdot\hat\pi(\Omega^1\mathfrak{a}) &= 0\;,\\ [\mathbb{c}^2\mathfrak{a},\hat\pi(\mathfrak{a}')] &= 0\;, & [\mathbb{c}^2\mathfrak{a},\hat\pi(\Omega^1\mathfrak{a})] &= 0\;. \end{aligned} \tag{3.121a}$$

The restriction to $\hat\pi(\mathfrak{a}')$ is due to possible problems with the boundary spaces. Due to (3.118) it is convenient to define

$$\mathbb{j}^0\mathfrak{a} := \mathbb{c}^0\mathfrak{a}\;,\qquad \mathbb{j}^1\mathfrak{a} := \mathbb{c}^1\mathfrak{a}\;,\qquad \mathbb{j}^2\mathfrak{a} := \mathbb{c}^2\mathfrak{a} + \hat\pi(\mathcal{J}^2\mathfrak{a}) + \{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\}\;. \tag{3.121b}$$

We recall that the commutator and the differential in the curvature $\theta = d\hat\rho + \frac{1}{2}[\hat\rho,\hat\rho]_g$ are indirectly defined via the graded Jacobi identity and the graded Leibniz rule (2.47b). The commutator and differential in $\pi(\Omega^*\mathfrak{g}) \mod \pi(\mathcal{J}^*\mathfrak{g})$ are given by (3.105) and (3.108). It is obvious that these formulae extend[12] to local elements of $\widehat{\mathcal{H}}^*\mathfrak{g}$. Only the map $\hat\sigma\circ\hat\pi^{-1}$ has to be extended to $\mathbb{r}^*\mathfrak{a}$ via the graded Leibniz rule:

$$\begin{aligned} &[\hat\sigma\circ\pi^{-1}(\eta^k)+\hat\pi(\mathcal{J}^{k+1}\mathfrak{a}),\hat\pi(\omega^l)+\hat\pi(\mathcal{J}^l\mathfrak{a})]_g\\ &\qquad := \hat\sigma\circ\pi^{-1}([\eta^k,\hat\pi(\omega^l)]_g)-(-1)^k[\eta^k,\hat\sigma(\omega^l)]_g+\hat\pi(\mathcal{J}^{k+l+1}\mathfrak{a})\;, \end{aligned} \tag{3.122}$$

for $\eta^k\in\mathbb{r}^k\mathfrak{a}$ and $\omega^l\in\Omega^l\mathfrak{a}$. Then we find for the curvature

$$\begin{aligned} \theta = \big(&(\mathbf{d}\otimes\mathbb{1}_F)(\rho)+\{\gamma\otimes -\mathrm{i}\mathcal{M}\;,\rho\}+\tfrac{1}{2}\{\rho,\rho\}\\ &+(1\otimes\hat\sigma\circ\pi^{-1})\circ\rho\circ(\gamma\otimes\mathbb{1}_F)\big) \mod \mathbb{J}^2\mathfrak{g}\;, \end{aligned} \tag{3.123}$$

where we recall that $\mathbb{J}^2\mathfrak{g} = \Lambda^0\otimes(\hat\pi(\mathcal{J}^2\mathfrak{a})+\{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\})+\tilde{\mathbb{c}}^2\mathfrak{g}$.

3.7.4 The Group of Local Gauge Transformations

The analysis of the group of local gauge transformations (3.113c) yields

$$\begin{aligned} \mathcal{U}_0(\mathfrak{g}) &= \exp(\Lambda^0\otimes\mathbb{u}_0(\mathfrak{a}))\;, \qquad \text{where}\\ \mathbb{u}_0(\mathfrak{a}) &= \{\,u_0\in\mathbb{r}^0\mathfrak{a}\;,\quad \hat\sigma\circ\hat\pi^{-1}(u_0)\subset\mathbb{c}^1\mathfrak{a}\,\}\;, \end{aligned} \tag{3.124}$$

see (2.48) and (2.54).

3.7.5 Bosonic and Fermionic Actions

In our case – $h = L^2(X,S)\otimes\mathbb{C}^F$ – we have $\mathscr{B}(h) = \mathscr{B}(L^2(X,S))\otimes M_F\mathbb{C}$. Then, the parameter d in (2.63) is equal to the dimension N of the manifold X, see [17]. Moreover, the trace theorem of Alain Connes [17, 62] says that in this case we have

$$\mathrm{Tr}_\omega((s^c\otimes \mathrm{m})\,|D|^{-N}) = \frac{1}{(\frac{N}{2})!(4\pi)^{\frac{N}{2}}}\int_X \mathrm{v}_g\,\mathrm{tr}_c(s^c)\,\mathrm{tr}(\mathrm{m})\;, \tag{3.125}$$

[12] provided that there exist non–vanishing spaces $\pi(\Omega^*\mathfrak{g})$ of sufficient degree

where we recall that v_g denotes the canonical volume form on X, tr_c denotes the trace in the Clifford algebra $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^N)$, normalized by $\mathrm{tr}_c(1) = 2^{N/2}$, and tr(m) is the matrix–trace of $\mathrm{m} \in \mathrm{M}_F\mathbb{C}$. We use the trace theorem (3.125) for the construction of $\mathfrak{e}(\theta)$, see (2.66). For the curvature θ of a local connection we have according to the above considerations a decomposition

$$\theta = \sum_\alpha \left(c_\alpha^2 \otimes (\tau_\alpha^0 + \mathbb{j}^0\mathfrak{a}) + c_\alpha^1 \gamma \otimes (\tau_\alpha^1 + \mathbb{j}^1\mathfrak{a}) + c_\alpha^0 \otimes (\tau_\alpha^2 + \mathbb{j}^2\mathfrak{a}) \right) , \qquad (3.126)$$

where $c_\alpha^i \in \Lambda^i$ and $\tau_\alpha^i \in \mathrm{M}_F\mathbb{C}$. Since $\Lambda^* = \bigoplus_{k=0}^N \Lambda^k$ is an orthogonal decomposition with respect to the scalar product (3.13) given by tr_c, we see that (2.66) is equivalent to finding for $i \in \{0,1,2\}$ and each α the elements $j_\alpha^i \in \mathbb{j}^i\mathfrak{a}$ satisfying

$$\mathrm{tr}(\tilde{j}^i(\tau_\alpha^i + j_\alpha^i)) = 0 , \quad \text{for all } \tilde{j}^i \in \mathbb{j}^i\mathfrak{a} . \qquad (3.127a)$$

These equations must be solved depending on the concrete L–cycle $(\mathfrak{a}, \mathbb{C}^F, \mathcal{M}, \hat{\pi}, \hat{\Gamma})$ and the concrete element τ_α^i, giving in the notation of (3.126)

$$\mathfrak{e}(\theta) = \sum_\alpha \left(c_\alpha^2 \otimes (\tau_\alpha^0 + j_\alpha^0) + c_\alpha^1 \gamma \otimes (\tau_\alpha^1 + j_\alpha^1) + c_\alpha^0 \otimes (\tau_\alpha^2 + j_\alpha^2) \right) . \qquad (3.127b)$$

Now, formula (2.64a) for the bosonic action takes the form (for an appropriate choice of constants)

$$S_B(\nabla) = \int_X \mathrm{v}_g \, \frac{1}{g_0^2 F} \, \mathrm{tr}_c(\mathfrak{e}(\theta)^2) . \qquad (3.128a)$$

Here, F is the complex dimension of $\mathbb{C}^F$ and g_0 a coupling constant, which is not determined by the theory and has to be fitted to experimental data. Moreover, tr_c contains both the traces in $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^N)$ and $\mathrm{M}_F\mathbb{C}$. For the fermionic action we obtain

$$S_F(\psi, \nabla) = \langle \psi, (D + \mathrm{i}\rho)\psi \rangle_h = \int_X \mathrm{v}_g \, \psi^*(D + \mathrm{i}\rho)\psi . \qquad (3.128b)$$

3.7.6 Summary

This finishes our prescription towards gauge field theories. Let us recall what the essential steps are. One starts to select the L–cycle from the physical data or assumptions. We have learned that the matrix part of the L–cycle contains the essential information. Hence, we must construct the spaces $\hat{\pi}(\Omega^n\mathfrak{a})$ and the ideal $\hat{\pi}(\mathcal{J}^n\mathfrak{a})$ up to second[13] order. This is necessary to compute the spaces $\mathbb{r}^0\mathfrak{a}, \mathbb{r}^1\mathfrak{a}$ and $\mathbb{j}^0\mathfrak{a}, \mathbb{j}^1\mathfrak{a}, \mathbb{j}^2\mathfrak{a}$ constituting the connection form ρ and the ideal $\mathbb{J}^2\mathfrak{g}$. Then we have to compute the curvature θ of the connection and to select its representative $\mathfrak{e}(\theta)$ orthogonal to $\mathbb{J}^2\mathfrak{g}$. Finally, we write down the bosonic and fermionic actions. This scheme can be applied to a large class of physical models. Among them are the SU(3)×SU(2)×U(1)–standard model, the flipped SU(5)×U(1)–Grand Unification model and – as a special case of the latter – the SU(5)–Grand Unification model, see the next sections.

[13] In some cases, one may need knowledge of $\hat{\pi}(\Omega^3\mathfrak{a})$, see (3.116d).

4 Electrodynamics and Standard Model

In this section we consider two almost trivial applications of non–associative geometry: the chiral spinor electrodynamics and the standard model. The description of electrodynamics is clumsy and of partial success only, because we cannot obtain the bosonic action. Also the formulation of the standard model is more direct in other Yang–Mills–Higgs models. But these two models have a high heuristic value: Any theory that claims to be an improvement must reproduce the examples accessible by the former theories, plus one additional model. Thus, electrodynamics and standard model are the compulsories for our approach. The free exercise is the Grand Unification model, which we start to investigate in Section 5.

4.1 Chiral Spinor Electrodynamics

4.1.1 The L–Cycle

We would like to reformulate the chiral spinor electrodynamics within our approach. According to Section 1.1 we must specify the group of local gauge transformations and the set of fermions. The other input data are not relevant, because we consider a massless theory where no symmetry breaking occurs. Of course, the group of local gauge transformations is $\mathscr{G} = C^\infty(X) \otimes \mathrm{U}(1)$, and its Lie algebra is

$$\mathfrak{g} = C^\infty(X) \otimes \mathrm{u}(1) \,. \tag{4.1a}$$

Here, $C^\infty(X)$ is the algebra of real–valued smooth functions on a four dimensional compact Euclidian space–time manifold X. We want to describe F fermions ψ_i of the electric charges $q_i \neq 0$. Therefore, the Hilbert space is

$$h = L^2(X,S) \otimes \mathbb{C}^F \,. \tag{4.1b}$$

The electric charges $\{q_i\}$ determine the representation $\tilde{\pi}$ of the gauge group $\mathscr{G}$, and the induced representation $\pi = \tilde{\pi}_*$ of its Lie algebra $\mathfrak{g}$ on h is

$$\pi(f \otimes \mathrm{i}) := f \otimes \mathrm{i}\,\mathrm{diag}(q_1, q_2, \dots, q_F) \,, \quad f \in C^\infty(X) \,. \tag{4.1c}$$

The generalized Dirac operator coincides with the classical one,

$$D = \mathsf{D} \otimes \mathbb{1}_F \,. \tag{4.1d}$$

We want to describe left–handed and right–handed fermions. Thus, the chirality operator is

$$\Gamma = \gamma^5 \otimes \mathrm{diag}(s_1, s_2, \dots, s_F) \,, \quad s_j = \pm 1 \,. \tag{4.1e}$$

It is obvious that $(\mathfrak{g}, h, D, \pi, \Gamma)$ is an L–cycle.

4.1.2 The Structure of $\pi(\Omega^*\mathfrak{g}), \pi(\mathcal{J}^*\mathfrak{g})$ and $\Omega_D^*\mathfrak{g}$

Now it is easy to find the structure of $\pi(\Omega^*\mathfrak{g}), \pi(\mathcal{J}^*\mathfrak{g})$ and $\Omega_D^*\mathfrak{g}$. Since there is no mass matrix $\mathcal{M}$ present we have $\hat{\pi}(\Omega^n\mathfrak{a}) \equiv 0$ for all $n > 0$. From (3.30) we get

$$\pi(\Omega^1\mathfrak{g}) = B^1 \otimes \mathrm{i}\,\mathrm{diag}(q_1, q_2, \ldots, q_F)\;. \tag{4.2a}$$

From Proposition 15 we get

$$\pi(\Omega^2\mathfrak{g}) = C^\infty(X) \otimes \mathrm{diag}(q_1^2, q_2^2, \ldots, q_F^2)\;, \qquad \pi(\Omega^n\mathfrak{g}) \equiv 0 \quad \text{for } n \geq 3\;. \tag{4.2b}$$

Now, Theorem 22 tells us that

$$\pi(\mathcal{J}^2\mathfrak{g}) = C^\infty(X) \otimes \mathrm{diag}(q_1^2, q_2^2, \ldots, q_F^2)\;, \qquad \pi(\mathcal{J}^n\mathfrak{g}) \equiv 0 \quad \text{for } n \geq 3\;. \tag{4.2c}$$

Therefore,

$$\Omega_D^0\mathfrak{g} = \pi(\mathfrak{g})\;, \qquad \Omega_D^1\mathfrak{g} = \pi(\Omega^1\mathfrak{g})\;, \qquad \Omega_D^n\mathfrak{g} = 0 \quad \text{for } n \geq 2\;. \tag{4.2d}$$

4.1.3 The Connection Form

We know from (3.114) that local connection forms ρ are given by

$$\rho \in \Gamma(C) \otimes \mathrm{M}_F\mathbb{C} \cap \mathcal{H}^1\mathfrak{g}\;, \tag{4.3a}$$

where $\Gamma(C)$ is the set of sections of the Clifford bundle. The condition $\rho \in \mathcal{H}^1\mathfrak{g}$ yields

$$[\rho, \pi(\mathfrak{g})] \subset \pi(\Omega^1\mathfrak{g})\;, \qquad \{\rho, \pi(\Omega^1\mathfrak{g})\} \subset \pi(\Omega^2\mathfrak{g})\;, \tag{4.3b}$$

see (2.44). Let us assume that the charges are ordered as

$$q_1 = \cdots = q_{n_1}\;,\;\; q_{n_1+1} = \cdots = q_{n_1+n_2}\;,\;\; \ldots\;,\;\; q_{F+1-n_k} = \cdots = q_F\;. \tag{4.4}$$

Then, the first equation (4.3b) implies

$$\rho = \textstyle\sum_\alpha A_\alpha \otimes \mathrm{diag}(m_\alpha^1, m_\alpha^2, \ldots, m_\alpha^k)\;, \quad A_\alpha \in \Gamma(C)\;,\;\; m_\alpha^i \in \mathrm{M}_{n_i}\mathbb{C}\;. \tag{4.5a}$$

We insert this result into the second equation (4.3b) and find

$$\begin{aligned}&\{\rho, b^1 \otimes \mathrm{i}\,\mathrm{diag}(q_1, \ldots, q_F)\}\\ &\qquad = \textstyle\sum_\alpha (b^1 A_\alpha + A_\alpha b^1) \otimes \mathrm{diag}(q_{n_1} m_\alpha^1, q_{n_2} m_\alpha^2, \ldots, q_{n_k} m_\alpha^k)\;,\end{aligned} \tag{4.5b}$$

where $b^1 \in B^1$. The first component of the tensor product must belong to $C^\infty(X)$, which yields $A_\alpha \in \Lambda^1$ for all α. It is very important that A_α belongs to Λ^1 and not necessarily to B^1. Moreover, all matrices $q_{n_i} m_\alpha^i$ must be diagonal. Due to $q_i \neq 0$, the matrices m_α^i must be diagonal themselves. Finally, the ratio for fixed α between m_α^i is fixed and given by the ratio of the charges q_i. Therefore, we get with $\rho = -\rho^*$ the final formula for the connection form

$$\rho = A \otimes \mathrm{i}\,\mathrm{diag}(q_1, q_2, \ldots, q_F)\;, \quad A \in \Lambda^1\;. \tag{4.5c}$$

Obviously, Γ anticommutes with ρ, and we have $[\rho, \pi(\mathcal{J}^*\mathfrak{g})]_g \subset \pi(\mathcal{J}^*\mathfrak{g})$. Comparing (2.45) with (4.3b) and (4.5c) we find $\tilde{\mathfrak{c}}^1\mathfrak{g} \equiv 0$. Therefore, $\hat{\rho} = \rho$.

4.1.4 There is no Bosonic Action

Now we are going to compute the curvature θ of our connection. Proposition 9 tells us that $\theta = d\rho + \frac{1}{2}[\rho,\rho]_g$. It is clear that $[\rho,\rho]_g \in \pi(\jmath^2\mathfrak{g})$. Therefore, modulo $\pi(\jmath^2\mathfrak{g})$ we have $\theta = d\rho$. One could think that $d\rho$ coincides with the application of the exterior differential to ρ. But this is not the case. Namely, we must use formula (2.47b) for the computation of $d\rho$, which yields

$$[d\rho, \pi(\mathfrak{g})] = 0 \mod \pi(\jmath^2\mathfrak{g}) , \qquad [d\rho, \mathbf{d}\pi(\mathfrak{g})] = 0 . \tag{4.6}$$

But these equations mean that $d\rho \mod \pi(\jmath^2\mathfrak{g})$ belongs to the graded centralizer of $\pi(\Omega^*\mathfrak{g})$. Therefore, the canonical representative $\mathfrak{e}(\theta)$ of the curvature is zero. Another representative would be $\mathbf{d}\rho$, but the use of $\mathbf{d}\rho$ is not compatible with our framework. *Thus, although our connection form has classically a non–vanishing curvature, we get a vanishing curvature in our formalism. The model for the chiral spinor electrodynamics is too simple for our approach. The only possibility to obtain a bosonic action for Abelian Lie algebras is to include additional semisimple Lie algebras. An example is the construction of the standard model in Section 4.2, where no problems occur. Another example is a mixture of the chiral electrodynamics with a* SU(3)*–colour symmetry for the quarks.*

4.1.5 The Fermionic Action

The fermionic action is the usual one,

$$S_F = \int_X dx \Big(\sum_{i=1}^{F} \langle \psi_i, (\mathsf{D} - q_i A)\psi_i \rangle \Big) . \tag{4.7}$$

We can perform a Wick rotation to Minkowski space and impose the chirality condition $\Gamma\psi = \psi$. Then, ψ_i describes a left–handed or right–handed fermion, respectively, if $s_i = -1$ or $s_i = +1$ in (4.1e).

4.2 The Standard Model

The second physical application of non–associative geometry is the formulation of the standard model. There exist already numerous formulations of the standard model within the context of non–commutative geometry, see [9–12,17,19–23,29,30,33–36,39, 40,42–44,48,57,59,62,64]. For a review comparing the different branches see [58]. Our formulation is no improvement, because the techniques do not differ very much. The essential point is that our approach can be applied to Grand Unification as well. The standard model is simply a consistency check.

We assume that there are three generations of fermions; however, the calculation works with an arbitrary number bigger than one of generations as well. A more essential assumption is that the fermionic mass spectrum is not degenerate. Otherwise we obtain no Higgs potential. If right neutrinos are present then the Dirac mass matrix for the neutrino sector must be invertible.

Under these assumptions we obtain the bosonic action of the standard model in a unified form. We find that the masses of the W and Higgs bosons are (on tree–level) determined by the mass of the top quark. Moreover, we get a tree–level prediction for the Weinberg angle. The fermionic action will not be displayed, because it is identical as in the classical formulation.

4.2.1 The Matrix L–Cycle for the Standard Model

The L–cycle for the $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$*–standard model follows immediately from the physical situation and the considerations of Section 1.1. Hence, the matrix part of that L–cycle takes the following form:*

The matrix Lie algebra of the standard model is – as one may expect –

$$\mathfrak{a} = \mathrm{su}(3) \oplus \mathrm{su}(2) \oplus \mathrm{u}(1)\,. \tag{4.8}$$

The Hilbert space is $\mathbb{C}^{48}$, because we want to include right neutrinos. We label elements of $\mathbb{C}^{48}$ in a suggestive way by the fermions of the first generation:

$$(\boldsymbol{u}_L\,,\,\boldsymbol{d}_L\,,\,\boldsymbol{u}_R\,,\,\boldsymbol{d}_R\,,\,\nu_L\,,\,e_L\,,\,\nu_R\,,\,e_R)^T \in \mathbb{C}^{48}\,, \tag{4.9}$$

where $\boldsymbol{u}_L, \boldsymbol{d}_L, \boldsymbol{u}_R, \boldsymbol{d}_R \in \mathbb{C}^3 \otimes \mathbb{C}^3$ and $\nu_L, e_L, \nu_R, e_R \in \mathbb{C}^3$. We can obtain a model without right neutrinos if we pass to the Hilbert space $\mathbb{C}^{45}$ by omitting $\nu_R \in \mathbb{C}^3$ in (4.9). Moreover, we must omit in all formulae below the rows and columns of matrices that interact with ν_R. However, in order to present the results for both cases – with and without right neutrinos – in one series of formulae, we put the rows and columns acting on ν_R equal to zero. The task to erase these zeroes is left to the reader.

The representation $\hat{\pi}$ of $\mathfrak{a}$ on $\mathbb{C}^{48}$ is

$$\begin{aligned}
&\hat{\pi}((a_3,a_2,a_1)) = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.10)\\
&\quad \mathrm{i} f_0 \,\mathrm{diag}\big(\tfrac{1}{3}\mathbb{1}_3\otimes\mathbb{1}_3\,,\,\tfrac{1}{3}\mathbb{1}_3\otimes\mathbb{1}_3\,,\,\tfrac{4}{3}\mathbb{1}_3\otimes\mathbb{1}_3\,,\,-\tfrac{2}{3}\mathbb{1}_3\otimes\mathbb{1}_3\,,\,-\mathbb{1}_3\,,\,-\mathbb{1}_3\,,\,0_3\,,\,-2\mathbb{1}_3\big) +\\
&\left(\begin{array}{c|c}
\begin{array}{cc|c|c}
(a_3+\mathrm{i}f_3\mathbb{1}_3)\otimes\mathbb{1}_3\,; & \mathrm{i}(f_1-\mathrm{i}f_2)\mathbb{1}_3\otimes\mathbb{1}_3 & 0 & 0\\
\mathrm{i}(f_1+\mathrm{i}f_2)\mathbb{1}_3\otimes\mathbb{1}_3\,; & (a_3-\mathrm{i}f_3\mathbb{1}_3)\otimes\mathbb{1}_3 & 0 & 0\\ \hline
0 & 0 & a_3\otimes\mathbb{1}_3 & 0\\ \hline
0 & 0 & 0 & a_3\otimes\mathbb{1}_3
\end{array} & \mathrm{O}\\ \hline
\mathrm{O} & \begin{array}{cc|c|c}
\mathrm{i}f_3\otimes\mathbb{1}_3\,; & \mathrm{i}(f_1-\mathrm{i}f_2)\otimes\mathbb{1}_3 & 0 & 0\\
\mathrm{i}(f_1+\mathrm{i}f_2)\otimes\mathbb{1}_3\,; & -\mathrm{i}f_3\otimes\mathbb{1}_3 & 0 & 0\\ \hline
0 & 0 & 0_3 & 0\\ \hline
0 & 0 & 0 & 0_3
\end{array}
\end{array}\right).
\end{aligned}$$

Here, the matrix $a_3 \in \mathrm{su}(3) \subset \mathrm{M}_3\mathbb{C}$ is written down in the standard matrix representation, $a_2 = \begin{pmatrix} \mathrm{i}f_3\,; & \mathrm{i}(f_1-\mathrm{i}f_2) \\ \mathrm{i}(f_1+\mathrm{i}f_2)\,; & -\mathrm{i}f_3 \end{pmatrix} \in \mathrm{su}(2)$, for $f_1,\,f_2,\,f_3 \in \mathbb{R}$, and $a_1 = \mathrm{i}f_0 \in \mathrm{u}(1) \equiv \mathrm{i}\mathbb{R}$.

The generalized Dirac operator is

$$\mathcal{M} = \begin{pmatrix} \begin{array}{cc|c|c} 0 & 0 & \mathbb{1}_3 \otimes M_u & 0 \\ 0 & 0 & 0 & \mathbb{1}_3 \otimes M_d \\ \hline \mathbb{1}_3 \otimes M_u^* & 0 & 0 & 0 \\ \hline 0 & \mathbb{1}_3 \otimes M_d^* & 0 & 0 \end{array} & \mathrm{O} \\ \mathrm{O} & \begin{array}{cc|c|c} 0 & 0 & M_\nu & 0 \\ 0 & 0 & 0 & M_e \\ \hline M_\nu^* & 0 & 0 & 0 \\ \hline 0 & M_e^* & 0 & 0 \end{array} \end{pmatrix}, \tag{4.11}$$

where $M_u, M_d, M_\nu, M_e \in \mathrm{M}_3\mathbb{C}$ are the mass matrices of the fermions. For $M_\nu \equiv 0$ the right neutrinos decouple completely from the rest of physics. The grading operator is

$$\hat{\Gamma} = \mathrm{diag}(-\mathbb{1}_3 \otimes \mathbb{1}_3\,, -\mathbb{1}_3 \otimes \mathbb{1}_3\,, \mathbb{1}_3 \otimes \mathbb{1}_3\,, \mathbb{1}_3 \otimes \mathbb{1}_3\,, -\mathbb{1}_3\,, -\mathbb{1}_3\,, \mathbb{1}_3\,, \mathbb{1}_3)\,. \tag{4.12}$$

One has $\hat{\Gamma}^2 = \mathbb{1}_{48}\,,\ [\hat{\Gamma}, \hat{\pi}(\mathfrak{a})] = 0$ and $\{\hat{\Gamma}, \mathcal{M}\} = 0\,.$

4.2.2 The Structure of $\hat{\pi}(\Omega^1\mathfrak{a})$ and $\hat{\pi}(\Omega^2\mathfrak{a})$

A simple calculation yields the structure of $\hat{\pi}(\Omega^1\mathfrak{a})$ and $\hat{\pi}(\Omega^2\mathfrak{a})$:

It is easy to see that for $a^i_\alpha = (a^i_{3,\alpha}, a^i_{2,\alpha}, a^i_{1,\alpha}) \in \mathfrak{a}$ one has

$$\tau^1 := \textstyle\sum_{\alpha,z\geq 0}[\hat{\pi}(a^z_\alpha), \dots [\hat{\pi}(a^1_\alpha), [-\mathrm{i}\mathcal{M}\,, \hat{\pi}(a^0_\alpha)]] \dots] = \tag{4.13a}$$

$$\mathrm{i}\begin{pmatrix} \begin{array}{cc|c|c} 0 & 0 & \bar{b}_2 \mathbb{1}_3 \otimes M_u & b_1 \mathbb{1}_3 \otimes M_d \\ 0 & 0 & -\bar{b}_1 \mathbb{1}_3 \otimes M_u & b_2 \mathbb{1}_3 \otimes M_d \\ \hline b_2 \mathbb{1}_3 \otimes M_u^*\,; & -b_1 \mathbb{1}_3 \otimes M_u^* & 0 & 0 \\ \hline \bar{b}_1 \mathbb{1}_3 \otimes M_d^*\,; & \bar{b}_2 \mathbb{1}_3 \otimes M_d^* & 0 & 0 \end{array} & \mathrm{O} \\ \mathrm{O} & \begin{array}{cc|c|c} 0 & 0 & \bar{b}_2 \otimes M_\nu & b_1 \otimes M_e \\ 0 & 0 & -\bar{b}_1 \otimes M_\nu & b_2 \otimes M_e \\ \hline b_2 \otimes M_\nu^*\,; & -b_1 \otimes M_\nu^* & 0 & 0 \\ \hline \bar{b}_1 \otimes M_e^*\,; & \bar{b}_2 \otimes M_e^* & 0 & 0 \end{array} \end{pmatrix},$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \sum_{\alpha,z\geq 0} a^z_{2,\alpha} a^z_{1,\alpha} \cdots a^1_{2,\alpha} a^1_{1,\alpha} a^0_{2,\alpha} a^0_{1,\alpha} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2\,. \tag{4.13b}$$

Due to (2.25), the matrix (4.13a) is the general form of an element $\tau^1 \in \hat{\pi}(\Omega^1\mathfrak{a})$.

According to (2.26), elements $\tau^2 \in \hat{\pi}(\Omega^2\mathfrak{a})$ are obtained by summing up anticommutators of two elements of $\hat{\pi}(\Omega^1\mathfrak{a})$:

$$\tau^2 := \textstyle\sum_\alpha \{\tau^1_\alpha, \tau^1_\alpha\}\,, \qquad \tau^1_\alpha \in \hat{\pi}(\Omega^1\mathfrak{a})\,.$$

Let

$$\begin{pmatrix} \mathrm{i}f_3\,; & \mathrm{i}(f_1-\mathrm{i}f_2) \\ \mathrm{i}(f_1+\mathrm{i}f_2)\,; & -\mathrm{i}f_3 \end{pmatrix} := \begin{pmatrix} \mathrm{i}(|b_2|^2-|b_1|^2)\,; & -2\mathrm{i}b_1\overline{b}_2 \\ -2\mathrm{i}\overline{b}_1 b_2\,; & -\mathrm{i}(|b_2|^2-|b_1|^2) \end{pmatrix} \in \mathrm{su}(2)\,, \tag{4.14a}$$

$$\begin{aligned} M_{ud} &= M_u M_u^* - M_d M_d^*\,, & M_{\nu e} &= M_\nu M_\nu^* - M_e M_e^*\,, \\ M_{\{ud\}} &= M_u M_u^* + M_d M_d^*\,, & M_{\{\nu e\}} &= M_\nu M_\nu^* + M_e M_e^*\,. \end{aligned} \tag{4.14b}$$

Then we have

$$\{\tau^1,\tau^1\} =$$

$$\mathrm{i}\left(\begin{array}{cc|c|c|cc|c|c} \mathrm{i}f_3\mathbb{1}_3{\otimes}M_{ud}\,; & \mathrm{i}(f_1-\mathrm{i}f_2)\mathbb{1}_3{\otimes}M_{ud} & 0 & 0 & & & & \\ \mathrm{i}(f_1+\mathrm{i}f_2)\mathbb{1}_3{\otimes}M_{ud}\,; & -\mathrm{i}f_3\mathbb{1}_3{\otimes}M_{ud} & 0 & 0 & & \mathrm{O} & & \\ \hline 0 & 0 & 0_9 & 0 & & & & \\ \hline 0 & 0 & 0 & 0_9 & & & & \\ \hline & & & & \mathrm{i}f_3{\otimes}M_{\nu e}\,; & \mathrm{i}(f_1-\mathrm{i}f_2){\otimes}M_{\nu e} & 0 & 0 \\ & \mathrm{O} & & & \mathrm{i}(f_1+\mathrm{i}f_2){\otimes}M_{\nu e}\,; & -\mathrm{i}f_3{\otimes}M_{\nu e} & 0 & 0 \\ \cline{5-8} & & & & 0 & 0 & 0_3 & 0 \\ \cline{5-8} & & & & 0 & 0 & 0 & 0_3 \end{array}\right) \tag{4.15a}$$

$$-(\|b_1\|^2+\|b_2\|^2)\,\mathrm{diag}\big(\mathbb{1}_3\otimes M_{\{ud\}}\,,\ \mathbb{1}_3\otimes M_{\{ud\}}\,,\ \mathbb{1}_3\otimes 2M_u^*M_u\,,\ \mathbb{1}_3\otimes 2M_d^*M_d\,, \\ M_{\{\nu e\}}\,,\ M_{\{\nu e\}}\,,\ 2M_\nu^*M_\nu\,,\ 2M_e^*M_e\big)\,. \tag{4.15b}$$

4.2.3 The Ideal $\hat{\pi}(\mathcal{J}^*\mathfrak{a})$

Also the ideal $\hat{\pi}(\mathcal{J}^*\mathfrak{a})$ is easy to derive. We find that $\hat{\sigma}(\Omega^n\mathfrak{a}) = \hat{\pi}(\mathcal{J}^{n+1}\mathfrak{a})$, the analogous relation as for even matrix K–cycles. Remarkably, $\Omega_D^n\mathfrak{a} = 0$ for all $n \geq 3$.

Next, for $\tau^1 = \hat{\pi}(\omega^1)$ given by (4.13a), $\omega^1 = \sum_{\alpha,z\geq 0}[\iota(a_\alpha^z),\dots[\iota(a_\alpha^1),\iota(da_\alpha^0)]\dots] \in \Omega^1\mathfrak{a}$, we obtain with (3.38)

$$\hat{\sigma}(\omega^1) \equiv \textstyle\sum_{\alpha,z\geq 0}[\hat{\pi}(a_\alpha^z),\dots[\hat{\pi}(a_\alpha^1),[\mathcal{M}^{\,2},\hat{\pi}(a_\alpha^0)]]\dots] = \tag{4.16a}$$

$$\mathrm{i}\left(\begin{array}{cc|c|c|cc|c|c} \mathrm{i}f_3\mathbb{1}_3{\otimes}M_{ud}\,; & \mathrm{i}(f_1-\mathrm{i}f_2)\mathbb{1}_3{\otimes}M_{ud} & 0 & 0 & & & & \\ \mathrm{i}(f_1+\mathrm{i}f_2)\mathbb{1}_3{\otimes}M_{ud}\,; & -\mathrm{i}f_3\mathbb{1}_3{\otimes}M_{ud} & 0 & 0 & & \mathrm{O} & & \\ \hline 0 & 0 & 0_9 & 0 & & & & \\ \hline 0 & 0 & 0 & 0_9 & & & & \\ \hline & & & & \mathrm{i}f_3{\otimes}M_{\nu e}\,; & \mathrm{i}(f_1-\mathrm{i}f_2){\otimes}M_{\nu e} & 0 & 0 \\ & \mathrm{O} & & & \mathrm{i}(f_1+\mathrm{i}f_2){\otimes}M_{\nu e}\,; & -\mathrm{i}f_3{\otimes}M_{\nu e} & 0 & 0 \\ \cline{5-8} & & & & 0 & 0 & 0_3 & 0 \\ \cline{5-8} & & & & 0 & 0 & 0 & 0_3 \end{array}\right),$$

$$\begin{pmatrix} \mathrm{i}f_3\,; & \mathrm{i}(f_1-\mathrm{i}f_2) \\ \mathrm{i}(f_1+\mathrm{i}f_2)\,; & -\mathrm{i}f_3 \end{pmatrix} := \sum_{\alpha,z\geq 0}\Big[a_{2,\alpha}^z,\dots\Big[a_{2,\alpha}^1,\Big[a_{2,\alpha}^0,\begin{pmatrix} \frac{\mathrm{i}}{2} & 0 \\ 0 & -\frac{\mathrm{i}}{2} \end{pmatrix}\Big]\Big]\dots\Big] \in \mathrm{su}(2)\,. \tag{4.16b}$$

Choosing

$$\omega_0^1 = \imath(da_2^0) + [\imath(a_2^1), [\imath(a_2^1), \imath(da_2^0)]],$$
$$a_2^0 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \in \mathrm{su}(2), \qquad a_2^1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \in \mathrm{su}(2),$$

we have $\omega_0^1 \in \ker \hat{\pi}$ due to $(a_2^0 + a_2^1 a_2^1 a_2^0)\binom{0}{1} = \binom{0}{0}$, see (4.13). On the other hand, $\hat{\sigma}(\omega_0^1) \neq 0$ is the matrix (4.16a), with $f_1 = 0$, $f_2 = -3$, $f_3 = 0$. Obviously, the matrix (4.16a) can be represented as $\hat{\sigma}(\omega^1)$, for $\omega^1 = \sum_{\alpha,z\geq 0}[\imath(a_\alpha^z), \dots [\imath(a_\alpha^0), \omega_0^1] \dots] \in \ker \hat{\pi}$. Therefore, each element of $\hat{\pi}(\jmath^2\mathfrak{a})$ is precisely of the form (4.16a), see (2.43b):

$$\hat{\sigma}(\Omega^1\mathfrak{a}) \equiv \hat{\pi}(\jmath^2\mathfrak{a}) \,. \tag{4.17}$$

Now, Proposition 7 and equation (2.42) imply for $n \geq 3$

$$\hat{\pi}(\jmath^n\mathfrak{a}) = \sum_{k=1}^{n-2} \underbrace{[\hat{\pi}(\Omega^1\mathfrak{a}), [\hat{\pi}(\Omega^1\mathfrak{a}), \dots [\hat{\pi}(\Omega^1\mathfrak{a}),}_{n-2-k} [\hat{\pi}(\jmath^2\mathfrak{a}), \hat{\pi}(\Omega^k\mathfrak{a})]_g]_g \dots]_g]_g \,. \tag{4.18}$$

Comparing the results (4.17) and (4.16a) with (4.15) we get

$$\begin{aligned}\{\tau^1, \tau^1\} = -(\|b_1\|^2 + \|b_2\|^2)\, \mathrm{diag}\,\big(&\mathbb{1}_3 \otimes M_{\{ud\}}, \mathbb{1}_3 \otimes M_{\{ud\}}, \mathbb{1}_3 \otimes 2M_u^* M_u, \mathbb{1}_3 \otimes 2M_d^* M_d, \\ &M_{\{\nu e\}}, M_{\{\nu e\}}, 2M_\nu^* M_\nu, 2M_e^* M_e\big) \mod \hat{\pi}(\jmath^2\mathfrak{a}) \,.\end{aligned} \tag{4.19}$$

It is clear that (4.19) is orthogonal to $\hat{\pi}(\jmath^2\mathfrak{a})$.

It is very instructive to compute both the commutators $\hat{\pi}(\Omega^3\mathfrak{a}) = [\hat{\pi}(\Omega^2\mathfrak{a}), \hat{\pi}(\Omega^1\mathfrak{a})]$ and $\hat{\pi}(\jmath^3\mathfrak{a}) = [\hat{\pi}(\jmath^2\mathfrak{a}), \hat{\pi}(\Omega^1\mathfrak{a})]$. One finds the remarkable result $\hat{\pi}(\Omega^3\mathfrak{a}) = \hat{\pi}(\jmath^3\mathfrak{a})$, which clearly extends to all higher orders:

$$\hat{\pi}(\Omega^n\mathfrak{a}) = \hat{\pi}(\jmath^n\mathfrak{a}) \,, \quad \text{for all } n \geq 3 \,. \tag{4.20}$$

Thus, $\Omega_D^n\mathfrak{a} \equiv 0$ for $n \geq 3$ and $\Omega_D^n\mathfrak{g} \equiv 0$ for $n \geq 7$. It is interesting that the highest non–vanishing space $\Omega_D^n\mathfrak{a}$ is $\Omega_D^2\mathfrak{a}$. If we also had $\Omega_D^2\mathfrak{a} = 0$, then the Higgs potential would not survive in our formulation.

4.2.4 The Structure of $\{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}$

We need the structure of $\{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}$ *for the computation of the ideal* $\pi(\jmath^2\mathfrak{g})$.

A simple calculation yields for elements of $\{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}$ the form

$\{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} \ni \mathrm{diag}(A_q + \Delta_q, A_\ell + \Delta_\ell)$, where

$$
A_q = \sum_\alpha \mathrm{i} \left(\begin{array}{cc|c|c}
\begin{bmatrix} (\frac{1}{3}\hat{\lambda}^0_\alpha + \lambda^3_\alpha + \lambda^0_\alpha) a_{3,\alpha} \\ + \frac{1}{3}\mathrm{i}\hat{\lambda}^3_\alpha \mathbb{1}_3 \end{bmatrix} & \begin{bmatrix} (\lambda^1_\alpha - \mathrm{i}\lambda^2_\alpha) a_{3,\alpha} + \\ \frac{1}{3}\mathrm{i}(\hat{\lambda}^1_\alpha - \mathrm{i}\hat{\lambda}^2_\alpha)\mathbb{1}_3 \end{bmatrix} & 0 & 0 \\
\begin{bmatrix} (\lambda^1_\alpha + \mathrm{i}\lambda^2_\alpha) a_{3,\alpha} + \\ \frac{1}{3}\mathrm{i}(\hat{\lambda}^1_\alpha + \mathrm{i}\hat{\lambda}^2_\alpha)\mathbb{1}_3 \end{bmatrix} & \begin{bmatrix} (\frac{1}{3}\hat{\lambda}^0_\alpha - \lambda^3_\alpha + \lambda^0_\alpha) a_{3,\alpha} \\ -\frac{1}{3}\mathrm{i}\hat{\lambda}^3_\alpha \mathbb{1}_3 \end{bmatrix} & 0 & 0 \\ \hline
0 & 0 & (\lambda^0_\alpha + \frac{4}{3}\hat{\lambda}^0_\alpha) a_{3,\alpha} & 0 \\ \hline
0 & 0 & 0 & (\lambda^0_\alpha - \frac{2}{3}\hat{\lambda}^0_\alpha) a_{3,\alpha}
\end{array}\right) \otimes \mathbb{1}_3 ,
$$

$$
A_\ell = \sum_\alpha \mathrm{i} \left(\begin{array}{cc|c|c}
-\mathrm{i}\hat{\lambda}^3_\alpha \mathbb{1}_3 ; & -\mathrm{i}(\hat{\lambda}^1_\alpha - \mathrm{i}\hat{\lambda}^2_\alpha)\mathbb{1}_3 & 0 & 0 \\
-\mathrm{i}(\hat{\lambda}^1_\alpha + \mathrm{i}\hat{\lambda}^2_\alpha)\mathbb{1}_3 & \mathrm{i}\hat{\lambda}^3_\alpha \mathbb{1}_3 & 0 & 0 \\ \hline
0 & 0 & 0_3 & 0 \\ \hline
0 & 0 & 0 & 0_3
\end{array}\right) ,
$$

$$
\begin{aligned}
\Delta_q &= \mathrm{diag}\big((\lambda+\tilde{\lambda}+\tfrac{1}{9}\hat{\lambda})\mathbb{1}_3, (\lambda+\tilde{\lambda}+\tfrac{1}{9}\hat{\lambda})\mathbb{1}_3, (\lambda+\tfrac{16}{9}\hat{\lambda})\mathbb{1}_3, (\lambda+\tfrac{4}{9}\hat{\lambda})\mathbb{1}_3\big) \otimes \mathbb{1}_3 , \\
\Delta_\ell &= \mathrm{diag}\big((\tilde{\lambda}+\hat{\lambda})\mathbb{1}_3 , (\tilde{\lambda}+\hat{\lambda})\mathbb{1}_3 , 0_3 , 4\hat{\lambda}\mathbb{1}_3\big) ,
\end{aligned} \tag{4.21}
$$

with $a_{3,\alpha} \in \mathrm{su}(3)$ and $\lambda^0_\alpha, \lambda^1_\alpha, \lambda^2_\alpha, \lambda^3_\alpha, \hat{\lambda}^0_\alpha, \hat{\lambda}^1_\alpha, \hat{\lambda}^2_\alpha, \hat{\lambda}^3_\alpha, \lambda, \tilde{\lambda}, \hat{\lambda} \in \mathbb{R}$.

4.2.5 The Structure of the Connection Form

Now we must solve equations (3.115), (3.116) *and* (3.117) *in order to find the structure of the connection form. This calculation has no analogue in non–commutative geometry. The result is simply* $\rho \in ((\Lambda^0 \otimes \hat{\pi}(\mathfrak{a})) \oplus (\Lambda^0\gamma^5 \otimes \hat{\pi}(\Omega^1\mathfrak{a})))$.

In order to write down the structure of the connection form we must find the spaces $\mathbb{r}^0\mathfrak{a}$ and $\mathbb{r}^1\mathfrak{a}$, see (3.115). From (3.116a) we get for elements $\eta^0 \in \mathbb{r}^0\mathfrak{a}$ the structure

$$
\eta^0 = \pi(a) + \mathrm{i}\,\mathrm{diag}(\mathbb{1}_3 \otimes m_1 , \mathbb{1}_3 \otimes m_1 , \mathbb{1}_3 \otimes m_2 , \mathbb{1}_3 \otimes m_3 , m_4 , m_4 , m_5 , m_6), \tag{4.22}
$$

where $a \in \mathfrak{a}$ and m_i are selfadjoint elements of $\mathrm{M}_3\mathbb{C}$. We put $m_5 \equiv 0$ for $M_\nu = 0$. We insert this structure into (3.116b) and (3.116c) and find with (4.10), (4.13a) and (4.21) the equations

$$
\begin{aligned}
m_1 M_u - M_u m_2 &= -\mathrm{i}\bar{\alpha} M_u , & m_1 M_d - M_d m_3 &= -\mathrm{i}\alpha M_d , \\
m_4 M_\nu - M_\nu m_5 &= -\mathrm{i}\bar{\alpha} M_\nu , & m_4 M_e - M_e m_6 &= -\mathrm{i}\alpha M_e ,
\end{aligned} \tag{4.23a}
$$

$$
\begin{aligned}
\tfrac{1}{3} m_1 &= \beta(\lambda + \tilde{\lambda} + \tfrac{1}{9}\hat{\lambda})\mathbb{1}_3 + \gamma M_{ud} , & -m_4 &= \beta(\tilde{\lambda} + \hat{\lambda})\mathbb{1}_3 + \gamma M_{\nu e} , \\
\tfrac{4}{3} m_2 &= \beta(\lambda + \tfrac{16}{9}\hat{\lambda})\mathbb{1}_3 + 2\gamma M_u^* M_u , & 0 &= 2\gamma M_\nu^* M_\nu , \\
-\tfrac{2}{3} m_3 &= \beta(\lambda + \tfrac{4}{9}\hat{\lambda})\mathbb{1}_3 + 2\gamma M_d^* M_d , & -2m_6 &= 4\beta\hat{\lambda}\mathbb{1}_3 + 2\gamma M_e^* M_e ,
\end{aligned} \tag{4.23b}
$$

for $\alpha \in \mathbb{C}$ and $\beta, \gamma \in \mathbb{R}$. Multiplying the first equation (4.23a) by M_u^* from the right and replacing $m_2 M_u^*$ by the conjugate of the first equation we get

$$
[m_1, M_u M_u^*] = -\mathrm{i}(\alpha + \bar{\alpha}) M_u M_u^* .
$$

Applying the trace and respecting $\operatorname{tr}(M_u M_u^*) > 0$ we get $\alpha = \mathrm{i}\delta$, for $\delta \in \mathbb{R}$. If $M_\nu \neq 0$ we clearly have $\gamma = 0$. For $M_\nu = 0$ we insert the equations for m_4 and m_6 in (4.23b) into the last equation (4.23a) and get

$$\beta(\hat{\lambda} - \tilde{\lambda}) M_e + 2\gamma M_e M_e^* M_e = \delta M_e \; .$$

For a mass matrix M_e that is not degenerate, this equation is only compatible with $\gamma = 0$. A different way would be to use the identity $[m_1, M_u M_u^*] = [m_1, M_d M_d^*]$. Since generically $M_u M_u^*$ and $M_d M_d^*$ are not diagonalizable at the same time, we conclude that m_1 is proportional to the identity matrix and $\gamma = 0$. Then, one gets for the system (4.23) the solution

$$\begin{aligned} m_1 &= \tfrac{1}{3}\delta \mathbb{1}_3 \,, & m_2 &= \tfrac{4}{3}\delta \mathbb{1}_3 \,, & m_3 &= -\tfrac{2}{3}\delta \mathbb{1}_3 \,, \\ m_4 &= -\delta \mathbb{1}_3 \,, & & & m_6 &= -2\delta \mathbb{1}_3 \,, \end{aligned} \tag{4.24}$$

and $\hat{\lambda} = \frac{\beta}{\delta}$, $\tilde{\lambda} = \lambda = 0$. For $M_\nu \neq 0$ we get $M_\nu M_5 = 0$ and, therefore, $m_5 = 0$ if M_ν is invertible. Since (3.116d) is now trivially satisfied, we come to the conclusion

$$\mathbb{r}^0 \mathfrak{a} \equiv \hat{\pi}(\mathfrak{a}) \; . \tag{4.25}$$

Next we compute $\mathbb{r}^1 \mathfrak{a}$. Formula (3.116e) yields with (4.10) the block structure

$$\mathbb{r}^1 \mathfrak{a} = \hat{\pi}(\Omega^1 \mathfrak{a}) + \mathrm{i}\, \operatorname{diag}(\mathbb{1}_3 \otimes m_1 \,,\, \mathbb{1}_3 \otimes m_1 \,,\, \mathbb{1}_3 \otimes m_2 \,,\, \mathbb{1}_3 \otimes m_3 \,, m_4 \,, m_4 \,, m_5 \,, m_6) , \tag{4.26}$$

where m_i are selfadjoint elements of $\mathrm{M}_3\mathbb{C}$. However, for elements $\eta^1 \in \mathbb{r}^1 \mathfrak{a}$ we have $\hat{\Gamma}\eta^1 + \eta^1 \hat{\Gamma} = 0$, which can only be fulfilled for $m_i \equiv 0$, see (4.12). Therefore,

$$\mathbb{r}^1 \mathfrak{a} \equiv \hat{\pi}(\Omega^1 \mathfrak{a}) \; . \tag{4.27}$$

It is clear that (3.117) is fulfilled.

4.2.6 The Ideal $\mathbb{J}^2\mathfrak{g}$

It remains find the ideal (3.121a). *The calculation for $M_\nu \neq 0$ and $M_\nu = 0$ differs considerably.*

For invertible matrices $M_{u,d,\nu,e}$, any non–vanishing element of $\hat{\pi}(\Omega^1 \mathfrak{a})$ is invertible, too. Therefore, the first two lines in (3.121a) yield

$$\mathbb{c}^0 \mathfrak{a} = 0 \,, \qquad\qquad \mathbb{c}^1 \mathfrak{a} = 0 \; . \tag{4.28}$$

For $M_\nu = 0$ we must proceed differently. It is possible to find $a_2 \in \mathrm{su}(2)$ and $\tau^1 \in \hat{\pi}(\Omega^1 \mathfrak{a})$ such that $\hat{\pi}(a_2) + \tau^1$ is invertible after omitting the row and column acting on ν_R. Now we conclude the same result (4.28) for $M_\nu = 0$.

The first equation of the last line in (3.121a) has the solution

$$\mathfrak{c}^2\mathfrak{a} \ni \mathrm{diag}(\mathbb{1}_3 \otimes m_1\,,\, \mathbb{1}_3 \otimes m_1\,,\, \mathbb{1}_3 \otimes m_2\,,\, \mathbb{1}_3 \otimes m_3\,,\, m_4\,,\, m_4\,,\, m_5\,,\, m_6)\,,$$

where m_i are selfadjoint elements of $\mathrm{M}_3\mathbb{C}$. We put $m_5 \equiv 0$ for $M_\nu = 0$. Then, the very last equation in (3.121a) yields

$$\begin{aligned} m_1 M_u - M_u m_2 &= 0\,, & m_1 M_d - M_d m_3 &= 0\,, \\ m_4 M_\nu - M_\nu m_5 &= 0\,, & m_4 M_e - M_e m_6 &= 0\,, \end{aligned} \tag{4.29}$$

from which we get that m_1 commutes with both $M_u M_u^*$ and $M_d M_d^*$ and m_4 commutes with both $M_\nu M_\nu^*$ and $M_e M_e^*$. Generically, $M_u M_u^*$ and $M_d M_d^*$ are not diagonalizable at the same time, which yields $m_1 = m_2 = m_3 = \tilde{\lambda}_1 \mathbb{1}_3$, for $\tilde{\lambda}_1 \in \mathbb{R}$. If $M_\nu \neq 0$ then we assume that also $M_\nu M_\nu^*$ and $M_e M_e^*$ are not diagonalizable at the same time, which yields $m_4 = m_5 = m_6 = \tilde{\lambda}_2 \mathbb{1}_3$, for $\tilde{\lambda}_2 \in \mathbb{R}$. Thus, we have with (3.121b)

$$\begin{aligned} \mathfrak{j}^2\mathfrak{a} &= \hat{\pi}(\jmath^2\mathfrak{a}) \oplus \big(\{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} + \mathrm{diag}(\mathbb{R}\mathbb{1}_{36}\,,\, \mathbb{R}\mathbb{1}_{12})\big) \\ &\ni J_2 \oplus \mathrm{diag}(A_q\,,\, A_\ell) \oplus \mathrm{diag}(J_q\,,\, J_\ell)\,, \\ J_q &= \mathrm{diag}((\lambda_1 + \tfrac{1}{9}\lambda_0)\mathbb{1}_3\,,\, (\lambda_1 + \tfrac{1}{9}\lambda_0)\mathbb{1}_3\,,\, (\lambda_1 + \lambda_3 + \tfrac{16}{9}\lambda_0)\mathbb{1}_3\,,\, (\lambda_1 + \lambda_3 + \tfrac{4}{9}\lambda_0)\mathbb{1}_3) \otimes \mathbb{1}_3\,, \\ J_\ell &= \mathrm{diag}((\lambda_2 + \lambda_0)\mathbb{1}_3\,,\, (\lambda_2 + \lambda_0)\mathbb{1}_3\,,\, (\lambda_2 + \lambda_3)\mathbb{1}_3\,,\, (\lambda_2 + \lambda_3 + 4\lambda_0)\mathbb{1}_3)\,, \end{aligned} \tag{4.30}$$

for $J_2 \in \hat{\pi}(\jmath^2\mathfrak{a})$ and $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. In the case $M_\nu = 0$, however, m_4 is arbitrary, and (4.30) must be replaced by

$$\begin{aligned} \mathfrak{j}^2\mathfrak{a} &= \hat{\pi}(\jmath^2\mathfrak{a}) \oplus \big(\{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} + \mathrm{diag}(\mathbb{R}\mathbb{1}_{36}\,,\, m_4, m_4, 0_3, M_e^{-1} m_4 M_e)\big) \\ &\ni J_2 \oplus \mathrm{diag}(A_q\,,\, A_\ell) \oplus \mathrm{diag}(J_q\,,\, J'_\ell)\,, \\ J'_\ell &= \mathrm{diag}(m_4 + \lambda_0 \mathbb{1}_3\,,\, m_4 + \lambda_0 \mathbb{1}_3\,,\, 0_3\,,\, M_e^{-1} m_4 M_e + (\lambda_3 + 4\lambda_0)\mathbb{1}_3)\,. \end{aligned} \tag{4.31}$$

4.2.7 The Factorization

In order to write down the bosonic action it is necessary to select the representative $\mathfrak{e}(\{\tau^1, \tau^1\})$ *of* $\{\tau^1, \tau^1\} + \mathfrak{j}^2\mathfrak{a}$ *orthogonal to* $\mathfrak{j}^2\mathfrak{a}$. *We solve this problem using computer algebra and quote only the result:*

For $M_\nu \neq 0$ let

$$\begin{aligned} \tilde{M}_{\{ud\}} &:= M_u M_u^* + M_d M_d^* - \tfrac{1}{3}\,\mathrm{tr}(M_u M_u^* + M_d M_d^*)\mathbb{1}_3\,, \\ \tilde{M}_{\{\nu e\}} &:= M_\nu M_\nu^* + M_e M_e^* - \tfrac{1}{3}\,\mathrm{tr}(M_\nu M_\nu^* + M_e M_e^*)\mathbb{1}_3\,, \\ \tilde{M}_{uu} &:= M_u^* M_u - \tfrac{1}{24}\,\mathrm{tr}(5M_u M_u^* + 3M_d M_d^* - M_\nu M_\nu^* + M_e M_e^*)\mathbb{1}_3\,, \\ \tilde{M}_{dd} &:= M_d^* M_d - \tfrac{1}{24}\,\mathrm{tr}(3M_u M_u^* + 5M_d M_d^* + M_\nu M_\nu^* - M_e M_e^*)\mathbb{1}_3\,, \\ \tilde{M}_{\nu\nu} &:= M_\nu^* M_\nu - \tfrac{1}{24}\,\mathrm{tr}(-3M_u M_u^* + 3M_d M_d^* + 7M_\nu M_\nu^* + M_e M_e^*)\mathbb{1}_3\,, \\ \tilde{M}_{ee} &:= M_e^* M_e - \tfrac{1}{24}\,\mathrm{tr}(3M_u M_u^* - 3M_d M_d^* + M_\nu M_\nu^* + 7M_e M_e^*)\mathbb{1}_3\,. \end{aligned} \tag{4.32a}$$

For $M_\nu = 0$ let

$$\begin{aligned}
\tilde{M}_{\{ud\}} &:= M_u M_u^* + M_d M_d^* - \tfrac{1}{60}\,\mathrm{tr}(17 M_u M_u^* + 23 M_d M_d^*)\mathbb{1}_3\,,\\
\tilde{M}_{\{\nu e\}} &:= -\tfrac{3}{20}\,\mathrm{tr}(M_d M_d^* - M_u M_u^*)\mathbb{1}_3\,,\\
\tilde{M}_{uu} &:= M_u^* M_u - \tfrac{1}{30}\,\mathrm{tr}(7 M_u M_u^* + 3 M_d M_d^*)\mathbb{1}_3\,,\\
\tilde{M}_{dd} &:= M_d^* M_d - \tfrac{1}{60}\,\mathrm{tr}(3 M_u M_u^* + 17 M_d M_d^*)\mathbb{1}_3\,,\\
\tilde{M}_{ee} &:= \tfrac{3}{20}\,\mathrm{tr}(M_d M_d^* - M_u M_u^*)\mathbb{1}_3\,, \qquad \tilde{M}_{\nu\nu} := 0_3\,.
\end{aligned} \tag{4.32b}$$

Then, the canonical embedding $\mathfrak{e}(\{\tau^1,\tau^1\})$ of $\{\tau^1,\tau^1\}$ into $\mathrm{M}_{48}\mathbb{C}$ is given by

$$\begin{aligned}
\mathfrak{e}(\{\tau^1,\tau^1\}) = -(\|b_1\|^2 + \|b_2\|^2)\,\mathrm{diag}\big(&\mathbb{1}_3\otimes\tilde{M}_{\{ud\}}, \mathbb{1}_3\otimes\tilde{M}_{\{ud\}}, \mathbb{1}_3\otimes 2\tilde{M}_{uu}, \mathbb{1}_3\otimes 2\tilde{M}_{dd},\\
&\tilde{M}_{\{\nu e\}}, \tilde{M}_{\{\nu e\}}, 2\tilde{M}_{\nu\nu}, 2\tilde{M}_{ee}\big)\,.
\end{aligned} \tag{4.33}$$

4.2.8 The Bosonic Action

Now we have collected all information to construct the bosonic action. We get the same unified form of the bosonic action as in non–commutative geometry.

We include the four dimensional Riemannian spin manifold X and choose a selfadjoint local basis $\{\gamma^\mu\}_{\mu=1,2,3,4}$ of Λ^1. Moreover, we denote the grading operator $\boldsymbol{\gamma}$ by $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$. The connection form ρ has due to (3.115), (4.10) and (4.13) the structure

$$\rho = \begin{pmatrix} \rho_q & 0 \\ 0 & \rho_\ell \end{pmatrix}\,, \tag{4.34}$$

$$\rho_q = \left(\begin{array}{cc|c|c}
(\mathbf{A} + \mathrm{i}(\tfrac{1}{3}A^0 + A^3)\mathbb{1}_3)\otimes\mathbb{1}_3; & \mathrm{i}(A^1 - \mathrm{i}A^2)\mathbb{1}_3\otimes\mathbb{1}_3 & -\mathrm{i}\gamma^5\overline{\Phi}_2\mathbb{1}_3\otimes M_u & -\mathrm{i}\gamma^5\Phi_1\mathbb{1}_3\otimes M_d\\
\mathrm{i}(A^1 + \mathrm{i}A^2)\mathbb{1}_3\otimes\mathbb{1}_3; & (\mathbf{A} + \mathrm{i}(\tfrac{1}{3}A^0 - A^3)\mathbb{1}_3)\otimes\mathbb{1}_3 & \mathrm{i}\gamma^5\overline{\Phi}_1\mathbb{1}_3\otimes M_u & -\mathrm{i}\gamma^5\Phi_2\mathbb{1}_3\otimes M_d\\
\hline
-\mathrm{i}\gamma^5\Phi_2\mathbb{1}_3\otimes M_u^* & \mathrm{i}\gamma^5\Phi_1\mathbb{1}_3\otimes M_u^* & (\mathbf{A} + \tfrac{4}{3}\mathrm{i}A^0\mathbb{1}_3)\otimes\mathbb{1}_3 & 0\\
\hline
-\mathrm{i}\gamma^5\overline{\Phi}_1\mathbb{1}_3\otimes M_d^* & -\mathrm{i}\gamma^5\overline{\Phi}_2\mathbb{1}_3\otimes M_d^* & 0 & (\mathbf{A} - \tfrac{2}{3}\mathrm{i}A^0\mathbb{1}_3)\otimes\mathbb{1}_3
\end{array}\right),$$

$$\rho_\ell = \left(\begin{array}{cc|c|c}
\mathrm{i}(-A^0 + A^3)\otimes\mathbb{1}_3 & \mathrm{i}(A^1 - \mathrm{i}A^2)\otimes\mathbb{1}_3 & -\mathrm{i}\gamma^5\overline{\Phi}_2\otimes M_\nu & -\mathrm{i}\gamma^5\Phi_1\otimes M_e\\
\mathrm{i}(A^1 + \mathrm{i}A^2)\otimes\mathbb{1}_3 & \mathrm{i}(-A^0 - A^3)\otimes\mathbb{1}_3 & \mathrm{i}\gamma^5\overline{\Phi}_1\otimes M_\nu & -\mathrm{i}\gamma^5\Phi_2\otimes M_e\\
\hline
-\mathrm{i}\gamma^5\Phi_2\otimes M_\nu^* & \mathrm{i}\gamma^5\Phi_1\otimes M_\nu^* & 0_3 & 0\\
\hline
-\mathrm{i}\gamma^5\overline{\Phi}_1\otimes M_e^* & -\mathrm{i}\gamma^5\overline{\Phi}_2\otimes M_e^* & 0 & -2\mathrm{i}A^0\otimes\mathbb{1}_3
\end{array}\right),$$

where $\mathbf{A} \in \Lambda^1\otimes\mathrm{su}(3)$, $\tilde{\mathbf{A}} := \begin{pmatrix} \mathrm{i}A^3; & \mathrm{i}(A^1 - \mathrm{i}A^2)\\ \mathrm{i}(A^1 + \mathrm{i}A^2); & -\mathrm{i}A^3 \end{pmatrix} \in \Lambda^1\otimes\mathrm{su}(2)$, $\mathrm{i}A^0 \in \Lambda^1\otimes\mathrm{u}(1)$ and $\Phi_1, \Phi_2 \in \Lambda^0\otimes\mathbb{C}$. From Proposition 9 and (3.108) we obtain with $\hat{\sigma}(\omega^1) = 0$ mod $\hat{\pi}(\jmath^2\mathfrak{a})$ the formula for the curvature

$$\theta = (\mathbf{d}\rho + \tfrac{1}{2}\{\rho - \mathrm{i}\gamma^5\otimes\mathcal{M}\,, \rho - \mathrm{i}\gamma^5\otimes\mathcal{M}\,\} + \tfrac{1}{2}\{\mathcal{M}\,, \mathcal{M}\,\}) \quad \mathrm{mod}\ \Lambda^0\otimes\mathbb{j}^2\mathfrak{a}\,. \tag{4.35}$$

Then, inserting (4.34) and using (4.32) we get from (3.128a), combined with an appropriate normalization factor, the bosonic action as

$$S_B = \tfrac{1}{48g_0^2}\int_X \mathrm{v}_g\ \mathrm{tr}_c(\mathfrak{e}(\theta)\,\mathfrak{e}(\theta)) = \int_X \mathrm{v}_g\,(\mathscr{L}_2+\mathscr{L}_1+\mathscr{L}_0)\,, \tag{4.36a}$$

$$\mathscr{L}_2 = \tfrac{1}{4g_0^2}\,\mathrm{tr}_c((\mathbf{dA}+\tfrac12\{\mathbf{A},\mathbf{A}\})^2)+\tfrac{1}{4g_0^2}\,\mathrm{tr}_c((\mathbf{d}\tilde{\mathbf{A}}+\tfrac12\{\tilde{\mathbf{A}},\tilde{\mathbf{A}}\})^2)+\tfrac{5}{6g_0^2}\,\mathrm{tr}_c((\mathbf{d}\,\mathrm{i}A^0)^2)\,, \tag{4.36b}$$

$$\begin{aligned}\mathscr{L}_1 = \tfrac{1}{24g_0^2}\,\mathrm{tr}_c(\,&|\mathbf{d}\Phi_1+\mathrm{i}(A^0+A^3)\Phi_1+\mathrm{i}(A^1-\mathrm{i}A^2)(\Phi_2+1)|^2+\\ &+|\mathbf{d}\Phi_2+\mathrm{i}(A^0-A^3)(\Phi_2+1)+\mathrm{i}(A^1+\mathrm{i}A^2)\Phi_1|^2)\times\\ &\times\mathrm{tr}(3M_uM_u^*+3M_dM_d^*+M_\nu M_\nu^*+M_eM_e^*)\,,\end{aligned} \tag{4.36c}$$

$$\begin{aligned}\mathscr{L}_0 = \tfrac{1}{192g_0^2}(|\Phi_1|^2+|\Phi_2+1|^2-1)^2\,\mathrm{tr}_c(1)\times&\\ \times\mathrm{tr}(6\tilde{M}^2_{\{ud\}}+12\tilde{M}^2_{uu}+12\tilde{M}^2_{dd}+2\tilde{M}^2_{\{\nu e\}}+4\tilde{M}^2_{\nu\nu}+4\tilde{M}^2_{ee})\,.&\end{aligned} \tag{4.36d}$$

4.2.9 Tree–Level Predictions

Our tree–level predictions hold for the simplest scalar product. In our model, general scalar products as discussed within NCG for instance in [35] are possible as well. They correspond to different weights of quark and lepton contributions to the entire bosonic action. However, our matter was to demonstrate that our approach is applicable to the standard model and not to find the most general form of predictions. In any case, it is an open question how the tree–level predictions are modified by quantum corrections. We do not have many arguments yet against the point of view that these "predictions" have no value at all. There are two facts which support the hope that the tree–level predictions are not completely wrong. First, the relation obtained between the mass of the top quark and the mass of the W boson is fairly well satisfied. Second, there exist examples where relations between parameters that are not stemming from a symmetry of the theory are respected on quantum level (reduction of couplings, see [52, 53, 60, 69]). The application of that method to the parameter predictions of the standard model will be the subject of forthcoming investigations. Future will tell.

We perform the reparametrizations

$$\begin{aligned}&\mathbf{A}=\textstyle\sum_{a=1}^8 \tfrac{\mathrm{i}g_0}{2}G^a_\mu\gamma^\mu\otimes\lambda^a\,, \qquad \tilde{\mathbf{A}}=\textstyle\sum_{a=1}^3 \tfrac{\mathrm{i}g_0}{2}W^a_\mu\gamma^\mu\otimes\sigma^a\,, \qquad \mathrm{i}A^0=\tfrac{\mathrm{i}g_0}{2}\sqrt{\tfrac35}W^0_\mu\gamma^\mu\,,\\ &\Phi_i=g_0\phi_i\,/\sqrt{\mathrm{tr}(M_uM_u^*+M_dM_d^*+\tfrac13 M_\nu M_\nu^*+\tfrac13 M_eM_e^*)}\,,\quad i=1,2\,,\end{aligned} \tag{4.37}$$

where $\{\sigma^a\}$ are the Pauli matrices and $\{\lambda^a\}$ the Gell-Mann matrices, see Appendix C. Using (C.3), (C.11) and (C.6) and performing a Wick rotation to Minkowski space we obtain for (4.36) precisely the bosonic action of the standard model, see [28]. Here, the Weinberg angle θ_W and the masses m_W, m_Z and m_H of the W, Z and Higgs bosons are given by

$$
\begin{aligned}
m_W &= \tfrac{1}{2}\sqrt{\operatorname{tr}(M_u M_u^* + M_d M_d^* + \tfrac{1}{3} M_\nu M_\nu^* + \tfrac{1}{3} M_e M_e^*)} = \tfrac{1}{2} m_t \;, \\
m_Z &= m_W / \cos\theta_W \;, \qquad \sin^2\theta_W = \tfrac{3}{8} \;, \\
m_H &= \sqrt{\frac{\operatorname{tr}(\tilde{M}^2_{\{ud\}} + 2\tilde{M}^2_{uu} + 2\tilde{M}^2_{dd} + \tfrac{1}{3}\tilde{M}^2_{\{\nu e\}} + \tfrac{2}{3} M^2_{\nu\nu} + \tfrac{2}{3} M_{ee})}{\operatorname{tr}(M_u M_u^* + M_d M_d^* + \tfrac{1}{3} M_\nu M_\nu^* + \tfrac{1}{3} M_e M_e^*)}} \\
&= \begin{cases} \tfrac{3}{2} m_t & \text{for } M_\nu \neq 0 \;, \\ \sqrt{\tfrac{43}{20}}\, m_t & \text{for } M_\nu = 0 \;, \end{cases}
\end{aligned}
\tag{4.38}
$$

where m_t is the mass of the top quark. We have neglected the other fermion masses against m_t. The analogous relations in non–commutative geometry read for the simplest scalar product [44]

$$
m_W = \tfrac{1}{2} m_t \;, \qquad m_H = \sqrt{\tfrac{69}{28}}\, m_t \;, \qquad \sin^2\theta_W = \tfrac{12}{29} \;. \tag{4.39}
$$

Inserting (4.34) and (4.37) into the fermionic action (3.128b), we arrive after a Wick rotation to Minkowski space and imposing the chirality condition $\Gamma h = h$ at the usual fermionic action of the standard model [28].

5 The Matrix Part of the Unification Model

Sections 5, 6 and 7 are devoted to the formulation of the flipped $\mathrm{SU}(5)\times\mathrm{U}(1)$*–Grand Unification model. The Lie algebra* $\mathfrak{g}$ *of the L–cycle* $(\mathfrak{g}, h, D, \pi, \Gamma)$ *is the tensor product of the function algebra and a matrix Lie algebra* $\mathfrak{a}$ *. We have learned in Section 3 that the structure of the relevant differential Lie algebra* $\Omega_D^*\mathfrak{g}$ *depends crucially on properties of the matrix L–cycle* $(\mathfrak{a}, \mathbb{C}^F, \mathcal{M}, \hat{\pi}, \hat{\Gamma})$*; the space–time part is almost trivial. Therefore, we start in Section 5 with the investigation of the matrix L–cycle and of those matrix structures, which we need to formulate our model.*

5.1 Representations of the Lie Algebra under Consideration

The matrix Lie algebra of our L–cycle is the simple Lie algebra $\mathfrak{a} = \mathrm{su}(5)$ *. Nevertheless, our formalism generates a* $\mathrm{u}(1)$*–part for the connection form and for gauge transformations. To fix notations, let us say first a few words on those representations of* $\mathrm{su}(5)$*, which we need for the construction.*

5.1.1 A Basis of su(5)

Let B_{ij} be the 5×5–matrix, whose entry at the intersection of the i-th row with the j-th column is 1 and whose all other entries are zero. Then we choose the matrices

$$\begin{aligned}
&\mathrm{i}(B_{ij}+B_{ji})\,, \quad (B_{ij}-B_{ji})\,, && i,j=1,\ldots,5\,,\ i\neq j\,,\\
&Y' = \mathrm{i}\,\mathrm{diag}\,(-\tfrac{2}{5},-\tfrac{2}{5},-\tfrac{2}{5},\tfrac{3}{5},\tfrac{3}{5})\,, && T_8 = \mathrm{i}\,\mathrm{diag}\,(\tfrac{1}{3},\tfrac{1}{3},-\tfrac{2}{3},0,0)\,,\\
&T_3 = \mathrm{i}\,\mathrm{diag}\,(\tfrac{1}{2},-\tfrac{1}{2},0,0,0)\,, && I_3 = \mathrm{i}\,\mathrm{diag}\,(0,0,0,\tfrac{1}{2},-\tfrac{1}{2})
\end{aligned} \tag{5.1}$$

as a basis of $\mathrm{su}(5)$, where $\{Y', T_8, T_3, I_3\}$ is a basis of the commutative Cartan subalgebra of $\mathrm{su}(5)$.

5.1.2 Representations of su(5)

Next, we consider irreducible representations of su(5) in the Hilbert space $\mathbb{C}^n$. As usual we label these representations by $\underline{n}$ and their dual representations by $\underline{n}^*$. For such a representation there exists a system of n orthogonal eigenvectors of elements of the Cartan subalgebra. The eigenvalues of $\{Y', T_3, T_8, I_3\}$ belong to $\mathrm{i}\mathbb{R}$. We label the imaginary part of the eigenvalues by small letters, i.e. $\mathrm{i}y'$ is the eigenvalue of Y' and so on. Moreover, we label these n eigenvectors by $|y', t_8, t_3, i_3\rangle$. We consider the following six irreducible representations of su(5) :

$$\underline{10}\,,\ \underline{5}^*\,,\ \underline{1}\,,\ \underline{10}^*\,,\ \underline{5}\,,\ \underline{1}^*\,. \tag{5.2}$$

Below we list the eigenvectors $|y', t_8, t_3, i_3\rangle$ of these representations and label them by the fundamental fermions of the first fermion generation:

$$\begin{array}{rllll}
\underline{1} & e_R = |0,0,0,0\rangle\,, & & & \\
\underline{5} & u_R^r = |-\frac{2}{5},\frac{1}{3},\frac{1}{2},0\rangle\,, & u_R^b = |-\frac{2}{5},\frac{1}{3},-\frac{1}{2},0\rangle\,, & u_R^g = |-\frac{2}{5},-\frac{2}{3},0,0\rangle\,, & \\
 & e_L = |-\frac{3}{5},0,0,\frac{1}{2}\rangle\,, & \nu_L = |-\frac{3}{5},0,0,-\frac{1}{2}\rangle\,, & & \\
\underline{5}^* & \text{opposite signs of the eigenvalues of } \underline{5}\,, & & & (5.3) \\
\underline{10} & u_L^r = |\frac{1}{5},\frac{1}{3},\frac{1}{2},\frac{1}{2}\rangle\,, & u_L^b = |\frac{1}{5},\frac{1}{3},-\frac{1}{2},\frac{1}{2}\rangle\,, & u_L^g = |\frac{1}{5},-\frac{2}{3},0,\frac{1}{2}\rangle\,, & \\
 & d_L^r = |\frac{1}{5},\frac{1}{3},\frac{1}{2},-\frac{1}{2}\rangle\,, & d_L^b = |\frac{1}{5},\frac{1}{3},-\frac{1}{2},-\frac{1}{2}\rangle\,, & d_L^g = |\frac{1}{5},-\frac{2}{3},0,-\frac{1}{2}\rangle\,, & \\
 & d_R^r = |-\frac{4}{5},-\frac{1}{3},-\frac{1}{2},0\rangle\,, & d_R^b = |-\frac{4}{5},-\frac{1}{3},\frac{1}{2},0\rangle\,, & d_R^g = |-\frac{4}{5},\frac{2}{3},0,0\rangle\,, & \\
 & \nu_R = |\frac{6}{5},0,0,0\rangle\,, & & & \\
\underline{10}^* & \text{opposite signs of the eigenvalues of } \underline{10}\,. & & &
\end{array}$$

Here, u and d stand for the u– and d–quarks, e for the electron and ν for the neutrino. The subscripts R and L refer to right–handed and left–handed helicity. The superscripts r, b, g of the quarks stand for their colours "red, blue, green".

5.1.3 The Decomposition Rules

For the further calculation it is necessary to know how homomorphisms between the representations (5.2) decompose into irreducible su(5)–representations:

$$\begin{array}{llllr}
\mathrm{End}(\underline{10}) & = \mathrm{End}(\underline{10}^*) & = \underline{10} \otimes \underline{10}^* & = \underline{1} \oplus \underline{24} \oplus \underline{75} & (5.4a) \\
\mathrm{End}(\underline{5}) & = \mathrm{End}(\underline{5}^*) & = \underline{5} \otimes \underline{5}^* & = \underline{1} \oplus \underline{24} & (5.4b) \\
\mathrm{End}(\underline{1}) & & & = \underline{1} & (5.4c) \\
\mathrm{Hom}(\underline{5},\underline{10}) & = \mathrm{Hom}(\underline{10}^*,\underline{5}^*) & = \underline{5}^* \otimes \underline{10} & = \underline{5} \oplus \underline{45}^* & (5.4d) \\
\mathrm{Hom}(\underline{5},\underline{10}^*) & = \mathrm{Hom}(\underline{10},\underline{5}^*) & = \underline{5}^* \otimes \underline{10}^* & = \underline{10} \oplus \underline{40}^* & (5.4e) \\
\mathrm{Hom}(\underline{5}^*,\underline{5}) & & = \underline{5} \otimes \underline{5} & = \underline{10} \oplus \underline{15} & (5.4f) \\
\mathrm{Hom}(\underline{10}^*,\underline{10}) & & = \underline{10} \otimes \underline{10} & = \underline{5}^* \oplus \underline{45} \oplus \underline{50} & (5.4g) \\
\mathrm{Hom}(\underline{1},\underline{5}) & = \mathrm{Hom}(\underline{5}^*,\underline{1}) & & = \underline{5} & (5.4h) \\
\mathrm{Hom}(\underline{1},\underline{10}) & = \mathrm{Hom}(\underline{10}^*,\underline{1}) & & = \underline{10} & (5.4i)
\end{array}$$

5.1.4 The $\underline{24}$–Representation

Now we continue the specification of our L–cycle. The $\underline{24}$*–representation yields the representation* $\hat{\pi}$ *of the matrix L–cycle, because the* su(5) *has 24 generators.*

Using the notation just introduced we consider the following matrix representation of elements $a \in \mathfrak{a}$:

$$
\begin{aligned}
a = \mathrm{i}(&f_1 Y' + f_2 I_3 + f_3 T_3 + f_8 T_8 + g_0 B_{45} + \bar{g}_0 B_{54} + g_1 B_{12} + \bar{g}_1 B_{21} \\
&+ g_2 B_{13} + \bar{g}_2 B_{31} + g_3 B_{23} + \bar{g}_3 B_{32} + g_4 B_{14} + \bar{g}_4 B_{41} + g_5 B_{15} + \bar{g}_5 B_{51} \\
&+ g_6 B_{24} + \bar{g}_6 B_{42} + g_7 B_{25} + \bar{g}_7 B_{52} + g_8 B_{34} + \bar{g}_8 B_{43} + g_9 B_{35} + \bar{g}_9 B_{53}) \\
= \mathrm{i}&\left(\begin{array}{ccc|c|c}
-\frac{2}{5} f_1 + \frac{1}{2} f_3 + \frac{1}{3} f_8 & g_1 & g_2 & g_4 & g_5 \\
\bar{g}_1 & -\frac{2}{5} f_1 - \frac{1}{2} f_3 + \frac{1}{3} f_8 & g_3 & g_6 & g_7 \\
\bar{g}_2 & \bar{g}_3 & -\frac{2}{5} f_1 - \frac{2}{3} f_8 & g_8 & g_9 \\
\hline
\bar{g}_4 & \bar{g}_6 & \bar{g}_8 & \frac{3}{5} f_1 + \frac{1}{2} f_2 & g_0 \\
\hline
\bar{g}_5 & \bar{g}_7 & \bar{g}_9 & \bar{g}_0 & \frac{3}{5} f_1 - \frac{1}{2} f_2
\end{array}\right),
\end{aligned}
\tag{5.5}
$$

where $f_i \in \mathbb{R}$ and $g_j \in \mathbb{C}$. The real numbers f_i, $i = 1,2,3,8$, are associated in this order to the generators of the Cartan subalgebra Y', I_3, T_3, T_8 of su(5). The natural involution * on $\mathfrak{a}$ is given by $a^* = -a$ for all $a \in \mathfrak{a}$. We choose the Hilbert space $\mathbb{C}^F = \mathbb{C}^{192}$ and represent it as

$$
\mathbb{C}^{192} = \left(\underline{10} \oplus \underline{5}^* \oplus \underline{1} \oplus \underline{10}^* \oplus \underline{5} \oplus \underline{1}\right) \otimes \mathbb{C}^2 \otimes \mathbb{C}^3 . \tag{5.6}
$$

Using the identities (5.4) we decompose End($\mathbb{C}^{192}$) into irreducible representations of su(5), each of them tensorized by $\mathbb{C}^2 \otimes \mathbb{C}^3$. Since the Lie algebra su(5) has 24 generators, see also (5.5), we get a natural representation $\hat{\pi}$ of $\mathfrak{a}$ in End($\mathbb{C}^{192}$) by selecting the $\underline{24}$–representations in (5.4): Let

$$
\pi_5(a) = a \qquad \text{as in (5.5)}, \tag{5.7a}
$$

$$
\pi_{10}(a) = \tag{5.7b}
$$

$$
\mathrm{i}\left(\begin{array}{ccc|ccc|ccc|c}
\begin{bmatrix}\frac{1}{5} f_1 + \frac{1}{2} f_3 \\ + \frac{1}{3} f_8 + \frac{1}{2} f_2\end{bmatrix}; & g_1; & g_2 & g_0 & 0 & 0 & 0 & \bar{g}_8 & -\bar{g}_6 & -g_5 \\
\bar{g}_1; & \begin{bmatrix}\frac{1}{5} f_1 - \frac{1}{2} f_3 \\ + \frac{1}{3} f_8 + \frac{1}{2} f_2\end{bmatrix}; & g_3 & 0 & g_0 & 0 & -\bar{g}_8 & 0 & \bar{g}_4 & -g_7 \\
\bar{g}_2; & \bar{g}_3; & \begin{bmatrix}\frac{1}{5} f_1 - \frac{2}{3} f_8 \\ + \frac{1}{2} f_2\end{bmatrix} & 0 & 0 & g_0 & \bar{g}_6 & -\bar{g}_4 & 0 & -g_9 \\
\hline
\bar{g}_0 & 0 & 0 & \begin{bmatrix}\frac{1}{5} f_1 + \frac{1}{2} f_3 \\ + \frac{1}{3} f_8 - \frac{1}{2} f_2\end{bmatrix}; & g_1; & g_2 & 0 & \bar{g}_9 & -\bar{g}_7 & g_4 \\
0 & \bar{g}_0 & 0 & \bar{g}_1; & \begin{bmatrix}\frac{1}{5} f_1 - \frac{1}{2} f_3 \\ + \frac{1}{3} f_8 - \frac{1}{2} f_2\end{bmatrix}; & g_3 & -\bar{g}_9 & 0 & \bar{g}_5 & g_6 \\
0 & 0 & \bar{g}_0 & \bar{g}_2; & \bar{g}_3; & \begin{bmatrix}\frac{1}{5} f_1 - \frac{2}{3} f_8 \\ - \frac{1}{2} f_2\end{bmatrix} & \bar{g}_7 & -\bar{g}_5 & 0 & g_8 \\
\hline
0 & -g_8 & g_6 & 0 & -g_9 & g_7 & \begin{bmatrix}-\frac{4}{5} f_1 - \frac{1}{2} f_3 \\ - \frac{1}{3} f_8\end{bmatrix}; & -\bar{g}_1; & -\bar{g}_2 & 0 \\
g_8 & 0 & -g_4 & g_9 & 0 & -g_5 & -g_1; & \begin{bmatrix}-\frac{4}{5} f_1 + \frac{1}{2} f_3 \\ - \frac{1}{3} f_8\end{bmatrix}; & -\bar{g}_3 & 0 \\
-g_6 & g_4 & 0 & -g_7 & g_5 & 0 & -g_2; & -g_3; & \begin{bmatrix}-\frac{4}{5} f_1 + \frac{2}{3} f_8\end{bmatrix} & 0 \\
\hline
-\bar{g}_5 & -\bar{g}_7 & -\bar{g}_9 & \bar{g}_4 & \bar{g}_6 & \bar{g}_8 & 0 & 0 & 0 & \frac{6}{5} f_1
\end{array}\right).
$$

Then we define

$$\hat\pi(a) := \left(\begin{array}{ccc|ccc} \pi_{10}(a) & 0 & 0 & & & \\ 0 & \overline{\pi_5(a)} & 0 & & \mathrm{O} & \\ 0 & 0 & 0_{3\times 3} & & & \\ \hline & & & \overline{\pi_{10}(a)} & 0 & 0 \\ & \mathrm{O} & & 0 & \pi_5(a) & 0 \\ & & & 0 & 0 & 0_{3\times 3} \end{array}\right) \otimes \mathbb{1}_6 \,. \tag{5.7c}$$

5.1.5 The $\underline{75}$–Representation

The $\underline{75}$–representation occurs in anticommutators of elements of $\underline{24}$–representations. It is a part of the ideal.

We use the following definition of the $\underline{75}$–representation occurring in the decomposition (5.4a): Let $\mathfrak{v}$ be the set of 10×10–matrices defined by

$$\mathfrak{v} := \{\, v \in \mathrm{su}(10)\,,\ \ \mathrm{tr}(v\,\pi_{10}(a)) = 0 \ \ \forall a \in \mathfrak{a} \,\}\,. \tag{5.8}$$

For $v \in \mathfrak{v}$ we define

$$\hat\pi(v) = \left(\begin{array}{ccc|ccc} v & 0 & 0 & & & \\ 0 & 0 & 0 & & \mathrm{O} & \\ 0 & 0 & 0 & & & \\ \hline & & & \bar v & 0 & 0 \\ & \mathrm{O} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{array}\right) \otimes \mathbb{1}_6 \,. \tag{5.9a}$$

One has

$$[\hat\pi(a), \hat\pi(v)] = \hat\pi([\pi_{10}(a), v]) \in \hat\pi(\mathfrak{v})\,, \quad a \in \mathfrak{a}\,,\ \ v \in \mathfrak{v}\,. \tag{5.9b}$$

5.1.6 The $\underline{5}$–Representation

The $\underline{5}$–representation is responsible for the basic mass terms.

Next, we consider the $\underline{5}$–representations occurring on the r.h.s. of (5.4). Let $\mathfrak{b} = \mathbb{C}^5$ be the vector space of matrices represented in the form

$$b = \mathrm{i}(b_1, b_2, b_3, b_4, b_5)^T\,, \qquad b_i \in \mathbb{C}\,. \tag{5.10}$$

We define a linear map $\hat\pi$ of $\mathfrak{b}$ in $\mathrm{End}(\mathbb{C}^{192})$, putting

$$\hat\pi(b) = \begin{pmatrix} 0 & \hat\pi_0(b) \otimes \mathbb{1}_6 \\ -\hat\pi_0(b)^* \otimes \mathbb{1}_6 & 0 \end{pmatrix} \tag{5.11a}$$

$$= \left(\begin{array}{ccc|ccc} & & & \pi_{10,10}(b) & \pi_{10,5}(b) & 0 \\ & \mathrm{O} & & \pi_{10,5}(b)^T & 0 & \pi_{5,1}(b) \\ & & & 0 & \pi_{5,1}(b)^T & 0 \\ \hline -\pi_{10,10}(b)^* & -\overline{\pi_{10,5}(b)} & 0 & & & \\ -\pi_{10,5}(b)^* & 0 & -\overline{\pi_{5,1}(b)} & & \mathrm{O} & \\ 0 & -\pi_{5,1}(b)^* & 0 & & & \end{array}\right) \otimes \mathbb{1}_6 \,,$$

$$\hat\pi_0(b) = \hat\pi_0(b)^T = \tag{5.11b}$$

$$\mathrm{i}\left(\begin{array}{ccc|ccc|ccc|c||ccc|cc||c}
0 & 0 & 0 & 0 & \bar b_3 & -\bar b_2 & \bar b_5 & 0 & 0 & 0 & b_4 & 0 & 0 & -b_1 & 0 & 0 \\
0 & 0 & 0 & -\bar b_3 & 0 & \bar b_1 & 0 & \bar b_5 & 0 & 0 & 0 & b_4 & 0 & -b_2 & 0 & 0 \\
0 & 0 & 0 & \bar b_2 & -\bar b_1 & 0 & 0 & 0 & \bar b_5 & 0 & 0 & 0 & b_4 & -b_3 & 0 & 0 \\
\hline
0 & -\bar b_3 & \bar b_2 & 0 & 0 & 0 & -\bar b_4 & 0 & 0 & 0 & b_5 & 0 & 0 & 0 & -b_1 & 0 \\
\bar b_3 & 0 & -\bar b_1 & 0 & 0 & 0 & 0 & -\bar b_4 & 0 & 0 & 0 & b_5 & 0 & 0 & -b_2 & 0 \\
-\bar b_2 & \bar b_1 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar b_4 & 0 & 0 & 0 & b_5 & 0 & -b_3 & 0 \\
\hline
\bar b_5 & 0 & 0 & -\bar b_4 & 0 & 0 & 0 & 0 & 0 & \bar b_1 & 0 & -b_3 & b_2 & 0 & 0 & 0 \\
0 & \bar b_5 & 0 & 0 & -\bar b_4 & 0 & 0 & 0 & 0 & \bar b_2 & b_3 & 0 & -b_1 & 0 & 0 & 0 \\
0 & 0 & \bar b_5 & 0 & 0 & -\bar b_4 & 0 & 0 & 0 & \bar b_3 & -b_2 & b_1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \bar b_1 & \bar b_2 & \bar b_3 & 0 & 0 & 0 & 0 & b_5 & -b_4 & 0 \\
\hline\hline
b_4 & 0 & 0 & b_5 & 0 & 0 & 0 & b_3 & -b_2 & 0 & 0 & 0 & 0 & 0 & 0 & \bar b_1 \\
0 & b_4 & 0 & 0 & b_5 & 0 & -b_3 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & \bar b_2 \\
0 & 0 & b_4 & 0 & 0 & b_5 & b_2 & -b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar b_3 \\
\hline
-b_1 & -b_2 & -b_3 & 0 & 0 & 0 & 0 & 0 & 0 & b_5 & 0 & 0 & 0 & 0 & 0 & \bar b_4 \\
0 & 0 & 0 & -b_1 & -b_2 & -b_3 & 0 & 0 & 0 & -b_4 & 0 & 0 & 0 & 0 & 0 & \bar b_5 \\
\hline\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar b_1 & \bar b_2 & \bar b_3 & \bar b_4 & \bar b_5 & 0
\end{array}\right) .$$

The matrices $\pi_{10,10}(b), \pi_{10,5}(b)$ and $\pi_{5,1}(b)$ are the embeddings of $b \in \underline{5}$ into $\underline{10} \otimes \underline{10}$, $\underline{5}^* \otimes \underline{10}$ and $\underline{1} \otimes \underline{5}^*$, see (5.4). Observe that

$$[\hat\pi(a), \hat\pi(b)] = \hat\pi(ab) \in \hat\pi(\mathfrak{b}) \,, \qquad a \in \mathfrak{a} \,,\ b \in \mathfrak{b} \,. \tag{5.12}$$

Due to the first three formulae of (5.4), the $\underline{24}$– and $\underline{1}$–parts of $\pi_{i,j}(b)\pi_{i,j}(b)^*$, respectively, must be correlated. Indeed, we find with

$$(b,b)' := bb^* - \tfrac{1}{5}\,\mathrm{tr}(bb^*)\mathbb{1}_5 \in \mathrm{i}\mathfrak{a} \tag{5.13a}$$

the identities

$$\begin{aligned}
\pi_{10,10}(b)\pi_{10,10}(b)^* &= \mathrm{i}\pi_{10}(\mathrm{i}(b,b)') + \tfrac{3}{5}(b^*b)\mathbb{1}_{10}\ ,\\
\pi_{10,5}(b)\pi_{10,5}(b)^* &= -\mathrm{i}\pi_{10}(\mathrm{i}(b,b)') + \tfrac{2}{5}(b^*b)\mathbb{1}_{10}\ ,\\
\pi_{10,5}(b)^*\pi_{10,5}(b) &= \mathrm{i}\pi_{5}(\mathrm{i}(b,b)') + \tfrac{4}{5}(b^*b)\mathbb{1}_{5}\ ,\\
\overline{\pi_{5,1}(b)}\pi_{5,1}(b)^T &= -\mathrm{i}\pi_{5}(\mathrm{i}(b,b)') + \tfrac{1}{5}(b^*b)\mathbb{1}_{5}\ ,\\
\pi_{5,1}(b)^T\overline{\pi_{5,1}(b)} &= (b^*b)\ .
\end{aligned} \tag{5.13b}$$

5.1.7 The $\underline{45}$–Representation

If there was only the $\underline{5}$–representation then the (u,c,t)–quarks and the neutrinos would receive the same masses – which would be desaster. The $\underline{45}$–representation breaks this degeneracy.

Let us comment on the $\underline{45}^*$–representation of su(5) occurring in (5.4d). It is the vector space $\mathfrak{w}$ of 10×5–matrices determined by

$$\mathfrak{w} := \{\ w \in \mathrm{Hom}(\mathbb{C}^5,\mathbb{C}^{10})\ ,\ \ \mathrm{tr}(w\,\pi_{10,5}(b)^*) = 0\ \ \forall b \in \underline{5}\ \}\ . \tag{5.14a}$$

Elements $w \in \mathfrak{w}$ have in terms of submatrices the matrix structure

$$w = \mathrm{i}\begin{pmatrix} w_A & w_a + w_b & w_c \\ w_B & w_d & w_a - w_b \\ w_C - \varepsilon(w_a) & \overline{w_e} & \overline{w_f} \\ w_g^* & -\mathrm{tr}(w_B) & \mathrm{tr}(w_A) \end{pmatrix}\ . \tag{5.14b}$$

Here, $w_a,\dots,w_g$ are complex 3×1–column matrices, w_A and w_B are complex 3×3–matrices, $w_C = w_C^T$ is a complex symmetrical 3×3–matrix and

$$(\varepsilon(w))_{\alpha\beta} = \textstyle\sum_{\gamma=1}^{3} \varepsilon_{\alpha\beta\gamma} w_\gamma\ . \tag{5.14c}$$

We define two representations $\hat{\pi}$ and $\hat{\pi}'$ of $\mathfrak{w}$ in $\mathrm{M}_{192}\mathbb{C}$ by

$$\hat{\pi}(w) := \left(\begin{array}{ccc|ccc} & & & 0 & w & 0 \\ & \mathrm{O} & & w^T & 0 & 0 \\ & & & 0 & 0 & 0 \\ \hline 0 & -\overline{w} & 0 & & & \\ -w^* & 0 & 0 & & \mathrm{O} & \\ 0 & 0 & 0 & & & \end{array}\right) \otimes \mathbb{1}_6\ , \tag{5.15a}$$

$$\hat{\pi}'(w) := \left(\begin{array}{ccc|ccc} & & & \pi_{10,10}(w) & 0 & 0 \\ & \mathrm{O} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \hline -\pi_{10,10}(w)^* & 0 & 0 & & & \\ 0 & 0 & 0 & & \mathrm{O} & \\ 0 & 0 & 0 & & & \end{array}\right) \otimes \mathbb{1}_6 \,, \qquad (5.15b)$$

where

$$\pi_{10,10}(w) = \mathrm{i} \begin{pmatrix} \varepsilon(\overline{w_d}) & \overline{w_C} - \varepsilon(\overline{w_b}) & w_B^* - \mathrm{tr}(w_B^*)\mathbb{1}_3 & w_e \\ -\overline{w_C} - \varepsilon(\overline{w_b}) & -\varepsilon(\overline{w_c}) & -w_A^* + \mathrm{tr}(w_A^*)\mathbb{1}_3 & w_f \\ -\overline{w_B} + \mathrm{tr}(\overline{w_B})\mathbb{1}_3 & \overline{w_A} - \mathrm{tr}(\overline{w_A})\mathbb{1}_3 & -\varepsilon(w_g) & 2\overline{w_a} \\ -w_e^T & -w_f^T & -2w_a^* & 0 \end{pmatrix} . \qquad (5.16)$$

One finds the properties

$$\hat{\pi}(a)\hat{\pi}(w) - \hat{\pi}(w)\hat{\pi}(a) = \hat{\pi}\big(\pi_{10}(a)w - w\pi_5(a)\big) \in \hat{\pi}(\mathfrak{w}) \,, \qquad (5.17a)$$

$$\hat{\pi}(a)\hat{\pi}'(w) - \hat{\pi}'(w)\hat{\pi}(a) = \hat{\pi}'\big(\pi_{10}(a)w - w\pi_5(a)\big) \in \hat{\pi}'(\mathfrak{w}) \,, \quad a \in \mathfrak{a},\ w \in \mathfrak{w} \,. \qquad (5.17b)$$

5.1.8 The $\underline{50}$–Representation

The $\underline{50}$–representation is responsible for the Majorana masses of the right neutrinos. Moreover, it gives a high mass to a neutral gauge field and makes it unobservable.

Finally, we consider the $\underline{50}$–representation of su(5) occurring in (5.4g). It is the vector space $\mathfrak{c}$ of symmetric complex 10×10–matrices determined by

$$\mathfrak{c} := \{\, c \in \mathrm{M}_{10}\mathbb{C} \,,\ \ c = c^T \,,\ \ \mathrm{tr}(c\,\pi_{10,10}(b)^*) = 0 \ \ \forall b \in \underline{5} \,\} \,. \qquad (5.18a)$$

Elements $c \in \mathfrak{c}$ have in terms of submatrices the matrix structure

$$c = \mathrm{i} \begin{pmatrix} \overline{c_A} & \overline{c_D} - \frac{1}{2}\varepsilon(\overline{c_c}) & (c_E^0)^* & c_a \\ \overline{c_D} + \frac{1}{2}\varepsilon(\overline{c_c}) & \overline{c_B} & (c_F^0)^* & c_b \\ \overline{c_E^0} & \overline{c_F^0} & c_C & \overline{c_c} \\ c_a^T & c_b^T & c_c^* & c_0 \end{pmatrix} . \qquad (5.18b)$$

Here, c_a, c_b, c_c are complex 3×1–column matrices, $c_A, \ldots, c_D$ are complex symmetrical 3×3–matrices, c_E^0, c_F^0 are complex tracefree 3×3–matrices and $c_0 \in \mathbb{C}$. We define a representation $\hat{\pi}$ of $\mathfrak{c}$ in $\mathrm{M}_{192}\mathbb{C}$ by

$$\hat{\pi}(c) := \left(\begin{array}{ccc|ccc} & & & c & 0 & 0 \\ & \mathrm{O} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \hline -c^* & 0 & 0 & & & \\ 0 & 0 & 0 & & \mathrm{O} & \\ 0 & 0 & 0 & & & \end{array}\right) \otimes \mathbb{1}_6 \tag{5.19a}$$

and find the property

$$\hat{\pi}(a)\hat{\pi}(c) - \hat{\pi}(c)\hat{\pi}(a) = \hat{\pi}\big(\pi_{10}(a)c + c\pi_{10}(a)^T\big) \in \hat{\pi}(\mathfrak{c}) , \quad a \in \mathfrak{a},\ c \in \mathfrak{c} . \tag{5.19b}$$

5.2 The Remaining Ingredients of the L–Cycle

Here we define the generalized Dirac operator $\mathcal{M}$, which differs from the fermionic mass matrix by the matrix m responsible for breaking the Grand Unification symmetry down to the symmetry of the standard model.

5.2.1 The Generalized Dirac Operator $\mathcal{M}$

Let

$$m := Y' \in \mathfrak{a} , \qquad n := \mathrm{i}(0,0,0,1,0)^T \in \mathfrak{b} , \tag{5.20}$$

$$m' := \mathrm{i}\begin{pmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & -1 \end{pmatrix} \in \mathfrak{c} , \quad n' := \mathrm{i}\begin{pmatrix} \mathbb{1}_3 & 0_{3\times1} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times1} & 3 \end{pmatrix} \in \mathfrak{w} .$$

Then we put

$$\mathcal{M} = \left(\begin{array}{ccc|ccc} \mathcal{M}_{10} & 0 & 0 & \mathcal{M}_{10,10} & \mathcal{M}_{10,5} & 0 \\ 0 & \overline{\mathcal{M}_5} & 0 & \mathcal{M}_{10,5}^T & 0 & \mathcal{M}_{5,1} \\ 0 & 0 & 0 & 0 & \mathcal{M}_{5,1}^T & 0 \\ \hline \mathcal{M}_{10,10}^* & \overline{\mathcal{M}_{10,5}} & 0 & \overline{\mathcal{M}_{10}} & 0 & 0 \\ \mathcal{M}_{10,5}^* & 0 & \overline{\mathcal{M}_{5,1}} & 0 & \mathcal{M}_5 & 0 \\ 0 & \mathcal{M}_{5,1}^* & 0 & 0 & 0 & 0 \end{array}\right) , \quad \text{where} \tag{5.21a}$$

$$\begin{aligned} \mathcal{M}_{10} &= \mathrm{i}\pi_{10}(m) \otimes M'_{10} , & \mathcal{M}_5 &= -\mathrm{i}\pi_5(m) \otimes M'_5 , \\ \mathcal{M}_{10,10} &= \mathrm{i}\pi_{10,10}(n) \otimes M'_d + \mathrm{i}m' \otimes M'_N , & \mathcal{M}_{5,1} &= \mathrm{i}\pi_{5,1}(n) \otimes M'_e , \\ \mathcal{M}_{10,5} &= \mathrm{i}\pi_{10,5}(n) \otimes M'_{\tilde{u}} + \mathrm{i}n' \otimes M'_{\tilde{n}} . \end{aligned} \tag{5.21b}$$

Here, $M'_{10}, M'_5, M'_N, M'_{\tilde{u}}, M'_d, M'_e, M'_{\tilde{n}}$ are 6×6–matrices of the following block structure:

$$M'_i = \begin{pmatrix} 0_3 & M_i \\ M_i^* & 0_3 \end{pmatrix}, \qquad M'_f = \begin{pmatrix} M_f & 0_3 \\ 0_3 & M_f \end{pmatrix}, \tag{5.22}$$

for $i \in \{5,10\}$ and $f \in \{\tilde{u}, d, e, \tilde{n}, N\}$. The only condition to the 3×3–mass matrices $M_{10}, M_5, M_{\tilde{u}}, M_d, M_e, M_{\tilde{n}}$ and M_N is

$$M_d = M_d^T, \qquad M_N = M_N^T. \tag{5.23}$$

5.2.2 The Grading Operator $\hat{\Gamma}$

The final input of our L–cycle is the grading operator $\hat{\Gamma}$, which we choose as

$$\hat{\Gamma} = \begin{pmatrix} -\mathbb{1}_{16} \otimes \hat{\Gamma}' & 0_{96} \\ 0_{96} & \mathbb{1}_{16} \otimes \hat{\Gamma}' \end{pmatrix}, \qquad \hat{\Gamma}' = \begin{pmatrix} \mathbb{1}_3 & 0 \\ 0 & -\mathbb{1}_3 \end{pmatrix}. \tag{5.24}$$

From (5.7c) and (5.24) there follows that $\hat{\Gamma}$ commutes with $\hat{\pi}(\mathfrak{a})$. The fact that $\hat{\Gamma}'$ commutes with $M'_{\tilde{u},d,e,\tilde{n},N}$ and anticommutes with $M'_{10,5}$ implies that $\hat{\Gamma}$ anticommutes with $\mathcal{M}$. Therefore, the tuple $(\mathfrak{a}, \mathbb{C}^{192}, \mathcal{M}, \hat{\pi}, \hat{\Gamma})$ is an L–cycle.

5.2.3 Summary

Let us summarize the block structure of this L–cycle, for instance in terms of 4×4–block matrices with entries in 48×48–matrices:

$$\hat{\pi}(a) = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & \bar{A} & 0 \\ 0 & 0 & 0 & \bar{A} \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 0 & \mathcal{M}_i & \mathcal{M}_f & 0 \\ \mathcal{M}_i^* & 0 & 0 & \overline{\mathcal{M}_f} \\ \mathcal{M}_f^* & 0 & 0 & \overline{\mathcal{M}_i} \\ 0 & \mathcal{M}_f^* & \mathcal{M}_i^T & 0 \end{pmatrix}, \quad \hat{\Gamma} = \begin{pmatrix} -\mathbb{1}_{48} & 0 & 0 & 0 \\ 0 & \mathbb{1}_{48} & 0 & 0 \\ 0 & 0 & \mathbb{1}_{48} & 0 \\ 0 & 0 & 0 & -\mathbb{1}_{48} \end{pmatrix},$$

with (5.25)

$$A := \operatorname{diag}\big(\pi_{10}(a) \otimes \mathbb{1}_3\,,\ \overline{\pi_5(a)} \otimes \mathbb{1}_3\,,\ 0_3\big)\,,$$

$$\mathcal{M}_i := \operatorname{diag}\big(\mathrm{i}\pi_{10}(m) \otimes M_{10}\,,\ \overline{-\mathrm{i}\pi_5(m) \otimes M_5}\,,\ 0_3\big)\,,$$

$$\mathcal{M}_f := \begin{pmatrix} \mathrm{i}\pi_{10,10}(n) \otimes M_d + \mathrm{i}m' \otimes M_N & \mathrm{i}\pi_{10,5}(n) \otimes M_{\tilde{u}} + \mathrm{i}n' \otimes M_{\tilde{n}} & 0 \\ \mathrm{i}\pi_{10,5}(n)^T \otimes M_{\tilde{u}}^T + \mathrm{i}n'^T \otimes M_{\tilde{n}}^T & 0 & \mathrm{i}\pi_{5,1}(n) \otimes M_e \\ 0 & \mathrm{i}\pi_{5,1}(n)^T \otimes M_e^T & 0 \end{pmatrix} \equiv \mathcal{M}_f^T.$$

The matrix $\mathcal{M}_f$ is the fermionic mass matrix and $\mathcal{M}_i$ leads to the symmetry breaking pattern $\mathrm{su}(5) \to \mathrm{su}(3) \oplus \mathrm{su}(2) \oplus \mathrm{u}(1)$. *Note that the induced representation of the* u(1)–*part is not the hypercharge of the standard model, because later an additional* u(1)–*part enters the game.*

5.3 The Structure of $\hat{\pi}(\Omega^1\mathfrak{a})$ and $\hat{\pi}(\Omega^2\mathfrak{a})$

It is very easy to compute the structure of $\hat{\pi}(\Omega^1\mathfrak{a})$ and $\hat{\pi}(\Omega^2\mathfrak{a})$:

5.3.1 The Structure of $\hat{\pi}(\Omega^1\mathfrak{a})$

We recall (2.25) that elements $\tau^1 \in \hat{\pi}(\Omega^1\mathfrak{a})$ are of the form

$$\tau^1 = \textstyle\sum_{\alpha,z\geq 0}[\hat{\pi}(a^z_\alpha),[\ldots[\hat{\pi}(a^1_\alpha),[-\mathrm{i}\mathcal{M}\ ,\hat{\pi}(a^0_\alpha)]]\ldots]]\,. \tag{5.26}$$

Using (5.12), (5.17a) and the fact that $\hat{\pi}(\mathfrak{a})$ is a representation we obtain the explicit structure of elements $\tau^1 \in \hat{\pi}(\Omega^1\mathfrak{a})$:

$$\tau^1 = \tag{5.27a}$$
$$\left(\begin{array}{ccc|ccc}
\pi_{10}(a)\otimes M'_{10} & 0 & 0 & \begin{bmatrix}\pi_{10,10}(b)\otimes M'_d \\ +c\otimes M'_N\end{bmatrix} & \begin{bmatrix}\pi_{10,5}(b)\otimes M'_{\tilde{u}} \\ +w\otimes M'_{\tilde{n}}\end{bmatrix} & 0 \\
0 & \overline{\pi_5(a)\otimes M'_5} & 0 & \begin{bmatrix}\pi_{10,5}(b)^T\otimes {M'_{\tilde{u}}}^T \\ +w^T\otimes {M'_{\tilde{n}}}^T\end{bmatrix} & 0 & \pi_{5,1}(b)\otimes M'_e \\
0 & 0 & 0 & 0 & \pi_{5,1}(b)^T\otimes {M'_e}^T & 0 \\
\hline
\begin{bmatrix}-\pi_{10,10}(b)^*\otimes {M'_d}^* \\ -c^*\otimes M'_N\end{bmatrix} & \begin{bmatrix}-\overline{\pi_{10,5}(b)\otimes M'_{\tilde{u}}} \\ -\overline{w\otimes M'_{\tilde{n}}}\end{bmatrix} & 0 & -\overline{\pi_{10}(a)\otimes M'_{10}} & 0 & 0 \\
\begin{bmatrix}-\pi_{10,5}(b)^*\otimes {M'_{\tilde{u}}}^* \\ -w^*\otimes {M'_{\tilde{n}}}^*\end{bmatrix} & 0 & -\overline{\pi_{5,1}(b)\otimes M'_e} & 0 & -\pi_5(a)\otimes M'_5 & 0 \\
0 & -\pi_{5,1}(b)^*\otimes {M'_e}^* & 0 & 0 & 0 & 0
\end{array}\right),$$

$$a = \textstyle\sum_{\alpha,z\geq 0}[a^z_\alpha,[\ldots[a^1_\alpha,[m,a^0_\alpha]]\ldots]] \in \mathfrak{a}\ , \tag{5.27b}$$
$$b = -\textstyle\sum_{\alpha,z\geq 0} a^z_\alpha \cdots a^1_\alpha a^0_\alpha n \in \mathfrak{b}\ , \tag{5.27c}$$
$$\hat{\pi}(w) = \textstyle\sum_{\alpha,z\geq 0}[\hat{\pi}(a^z_\alpha),[\ldots[\hat{\pi}(a^1_\alpha),[\hat{\pi}(n'),\hat{\pi}(a^0_\alpha)]]\ldots]] \in \hat{\pi}(\mathfrak{w})\ , \tag{5.27d}$$
$$\hat{\pi}(c) = \textstyle\sum_{\alpha,z\geq 0}[\hat{\pi}(a^z_\alpha),[\ldots[\hat{\pi}(a^1_\alpha),[\hat{\pi}(m'),\hat{\pi}(a^0_\alpha)]]\ldots]] \in \hat{\pi}(\mathfrak{c})\ . \tag{5.27e}$$

It is obvious that a,b,c,w are independent as elements of different irreducible representations of su(5).

5.3.2 The Structure of $\hat{\pi}(\Omega^2\mathfrak{a})$

Next, we are going to construct $\hat{\pi}(\Omega^2\mathfrak{a})$. According to (2.26), elements $\tau^2 \in \hat{\pi}(\Omega^2\mathfrak{a})$ are obtained by summing up anticommutators of the type[14]

$$\tau^2 := \tfrac{1}{2}\{\tau^1,\tau^1\}\,. \tag{5.28a}$$

[14]Note that $\{\tau_1,\tilde{\tau}_1\} = \frac{1}{4}\{\tau_1+\tilde{\tau}_1,\tau_1+\tilde{\tau}_1\} - \frac{1}{4}\{\tau_1-\tilde{\tau}_1,\tau_1-\tilde{\tau}_1\}$.

Thus, using (5.13b) we get from (5.27a) the structure

$$\tau^2 = \left(\begin{array}{ccc|ccc} \tau_{10} & \tau_{\widetilde{10,5}} & \tau_{10,1} & \tau_{10,10} & \tau_{10,5} & 0 \\ \tau^*_{\widetilde{10,5}} & \tau^T_5 & 0 & \tau^T_{10,5} & 0 & \tau_{5,1} \\ \tau^*_{10,1} & 0 & \tau_1 & 0 & \tau^T_{5,1} & 0 \\ \hline \tau^*_{10,10} & \overline{\tau_{10,5}} & 0 & \tau^T_{10} & \overline{\tau_{\widetilde{10,5}}} & \overline{\tau_{10,1}} \\ \tau^*_{10,5} & 0 & \overline{\tau_{5,1}} & \tau^T_{\widetilde{10,5}} & \tau_5 & 0 \\ 0 & \tau^*_{5,1} & 0 & \tau^T_{10,1} & 0 & \tau^T_1 \end{array}\right) , \quad \text{where} \tag{5.28b}$$

$$\begin{aligned} \tau_{10} = {}& \mathrm{i}\pi_{10}(\mathrm{i}(b,b)') \otimes (M'_{\tilde{u}}M'^*_{\tilde{u}} - M'_d M'^*_d) - (b^*b)\mathbb{1}_{10} \otimes (\tfrac{2}{5}M'_{\tilde{u}}M'^*_{\tilde{u}} + \tfrac{3}{5}M'_d M'^*_d) \\ & + \tfrac{1}{2}\{\pi_{10}(a), \pi_{10}(a)\} \otimes M'^2_{10} \\ & - ww^* \otimes M'_{\tilde{n}}M'^*_{\tilde{n}} - w\pi_{10,5}(b)^* \otimes M'_{\tilde{n}}M'^*_{\tilde{u}} - \pi_{10,5}(b)w^* \otimes M'_{\tilde{u}}M'^*_{\tilde{n}} \\ & - cc^* \otimes M'_N M'^*_N - c\pi_{10,10}(b)^* \otimes M'_N M'^*_d - \pi_{10,10}(b)c^* \otimes M'_d M'^*_N \end{aligned}$$

$$\begin{aligned} \tau_5 = {}& \mathrm{i}\pi_5(\mathrm{i}(b,b)') \otimes (\overline{M}'_e M'^T_e - M'^*_{\tilde{u}} M'_{\tilde{u}}) - (b^*b)\mathbb{1}_5 \otimes (\tfrac{4}{5}M'^*_{\tilde{u}}M'_{\tilde{u}} + \tfrac{1}{5}\overline{M}'_e M'^T_e) \\ & + \tfrac{1}{2}\{\pi_5(a), \pi_5(a)\} \otimes M'^2_5 \\ & - w^*w \otimes M'^*_{\tilde{n}}M'_{\tilde{n}} - w^*\pi_{10,5}(b) \otimes M'^*_{\tilde{n}}M'_{\tilde{u}} - \pi_{10,5}(b)^* w \otimes M'^*_u M'_{\tilde{n}} , \\ \tau_1 = {}& -b^*b \otimes M'^T_e \overline{M}'_e , \end{aligned} \tag{5.28c}$$

$$\begin{aligned} \tau_{10,10} = {}& \pi_{10,10}(ab) \otimes \tfrac{1}{2}(M'_{10}M'_d + M'_d M'^T_{10}) \\ & + (\pi_{10}(a)\pi_{10,10}(b) - \pi_{10,10}(b)\pi_{10}(a)^T) \otimes \tfrac{1}{2}(M'_{10}M'_d - M'_d M'^T_{10}) \\ & + (\pi_{10}(a)c + c\pi_{10}(a)^T) \otimes \tfrac{1}{2}(M'_{10}M'_N + M'_N M'^T_{10}) \\ & + (\pi_{10}(a)c - c\pi_{10}(a)^T) \otimes \tfrac{1}{2}(M'_{10}M'_N - M'_N M'^T_{10}) , \\ \tau_{10,5} = {}& \pi_{10}(a)\pi_{10,5}(b) \otimes M'_{10}M'_{\tilde{u}} - \pi_{10,5}(b)\pi_5(a) \otimes M'_{\tilde{u}}M'_5 \\ & + \pi_{10}(a)w \otimes M'_{10}M'_{\tilde{n}} - w\pi_5(a) \otimes M'_{\tilde{n}}M'_5 , \\ \tau_{\widetilde{10,5}} = {}& -\pi_{10,10}(b)\overline{w} \otimes M'_d \overline{M}'_{\tilde{n}} - c\overline{w} \otimes M'_N \overline{M}'_{\tilde{n}} - c\overline{\pi_{10,5}(b)} \otimes M'_N \overline{M}'_{\tilde{u}} , \\ \tau_{5,1} = {}& \pi_{5,1}(ab) \otimes M'^T_5 M'_e , \qquad \tau_{10,1} = -w\overline{\pi_{5,1}(b)} \otimes M'_{\tilde{n}} \overline{M}'_e . \end{aligned} \tag{5.28d}$$

5.4 Towards the Structure of the Connection Form

Here we must perform a rather awkward calculation to find the structure of the connection form. Nevertheless, the reader is encouraged to read at least Section 5.4.1, because the result has far–reaching consequences: We find for generic mass matrices an additional u(1)*–gauge field, together with its representation on the fermionic Hilbert space. Remarkably, this representation is realized in nature. This is a purely algebraic result, for which I have no geometric interpretation. Technically, we study the equations* (3.116) *for* $\mathbb{r}^0\mathfrak{a}$ *in Section 5.4.1 and for* $\mathbb{r}^1\mathfrak{a}$ *in Section 5.4.2. But this is only one part of the job. One also has to check that the equations* (3.117) *are satisfied. We will postpone this task to Section 5.5.6.*

5.4.1 The Construction of $\mathfrak{r}^0\mathfrak{a}$

To compute the structure of elements $\eta^0 \in \mathfrak{r}^0\mathfrak{a}$ we first decompose η^0 according to (5.4) into irreducible su(5)–representations, each of them tensorized by $\mathrm{M}_6\mathbb{C}$. Then, the condition (3.116a) yields the block structure

$$\eta^0 = \hat{\pi}(a) + \mathrm{i}\,\mathrm{diag}(\mathbb{1}_{10}\otimes m'_{10}\,,\ \mathbb{1}_5\otimes m'_{\tilde{5}}\,,\ m'_1\,,\ \mathbb{1}_{10}\otimes m'_{\widetilde{10}}\,,\ \mathbb{1}_5\otimes m'_5\,,\ m'_{\tilde{1}})\,,$$

where $a\in\mathfrak{a}$ and $m'_{10,\tilde{5},1,\widetilde{10},5,\tilde{1}}$ are selfadjoint elements of $\mathrm{M}_6\mathbb{C}$. Due to (2.44) we have $m'_i = \mathrm{diag}(m_i, \hat{m}_i)$, for $m_i, \hat{m}_i \in \mathrm{M}_3\mathbb{C}$.

We insert this structure into the condition (3.116b). Using (5.27a), (5.11), (5.17a) and (5.19) we obtain from the off–diagonal blocks the equations

$$\begin{aligned} m_{10}M_d - M_d m_{\widetilde{10}} &= -\mathrm{i}\bar{\alpha}M_d\,, & m_{10}M_N - M_N m_{\widetilde{10}} &= -\mathrm{i}\bar{\alpha}'M_N\,,\\ m_{10}M_{\tilde{u}} - M_{\tilde{u}}m_5 &= -\mathrm{i}\alpha M_{\tilde{u}}\,, & m_{10}M_{\tilde{n}} - M_{\tilde{n}}m_5 &= -\mathrm{i}\alpha'' M_{\tilde{n}}\,,\\ m_{\tilde{5}}M^T_{\tilde{u}} - M^T_{\tilde{u}}m_{\widetilde{10}} &= -\mathrm{i}\alpha M^T_{\tilde{u}}\,, & m_{\tilde{5}}M^T_{\tilde{n}} - M^T_{\tilde{n}}m_{\widetilde{10}} &= -\mathrm{i}\alpha'' M^T_{\tilde{n}}\,,\\ m_{\tilde{5}}M_e - M_e m_{\tilde{1}} &= -\mathrm{i}\bar{\alpha}M_e\,, & m_1 M^T_e - M^T_e m_5 &= -\mathrm{i}\bar{\alpha}M^T_e\,, \end{aligned} \tag{5.29a}$$

for $\alpha, \alpha', \alpha'' \in \mathbb{C}$. The same equations hold for $\hat{m}_i$, with the same parameters $\alpha, \alpha', \alpha''$. Multiplying the first equation by M^*_d from the right and subtracting the Hermitian conjugate of the resulting equation we get for instance

$$[m_{10}, M_d M^*_d] = -\mathrm{i}(\alpha+\bar{\alpha})M_d M^*_d\,.$$

Applying the trace and respecting $\mathrm{tr}(M_d M^*_d) > 0$ we get $\alpha = \mathrm{i}\lambda$, for $\lambda\in\mathbb{R}$. Analogously, we have $\alpha' = \mathrm{i}\lambda'$ and $\alpha'' = \mathrm{i}\lambda''$. Thus, we find the equations

$$[m_{10}, M_d M^*_d] = [m_{10}, M_N M^*_N] = [m_{10}, M_{\tilde{u}} M^*_{\tilde{u}}] = [m_{10}, M_{\tilde{n}} M^*_{\tilde{n}}] = 0\,. \tag{5.29b}$$

For generic mass matrices $M_{d,N,\tilde{u},\tilde{n}}$, these equations can only be satisfied for $m_{10} = (\nu - \frac{1}{2}\lambda)\mathbb{1}_3$, for $\nu\in\mathbb{R}$. We assume that $M_{d,\tilde{u},e}$ are invertible and find the solution

$$\begin{aligned} m_{10} &= (\nu - \tfrac{1}{2}\lambda)\mathbb{1}_3\,, & m_5 &= (\nu - \tfrac{3}{2}\lambda)\mathbb{1}_3\,, & m_1 &= (\nu - \tfrac{5}{2}\lambda)\mathbb{1}_3\,,\\ m_{\widetilde{10}} &= (\nu + \tfrac{1}{2}\lambda)\mathbb{1}_3\,, & m_{\tilde{5}} &= (\nu + \tfrac{3}{2}\lambda)\mathbb{1}_3\,, & m_{\tilde{1}} &= (\nu + \tfrac{5}{2}\lambda)\mathbb{1}_3\,, \end{aligned} \tag{5.29c}$$

where $\nu,\lambda\in\mathbb{R}$. For $\hat{m}_i$ we get the same equations, with the same λ but possibly a different $\hat{\nu}$ instead of ν. Inserting this result into the π_{10}–block we get the equations

$$(\nu-\hat{\nu})M_{10} = \beta M_{10}\,, \qquad (\nu-\hat{\nu})M^*_{10} = -\beta M^*_{10}\,,$$

which are only compatible with $\nu = \hat{\nu}$. Thus, we obtain with (5.7c) the preliminary result

$$\eta^0 = \hat{\pi}(a) + \hat{\pi}(\mathrm{u}(1)) + \mathrm{i}\nu\mathbb{1}_{192}\,, \tag{5.30a}$$

$$\hat{\pi}(\mathrm{i}\lambda) := \mathrm{i}\lambda\,\mathrm{diag}(-\tfrac{1}{2}\mathbb{1}_{10}\,,\ \tfrac{3}{2}\mathbb{1}_5\,,\ -\tfrac{5}{2}\,,\ \tfrac{1}{2}\mathbb{1}_{10}\,,\ -\tfrac{3}{2}\mathbb{1}_5\,,\ \tfrac{5}{2})\otimes\mathbb{1}_6\,. \tag{5.30b}$$

It is clear that for the part $\hat{\pi}(a)$ of η^0 the conditions (3.116c) and (3.116d) are satisfied. For the part $\mathrm{i}\nu\mathbb{1}_{192}$ of η^0 the condition (3.116c) is equivalent to the question, whether $\mathrm{i}\hat{\pi}(\mathfrak{a}) \subset \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} + \hat{\pi}(\Omega^2\mathfrak{a})$ or not. At this point we need for the first time some important identities between $\{\pi_{10}(\mathfrak{a}), \pi_{10}(\mathfrak{a})\}$ and $\{\pi_5(\mathfrak{a}), \pi_5(\mathfrak{a})\}$. For the traces one finds

$$\begin{aligned} \mathrm{tr}(\pi_{10}(a)\,\pi_{10}(a)) &= \mathrm{tr}(\overline{\pi_{10}(a)}\,\overline{\pi_{10}(a)}) = 3\,\mathrm{tr}(aa)\,, \\ \mathrm{tr}(\pi_5(a)\,\pi_5(a)) &= \mathrm{tr}(\overline{\pi_5(a)}\,\overline{\pi_5(a)}) \quad = \quad \mathrm{tr}(aa)\,, \quad a \in \mathfrak{a}\,. \end{aligned} \tag{5.31}$$

Moreover, due to the decomposition

$$\underline{24} \otimes \underline{24} = \underline{200} \oplus \underline{126} \oplus \underline{126}^* \oplus \underline{75} \oplus \underline{24} \oplus \underline{24} \oplus \underline{1}\,, \tag{5.32}$$

there is a chance that the $\underline{24}$–parts $\{\pi_{10}(a), \pi_{10}(a)\}_{\underline{24}} \in \mathrm{i}\pi_{10}(\mathfrak{a})$ and $\{\pi_5(a), \pi_5(a)\}_{\underline{24}} \in \mathrm{i}\mathfrak{a}$ of $\{\pi_{10}(a), \pi_{10}(a)\}$ and $\{\pi_5(a), \pi_5(a)\}$, respectively, are correlated. Indeed, one finds

$$\mathrm{i}\{\pi_{10}(a), \pi_{10}(a)\}_{\underline{24}} = \tfrac{1}{3}\pi_{10}\big(\mathrm{i}\{\pi_5(a), \pi_5(a)\}_{\underline{24}}\big)\,. \tag{5.33}$$

Now, it is not difficult to see that

$$\begin{aligned} &\mathrm{i}\hat{\pi}(\mathfrak{a}) \subset \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} + \hat{\pi}(\Omega^2\mathfrak{a}) \quad \text{iff} \quad M_{10} = \lambda_{10}u_{10}\,,\ M_5 = \lambda_5 u_5\,, \\ &\qquad u_{10}, u_5 \in \mathrm{U}(3)\,,\ \lambda_{10}, \lambda_5 \in \mathbb{R}\,,\ \lambda_{10}^2 \neq \lambda_5^2\,. \end{aligned} \tag{5.34}$$

In this case also the condition $\mathrm{i}\hat{\pi}(\Omega^1\mathfrak{a}) \subset \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\Omega^1\mathfrak{a})\} + \hat{\pi}(\Omega^3\mathfrak{a})$ is satisfied.

Finally, we must check the conditions (3.116c) and (3.116d) for $\hat{\pi}(\mathrm{u}(1)) \subset \mathfrak{r}^0\mathfrak{a}$. It is easy to see (5.33) that

$$\{\hat{\pi}(\mathrm{u}(1)), \hat{\pi}(\mathfrak{a})\} \subset \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}\,, \tag{5.35a}$$

$$\{\hat{\pi}(\mathrm{u}(1)), \hat{\pi}(\Omega^1\mathfrak{a})\}_{\text{diagonal blocks}} \subset \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\Omega^1\mathfrak{a})\}_{\text{diagonal blocks}}\,. \tag{5.35b}$$

In the off–diagonal blocks we have

$$\{\hat{\pi}(\mathrm{i}), \tau^1\}_{10,10} = 0\,, \quad \{\hat{\pi}(\mathrm{i}), \tau^1\}_{10,5} = -2\mathrm{i}\tau_{10,5}\,, \quad \{\hat{\pi}(\mathrm{i}), \tau^1\}_{5,1} = 4\mathrm{i}\tau_{5,1}\,. \tag{5.35c}$$

We use the identities

$$\begin{aligned} (\pi_{10}(a)\pi_{10,5}(b))_{\underline{5}} &= \tfrac{3}{4}\pi_{10,5}(ab)\,, & (\pi_{10,5}(b)\pi_5(a))_{\underline{5}} &= -\tfrac{1}{4}\pi_{10,5}(ab)\,, \\ (\pi_{10}(a)\pi_{10,5}(b))_{\underline{45}} &= (\pi_{10,5}(b)\pi_5(a))_{\underline{45}}\,, & (\pi_{10}(a)w)_{\underline{5}} &= (w\pi_5(a))_{\underline{5}}\,, \end{aligned} \tag{5.36}$$

for $a \in \mathfrak{a}$, $b \in \mathfrak{b}$ and $w \in \mathfrak{w}$. We have

$$\begin{aligned} \textstyle\sum_\alpha \{\hat{\pi}(a_\alpha), \tau_\alpha^1\}_{10,10} &= \textstyle\sum_\alpha \left(\pi_{10}(a_\alpha)\pi_{10,10}(b_\alpha) + \pi_{10,10}(b_\alpha)\overline{\pi_{10}(a_\alpha)}\right)_{\underline{45}} \otimes M'_d \\ &\quad + \textstyle\sum_\alpha \left(\pi_{10}(a_\alpha)c + c\overline{\pi_{10}(a_\alpha)}\right)_{\underline{45}} \otimes M'_N\,, \end{aligned} \tag{5.37a}$$

$$\begin{aligned} \textstyle\sum_\alpha \{\hat{\pi}(a_\alpha), \tau_\alpha^1\}_{10,5} &= \tfrac{1}{2}\textstyle\sum_\alpha \pi_{10,5}(a_\alpha b_\alpha) \otimes M'_{\tilde{u}} + 2\textstyle\sum_\alpha \left(\pi_{10,5}(b_\alpha)\pi_5(a_\alpha)\right)_{\underline{45}} \otimes M'_{\tilde{u}} \\ &\quad + 2\textstyle\sum_\alpha \left(w_\alpha\pi_5(a_\alpha)\right)_{\underline{5}} \otimes M'_{\tilde{n}} + \textstyle\sum_\alpha \left(\pi_{10}(a_\alpha)w_\alpha + w_\alpha\pi_5(a_\alpha)\right)_{\underline{45}} \otimes M'_{\tilde{n}}\,, \\ \textstyle\sum_\alpha \{\hat{\pi}(a_\alpha), \tau_\alpha^1\}_{5,1} &= \textstyle\sum_\alpha \pi_{5,1}(a_\alpha b_\alpha) \otimes M'_e\,. \end{aligned}$$

Here, a term $(\;)_{\underline{n}}$ belongs to the $\underline{n}$–representation of su(5). We put $c_\alpha \equiv 0$ and choose $a_\alpha, b_\alpha, w_\alpha$ such that

$$\textstyle\sum_\alpha \big(\pi_{10}(a_\alpha)\pi_{10,10}(b_\alpha)+\pi_{10,10}(b_\alpha)\overline{\pi_{10}(a_\alpha)}\big)=0\;, \qquad \sum_\alpha \big(w_\alpha \pi_5(a_\alpha)\big)_{\underline{5}}=0\;. \tag{5.37b}$$

Then, (5.37a) is precisely of the form (5.35c) if we put

$$b=\tfrac{1}{4}\mathrm{i}\textstyle\sum_\alpha a_\alpha b_\alpha\;, \qquad w=\tfrac{1}{2}\mathrm{i}\sum_\alpha \big(\pi_{10}(a_\alpha)w_\alpha + w_\alpha\pi_5(a_\alpha)\big)_{\underline{45}} \tag{5.37c}$$

in (5.35c). This proves $\{\hat{\pi}(\mathrm{u}(1)),\hat{\pi}(\Omega^1\mathfrak{a})\}\subset\{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\Omega^1\mathfrak{a})\}$. It remains to check the conditions (3.117) for the part $\hat{\pi}(\mathrm{u}(1))$. We will do this after having derived the structure of $\hat{\pi}(\jmath^2\mathfrak{a})$. We will find no further obstruction. In conclusion:

Lemma 26. *If M_{10} or M_5 are not multiples of unitary 3×3–matrices or if $M_{10}=\pm M u_{10}$, $M_5=Mu_5$, for $M\in\mathbb{R}$ and $u_{10},u_5\in\mathrm{U}(3)$, then the only solution of (3.116a)–(3.116d) is $\mathfrak{r}^0\mathfrak{a}=\hat{\pi}(\mathfrak{a})+\hat{\pi}(\mathrm{u}(1))$. For $M_{10}=\lambda_{10}u_{10}$ and $M_5=\lambda_5u_5$, with $\lambda_{10}^2\neq\lambda_5^2$, we have $\mathfrak{r}^0\mathfrak{a}=\hat{\pi}(\mathfrak{a})+\hat{\pi}(\mathrm{u}(1))+\mathrm{i}\mathbb{R}\mathbb{1}_{192}$.* □

We see that the connection form contains an additional u(1)–gauge field due to the term $\hat{\pi}(\mathrm{u}(1))$ defined in (5.30b).

5.4.2 The Construction of $\mathfrak{r}^1\mathfrak{a}$

The next step is to compute the structure of elements $\eta^1\in\mathfrak{r}^1\mathfrak{a}$ from (3.116e) and (3.116f). For that purpose we decompose η^1 according to (5.4) into irreducible su(5)–representations, each of them tensorized by $\mathrm{M}_6\mathbb{C}$. Then, the condition (3.116e) yields the block structure $\eta^1=\tau^1+\eta^1_0$, where $\tau^1\in\hat{\pi}(\Omega^1\mathfrak{a})$ and

$$\eta^1_0=\mathrm{i}\,\mathrm{diag}(\mathbb{1}_{10}\otimes m'_{10}\,,\,\mathbb{1}_5\otimes m'_{\tilde{5}}\,,\,m'_1\,,\,-\mathbb{1}_{10}\otimes m'_{\widetilde{10}}\,,\,-\mathbb{1}_5\otimes m'_5\,,\,-m'_{\tilde{1}})\,.$$

In this formula, $m'_{10,\tilde{5},1,\widetilde{10},5,\tilde{1}}$ are selfadjoint elements of $\mathrm{M}_6\mathbb{C}$. Due to (3.115) we have $m'_i=\big(\begin{smallmatrix}0 & m_i\\ m_i^* & 0\end{smallmatrix}\big)$, for $m_i\in\mathrm{M}_3\mathbb{C}$.

We insert this structure into the condition (3.116f). Using (5.27a), (5.28d) and the identities (5.36) we obtain the following conditions for the off–diagonal blocks of $\{\eta^1_0,\tau^1\}$:

$$\begin{aligned}
m_{10}M_d-M_dm_{\widetilde{10}}&=-\mathrm{i}\bar{\alpha}\tfrac{1}{2}(M_{10}M_d+M_dM_{10}^T)\,,\\
m_{10}M_N-M_Nm_{\widetilde{10}}&=-\mathrm{i}\bar{\alpha}'\tfrac{1}{2}(M_{10}M_N+M_NM_{10}^T)\,,\\
m_{10}M_{\tilde{u}}-M_{\tilde{u}}m_5&=-\tfrac{3}{4}\mathrm{i}\alpha M_{10}M_{\tilde{u}}-\tfrac{1}{4}\mathrm{i}\alpha M_{\tilde{u}}M_5-\mathrm{i}\beta(M_{10}M_{\tilde{n}}-M_{\tilde{n}}M_5)\,,\\
m_{10}M_{\tilde{n}}-M_{\tilde{n}}m_5&=-\mathrm{i}\alpha''(M_{10}M_{\tilde{u}}-M_{\tilde{u}}M_5)-\mathrm{i}\beta'M_{10}M_{\tilde{n}}-\mathrm{i}\beta''M_{\tilde{n}}M_5\,,\\
m_{\tilde{5}}M_{\tilde{u}}^T-M_{\tilde{u}}^Tm_{\widetilde{10}}&=-\tfrac{3}{4}\mathrm{i}\alpha M_{\tilde{u}}^TM_{10}^T-\tfrac{1}{4}\mathrm{i}\alpha M_5^TM_{\tilde{u}}^T-\mathrm{i}\beta(M_{\tilde{n}}^TM_{10}^T-M_5^TM_{\tilde{n}}^T)\,,\\
m_{\tilde{5}}M_{\tilde{n}}^T-M_{\tilde{n}}^Tm_{\widetilde{10}}&=-\mathrm{i}\alpha''(M_{\tilde{u}}^TM_{10}^T-M_5^TM_{\tilde{u}}^T)-\mathrm{i}\beta'M_{\tilde{n}}^TM_{10}^T-\mathrm{i}\beta''M_5^TM_{\tilde{n}}^T\,,\\
m_{\tilde{5}}M_e-M_em_{\tilde{1}}&=-\mathrm{i}\bar{\alpha}M_5^TM_e\,,\\
m_1M_e^T-M_e^Tm_5&=-\mathrm{i}\bar{\alpha}M_e^TM_5\,.
\end{aligned}\tag{5.38}$$

Moreover, there are eight further equations derived from (5.38) by the replacements $m_{10,\tilde{5},1,\widetilde{10},5,\tilde{1}} \mapsto m^*_{10,\tilde{5},1,\widetilde{10},5,\tilde{1}}$ and $M_{10,5} \mapsto M^*_{10,5}$, with the same $\alpha, \alpha', \alpha'', \beta, \beta', \beta''$. Thus, we must solve a system of 144 complex equations for 60 complex variables, which has generically (in particular for $M_{10}M_{\tilde{u}}$ not proportional to $M_{\tilde{u}}M_5$) only the trivial solution $\alpha = \alpha' = \alpha'' = \beta = \beta' = \beta'' = 0$ and $m_{10} = m_5 = m_1 = m_{\widetilde{10}} = m_{\tilde{5}} = m_{\tilde{1}} = \gamma\mathbb{1}_3$, for $\gamma \in \mathbb{C}$. We insert this solution into the diagonal blocks $\{\eta^1_0, \tau^1\}_i$ and get with (5.27a) from (5.28b) the conditions

$$\begin{aligned}
\{\eta^1_0, \tau^1\}_{10} &= \mathrm{i}\pi_{10}(a) \otimes \mathrm{diag}(\gamma M^*_{10} + \bar{\gamma} M_{10}, \gamma M^*_{10} + \bar{\gamma} M_{10}) \\
&= \big(\mathrm{i}\overline{\pi_{10}(a)} \otimes \mathrm{diag}(\gamma M^T_{10} + \overline{\bar{\gamma} M_{10}}, \gamma M^T_{10} + \overline{\bar{\gamma} M_{10}})\big)^T , \\
\{\eta^1_0, \tau^1\}_5 &= \mathrm{i}\pi_5(a) \otimes \mathrm{diag}(\gamma M^*_5 + \bar{\gamma} M_5, \gamma M^*_5 + \bar{\gamma} M_5) \\
&= \big(\mathrm{i}\overline{\pi_5(a)} \otimes \mathrm{diag}(\gamma M^T_5 + \overline{\bar{\gamma} M_5}, \gamma M^T_5 + \overline{\bar{\gamma} M_5})\big)^T .
\end{aligned} \tag{5.39}$$

Due to $a = -a^* \in \mathfrak{a}$ we find from (5.28c) and considerations of Section 5.6.2 the conditions

$$\begin{aligned}
\gamma M^*_{10} + \bar{\gamma} M_{10} = -\bar{\gamma} M^*_{10} - \gamma M_{10} &= \lambda_1 \mathbb{1}_3 + \lambda_2 M_{10} M^*_{10} + \lambda_3 (M'_{\tilde{u}} {M'_{\tilde{n}}}^* + M'_{\tilde{n}} {M'_{\tilde{u}}}^*) \\
&\quad + \lambda_4 M'_{\tilde{n}} {M'_{\tilde{n}}}^* + \mathrm{i}\lambda_5 (M'_{\tilde{u}} {M'_{\tilde{n}}}^* - M'_{\tilde{n}} {M'_{\tilde{u}}}^*) + \lambda_6 M'_N {M'_N}^* \\
&\quad + \lambda_7 (M'_N {M'_d}^* + M'_d {M'_N}^*) + \mathrm{i}\lambda_8 (M'_N {M'_d}^* - M'_d {M'_N}^*) , \\
\gamma M^*_5 + \bar{\gamma} M_5 = -\bar{\gamma} M^*_5 - \gamma M_5 &= 3\lambda_1 \mathbb{1}_3 + 3\lambda_2 M_5 M^*_5 + 3\lambda_3 ({M'_{\tilde{n}}}^* M'_{\tilde{u}} + {M'_{\tilde{u}}}^* M'_{\tilde{n}}) \\
&\quad + \lambda_9 {M'_{\tilde{n}}}^* M'_{\tilde{n}} + 3\mathrm{i}\lambda_5 ({M'_{\tilde{n}}}^* M'_{\tilde{u}} - {M'_{\tilde{u}}}^* M'_{\tilde{n}}) ,
\end{aligned} \tag{5.40}$$

for $\lambda_1, \dots, \lambda_9 \in \mathbb{R}$. We assume that there exists no solution $\{\gamma, \lambda_1, \dots, \lambda_9\}$ with $\gamma \neq 0$. In summary:

Lemma 27. *For generic mass matrices $M_{10,5,d,\tilde{u},\tilde{v},e}$, in particular if $M_{10}M_{\tilde{u}}$ is not proportional to $M_{\tilde{u}}M_5$ and* (5.40) *has no solution, the only solution of* (3.116e) *and* (3.116f) *is $\mathbb{r}^1\mathfrak{a} \equiv \hat{\pi}(\Omega^1\mathfrak{a})$.* □

We see that in the generic case, in particular, if non of the conditions

1) *M_{10} and M_5 are multiples of unitary matrices,*
2) *$M_{10}M_{\tilde{u}}$ is proportional to $M_{\tilde{u}}M_5$*
3) *there exists a solution $0 \neq \gamma \in \mathbb{C}$, $\lambda_1, \dots, \lambda_9 \in \mathbb{R}$ for* (5.40)

is fulfilled, we simply have

$$\mathbb{r}^0\mathfrak{a} = \hat{\pi}(\mathfrak{a}) + \hat{\pi}(\mathrm{u}(1)) , \qquad \mathbb{r}^1\mathfrak{a} = \hat{\pi}(\Omega^1\mathfrak{a}) . \tag{5.41}$$

We can always reach the generic case by adding a small perturbation to M_{10} and M_5,

$$M_{10} \mapsto M_{10} + \varepsilon_{10}\tilde{M}_{10} , \qquad M_5 \mapsto M_5 + \varepsilon_5\tilde{M}_5 , \tag{5.42}$$

and take the limit $\varepsilon_{10}, \varepsilon_5 \mapsto 0$ if we wish. This procedure does not change the predictions of the model, but it guarantees equations (5.41).

5.5 The Ideal $\mathbb{J}^2\mathfrak{g}$

The ideal $\mathbb{J}^2\mathfrak{g}$ consists of several parts, which we will compute step by step: The ideal $\hat{\pi}(\mathcal{J}^2\mathfrak{a})$, the anticommutator $\{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\mathfrak{a})\}$ and the centralizer of $\pi(\Omega^*\mathfrak{g})$.

5.5.1 The Structure of $\mathcal{M}^2$

The search for $\hat{\pi}(\mathcal{J}^2\mathfrak{a})$ is easier than one probably thinks. It suffices to study the square of the mass matrix $\mathcal{M}$. More precisely, one must decompose $\mathcal{M}^2$ into generators of irreducible representations and compare it with the generators of irreducible representations occurring in $-\mathrm{i}\mathcal{M}$. Generators of $\mathcal{M}^2$ occurring also in $-\mathrm{i}\mathcal{M}$ contribute to $\hat{\sigma}(\Omega^1\mathfrak{a})$. The rest generates the ideal $\hat{\pi}(\mathcal{J}^2\mathfrak{a})$. The result will be given in formula (5.49).

We recall (2.43b) that for the analysis of $\hat{\pi}(\mathcal{J}^2\mathfrak{a})$ we must find the space of elements $\hat{\sigma}(\omega^1)$, where $\omega^1 \in \Omega^1\mathfrak{a}\cap\ker\hat{\pi}$. For the computation of $\hat{\sigma}(\omega^1)$ we need knowledge of $\mathcal{M}^2$, see (3.38). We define

$$v_0 := \mathrm{i}\ \mathrm{diag}(-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}\mid -\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}\mid \tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}\mid 1) \in \mathfrak{v} \tag{5.43}$$

and abbreviate

$$M'_u := M'_{\tilde{u}} + M'_{\tilde{n}}\,, \qquad\qquad M'_n := M'_{\tilde{u}} - 3M'_{\tilde{n}}\,, \tag{5.44a}$$

analogously for the primeless matrices $M_{u,n,\tilde{u},\tilde{n}}$. The inverse transformation is given by

$$M'_{\tilde{u}} = \tfrac{1}{4}(3M'_u + M'_n)\,, \qquad\qquad M'_{\tilde{n}} = \tfrac{1}{4}(M'_u - M'_n)\,. \tag{5.44b}$$

Then, using (5.21) and (5.20), we find the following formula for $\mathcal{M}^2$:

$$\mathcal{M}^2 = \left(\begin{array}{ccc|ccc}
(\mathcal{M}^2)_{10} & (\mathcal{M}^2)_{\widetilde{10,5}} & 0 & (\mathcal{M}^2)_{10,10} & (\mathcal{M}^2)_{10,5} & 0 \\
(\mathcal{M}^2)^*_{\widetilde{10,5}} & (\mathcal{M}^2)^T_5 & 0 & (\mathcal{M}^2)^T_{10,5} & 0 & (\mathcal{M}^2)_{5,1} \\
0 & 0 & (\mathcal{M}^2)_1 & 0 & (\mathcal{M}^2)^T_{5,1} & 0 \\
\hline
(\mathcal{M}^2)^*_{10,10} & \overline{(\mathcal{M}^2)_{10,5}} & 0 & (\mathcal{M}^2)^T_{10} & \overline{(\mathcal{M}^2)_{\widetilde{10,5}}} & 0 \\
(\mathcal{M}^2)^*_{10,5} & 0 & \overline{(\mathcal{M}^2)_{5,1}} & (\mathcal{M}^2)^T_{\widetilde{10,5}} & (\mathcal{M}^2)_5 & 0 \\
0 & (\mathcal{M}^2)^*_{5,1} & 0 & 0 & 0 & (\mathcal{M}^2)^T_1
\end{array}\right), \tag{5.45a}$$

where

$$\begin{aligned}
(\mathcal{M}^2)_{10} = \mathbb{1}_{10}\otimes(&\tfrac{9}{25}{M'_{10}}^2 + \tfrac{4}{10}M'_{\tilde{u}}{M'_{\tilde{u}}}^* + \tfrac{6}{10}M'_d{M'_d}^* + \tfrac{12}{10}M'_{\tilde{n}}{M'_{\tilde{n}}}^* + \tfrac{1}{10}M'_N{M'_N}^*) \\
&-\mathrm{i}v_0\otimes({M'_{10}}^2 - 2(M'_{\tilde{u}}{M'_{\tilde{n}}}^* + M'_{\tilde{n}}{M'_{\tilde{u}}}^*) + 4M'_{\tilde{n}}{M'_{\tilde{n}}}^* + \tfrac{1}{2}M'_N{M'_N}^*) \\
&-\mathrm{i}\pi_{10}(\tfrac{Y'}{2} + I_3)\otimes(M'_u{M'_u}^* - M'_d{M'_d}^*) \\
&-\tfrac{1}{3}\mathrm{i}\pi_{10}(m)(\tfrac{1}{5}{M'_{10}}^2 - 4(M'_{\tilde{u}}{M'_{\tilde{n}}}^* + M'_{\tilde{n}}{M'_{\tilde{u}}}^*) + 8M'_{\tilde{n}}{M'_{\tilde{n}}}^* + M'_N{M'_N}^*),
\end{aligned}$$

$$
\begin{aligned}
(\mathcal{M}^{\,2})_5 &= \mathbb{1}_5 \otimes (\tfrac{6}{25}{M'_5}^2 + \tfrac{12}{5}{M'_{\tilde n}}^* M'_{\tilde n} + \tfrac{4}{5}{M'_{\tilde u}}^* M'_{\tilde u} + \tfrac{1}{5}\overline{M'_e}{M'_e}^T) \\
&\quad -\mathrm{i}\pi_5(\tfrac{Y'}{2} + I_3) \otimes (\overline{M'_e}{M'_e}^T - {M'_n}^* M'_n) \qquad (5.45b)\\
&\quad -\mathrm{i}\pi_5(m)(\tfrac{1}{5}{M'_5}^2 - 4({M'_{\tilde u}}^* M'_{\tilde n} + {M'_{\tilde n}}^* M'_{\tilde u}) + 8{M'_{\tilde n}}^* M'_{\tilde n}) ,\\
(\mathcal{M}^{\,2})_1 &= {M'_e}^T \overline{M'_e} ,\\
(\mathcal{M}^{\,2})_{10,10} &= \tfrac{3\mathrm{i}}{5}\pi_{10,10}(n) \otimes \tfrac{1}{2}(M'_{10}M'_d + M_d {M'_{10}}^T) + \tfrac{\mathrm{i}}{2}\pi_{10,10}(n') \otimes \tfrac{1}{2}(M'_{10}M'_d - M_d {M'_{10}}^T) \\
&\quad -\tfrac{12\mathrm{i}}{5} m' \otimes \tfrac{1}{2}(M'_{10}M'_N + M'_N \overline{M'_{10}}) ,\\
(\mathcal{M}^{\,2})_{5,1} &= \tfrac{3\mathrm{i}}{5}\pi_{5,1}(n) \otimes {M'_5}^T M'_e ,\\
(\mathcal{M}^{\,2})_{10,5} &= -\mathrm{i}\pi_{10,5}(n) \otimes (\tfrac{9}{20}M'_{10}M'_{\tilde u} + \tfrac{3}{20}M'_{\tilde u}M'_5 - \tfrac{3}{4}M'_{10}M'_{\tilde n} + \tfrac{3}{4}M'_{\tilde n}M'_5) \\
&\quad -\mathrm{i}n' \otimes (-\tfrac{1}{4}M'_{10}M'_{\tilde u} + \tfrac{1}{4}M'_{\tilde u}M'_5 + \tfrac{19}{20}M'_{10}M'_{\tilde n} - \tfrac{7}{20}M'_{\tilde n}M'_5) ,\\
(\mathcal{M}^{\,2})_{\widetilde{10,5}} &= -\mathrm{i}n'' \otimes M'_N \overline{M'_n} .
\end{aligned}
$$

Here, n'' is a generator of the $\underline{40}^*$–representation of su(5) occurring in the decomposition (5.4e):

$$
n'' := \mathrm{i}\begin{pmatrix} 0_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times1} & 1 \end{pmatrix} \in \underline{40}^* . \qquad (5.46)
$$

5.5.2 Discussion of $\hat\pi(\mathcal{J}^2\mathfrak{a})$

Due to (2.43b), the ideal $\hat\pi(\mathcal{J}^2\mathfrak{a})$ is given as the set of elements of the form

$$
\begin{aligned}
j_2 &= \textstyle\sum_{\alpha,z\geq 0}[\hat\pi(a^z_\alpha),[\dots[\hat\pi(a^1_\alpha),[\mathcal{M}^{\,2},\hat\pi(a^0_\alpha)]]\dots]] , \quad \text{where} \qquad (5.47a)\\
0 &= \textstyle\sum_{\alpha,z\geq 0}[\hat\pi(a^z_\alpha),[\dots[\hat\pi(a^1_\alpha),[-\mathrm{i}\mathcal{M},\hat\pi(a^0_\alpha)]]\dots]] . \qquad (5.47b)
\end{aligned}
$$

Obviously, terms in $\mathcal{M}^{\,2}$ proportional to the identities $\mathbb{1}_{10}, \mathbb{1}_5, 1$ do not contribute to j_2. Next, the term $(\mathcal{M}^{\,2})_{5,1} = \frac{3\mathrm{i}}{5}\pi_{5,1}(n) \otimes {M'_5}^T M'_e$ gives a contribution to j_2, which is $\frac{3\mathrm{i}}{5} \otimes {M'_5}^T$ times (from the left) the contribution of $-\mathrm{i}\mathcal{M}_{5,1} = \pi_{5,1}(n) \otimes M'_e$ to (5.47b), and hence equals zero. For the same argument, all terms in $(\mathcal{M}^{\,2})_{10,10}$ and $(\mathcal{M}^{\,2})_{10,5}$ do not contribute to j_2. The same is true for the terms proportional to $\pi_{10}(m)$ and $\pi_5(m)$. Thus, there remain only contributions from the terms $-\mathrm{i}\pi_{10}(\frac{Y'}{2} + I_3) \otimes M^2_{A,10}$, $-\mathrm{i}\pi_5(\frac{Y'}{2} + I_3) \otimes M^2_{A,5}$, $-\mathrm{i}v_0 \otimes M^2_V$ and $-\mathrm{i}n'' \otimes M'_N \overline{M'_n}$, where

$$
\begin{aligned}
M^2_V &:= {M'_{10}}^2 - 2(M'_{\tilde u}{M'_{\tilde n}}^* + M'_{\tilde n}{M'_{\tilde u}}^*) + 4M'_{\tilde n}{M'_{\tilde n}}^* + \tfrac{1}{2}M'_N{M'_N}^* , \qquad (5.48a)\\
M^2_{A,10} &:= M'_u{M'_u}^* - M'_d{M'_d}^* , \qquad M^2_{A,5} := \overline{M'_e}{M'_e}^T - {M'_n}^* M'_n . \qquad (5.48b)
\end{aligned}
$$

Since the irreducible representations $\underline{24}, \underline{75}, \underline{5}, \underline{45}^*, \underline{50}, \underline{40}^*$ are independent, it is always possible to fulfil (5.47b) and to generate by the commutators (5.47a) representations of arbitrary elements of $\underline{75}$ and $\underline{40}^*$. Moreover, the generator $\frac{Y'}{2}+I_3$ occurring in $\mathcal{M}^2$ generates independent elements of the $\underline{24}$–representation. To see this, put $a^0_\alpha = 2I_3 + \mathrm{i}B_{45} + \mathrm{i}B_{54}$ and $a^1_\alpha = -\frac{5}{3}Y' + \mathrm{i}B_{45} + \mathrm{i}B_{54}$ for all α and z. Then, (5.47b) is identically fulfilled due to

$$
\begin{aligned}
&[\hat{\pi}(a^1_\alpha), [-\mathrm{i}\mathcal{M}, \hat{\pi}(a^0_\alpha)]] = 0\,, \qquad \text{whereas} \\
&[\pi_j(a^1_\alpha)\otimes \mathbb{1}_6, [-\mathrm{i}\pi_j(\tfrac{Y'}{2}+I_3)\otimes M^2_{A,j}, \pi_j(a^0_\alpha)\otimes \mathbb{1}_6]] = -4\mathrm{i}\pi_j(I_3)\otimes M^2_{A,j} \neq 0\,,
\end{aligned}
$$

for $j \in \{10, 5\}$. Thus, elements $j_2 \in J_2 := \hat{\pi}(\mathcal{J}^2\mathfrak{a})$ are of the form

$$
j_2 = \tag{5.49}
$$
$$
\left(\begin{array}{ccc|ccc}
\begin{bmatrix} \mathrm{i}\pi_{10}(a)\otimes M^2_{A,10} \\ +\mathrm{i}v\otimes M^2_V \end{bmatrix} & \mathrm{i}c''\otimes M'_N\overline{M'_n} & 0 & & & \\
(\mathrm{i}c''\otimes M'_N\overline{M'_n})^* & (\mathrm{i}\pi_5(a)\otimes M^2_{A,5})^T & 0 & & \mathrm{O} & \\
0 & 0 & 0 & & & \\
\hline
 & & & \begin{bmatrix} (\mathrm{i}\pi_{10}(a)\otimes M^2_{A,10} \\ +\mathrm{i}v\otimes M^2_V)^T \end{bmatrix} & \overline{\mathrm{i}c''\otimes M'_N\overline{M'_n}} & 0 \\
 & \mathrm{O} & & (\mathrm{i}c''\otimes M'_N\overline{M'_n})^T & \mathrm{i}\pi_5(a)\otimes M^2_{A,5} & 0 \\
 & & & 0 & 0 & 0
\end{array}\right),
$$

where $a \in \mathfrak{a}$, $v \in \mathfrak{v}$ and $c'' \in \underline{40}^*$.

5.5.3 The Structure of $\{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}$

Let $J_0 := \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}$. From (5.7c) we obtain for elements $j_0 \in J_0$ the structure

$$
j_0 = \tag{5.50}
$$
$$
\left(\begin{array}{ccc|ccc}
\sum_\alpha\{\pi_{10}(a_\alpha), \pi_{10}(a_\alpha)\} & 0 & 0 & & & \\
0 & \sum_\alpha\{\pi_5(a_\alpha), \pi_5(a_\alpha)\}^T & 0 & & \mathrm{O} & \\
0 & 0 & 0 & & & \\
\hline
 & & & \sum_\alpha\{\pi_{10}(a_\alpha), \pi_{10}(a_\alpha)\}^T & 0 & 0 \\
 & \mathrm{O} & & 0 & \sum_\alpha\{\pi_5(a_\alpha), \pi_5(a_\alpha)\} & 0 \\
 & & & 0 & 0 & 0
\end{array}\right) \otimes \mathbb{1}_6\,,
$$

where $a_\alpha \in \mathfrak{a}$. In the non–vanishing blocks we separate the part proportional to the identity matrices and the trace–free parts, using (5.31) and (5.33). Then, we obtain for elements $j_0 \in J_0$ the structure

$$j_0 = \tag{5.51}$$
$$\left(\begin{array}{ccc|ccc} \frac{3}{5}\alpha\mathbb{1}_{10}+\frac{1}{3}\mathrm{i}\pi_{10}(a)+\mathrm{i}v & 0 & 0 & & & \\ 0 & \frac{2}{5}\alpha\mathbb{1}_5+\mathrm{i}\pi_5(a)^T & 0 & & \mathrm{O} & \\ 0 & 0 & 0 & & & \\ \hline & & & \frac{3}{5}\alpha\mathbb{1}_{10}+\frac{1}{3}\mathrm{i}\pi_{10}(a)^T+\mathrm{i}v^T & 0 & 0 \\ & \mathrm{O} & & 0 & \frac{2}{5}\alpha\mathbb{1}_5+\mathrm{i}\pi_5(a) & 0 \\ & & & 0 & 0 & 0 \end{array}\right) \otimes \mathbb{1}_6 ,$$

where $\alpha \in \mathbb{R}$, $a \in \mathfrak{a}$ and $v \in \mathfrak{v}$.

5.5.4 Discussion of $\mathbb{c}^0\mathfrak{a}, \mathbb{c}^1\mathfrak{a}$ and $\mathbb{c}^2\mathfrak{a}$

The graded centralizer of $\pi(\Omega^*\mathfrak{g})$ is trivial. It has the very simple form $\tilde{\mathbb{c}}^2\mathfrak{g} = \Lambda^0 \otimes \mathbb{1}_{192}$, as we show below.

Now we construct the spaces $\mathbb{c}^0\mathfrak{a}, \mathbb{c}^1\mathfrak{a}, \mathbb{c}^2\mathfrak{a}$ characterized in (3.121a). Since for invertible matrices $M_{\tilde{u},d,e}$ there exist invertible elements of $\hat{\pi}(\Omega^1\mathfrak{a})$, we conclude immediately

$$\mathbb{c}^0\mathfrak{a} = 0 , \qquad\qquad \mathbb{c}^1\mathfrak{a} = 0 . \tag{5.52}$$

The first equation of the last line in (3.121a) yields as before

$$\mathbb{c}^2\mathfrak{a} \ni \mathrm{diag}(\mathbb{1}_{10} \otimes m'_{10},\ \mathbb{1}_5 \otimes m'_{\tilde{5}},\ m'_1,\ \mathbb{1}_{10} \otimes m'_{\widetilde{10}},\ \mathbb{1}_5 \otimes m'_5,\ m'_{\tilde{1}}), \tag{5.53}$$

where $m'_{10,\tilde{5},1,\widetilde{10},5,\tilde{1}}$ are selfadjoint elements of $\mathrm{M}_6\mathbb{C}$ of the form $m_i = \mathrm{diag}(m_i, \hat{m}_i)$, with $m_i, \hat{m}_i \in \mathrm{M}_3\mathbb{C}$. Inserting this into the very last equation of (3.121a), we get for the off–diagonal blocks

$$\begin{aligned} m_{10}M_d - M_d m_{\widetilde{10}} &= 0 , & m_{10}M_N - M_N m_{\widetilde{10}} &= 0 , \\ m_{10}M_{\tilde{u}} - M_{\tilde{u}} m_5 &= 0 , & m_{10}M_{\tilde{n}} - M_{\tilde{n}} m_5 &= 0 , \\ m_{\tilde{5}}M^T_{\tilde{u}} - M^T_{\tilde{u}} m_{\widetilde{10}} &= 0 , & m_{\tilde{5}}M^T_{\tilde{n}} - M^T_{\tilde{n}} m_{\widetilde{10}} &= 0 , \\ m_{\tilde{5}}M_e - M_e m_{\tilde{1}} &= 0 , & m_1 M^T_e - M^T_e m_5 &= 0 . \end{aligned} \tag{5.54a}$$

We have the same equations for $\hat{m}_i$ instead of m_i. For generic mass matrices $M_{d,N,\tilde{u},\tilde{n},e}$, the only solution is

$$\begin{aligned} m_{10} = m_{\tilde{5}} = m_1 = m_{\widetilde{10}} = m_5 = m_{\tilde{1}} = \lambda_1 \mathbb{1}_3 , \\ \hat{m}_{10} = \hat{m}_{\tilde{5}} = \hat{m}_1 = \hat{m}_{\widetilde{10}} = \hat{m}_5 = \hat{m}_{\tilde{1}} = \lambda_2 \mathbb{1}_3 , \end{aligned} \tag{5.54b}$$

for $\lambda_1, \lambda_2 \in \mathbb{R}$. However, the diagonal blocks of the last equation of (3.121a) give $\lambda_1 = \lambda_2$ and

$$\mathbb{c}^2\mathfrak{a} = \mathbb{R}\mathbb{1}_{192} . \tag{5.55}$$

In summary, we have with (3.121b) and (2.47a):

Lemma 28. *The ideal $\mathbb{J}^2\mathfrak{g}$ commuting with functions has the structure*

$$\mathbb{J}^2\mathfrak{g} = \Lambda^0 \otimes \big(\hat{\pi}(\jmath^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} + \mathbb{R}\mathbb{1}_{192}\big)\;. \qquad \square$$

We denote $\hat{\pi}(\jmath^2\mathfrak{a}) \equiv J_2$, $\{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} \equiv J_0$ and $\mathbb{R}\mathbb{1}_{192} \equiv J_3$ for simplicity.

5.5.5 An Orthogonal Decomposition of the Ideal

It is advantageous to construct an orthogonal decomposition of the matrix part $J_0 + J_2 + J_3$ of our ideal $\mathbb{J}^2\mathfrak{g}$, which leads to a considerable simplification of the factorization procedure in Section 5.6.

We choose $J_0 + J_3$ as the reference space. The space J_3 is already orthogonal to J_2. Therefore, we search for a subspace $J_2' \subset J_0 + J_2$ orthogonal to J_0, preserving the orthogonality of J_2' with J_3. For this purpose we add to a given element $j_2 \in J_2$ as in (5.49) the element $j_0 \in J_0$ given by $\alpha = 0$, $a \mapsto \beta a$ and $v \mapsto \gamma v$. We are going to determine β and γ such that $j_2' = j_0 + j_2$ is orthogonal to any $\tilde{j}_0 \in J_0$:

$$\begin{aligned} 0 = \mathrm{tr}(\tilde{j}_0^* j_2') &= 2\;\mathrm{tr}(\tfrac{1}{3}\mathrm{i}\pi_{10}(\tilde{a})\,\mathrm{i}\pi_{10}(a))\;\mathrm{tr}(M_u' M_u'^{\,*} - M_d' M_d'^{\,*} + \tfrac{1}{3}\beta\mathbb{1}_6) \\ &\quad + 2\;\mathrm{tr}(\mathrm{i}\pi_5(\tilde{a})\,\mathrm{i}\pi_5(a))\;\mathrm{tr}(\overline{M}_e' M_e'^{\,T} - M_n'^{\,*} M_n' + \beta\mathbb{1}_6) \\ &\quad + 2\;\mathrm{tr}(\mathrm{i}\tilde{v}\,\mathrm{i}v)\;\mathrm{tr}(M_V^2 + \gamma\mathbb{1}_6)\;. \end{aligned}$$

The solution of this equation is

$$\beta = -\tfrac{1}{8}\,\mathrm{tr}(M_u' M_u'^{\,*} - M_d' M_d'^{\,*} + \overline{M}_e' M_e'^{\,T} - M_n'^{\,*} M_n')\;, \qquad \gamma = -\tfrac{1}{6}\,\mathrm{tr}(M_V^2)\;. \tag{5.56}$$

Putting

$$\begin{aligned} M_{ud}^2 &:= (M_u' M_u'^{\,*} - M_d' M_d'^{\,*}) - \tfrac{1}{24}\,\mathrm{tr}(M_u' M_u'^{\,*} - M_d' M_d'^{\,*} + \overline{M}_e' M_e'^{\,T} - M_n'^{\,*} M_n')\mathbb{1}_6\;, \\ M_{en}^2 &:= (\overline{M}_e' M_e'^{\,T} - M_n'^{\,*} M_n') - \tfrac{1}{8}\,\mathrm{tr}(M_u' M_u'^{\,*} - M_d' M_d'^{\,*} + \overline{M}_e' M_e'^{\,T} - M_n'^{\,*} M_n')\mathbb{1}_6\;, \end{aligned} \tag{5.57a}$$

$$\tilde{M}_V^2 := M_V^2 - \tfrac{1}{6}\,\mathrm{tr}(M_V^2)\mathbb{1}_6\;, \tag{5.57b}$$

we get the following structure of elements $j_2' \in J_2'$:

$$j_2' = \left(\begin{array}{ccc|ccc} \begin{bmatrix} \mathrm{i}\pi_{10}(a)\otimes M_{ud}^2 \\ +\mathrm{i}v\otimes\tilde{M}_V^2 \end{bmatrix} & \mathrm{i}c''\otimes M_N'\overline{M_n'} & 0 & & & \\ (\mathrm{i}c''\otimes M_N'\overline{M_n'})^* & (\mathrm{i}\pi_5(a)\otimes M_{en}^2)^T & 0 & & \mathrm{O} & \\ 0 & 0 & 0 & & & \\ \hline & & & \begin{bmatrix} (\mathrm{i}\pi_{10}(a)\otimes M_{ud}^2 \\ +\mathrm{i}v\otimes\tilde{M}_V^2)^T \end{bmatrix} & \overline{\mathrm{i}c''\otimes M_N'\overline{M_n'}} & 0 \\ & \mathrm{O} & & (\mathrm{i}c''\otimes M_N'\overline{M_n'})^T & \mathrm{i}\pi_5(a)\otimes M_{en}^2 & 0 \\ & & & 0 & 0 & 0 \end{array}\right). \tag{5.58}$$

That J_0 and J_3 are not orthogonal is not a problem.

5.5.6 A Verification

We still have to provide a supplement to the proof of Lemma 26, namely to verify formulae (3.117) *for* $\mathbb{r}^0\mathfrak{a} = \hat{\pi}(\mathrm{u}(1))$. *The method is to prove*

$$\hat{\sigma}(\mathrm{i}) = 0 \,, \tag{5.59}$$

where $\hat{\sigma}(\mathrm{i})$ *was defined in* (3.122). *As a by–product we get the explicit form of* $\hat{\sigma}(\Omega^1\mathfrak{a})$ *modulo the ideal* $\hat{\pi}(\mathcal{J}^2\mathfrak{a})$, *see* (5.62). *This result is an important input in the construction of the bosonic action.*

Putting $l = 1$ in (3.122), we must show

$$\hat{\sigma}\circ\pi^{-1}([\hat{\pi}(\mathrm{i}),\tau^1]) - [\hat{\pi}(\mathrm{i}),\hat{\sigma}\circ\hat{\pi}^{-1}(\tau^1)]_g \in \hat{\pi}(\mathcal{J}^2\mathfrak{a}) \,. \tag{5.60}$$

Calculating $\hat{\sigma}\circ\hat{\pi}^{-1}(\tau^1)$ means to calculate j_2 in (5.47a), however with the r.h.s. of (5.47b) equal to the given element τ^1 and not equal to zero. We have listed the matrix elements of $\mathcal{M}^2$ in (5.45). Again, terms in $\mathcal{M}^2$ proportional to the identities $\mathbb{1}_{10}, \mathbb{1}_5, 1$ do not contribute to j_2. Next, the terms proportional to $-\mathrm{i}v_0$, $-\mathrm{i}\pi_{10;5}(\frac{Y'}{2}+I_3)$ and $-\mathrm{i}n''$ contribute to the ideal $\hat{\pi}(\mathcal{J}^2\mathfrak{a})$, as explained above. Since we regard $\hat{\sigma}\circ\hat{\pi}^{-1}(\tau^1)$ modulo $\hat{\pi}(\mathcal{J}^2\mathfrak{a})$, it is not necessary to consider these terms. Therefore, there remain only the terms

$$-\mathrm{i}\pi_{10}(m)\otimes\tfrac{1}{3}(\tfrac{1}{5}M_{10}'^{\,2} - 4M'_{\tilde{u}}M_{\tilde{n}}'^{\,*} - 4M'_{\tilde{n}}M_{\tilde{u}}'^{\,*} + 8M'_{\tilde{n}}M_{\tilde{n}}'^{\,*} + M'_N M_N'^{\,*}) \,, \tag{5.61a}$$
$$-\mathrm{i}\pi_5(m)\otimes(\tfrac{1}{5}M_5'^{\,2} - 4M_{\tilde{n}}'^{\,*}M'_{\tilde{u}} - 4M_{\tilde{u}}'^{\,*}M'_{\tilde{n}} + 8M_{\tilde{n}}'^{\,*}M'_{\tilde{n}}) \tag{5.61b}$$

in the diagonal blocks $(\mathcal{M}^2)_{10}$ and $(\mathcal{M}^2)_5$ as well as the off–diagonal blocks $(\mathcal{M}^2)_{i,j}$, which give a contribution to $\hat{\sigma}\circ\hat{\pi}^{-1}(\tau^1)$. As we have already noticed, the contribution of $(\mathcal{M}^2)_{5,1}$ is $\frac{3\mathrm{i}}{5}\otimes M_5'^{\,T}$ times the contribution of $\pi_{5,1}(n)\otimes M'_e$ to (5.47b). We get analogous contributions from the other terms $(\mathcal{M}^2)_{i,j}$ and $(\mathcal{M}^2)_i$. Then we obtain in the same notations as in (5.27a) the formula

$$\hat{\sigma}\circ\hat{\pi}^{-1}(\tau^1) = \left(\begin{array}{ccc|ccc} \sigma_{10} & 0 & 0 & \sigma_{10,10} & \sigma_{10,5} & 0 \\ 0 & \sigma_5^T & 0 & \sigma_{10,5}^T & 0 & \sigma_{5,1} \\ 0 & 0 & 0 & 0 & \sigma_{5,1}^T & 0 \\ \hline \sigma_{10,10}^* & \overline{\sigma_{10,5}} & 0 & \sigma_{10}^T & 0 & 0 \\ \sigma_{10,5}^* & 0 & \overline{\sigma_{5,1}} & 0 & \sigma_5 & 0 \\ 0 & \sigma_{5,1}^* & 0 & 0 & 0 & 0 \end{array}\right) , \quad \text{where} \tag{5.62}$$

$$\begin{aligned}
\sigma_{10} &= -\mathrm{i}\pi_{10}(a)\otimes\tfrac{1}{3}(\tfrac{1}{5}M_{10}'^{\,2} - 4M'_{\tilde{u}}M_{\tilde{n}}'^{\,*} - 4M'_{\tilde{n}}M_{\tilde{u}}'^{\,*} + 8M'_{\tilde{n}}M_{\tilde{n}}'^{\,*} + M'_N M_N'^{\,*}) \,, \\
\sigma_5 &= -\mathrm{i}\pi_5(a)\otimes(\tfrac{1}{5}M_5'^{\,2} - 4M_{\tilde{n}}'^{\,*}M'_{\tilde{u}} - 4M_{\tilde{u}}'^{\,*}M'_{\tilde{n}} + 8M_{\tilde{n}}'^{\,*}M'_{\tilde{n}}) \,, \\
\sigma_{10,10} &= \tfrac{3\mathrm{i}}{5}\pi_{10,10}(b)\otimes\tfrac{1}{2}(M'_{10}M'_d + M'_d M_{10}'^{\,T}) + \tfrac{\mathrm{i}}{2}\pi_{10,10}(w)\otimes\tfrac{1}{2}(M'_{10}M'_d - M'_d M_{10}'^{\,T}) \\
&\quad -\tfrac{12\mathrm{i}}{5}c\otimes\tfrac{1}{2}(M'_{10}M'_N + M'_N\overline{M'_{10}}) \,,
\end{aligned}$$

$$\begin{aligned}
\sigma_{5,1} &= \tfrac{3\mathrm{i}}{5}\pi_{5,1}(b)\otimes M_5'^{\,T}M_e' \,, \\
\sigma_{10,5} &= \mathrm{i}\pi_{10,5}(b)\otimes(-\tfrac{9}{20}M_{10}'M_{\tilde u}' - \tfrac{3}{20}M_{\tilde u}'M_5' + \tfrac{3}{4}M_{10}'M_{\tilde n}' - \tfrac{3}{4}M_{\tilde n}'M_5') \\
&\quad + \mathrm{i}w\otimes(\tfrac{1}{4}M_{10}'M_{\tilde u}' - \tfrac{1}{4}M_{\tilde u}'M_5' - \tfrac{19}{20}M_{10}'M_{\tilde n}' + \tfrac{7}{20}M_{\tilde n}'M_5') \,.
\end{aligned}$$

Now, it is convenient to denote the matrix (5.27a) by $\tau^1\langle a,b,c,w\rangle$, the matrix (5.62) by $(\hat\sigma\circ\hat\pi^{-1}(\tau^1))\langle a,b,c,w\rangle$ and the matrix (5.49) by $j_2\langle a,c''\rangle$. The arguments in the brackets characterize the matrices. Then, it is easy to see that

$$\begin{aligned}
[\hat\pi(\mathrm{i}),\tau^1\langle a,b,c,w\rangle] &\equiv \tau^1\langle 0,\mathrm{i}b,-\mathrm{i}c,\mathrm{i}w\rangle \,, \\
[\hat\pi(\mathrm{i}),(\hat\sigma\circ\hat\pi^{-1}(\tau^1))\langle a,b,c,w\rangle] &\equiv (\hat\sigma\circ\hat\pi^{-1}(\tau^1))\langle 0,\mathrm{i}b,-\mathrm{i}c,\mathrm{i}w\rangle \,, \\
[\hat\pi(\mathrm{i}),j_2\langle a,c''\rangle] &\equiv j_2\langle 0,-2\mathrm{i}c''\rangle \,,
\end{aligned} \tag{5.63}$$

which proves formula (5.60).

Now, let us assume that we have verified (5.59) until degree k in (3.122). Using (2.42) we compute for $\omega_\alpha^1\in\Omega^1\mathfrak{a}$ and $\tilde\omega_\alpha^k\in\Omega^k\mathfrak{a}$

$$\begin{aligned}
&[\hat\pi(\mathrm{i}),\,\hat\sigma(\textstyle\sum_\alpha[\omega_\alpha^1,\tilde\omega_\alpha^k]_g)] = \sum_\alpha\big([\hat\pi(\mathrm{i}),[\hat\sigma(\omega_\alpha^1),\hat\pi(\tilde\omega_\alpha^k)]_g] - [\hat\pi(\mathrm{i}),[\hat\pi(\omega_\alpha^1),\hat\sigma(\tilde\omega_\alpha^k)]_g]\big) \\
&\quad = \textstyle\sum_\alpha\big([[\hat\pi(\mathrm{i}),\hat\sigma(\omega_\alpha^1)],\hat\pi(\tilde\omega_\alpha^k)] + [\hat\sigma(\omega_\alpha^1),[\hat\pi(\mathrm{i}),\hat\pi(\tilde\omega_\alpha^k)]] \\
&\qquad -[[\hat\pi(\mathrm{i}),\hat\pi(\omega_\alpha^1)],\hat\sigma(\tilde\omega_\alpha^k)]_g - [\hat\pi(\omega_\alpha^1),[\hat\pi(\mathrm{i}),\hat\sigma(\tilde\omega_\alpha^k)]]_g\big) \\
&\quad = \textstyle\sum_\alpha\big([\hat\sigma\circ\hat\pi^{-1}([\hat\pi(\mathrm{i}),\hat\pi(\omega_\alpha^1)]) + \hat\pi(\jmath^2\mathfrak{a}),\hat\pi(\tilde\omega_\alpha^k)] + [\hat\sigma(\omega_\alpha^1),[\hat\pi(\mathrm{i}),\hat\pi(\tilde\omega_\alpha^k)]] \\
&\qquad -[[\hat\pi(\mathrm{i}),\hat\pi(\omega_\alpha^1)],\hat\sigma(\tilde\omega_\alpha^k)]_g - [\hat\pi(\omega_\alpha^1),\hat\sigma\circ\hat\pi^{-1}([\hat\pi(\mathrm{i}),\hat\pi(\tilde\omega_\alpha^k)]) + \hat\pi(\jmath^{k+1}\mathfrak{a})]_g\big) \\
&\quad = \textstyle\sum_\alpha\big(\hat\sigma\circ\hat\pi^{-1}([[\hat\pi(\mathrm{i}),\hat\pi(\omega_\alpha^1)],\hat\pi(\tilde\omega_\alpha^k)]_g + [[\hat\pi(\omega_\alpha^1),[\hat\pi(\mathrm{i}),\hat\pi(\tilde\omega_\alpha^k)]]_g) + \hat\pi(\jmath^{k+2}\mathfrak{a})\big) \\
&\quad = \textstyle\sum_\alpha\big(\hat\sigma\circ\hat\pi^{-1}([\hat\pi(\mathrm{i}),[\hat\pi(\omega_\alpha^1),\hat\pi(\tilde\omega_\alpha^k)]_g] + \hat\pi(\jmath^{k+2}\mathfrak{a})\big) \,.
\end{aligned}$$

This finishes the proof of (5.59). As an immediate consequence we get for any $\omega^{n-1}\in\ker\hat\pi\cap\Omega^{n-1}\mathfrak{a}$

$$[\hat\pi(\mathrm{i}),\hat\pi(\jmath^n\mathfrak{a})] \ni [\hat\pi(\mathrm{i}),\hat\sigma(\omega^{n-1})] = \hat\sigma\circ\hat\pi^{-1}([\hat\pi(\mathrm{i}),\hat\pi(\omega^{n-1})]) + \hat\pi(\jmath^n\mathfrak{a}) = \hat\pi(\jmath^n\mathfrak{a}) \,. \tag{5.64}$$

Therefore, (3.117a) holds. Moreover, $\{\hat\pi(\mathrm{i}),\hat\pi(\jmath^n\mathfrak{a})\}\subset\{\hat\pi(\mathrm{i}),\hat\pi(\Omega^n\mathfrak{a})\}$ can be transformed into a subset of $\hat\pi(T_*^n\mathfrak{a})$ using $\{\hat\pi(\mathrm{i}),\hat\pi(\mathfrak{a})\}\subset\{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\}$ and $\{\hat\pi(\mathrm{i}),\hat\pi(\Omega^1\mathfrak{a})\}\subset\{\hat\pi(\mathfrak{a}),\hat\pi(\Omega^1\mathfrak{a})\}$. Finally, conditions (3.117d) and (3.117e) are satisfied due to (5.59).

5.6 The Factorization

There remains one step to do before we can compute physical quantities: We must find the representative $\mathfrak{e}(\theta)$ of the curvature θ orthogonal to the ideal $\mathbb{J}^2\mathfrak{g}$ just obtained in Section 5.5. In our case, this problem reduces to the search for the representative of $\tau^2+J_0+J_2'+J_3$ orthogonal to $J_0+J_2'+J_3$, for $\tau^2\in\hat\pi(\Omega^2\mathfrak{a})$. Technically, we decompose τ^2 and $J_0+J_2'+J_3$ into independent irreducible components, see Sections 5.6.2 *and 5.6.5. It turns out that the ideal only affects the generation space part (the space where products of $M_{u,d,e,n,N}$ and $M_{10,5}$ occur), see Section 5.6.3. Now it is easy to compute the orthogonal representative in Section 5.6.4. We extend this factorization to $\hat{\sigma}(\Omega^1\mathfrak{a})$ in Section 5.6.6.*

5.6.1 Introduction

Due to (5.41) and Lemma 28, the connection form and the curvature are given by

$$\rho \in (\Lambda^1 \otimes \hat{\pi}(\mathfrak{a})) \oplus (\Lambda^1 \otimes \hat{\pi}(\mathrm{u}(1))) \oplus (\Lambda^0\gamma \otimes \hat{\pi}(\Omega^1\mathfrak{a}))\,, \tag{5.65a}$$

$$\begin{aligned} \theta &= d\rho + \tfrac{1}{2}[\rho,\rho]_g \mod \mathbb{J}^2\mathfrak{g} \\ &\in (\Lambda^2 \otimes \hat{\pi}(\mathfrak{a})) \oplus (\Lambda^2 \otimes \hat{\pi}(\mathrm{u}(1))) \oplus (\Lambda^1\gamma \otimes \hat{\pi}(\Omega^1\mathfrak{a})) \oplus (\Lambda^0 \otimes (\hat{\pi}(\Omega^2\mathfrak{a}) \mod J))\,, \end{aligned} \tag{5.65b}$$

where $J = J_2' \oplus (J_0 + J_3)$. The explicit form of the differential and the commutator was given in (3.123). In order to compute the canonical representative $\mathfrak{e}(\theta)$ occurring in the bosonic action we must solve equations (3.127): We have to find for each given $\tau^2 \in \hat{\pi}(\Omega^2\mathfrak{a})$ an element $j \in J$ such that

$$\mathrm{tr}(\tilde{j}^*(\tau^2 + j)) = 0\,, \quad \forall\, \tilde{j} \in J\,. \tag{5.66}$$

In our application, τ^2 is the matrix part of the $\Lambda^0 \otimes \hat{\pi}(\Omega^2\mathfrak{a})$–component of θ.

5.6.2 The Decomposition of the Diagonal Blocks

Here we are going to study the contribution of the diagonal blocks τ_i of τ^2 to the trace (5.66). For this purpose, observe that in the parts $\pi_{10;5}(\mathrm{i}(b,b)')$ we can (and must) modulo J_2 replace

$$\begin{aligned} &M'_{\tilde{u}}{M'_{\tilde{u}}}^* - M'_d{M'_d}^* \mapsto M'_{\tilde{u}}{M'_{\tilde{u}}}^* - M'_u{M'_u}^* = -M'_{\tilde{u}}{M'_{\tilde{n}}}^* - M'_{\tilde{n}}{M'_{\tilde{u}}}^* - M'_{\tilde{n}}{M'_{\tilde{n}}}^*\,, \\ &\bar{M}'_e{M'_e}^T - {M'_{\tilde{u}}}^*M'_{\tilde{u}} \mapsto {M'_n}^*M'_n - {M'_{\tilde{u}}}^*M'_{\tilde{u}} = -3{M'_{\tilde{u}}}^*M'_{\tilde{n}} - 3{M'_{\tilde{n}}}^*M'_{\tilde{u}} + 9{M'_{\tilde{n}}}^*M'_{\tilde{n}}\,. \end{aligned} \tag{5.67}$$

In the diagonal part (5.28c) of τ^2 let us define

$$\begin{aligned} A^{10} &:= \tfrac{1}{2}\{\pi_{10}(a),\pi_{10}(a)\}\,, & A^5 &:= \tfrac{1}{2}\{\pi_5(a),\pi_5(a)\}\,, \\ B &:= -b^*b\,, & (b,b)' &:= bb^* - \tfrac{1}{5}\,\mathrm{tr}(bb^*)\mathbb{1}_5\,, \\ U^{10} &:= -cc^*\,, & \tilde{U}^{10} &:= -c\pi_{10,10}(b)^*\,, \\ \tilde{V}^{10} &:= -ww^*\,, & V^{10} &:= \tilde{V}^{10} - \mathrm{i}\pi_{10}(\mathrm{i}(b,b)')\,, \\ \tilde{V}^5 &:= -w^*w\,, & V^5 &:= \tilde{V}^5 + 9\mathrm{i}\pi_5(\mathrm{i}(b,b)')\,, \\ \tilde{W}^{10} &:= -\pi_{10,5}(b)w^*\,, & W^{10} &:= \tilde{W}^{10} - \mathrm{i}\pi_{10}(\mathrm{i}(b,b)')\,, \\ \tilde{W}^5 &:= -w^*\pi_{10,5}(b)\,, & W^5 &:= \tilde{W}^5 - 3\mathrm{i}\pi_5(\mathrm{i}(b,b)')\,. \end{aligned} \tag{5.68}$$

It is necessary to split $A^{10}, U^{10}, \tilde{U}^{10}, V^{10}$ and W^{10} according to (5.4a) and A^5, V^5 and W^5 according to (5.4b) into irreducible components. We clearly have

$$\mathrm{tr}(\tilde{U}^{10}) = \mathrm{tr}(\tilde{W}^{10}) = \mathrm{tr}(W^{10}) = \mathrm{tr}(\tilde{W}^5) = \mathrm{tr}(W^5) = 0\;. \tag{5.69a}$$

Moreover, due to $\underline{50} \otimes \underline{5} = \underline{175} \oplus \underline{75}$ we also have

$$\tilde{U}^{10}_{\underline{24}} \equiv 0\;. \tag{5.69b}$$

Thus, the non–vanishing components are

$$\begin{aligned} A^{10} &= A^{10}_{\underline{1}} \oplus A^{10}_{\underline{24}} \oplus A^{10}_{\underline{75}}\;, & A^5 &= A^5_{\underline{1}} \oplus A^5_{\underline{24}}\;, \\ U^{10} &= U^{10}_{\underline{1}} \oplus U^{10}_{\underline{24}} \oplus U^{10}_{\underline{75}}\;, & \tilde{U}^{10} &= \tilde{U}^{10}_{\underline{75}}\;, \\ V^{10} &= V^{10}_{\underline{1}} \oplus V^{10}_{\underline{24}} \oplus V^{10}_{\underline{75}}\;, & V^5 &= V^5_{\underline{1}} \oplus V^5_{\underline{24}}\;, \\ W^{10} &= W^{10}_{\underline{24}} \oplus W^{10}_{\underline{75}}\;, & W^5 &= W^5_{\underline{24}}\;. \end{aligned} \tag{5.70}$$

For these components we find

$$\begin{aligned} &A^{10}_{\underline{1}} = \tfrac{3}{10}\,\mathrm{tr}(A^5)\mathbb{1}_{10}\;, && A^5_{\underline{1}} = \tfrac{1}{5}\,\mathrm{tr}(A^5)\mathbb{1}_5\;, && A^{10}_{\underline{75}} = A^{10} - A^{10}_{\underline{24}} - A^{10}_{\underline{1}}\;, \\ &A^5_{\underline{24}} = A^5 - \tfrac{1}{5}\,\mathrm{tr}(A^5)\mathbb{1}_5\;, && && A^{10}_{\underline{24}} = -\tfrac{1}{3}\mathrm{i}\pi_{10}(\mathrm{i}A^5_{\underline{24}})\;, \\ &U^{10}_{\underline{1}} = \tfrac{1}{10}\,\mathrm{tr}(U^{10})\mathbb{1}_{10}\;, && W^{10}_{\underline{75}} = W^{10} - W^{10}_{\underline{24}}\;, && U^{10}_{\underline{75}} \equiv U_{10} - U^{10}_{\underline{1}} - U^{10}_{\underline{24}}\;, \\ &V^{10}_{\underline{1}} = \tfrac{1}{10}\,\mathrm{tr}(\tilde{V}^5)\mathbb{1}_{10}\;, && V^5_{\underline{1}} = \tfrac{1}{5}\,\mathrm{tr}(\tilde{V}^5)\mathbb{1}_5\;, && V^{10}_{\underline{75}} = V^{10} - V^{10}_{\underline{24}} - V^{10}_{\underline{1}}\;, \\ &V^5_{\underline{24}} = \tilde{V}^5 - \tfrac{1}{5}\,\mathrm{tr}(\tilde{V}^5)\mathbb{1}_5 + 9\mathrm{i}\pi_5(\mathrm{i}(b,b)')\;, && && V^{10}_{\underline{24}} = -\mathrm{i}\pi_{10}(\mathrm{i}\tilde{V}'_{\underline{24}} + \mathrm{i}(b,b)')\;, \\ &W^{10}_{\underline{24}} + W^{10*}_{\underline{24}} = -\tfrac{1}{3}\mathrm{i}\pi_{10}(\mathrm{i}W^5 + \mathrm{i}W^{5*})\;, && && W^{10}_{\underline{24}} - W^{10*}_{\underline{24}} = \tfrac{1}{3}\pi_{10}(W^5 - W^{5*})\;. \end{aligned} \tag{5.71}$$

Here, the term $\tilde{V}'_{\underline{24}} \in \mathrm{i}\mathfrak{a}$ is obtained as follows: Let $\{\beta_j\}_{j=1,\dots,24}$ be an orthonormal basis of $\mathfrak{a}$, $\mathrm{tr}(\beta_i\beta_j) = -\delta_{ij}$. We decompose $\mathrm{i}\tilde{V}'_{\underline{24}} := \sum_{j=1}^{24} a_j\beta_j$, where $a_j \in \mathbb{R}$. One finds the equation $a_j = -\frac{1}{3}\,\mathrm{tr}(\pi_{10}(\mathrm{i}\tilde{V}'_{\underline{24}})\pi_{10}(\beta_j)) \equiv -\frac{1}{3}\,\mathrm{tr}(\mathrm{i}\tilde{V}^{10}\pi_{10}(\beta_j))$, therefore,

$$\mathrm{i}\tilde{V}'_{\underline{24}} = -\tfrac{1}{3}\textstyle\sum_{j=1}^{24}\mathrm{tr}(\mathrm{i}\tilde{V}^{10}\pi_{10}(\beta_j))\beta_j\;. \tag{5.72}$$

Formula (5.72) shows the way how to obtain the other formulae of (5.71). One can prove

$$\begin{aligned} \textstyle\sum_{j=1}^{24}\mathrm{tr}(\mathrm{i}A^{10}\pi_{10}(\beta_j)) &\equiv \textstyle\sum_{j=1}^{24}\mathrm{tr}(\mathrm{i}A^5\pi_5(\beta_j))\;, \\ \textstyle\sum_{j=1}^{24}\mathrm{tr}(\mathrm{i}\tilde{W}^{10}\pi_{10}(\beta_j)) &\equiv \textstyle\sum_{j=1}^{24}\mathrm{tr}(\mathrm{i}\tilde{W}^5\pi_5(\beta_j))\;. \end{aligned} \tag{5.73}$$

However, there is no such simple relation between the 10– and 5–components of the $\tilde{V}$–part.

Let us summarize the structure of the diagonal matrix elements $\tau_{10}, \tau_5, \tau_1$ of (5.28b) in terms of the above decompositions:

$$
\begin{aligned}
\tau_{10} = {} & \mathrm{tr}(A^5)\mathbb{1}_{10} \otimes \tfrac{3}{10} {M'_{10}}^2 + \mathrm{tr}(V^5)\mathbb{1}_{10} \otimes \tfrac{1}{10} M'_{\tilde{n}} {M'_{\tilde{n}}}^* + \mathrm{tr}(U^{10})\mathbb{1}_{10} \otimes \tfrac{1}{10} M'_N {M'_N}^* \\
& + B\mathbb{1}_{10} \otimes (\tfrac{2}{5} M'_{\tilde{u}} {M'_{\tilde{u}}}^* + \tfrac{3}{5} M'_d {M'_d}^*) \\
& + A^{10}_{\underline{24}} \otimes {M'_{10}}^2 + V^{10}_{\underline{24}} \otimes M'_{\tilde{n}} {M'_{\tilde{n}}}^* + (W^{10}_{\underline{24}} + W^{10*}_{\underline{24}}) \otimes \tfrac{1}{2} (M'_{\tilde{u}} {M'_{\tilde{n}}}^* + M'_{\tilde{n}} {M'_{\tilde{u}}}^*) \\
& + U^{10}_{\underline{24}} \otimes M'_N {M'_N}^* + \mathrm{i}(W^{10}_{\underline{24}} - W^{10*}_{\underline{24}}) \otimes \tfrac{1}{2\mathrm{i}} (M'_{\tilde{u}} {M'_{\tilde{n}}}^* - M'_{\tilde{n}} {M'_{\tilde{u}}}^*) \\
& + A^{10}_{\underline{75}} \otimes {M'_{10}}^2 + V^{10}_{\underline{75}} \otimes M'_{\tilde{n}} {M'_{\tilde{n}}}^* + (W^{10}_{\underline{75}} + W^{10*}_{\underline{75}}) \otimes \tfrac{1}{2} (M'_{\tilde{u}} {M'_{\tilde{n}}}^* + M'_{\tilde{n}} {M'_{\tilde{u}}}^*) \\
& + U^{10}_{\underline{75}} \otimes M'_N {M'_N}^* + (\tilde{U}^{10}_{\underline{75}} + \tilde{U}^{10*}_{\underline{75}}) \otimes \tfrac{1}{2} (M'_N {M'_d}^* + M'_d {M'_N}^*) \\
& + \mathrm{i}(\tilde{U}^{10}_{\underline{75}} - \tilde{U}^{10*}_{\underline{75}}) \otimes \tfrac{1}{2\mathrm{i}} (M'_N {M'_d}^* - M'_d {M'_N}^*) \\
& + \mathrm{i}(W^{10}_{\underline{75}} - W^{10*}_{\underline{75}}) \otimes \tfrac{1}{2\mathrm{i}} (M'_{\tilde{u}} {M'_{\tilde{n}}}^* - M'_{\tilde{n}} {M'_{\tilde{u}}}^*) \; , \\
\tau_5 = {} & \mathrm{tr}(A^5)\mathbb{1}_5 \otimes \tfrac{1}{5} {M'_5}^2 + B\mathbb{1}_5 \otimes (\tfrac{4}{5} {M'_{\tilde{u}}}^* M'_{\tilde{u}} + \tfrac{1}{5} \overline{M}'_e {M'_e}^T) + \mathrm{tr}(V^5)\mathbb{1}_5 \otimes \tfrac{1}{5} {M'_{\tilde{n}}}^* M'_{\tilde{n}} \\
& + A^5_{\underline{24}} \otimes {M'_5}^2 + V^5_{\underline{24}} \otimes {M'_{\tilde{n}}}^* M'_{\tilde{n}} + (W^5 + W^{5*}) \otimes \tfrac{1}{2} ({M'_{\tilde{n}}}^* M'_{\tilde{u}} + {M'_{\tilde{u}}}^* M'_{\tilde{n}}) \\
& + \mathrm{i}(W^5 - W^{5*}) \otimes \tfrac{1}{2\mathrm{i}} ({M'_{\tilde{n}}}^* M'_{\tilde{u}} - {M'_{\tilde{u}}}^* M'_{\tilde{n}}) \; , \\
\tau_1 = {} & B \otimes {M'_e}^T \overline{M}'_e \; .
\end{aligned}
\tag{5.74}
$$

5.6.3 The Diagonal Blocks Modulo the Ideal

First, modulo J_2 we can replace $A^{10}_{\underline{75}} \otimes {M'_{10}}^2$ by

$$
A^{10}_{\underline{75}} \otimes (2M'_{\tilde{n}} {M'_{\tilde{u}}}^* + 2M'_{\tilde{u}} {M'_{\tilde{n}}}^* - 4M'_{\tilde{n}} {M'_{\tilde{n}}}^* - \tfrac{1}{2} M'_N {M'_N}^*) \; , \tag{5.75}
$$

see (5.48a). Now we add to τ^2 the element $j_0 \in J_0$ given by

$$
\begin{aligned}
\alpha &= \mathrm{tr}(A^5)\alpha_A + B\alpha_B + \mathrm{tr}(U^{10})\alpha_U + \mathrm{tr}(V^5)\alpha_V \; , \\
\mathrm{i}a &= A^5_{\underline{24}}\beta_A - \mathrm{i}\pi^{-1}_{10}(\mathrm{i}U^{10}_{\underline{24}})\beta_U + V^5_{\underline{24}}\check{\beta}_V - \mathrm{i}\pi^{-1}_{10}(\mathrm{i}V^{10}_{\underline{24}})\beta_V + (W^5 + W^{5*})\beta_W + \mathrm{i}(W^5 - W^{5*})\beta'_W \; , \\
\mathrm{i}v &= (V^{10}_{\underline{75}} - 4A^{10}_{\underline{75}})\gamma_V + (U^{10}_{\underline{75}} - \tfrac{1}{2}A^{10}_{\underline{75}})\gamma_U + (\tilde{U}^{10}_{\underline{75}} + \tilde{U}^{10*}_{\underline{75}})\tilde{\gamma}_U + \mathrm{i}(\tilde{U}^{10}_{\underline{75}} - \tilde{U}^{10*}_{\underline{75}})\tilde{\gamma}'_U \\
&\quad + (W^{10}_{\underline{75}} + W^{10*}_{\underline{75}} + 4A^{10}_{\underline{75}})\gamma_W + \mathrm{i}(W^{10}_{\underline{75}} - W^{10*}_{\underline{75}})\gamma'_W \; .
\end{aligned}
\tag{5.76a}
$$

Moreover, we add the element $j_2 \in J'_2$ given by

$$
\begin{aligned}
\mathrm{i}a &= A^5_{\underline{24}}\delta_A - \mathrm{i}\pi^{-1}_{10}(\mathrm{i}U^{10}_{\underline{24}})\delta_U + V^5_{\underline{24}}\check{\epsilon}_V - \mathrm{i}\pi^{-1}_{10}(\mathrm{i}V^{10}_{\underline{24}})\delta_V + (W^5 + W^{5*})\delta_W + \mathrm{i}(W^5 - W^{5*})\delta'_W \; , \\
\mathrm{i}v &= (V^{10}_{\underline{75}} - 4A^{10}_{\underline{75}})\epsilon_V + (U^{10}_{\underline{75}} - \tfrac{1}{2}A^{10}_{\underline{75}})\epsilon_U + (U^{10}_{\underline{75}} + U^{10*}_{\underline{75}})\tilde{\epsilon}_U + \mathrm{i}(U^{10}_{\underline{75}} - U^{10*}_{\underline{75}})\tilde{\epsilon}'_U \\
&\quad + (W^{10}_{\underline{75}} + W^{10*}_{\underline{75}} + 4A^{10}_{\underline{75}})\epsilon_W + \mathrm{i}(W^{10}_{\underline{75}} - W^{10*}_{\underline{75}})\epsilon'_W \; ,
\end{aligned}
\tag{5.76b}
$$

and the element $j_3 \in J_3$ given by

$$
v = \mathrm{tr}(A^5)\zeta_A + B\zeta_B + \mathrm{tr}(U^{10})\zeta_U + \mathrm{tr}(V^5)\zeta_V \; . \tag{5.76c}
$$

As a result, the matrix elements $\hat{\tau}_{10}, \hat{\tau}_5, \hat{\tau}_1$ of $\hat{\tau}^2 = \tau^2 + j_0 + j_2' + j_3$ take the form

$$\begin{aligned}
\hat{\tau}_{10} &= \mathrm{tr}(A^5)\mathbb{1}_{10}\otimes\hat{M}^{10}_{aa} + \mathrm{tr}(U^{10})\mathbb{1}_{10}\otimes\hat{M}^{10}_{cc} + \mathrm{tr}(V^5)\mathbb{1}_{10}\otimes\hat{M}^{10}_{nn} + B\mathbb{1}_{10}\otimes\hat{M}^{10}_{bb} \\
&\quad - \tfrac{1}{3}\mathrm{i}\pi_{10}(\mathrm{i}A^5_{\underline{24}})\otimes M^{10}_{aa} + U^{10}_{\underline{24}}\otimes M^{10}_{cc} + V^{10}_{\underline{24}}\otimes M^{10}_{nn} - \mathrm{i}\pi_{10}(\mathrm{i}V^5_{\underline{24}})\otimes\check{M}^{10}_{nn} \\
&\quad - \tfrac{1}{3}\mathrm{i}\pi_{10}(\mathrm{i}W^5 + \mathrm{i}W^{5*})\otimes M^{10}_{\{un\}} + \tfrac{1}{3}\mathrm{i}\pi_{10}(W^5 - W^{5*})\otimes M^{10}_{[un]} \\
&\quad + (V^{10}_{\underline{75}} - 4A^{10}_{\underline{75}})\otimes\tilde{M}^{10}_{nn} + (U^{10}_{\underline{75}} - \tfrac{1}{2}A^{10}_{\underline{75}})\otimes\tilde{M}^{10}_{cc} + (\tilde{U}^{10}_{\underline{75}} + \tilde{U}^{10}_{\underline{75}})\otimes\tilde{M}^{10}_{\{cd\}} \\
&\quad + \mathrm{i}(\tilde{U}^{10}_{\underline{75}} - \tilde{U}^{10}_{\underline{75}})\otimes\tilde{M}^{10}_{[cd]} + (W^{10}_{\underline{75}} + W^{10*}_{\underline{75}} + 4A^{10}_{\underline{75}})\otimes\tilde{M}^{10}_{\{un\}} \\
&\quad + \mathrm{i}(W^{10}_{\underline{75}} - W^{10*}_{\underline{75}})\otimes\tilde{M}^{10}_{[un]}\,, \qquad (5.77)\\
\hat{\tau}_5 &= \mathrm{tr}(A^5)\mathbb{1}_5\otimes\hat{M}^5_{aa} + \mathrm{tr}(U^{10})\mathbb{1}_5\otimes\hat{M}^5_{cc} + \mathrm{tr}(V^5)\mathbb{1}_5\otimes\hat{M}^5_{nn} + B\mathbb{1}_5\otimes\hat{M}^5_{bb} \\
&\quad + A^5_{\underline{24}}\otimes M^5_{aa} + V^5_{\underline{24}}\otimes\check{M}^5_{nn} - \mathrm{i}\pi^{-1}_{10}(\mathrm{i}V^{10}_{\underline{24}})\otimes M^5_{nn} - \mathrm{i}\pi^{-1}_{10}(\mathrm{i}U^{10}_{\underline{24}})\otimes M^5_{cc} \\
&\quad + (W^5 + W^{5*})\otimes M^5_{\{un\}} + \mathrm{i}(W^5 - W^{5*})\otimes M^5_{[un]}\,, \\
\hat{\tau}_1 &= \mathrm{tr}(A^5)\otimes\hat{M}^1_{aa} + \mathrm{tr}(U^{10})\otimes\hat{M}^1_{cc} + B\otimes\hat{M}^1_{bb} + \mathrm{tr}(V^5)\otimes\hat{M}^1_{nn}\,,
\end{aligned}$$

where

$$\begin{aligned}
&\hat{M}^{10}_{aa} := \tfrac{3}{10}{M'_{10}}^2 + (\tfrac{3}{5}\alpha_A + \zeta_A)\mathbb{1}_6\,, && \hat{M}^{10}_{cc} := \tfrac{1}{10}M'_N{M'_N}^* + (\tfrac{3}{5}\alpha_U + \zeta_U)\mathbb{1}_6\,, \\
&\hat{M}^{10}_{nn} = \tfrac{1}{10}M'_{\tilde{n}}{M'_{\tilde{n}}}^* + (\tfrac{3}{5}\alpha_V + \zeta_V)\mathbb{1}_6\,, \\
&\hat{M}^{10}_{bb} := \tfrac{2}{5}M'_{\tilde{u}}{M'_{\tilde{u}}}^* + \tfrac{3}{5}M'_d{M'_d}^* + (\tfrac{3}{5}\alpha_B + \zeta_B)\mathbb{1}_6\,, \\
&M^{10}_{aa} := {M'_{10}}^2 + \beta_A\mathbb{1}_6 + 3\delta_A M^2_{ud}\,, && M^{10}_{nn} := M'_{\tilde{n}}{M'_{\tilde{n}}}^* + \tfrac{1}{3}\beta_V\mathbb{1}_6 + \delta_V M^2_{ud}\,, \\
&M^{10}_{cc} := M'_N{M'_N}^* + \tfrac{1}{3}\beta_U\mathbb{1}_6 + \delta_U M^2_{ud}\,, && \check{M}^{10}_{nn} := \tfrac{1}{3}\check{\beta}_V\mathbb{1}_6 + \check{\delta}_V M^2_{ud}\,, \\
&M^{10}_{\{un\}} := \tfrac{1}{2}(M'_{\tilde{u}}{M'_{\tilde{n}}}^* + M'_{\tilde{n}}{M'_{\tilde{u}}}^*) + \beta_W\mathbb{1}_6 + 3\delta_W M^2_{ud}\,, \\
&M^{10}_{[un]} := \tfrac{1}{2\mathrm{i}}(M'_{\tilde{u}}{M'_{\tilde{n}}}^* - M'_{\tilde{n}}{M'_{\tilde{u}}}^*) + \beta'_W\mathbb{1}_6 + 3\delta'_W M^2_{ud}\,, \\
&\tilde{M}^{10}_{nn} := M'_{\tilde{n}}{M'_{\tilde{n}}}^* + \gamma_V\mathbb{1}_6 + \epsilon_V\tilde{M}^2_V\,, && \tilde{M}^{10}_{cc} := M'_N{M'_N}^* + \gamma_U\mathbb{1}_6 + \epsilon_U\tilde{M}^2_V\,, \\
&\tilde{M}^{10}_{\{cd\}} := \tfrac{1}{2}(M'_N{M'_d}^* + M'_d{M'_N}^*) + \tilde{\gamma}_U\mathbb{1}_6 + \tilde{\epsilon}_U\tilde{M}^2_V\,, \\
&\tilde{M}^{10}_{[cd]} := \tfrac{1}{2\mathrm{i}}(M'_N{M'_d}^* - M'_d{M'_N}^*) + \tilde{\gamma}'_U\mathbb{1}_6 + \tilde{\epsilon}'_U\tilde{M}^2_V\,, \\
&\tilde{M}^{10}_{\{un\}} := \tfrac{1}{2}(M'_{\tilde{u}}{M'_{\tilde{n}}}^* + M'_{\tilde{n}}{M'_{\tilde{u}}}^*) + \gamma_W\mathbb{1}_6 + \epsilon_W\tilde{M}^2_V\,, \\
&\tilde{M}^{10}_{[un]} := \tfrac{1}{2\mathrm{i}}(M'_{\tilde{u}}{M'_{\tilde{n}}}^* - M'_{\tilde{n}}{M'_{\tilde{u}}}^*) + \gamma'_W\mathbb{1}_6 + \epsilon'_W\tilde{M}^2_V\,, \qquad (5.78)\\
&\hat{M}^5_{aa} := \tfrac{1}{5}{M'_5}^2 + (\tfrac{2}{5}\alpha_A + \zeta_A)\mathbb{1}_6\,, && \hat{M}^5_{cc} := (\tfrac{2}{5}\alpha_U + \zeta_U)\mathbb{1}_6\,, \\
&\hat{M}^5_{nn} := \tfrac{1}{5}{M'_{\tilde{n}}}^*M'_{\tilde{n}} + (\tfrac{2}{5}\alpha_V + \zeta_V)\mathbb{1}_6\,, \\
&\hat{M}^5_{bb} := \tfrac{4}{5}{M'_{\tilde{u}}}^*M'_{\tilde{u}} + \tfrac{1}{5}\bar{M}'_e{M'_e}^T + (\tfrac{2}{5}\alpha_B + \zeta_B)\mathbb{1}_6\,, \\
&M^5_{aa} := {M'_5}^2 + \beta_A\mathbb{1}_6 + \delta_A M^2_{en}\,, && \check{M}^5_{nn} := {M'_{\tilde{n}}}^*M'_{\tilde{n}} + \check{\beta}_V\mathbb{1}_6 + \check{\delta}_V M^2_{en}\,, \\
&M^5_{nn} := \beta_V\mathbb{1}_6 + \delta_V M^2_{en}\,, && M^5_{cc} := \beta_U\mathbb{1}_6 + \delta_U M^2_{en}\,, \\
&M^5_{\{un\}} := \tfrac{1}{2}({M'_{\tilde{n}}}^*M'_{\tilde{u}} + {M'_{\tilde{u}}}^*M'_{\tilde{n}}) + \beta_W\mathbb{1}_6 + \delta_W M^2_{en}\,, \\
&M^5_{[un]} := \tfrac{1}{2\mathrm{i}}({M'_{\tilde{n}}}^*M'_{\tilde{u}} - {M'_{\tilde{u}}}^*M'_{\tilde{n}}) + \beta'_W\mathbb{1}_6 + \delta'_W M^2_{en}\,, \\
&\hat{M}^1_{aa} := \zeta_A\mathbb{1}_6\,, && \hat{M}^1_{nn} := \zeta_V\mathbb{1}_6\,, \\
&\hat{M}^1_{cc} := \zeta_U\mathbb{1}_6\,, && \hat{M}^1_{bb} := {M'_e}^T\bar{M}'_e + \zeta_B\mathbb{1}_6\,.
\end{aligned}$$

5.6.4 The Explicit Calculation

The components $\mathrm{tr}(A^5), \ldots, \mathrm{i}(W^5 - W^{5*})$ occurring in (5.77) are independent. The real constants $\alpha_A, \ldots \zeta_V$ are determined by equation (5.66). Thus, we must solve the system

$$\begin{aligned}
0 &= 10 \cdot \tfrac{3}{5}(\tfrac{3}{10}\,\mathrm{tr}({M'_{10}}^2) + 6(\tfrac{3}{5}\alpha_A + \zeta_A)) + 5 \cdot \tfrac{2}{5}(\tfrac{1}{5}\,\mathrm{tr}({M'_5}^2) + 6(\tfrac{2}{5}\alpha_A + \zeta_A))\,, \\
0 &= 10(\tfrac{3}{10}\,\mathrm{tr}({M'_{10}}^2) + 6(\tfrac{3}{5}\alpha_A + \zeta_A)) + 5(\tfrac{1}{5}\,\mathrm{tr}({M'_5}^2) + 6(\tfrac{2}{5}\alpha_A + \zeta_A)) + 6\zeta_A\,, \\
0 &= 10 \cdot \tfrac{3}{5}(\tfrac{1}{10}\,\mathrm{tr}(M'_N {M'_N}^*) + 6(\tfrac{3}{5}\alpha_U + \zeta_U)) + 5 \cdot \tfrac{2}{5} \cdot 6(\tfrac{2}{5}\alpha_U + \zeta_U)\,, \\
0 &= 10(\tfrac{1}{10}\,\mathrm{tr}(M'_N {M'_N}^*) + 6(\tfrac{3}{5}\alpha_U + \zeta_U)) + 5 \cdot 6(\tfrac{2}{5}\alpha_U + \zeta_U) + 6\zeta_U\,, \\
0 &= 10 \cdot \tfrac{3}{5}(\tfrac{1}{10}\,\mathrm{tr}(M'_{\tilde n} {M'_{\tilde n}}^*) + 6(\tfrac{3}{5}\alpha_V + \zeta_V)) + 5 \cdot \tfrac{2}{5}(\tfrac{1}{5}\,\mathrm{tr}(M'_{\tilde n} {M'_{\tilde n}}^*) + 6(\tfrac{2}{5}\alpha_V + \zeta_V))\,, \\
0 &= 10(\tfrac{1}{10}\,\mathrm{tr}(M'_{\tilde n} {M'_{\tilde n}}^*) + 6(\tfrac{3}{5}\alpha_V + \zeta_V)) + 5(\tfrac{1}{5}\,\mathrm{tr}(M'_{\tilde n} {M'_{\tilde n}}^*) + 6(\tfrac{2}{5}\alpha_V + \zeta_V)) + 6\zeta_V\,, \\
0 &= 10 \cdot \tfrac{3}{5}(\tfrac{2}{5}\,\mathrm{tr}(M'_{\tilde u} {M'_{\tilde u}}^*) + \tfrac{3}{5}\,\mathrm{tr}(M'_d {M'_d}^*) + 6(\tfrac{3}{5}\alpha_B + \zeta_B)) \\
&\quad + 5 \cdot \tfrac{2}{5}(\tfrac{4}{5}\,\mathrm{tr}({M'_{\tilde u}}^* M'_{\tilde u}) + \tfrac{1}{5}\,\mathrm{tr}(\bar{M}'_e {M'_e}^T) + 6(\tfrac{2}{5}\alpha_B + \zeta_B))\,, \\
0 &= 10(\tfrac{2}{5}\,\mathrm{tr}(M'_{\tilde u} {M'_{\tilde u}}^*) + \tfrac{3}{5}\,\mathrm{tr}(M'_d {M'_d}^*) + 6(\tfrac{3}{5}\alpha_B + \zeta_B)) \\
&\quad + 5(\tfrac{4}{5}\,\mathrm{tr}({M'_{\tilde u}}^* M'_{\tilde u}) + \tfrac{1}{5}\,\mathrm{tr}(\bar{M}'_e {M'_e}^T) + 6(\tfrac{2}{5}\alpha_B + \zeta_B)) + (\mathrm{tr}({M'_e}^T M'_e) + 6\zeta_B)\,, \\[4pt]
0 &= \tfrac{1}{3}(\mathrm{tr}({M'_{10}}^2) + 6\beta_A) + \mathrm{tr}({M'_5}^2) + 6\beta_A\,, \\
0 &= \mathrm{tr}({M'_{10}}^2 M_{ud}^2) + 3\delta_A\,\mathrm{tr}((M_{ud}^2)^2) + \mathrm{tr}({M'_5}^2 M_{en}^2) + \delta_A\,\mathrm{tr}((M_{en}^2)^2)\,, \\
0 &= (\mathrm{tr}(M'_N {M'_N}^*) + 2\beta_U) + 6\beta_U\,, \\
0 &= 3\,\mathrm{tr}(M'_N {M'_N}^* M_{ud}^2) + 3\delta_U\,\mathrm{tr}((M_{ud}^2)^2) + \delta_U\,\mathrm{tr}((M_{en}^2)^2)\,, \\
0 &= \mathrm{tr}(M'_{\tilde n} {M'_{\tilde n}}^*) + \tfrac{1}{3} \cdot 6\beta_V + 6\beta_V\,, \\
0 &= 3\,\mathrm{tr}(M'_{\tilde n} {M'_{\tilde n}}^* M_{ud}^2) + 3\delta_V\,\mathrm{tr}((M_{ud}^2)^2) + \delta_V\,\mathrm{tr}((M_{en}^2)^2)\,, \\
0 &= \tfrac{1}{3} \cdot 6\check\beta_V + \mathrm{tr}({M'_{\tilde n}}^* M'_{\tilde n}) + 6\check\beta_V\,, \qquad (5.79) \\
0 &= 3\check\delta_V\,\mathrm{tr}((M_{ud}^2)^2) + \mathrm{tr}({M'_{\tilde n}}^* M'_{\tilde n} M_{en}^2) + \check\delta_V\,\mathrm{tr}((M_{en}^2)^2)\,, \\
0 &= \tfrac{1}{3}(\tfrac{1}{2}\,\mathrm{tr}(M'_{\tilde u} {M'_{\tilde n}}^* + M'_{\tilde n} {M'_{\tilde u}}^*) + 6\beta_W) + \tfrac{1}{2}\,\mathrm{tr}({M'_{\tilde n}}^* M'_{\tilde u} + {M'_{\tilde u}}^* M'_{\tilde n}) + 6\beta_W\,, \\
0 &= \tfrac{1}{3}(\tfrac{1}{2\mathrm{i}}\,\mathrm{tr}(M'_{\tilde u} {M'_{\tilde n}}^* - M'_{\tilde n} {M'_{\tilde u}}^*) + 6\beta'_W) + \tfrac{1}{2\mathrm{i}}\,\mathrm{tr}({M'_{\tilde n}}^* M'_{\tilde u} - {M'_{\tilde u}}^* M'_{\tilde n}) + 6\beta'_W\,, \\
0 &= \tfrac{1}{2}\,\mathrm{tr}((M'_{\tilde u} {M'_{\tilde n}}^* + M'_{\tilde n} {M'_{\tilde u}}^*) M_{ud}^2) + 3\delta_W\,\mathrm{tr}((M_{ud}^2)^2) \\
&\qquad + \tfrac{1}{2}\,\mathrm{tr}(({M'_{\tilde n}}^* M'_{\tilde u} + {M'_{\tilde u}}^* M'_{\tilde n}) M_{en}^2) + \delta_W\,\mathrm{tr}((M_{en}^2)^2)\,, \\
0 &= \tfrac{1}{2\mathrm{i}}\,\mathrm{tr}((M'_{\tilde u} {M'_{\tilde n}}^* - M'_{\tilde n} {M'_{\tilde u}}^*) M_{ud}^2) + 3\delta'_W\,\mathrm{tr}((M_{ud}^2)^2) \\
&\qquad + (\tfrac{1}{2\mathrm{i}}\,\mathrm{tr}(({M'_{\tilde n}}^* M'_{\tilde u} - {M'_{\tilde u}}^* M'_{\tilde n}) M_{en}^2) + \delta'_W\,\mathrm{tr}((M_{en}^2)^2)\,, \\
0 &= \mathrm{tr}(M'_N {M'_N}^*) + 6\gamma_U\,, \\
0 &= \tfrac{1}{2}\,\mathrm{tr}(M'_N {M'_d}^* + M'_d {M'_N}^*) + 6\tilde\gamma_U\,, \\
0 &= \tfrac{1}{2\mathrm{i}}\,\mathrm{tr}(M'_N {M'_d}^* - M'_d {M'_N}^*) + 6\tilde\gamma'_U\,, \\
0 &= \mathrm{tr}(M'_{\tilde n} {M'_{\tilde n}}^*) + 6\gamma_V\,, \\
0 &= \tfrac{1}{2}\,\mathrm{tr}(M'_{\tilde u} {M'_{\tilde n}}^* + M'_{\tilde n} {M'_{\tilde u}}^*) + 6\gamma_W\,,
\end{aligned}$$

$$
\begin{aligned}
0 &= \tfrac{1}{2\mathrm{i}}\operatorname{tr}(M'_{\tilde u}M'^{\,*}_{\tilde n} - M'_{\tilde n}M'^{\,*}_{\tilde u}) + 6\gamma'_W \;, \\
0 &= \operatorname{tr}(M'_N M'^{\,*}_N \tilde M_V^2) + \epsilon_U \operatorname{tr}((\tilde M_V^2)^2) \;, \\
0 &= \tfrac{1}{2}\operatorname{tr}((M'_N M'^{\,*}_d + M'_d M'^{\,*}_N)\tilde M_V^2) + \tilde\epsilon_U \operatorname{tr}((\tilde M_V^2)^2) \;, \\
0 &= \tfrac{1}{2\mathrm{i}}\operatorname{tr}((M'_N M'^{\,*}_d - M'_d M'^{\,*}_N)\tilde M_V^2) + \tilde\epsilon'_U \operatorname{tr}((\tilde M_V^2)^2) \;, \\
0 &= \operatorname{tr}(M'_{\tilde n} M'^{\,*}_{\tilde n} \tilde M_V^2) + \epsilon_V \operatorname{tr}((\tilde M_V^2)^2) \;, \\
0 &= \tfrac{1}{2}\operatorname{tr}((M'_{\tilde u}M'^{\,*}_{\tilde n} + M'_{\tilde n}M'^{\,*}_{\tilde u})\tilde M_V^2) + \epsilon_W \operatorname{tr}((\tilde M_V^2)^2) \;, \\
0 &= \tfrac{1}{2\mathrm{i}}\operatorname{tr}((M'_{\tilde u}M'^{\,*}_{\tilde n} - M'_{\tilde n}M'^{\,*}_{\tilde u})\tilde M_V^2) + \epsilon'_W \operatorname{tr}((\tilde M_V^2)^2) \;.
\end{aligned}
$$

The solution is

$$
\begin{aligned}
&\alpha_A = -\tfrac{1}{8}\operatorname{tr}(M'^{\,2}_{10}) + \tfrac{1}{24}\operatorname{tr}(M'^{\,2}_5) \;, &&\alpha_B = -\tfrac{1}{4}\operatorname{tr}(M'_d M'^{\,*}_d) + \tfrac{1}{4}\operatorname{tr}(M'_e M'^{\,*}_e) \;, \\
&\alpha_U = -\tfrac{1}{24}\operatorname{tr}(M'_N M'^{\,*}_N) \;, &&\alpha_V = 0 \;, \\
&\zeta_A = \tfrac{1}{32}\operatorname{tr}(M'^{\,2}_{10}) - \tfrac{1}{32}\operatorname{tr}(M'^{\,2}_5) \;, &&\zeta_V = -\tfrac{1}{48}\operatorname{tr}(M'_{\tilde n} M'^{\,*}_{\tilde n}) \;, \\
&\zeta_B = -\tfrac{1}{12}\operatorname{tr}(M'_{\tilde u} M'^{\,*}_{\tilde u}) + \tfrac{1}{16}\operatorname{tr}(M'_d M'^{\,*}_d) - \tfrac{7}{48}\operatorname{tr}(M'_e M'^{\,*}_e) \;, \\
&\zeta_U = \tfrac{1}{96}\operatorname{tr}(M'_N M'^{\,*}_N) \;, \\
&\beta_A = -\tfrac{1}{8}\operatorname{tr}(M'^{\,2}_5) - \tfrac{1}{24}\operatorname{tr}(M'^{\,2}_{10}) \;, &&\delta_A = -\frac{\operatorname{tr}(M'^{\,2}_{10} M^2_{ud} + M'^{\,2}_5 M^2_{en})}{\operatorname{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;, \\
&\beta_U = -\tfrac{1}{8}\operatorname{tr}(M'_N M'^{\,*}_N) \;, &&\delta_U = -3\frac{\operatorname{tr}(M'_N M'^{\,*}_N M^2_{ud})}{\operatorname{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;, \\
&\beta_V = -\tfrac{1}{8}\operatorname{tr}(M'_{\tilde n} M'^{\,*}_{\tilde n}) \;, &&\check\beta_V = -\tfrac{1}{8}\operatorname{tr}(M'_{\tilde n} M'^{\,*}_{\tilde n}) \;, \\
&\delta_V = -\frac{\operatorname{tr}(3M'_{\tilde n} M'^{\,*}_{\tilde n} M^2_{ud})}{\operatorname{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;, &&\check\delta_V = -\frac{\operatorname{tr}(M'^{\,*}_{\tilde n} M'_{\tilde n} M^2_{en})}{\operatorname{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;, \\
&\beta_W = -\tfrac{1}{12}\operatorname{tr}(M'_{\tilde u} M'^{\,*}_{\tilde n} + M'_{\tilde n} M'^{\,*}_{\tilde u}) \;, &&\beta'_W = -\tfrac{1}{12\mathrm{i}}\operatorname{tr}(M'_{\tilde u} M'^{\,*}_{\tilde n} - M'_{\tilde n} M'^{\,*}_{\tilde u}) \;, \\
&\delta_W = -\frac{\operatorname{tr}((M'_{\tilde u} M'^{\,*}_{\tilde n} + M'_{\tilde n} M'^{\,*}_{\tilde u}) M^2_{ud} + (M'^{\,*}_{\tilde n} M'_{\tilde u} + M'^{\,*}_{\tilde u} M'_{\tilde n}) M^2_{en})}{2\operatorname{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;, \\
&\delta'_W = -\frac{\operatorname{tr}((M'_{\tilde u} M'^{\,*}_{\tilde n} - M'_{\tilde n} M'^{\,*}_{\tilde u}) M^2_{ud} + (M'^{\,*}_{\tilde n} M'_{\tilde u} - M'^{\,*}_{\tilde u} M'_{\tilde n}) M^2_{en})}{2\mathrm{i}\operatorname{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;, \\
&\gamma_V = -\tfrac{1}{6}\operatorname{tr}(M'_{\tilde n} M'^{\,*}_{\tilde n}) \;, &&\epsilon_V = -\frac{\operatorname{tr}(M'_{\tilde n} M'^{\,*}_{\tilde n} \tilde M^2_V)}{\operatorname{tr}((\tilde M^2_V)^2)} \;, \\
&\gamma_W = -\tfrac{1}{12}\operatorname{tr}(M'_{\tilde u} M'^{\,*}_{\tilde n} + M'_{\tilde n} M'^{\,*}_{\tilde u}) \;, &&\epsilon_W = -\frac{\operatorname{tr}((M'_{\tilde u} M'^{\,*}_{\tilde n} + M'_{\tilde n} M'^{\,*}_{\tilde u})\tilde M^2_V)}{2\operatorname{tr}((\tilde M^2_V)^2)} \;, \\
&\gamma'_W = -\tfrac{1}{12\mathrm{i}}\operatorname{tr}(M'_{\tilde u} M'^{\,*}_{\tilde n} - M'_{\tilde n} M'^{\,*}_{\tilde u}) \;, &&\epsilon'_W = -\frac{\operatorname{tr}((M'_{\tilde u} M'^{\,*}_{\tilde n} - M'_{\tilde n} M'^{\,*}_{\tilde u})\tilde M^2_V)}{2\mathrm{i}\operatorname{tr}((\tilde M^2_V)^2)} \;, \\
&\gamma_U = -\tfrac{1}{6}\operatorname{tr}(M'_N M'^{\,*}_N) \;, &&\epsilon_U = -\frac{\operatorname{tr}(M'_N M'^{\,*}_N \tilde M^2_V)}{\operatorname{tr}((\tilde M^2_V)^2)} \;, \\
&\tilde\gamma_U = -\tfrac{1}{12}\operatorname{tr}(M'_N M'^{\,*}_d + M'_d M'^{\,*}_N) \;, &&\tilde\epsilon_U = -\frac{\operatorname{tr}((M'_N M'^{\,*}_d + M'_d M'^{\,*}_N)\tilde M^2_V)}{2\operatorname{tr}((\tilde M^2_V)^2)} \;, \\
&\tilde\gamma'_U = -\tfrac{1}{12\mathrm{i}}\operatorname{tr}(M'_N M'^{\,*}_d - M'_d M'^{\,*}_N) \;, &&\tilde\epsilon'_U = -\frac{\operatorname{tr}((M'_N M'^{\,*}_d - M'_d M'^{\,*}_N)\tilde M^2_V)}{2\mathrm{i}\operatorname{tr}((\tilde M^2_V)^2)} \;.
\end{aligned}
\tag{5.80}
$$

In summary,

$$\begin{aligned}
&\hat{M}^{10}_{aa} = \tfrac{3}{10}M'^{\,2}_{10} - \tfrac{1}{160}\,\mathrm{tr}(7M'^{\,2}_{10} + M'^{\,2}_{5})\mathbb{1}_6\,, \qquad \hat{M}^{10}_{cc} = \tfrac{1}{10}M'_N M'^{\,*}_N - \tfrac{7}{480}\,\mathrm{tr}(M'_N M'^{\,*}_N)\,,\\
&\hat{M}^{10}_{nn} = \tfrac{1}{10}M'_{\tilde{n}} M'^{\,*}_{\tilde{n}} - \tfrac{1}{48}\,\mathrm{tr}(M'_{\tilde{n}} M'^{\,*}_{\tilde{n}})\mathbb{1}_6\,,\\
&\hat{M}^{10}_{bb} = \tfrac{2}{5}M'_{\tilde{u}} M'^{\,*}_{\tilde{u}} + \tfrac{3}{5}M'_d M'^{\,*}_d - \mathrm{tr}(\tfrac{1}{12}M'_{\tilde{u}} M'^{\,*}_{\tilde{u}} + \tfrac{7}{80}M'_d M'^{\,*}_d - \tfrac{1}{240}M'_e M'^{\,*}_e)\mathbb{1}_6\,,\\
&\hat{M}^{5}_{aa} = \tfrac{1}{5}M'^{\,2}_{5} - \tfrac{1}{480}\,\mathrm{tr}(9M'^{\,2}_{10} + 7M'^{\,2}_{5})\mathbb{1}_6)\,, \qquad \hat{M}^{5}_{cc} = -\tfrac{3}{480}\,\mathrm{tr}(M'_N M'^{\,*}_N)\,,\\
&\hat{M}^{5}_{nn} = \tfrac{1}{5}M'^{\,*}_{\tilde{n}} M'_{\tilde{n}} - \tfrac{1}{48}\,\mathrm{tr}(M'_{\tilde{n}} M'^{\,*}_{\tilde{n}})\mathbb{1}_6\,,\\
&\hat{M}^{5}_{bb} = \tfrac{4}{5}M'^{\,*}_{\tilde{u}} M'_{\tilde{u}} + \tfrac{1}{5}\overline{M'_e} M'^{\,T}_e - \mathrm{tr}(\tfrac{1}{12}M'_{\tilde{u}} M'^{\,*}_{\tilde{u}} + \tfrac{3}{80}M'_d M'^{\,*}_d + \tfrac{11}{240}M'_e M'^{\,*}_e)\mathbb{1}_6\,,\\
&\hat{M}^{1}_{aa} = \tfrac{1}{32}\,\mathrm{tr}(M'^{\,2}_{10} - M'^{\,2}_{5})\mathbb{1}_6\,, \qquad \hat{M}^{1}_{cc} = \tfrac{1}{96}\,\mathrm{tr}(M'_N M'^{\,*}_N)\,,\\
&\hat{M}^{1}_{nn} = -\tfrac{1}{48}\,\mathrm{tr}(M'_{\tilde{n}} M'^{\,*}_{\tilde{n}})\mathbb{1}_6\,,\\
&\hat{M}^{1}_{bb} = M'^{\,T}_e \overline{M'_e} - \mathrm{tr}(\tfrac{1}{12}M'_{\tilde{u}} M'^{\,*}_{\tilde{u}} - \tfrac{1}{16}M'_d M'^{\,*}_d + \tfrac{7}{48}M'_e M'^{\,*}_e)\mathbb{1}_6\,.
\end{aligned} \tag{5.81}$$

5.6.5 The $\underline{10} \oplus \underline{40}^*$–Decomposition

We must also factorize the $\tau_{\widetilde{10,5}}$–blocks in (5.28b) with respect to the $\underline{40}^*$–part of $\hat{\pi}(\jmath^2\mathfrak{a})$ determined by c'' in (5.58). For this purpose we must decompose $\tau_{\widetilde{10,5}}$ according to (5.4e):

$$\begin{aligned}
\pi_{10,10}(b)\overline{w} &\equiv (\pi_{10,10}(b)\overline{w})_{\underline{40}} \oplus (\pi_{10,10}(b)\overline{w})_{\underline{10}}\,,\\
c\overline{w} &\equiv (c\overline{w})_{\underline{40}} \oplus (c\overline{w})_{\underline{10}}\,,\\
\overline{c\pi_{10,5}(b)} &\equiv (\overline{c\pi_{10,5}(b)})_{\underline{40}}\,.
\end{aligned} \tag{5.82}$$

We have $(\overline{c\pi_{10,5}(b)})_{\underline{10}} \equiv 0$ due to $\underline{50} \otimes \underline{5}^* = \underline{210} \oplus \underline{40}$. The $\underline{10}$–representation remains unaffected by the factorization, whereas the $\underline{40}^*$–part must be orthogonal to $c'' \otimes M'_N \overline{M'_n}$. This yields with (5.44b) the result

$$\begin{aligned}
\hat{\tau}_{\widetilde{10,5}} = &-(\pi_{10,10}(b)\overline{w})_{\underline{10}} \otimes M'_d \overline{M'_{\tilde{n}}} - (c\overline{w})_{\underline{10}} \otimes M'_N \overline{M'_{\tilde{n}}}\\
&-(\pi_{10,10}(b)\overline{w})_{\underline{40}} \otimes M'_{d\tilde{n}} - (\tfrac{1}{4}(c\overline{w})_{\underline{40}} + \tfrac{3}{4}\overline{c\pi_{10,5}(b)}) \otimes M'_{Nu}\,,
\end{aligned} \tag{5.83a}$$

where

$$\begin{aligned}
M'_{d\tilde{n}} &:= M'_d \overline{M'_{\tilde{n}}} - \frac{\mathrm{tr}(M'_d \overline{M'_{\tilde{n}}}(M'_N \overline{M'_n})^*)}{\mathrm{tr}((M'_N \overline{M'_n})(M'_N \overline{M'_n})^*)} M'_N \overline{M'_n}\,,\\
M'_{Nu} &:= M'_N \overline{M'_u} - \frac{\mathrm{tr}(M'_N \overline{M'_u}(M'_N \overline{M'_n})^*}{\mathrm{tr}((M'_N \overline{M'_n})(M'_N \overline{M'_n})^*)} M'_N \overline{M'_n}\,.
\end{aligned} \tag{5.83b}$$

5.6.6 The Differentiation Rule

The last step before including the functions algebra is to find the differentiation rule for elements $\tau^1 \in \Omega^1_D\mathfrak{a}$. According to (2.37b) and Proposition 7 we have

$$d\tau^1 = \{-\mathrm{i}\mathcal{M}\,, \tau^1\} + \hat{\sigma} \circ \hat{\pi}^{-1}(\tau^1) \quad \mathrm{mod}\ \mathbb{j}^2\mathfrak{a}\,. \tag{5.84}$$

We have computed $\hat{\sigma} \circ \hat{\pi}^{-1}(\tau^1)$ in (5.62); it remains to perform the factorization in the diagonal blocks (5.61a) and (5.61b). The same method as before shows that the representatives orthogonal to $J_2' \oplus (J_0 + J_3)$ are

$$(5.61\text{a}) \mapsto -\tfrac{1}{3}\mathrm{i}\pi_{10}(a) \otimes (\tfrac{1}{5}M^{10}_{aa} - 8M^{10}_{\{un\}} + 8M^{10}_{nn} + 24\check{M}^{10}_{nn} + M^{10}_{cc}) \,, \tag{5.85a}$$

$$(5.61\text{b}) \mapsto -\mathrm{i}\pi_5(a) \otimes (\tfrac{1}{5}M^5_{aa} - 8M^5_{\{un\}} + \tfrac{8}{3}M^5_{nn} + 8\check{M}^5_{nn} + \tfrac{1}{3}M^5_{cc}) \,. \tag{5.85b}$$

6 The Action of the Flipped SU(5)×U(1)–Model

Now we come to the physical highlights of the thesis – the derivation of the action of the flipped SU(5)×U(1)*–Grand Unification model. We do not need many new ideas and all calculations are fairly easy, in principle. However, there occur huge matrices, which could produce a horrible picture of our model. In such cases, the reader may fly through the pages. Optionally, she or he may randomly pick out certain terms to check. The rest can be considered as an encyclopaedia of the model. We start to investigate the curvature in Section 6.1 and compute symbolically the bosonic action as the trace of the squared curvature in Section 6.2. Then we express in Section 6.3 the Yang–Mills part and the covariant derivatives of the Higgs fields in terms of local coordinates. The Higgs potential will be studied in Section 7. In Section 6.4 we compute the fermionic action.*

6.1 The Curvature

Before we can study the curvature we must think about the structure of the connection form ρ in Section 6.1.1. Then we apply our general formulae to write down symbolically the curvature θ, see Section 6.1.2. It is convenient to pass to a different configuration $\tilde{H}$ for the Higgs fields by adding the mass matrix $\mathcal{M}$ to the old configuration H. This steps corresponds to the shift from the minimum configuration of the Higgs potential to the configuration where the Higgs fields transform homogeneously under gauge transformations. (In classical formulations one obtains the Higgs fields in their homogeneous phase $\tilde{H}$, whereas non–commutative and non–associative geometry yield Higgs fields in their broken phase H.) In Section 6.1.3 we decompose the canonical representative $\mathfrak{e}(\theta)$ of the curvature into irreducible matrix components. From there on the formulae become lengthy.

6.1.1 The Structure of the Connection Form

We choose X to be a four dimensional, Riemannian spin manifold, i.e. $N = 4$. When using a local basis $\{\gamma^\mu\}_{\mu=1,2,3,4}$ of Λ^1, then the basis elements γ^μ are selfadjoint as complex sections of the Clifford bundle. Elements of Λ^1 defined in Section 3.1 are locally represented by real linear combinations of $\{\gamma^\mu\}_{\mu=1,2,3,4}$. The grading operator is $\gamma \equiv \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$.

The first step is to write down the connection form ρ, which is an arbitrary element of $(\Lambda^1 \otimes \pi(\mathfrak{a})) \oplus (\Lambda^1 \otimes \pi(\mathrm{u}(1))) \oplus (\Lambda^0\gamma^5 \otimes \pi(\Omega^1\mathfrak{a}))$, see (3.115) and (5.41). Thus, the connection form has the structure

$$\begin{aligned}&\rho = \pi(A) + \pi(A'') + \gamma^5\pi(H) ,\\ &A \in \Lambda^1 \otimes \mathrm{su}(5) , \qquad A'' \in \Lambda^1 \otimes \mathrm{u}(1) , \qquad H \in \Lambda^0 \otimes \Omega^1\mathfrak{a} ,\end{aligned} \tag{6.1}$$

where $\pi = \mathrm{id}\otimes\hat{\pi}$ and γ^5 acts componentwise. Elements of $\hat{\pi}(\Omega^1\mathfrak{a})$ are specified by elements of $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and $\mathfrak{w}$, see (5.27a). Thus, we consider H as a sum

$$\begin{aligned} & H = \Psi + \Phi + \Upsilon + \Xi \;, \\ & \Psi \in \Lambda^0 \otimes \mathfrak{a}\;, \quad \Phi \in \Lambda^0 \otimes \mathfrak{b}\;, \quad \Xi \in \Lambda^0 \otimes \mathfrak{c}\;, \quad \Upsilon \in \Lambda^0 \otimes \mathfrak{w}\;, \end{aligned} \tag{6.2}$$

and identify $\pi(H)$ with the pointwise embedding of $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and $\mathfrak{w}$ into $\hat{\pi}(\Omega^1\mathfrak{a})$.

6.1.2 The Calculation of the Curvature

We recall (Proposition 9 and Lemma 28) that the curvature θ of the connection form ρ is obtained from

$$\theta = d\rho + \tfrac{1}{2}\{\rho,\rho\} \quad \mathrm{mod}\ \Lambda^0 \otimes \mathbb{j}^2\mathfrak{a}\;. \tag{6.3a}$$

Inserting (6.1) and (6.2), we find with (5.62)

$$\begin{aligned} \theta = {} & \mathbf{d}\pi(A) + \mathbf{d}\pi(A'') + \tfrac{1}{2}\{\pi(A),\pi(A)\} \\ & - \gamma^5\big(\mathbf{d}\pi(\Psi) + \mathbf{d}\pi(\Phi) + \mathbf{d}\pi(\Xi) + \mathbf{d}\pi(\Upsilon) \\ & \quad + [\pi(A){+}\pi(A''), \pi(\Psi){+}\pi(\Phi){+}\pi(\Xi){+}\pi(\Upsilon){-}\mathrm{i}\mathcal{M}\ ]\big) \\ & + \big(\tfrac{1}{2}\{\pi(\Psi){+}\pi(\Phi){+}\pi(\Xi){+}\pi(\Upsilon), \pi(\Psi){+}\pi(\Phi){+}\pi(\Xi){+}\pi(\Upsilon)\} \\ & \quad + \{\pi(\Psi){+}\pi(\Phi){+}\pi(\Xi){+}\pi(\Upsilon), {-}\mathrm{i}\mathcal{M}\ \} + \hat{\sigma}_{\mathfrak{g}}(\rho) \quad \mathrm{mod}\ \Lambda^0 \otimes \mathbb{j}^2\mathfrak{a}\big)\;, \end{aligned} \tag{6.3b}$$

where

$$\begin{aligned} \hat{\sigma}_{\mathfrak{g}}(\rho) := & -\tfrac{12\mathrm{i}}{5}\pi(\Xi \otimes \tfrac{1}{2}(M'_{10}M'_N{+}M'_N{M'_{10}}^T)) + \tfrac{\mathrm{i}}{2}\pi(\pi_{10,10}(\Upsilon) \otimes \tfrac{1}{2}(M'_{10}M'_d{-}M'_d{M'_{10}}^T)) \\ & + \tfrac{3\mathrm{i}}{5}\pi(\pi_{10,10}(\Phi) \otimes \tfrac{1}{2}(M'_{10}M'_d{+}M'_d{M'_{10}}^T) + \tfrac{3\mathrm{i}}{5}\pi(\pi_{5,1}(\Phi) \otimes {M'_5}^T M'_e) \\ & - \mathrm{i}\pi(\pi_{10,5}(\Phi) \otimes (\tfrac{9}{20}M'_{10}M'_{\tilde{u}} + \tfrac{3}{20}M'_{\tilde{u}}M'_5 - \tfrac{3}{4}M'_{10}M'_{\tilde{n}} + \tfrac{3}{4}M'_{\tilde{n}}M'_5)) \\ & - \mathrm{i}\pi(\pi_{10,5}(\Upsilon) \otimes (-\tfrac{1}{4}M'_{10}M'_{\tilde{u}} + \tfrac{1}{4}M'_{\tilde{u}}M'_5 + \tfrac{19}{20}M'_{10}M'_{\tilde{n}} - \tfrac{7}{20}M'_{\tilde{n}}M'_5)) \\ & - \tfrac{1}{3}\mathrm{i}\pi(\pi_{10}(\Psi) \otimes (\tfrac{1}{5}M^{10}_{aa} - 8M^{10}_{\{un\}} + 8M^{10}_{nn} + 24\check{M}^{10}_{nn} + M^{10}_{cc})) \\ & - \mathrm{i}\pi(\pi_5(\Psi) \otimes (\tfrac{1}{5}M^5_{aa} - 8M^5_{\{un\}} + \tfrac{8}{3}M^5_{nn} + 8\check{M}^5_{nn} + \tfrac{1}{3}M^5_{cc}))\;. \end{aligned} \tag{6.3c}$$

Here we have denoted by π the embedding of the selected matrix elements of (5.62) into the matrix (5.62). We have

$$\begin{aligned} & \tfrac{1}{2}\{\pi(\Psi){+}\ \pi(\Phi){+}\pi(\Xi){+}\pi(\Upsilon), \pi(\Psi){+}\pi(\Phi){+}\pi(\Xi){+}\pi(\Upsilon)\} \\ & \qquad + \{\pi(\Psi){+}\pi(\Phi){+}\pi(\Xi){+}\pi(\Upsilon), {-}\mathrm{i}\mathcal{M}\ \} \\ & \quad = \tfrac{1}{2}\{\pi(\tilde{\Psi}){+}\pi(\tilde{\Phi}){+}\pi(\tilde{\Xi}){+}\pi(\tilde{\Upsilon}), \pi(\tilde{\Psi}){+}\pi(\tilde{\Phi}){+}\pi(\tilde{\Xi}){+}\pi(\tilde{\Upsilon})\} + \mathcal{M}^{\ 2}\;, \end{aligned} \tag{6.4a}$$

where

$$\tilde{\Psi} := \Psi + m\;, \qquad \tilde{\Phi} := \Phi + n\;, \qquad \tilde{\Xi} := \Xi + m'\;, \qquad \tilde{\Upsilon} := \Upsilon + n'\;. \tag{6.4b}$$

Let

$$\hat{\sigma}_{\mathfrak{g}}(\tilde{\rho}) := \text{ formula (6.3c) with } \Psi \mapsto \tilde{\Psi}\,,\;\; \Phi \mapsto \tilde{\Phi}\,,\;\; \Xi \mapsto \tilde{\Xi}\,,\;\; \Upsilon \mapsto \tilde{\Upsilon}\,. \qquad (6.5)$$

Then we obtain from (6.3b) and (5.45)

$$\begin{aligned}
\theta = {} & \mathbf{d}\pi(A) + \tfrac{1}{2}\{\pi(A),\pi(A)\} \qquad (6.6)\\
& - \gamma^5(\mathbf{d}\pi(\tilde{\Psi}) + \mathbf{d}\pi(\tilde{\Phi}) + \mathbf{d}\pi(\tilde{\Upsilon}) + \mathbf{d}\pi(\tilde{\Xi})\\
& + [\pi(A)+\pi(A''), \pi(\tilde{\Psi}) + \pi(\tilde{\Phi}) + \pi(\tilde{\Xi}) + \pi(\tilde{\Upsilon})])\\
& + \big(\tfrac{1}{2}\{\pi(\tilde{\Psi}) + \pi(\tilde{\Phi}) + \pi(\tilde{\Xi}) + \pi(\tilde{\Upsilon}), \pi(\tilde{\Psi}) + \pi(\tilde{\Phi}) + \pi(\tilde{\Xi}) + \pi(\tilde{\Upsilon})\} + \hat{\sigma}_{\mathfrak{g}}(\tilde{\rho})\\
& + \mathrm{diag}\,\big(\mathbb{1}_{10} \otimes (\tfrac{6}{5}\hat{M}^{10}_{aa} + \hat{M}^{10}_{bb} + 12\hat{M}^{10}_{nn} + \hat{M}^{10}_{cc}),\; \mathbb{1}_5 \otimes (\tfrac{6}{5}\hat{M}^{5}_{aa} + \hat{M}^{5}_{bb} + 12\hat{M}^{5}_{nn} + \hat{M}^{5}_{cc})^T,\\
& \qquad \tfrac{6}{5}\hat{M}^{1}_{aa} + \hat{M}^{1}_{bb} + 12\hat{M}^{1}_{nn} + \hat{M}^{1}_{cc},\; \mathbb{1}_{10} \otimes (\tfrac{6}{5}\hat{M}^{10}_{aa} + \hat{M}^{10}_{bb} + 12\hat{M}^{10}_{nn} + \hat{M}^{10}_{cc})^T,\\
& \qquad \mathbb{1}_5 \otimes (\tfrac{6}{5}\hat{M}^{5}_{aa} + \hat{M}^{5}_{bb} + 12\hat{M}^{5}_{nn} + \hat{M}^{5}_{cc}),\; (\tfrac{6}{5}\hat{M}^{1}_{aa} + \hat{M}^{1}_{bb} + 12\hat{M}^{1}_{nn} + \hat{M}^{1}_{cc})^T\;\big)\\
& \qquad\qquad \bmod \Lambda^0 \otimes \mathbb{j}^2\mathfrak{a}\big)\,.
\end{aligned}$$

It is very important that the non–diagonal part of $\mathcal{M}^{\,2}$ marries $\hat{\sigma}_{\mathfrak{g}}(\rho)$ to give precisely $\hat{\sigma}_{\mathfrak{g}}(\tilde{\rho})$. Therefore, we can express the Higgs potential (which originally is given in the minimum configuration) completely in terms of the homogeneous configuration of the Higgs fields. That becomes apparent in Section 6.2.2 when we consider gauge transformations.

6.1.3 The Matrix Decomposition of the Curvature

Now, using (5.77) *we decompose the short formula* (6.6) *into matrix components. The result is surprisingly long. It comes even worse: We must compute the square of that huge matrix to obtain the bosonic Lagrangian, see Section 6.2.1.*

We define

$$\check{\tilde{\Phi}} := \pi_{10,5}(\tilde{\Phi})\,, \qquad \hat{\tilde{\Phi}} := \pi_{10,10}(\tilde{\Phi})\,,$$
$$(\tilde{\Phi},\tilde{\Phi})' := \tilde{\Phi}\tilde{\Phi}^* - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\Phi}\tilde{\Phi}^*)\mathbb{1}_5\,, \qquad (\tilde{\Xi}\tilde{\Xi}^*)' := -\mathrm{i}\pi_{10}^{-1}(\mathrm{i}(\tilde{\Xi}\tilde{\Xi}^*)_{\underline{24}})\,.$$

Using (5.77) we obtain the following matrix representation of $\mathfrak{e}(\theta)$:

$$\mathfrak{e}(\theta) = \left(\begin{array}{ccc|ccc}
\theta_{10} & \theta_{\widetilde{10,5}} & \theta_{10,1} & \theta_{10,10} & \theta_{10,5} & 0\\
\theta^*_{\widetilde{10,5}} & \theta^T_5 & 0 & \theta^T_{10,5} & 0 & \theta_{5,1}\\
\theta^*_{10,1} & 0 & \theta_1 & 0 & \theta^T_{5,1} & 0\\
\hline
\theta^*_{10,10} & \overline{\theta_{10,5}} & 0 & \theta^T_{10} & \overline{\theta_{\widetilde{10,5}}} & \overline{\theta_{10,1}}\\
\theta^*_{10,5} & 0 & \overline{\theta_{5,1}} & \theta^T_{\widetilde{10,5}} & \theta_5 & 0\\
0 & \theta^*_{5,1} & 0 & \theta^T_{10,1} & 0 & \theta^T_1
\end{array}\right)\,, \text{ where} \qquad (6.7a)$$

$$
\begin{aligned}
\theta_{10} = {} & \pi_{10}(\mathbf{d}A + \tfrac{1}{2}\{A,A\}) \otimes \mathbb{1}_6 - \tfrac{1}{2}\mathbf{d}A''\mathbb{1}_{10} \otimes \mathbb{1}_6 - \gamma^5 \pi_{10}(\mathbf{d}\tilde{\Psi} + [A,\tilde{\Psi}]) \otimes M'_{10} \\
& + (\tfrac{6}{5} + \mathrm{tr}(\tilde{\Psi}^2))\mathbb{1}_{10} \otimes \hat{M}^{10}_{aa} + (1 - \tilde{\boldsymbol{\Phi}}^* \tilde{\boldsymbol{\Phi}})\mathbb{1}_{10} \otimes \hat{M}^{10}_{bb} \\
& + (1 - \mathrm{tr}(\tilde{\Xi}\tilde{\Xi}^*))\mathbb{1}_{10} \otimes \hat{M}^{10}_{cc} + (12 - \mathrm{tr}(\tilde{\Upsilon}\tilde{\Upsilon}^*))\mathbb{1}_{10} \otimes \hat{M}^{10}_{nn} \\
& - \tfrac{1}{3}\mathrm{i}\pi_{10}(\mathrm{i}(\tilde{\Psi}^2 - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\Psi}^2)\mathbb{1}_5 - \tfrac{1}{5}\mathrm{i}\tilde{\Psi})) \otimes M^{10}_{aa} + \mathrm{i}\pi_{10}(\mathrm{i}(\tilde{\Xi}\tilde{\Xi}^*)' - \tfrac{1}{3}\tilde{\Psi}) \otimes M^{10}_{cc} \\
& + \mathrm{i}\pi_{10}(\mathrm{i}((\tilde{\Upsilon}\tilde{\Upsilon}^*)' + \tfrac{8}{3}\mathrm{i}\tilde{\Psi} - (\tilde{\boldsymbol{\Phi}},\tilde{\boldsymbol{\Phi}})')) \otimes M^{10}_{nn} \\
& + \mathrm{i}\pi_{10}(\mathrm{i}(\tilde{\Upsilon}^*\tilde{\Upsilon} - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\Upsilon}^*\tilde{\Upsilon})\mathbb{1}_5 + 8\mathrm{i}\tilde{\Psi} + 9(\tilde{\boldsymbol{\Phi}},\tilde{\boldsymbol{\Phi}})')) \otimes \check{M}^{10}_{nn} \\
& + \tfrac{1}{3}\mathrm{i}\pi_{10}(\mathrm{i}(\tilde{\Upsilon}^*\check{\tilde{\boldsymbol{\Phi}}} + \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\Upsilon} - 8\mathrm{i}\tilde{\Psi} - 6(\tilde{\boldsymbol{\Phi}},\tilde{\boldsymbol{\Phi}})')) \otimes M^{10}_{\{un\}} - \tfrac{1}{3}\mathrm{i}\pi_{10}(\tilde{\Upsilon}^*\check{\tilde{\boldsymbol{\Phi}}} - \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\Upsilon}) \otimes M^{10}_{[un]} \\
& - \big(\tilde{\Xi}\tilde{\Xi}^* + \tfrac{1}{2}(\pi_{10}(\tilde{\Psi}))^2 - \tfrac{1}{10}(\mathrm{tr}(\tilde{\Xi}\tilde{\Xi}^*) + \tfrac{3}{2}\,\mathrm{tr}(\tilde{\Psi}^2))\mathbb{1}_{10} \\
& \quad + \mathrm{i}\pi_{10}(\mathrm{i}(\tilde{\Xi}\tilde{\Xi}^*)' + \tfrac{1}{6}\mathrm{i}(\tilde{\Psi}^2 - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\Psi}^2)\mathbb{1}_5))\big) \otimes \tilde{M}^{10}_{cc} \\
& - \big(\tilde{\Xi}\hat{\tilde{\boldsymbol{\Phi}}}^* + \hat{\tilde{\boldsymbol{\Phi}}}\tilde{\Xi}^*\big) \otimes \tilde{M}^{10}_{\{cd\}} - \mathrm{i}\big(\tilde{\Xi}\hat{\tilde{\boldsymbol{\Phi}}}^* - \hat{\tilde{\boldsymbol{\Phi}}}\tilde{\Xi}^*\big) \otimes \tilde{M}^{10}_{[cd]} \\
& - \big(\tilde{\Upsilon}\tilde{\Upsilon}^* + 4(\pi_{10}(\tilde{\Psi}))^2 - \tfrac{1}{10}\,\mathrm{tr}(\tilde{\Upsilon}^*\tilde{\Upsilon} + 12\tilde{\Psi}^2)\mathbb{1}_{10} \\
& \quad + \mathrm{i}\pi_{10}(\mathrm{i}((\tilde{\Upsilon}\tilde{\Upsilon}^*)' + \tfrac{4}{3}\tilde{\Psi}^2 - \tfrac{4}{15}\,\mathrm{tr}(\tilde{\Psi}^2)\mathbb{1}_5))\big) \otimes \tilde{M}^{10}_{nn} \\
& - \big(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\Upsilon}^* + \tilde{\Upsilon}\check{\tilde{\boldsymbol{\Phi}}}^* - 4(\pi_{10}(\tilde{\Psi}))^2 + \tfrac{6}{5}\,\mathrm{tr}(\tilde{\Psi}^2)\mathbb{1}_{10} \\
& \quad + \tfrac{1}{3}\mathrm{i}\pi_{10}(\mathrm{i}(\tilde{\Upsilon}^*\check{\tilde{\boldsymbol{\Phi}}} + \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\Upsilon} - 4\tilde{\Psi}^2 + \tfrac{4}{5}\,\mathrm{tr}(\tilde{\Psi}^2)\mathbb{1}_5))\big) \otimes \tilde{M}^{10}_{\{un\}} \\
& - \mathrm{i}\big(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\Upsilon}^* - \tilde{\Upsilon}\check{\tilde{\boldsymbol{\Phi}}}^* - \tfrac{1}{3}\pi_{10}(\tilde{\Upsilon}^*\check{\tilde{\boldsymbol{\Phi}}} - \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\Upsilon})\big) \otimes \tilde{M}^{10}_{[un]}\,,
\end{aligned}
\tag{6.7b}
$$

$$
\begin{aligned}
\theta_5 = {} & \pi_5(\mathbf{d}A + \tfrac{1}{2}\{A,A\}) \otimes \mathbb{1}_6 - \tfrac{3}{2}\mathbf{d}A''\mathbb{1}_5 \otimes \mathbb{1}_6 + \gamma^5 \pi_5(\mathbf{d}\tilde{\Psi} + [A,\tilde{\Psi}]) \otimes M'_5 \\
& + (\tfrac{6}{5} + \mathrm{tr}(\tilde{\Psi}^2))\mathbb{1}_5 \otimes \hat{M}^5_{aa} + (1 - \tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}})\mathbb{1}_5 \otimes \hat{M}^5_{bb} \\
& + (1 - \mathrm{tr}(\tilde{\Xi}\tilde{\Xi}^*))\mathbb{1}_5 \otimes \hat{M}^5_{cc} + (12 - \mathrm{tr}(\tilde{\Upsilon}\tilde{\Upsilon}^*))\mathbb{1}_{10} \otimes \hat{M}^5_{nn} \\
& - \mathrm{i}\pi_5(\mathrm{i}(\tilde{\Psi}^2 - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\Psi}^2)\mathbb{1}_5 - \tfrac{1}{5}\mathrm{i}\tilde{\Psi})) \otimes M^5_{aa} + \mathrm{i}\pi_5(\mathrm{i}(\tilde{\Xi}\tilde{\Xi}^*)' - \tfrac{1}{3}\tilde{\Psi}) \otimes M^5_{cc} \\
& + \mathrm{i}\pi_5(\mathrm{i}((\tilde{\Upsilon}\tilde{\Upsilon}^*)' + \tfrac{8}{3}\mathrm{i}\tilde{\Psi} - (\tilde{\boldsymbol{\Phi}},\tilde{\boldsymbol{\Phi}})')) \otimes M^5_{nn} \\
& + \mathrm{i}\pi_5(\mathrm{i}(\tilde{\Upsilon}^*\tilde{\Upsilon} - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\Upsilon}^*\tilde{\Upsilon})\mathbb{1}_5 + 8\mathrm{i}\tilde{\Psi} + 9(\tilde{\boldsymbol{\Phi}},\tilde{\boldsymbol{\Phi}})')) \otimes \check{M}^5_{nn} \\
& + \mathrm{i}\pi_5(\mathrm{i}(\tilde{\Upsilon}^*\check{\tilde{\boldsymbol{\Phi}}} + \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\Upsilon} - 8\mathrm{i}\tilde{\Psi} - 6(\tilde{\boldsymbol{\Phi}},\tilde{\boldsymbol{\Phi}})')) \otimes M^5_{\{un\}} - \mathrm{i}\pi_5(\tilde{\Upsilon}^*\check{\tilde{\boldsymbol{\Phi}}} - \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\Upsilon}) \otimes M^5_{[un]}\,,
\end{aligned}
\tag{6.7c}
$$

$$
\begin{aligned}
\theta_1 = {} & -\tfrac{5}{2}\mathbf{d}A''\mathbb{1}_6 + (\tfrac{6}{5} + \mathrm{tr}(\tilde{\Psi}^2)) \otimes \hat{M}^1_{aa} + (1 - \tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}) \otimes \hat{M}^1_{bb} \\
& + (1 - \mathrm{tr}(\tilde{\Xi}\tilde{\Xi}^*)) \otimes \hat{M}^1_{cc} + \mathrm{tr}(12 - \mathrm{tr}(\Upsilon\Upsilon^*)) \otimes \hat{M}^1_{nn}\,,
\end{aligned}
\tag{6.7d}
$$

$$
\begin{aligned}
\theta_{10,10} = {} & -\gamma^5 \pi_{10,10}(\mathbf{d}\tilde{\boldsymbol{\Phi}} + (A + A''\mathbb{1}_5)\tilde{\boldsymbol{\Phi}}) \otimes M'_d \\
& - \gamma^5(\mathbf{d}\tilde{\Xi} + \pi_{10}(A)\tilde{\Xi} + \tilde{\Xi}\pi_{10}(A)^T - A''\tilde{\Xi}) \otimes M'_N \\
& + \pi_{10,10}(\tilde{\Psi}\tilde{\boldsymbol{\Phi}} - \tfrac{3\mathrm{i}}{5}\tilde{\boldsymbol{\Phi}}) \otimes \tfrac{1}{2}(M'_{10}M'_d + M'_d {M'_{10}}^T) \\
& + (\pi_{10}(\tilde{\Psi})\pi_{10,10}(\tilde{\boldsymbol{\Phi}}) - \pi_{10,10}(\tilde{\boldsymbol{\Phi}})\pi_{10}(\tilde{\Psi})^T + \tfrac{\mathrm{i}}{2}\pi_{10,10}(\tilde{\Upsilon})) \otimes \tfrac{1}{2}(M'_{10}M'_d - M'_d {M'_{10}}^T) \\
& + (\pi_{10}(\tilde{\Psi})\tilde{\Xi} + \tilde{\Xi}\pi_{10}(\tilde{\Psi})^T - \tfrac{12\mathrm{i}}{5}\tilde{\Xi}) \otimes \tfrac{1}{2}(M'_{10}M'_N + M'_N {M'_{10}}^T) \\
& + (\pi_{10}(\tilde{\Psi})\tilde{\Xi} - \tilde{\Xi}\pi_{10}(\tilde{\Psi})^T) \otimes \tfrac{1}{2}(M'_{10}M'_N - M'_N {M'_{10}}^T)\,,
\end{aligned}
\tag{6.7e}
$$

$$
\begin{aligned}
\theta_{10,5} = {} & -\gamma^5 \pi_{10,5}(\mathbf{d}\tilde{\boldsymbol{\Phi}} + (A + A''\mathbb{1}_5)\tilde{\boldsymbol{\Phi}}) \otimes M'_{\tilde{u}} \\
& - \gamma^5(\mathbf{d}\tilde{\Upsilon} + \pi_{10}(A)\tilde{\Upsilon} - \tilde{\Upsilon}\pi_5(A) + A''\tilde{\Upsilon})) \otimes M'_{\tilde{n}} \\
& + (\pi_{10}(\tilde{\Psi})\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{9\mathrm{i}}{20}\check{\tilde{\boldsymbol{\Phi}}} + \tfrac{\mathrm{i}}{4}\tilde{\Upsilon}) \otimes M'_{10}M'_{\tilde{u}} - (\check{\tilde{\boldsymbol{\Phi}}}\pi_5(\tilde{\Psi}) + \tfrac{3\mathrm{i}}{20}\check{\tilde{\boldsymbol{\Phi}}} + \tfrac{\mathrm{i}}{4}\tilde{\Upsilon}) \otimes M'_{\tilde{u}}M'_5 \\
& + (\pi_{10}(\tilde{\Psi})\tilde{\Upsilon} + \tfrac{3\mathrm{i}}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{19\mathrm{i}}{20}\tilde{\Upsilon}) \otimes M'_{10}M'_{\tilde{n}} - (\tilde{\Upsilon}\pi_5(\tilde{\Psi}) + \tfrac{3\mathrm{i}}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{7\mathrm{i}}{20}\tilde{\Upsilon}) \otimes M'_{\tilde{n}}M'_5\,,
\end{aligned}
\tag{6.7f}
$$

$$
\begin{aligned}
\theta_{5,1} &= -\gamma^5 \pi_{5,1}(\mathbf{d}\tilde{\Phi} + (A{+}A''\mathbb{1}_5)\tilde{\Phi}) \otimes M'_e + \pi_{5,1}(\tilde{\Psi}\tilde{\Phi} - \tfrac{3\mathrm{i}}{5}\tilde{\Phi}) \otimes M_5'^{\,T} M'_e \,, && (6.7g)\\
\theta_{10,1} &= -\tilde{\Upsilon}\tilde{\Phi} \otimes M'_{\tilde{n}}\bar{M}'_e \,, && (6.7h)\\
\theta_{\widetilde{10,5}} &= -(\tilde{\Upsilon}^* \hat{\tilde{\Phi}})^T_{\underline{10}} \otimes M'_d \bar{M}'_{\tilde{n}} - (\tilde{\Upsilon}^* \tilde{\Xi})^T_{\underline{10}} \otimes M'_N \bar{M}'_{\tilde{n}} && (6.7i)\\
&\quad - (\tilde{\Upsilon}^* \hat{\tilde{\Phi}})^T_{\underline{40}} \otimes M'_{d\tilde{n}} - (\tfrac{1}{4}(\tilde{\Upsilon}^* \tilde{\Xi})_{\underline{40}} + \tfrac{3}{4}(\check{\tilde{\Phi}}\tilde{\Xi}))^T \otimes M'_{Nu} \,.
\end{aligned}
$$

6.2 The Bosonic Action

Now there is no way to avoid the jump into the icy water: We must compute the square of the matrix (6.7)*. The Higgs potential is a monster. But this is no surprise. We have such a rich Higgs structure that there must exist a huge number of gauge invariant combinations of the fields.*

6.2.1 The Calculation of the Action

It is convenient to put

$$
\begin{aligned}
&\tilde{\mathbf{\Psi}} := -\mathrm{i}\tilde{\Psi} \,, && \check{\tilde{\mathbf{\Psi}}} := -\mathrm{i}\pi_{10}(\tilde{\Psi}) \,, && \tilde{\mathbf{\Upsilon}} := -\mathrm{i}\tilde{\Upsilon} \,,\\
&\hat{\tilde{\mathbf{\Upsilon}}} := -\mathrm{i}\pi_{10,10}(\tilde{\Upsilon}) \,, && \tilde{\mathbf{\Phi}} := -\mathrm{i}\tilde{\Phi} \,, && \check{\tilde{\mathbf{\Phi}}} := -\mathrm{i}\pi_{10,5}(\tilde{\Phi}) \,, \qquad (6.8)\\
&\hat{\tilde{\mathbf{\Phi}}} := -\mathrm{i}\pi_{10,10}(\tilde{\Phi}) \,, && \tilde{\mathbf{\Xi}} := -\mathrm{i}\tilde{\Xi} \,, && \check{A} := \pi_{10}(A) \,.
\end{aligned}
$$

It turns out that the computation of the bosonic action is not difficult now. The only problem is the length. All what one needs are the orthogonality of different irreducible representations and the relations

$$
\begin{aligned}
&\mathrm{tr}(\pi_{10}(a)\pi_{10}(\tilde{a})) = 3\,\mathrm{tr}(\pi_5(a)\pi_5(\tilde{a})) = 3\,\mathrm{tr}(a\tilde{a}) \,,\\
&\mathrm{tr}\big((A - \tfrac{1}{10}\mathrm{tr}(A)\mathbb{1}_{10} - A_{\underline{24}})(\tilde{A} - \tfrac{1}{10}\mathrm{tr}(\tilde{A})\mathbb{1}_{10} - \tilde{A}_{\underline{24}})\big) \qquad (6.9)\\
&\qquad = \mathrm{tr}(A\tilde{A}) - \tfrac{1}{10}\mathrm{tr}(A)\,\mathrm{tr}(\tilde{A}) - \mathrm{tr}(A_{\underline{24}}\tilde{A}_{\underline{24}}) \,,
\end{aligned}
$$

for $a,\tilde{a} \in \mathfrak{a}$ and skew–adjoint $A,\tilde{A} \in \mathrm{M}_{10}\mathbb{C}$. We compute the bosonic Lagrangian $\mathscr{L} = \frac{1}{192 g_0^2}\,\mathrm{tr}_c((\mathfrak{e}(\theta))^2)$, where g_0 is a coupling constant and tr_c the combination of the trace over the matrix structure with the trace in the Clifford algebra. For functions $f \in C^\infty(X)$ we have $\mathrm{tr}_c(f) = 4f$. We find:

$$
\begin{aligned}
&\tfrac{1}{192 g_0^2}\,\mathrm{tr}_c((\mathfrak{e}(\theta))^2) = \mathscr{L}_2 + \mathscr{L}_1 + \mathscr{L}_0 \,, && (6.10a)\\
&\mathscr{L}_2 = \tfrac{1}{4g_0^2}\,\mathrm{tr}_c((\mathbf{d}A + \tfrac{1}{2}\{A,A\})^2) + \tfrac{5}{4g_0^2}\,\mathrm{tr}_c((\mathbf{d}A'')^2) \,, && (6.10b)\\
&\mathscr{L}_1 = \tfrac{1}{g_0^2}\mu_0\,\mathrm{tr}_c((\mathbf{d}\check{\tilde{\mathbf{\Psi}}} + [A,\check{\tilde{\mathbf{\Psi}}}])^2) && (6.10c)\\
&\quad + \tfrac{1}{g_0^2}\mu_1\,\mathrm{tr}_c((\mathbf{d}\tilde{\mathbf{\Phi}} + (A + A''\mathbb{1}_5)\tilde{\mathbf{\Phi}})^*(\mathbf{d}\tilde{\mathbf{\Phi}} + (A + A''\mathbb{1}_5)\tilde{\mathbf{\Phi}}))\\
&\quad + \tfrac{1}{g_0^2}\mu_2\,\mathrm{tr}_c((\mathbf{d}\tilde{\mathbf{\Upsilon}} + \check{A}\tilde{\mathbf{\Upsilon}} - \tilde{\mathbf{\Upsilon}}A + A''\tilde{\mathbf{\Upsilon}})^*(\mathbf{d}\tilde{\mathbf{\Upsilon}} + \check{A}\tilde{\mathbf{\Upsilon}} - \tilde{\mathbf{\Upsilon}}A + A''\tilde{\mathbf{\Upsilon}}))\\
&\quad + \tfrac{1}{g_0^2}\mu_3\,\mathrm{tr}_c((\mathbf{d}\tilde{\mathbf{\Xi}} + \check{A}\tilde{\mathbf{\Xi}} + \tilde{\mathbf{\Xi}}\check{A}^T - A''\tilde{\mathbf{\Xi}})^*(\mathbf{d}\tilde{\mathbf{\Xi}} + \check{A}\tilde{\mathbf{\Xi}} + \tilde{\mathbf{\Xi}}\check{A}^T - A''\tilde{\mathbf{\Xi}})) \,,
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{L}_0 = \tfrac{1}{24g_0^2}\big\{ & \mu^{\mathrm{a}}(\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)-\tfrac{6}{5})^2+\mu^{\mathrm{b}}(\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}-1)^2+\mu^{\mathrm{c}}(\mathrm{tr}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}})-12)^2 \qquad (6.10\mathrm{d})\\
&+\mu^{\mathrm{d}}(\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)-\tfrac{6}{5})(\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}-1)+\mu^{\mathrm{e}}(\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)-\tfrac{6}{5})(\mathrm{tr}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}})-12)\\
&+\mu^{\mathrm{f}}(\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}-1)(\mathrm{tr}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}})-12)\\
&+\breve{\mu}^{\mathrm{a}}(\mathrm{tr}(\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)-1)^2+\breve{\mu}^{\mathrm{b}}(\mathrm{tr}(\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)-1)(\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)-\tfrac{6}{5})\\
&+\breve{\mu}^{\mathrm{c}}(\mathrm{tr}(\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)-1)(\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}-1)+\breve{\mu}^{\mathrm{d}}(\mathrm{tr}(\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)-1)(\mathrm{tr}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}})-12)\\
&+\breve{\mu}^{\mathrm{e}}\,\mathrm{tr}((\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Xi}}+\tilde{\boldsymbol{\Xi}}\check{\tilde{\boldsymbol{\Psi}}}^T-\tfrac{12}{5}\tilde{\boldsymbol{\Xi}})(\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Xi}}+\tilde{\boldsymbol{\Xi}}\check{\tilde{\boldsymbol{\Psi}}}^T-\tfrac{12}{5}\tilde{\boldsymbol{\Xi}})^*)\\
&+\breve{\mu}^{\mathrm{f}}\,\mathrm{tr}((\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Xi}}-\tilde{\boldsymbol{\Xi}}\check{\tilde{\boldsymbol{\Psi}}}^T)(\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Xi}}-\tilde{\boldsymbol{\Xi}}\check{\tilde{\boldsymbol{\Psi}}}^T)^*)\\
&+\breve{\mu}^{\mathrm{g}}\,\mathrm{Re}(\mathrm{tr}((\check{\tilde{\boldsymbol{\Psi}}}\hat{\tilde{\boldsymbol{\Phi}}}-\hat{\tilde{\boldsymbol{\Phi}}}\check{\tilde{\boldsymbol{\Psi}}}^T+\tfrac{1}{2}\hat{\tilde{\boldsymbol{\Upsilon}}})(\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Xi}}-\tilde{\boldsymbol{\Xi}}\check{\tilde{\boldsymbol{\Psi}}}^T)^*))\\
&+\breve{\mu}^{\mathrm{h}}\,\mathrm{Im}(\mathrm{tr}((\check{\tilde{\boldsymbol{\Psi}}}\hat{\tilde{\boldsymbol{\Phi}}}-\hat{\tilde{\boldsymbol{\Phi}}}\check{\tilde{\boldsymbol{\Psi}}}^T+\tfrac{1}{2}\hat{\tilde{\boldsymbol{\Upsilon}}})(\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Xi}}-\tilde{\boldsymbol{\Xi}}\check{\tilde{\boldsymbol{\Psi}}}^T)^*))\\
&+\mu^{\mathrm{g}}\,\tilde{\boldsymbol{\Phi}}^*(\tilde{\boldsymbol{\Psi}}-\tfrac{3}{5}\mathbb{1}_5)^2\tilde{\boldsymbol{\Phi}}+\mu^{\mathrm{h}}\,\mathrm{tr}((\check{\tilde{\boldsymbol{\Psi}}}\hat{\tilde{\boldsymbol{\Phi}}}-\hat{\tilde{\boldsymbol{\Phi}}}\check{\tilde{\boldsymbol{\Psi}}}^T+\tfrac{1}{2}\hat{\tilde{\boldsymbol{\Upsilon}}})(\check{\tilde{\boldsymbol{\Psi}}}\hat{\tilde{\boldsymbol{\Phi}}}-\hat{\tilde{\boldsymbol{\Phi}}}\check{\tilde{\boldsymbol{\Psi}}}^T+\tfrac{1}{2}\hat{\tilde{\boldsymbol{\Upsilon}}})^*)\\
&+\mu^{\mathrm{i}}\,\mathrm{tr}((\check{\tilde{\boldsymbol{\Psi}}}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{9}{20}\check{\tilde{\boldsymbol{\Phi}}}+\tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\check{\tilde{\boldsymbol{\Psi}}}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{9}{20}\check{\tilde{\boldsymbol{\Phi}}}+\tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}}))\\
&+\mu^{\mathrm{j}}\,\mathrm{tr}((\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Psi}}+\tfrac{3}{20}\check{\tilde{\boldsymbol{\Phi}}}+\tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Psi}}+\tfrac{3}{20}\check{\tilde{\boldsymbol{\Phi}}}+\tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}}))\\
&+\mu^{\mathrm{k}}\,\mathrm{tr}((\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Upsilon}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{19}{20}\tilde{\boldsymbol{\Upsilon}})^*(\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Upsilon}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{19}{20}\tilde{\boldsymbol{\Upsilon}}))\\
&+\mu^{\mathrm{l}}\,\mathrm{tr}((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Psi}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{7}{20}\tilde{\boldsymbol{\Upsilon}})^*(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Psi}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{7}{20}\tilde{\boldsymbol{\Upsilon}}))\\
&-\mu^{\mathrm{m}}\,\mathrm{Re}(\mathrm{tr}((\check{\tilde{\boldsymbol{\Psi}}}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{9}{20}\check{\tilde{\boldsymbol{\Phi}}}+\tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Psi}}+\tfrac{3}{20}\check{\tilde{\boldsymbol{\Phi}}}+\tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})))\\
&+\mu^{\mathrm{n}}\,\mathrm{Re}(\mathrm{tr}((\check{\tilde{\boldsymbol{\Psi}}}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{9}{20}\check{\tilde{\boldsymbol{\Phi}}}+\tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Upsilon}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{19}{20}\tilde{\boldsymbol{\Upsilon}})))\\
&+\mu^{\mathrm{o}}\,\mathrm{Im}(\mathrm{tr}((\check{\tilde{\boldsymbol{\Psi}}}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{9}{20}\check{\tilde{\boldsymbol{\Phi}}}+\tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Upsilon}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{19}{20}\tilde{\boldsymbol{\Upsilon}})))\\
&-\mu^{\mathrm{p}}\,\mathrm{Re}(\mathrm{tr}((\check{\tilde{\boldsymbol{\Psi}}}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{9}{20}\check{\tilde{\boldsymbol{\Phi}}}+\tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Psi}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{7}{20}\tilde{\boldsymbol{\Upsilon}})\\
&\qquad\qquad+(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Psi}}+\tfrac{3}{20}\check{\tilde{\boldsymbol{\Phi}}}+\tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Upsilon}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{19}{20}\tilde{\boldsymbol{\Upsilon}})))\\
&-\mu^{\mathrm{q}}\,\mathrm{Im}(\mathrm{tr}((\check{\tilde{\boldsymbol{\Psi}}}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{9}{20}\check{\tilde{\boldsymbol{\Phi}}}+\tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Psi}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{7}{20}\tilde{\boldsymbol{\Upsilon}})\\
&\qquad\qquad+(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Psi}}+\tfrac{3}{20}\check{\tilde{\boldsymbol{\Phi}}}+\tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Upsilon}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{19}{20}\tilde{\boldsymbol{\Upsilon}})))\\
&+\mu^{\mathrm{r}}\,\mathrm{Re}(\mathrm{tr}((\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Psi}}+\tfrac{3}{20}\check{\tilde{\boldsymbol{\Phi}}}+\tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Psi}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{7}{20}\tilde{\boldsymbol{\Upsilon}})))\\
&+\mu^{\mathrm{s}}\,\mathrm{Im}(\mathrm{tr}((\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Psi}}+\tfrac{3}{20}\check{\tilde{\boldsymbol{\Phi}}}+\tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Psi}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{7}{20}\tilde{\boldsymbol{\Upsilon}})))\\
&-\mu^{\mathrm{t}}\,\mathrm{Re}(\mathrm{tr}((\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Upsilon}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{19}{20}\tilde{\boldsymbol{\Upsilon}})^*(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Psi}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}-\tfrac{7}{20}\tilde{\boldsymbol{\Upsilon}})))\\
&+\mu^{\mathrm{u}}\,\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Phi}}+\mu^{\mathrm{v}}\,\mathrm{tr}((\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})_{\underline{10}}(\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})^*_{\underline{10}})+\mu^{\mathrm{w}}\,\mathrm{tr}((\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})_{\underline{40}}(\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})^*_{\underline{40}})\\
&+\breve{\mu}^{\mathrm{i}}\,\mathrm{tr}((\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})_{\underline{10}}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})^*_{\underline{10}})+\breve{\mu}^{\mathrm{l}}\,\mathrm{tr}((\tfrac{1}{4}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})_{\underline{40}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}})(\tfrac{1}{4}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})_{\underline{40}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}})^*)\\
&+\breve{\mu}^{\mathrm{j}}\,\mathrm{Re}(\mathrm{tr}((\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})_{\underline{10}}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})^*_{\underline{10}}))+\breve{\mu}^{\mathrm{m}}\,\mathrm{Re}(\mathrm{tr}((\tfrac{1}{4}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})_{\underline{40}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}})(\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})^*_{\underline{40}}))\\
&+\breve{\mu}^{\mathrm{k}}\,\mathrm{Im}(\mathrm{tr}((\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})_{\underline{10}}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})^*_{\underline{10}}))+\breve{\mu}^{\mathrm{n}}\,\mathrm{Im}(\mathrm{tr}((\tfrac{1}{4}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})_{\underline{40}}+\tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}})(\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})^*_{\underline{40}}))\\
&+\tilde{\mu}^{\mathrm{a}}\,\mathrm{tr}((\tilde{\boldsymbol{\Psi}}^2-\tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5-\tfrac{1}{5}\tilde{\boldsymbol{\Psi}})^2)\\
&+\tilde{\mu}^{\mathrm{b}}\,\mathrm{tr}((\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}}-\tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}})\mathbb{1}_5-8\tilde{\boldsymbol{\Psi}}+9\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^*-\tfrac{9}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)^2)\\
&+\tilde{\mu}^{\mathrm{c}}\,\mathrm{tr}(((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*)'-\tfrac{8}{3}\tilde{\boldsymbol{\Psi}}-\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^*+\tfrac{1}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)^2)\\
&+\tilde{\mu}^{\mathrm{d}}\,\mathrm{tr}((\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}}+\check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}}+8\tilde{\boldsymbol{\Psi}}-6\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^*+\tfrac{6}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)^2)+\tilde{\mu}^{\mathrm{e}}\,\mathrm{tr}(-(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}}-\check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}})^2)\\
&+\tilde{\mu}^{\mathrm{f}}\,\mathrm{tr}((\tilde{\boldsymbol{\Psi}}^2-\tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5-\tfrac{1}{5}\tilde{\boldsymbol{\Psi}})(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}}-\tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}})\mathbb{1}_5-8\tilde{\boldsymbol{\Psi}}+9\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^*-\tfrac{9}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5))
\end{aligned}
$$

$$
\begin{aligned}
&+\tilde{\mu}^{\mathrm{g}}\operatorname{tr}\big((\tilde{\boldsymbol\Psi}^2-\tfrac15\operatorname{tr}(\tilde{\boldsymbol\Psi}^2)\mathbb{1}_5-\tfrac15\tilde{\boldsymbol\Psi})((\tilde{\boldsymbol\Upsilon}\tilde{\boldsymbol\Upsilon}^*)'-\tfrac83\tilde{\boldsymbol\Psi}-\tilde{\boldsymbol\Phi}\tilde{\boldsymbol\Phi}^*+\tfrac15\tilde{\boldsymbol\Phi}^*\tilde{\boldsymbol\Phi}\mathbb{1}_5)\big)\\
&+\tilde{\mu}^{\mathrm{h}}\operatorname{tr}\big((\tilde{\boldsymbol\Psi}^2-\tfrac15\operatorname{tr}(\tilde{\boldsymbol\Psi}^2)\mathbb{1}_5-\tfrac15\tilde{\boldsymbol\Psi})(\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}+\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon}+8\tilde{\boldsymbol\Psi}-6\tilde{\boldsymbol\Phi}\tilde{\boldsymbol\Phi}^*+\tfrac65\tilde{\boldsymbol\Phi}^*\tilde{\boldsymbol\Phi}\mathbb{1}_5)\big)\\
&+\tilde{\mu}^{\mathrm{i}}\,\mathrm{i}\operatorname{tr}\big((\tilde{\boldsymbol\Psi}^2-\tfrac15\operatorname{tr}(\tilde{\boldsymbol\Psi}^2)\mathbb{1}_5-\tfrac15\tilde{\boldsymbol\Psi})(\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}-\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon})\big)\\
&+\tilde{\mu}^{\mathrm{j}}\operatorname{tr}\big((\tilde{\boldsymbol\Upsilon}^*\tilde{\boldsymbol\Upsilon}-\tfrac15\operatorname{tr}(\tilde{\boldsymbol\Upsilon}^*\tilde{\boldsymbol\Upsilon})\mathbb{1}_5-8\tilde{\boldsymbol\Psi}+9\tilde{\boldsymbol\Phi}\tilde{\boldsymbol\Phi}^*-\tfrac95\tilde{\boldsymbol\Phi}^*\tilde{\boldsymbol\Phi}\mathbb{1}_5)\times\\
&\qquad\times((\tilde{\boldsymbol\Upsilon}\tilde{\boldsymbol\Upsilon}^*)'-\tfrac83\tilde{\boldsymbol\Psi}-\tilde{\boldsymbol\Phi}\tilde{\boldsymbol\Phi}^*+\tfrac15\tilde{\boldsymbol\Phi}^*\tilde{\boldsymbol\Phi}\mathbb{1}_5)\big)\\
&+\tilde{\mu}^{\mathrm{k}}\operatorname{tr}\big((\tilde{\boldsymbol\Upsilon}^*\tilde{\boldsymbol\Upsilon}-\tfrac15\operatorname{tr}(\tilde{\boldsymbol\Upsilon}^*\tilde{\boldsymbol\Upsilon})\mathbb{1}_5-8\tilde{\boldsymbol\Psi}+9\tilde{\boldsymbol\Phi}\tilde{\boldsymbol\Phi}^*-\tfrac95\tilde{\boldsymbol\Phi}^*\tilde{\boldsymbol\Phi}\mathbb{1}_5)\times\\
&\qquad\times(\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}+\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon}+8\tilde{\boldsymbol\Psi}-6\tilde{\boldsymbol\Phi}\tilde{\boldsymbol\Phi}^*+\tfrac65\tilde{\boldsymbol\Phi}^*\tilde{\boldsymbol\Phi}\mathbb{1}_5)\big)\\
&+\tilde{\mu}^{\mathrm{l}}\,\mathrm{i}\operatorname{tr}\big((\tilde{\boldsymbol\Upsilon}^*\tilde{\boldsymbol\Upsilon}-\tfrac15\operatorname{tr}(\tilde{\boldsymbol\Upsilon}^*\tilde{\boldsymbol\Upsilon})\mathbb{1}_5-8\tilde{\boldsymbol\Psi}+9\tilde{\boldsymbol\Phi}\tilde{\boldsymbol\Phi}^*-\tfrac95\tilde{\boldsymbol\Phi}^*\tilde{\boldsymbol\Phi}\mathbb{1}_5)(\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}-\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon})\big)\\
&+\tilde{\mu}^{\mathrm{m}}\operatorname{tr}\big(((\tilde{\boldsymbol\Upsilon}\tilde{\boldsymbol\Upsilon}^*)'-\tfrac83\tilde{\boldsymbol\Psi}-\tilde{\boldsymbol\Phi}\tilde{\boldsymbol\Phi}^*+\tfrac15\tilde{\boldsymbol\Phi}^*\tilde{\boldsymbol\Phi}\mathbb{1}_5)\times\\
&\qquad\quad\times(\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}+\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon}+8\tilde{\boldsymbol\Psi}-6\tilde{\boldsymbol\Phi}\tilde{\boldsymbol\Phi}^*+\tfrac65\tilde{\boldsymbol\Phi}^*\tilde{\boldsymbol\Phi}\mathbb{1}_5)\big)\\
&+\tilde{\mu}^{\mathrm{n}}\,\mathrm{i}\operatorname{tr}\big(((\tilde{\boldsymbol\Upsilon}\tilde{\boldsymbol\Upsilon}^*)'-\tfrac83\tilde{\boldsymbol\Psi}-\tilde{\boldsymbol\Phi}\tilde{\boldsymbol\Phi}^*+\tfrac15\tilde{\boldsymbol\Phi}^*\tilde{\boldsymbol\Phi}\mathbb{1}_5)(\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}-\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon})\big)\\
&+\tilde{\mu}^{\mathrm{o}}\,\mathrm{i}\operatorname{tr}\big((\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}+\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon}+8\tilde{\boldsymbol\Psi}-6\tilde{\boldsymbol\Phi}\tilde{\boldsymbol\Phi}^*+\tfrac65\tilde{\boldsymbol\Phi}^*\tilde{\boldsymbol\Phi}\mathbb{1}_5)(\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}-\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon})\big)\\
&+\hat{\mu}^{\mathrm{a}}\operatorname{tr}\big(((\tilde{\boldsymbol\Xi}\tilde{\boldsymbol\Xi}^*)'-\tfrac13\tilde{\boldsymbol\Psi})^2\big)+\hat{\mu}^{\mathrm{b}}\operatorname{tr}\big(((\tilde{\boldsymbol\Xi}\tilde{\boldsymbol\Xi}^*)'-\tfrac13\tilde{\boldsymbol\Psi})(\tilde{\boldsymbol\Psi}^2-\tfrac15\operatorname{tr}(\tilde{\boldsymbol\Psi}^2)\mathbb{1}_5-\tfrac15\tilde{\boldsymbol\Psi})\big)\\
&+\hat{\mu}^{\mathrm{c}}\operatorname{tr}\big(((\tilde{\boldsymbol\Xi}\tilde{\boldsymbol\Xi}^*)'-\tfrac13\tilde{\boldsymbol\Psi})(\tilde{\boldsymbol\Upsilon}^*\tilde{\boldsymbol\Upsilon}-\tfrac15\operatorname{tr}(\tilde{\boldsymbol\Upsilon}^*\tilde{\boldsymbol\Upsilon})\mathbb{1}_5-8\tilde{\boldsymbol\Psi}+9\tilde{\boldsymbol\Phi}\tilde{\boldsymbol\Phi}^*-\tfrac95\tilde{\boldsymbol\Phi}^*\tilde{\boldsymbol\Phi}\mathbb{1}_5)\big)\\
&+\hat{\mu}^{\mathrm{d}}\operatorname{tr}\big(((\tilde{\boldsymbol\Xi}\tilde{\boldsymbol\Xi}^*)'-\tfrac13\tilde{\boldsymbol\Psi})((\tilde{\boldsymbol\Upsilon}\tilde{\boldsymbol\Upsilon}^*)'-\tfrac83\tilde{\boldsymbol\Psi}-\tilde{\boldsymbol\Phi}\tilde{\boldsymbol\Phi}^*+\tfrac15\tilde{\boldsymbol\Phi}^*\tilde{\boldsymbol\Phi}\mathbb{1}_5)\big)\\
&+\hat{\mu}^{\mathrm{e}}\operatorname{tr}\big(((\tilde{\boldsymbol\Xi}\tilde{\boldsymbol\Xi}^*)'-\tfrac13\tilde{\boldsymbol\Psi})(\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}+\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon}+8\tilde{\boldsymbol\Psi}-6\tilde{\boldsymbol\Phi}\tilde{\boldsymbol\Phi}^*+\tfrac65\tilde{\boldsymbol\Phi}^*\tilde{\boldsymbol\Phi}\mathbb{1}_5)\big)\\
&+\hat{\mu}^{\mathrm{f}}\,\mathrm{i}\operatorname{tr}\big(((\tilde{\boldsymbol\Xi}\tilde{\boldsymbol\Xi}^*)'-\tfrac13\tilde{\boldsymbol\Psi})(\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}-\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon})\big)\\
&+\tilde{\mu}^{\mathrm{p}}\Big(\operatorname{tr}((\tilde{\boldsymbol\Upsilon}\tilde{\boldsymbol\Upsilon}^*-4\check{\tilde{\boldsymbol\Psi}}^2)^2)-\tfrac{1}{10}(\operatorname{tr}(\tilde{\boldsymbol\Upsilon}^*\tilde{\boldsymbol\Upsilon}-12\tilde{\boldsymbol\Psi}^2))^2-\\
&\qquad\qquad-3\operatorname{tr}(((\tilde{\boldsymbol\Upsilon}\tilde{\boldsymbol\Upsilon}^*)'-\tfrac43\tilde{\boldsymbol\Psi}^2+\tfrac{4}{15}\operatorname{tr}(\tilde{\boldsymbol\Psi}^2)\mathbb{1}_5)^2)\Big)\\
&+\tilde{\mu}^{\mathrm{q}}\Big(\operatorname{tr}((\check{\tilde{\boldsymbol\Phi}}\tilde{\boldsymbol\Upsilon}^*+\tilde{\boldsymbol\Upsilon}\check{\tilde{\boldsymbol\Phi}}^*+4\check{\tilde{\boldsymbol\Psi}}^2)^2)-\tfrac{72}{5}(\operatorname{tr}(\tilde{\boldsymbol\Psi}^2))^2-\\
&\qquad\qquad-\tfrac13\operatorname{tr}((\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}+\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon}+4\tilde{\boldsymbol\Psi}^2-\tfrac45\operatorname{tr}(\tilde{\boldsymbol\Psi}^2)\mathbb{1}_5)^2)\Big)\\
&+\tilde{\mu}^{\mathrm{r}}\Big(\operatorname{tr}(-(\check{\tilde{\boldsymbol\Phi}}\tilde{\boldsymbol\Upsilon}^*-\tilde{\boldsymbol\Upsilon}\check{\tilde{\boldsymbol\Phi}}^*)^2)+\tfrac13\operatorname{tr}((\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}-\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon})^2)\Big)\\
&+\tilde{\mu}^{\mathrm{s}}\Big(\operatorname{tr}((\tilde{\boldsymbol\Upsilon}\tilde{\boldsymbol\Upsilon}^*-4\check{\tilde{\boldsymbol\Psi}}^2)(\check{\tilde{\boldsymbol\Phi}}\tilde{\boldsymbol\Upsilon}^*+\tilde{\boldsymbol\Upsilon}\check{\tilde{\boldsymbol\Phi}}^*+4\check{\tilde{\boldsymbol\Psi}}^2))-\tfrac65\operatorname{tr}(\tilde{\boldsymbol\Psi}^2)\operatorname{tr}(\tilde{\boldsymbol\Upsilon}^*\tilde{\boldsymbol\Upsilon}-12\tilde{\boldsymbol\Psi}^2)-\\
&\qquad-\operatorname{tr}(((\tilde{\boldsymbol\Upsilon}\tilde{\boldsymbol\Upsilon}^*)'-\tfrac43\tilde{\boldsymbol\Psi}^2+\tfrac{4}{15}\operatorname{tr}(\tilde{\boldsymbol\Psi}^2)\mathbb{1}_5)(\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}+\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon}+4\tilde{\boldsymbol\Psi}^2-\tfrac45\operatorname{tr}(\tilde{\boldsymbol\Psi}^2)\mathbb{1}_5))\Big)\\
&+\tilde{\mu}^{\mathrm{t}}\,\mathrm{i}\Big(\operatorname{tr}((\tilde{\boldsymbol\Upsilon}\tilde{\boldsymbol\Upsilon}^*-4\check{\tilde{\boldsymbol\Psi}}^2)(\check{\tilde{\boldsymbol\Phi}}\tilde{\boldsymbol\Upsilon}^*-\tilde{\boldsymbol\Upsilon}\check{\tilde{\boldsymbol\Phi}}^*))-\operatorname{tr}(((\tilde{\boldsymbol\Upsilon}\tilde{\boldsymbol\Upsilon}^*)'-\tfrac43\tilde{\boldsymbol\Psi}^2)(\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}-\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon}))\Big)\\
&+\tilde{\mu}^{\mathrm{u}}\,\mathrm{i}\Big(\operatorname{tr}((\check{\tilde{\boldsymbol\Phi}}\tilde{\boldsymbol\Upsilon}^*+\tilde{\boldsymbol\Upsilon}\check{\tilde{\boldsymbol\Phi}}^*+4\check{\tilde{\boldsymbol\Psi}}^2)(\check{\tilde{\boldsymbol\Phi}}\tilde{\boldsymbol\Upsilon}^*-\tilde{\boldsymbol\Upsilon}\check{\tilde{\boldsymbol\Phi}}^*))-\\
&\qquad\qquad-\tfrac13\operatorname{tr}((\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}+\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon}+4\tilde{\boldsymbol\Psi}^2)(\tilde{\boldsymbol\Upsilon}^*\check{\tilde{\boldsymbol\Phi}}-\check{\tilde{\boldsymbol\Phi}}^*\tilde{\boldsymbol\Upsilon}))\Big)\\
&+\hat{\mu}^{\mathrm{g}}\Big(\operatorname{tr}((\tilde{\boldsymbol\Xi}\tilde{\boldsymbol\Xi}^*-\tfrac12\check{\tilde{\boldsymbol\Psi}}^2)^2)-\tfrac{1}{10}(\operatorname{tr}(\tilde{\boldsymbol\Xi}\tilde{\boldsymbol\Xi}^*)-\tfrac32\operatorname{tr}(\tilde{\boldsymbol\Psi}^2))^2-\\
&\qquad\qquad-3\operatorname{tr}(((\tilde{\boldsymbol\Xi}\tilde{\boldsymbol\Xi}^*)'-\tfrac16\tilde{\boldsymbol\Psi}^2+\tfrac{1}{30}\operatorname{tr}(\tilde{\boldsymbol\Psi}^2)\mathbb{1}_5)^2)\Big)\\
&+\hat{\mu}^{\mathrm{h}}\operatorname{tr}((\tilde{\boldsymbol\Xi}\hat{\tilde{\boldsymbol\Phi}}^*+\hat{\tilde{\boldsymbol\Phi}}\tilde{\boldsymbol\Xi}^*)^2)-\hat{\mu}^{\mathrm{i}}\operatorname{tr}((\tilde{\boldsymbol\Xi}\hat{\tilde{\boldsymbol\Phi}}^*-\hat{\tilde{\boldsymbol\Phi}}\tilde{\boldsymbol\Xi}^*)^2)+\hat{\mu}^{\mathrm{j}}\operatorname{tr}((\tilde{\boldsymbol\Xi}\tilde{\boldsymbol\Xi}^*-\tfrac12\check{\tilde{\boldsymbol\Psi}}^2)(\tilde{\boldsymbol\Xi}\hat{\tilde{\boldsymbol\Phi}}^*+\hat{\tilde{\boldsymbol\Phi}}\tilde{\boldsymbol\Xi}^*))\\
&+\hat{\mu}^{\mathrm{k}}\,\mathrm{i}\operatorname{tr}((\tilde{\boldsymbol\Xi}\tilde{\boldsymbol\Xi}^*-\tfrac12\check{\tilde{\boldsymbol\Psi}}^2)(\tilde{\boldsymbol\Xi}\hat{\tilde{\boldsymbol\Phi}}^*-\hat{\tilde{\boldsymbol\Phi}}\tilde{\boldsymbol\Xi}^*))+\hat{\mu}^{\mathrm{l}}\,\mathrm{i}\operatorname{tr}((\tilde{\boldsymbol\Xi}\hat{\tilde{\boldsymbol\Phi}}^*+\hat{\tilde{\boldsymbol\Phi}}\tilde{\boldsymbol\Xi}^*)(\tilde{\boldsymbol\Xi}\hat{\tilde{\boldsymbol\Phi}}^*-\hat{\tilde{\boldsymbol\Phi}}\tilde{\boldsymbol\Xi}^*))\\
&+\hat{\mu}^{\mathrm{m}}\Big(\operatorname{tr}((\tilde{\boldsymbol\Xi}\tilde{\boldsymbol\Xi}^*-\tfrac12\check{\tilde{\boldsymbol\Psi}}^2)(\tilde{\boldsymbol\Upsilon}\tilde{\boldsymbol\Upsilon}^*-4\check{\tilde{\boldsymbol\Psi}}^2))-\tfrac{1}{10}(\operatorname{tr}(\tilde{\boldsymbol\Xi}\tilde{\boldsymbol\Xi}^*)-\tfrac32\operatorname{tr}(\tilde{\boldsymbol\Psi}^2))\operatorname{tr}(\tilde{\boldsymbol\Upsilon}^*\tilde{\boldsymbol\Upsilon}-12\tilde{\boldsymbol\Psi}^2)-\\
&\qquad\qquad-3\operatorname{tr}(((\tilde{\boldsymbol\Xi}\tilde{\boldsymbol\Xi}^*)'-\tfrac16\tilde{\boldsymbol\Psi}^2+\tfrac{1}{30}\operatorname{tr}(\tilde{\boldsymbol\Psi}^2)\mathbb{1}_5)((\tilde{\boldsymbol\Upsilon}\tilde{\boldsymbol\Upsilon}^*)'-\tfrac43\tilde{\boldsymbol\Psi}^2+\tfrac{4}{15}\operatorname{tr}(\tilde{\boldsymbol\Psi}^2)\mathbb{1}_5))\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\hat{\mu}^{\mathrm{n}}\Big(\mathrm{tr}((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^{*}-\tfrac{1}{2}\check{\tilde{\boldsymbol{\Psi}}}^{2})(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^{*}+\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^{*}+4\check{\tilde{\boldsymbol{\Psi}}}^{2}))-\tfrac{12}{10}(\mathrm{tr}(\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^{*})-\tfrac{3}{2}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^{2}))\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^{2})-\\
&\qquad\qquad -\mathrm{tr}(((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^{*})'-\tfrac{1}{6}\tilde{\boldsymbol{\Psi}}^{2}+\tfrac{1}{30}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^{2})\mathbb{1}_{5})(\tilde{\boldsymbol{\Upsilon}}^{*}\check{\tilde{\boldsymbol{\Phi}}}+\check{\tilde{\boldsymbol{\Phi}}}^{*}\tilde{\boldsymbol{\Upsilon}}+4\tilde{\boldsymbol{\Psi}}^{2}-\tfrac{4}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^{2})\mathbb{1}_{5}))\Big)\\
&+\hat{\mu}^{\mathrm{o}}\Big(\mathrm{i}\,\mathrm{tr}((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^{*}-\tfrac{1}{2}\check{\tilde{\boldsymbol{\Psi}}}^{2})(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^{*}-\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^{*}))-\\
&\qquad\qquad -\mathrm{i}\,\mathrm{tr}(((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^{*})'-\tfrac{1}{6}\tilde{\boldsymbol{\Psi}}^{2}+\tfrac{1}{30}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^{2})\mathbb{1}_{5})(\tilde{\boldsymbol{\Upsilon}}^{*}\check{\tilde{\boldsymbol{\Phi}}}-\check{\tilde{\boldsymbol{\Phi}}}^{*}\tilde{\boldsymbol{\Upsilon}}))\Big)\\
&+\hat{\mu}^{\mathrm{p}}\,\mathrm{tr}((\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^{*}+\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^{*})(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^{*}-4\check{\tilde{\boldsymbol{\Psi}}}^{2}))+\hat{\mu}^{\mathrm{q}}\,\mathrm{tr}((\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^{*}+\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^{*})(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^{*}+\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^{*}+4\check{\tilde{\boldsymbol{\Psi}}}^{2}))\\
&+\hat{\mu}^{\mathrm{r}}\,\mathrm{i}\,\mathrm{tr}((\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^{*}+\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^{*})(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^{*}-\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^{*}))+\hat{\mu}^{\mathrm{s}}\,\mathrm{i}\,\mathrm{tr}((\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^{*}-\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^{*})(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^{*}-4\check{\tilde{\boldsymbol{\Psi}}}^{2}))\\
&+\hat{\mu}^{\mathrm{t}}\,\mathrm{i}\,\mathrm{tr}((\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^{*}-\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^{*})(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^{*}+\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^{*}+4\check{\tilde{\boldsymbol{\Psi}}}^{2}))-\hat{\mu}^{\mathrm{u}}\,\mathrm{tr}((\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^{*}-\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^{*})(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^{*}-\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^{*}))\Big\}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
&\mu_0=\tfrac{1}{96}\,\mathrm{tr}(3M_{10}'^{\,2}+M_5'^{\,2})\,, && \mu_2=\tfrac{1}{48}\,\mathrm{tr}(M'_{\tilde{n}}M_{\tilde{n}}'^{\,*})\,,\\
&\mu_1=\mathrm{tr}(\tfrac{1}{16}M'_dM_d'^{\,*}+\tfrac{1}{12}M'_{\tilde{u}}M_{\tilde{u}}'^{\,*}+\tfrac{1}{48}M'_eM_e'^{\,*})\,, && \mu_3=\tfrac{1}{96}\,\mathrm{tr}(M'_NM_N'^{\,*})\,,
\end{aligned}
\tag{6.10e}
$$

$$
\begin{aligned}
&\mu^{\mathrm{a}}=\mathrm{tr}(10(\hat{M}^{10}_{aa})^2+5(\hat{M}^{5}_{aa})^2+(\hat{M}^{1}_{aa})^2)\,,\\
&\mu^{\mathrm{b}}=\mathrm{tr}(10(\hat{M}^{10}_{bb})^2+5(\hat{M}^{5}_{bb})^2+(\hat{M}^{1}_{bb})^2)\,,\\
&\mu^{\mathrm{c}}=\mathrm{tr}(10(\hat{M}^{10}_{nn})^2+5(\hat{M}^{5}_{nn})^2+(\hat{M}^{1}_{nn})^2)\,,\\
&\mu^{\mathrm{d}}=\mathrm{tr}(20\hat{M}^{10}_{aa}\hat{M}^{10}_{bb}+10\hat{M}^{5}_{aa}\hat{M}^{5}_{bb}+2\hat{M}^{1}_{aa}\hat{M}^{1}_{bb})\,,\\
&\mu^{\mathrm{e}}=\mathrm{tr}(20\hat{M}^{1}_{aa}\hat{M}^{10}_{nn}+10\hat{M}^{5}_{aa}\hat{M}^{5}_{nn}+2\hat{M}^{1}_{aa}\hat{M}^{1}_{nn})\\
&\mu^{\mathrm{f}}=\mathrm{tr}(20\hat{M}^{10}_{bb}\hat{M}^{10}_{nn}+10\hat{M}^{5}_{bb}\hat{M}^{5}_{nn}+2\hat{M}^{1}_{bb}\hat{M}^{1}_{nn})\,,\\
&\mu^{\mathrm{g}}=\mathrm{tr}(\tfrac{3}{2}(M'_{10}M'_d+M'_dM_{10}{}^{T})(M'_{10}M'_d+M'_dM_{10}{}^{T})^{*}+2(M_5'^{\,T})^2M'_eM_e'^{\,*})\,,\\
&\mu^{\mathrm{h}}=\tfrac{1}{4}\,\mathrm{tr}((M'_{10}M'_d-M'_dM_{10}'^{\,T})(M'_{10}M'_d-M'_dM_{10}'^{\,T})^{*})\,,\\
&\mu^{\mathrm{i}}=\mathrm{tr}(2M'_{\tilde{u}}M_{\tilde{u}}'^{\,*}M_{10}'^{\,2})\,, && \mu^{\mathrm{j}}=\mathrm{tr}(2M_{\tilde{u}}'^{\,*}M'_{\tilde{u}}M_5'^{\,2})\,,\\
&\mu^{\mathrm{k}}=\mathrm{tr}(2M'_{\tilde{n}}M_{\tilde{n}}'^{\,*}M_{10}'^{\,2})\,, && \mu^{\mathrm{l}}=\mathrm{tr}(2M_{\tilde{n}}'^{\,*}M'_{\tilde{n}}M_5'^{\,2})\,,\\
&\mu^{\mathrm{m}}=\mathrm{tr}(4M'_{\tilde{u}}M'_5M_{\tilde{u}}'^{\,*}M'_{10})\,, && \mu^{\mathrm{n}}=\mathrm{Re}(\mathrm{tr}(4M'_{\tilde{u}}M_{\tilde{n}}'^{\,*}M_{10}'^{\,2}))\,,\\
&\mu^{\mathrm{o}}=\mathrm{Im}(\mathrm{tr}(4M'_{\tilde{u}}M_{\tilde{n}}'^{\,*}M_{10}'^{\,2}))\,, && \mu^{\mathrm{p}}=\mathrm{Re}(\mathrm{tr}(4M'_{\tilde{u}}M'_5M_{\tilde{n}}'^{\,*}M'_{10}))\,,\\
&\mu^{\mathrm{q}}=\mathrm{Im}(\mathrm{tr}(4M'_{\tilde{u}}M'_5M_{\tilde{n}}'^{\,*}M'_{10}))\,, && \mu^{\mathrm{r}}=\mathrm{Re}(\mathrm{tr}(4M_{\tilde{n}}'^{\,*}M'_{\tilde{u}}M_5'^{\,2}))\,,\\
&\mu^{\mathrm{s}}=\mathrm{Im}(\mathrm{tr}(4M_{\tilde{n}}'^{\,*}M'_{\tilde{u}}M_5'^{\,2}))\,, && \mu^{\mathrm{t}}=\mathrm{tr}(4M'_{\tilde{n}}M'_5M_{\tilde{n}}'^{\,*}M'_{10})\,,\\
&\mu^{\mathrm{u}}=\mathrm{tr}(2M_{\tilde{n}}'^{\,*}M'_{\tilde{n}}\overline{M}'_eM_e'^{\,T})\,, && \mu^{\mathrm{v}}=\mathrm{tr}(2M'_{\tilde{n}}M_{\tilde{n}}'^{\,*}M'_dM_d'^{\,*})\,,\\
&\mu^{\mathrm{w}}=\mathrm{tr}(2M'_{d\tilde{n}}M_{d\tilde{n}}'^{\,*})\,,
\end{aligned}
\tag{6.10f}
$$

$$
\begin{aligned}
&\breve{\mu}^{\mathrm{a}}=\mathrm{tr}(10(\hat{M}^{10}_{cc})^2+5(\hat{M}^{5}_{cc})^2+(\hat{M}^{1}_{cc})^2)\,,\\
&\breve{\mu}^{\mathrm{b}}=\mathrm{tr}(20\hat{M}^{10}_{cc}\hat{M}^{10}_{aa}+10\hat{M}^{5}_{cc}\hat{M}^{5}_{aa}+2\hat{M}^{1}_{cc}\hat{M}^{1}_{aa})\,,\\
&\breve{\mu}^{\mathrm{c}}=\mathrm{tr}(20\hat{M}^{1}_{cc}\hat{M}^{10}_{bb}+10\hat{M}^{5}_{cc}\hat{M}^{5}_{bb}+2\hat{M}^{1}_{cc}\hat{M}^{1}_{bb})\\
&\breve{\mu}^{\mathrm{d}}=\mathrm{tr}(20\hat{M}^{10}_{cc}\hat{M}^{10}_{nn}+10\hat{M}^{5}_{cc}\hat{M}^{5}_{nn}+2\hat{M}^{1}_{cc}\hat{M}^{1}_{nn})\,,\\
&\breve{\mu}^{\mathrm{e}}=\tfrac{1}{4}\,\mathrm{tr}((M'_{10}M'_N+M'_NM_{10}'^{\,T})(M'_{10}M'_N+M'_NM_{10}'^{\,T})^{*})\,,\\
&\breve{\mu}^{\mathrm{f}}=\tfrac{1}{4}\,\mathrm{tr}((M'_{10}M'_N-M'_NM_{10}'^{\,T})(M'_{10}M'_N-M'_NM_{10}'^{\,T})^{*})\,,
\end{aligned}
\tag{6.10g}
$$

$$\begin{aligned}
&\check{\mu}^{\mathrm{g}} = \tfrac{1}{2}\mathrm{Re}(\mathrm{tr}((M'_{10}M'_N - M'_N{M'_{10}}^T)(M'_{10}M'_d - M'_d{M'_{10}}^T)^*)) \,,\\
&\check{\mu}^{\mathrm{h}} = \tfrac{1}{2}\mathrm{Im}(\mathrm{tr}((M'_{10}M'_N - M'_N{M'_{10}}^T)(M'_{10}M'_d - M'_d{M'_{10}}^T)^*)) \,,\\
&\check{\mu}^{\mathrm{i}} = \mathrm{tr}(2M'_{\tilde{n}}{M'_{\tilde{n}}}^*M'_N{M'_N}^*) \,, &&\check{\mu}^{\mathrm{j}} = \mathrm{Re}(\mathrm{tr}(4M'_{\tilde{n}}{M'_{\tilde{n}}}^*M'_N{M'_d}^*)) \,,\\
&\check{\mu}^{\mathrm{k}} = \mathrm{Im}(\mathrm{tr}(4M'_{\tilde{n}}{M'_{\tilde{n}}}^*M'_N{M'_d}^*)) \,, &&\check{\mu}^{\mathrm{l}} = \mathrm{tr}(2M'_{Nu}{M'_{Nu}}^*) \,,\\
&\check{\mu}^{\mathrm{m}} = \mathrm{Re}(\mathrm{tr}(4M'_{d\tilde{n}}{M'_{Nu}}^*)) \,, &&\check{\mu}^{\mathrm{n}} = \mathrm{Im}(\mathrm{tr}(4M'_{d\tilde{n}}{M'_{Nu}}^*)) \,,
\end{aligned}$$

$$\begin{aligned}
&\tilde{\mu}^{\mathrm{a}} = \mathrm{tr}(\tfrac{1}{3}(M^{10}_{aa})^2 + (M^5_{aa})^2) \,, &&\tilde{\mu}^{\mathrm{b}} = \mathrm{tr}(3(\check{M}^{10}_{nn})^2 + (\check{M}^5_{nn})^2) \,,\\
&\tilde{\mu}^{\mathrm{c}} = \mathrm{tr}(3(M^{10}_{nn})^2 + (M^5_{nn})^2) \,, &&\tilde{\mu}^{\mathrm{d}} = \mathrm{tr}(\tfrac{1}{3}(M^{10}_{\{un\}})^2 + (M^5_{\{un\}})^2) \,,\\
&\tilde{\mu}^{\mathrm{e}} = \mathrm{tr}(\tfrac{1}{3}(M^{10}_{[un]})^2 + (M^5_{[un]})^2) \,, &&\tilde{\mu}^{\mathrm{f}} = \mathrm{tr}(2M^{10}_{aa}\check{M}^{10}_{nn} + 2M^5_{aa}\check{M}^5_{nn}) \,,\\
&\tilde{\mu}^{\mathrm{g}} = \mathrm{tr}(2M^{10}_{aa}M^{10}_{nn} + 2M^5_{aa}M^5_{nn}) \,, &&\tilde{\mu}^{\mathrm{h}} = \mathrm{tr}(\tfrac{2}{3}M^{10}_{aa}M^{10}_{\{un\}} + 2M^5_{aa}M^5_{\{un\}}) \,,\\
&\tilde{\mu}^{\mathrm{i}} = \mathrm{tr}(\tfrac{2}{3}M^{10}_{aa}M^{10}_{[un]} + 2M^5_{aa}M^5_{[un]}) \,, &&\tilde{\mu}^{\mathrm{j}} = \mathrm{tr}(6\check{M}^{10}_{nn}M^{10}_{nn} + 2\check{M}^5_{nn}M^5_{nn}) \,,\\
&\tilde{\mu}^{\mathrm{k}} = \mathrm{tr}(2\check{M}^{10}_{nn}M^{10}_{\{un\}} + 2\check{M}^5_{nn}M^5_{\{un\}}) \,, &&\tilde{\mu}^{\mathrm{l}} = \mathrm{tr}(2\check{M}^{10}_{nn}M^{10}_{[un]} + 2\check{M}^5_{nn}M^5_{[un]}) \,,\\
&\tilde{\mu}^{\mathrm{m}} = \mathrm{tr}(2M^{10}_{nn}M^{10}_{\{un\}} + 2M^5_{nn}M^5_{\{un\}}) \,, &&\tilde{\mu}^{\mathrm{n}} = \mathrm{tr}(2M^{10}_{nn}M^{10}_{[un]} + 2M^5_{nn}M^5_{[un]}) \,,\\
&\tilde{\mu}^{\mathrm{o}} = \mathrm{tr}(\tfrac{2}{3}M^{10}_{\{un\}}M^{10}_{[un]} + 2M^5_{\{un\}}M^5_{[un]}) \,,\\
&\tilde{\mu}^{\mathrm{p}} = \mathrm{tr}((\tilde{M}^{10}_{nn})^2) \,, &&\tilde{\mu}^{\mathrm{q}} = \mathrm{tr}((\tilde{M}^{10}_{\{un\}})^2) \,,\\
&\tilde{\mu}^{\mathrm{r}} = \mathrm{tr}((\tilde{M}^{10}_{[un]})^2) \,, &&\tilde{\mu}^{\mathrm{s}} = \mathrm{tr}(2\tilde{M}^{10}_{nn}\tilde{M}^{10}_{\{un\}}) \,,\\
&\tilde{\mu}^{\mathrm{t}} = \mathrm{tr}(2\tilde{M}^{10}_{nn}\tilde{M}^{10}_{[un]}) \,, &&\tilde{\mu}^{\mathrm{u}} = \mathrm{tr}(2\tilde{M}^{10}_{\{un\}}\tilde{M}^{10}_{[un]}) \,,
\end{aligned} \tag{6.10h}$$

$$\begin{aligned}
&\hat{\mu}^{\mathrm{a}} = \mathrm{tr}(3(M^{10}_{cc})^2 + (M^5_{cc})^2) \,, &&\hat{\mu}^{\mathrm{b}} = \mathrm{tr}(2M^{10}_{cc}M^{10}_{aa} + 2M^5_{cc}M^5_{aa}) \,,\\
&\hat{\mu}^{\mathrm{c}} = \mathrm{tr}(6M^{10}_{cc}\check{M}^{10}_{nn} + 2M^5_{cc}\check{M}^5_{nn}) \,, &&\hat{\mu}^{\mathrm{d}} = \mathrm{tr}(6M^{10}_{cc}M^{10}_{nn} + 2M^5_{cc}M^5_{nn}) \,,\\
&\hat{\mu}^{\mathrm{e}} = \mathrm{tr}(2M^{10}_{cc}M^{10}_{\{un\}} + 2M^5_{cc}M^5_{\{un\}}) \,, &&\hat{\mu}^{\mathrm{f}} = \mathrm{tr}(2M^{10}_{cc}M^{10}_{[un]} + 2M^5_{cc}M^5_{[un]}) \,,
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}^{\mathrm{g}} = \mathrm{tr}((\tilde{M}^{10}_{cc})^2) \,, &&\hat{\mu}^{\mathrm{h}} = \mathrm{tr}((\tilde{M}^{10}_{\{cd\}})^2) \,,\\
&\hat{\mu}^{\mathrm{i}} = \mathrm{tr}((\tilde{M}^{10}_{[cd]})^2) \,, &&\hat{\mu}^{\mathrm{j}} = \mathrm{tr}(2\tilde{M}^{10}_{cc}\tilde{M}^{10}_{\{cd\}}) \,,\\
&\hat{\mu}^{\mathrm{k}} = \mathrm{tr}(2\tilde{M}^{10}_{cc}\tilde{M}^{10}_{[cd]}) \,, &&\hat{\mu}^{\mathrm{l}} = \mathrm{tr}(2\tilde{M}^{10}_{\{cd\}}\tilde{M}^{10}_{[cd]}) \,,\\
&\hat{\mu}^{\mathrm{m}} = \mathrm{tr}(2\tilde{M}^{10}_{cc}\tilde{M}^{10}_{nn}) \,, &&\hat{\mu}^{\mathrm{n}} = \mathrm{tr}(2\tilde{M}^{10}_{cc}\tilde{M}^{10}_{\{un\}}) \,,\\
&\hat{\mu}^{\mathrm{o}} = \mathrm{tr}(2\tilde{M}^{10}_{cc}\tilde{M}^{10}_{[un]}) \,, &&\hat{\mu}^{\mathrm{p}} = \mathrm{tr}(2\tilde{M}^{10}_{\{cd\}}\tilde{M}^{10}_{nn}) \,,\\
&\hat{\mu}^{\mathrm{q}} = \mathrm{tr}(2\tilde{M}^{10}_{\{cd\}}\tilde{M}^{10}_{\{un\}}) \,, &&\hat{\mu}^{\mathrm{r}} = \mathrm{tr}(2\tilde{M}^{10}_{\{cd\}}\tilde{M}^{10}_{[un]}) \,,\\
&\hat{\mu}^{\mathrm{s}} = \mathrm{tr}(2\tilde{M}^{10}_{[cd]}\tilde{M}^{10}_{nn}) \,, &&\hat{\mu}^{\mathrm{t}} = \mathrm{tr}(2\tilde{M}^{10}_{[cd]}\tilde{M}^{10}_{\{un\}}) \,,\\
&\hat{\mu}^{\mathrm{u}} = \mathrm{tr}(2\tilde{M}^{10}_{[cd]}\tilde{M}^{10}_{[un]}) \,.
\end{aligned} \tag{6.10i}$$

6.2.2 Gauge Transformations

It is interesting to study gauge transformations of the connection form ρ. These transformations are rather complicated, but $\tilde{\rho} := \rho - \mathrm{i}\gamma^5\mathcal{M}$ transforms as a classical connection form. Now, the gauge invariance of the bosonic action can be directly verified.

We recall (Definition 10, Definition 25 and formulae (3.124) and (5.59)) that the group of local gauge transformations of our model is the group

$$
\begin{aligned}
\mathcal{U}_0(\mathfrak{g}) &= \exp\big(\pi(C^\infty(X)\otimes(\mathrm{su}(5)\oplus\mathrm{u}(1)))\big) \\
&\cong \exp(C^\infty(X)\otimes\mathrm{su}(5))\times\exp(C^\infty(X)\otimes\mathrm{u}(1)) \cong C^\infty(X)\otimes(\mathrm{SU}(5)\times\mathrm{U}(1))\;.
\end{aligned}
\tag{6.11}
$$

Here, the second isomorphism is understood locally between appropriate charts of the manifold. Moreover, we recall (Proposition 9) that the (bosonic) connection has the structure $\nabla = d+[\rho,\ .\ ]_g$, or with (3.108)

$$
\nabla(\ .\ ) = \mathbf{d}(\ .\ ) + [\rho - \mathrm{i}\gamma^5\mathcal{M}\ ,\ .\ ]_g + \hat{\sigma}_{\mathfrak{g}}\circ\pi^{-1}(\ .\ )\gamma^5\;.
$$

Since $\hat{\sigma}_{\mathfrak{g}}\circ\pi^{-1}(\ .\ )\gamma^5$ is invariant under gauge transformations introduced in Definition 10, we get from (2.61) and (3.108)

$$
\begin{aligned}
\nabla'(\ .\ ) &= \mathbf{d}(\ .\ ) + [\gamma_u(\rho) - \mathrm{i}\gamma^5\mathcal{M}\ ,\ .\ ]_g + \hat{\sigma}_{\mathfrak{g}}\circ\pi^{-1}(\ .\ )\gamma^5 \\
&= \mathbf{d}(\ .\ ) + [u\mathbf{d}(u^*) + u(\rho - \mathrm{i}\gamma^5\mathcal{M}\ )u^*,\ .\ ]_g + \hat{\sigma}_{\mathfrak{g}}\circ\pi^{-1}(\ .\ )\gamma^5\;,
\end{aligned}
$$

for $u\in\mathcal{U}_0(\mathfrak{g})$. Thus, the linear combination $\tilde{\rho} := \rho - \mathrm{i}\gamma^5\mathcal{M}$ transforms as a classical connection form with respect to the reference connection $\mathbf{d}$,

$$
\gamma_u(\tilde{\rho}) = u\mathbf{d}(u^*) + u\tilde{\rho}u^*\;. \tag{6.12}
$$

In our model, the step from ρ to $\tilde{\rho}$ is just the step from $\{\Psi,\Phi,\Xi,\Upsilon\}$ to $\{\tilde{\Psi},\tilde{\Phi},\tilde{\Xi},\tilde{\Upsilon}\}$, see (6.4b) and (5.21). If we expand (6.12) we find the following gauge transformations for the physical fields:

$$
\begin{aligned}
&\gamma_u(A) = u_5\mathbf{d}u_5^* + u_5Au_5^*, && \gamma_u(\check{A}) = u_{10}\mathbf{d}u_{10}^* + u_{10}\check{A}u_{10}^*,\\
&\gamma_u(A'') = u_1\mathbf{d}u_1^* + A'', && \\
&\gamma_u(\tilde{\mathbf{\Upsilon}}) = u_1u_{10}\tilde{\mathbf{\Upsilon}}u_5^*\;, && \gamma_u(\hat{\tilde{\mathbf{\Upsilon}}}) = u_1^*u_{10}\hat{\tilde{\mathbf{\Upsilon}}}u_{10}^T\;,\\
&\gamma_u(\tilde{\mathbf{\Psi}}) = u_5\tilde{\mathbf{\Psi}}u_5^*\;, && \gamma_u(\check{\tilde{\mathbf{\Psi}}}) = u_{10}\check{\tilde{\mathbf{\Psi}}}u_{10}^*\;,\\
&\gamma_u(\tilde{\mathbf{\Phi}}) = u_1u_5\tilde{\mathbf{\Phi}}\;, && \gamma_u(\check{\tilde{\mathbf{\Phi}}}) = u_1u_{10}\check{\tilde{\mathbf{\Phi}}}u_5^*\;,\\
&\gamma_u(\hat{\tilde{\mathbf{\Phi}}}) = u_1^*u_{10}\hat{\tilde{\mathbf{\Phi}}}u_{10}^T\;, && \gamma_u(\tilde{\mathbf{\Xi}}) = u_1^*u_{10}\tilde{\mathbf{\Xi}}u_{10}^T\;,
\end{aligned}
\tag{6.13a}
$$

where

$$
\begin{aligned}
&u_5 = \exp(t_5)\;, && u_{10} = \exp(\pi_{10}(t_5))\;, && t_5\in C^\infty(X)\otimes\mathrm{su}(5)\;,\\
&u_1 = \exp(t_1)\;, && && t_1\in C^\infty(X)\otimes\mathrm{u}(1)\;.
\end{aligned}
\tag{6.13b}
$$

Note that – due to (2.55) and the fact that π_{10} is a representation – we find for $a\in C^\infty(X)\otimes\mathfrak{a}$ the identity

$$
\pi_{10}(\exp(t_5)a\exp(-t_5)) \equiv \exp(\pi_{10}(t_5))\pi_{10}(a)\exp(-\pi_{10}(t_5))\,.
$$

Moreover, due to (5.12), (5.16) and (5.17b) we get for $b \in C^\infty(X) \otimes \mathfrak{b}$ and $w \in C^\infty(X) \otimes \mathfrak{w}$ the identities

$$\begin{aligned}
\pi_{10,5}(\exp(t_5)b) &\equiv \exp(\pi_{10}(t_5))\pi_{10,5}(b)\exp(-\pi_5(t_5))\,,\\
\pi_{10,5}(\exp(t_1)b) &\equiv \exp(t_1)\pi_{10,5}(b)\,,\\
\pi_{10,10}(\exp(t_5)b) &\equiv \exp(\pi_{10}(t_5))\pi_{10,10}(b)\exp(\pi_{10}(t_5))^T\,,\\
\pi_{10,10}(\exp(t_1)b) &\equiv \exp(t_1)^*\pi_{10,10}(b)\,,\\
\pi_{10,10}(\pi_{10}(t_5)w\exp(-\pi_5(t_5))) &\equiv \exp(\pi_{10}(t_5))\pi_{10,10}(w)\exp(\pi_{10}(t_5))^T\,,\\
\pi_{10,10}(\exp(t_1)w) &\equiv \exp(t_1)^*\pi_{10,10}(w)\,.
\end{aligned}$$

Therefore, (6.13) is consistent. By construction, the Lagrangian (6.10) is invariant under the gauge transformation (6.13), which can be checked by a direct calculation as well.

6.2.3 Remarks on the Higgs Potential

Our goal is to find a local minimum of the Higgs potential $\mathscr{L}_0$. A direct calculation of all the traces in (6.10d) *is not practicable. Due to the 90 real parameters of $\tilde{\Upsilon}$ and the 100 real parameters of $\tilde{\Xi}$, a complete expansion of* (6.10d) *would be extremely large. From such a monster one had to compute the partial derivatives and to solve the resulting equations in order to find possible candidates for a minimum. Then one had to compute the second derivatives in order to check that one has indeed a local minimum and not a saddle point or a local maximum – a hopeless undertaking. Fortunately, there is a theory behind this Higgs potential. We know that, applying the transformation* (6.4b) *in the other direction, the Λ^0–part of the curvature $\mathfrak{e}(\theta)$ (and hence the Higgs potential $\mathscr{L}_0$) is zero for*

$$\Psi = 0\,,\ \ \Phi = 0\,,\ \ \Xi = 0\,,\ \ \Upsilon = 0 \qquad \textit{or} \qquad \tilde{\Psi} = m\,,\ \ \tilde{\Phi} = n\,,\ \ \tilde{\Xi} = m'\,,\ \ \tilde{\Upsilon} = n'\,. \tag{6.14}$$

Since the Higgs potential $\mathscr{L}_0$ is not negative as the trace of the square of the Λ^0–part of the selfadjoint matrix $\mathfrak{e}(\theta)$, the point (6.14) *is a global minimum of $\mathscr{L}_0$. But* (6.14) *is clearly a local minimum as well. In the vicinity of* (6.14)*, the Λ^0–part of $\mathfrak{e}(\theta)$ is linear in the components of Ψ, Φ, Ξ and Υ so that the Higgs potential $\mathscr{L}_0$ is in lowest order quadratic in these components.*

If one modified the coefficients in front of $\tilde{\Psi}, \tilde{\Phi}, \tilde{\Xi}$ or $\tilde{\Upsilon}$ in one or more[15] *terms in* (6.7) *then* (6.14) *would no longer be the minimum. In particular, the linear terms in* (6.7) *leading to cubic terms in* (6.10d) *are essential. A modification of the coefficients $\mu^i, \check{\mu}^i, \tilde{\mu}^i, \hat{\mu}^i$ given by the traces over the mass matrices preserves the minimum* (6.14)*. But the relative coefficients of the fields $\tilde{\Psi}, \tilde{\Phi}, \tilde{\Xi}, \tilde{\Upsilon}$ are distinguished. Thus, our model provides a machinery to construct these distinguished coefficients.*

[15] A global rescaling of $\tilde{\Psi}, \tilde{\Phi}, \tilde{\Xi}$ or $\tilde{\Upsilon}$ in all terms is possible, of course.

6.3 The Bosonic Lagrangian in Local Coordinates

In this subsection we will write down the Lagrangians $\mathscr{L}_2$ and $\mathscr{L}_1$ in (6.10) *in terms of local coordinates. The Higgs potential $\mathscr{L}_0$ will be studied later.*

6.3.1 Spontaneous Symmetry Breaking

We restrict the symmetry group of our model to the symmetry group of the Higgs vacuum by gauging away certain Higgs fields. We give the explicit matrix form for the Higgs fields. We need this explicit form when we select the quadratic terms of the Higgs potential in Section 7.

Let us introduce in the same way as in (6.8) the bold matrices

$$
\begin{aligned}
&\mathbf{m} := -\mathrm{i}\pi_5(m) \equiv \mathrm{diag}(-\tfrac{2}{5},-\tfrac{2}{5},-\tfrac{2}{5},\tfrac{3}{5},\tfrac{3}{5})\,,\\
&\check{\mathbf{m}} := -\mathrm{i}\pi_{10}(m) \equiv \mathrm{diag}(\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5},-\tfrac{4}{5},-\tfrac{4}{5},-\tfrac{4}{5},\tfrac{6}{5})\,, \qquad (6.15)\\
&\mathbf{n}' := -\mathrm{i}n' \equiv \begin{pmatrix} \mathbb{1}_3 & 0_{3\times1} & 0_{3\times1}\\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1}\\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1}\\ 0_{1\times3} & 0_{1\times1} & 3 \end{pmatrix}, \quad \check{\mathbf{n}} := -\mathrm{i}\pi_{10,5}(n) \equiv \begin{pmatrix} \mathbb{1}_3 & 0_{3\times1} & 0_{3\times1}\\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1}\\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1}\\ 0_{1\times3} & 0_{1\times1} & -1 \end{pmatrix},\\
&\mathbf{n} := \overline{-\mathrm{i}\pi_{5,1}(n)} \equiv \begin{pmatrix} 0_{3\times1}\\ 1\\ 0_{1\times1} \end{pmatrix}, \quad \hat{\mathbf{n}} := -\mathrm{i}\pi_{10,10}(n) \equiv \begin{pmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1}\\ 0_{3\times3} & 0_{3\times3} & -\mathbb{1}_3 & 0_{3\times1}\\ 0_{3\times3} & -\mathbb{1}_3 & 0_{3\times3} & 0_{3\times1}\\ 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & 0_{1\times1} \end{pmatrix},\\
&\mathbf{m}' := -\mathrm{i}m' \equiv \begin{pmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1}\\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1}\\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1}\\ 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & -1 \end{pmatrix},\\
&\hat{\mathbf{n}}' := -\mathrm{i}\pi_{10,10}(n') \equiv \begin{pmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1}\\ 0_{3\times3} & 0_{3\times3} & 2\mathbb{1}_3 & 0_{3\times1}\\ 0_{3\times3} & -2\mathbb{1}_3 & 0_{3\times3} & 0_{3\times1}\\ 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & 0_{1\times1} \end{pmatrix},
\end{aligned}
$$

see (5.20). We shall write our formulae in terms of the "physical" fields $\boldsymbol{\Psi},\boldsymbol{\Phi},\boldsymbol{\Xi},\boldsymbol{\Upsilon}$ given by

$$
\tilde{\boldsymbol{\Psi}} = \boldsymbol{\Psi} + \mathbf{m}\,, \qquad \tilde{\boldsymbol{\Phi}} = \boldsymbol{\Phi} + \mathbf{n}\,, \qquad \tilde{\boldsymbol{\Xi}} = \boldsymbol{\Xi} + \mathbf{m}'\,, \qquad \tilde{\boldsymbol{\Upsilon}} = \boldsymbol{\Upsilon} + \mathbf{n}'\,. \qquad (6.16)
$$

The subgroup of $C^\infty(X)\otimes(\mathrm{SU}(5)\times\mathrm{U}(1))$, which leaves (6.14) invariant, is the group $C^\infty(X)\otimes(\mathrm{SU}(3)_C\times\mathrm{U}(1)_{EM})$. Elements $u_0=u_1u_5$ of this subgroup have the matrix representation

$$u_0=\mathrm{e}^{12\mathrm{i}\phi}\begin{pmatrix}\mathrm{e}^{-2\mathrm{i}\phi}u_3 & 0\\ 0 & \mathrm{e}^{3\mathrm{i}\phi}u_2\end{pmatrix}\,,\quad u_2=\begin{pmatrix}\mathrm{e}^{-15\mathrm{i}\phi} & 0\\ 0 & \mathrm{e}^{15\mathrm{i}\phi}\end{pmatrix}\in C^\infty(X)\otimes\mathrm{SU}(2)\,,\tag{6.17}$$

where $u_3\in C^\infty(X)\otimes\mathrm{SU}(3)_C$ and $\mathrm{e}^{\mathrm{i}\phi}\in C^\infty(X)\otimes\mathrm{U}(1)$. This means that the vacuum (the minimum of $\mathscr{L}_0$) breaks the symmetry group $C^\infty(X)\otimes(\mathrm{SU}(5)\times\mathrm{U}(1))$ of $\mathscr{L}$ down to $C^\infty(X)\otimes(\mathrm{SU}(3)_C\times\mathrm{U}(1)_{EM})$. The Higgs mechanism consists in reducing the symmetry of the whole theory to the symmetry of the vacuum by fixing the gauge group in a certain way: First, using a $C^\infty(X)\otimes(\mathrm{SU}(5)\times\mathrm{U}(1))$–transformation (6.13) we bring $\tilde{\mathbf{\Psi}}$ into block–diagonal form $\tilde{\mathbf{\Psi}}=\mathbf{\Psi}+\mathbf{m}$,

$$\mathbf{\Psi}=\begin{pmatrix}-\sqrt{\tfrac{4}{15}}\Psi_0\mathbb{1}_3+\mathbf{\Psi}_g & 0\\ 0 & \sqrt{\tfrac{3}{5}}\Psi_0+\mathbf{\Psi}_w\end{pmatrix}\,,\tag{6.18a}$$

$$\mathbf{\Psi}_g=\begin{pmatrix}\sqrt{\tfrac13}\Psi_8+\Psi_3 & \Psi_1-\mathrm{i}\Psi_2 & \Psi_4-\mathrm{i}\Psi_5\\ \Psi_1+\mathrm{i}\Psi_2 & \sqrt{\tfrac13}\Psi_8-\Psi_3 & \Psi_6-\mathrm{i}\Psi_7\\ \Psi_4+\mathrm{i}\Psi_5 & \Psi_6+\mathrm{i}\Psi_7 & -\sqrt{\tfrac43}\Psi_8\end{pmatrix}=\sum_{a=1}^{8}\Psi_a\lambda^a\,,\quad \Psi_a\in C^\infty(X)\,,\tag{6.18b}$$

$$\mathbf{\Psi}_w=\begin{pmatrix}\Psi_3' & \Psi_1'-\mathrm{i}\Psi_2'\\ \Psi_1'+\mathrm{i}\Psi_2' & -\Psi_3'\end{pmatrix}=\sum_{a=1}^{3}\Psi_a'\sigma^a\,,\quad \Psi_a'\in C^\infty(X)\,.\tag{6.18c}$$

Here, λ^a are the Gell–Mann matrices and σ^a the Pauli matrices, see Appendix C. Next, using a $C^\infty(X)\otimes(\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1))$–transformation $\tilde{\mathbf{\Psi}}\mapsto\tilde{u}\tilde{\mathbf{\Psi}}\tilde{u}^*$, $\tilde{\mathbf{\Phi}}\mapsto\tilde{u}\tilde{\mathbf{\Phi}}$,

$$\tilde{u}=\mathrm{e}^{\mathrm{i}\phi'}\begin{pmatrix}\mathrm{e}^{-2\mathrm{i}\phi}\mathbb{1}_3 & 0\\ 0 & \mathrm{e}^{3\mathrm{i}\phi}u_2\end{pmatrix}\,,\quad u_2\in C^\infty(X)\otimes\mathrm{SU}(2)\,,\tag{6.19}$$

we transform $\mathbf{\Phi}=\tilde{\mathbf{\Phi}}-\mathbf{n}$ into

$$\mathbf{\Phi}=\begin{pmatrix}\mathbf{\Phi}_g\\ \mathbf{\Phi}_w\end{pmatrix}\,,\quad \mathbf{\Phi}_g=\begin{pmatrix}\Phi_1+\mathrm{i}\Phi_4\\ \Phi_2+\mathrm{i}\Phi_5\\ \Phi_3+\mathrm{i}\Phi_6\end{pmatrix}\,,\quad \mathbf{\Phi}_w=\begin{pmatrix}\Phi_0\\ 0\end{pmatrix}\,,\quad \Phi_a\in C^\infty(X)\,.\tag{6.20}$$

This transformation also changes $\mathbf{\Psi}$, but it preserves the block–matrix structure of (6.18a). Finally, using a $C^\infty(X)\otimes(\mathrm{U}(1)\times\mathrm{U}(1))$–transformation $\tilde{\mathbf{\Psi}}\mapsto\tilde{u}\tilde{\mathbf{\Psi}}\tilde{u}^*$, $\tilde{\mathbf{\Phi}}\mapsto\tilde{u}\tilde{\mathbf{\Phi}}$, $\tilde{\mathbf{\Xi}}\mapsto\tilde{u}\tilde{\mathbf{\Xi}}\tilde{u}^T$,

$$\tilde{u}=\mathrm{e}^{-3\mathrm{i}\phi}\begin{pmatrix}\mathrm{e}^{-2\mathrm{i}\phi}\mathbb{1}_3 & 0\\ 0 & \mathrm{e}^{3\mathrm{i}\phi}\mathbb{1}_2\end{pmatrix}\,,\tag{6.21}$$

we transform $\Xi = \tilde{\Xi} - \mathbf{m}'$ into

$$\Xi = \begin{pmatrix} \overline{\Xi_A} & \overline{\Xi_D} - \frac{1}{2}\varepsilon(\overline{\Xi_c}) & (\Xi_E^0)^* & \Xi_a \\ \overline{\Xi_D} + \frac{1}{2}\varepsilon(\overline{\Xi_c}) & \overline{\Xi_B} & (\Xi_F^0)^* & \Xi_b \\ \overline{\Xi_E^0} & \overline{\Xi_F^0} & \Xi_C & \overline{\Xi_c} \\ \Xi_a^T & \Xi_b^T & \Xi_c^* & -\Xi_0 \end{pmatrix} \tag{6.22a}$$

such that $\Xi_0 \in C^\infty(X)$ is a *real* function. The explicit form of this matrix is presented in (6.22b), where $\Xi_i \in C^\infty(X)$, $i = 0, \ldots, 98$. The transformation (6.21) changes $\mathbf{\Psi}$ and $\mathbf{\Phi}$ but preserves the block–matrix structures of (6.18a) and (6.20). Without loss of generality we assume that $\mathbf{\Psi}, \mathbf{\Phi}$ and Ξ are given by (6.18), (6.20) and (6.22), respectively. Then, the matrix $\mathbf{\Upsilon}$ is an arbitrary element of $\mathrm{i}\mathfrak{w}$ as displayed in (6.23b), where $\Upsilon_i \in C^\infty(X)$, $i = 0, \ldots, 89$. In terms of block matrices we have

$$\mathbf{\Upsilon} = \begin{pmatrix} \mathbf{\Upsilon}_A & \mathbf{\Upsilon}_a + \mathbf{\Upsilon}_b & \mathbf{\Upsilon}_c \\ \mathbf{\Upsilon}_B & \mathbf{\Upsilon}_d & \mathbf{\Upsilon}_a - \mathbf{\Upsilon}_b \\ \mathbf{\Upsilon}_C - \varepsilon(\mathbf{\Upsilon}_a) & \overline{\mathbf{\Upsilon}_e} & \overline{\mathbf{\Upsilon}_f} \\ \mathbf{\Upsilon}_g^* & -\mathrm{tr}(\mathbf{\Upsilon}_B) & \mathrm{tr}(\mathbf{\Upsilon}_A) \end{pmatrix} . \tag{6.23a}$$

6.3.2 The Gauge Fields in Local Coordinates

Here we present the explicit matrix form of the gauge fields.

Following (5.5) we make for A and A'' the ansatz

$$A = \frac{\mathrm{i}g_0}{2} \begin{pmatrix} \sqrt{\frac{4}{15}} A' \mathbb{1}_3 + \mathbf{G} & \mathbf{X} \\ \mathbf{X}^* & -\sqrt{\frac{3}{5}} A' \mathbb{1}_2 + \mathbf{W} \end{pmatrix} , \quad A' \in \Lambda^1 , \tag{6.24a}$$

$$A'' = \tfrac{\mathrm{i}g_0}{2} \sqrt{\tfrac{2}{5}} \tilde{A} , \quad \tilde{A} \in \Lambda^1 , \tag{6.24b}$$

$$\mathbf{G} = \begin{pmatrix} \sqrt{\frac{1}{3}} G^8 + G^3 & G^1 - \mathrm{i}G^2 & G^4 - \mathrm{i}G^5 \\ G^1 + \mathrm{i}G^2 & \sqrt{\frac{1}{3}} G^8 - G^3 & G^6 - \mathrm{i}G^7 \\ G^4 + \mathrm{i}G^5 & G^6 + \mathrm{i}G^7 & -\sqrt{\frac{4}{3}} G^8 \end{pmatrix} = \sum_{a=1}^{8} G^a \lambda^a , \quad G^a \in \Lambda^1 , \tag{6.24c}$$

$$\mathbf{W} = \begin{pmatrix} W^3 & W^1 - \mathrm{i}W^2 \\ W^1 + \mathrm{i}W^2 & -W^3 \end{pmatrix} = \sum_{a=1}^{3} W^a \sigma^a , \quad W^a \in \Lambda^1 , \tag{6.24d}$$

$$\mathbf{X} = \Big(X , Y \Big) , \quad X = \begin{pmatrix} X^1 - \mathrm{i}X^2 \\ X^3 - \mathrm{i}X^4 \\ X^5 - \mathrm{i}X^6 \end{pmatrix} , \quad Y = \begin{pmatrix} Y^1 - \mathrm{i}Y^2 \\ Y^3 - \mathrm{i}Y^4 \\ Y^5 - \mathrm{i}Y^6 \end{pmatrix} , \quad X^a, Y^a \in \Lambda^1 . \tag{6.24e}$$

$$\boldsymbol{\Xi} = \tag{6.22b}$$

$$\left(\begin{array}{ccc|ccc|ccc|c}
\begin{bmatrix}\frac{1}{\sqrt{3}}(\Xi_1-\mathrm{i}\Xi_{50})\\+\frac{1}{\sqrt{2}}(\Xi_2-\mathrm{i}\Xi_{51})\\+\frac{1}{\sqrt{6}}(\Xi_3-\mathrm{i}\Xi_{52})\end{bmatrix} & \frac{1}{\sqrt{2}}(\Xi_4-\mathrm{i}\Xi_{53}) & \frac{1}{\sqrt{2}}(\Xi_5-\mathrm{i}\Xi_{54}) & \begin{bmatrix}\frac{1}{\sqrt{6}}(\Xi_{19}-\mathrm{i}\Xi_{68})\\+\frac{1}{2}(\Xi_{20}-\mathrm{i}\Xi_{69})\\+\frac{1}{\sqrt{12}}(\Xi_{21}-\mathrm{i}\Xi_{70})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{22}-\mathrm{i}\Xi_{71})\\-\frac{1}{\sqrt{12}}(\Xi_{49}-\mathrm{i}\Xi_{98})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{23}-\mathrm{i}\Xi_{72})\\+\frac{1}{\sqrt{12}}(\Xi_{48}-\mathrm{i}\Xi_{97})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{27}-\mathrm{i}\Xi_{76})\\+\frac{1}{\sqrt{12}}(\Xi_{32}-\mathrm{i}\Xi_{81})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{25}-\mathrm{i}\Xi_{26})\\-\frac{\mathrm{i}}{2}(\Xi_{74}-\mathrm{i}\Xi_{75})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{28}-\mathrm{i}\Xi_{29})\\-\frac{\mathrm{i}}{2}(\Xi_{77}-\mathrm{i}\Xi_{78})\end{bmatrix} & \frac{1}{\sqrt{2}}(\Xi_{41}+\mathrm{i}\Xi_{90}) \\
\frac{1}{\sqrt{2}}(\Xi_4-\mathrm{i}\Xi_{53}) & \begin{bmatrix}\frac{1}{\sqrt{3}}(\Xi_1-\mathrm{i}\Xi_{50})\\-\frac{1}{\sqrt{2}}(\Xi_2-\mathrm{i}\Xi_{51})\\+\frac{1}{\sqrt{6}}(\Xi_3-\mathrm{i}\Xi_{52})\end{bmatrix} & \frac{1}{\sqrt{2}}(\Xi_6-\mathrm{i}\Xi_{55}) & \begin{bmatrix}\frac{1}{2}(\Xi_{22}-\mathrm{i}\Xi_{71})\\+\frac{1}{\sqrt{12}}(\Xi_{49}-\mathrm{i}\Xi_{98})\end{bmatrix} & \begin{bmatrix}\frac{1}{\sqrt{6}}(\Xi_{19}-\mathrm{i}\Xi_{68})\\-\frac{1}{2}(\Xi_{20}-\mathrm{i}\Xi_{69})\\+\frac{1}{\sqrt{12}}(\Xi_{21}-\mathrm{i}\Xi_{70})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{24}-\mathrm{i}\Xi_{73})\\-\frac{1}{\sqrt{12}}(\Xi_{47}-\mathrm{i}\Xi_{96})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{25}+\mathrm{i}\Xi_{26})\\-\frac{\mathrm{i}}{2}(\Xi_{74}+\mathrm{i}\Xi_{75})\end{bmatrix} & \begin{bmatrix}-\frac{1}{2}(\Xi_{27}-\mathrm{i}\Xi_{76})\\+\frac{1}{\sqrt{12}}(\Xi_{32}-\mathrm{i}\Xi_{81})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{30}-\mathrm{i}\Xi_{31})\\-\frac{\mathrm{i}}{2}(\Xi_{79}-\mathrm{i}\Xi_{80})\end{bmatrix} & \frac{1}{\sqrt{2}}(\Xi_{42}+\mathrm{i}\Xi_{91}) \\
\frac{1}{\sqrt{2}}(\Xi_5-\mathrm{i}\Xi_{54}) & \frac{1}{\sqrt{2}}(\Xi_6-\mathrm{i}\Xi_{55}) & \begin{bmatrix}\frac{1}{\sqrt{3}}(\Xi_1-\mathrm{i}\Xi_{50})\\-\frac{2}{\sqrt{6}}(\Xi_3-\mathrm{i}\Xi_{52})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{23}-\mathrm{i}\Xi_{72})\\-\frac{1}{\sqrt{12}}(\Xi_{48}-\mathrm{i}\Xi_{97})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{24}-\mathrm{i}\Xi_{73})\\+\frac{1}{\sqrt{12}}(\Xi_{47}-\mathrm{i}\Xi_{96})\end{bmatrix} & \begin{bmatrix}\frac{1}{\sqrt{6}}(\Xi_{19}-\mathrm{i}\Xi_{68})\\-\frac{2}{\sqrt{12}}(\Xi_{21}-\mathrm{i}\Xi_{70})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{28}+\mathrm{i}\Xi_{29})\\-\frac{\mathrm{i}}{2}(\Xi_{77}+\mathrm{i}\Xi_{78})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{30}+\mathrm{i}\Xi_{31})\\-\frac{\mathrm{i}}{2}(\Xi_{79}+\mathrm{i}\Xi_{80})\end{bmatrix} & -\frac{2}{\sqrt{12}}(\Xi_{32}-\mathrm{i}\Xi_{81}) & \frac{1}{\sqrt{2}}(\Xi_{43}+\mathrm{i}\Xi_{92}) \\
\hline
\begin{bmatrix}\frac{1}{\sqrt{6}}(\Xi_{19}-\mathrm{i}\Xi_{68})\\+\frac{1}{2}(\Xi_{20}-\mathrm{i}\Xi_{69})\\+\frac{1}{\sqrt{12}}(\Xi_{21}-\mathrm{i}\Xi_{70})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{22}-\mathrm{i}\Xi_{71})\\+\frac{1}{\sqrt{12}}(\Xi_{49}-\mathrm{i}\Xi_{98})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{23}-\mathrm{i}\Xi_{72})\\-\frac{1}{\sqrt{12}}(\Xi_{48}-\mathrm{i}\Xi_{97})\end{bmatrix} & \begin{bmatrix}\frac{1}{\sqrt{3}}(\Xi_7-\mathrm{i}\Xi_{56})\\+\frac{1}{\sqrt{2}}(\Xi_8-\mathrm{i}\Xi_{57})\\+\frac{1}{\sqrt{6}}(\Xi_9-\mathrm{i}\Xi_{58})\end{bmatrix} & \frac{1}{\sqrt{2}}(\Xi_{10}-\mathrm{i}\Xi_{59}) & \frac{1}{\sqrt{2}}(\Xi_{11}-\mathrm{i}\Xi_{60}) & \begin{bmatrix}\frac{1}{2}(\Xi_{35}-\mathrm{i}\Xi_{84})\\+\frac{1}{\sqrt{12}}(\Xi_{40}-\mathrm{i}\Xi_{89})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{33}-\mathrm{i}\Xi_{34})\\-\frac{\mathrm{i}}{2}(\Xi_{82}-\mathrm{i}\Xi_{83})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{36}-\mathrm{i}\Xi_{37})\\-\frac{\mathrm{i}}{2}(\Xi_{85}-\mathrm{i}\Xi_{86})\end{bmatrix} & \frac{1}{\sqrt{2}}(\Xi_{44}+\mathrm{i}\Xi_{93}) \\
\begin{bmatrix}\frac{1}{2}(\Xi_{22}-\mathrm{i}\Xi_{71})\\-\frac{1}{\sqrt{12}}(\Xi_{49}-\mathrm{i}\Xi_{98})\end{bmatrix} & \begin{bmatrix}\frac{1}{\sqrt{6}}(\Xi_{19}-\mathrm{i}\Xi_{68})\\-\frac{1}{2}(\Xi_{20}-\mathrm{i}\Xi_{69})\\+\frac{1}{\sqrt{12}}(\Xi_{21}-\mathrm{i}\Xi_{70})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{24}-\mathrm{i}\Xi_{73})\\+\frac{1}{\sqrt{12}}(\Xi_{47}-\mathrm{i}\Xi_{96})\end{bmatrix} & \frac{1}{\sqrt{2}}(\Xi_{10}-\mathrm{i}\Xi_{59}) & \begin{bmatrix}\frac{1}{\sqrt{3}}(\Xi_7-\mathrm{i}\Xi_{56})\\-\frac{1}{\sqrt{2}}(\Xi_8-\mathrm{i}\Xi_{57})\\+\frac{1}{\sqrt{6}}(\Xi_9-\mathrm{i}\Xi_{58})\end{bmatrix} & \frac{1}{\sqrt{2}}(\Xi_{12}-\mathrm{i}\Xi_{61}) & \begin{bmatrix}\frac{1}{2}(\Xi_{33}+\mathrm{i}\Xi_{34})\\-\frac{\mathrm{i}}{2}(\Xi_{82}+\mathrm{i}\Xi_{83})\end{bmatrix} & \begin{bmatrix}-\frac{1}{2}(\Xi_{35}-\mathrm{i}\Xi_{84})\\+\frac{1}{\sqrt{12}}(\Xi_{40}-\mathrm{i}\Xi_{89})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{38}-\mathrm{i}\Xi_{39})\\-\frac{\mathrm{i}}{2}(\Xi_{87}-\mathrm{i}\Xi_{88})\end{bmatrix} & \frac{1}{\sqrt{2}}(\Xi_{45}+\mathrm{i}\Xi_{94}) \\
\begin{bmatrix}\frac{1}{2}(\Xi_{23}-\mathrm{i}\Xi_{72})\\+\frac{1}{\sqrt{12}}(\Xi_{48}-\mathrm{i}\Xi_{97})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{24}-\mathrm{i}\Xi_{73})\\-\frac{1}{\sqrt{12}}(\Xi_{47}-\mathrm{i}\Xi_{96})\end{bmatrix} & \begin{bmatrix}\frac{1}{\sqrt{6}}(\Xi_{19}-\mathrm{i}\Xi_{68})\\-\frac{2}{\sqrt{12}}(\Xi_{21}-\mathrm{i}\Xi_{70})\end{bmatrix} & \frac{1}{\sqrt{2}}(\Xi_{11}-\mathrm{i}\Xi_{60}) & \frac{1}{\sqrt{2}}(\Xi_{12}-\mathrm{i}\Xi_{61}) & \begin{bmatrix}\frac{1}{\sqrt{3}}(\Xi_7-\mathrm{i}\Xi_{56})\\-\frac{2}{\sqrt{6}}(\Xi_9-\mathrm{i}\Xi_{58})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{36}+\mathrm{i}\Xi_{37})\\-\frac{\mathrm{i}}{2}(\Xi_{85}+\mathrm{i}\Xi_{86})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{38}+\mathrm{i}\Xi_{39})\\-\frac{\mathrm{i}}{2}(\Xi_{87}+\mathrm{i}\Xi_{88})\end{bmatrix} & -\frac{2}{\sqrt{12}}(\Xi_{40}-\mathrm{i}\Xi_{89}) & \frac{1}{\sqrt{2}}(\Xi_{46}+\mathrm{i}\Xi_{95}) \\
\hline
\begin{bmatrix}\frac{1}{2}(\Xi_{27}-\mathrm{i}\Xi_{76})\\+\frac{1}{\sqrt{12}}(\Xi_{32}-\mathrm{i}\Xi_{81})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{25}+\mathrm{i}\Xi_{26})\\-\frac{\mathrm{i}}{2}(\Xi_{74}+\mathrm{i}\Xi_{75})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{28}+\mathrm{i}\Xi_{29})\\-\frac{\mathrm{i}}{2}(\Xi_{77}+\mathrm{i}\Xi_{78})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{35}-\mathrm{i}\Xi_{84})\\+\frac{1}{\sqrt{12}}(\Xi_{40}-\mathrm{i}\Xi_{89})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{33}+\mathrm{i}\Xi_{34})\\-\frac{\mathrm{i}}{2}(\Xi_{82}+\mathrm{i}\Xi_{83})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{36}+\mathrm{i}\Xi_{37})\\-\frac{1}{2}(\Xi_{85}+\mathrm{i}\Xi_{86})\end{bmatrix} & \begin{bmatrix}\frac{1}{\sqrt{3}}(\Xi_{13}+\mathrm{i}\Xi_{62})\\+\frac{1}{\sqrt{2}}(\Xi_{14}+\mathrm{i}\Xi_{63})\\+\frac{1}{\sqrt{6}}(\Xi_{15}+\mathrm{i}\Xi_{64})\end{bmatrix} & \frac{1}{\sqrt{2}}(\Xi_{16}+\mathrm{i}\Xi_{65}) & \frac{1}{\sqrt{2}}(\Xi_{17}+\mathrm{i}\Xi_{66}) & \frac{1}{\sqrt{3}}(\Xi_{47}-\mathrm{i}\Xi_{96}) \\
\begin{bmatrix}\frac{1}{2}(\Xi_{25}-\mathrm{i}\Xi_{26})\\-\frac{\mathrm{i}}{2}(\Xi_{74}-\mathrm{i}\Xi_{75})\end{bmatrix} & \begin{bmatrix}-\frac{1}{2}(\Xi_{27}-\mathrm{i}\Xi_{76})\\+\frac{1}{\sqrt{12}}(\Xi_{32}-\mathrm{i}\Xi_{81})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{30}+\mathrm{i}\Xi_{31})\\-\frac{\mathrm{i}}{2}(\Xi_{79}+\mathrm{i}\Xi_{80})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{33}-\mathrm{i}\Xi_{34})\\-\frac{\mathrm{i}}{2}(\Xi_{82}-\mathrm{i}\Xi_{83})\end{bmatrix} & \begin{bmatrix}-\frac{1}{2}(\Xi_{35}-\mathrm{i}\Xi_{84})\\+\frac{1}{\sqrt{12}}(\Xi_{40}-\mathrm{i}\Xi_{89})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{38}+\mathrm{i}\Xi_{89})\\-\frac{\mathrm{i}}{2}(\Xi_{87}+\mathrm{i}\Xi_{88})\end{bmatrix} & \frac{1}{\sqrt{2}}(\Xi_{16}+\mathrm{i}\Xi_{65}) & \begin{bmatrix}\frac{1}{\sqrt{3}}(\Xi_{13}+\mathrm{i}\Xi_{62})\\-\frac{1}{\sqrt{2}}(\Xi_{14}+\mathrm{i}\Xi_{63})\\+\frac{1}{\sqrt{6}}(\Xi_{15}+\mathrm{i}\Xi_{64})\end{bmatrix} & \frac{1}{\sqrt{2}}(\Xi_{18}+\mathrm{i}\Xi_{67}) & \frac{1}{\sqrt{3}}(\Xi_{48}-\mathrm{i}\Xi_{97}) \\
\begin{bmatrix}\frac{1}{2}(\Xi_{28}-\mathrm{i}\Xi_{29})\\-\frac{\mathrm{i}}{2}(\Xi_{77}-\mathrm{i}\Xi_{78})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{30}-\mathrm{i}\Xi_{31})\\-\frac{\mathrm{i}}{2}(\Xi_{79}-\mathrm{i}\Xi_{80})\end{bmatrix} & -\frac{2}{\sqrt{12}}(\Xi_{32}-\mathrm{i}\Xi_{81}) & \begin{bmatrix}\frac{1}{2}(\Xi_{36}-\mathrm{i}\Xi_{37})\\-\frac{\mathrm{i}}{2}(\Xi_{85}-\mathrm{i}\Xi_{86})\end{bmatrix} & \begin{bmatrix}\frac{1}{2}(\Xi_{38}-\mathrm{i}\Xi_{39})\\-\frac{\mathrm{i}}{2}(\Xi_{87}-\mathrm{i}\Xi_{88})\end{bmatrix} & -\frac{2}{\sqrt{12}}(\Xi_{40}-\mathrm{i}\Xi_{89}) & \frac{1}{\sqrt{2}}(\Xi_{17}+\mathrm{i}\Xi_{66}) & \frac{1}{\sqrt{2}}(\Xi_{18}+\mathrm{i}\Xi_{67}) & \begin{bmatrix}\frac{1}{\sqrt{3}}(\Xi_{13}+\mathrm{i}\Xi_{62})\\-\frac{2}{\sqrt{6}}(\Xi_{15}+\mathrm{i}\Xi_{64})\end{bmatrix} & \frac{1}{\sqrt{3}}(\Xi_{49}-\mathrm{i}\Xi_{98}) \\
\hline
\frac{1}{\sqrt{2}}(\Xi_{41}+\mathrm{i}\Xi_{90}) & \frac{1}{\sqrt{2}}(\Xi_{42}+\mathrm{i}\Xi_{91}) & \frac{1}{\sqrt{2}}(\Xi_{43}+\mathrm{i}\Xi_{92}) & \frac{1}{\sqrt{2}}(\Xi_{44}+\mathrm{i}\Xi_{93}) & \frac{1}{\sqrt{2}}(\Xi_{45}+\mathrm{i}\Xi_{94}) & \frac{1}{\sqrt{2}}(\Xi_{46}+\mathrm{i}\Xi_{95}) & \frac{1}{\sqrt{3}}(\Xi_{47}-\mathrm{i}\Xi_{96}) & \frac{1}{\sqrt{3}}(\Xi_{48}-\mathrm{i}\Xi_{97}) & \frac{1}{\sqrt{3}}(\Xi_{49}-\mathrm{i}\Xi_{98}) & -\Xi_0
\end{array}\right)$$

$$
\boldsymbol{\Upsilon} = \tag{6.23b}
$$

$$
\left(\begin{array}{ccc|cc}
\begin{bmatrix} \frac{1}{\sqrt{6}}(\Upsilon_0+\mathrm{i}\Upsilon_{45}) \\ +\Upsilon_3+\mathrm{i}\Upsilon_{48} \\ +\frac{1}{\sqrt{3}}(\Upsilon_8+\mathrm{i}\Upsilon_{53}) \end{bmatrix} &
\begin{bmatrix} (\Upsilon_1-\mathrm{i}\Upsilon_2) \\ +\mathrm{i}(\Upsilon_{46}-\mathrm{i}\Upsilon_{47}) \end{bmatrix} &
\begin{bmatrix} (\Upsilon_4-\mathrm{i}\Upsilon_5) \\ +\mathrm{i}(\Upsilon_{49}-\mathrm{i}\Upsilon_{50}) \end{bmatrix} &
\begin{bmatrix} \frac{1}{\sqrt{2}}(\Upsilon_9+\mathrm{i}\Upsilon_{54}) \\ +\Upsilon_{12}+\mathrm{i}\Upsilon_{57} \end{bmatrix} &
\sqrt{2}(\Upsilon_{15}+\mathrm{i}\Upsilon_{60}) \\
\begin{bmatrix} (\Upsilon_1+\mathrm{i}\Upsilon_2) \\ +\mathrm{i}(\Upsilon_{46}+\mathrm{i}\Upsilon_{47}) \end{bmatrix} &
\begin{bmatrix} \frac{1}{\sqrt{6}}(\Upsilon_0+\mathrm{i}\Upsilon_{45}) \\ -\Upsilon_3-\mathrm{i}\Upsilon_{48} \\ +\frac{1}{\sqrt{3}}(\Upsilon_8+\mathrm{i}\Upsilon_{53}) \end{bmatrix} &
\begin{bmatrix} (\Upsilon_6-\mathrm{i}\Upsilon_7) \\ +\mathrm{i}(\Upsilon_{51}-\mathrm{i}\Upsilon_{52}) \end{bmatrix} &
\begin{bmatrix} \frac{1}{\sqrt{2}}(\Upsilon_{10}+\mathrm{i}\Upsilon_{55}) \\ +\Upsilon_{13}+\mathrm{i}\Upsilon_{58} \end{bmatrix} &
\sqrt{2}(\Upsilon_{16}+\mathrm{i}\Upsilon_{61}) \\
\begin{bmatrix} (\Upsilon_4+\mathrm{i}\Upsilon_5) \\ +\mathrm{i}(\Upsilon_{49}+\mathrm{i}\Upsilon_{50}) \end{bmatrix} &
\begin{bmatrix} (\Upsilon_6+\mathrm{i}\Upsilon_7) \\ +\mathrm{i}(\Upsilon_{51}+\mathrm{i}\Upsilon_{52}) \end{bmatrix} &
\begin{bmatrix} \frac{1}{\sqrt{6}}(\Upsilon_0+\mathrm{i}\Upsilon_{45}) \\ -\frac{2}{\sqrt{3}}(\Upsilon_8+\mathrm{i}\Upsilon_{53}) \end{bmatrix} &
\begin{bmatrix} \frac{1}{\sqrt{2}}(\Upsilon_{11}+\mathrm{i}\Upsilon_{56}) \\ +\Upsilon_{14}+\mathrm{i}\Upsilon_{59} \end{bmatrix} &
\sqrt{2}(\Upsilon_{17}+\mathrm{i}\Upsilon_{62}) \\
\hline
\begin{bmatrix} \frac{1}{\sqrt{6}}(\Upsilon_{18}+\mathrm{i}\Upsilon_{63}) \\ +\Upsilon_{21}+\mathrm{i}\Upsilon_{66} \\ +\frac{1}{\sqrt{3}}(\Upsilon_{26}+\mathrm{i}\Upsilon_{71}) \end{bmatrix} &
\begin{bmatrix} (\Upsilon_{19}-\mathrm{i}\Upsilon_{20}) \\ +\mathrm{i}(\Upsilon_{64}-\mathrm{i}\Upsilon_{65}) \end{bmatrix} &
\begin{bmatrix} (\Upsilon_{22}-\mathrm{i}\Upsilon_{23}) \\ +\mathrm{i}(\Upsilon_{67}-\mathrm{i}\Upsilon_{68}) \end{bmatrix} &
\sqrt{2}(\Upsilon_{27}+\mathrm{i}\Upsilon_{72}) &
\begin{bmatrix} \frac{1}{\sqrt{2}}(\Upsilon_9+\mathrm{i}\Upsilon_{54}) \\ -\Upsilon_{12}-\mathrm{i}\Upsilon_{57} \end{bmatrix} \\
\begin{bmatrix} (\Upsilon_{19}+\mathrm{i}\Upsilon_{20}) \\ +\mathrm{i}(\Upsilon_{64}+\mathrm{i}\Upsilon_{65}) \end{bmatrix} &
\begin{bmatrix} \frac{1}{\sqrt{6}}(\Upsilon_{18}+\mathrm{i}\Upsilon_{63}) \\ -\Upsilon_{21}-\mathrm{i}\Upsilon_{66} \\ +\frac{1}{\sqrt{3}}(\Upsilon_{26}+\mathrm{i}\Upsilon_{71}) \end{bmatrix} &
\begin{bmatrix} (\Upsilon_{24}-\mathrm{i}\Upsilon_{25}) \\ +\mathrm{i}(\Upsilon_{69}-\mathrm{i}\Upsilon_{70}) \end{bmatrix} &
\sqrt{2}(\Upsilon_{28}+\mathrm{i}\Upsilon_{73}) &
\begin{bmatrix} \frac{1}{\sqrt{2}}(\Upsilon_{10}+\mathrm{i}\Upsilon_{55}) \\ -\Upsilon_{13}-\mathrm{i}\Upsilon_{58} \end{bmatrix} \\
\begin{bmatrix} (\Upsilon_{22}+\mathrm{i}\Upsilon_{23}) \\ +\mathrm{i}(\Upsilon_{67}+\mathrm{i}\Upsilon_{68}) \end{bmatrix} &
\begin{bmatrix} (\Upsilon_{24}+\mathrm{i}\Upsilon_{25}) \\ +\mathrm{i}(\Upsilon_{69}+\mathrm{i}\Upsilon_{70}) \end{bmatrix} &
\begin{bmatrix} \frac{1}{\sqrt{6}}(\Upsilon_{18}+\mathrm{i}\Upsilon_{63}) \\ -\frac{2}{\sqrt{3}}(\Upsilon_{26}+\mathrm{i}\Upsilon_{71}) \end{bmatrix} &
\sqrt{2}(\Upsilon_{29}+\mathrm{i}\Upsilon_{74}) &
\begin{bmatrix} \frac{1}{\sqrt{2}}(\Upsilon_{11}+\mathrm{i}\Upsilon_{56}) \\ -\Upsilon_{14}-\mathrm{i}\Upsilon_{59} \end{bmatrix} \\
\hline
\begin{bmatrix} \sqrt{\frac{2}{3}}(\Upsilon_{30}+\mathrm{i}\Upsilon_{75}) \\ +\Upsilon_{31}+\mathrm{i}\Upsilon_{76} \\ +\frac{1}{\sqrt{3}}(\Upsilon_{32}+\mathrm{i}\Upsilon_{77}) \end{bmatrix} &
\begin{bmatrix} \Upsilon_{33}+\mathrm{i}\Upsilon_{78} \\ -\frac{1}{\sqrt{2}}(\Upsilon_{11}+\mathrm{i}\Upsilon_{56}) \end{bmatrix} &
\begin{bmatrix} \Upsilon_{34}+\mathrm{i}\Upsilon_{79} \\ +\frac{1}{\sqrt{2}}(\Upsilon_{10}+\mathrm{i}\Upsilon_{55}) \end{bmatrix} &
\sqrt{2}(\Upsilon_{36}-\mathrm{i}\Upsilon_{81}) &
\sqrt{2}(\Upsilon_{39}-\mathrm{i}\Upsilon_{84}) \\
\begin{bmatrix} \Upsilon_{33}+\mathrm{i}\Upsilon_{78} \\ +\frac{1}{\sqrt{2}}(\Upsilon_{11}+\mathrm{i}\Upsilon_{56}) \end{bmatrix} &
\begin{bmatrix} \sqrt{\frac{2}{3}}(\Upsilon_{30}+\mathrm{i}\Upsilon_{75}) \\ -\Upsilon_{31}-\mathrm{i}\Upsilon_{76} \\ +\frac{1}{\sqrt{3}}(\Upsilon_{32}+\mathrm{i}\Upsilon_{77}) \end{bmatrix} &
\begin{bmatrix} \Upsilon_{35}+\mathrm{i}\Upsilon_{80} \\ -\frac{1}{\sqrt{2}}(\Upsilon_9+\mathrm{i}\Upsilon_{54}) \end{bmatrix} &
\sqrt{2}(\Upsilon_{37}-\mathrm{i}\Upsilon_{82}) &
\sqrt{2}(\Upsilon_{40}-\mathrm{i}\Upsilon_{85}) \\
\begin{bmatrix} \Upsilon_{34}+\mathrm{i}\Upsilon_{79} \\ -\frac{1}{\sqrt{2}}(\Upsilon_{10}+\mathrm{i}\Upsilon_{55}) \end{bmatrix} &
\begin{bmatrix} \Upsilon_{35}+\mathrm{i}\Upsilon_{80} \\ +\frac{1}{\sqrt{2}}(\Upsilon_9+\mathrm{i}\Upsilon_{54}) \end{bmatrix} &
\begin{bmatrix} \sqrt{\frac{2}{3}}(\Upsilon_{30}+\mathrm{i}\Upsilon_{75}) \\ -\frac{2}{\sqrt{3}}(\Upsilon_{32}+\mathrm{i}\Upsilon_{77}) \end{bmatrix} &
\sqrt{2}(\Upsilon_{38}-\mathrm{i}\Upsilon_{83}) &
\sqrt{2}(\Upsilon_{41}-\mathrm{i}\Upsilon_{88}) \\
\hline
\sqrt{2}(\Upsilon_{42}-\mathrm{i}\Upsilon_{87}) &
\sqrt{2}(\Upsilon_{43}-\mathrm{i}\Upsilon_{88}) &
\sqrt{2}(\Upsilon_{44}-\mathrm{i}\Upsilon_{89}) &
-\frac{3}{\sqrt{6}}(\Upsilon_{18}+\mathrm{i}\Upsilon_{63}) &
\frac{3}{\sqrt{6}}(\Upsilon_0+\mathrm{i}\Upsilon_{45})
\end{array}\right)
$$

In terms of the local basis $\{\gamma^\mu\}_{\mu=1,2,3,4}$ of Λ^1 we put

$$\begin{array}{lllll} \mathbf{G}=\mathbf{G}_\mu\gamma^\mu\,, & G^a=G^a_\mu\gamma^\mu\,, & W^a=W^a_\mu\gamma^\mu\,, & A'=A'_\mu\gamma^\mu\,, & \tilde{A}=\tilde{A}_\mu\gamma^\mu\,, \\ \mathbf{X}=\mathbf{X}_\mu\gamma^\mu\,, & X=X_\mu\gamma^\mu\,, & X^a=X^a_\mu\gamma^\mu\,, & Y=Y_\mu\gamma^\mu\,, & Y^a=Y^a_\mu\gamma^\mu\,. \end{array} \tag{6.25}$$

6.3.3 The Gauge Field Lagrangian

The gauge field Lagrangian $\mathscr{L}_2$ is easy to compute:

We introduce the abbreviation

$$S_{[\mu}T_{\nu]} := S_\mu T_\nu - S_\nu T_\mu\,. \tag{6.26}$$

Then, we obtain in terms of the local basis $\gamma^\mu\wedge\gamma^\nu$ of Λ^2

$$\mathbf{d}A+\tfrac{1}{2}\{A,A\}=\frac{\mathrm{i}g_0}{4}\begin{pmatrix} \begin{bmatrix} \frac{2}{3}(\sqrt{\frac{3}{5}}A'_{\mu\nu}-X^0_{\mu\nu})\mathbb{1}_3 \\ +\sum_{a=1}^8(G^a_{\mu\nu}-X^a_{\mu\nu})\lambda^a \end{bmatrix} & (D\mathbf{X})_{\mu\nu} \\ (D\mathbf{X})^*_{\mu\nu} & \begin{bmatrix} (-\sqrt{\frac{3}{5}}A'_{\mu\nu}+X^0_{\mu\nu})\mathbb{1}_2 \\ +\sum_{a=1}^3(W^a_{\mu\nu}-\tilde{X}^a_{\mu\nu})\sigma^a \end{bmatrix} \end{pmatrix}\gamma^\mu\wedge\gamma^\nu\,, \tag{6.27}$$

where

$$\begin{aligned}
G^a_{\mu\nu}&=\partial_{[\mu}G^a_{\nu]}-g_0\textstyle\sum_{b,c=1}^8 f_{abc}G^b_\mu G^c_\nu\,, \qquad W^a_{\mu\nu}=\partial_{[\mu}W^a_{\nu]}-g_0\sum_{b,c=1}^3\varepsilon_{abc}W^b_\mu W^c_\nu\,,\\
A'_{\mu\nu}&=\partial_{[\mu}A'_{\nu]}\,, \qquad (D\mathbf{X})_{\mu\nu}=\big(\,(DX)_{\mu\nu}\,,\,(DY)_{\mu\nu}\,\big)\,,\\
(DX)_{\mu\nu}&=\partial_{[\mu}X_{\nu]}+\mathrm{i}\tfrac{g_0}{2}(\mathbf{G}_{[\mu}\cdot X_{\nu]}+(\sqrt{\tfrac{5}{3}}A'-W^3)_{[\mu}X_{\nu]}-(W^1+\mathrm{i}W^2)_{[\mu}Y_{\nu]})\,,\\
(DY)_{\mu\nu}&=\partial_{[\mu}Y_{\nu]}+\mathrm{i}\tfrac{g_0}{2}(\mathbf{G}_{[\mu}\cdot Y_{\nu]}+(\sqrt{\tfrac{5}{3}}A'+W^3)_{[\mu}Y_{\nu]}-(W^1-\mathrm{i}W^2)_{[\mu}X_{\nu]})\,,\\
X^a_{\mu\nu}&=\tfrac{g_0}{2}\textstyle\sum_{b,c=1}^6 e^{abc}(X^b_\mu X^c_\nu+Y^b_\mu Y^c_\nu)\,, \quad a=0,\dots,8\,, \qquad e^{abc}=-e^{acb}\,,\\
\tilde{X}^1_{\mu\nu}&=\tfrac{g_0}{2}(-X^1_{[\mu}Y^2_{\nu]}+X^2_{[\mu}Y^1_{\nu]}-X^3_{[\mu}Y^4_{\nu]}+X^4_{[\mu}Y^3_{\nu]}-X^5_{[\mu}Y^6_{\nu]}+X^6_{[\mu}Y^5_{\nu]})\,,\\
\tilde{X}^2_{\mu\nu}&=\tfrac{g_0}{2}\textstyle\sum_{a=1}^6 X^a_{[\mu}Y^a_{\nu]}\,,\\
\tilde{X}^3_{\mu\nu}&=\tfrac{g_0}{2}\textstyle\sum_{b,c=1}^6 e^{0bc}(Y^b_\mu Y^c_\nu-X^b_\mu X^c_\nu)\,,\\
1&=e^{012}=e^{034}=e^{056}=e^{114}=e^{132}=e^{213}=e^{224}=e^{312}=-e^{334}=e^{416}=e^{452}\\
&=e^{515}=e^{526}=e^{636}=e^{654}=e^{735}=e^{746}=\tfrac{\sqrt{3}}{2}e^{812}=\tfrac{\sqrt{3}}{2}e^{834}=-\sqrt{3}e^{856}\,.
\end{aligned} \tag{6.28}$$

All other e^{abc} vanish. Moreover,

$$\mathbf{d}A''=\mathrm{i}\frac{g_0}{4}\sqrt{\frac{2}{5}}\tilde{A}_{\mu\nu}\,, \qquad \tilde{A}_{\mu\nu}:=\partial_{[\mu}\tilde{A}_{\nu]}\,. \tag{6.29}$$

Then, using (C.3) and (C.6) we obtain for (6.10b)

$$\begin{aligned}
\mathscr{L}_2=\tfrac{1}{4}\delta^{\kappa\lambda}\delta^{\mu\nu}\Big(&A'_{\kappa\lambda}A'_{\mu\nu}+\tilde{A}_{\kappa\lambda}\tilde{A}'_{\mu\nu}+\sum_{a=1}^8 G^a_{\kappa\lambda}G^a_{\mu\nu}+\sum_{a=1}^3 W^a_{\kappa\lambda}W^a_{\mu\nu}\\
&+(DX)^*_{\kappa\lambda}\cdot(DX)_{\mu\nu}+(DY)^*_{\kappa\lambda}\cdot(DY)_{\mu\nu}+(\tfrac{5}{3}X^0_{\kappa\lambda}X^0_{\mu\nu}-2\sqrt{\tfrac{5}{3}}X^0_{\kappa\lambda}A'_{\mu\nu})\\
&+\sum_{a=1}^8(X^a_{\kappa\lambda}X^a_{\mu\nu}-2X^a_{\kappa\lambda}G^a_{\mu\nu})+\sum_{a=1}^3(\tilde{X}^a_{\kappa\lambda}\tilde{X}^a_{\mu\nu}-2\tilde{X}^a_{\kappa\lambda}W^a_{\mu\nu})\Big)\,.
\end{aligned} \tag{6.30}$$

6.3.4 The Covariant Derivatives of the Higgs Fields

We compute the covariant derivatives of the Higgs fields. The formulae are rather long. But for the discussion of the physical properties we will only need the partial derivatives and the couplings to the electrically neutral and colour–neutral gauge fields $W^3, A', \tilde{A}$. The reader should focus the attention to these terms.

Using (6.18a) and (6.24a) we get

$$\begin{aligned}
&\mathbf{d}\mathbf{\Psi} + [A, \mathbf{\Psi} + \mathbf{m}] = \\
&\begin{pmatrix} -\sqrt{\frac{4}{15}}\partial_\mu \Psi_0 \mathbb{1}_3 + \sum_{a=1}^8 D_\mu \mathbf{\Psi}_a \lambda^a & (D\mathbf{X})_\mu \\ (D\mathbf{X})^*_\mu & \sqrt{\frac{3}{5}}\partial_\mu \Psi_0 \mathbb{1}_2 + \sum_{a=1}^3 D_\mu \mathbf{\Psi}'_a \sigma^a \end{pmatrix} \gamma^\mu , \\
&D_\mu \mathbf{\Psi}_a = \partial_\mu \mathbf{\Psi}_a - g_0 \textstyle\sum_{b,c=1}^8 f_{abc} G^b_\mu \Psi_c , \\
&D_\mu \mathbf{\Psi}'_a = \partial_\mu \mathbf{\Psi}'_a - g_0 \textstyle\sum_{b,c=1}^3 \varepsilon_{abc} W^b_\mu \Psi'_c , \\
&(D\mathbf{X})_\mu = \mathrm{i}\tfrac{g_0}{2}(-\mathbf{\Psi}_g \cdot \mathbf{X}_\mu + \mathbf{X}_\mu \cdot \mathbf{\Psi}_w + (\sqrt{\tfrac{5}{3}}\Psi_0 + 1)\mathbf{X}_\mu) .
\end{aligned} \tag{6.31}$$

With (C.6) and (C.3) we obtain

$$\begin{aligned}
\frac{\mu_0}{g_0^2}\,\mathrm{tr}_c((\mathbf{d}\mathbf{\Psi} + [A, \mathbf{\Psi}])^2) = \frac{8\mu_0}{g_0^2}\delta^{\mu\nu}\Big(&\sum_{a=1}^8 (D_\mu \mathbf{\Psi}_a)(D_\nu \mathbf{\Psi}_a) + \sum_{a=1}^3 (D_\mu \mathbf{\Psi}'_a)(D_\nu \mathbf{\Psi}'_a) \\
&+ (\partial_\mu \Psi_0)(\partial_\nu \Psi_0) + \mathrm{tr}((D\mathbf{X})^*_\mu (D\mathbf{X})_\nu)\Big) .
\end{aligned} \tag{6.32}$$

Now, using (6.24a) and (6.20) we calculate

$$\begin{aligned}
&\mathbf{d}\mathbf{\Phi} + (A + A''\mathbb{1}_5)(\mathbf{\Phi} + \mathbf{n}) = \begin{pmatrix} D_\mu \mathbf{\Phi}_g \\ D_\mu \mathbf{\Phi}_w \end{pmatrix} \gamma^\mu , \\
&D_\mu \mathbf{\Phi}_g = \partial_\mu \mathbf{\Phi}_g + \mathrm{i}\tfrac{g_0}{2}(\mathbf{G}_\mu \cdot \mathbf{\Phi}_g + (\tfrac{2}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu)\mathbf{\Phi}_g + X_\mu(\Phi_0 + 1)) , \\
&D_\mu \mathbf{\Phi}_w = \begin{pmatrix} \partial_\mu \Phi_0 + \mathrm{i}\frac{g_0}{2}((W^3_\mu - \sqrt{\frac{3}{5}}A'_\mu + \sqrt{\frac{2}{5}}\tilde{A}_\mu)(\Phi_0 + 1) + X^*_\mu \cdot \mathbf{\Phi}_g) \\ \mathrm{i}\frac{g_0}{2}((W^1_\mu + \mathrm{i}W^2_\mu)(\Phi_0 + 1) + Y^*_\mu \cdot \mathbf{\Phi}_g) \end{pmatrix} .
\end{aligned} \tag{6.33}$$

This gives

$$\begin{aligned}
&\frac{\mu_1}{g_0^2}\,\mathrm{tr}_c((\mathbf{d}\mathbf{\Phi} + (A + A''\mathbb{1}_5)(\mathbf{\Phi} + \mathbf{n}))^*(\mathbf{d}\mathbf{\Phi} + (A + A''\mathbb{1}_5)(\mathbf{\Phi} + \mathbf{n}))) \\
&\quad = \frac{4\mu_1}{g_0^2}\delta^{\mu\nu}\{(D_\mu \mathbf{\Phi}_g)^*(D_\nu \mathbf{\Phi}_g) + (D_\mu \mathbf{\Phi}_w)^*(D_\nu \mathbf{\Phi}_w)\} .
\end{aligned} \tag{6.34}$$

Next, using (6.24a) and (6.23a) we calculate

$$\begin{aligned}
&\mathbf{d}\mathbf{\Upsilon} + \check{A}(\mathbf{\Upsilon} + \mathbf{n}') - (\mathbf{\Upsilon} + \mathbf{n}')A + A''(\mathbf{\Upsilon} + \mathbf{n}') \\
&\quad = \begin{pmatrix} D_\mu \mathbf{\Upsilon}_A & D_\mu \mathbf{\Upsilon}_a + D_\mu \mathbf{\Upsilon}_b & D_\mu \mathbf{\Upsilon}_c \\ D_\mu \mathbf{\Upsilon}_B & D_\mu \mathbf{\Upsilon}_d & D_\mu \mathbf{\Upsilon}_a - D_\mu \mathbf{\Upsilon}_b \\ D_\mu \mathbf{\Upsilon}_C - \varepsilon(D_\mu \mathbf{\Upsilon}_a) & \overline{D_\mu \mathbf{\Upsilon}_e} & \overline{D_\mu \mathbf{\Upsilon}_f} \\ (D_\mu \mathbf{\Upsilon}_g)^* & -\mathrm{tr}(D_\mu \mathbf{\Upsilon}_B) & \mathrm{tr}(D_\mu \mathbf{\Upsilon}_A) \end{pmatrix} \gamma^\mu ,
\end{aligned} \tag{6.35}$$

$$
\begin{aligned}
D_\mu \boldsymbol{\Upsilon}_A := {}& \partial_\mu \boldsymbol{\Upsilon}_A + \mathrm{i}\tfrac{g_0}{2}([\mathbf{G}_\mu, \boldsymbol{\Upsilon}_A] + (-\tfrac{3}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu + W^3_\mu)(\boldsymbol{\Upsilon}_A + \mathbb{1}_3) \\
& + (W^1_\mu - \mathrm{i}W^2_\mu)\boldsymbol{\Upsilon}_B + \epsilon(\bar{X}_\mu)\boldsymbol{\Upsilon}_C - Y_\mu \boldsymbol{\Upsilon}^*_g - \boldsymbol{\Upsilon}_b X^*_\mu - (X^*_\mu \boldsymbol{\Upsilon}_a)\mathbb{1}_3 - \boldsymbol{\Upsilon}_c Y^*_\mu \,, \\
D_\mu \boldsymbol{\Upsilon}_B := {}& \partial_\mu \boldsymbol{\Upsilon}_B + \mathrm{i}\tfrac{g_0}{2}([\mathbf{G}_\mu, \boldsymbol{\Upsilon}_B] + (-\tfrac{3}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu - W^3_\mu)\boldsymbol{\Upsilon}_B \\
& + (W^1_\mu + \mathrm{i}W^2_\mu)(\boldsymbol{\Upsilon}_A + \mathbb{1}_3) + \epsilon(\bar{Y}_\mu)\boldsymbol{\Upsilon}_C + X_\mu \boldsymbol{\Upsilon}^*_g + \boldsymbol{\Upsilon}_b Y^*_\mu - (Y^*_\mu \boldsymbol{\Upsilon}_a)\mathbb{1}_3 - \boldsymbol{\Upsilon}_d X^*_\mu) \,, \\
D_\mu \boldsymbol{\Upsilon}_C := {}& \tfrac{1}{2}\big(\partial_\mu \boldsymbol{\Upsilon}_C + \mathrm{i}\tfrac{g_0}{2}(-2\boldsymbol{\Upsilon}_C \mathbf{G}_\mu + (\tfrac{2}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu)\boldsymbol{\Upsilon}_C \\
& - \epsilon(X_\mu)(\boldsymbol{\Upsilon}_A + \mathbb{1}_3) - \epsilon(Y_\mu)\boldsymbol{\Upsilon}_B - \overline{\boldsymbol{\Upsilon}_e} X^*_\mu - \overline{\boldsymbol{\Upsilon}_f} Y^*_\mu)\big) + \text{ Transpose} \,, \\
D_\mu \boldsymbol{\Upsilon}_a := {}& \partial_\mu \boldsymbol{\Upsilon}_a + \mathrm{i}\tfrac{g_0}{2}(\mathbf{G}_\mu \boldsymbol{\Upsilon}_a + (\tfrac{2}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu)\boldsymbol{\Upsilon}_a - \tfrac{1}{2}(\boldsymbol{\Upsilon}_A - (\mathrm{tr}(\boldsymbol{\Upsilon}_A) + 2)\mathbb{1}_3)X_\mu \\
& - \tfrac{1}{2}(\boldsymbol{\Upsilon}_B - \mathrm{tr}(\boldsymbol{\Upsilon}_B)\mathbb{1}_3)Y_\mu + \tfrac{1}{2}\epsilon(\bar{X}_\mu)\overline{\boldsymbol{\Upsilon}_e} + \tfrac{1}{2}\epsilon(\bar{Y}_\mu)\overline{\boldsymbol{\Upsilon}_f}) \,, \\
D_\mu \boldsymbol{\Upsilon}_b := {}& \partial_\mu \boldsymbol{\Upsilon}_b + \mathrm{i}\tfrac{g_0}{2}(\mathbf{G}_\mu \boldsymbol{\Upsilon}_b + (\tfrac{2}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu)\boldsymbol{\Upsilon}_b \\
& - \tfrac{1}{2}X_\mu(\boldsymbol{\Upsilon}_A + (\mathrm{tr}(\boldsymbol{\Upsilon}_A) + 4)\mathbb{1}_3) + \tfrac{1}{2}Y_\mu(\boldsymbol{\Upsilon}_B + \mathrm{tr}(\boldsymbol{\Upsilon}_B)\mathbb{1}_3) + \tfrac{1}{2}\epsilon(\bar{X}_\mu)\overline{\boldsymbol{\Upsilon}_e} \\
& - \tfrac{1}{2}\epsilon(\bar{Y}_\mu)\overline{\boldsymbol{\Upsilon}_f} - \boldsymbol{\Upsilon}_c(W^1_\mu + \mathrm{i}W^2_\mu) + \boldsymbol{\Upsilon}_d(W^1_\mu - \mathrm{i}W^2_\mu)) \,, \\
D_\mu \boldsymbol{\Upsilon}_c := {}& \partial_\mu \boldsymbol{\Upsilon}_c + \mathrm{i}\tfrac{g_0}{2}(\mathbf{G}_\mu \boldsymbol{\Upsilon}_c + (\tfrac{2}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu + 2W^3_\mu)\boldsymbol{\Upsilon}_c \\
& - Y_\mu(\boldsymbol{\Upsilon}_A + (\mathrm{tr}(\boldsymbol{\Upsilon}_A) + 4)\mathbb{1}_3) + \epsilon(\bar{X}_\mu)\overline{\boldsymbol{\Upsilon}_f} - 2\boldsymbol{\Upsilon}_b(W^1_\mu - \mathrm{i}W^2_\mu)) \,, \\
D_\mu \boldsymbol{\Upsilon}_d := {}& \partial_\mu \boldsymbol{\Upsilon}_d + \mathrm{i}\tfrac{g_0}{2}(\mathbf{G}_\mu \boldsymbol{\Upsilon}_d + (\tfrac{2}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu - 2W^3_\mu)\boldsymbol{\Upsilon}_d \\
& - X_\mu(\boldsymbol{\Upsilon}_B + \mathrm{tr}(\boldsymbol{\Upsilon}_B)\mathbb{1}_3) + \epsilon(\bar{Y}_\mu)\overline{\boldsymbol{\Upsilon}_e} + 2\boldsymbol{\Upsilon}_b(W^1_\mu + \mathrm{i}W^2_\mu)) \,, \\
D_\mu \boldsymbol{\Upsilon}_e := {}& \partial_\mu \boldsymbol{\Upsilon}_e + \mathrm{i}\tfrac{g_0}{2}(\mathbf{G}_\mu \boldsymbol{\Upsilon}_e - (\tfrac{7}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu - W^3_\mu)\boldsymbol{\Upsilon}_e + \epsilon(\bar{X}_\mu)\overline{\boldsymbol{\Upsilon}_b} + \boldsymbol{\Upsilon}^*_C \bar{X}_\mu \\
& + \boldsymbol{\Upsilon}_f(W^1_\mu - \mathrm{i}W^2_\mu) + \epsilon(\bar{Y}_\mu)\overline{\boldsymbol{\Upsilon}_d} + 2\epsilon(\bar{X}_\mu)\overline{\boldsymbol{\Upsilon}_a}) \,, \\
D_\mu \boldsymbol{\Upsilon}_f := {}& \partial_\mu \boldsymbol{\Upsilon}_f + \mathrm{i}\tfrac{g_0}{2}(\mathbf{G}_\mu \boldsymbol{\Upsilon}_f - (\tfrac{7}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu + W^3_\mu)\boldsymbol{\Upsilon}_f - \epsilon(\bar{Y}_\mu)\overline{\boldsymbol{\Upsilon}_b} + \boldsymbol{\Upsilon}^*_C \bar{Y}_\mu \\
& + \boldsymbol{\Upsilon}_e(W^1_\mu + \mathrm{i}W^2_\mu) + \epsilon(\bar{X}_\mu)\overline{\boldsymbol{\Upsilon}_c} + 2\epsilon(\bar{Y}_\mu)\overline{\boldsymbol{\Upsilon}_a}) \,, \\
D_\mu \boldsymbol{\Upsilon}_g := {}& \partial_\mu \boldsymbol{\Upsilon}_g + \mathrm{i}\tfrac{g_0}{2}(\mathbf{G}_\mu \boldsymbol{\Upsilon}_g + (\tfrac{8}{\sqrt{15}}A'_\mu - \sqrt{\tfrac{2}{5}}\tilde{A}_\mu)\boldsymbol{\Upsilon}_g + (\boldsymbol{\Upsilon}^*_A + (\mathrm{tr}(\boldsymbol{\Upsilon}^*_A) + 4)\mathbb{1}_3)Y_\mu \\
& - (\boldsymbol{\Upsilon}^*_B + \mathrm{tr}(\boldsymbol{\Upsilon}^*_B)\mathbb{1}_3)X_\mu) \,.
\end{aligned}
$$

This gives

$$
\begin{aligned}
& \tfrac{\mu_2}{g_0^2}\,\mathrm{tr}_c((\mathbf{d}\boldsymbol{\Upsilon} + \check{A}(\boldsymbol{\Upsilon} + \mathbf{n}') - (\boldsymbol{\Upsilon} + \mathbf{n}')A + A''(\boldsymbol{\Upsilon} + \mathbf{n}'))^* \times \\
& \qquad\qquad \times (\mathbf{d}\boldsymbol{\Upsilon} + \check{A}(\boldsymbol{\Upsilon} + \mathbf{n}') - (\boldsymbol{\Upsilon} + \mathbf{n}')A + A''(\boldsymbol{\Upsilon} + \mathbf{n}'))) \qquad (6.36) \\
& = \tfrac{4\mu_2}{g_0^2}\delta^{\mu\nu}\big(\mathrm{tr}((D_\mu \boldsymbol{\Upsilon}_A)^*(D_\nu \boldsymbol{\Upsilon}_A)) + \mathrm{tr}((D_\mu \boldsymbol{\Upsilon}_B)^*(D_\nu \boldsymbol{\Upsilon}_B)) + \mathrm{tr}((D_\mu \boldsymbol{\Upsilon}_C)^*(D_\nu \boldsymbol{\Upsilon}_C)) \\
& + 4(D_\mu \boldsymbol{\Upsilon}_a)^*(D_\nu \boldsymbol{\Upsilon}_a) + 2(D_\mu \boldsymbol{\Upsilon}_b)^*(D_\nu \boldsymbol{\Upsilon}_b) + (D_\mu \boldsymbol{\Upsilon}_c)^*(D_\nu \boldsymbol{\Upsilon}_c) \\
& + (D_\mu \boldsymbol{\Upsilon}_d)^*(D_\nu \boldsymbol{\Upsilon}_d) + (D_\mu \boldsymbol{\Upsilon}_e)^*(D_\nu \boldsymbol{\Upsilon}_e) + (D_\mu \boldsymbol{\Upsilon}_f)^*(D_\nu \boldsymbol{\Upsilon}_f) \\
& + (D_\mu \boldsymbol{\Upsilon}_g)^*(D_\nu \boldsymbol{\Upsilon}_g) + (\mathrm{tr}(D_\mu \boldsymbol{\Upsilon}_A))^*\,\mathrm{tr}(D_\nu \boldsymbol{\Upsilon}_A) + (\mathrm{tr}(D_\mu \boldsymbol{\Upsilon}_B))^*(D_\nu \boldsymbol{\Upsilon}_B))\big) \,.
\end{aligned}
$$

Finally, using (6.24a) and (6.22a) we calculate

$$
\begin{aligned}
&\mathbf{d}\Xi + \breve{A}(\Xi+\mathbf{n}') + (\Xi+\mathbf{m}')\breve{A}^T - A''(\Xi+\mathbf{m}') \\
&= \begin{pmatrix} \overline{D_\mu\Xi_A} & \overline{D_\mu\Xi_D} - \frac{1}{2}\varepsilon(\overline{D_\mu\Xi_c}) & (D_\mu\Xi_E^0)^* & D_\mu\Xi_a \\ \overline{D_\mu\Xi_D} + \frac{1}{2}\varepsilon(\overline{D_\mu\Xi_c}) & \overline{D_\mu\Xi_B} & (D_\mu\Xi_F^0)^* & D_\mu\Xi_b \\ \overline{D_\mu\Xi_E^0} & \overline{D_\mu\Xi_F^0} & D_\mu\Xi_C & \overline{D_\mu\Xi_c} \\ (D_\mu\Xi_a)^T & (D_\mu\Xi_b)^T & (D_\mu\Xi_c)^* & -D_\mu\Xi_0 \end{pmatrix}\gamma^\mu , \qquad (6.37)\\
&D_\mu\Xi_A := \partial_\mu\Xi_A + \mathrm{i}\tfrac{g_0}{2}\big(-\mathbf{G}_\mu^T\Xi_A - \Xi_A\mathbf{G}_\mu + (\tfrac{2}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu - 2W_\mu^3)\Xi_A \\
&\qquad -2(W_\mu^1 + \mathrm{i}W_\mu^2)\Xi_D + \overline{Y}_\mu\Xi_a^* + \overline{\Xi_a}Y_\mu^* + (\Xi_E^0)^T\epsilon(X_\mu) - \epsilon(X_\mu)\Xi_E^0\big) , \\
&D_\mu\Xi_B := \partial_\mu\Xi_B + \mathrm{i}\tfrac{g_0}{2}\big(-\mathbf{G}_\mu^T\Xi_B - \Xi_B\mathbf{G}_\mu + (\tfrac{2}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu + 2W_\mu^3)\Xi_B \\
&\qquad -2(W_\mu^1 - \mathrm{i}W_\mu^2)\Xi_D - \overline{X}_\mu\Xi_b^* - \overline{\Xi_b}X_\mu^* + (\Xi_F^0)^T\epsilon(Y_\mu) - \epsilon(Y_\mu)\Xi_F^0\big) , \\
&D_\mu\Xi_C := \partial_\mu\Xi_C + \mathrm{i}\tfrac{g_0}{2}\big(-\mathbf{G}_\mu^T\Xi_C - \Xi_C\mathbf{G}_\mu + (\tfrac{8}{\sqrt{15}}A'_\mu - \sqrt{\tfrac{2}{5}}\tilde{A}_\mu)\Xi_C \\
&\qquad +\overline{\Xi_E^0}\epsilon(X_\mu) - \epsilon(X_\mu)(\Xi_E^0)^* + \overline{\Xi_F^0}\epsilon(Y_\mu) - \epsilon(Y_\mu)(\Xi_F^0)^*\big) , \\
&D_\mu\Xi_D := \partial_\mu\Xi_D + \mathrm{i}\tfrac{g_0}{2}\big(-\mathbf{G}_\mu^T\Xi_D - \Xi_D\mathbf{G}_\mu + (\tfrac{2}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu)\Xi_D - (W_\mu^1 + \mathrm{i}W_\mu^2)\Xi_B \\
&\qquad -(W_\mu^1 - \mathrm{i}W_\mu^2)\Xi_A + \tfrac{1}{2}\overline{Y}_\mu\Xi_b^* + \tfrac{1}{2}\overline{\Xi_b}Y_\mu^* - \tfrac{1}{2}\overline{X}_\mu\Xi_a^* - \tfrac{1}{2}\overline{\Xi_a}X_\mu^* \\
&\qquad +\tfrac{1}{2}(\Xi_F^0)^T\epsilon(X_\mu) - \tfrac{1}{2}\epsilon(X_\mu)\Xi_F^0 + \tfrac{1}{2}(\Xi_E^0)^T\epsilon(Y_\mu) - \tfrac{1}{2}\epsilon(Y_\mu)\Xi_E^0\big) , \\
&D_\mu\Xi_E^0 := \partial_\mu\Xi_E^0 + \mathrm{i}\tfrac{g_0}{2}\big([\mathbf{G}_\mu,\Xi_E^0] + (\tfrac{3}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu - W_\mu^3)\Xi_E^0 - (W_\mu^1 + \mathrm{i}W_\mu^2)\Xi_F^0 \\
&\qquad +\epsilon(\overline{X}_\mu)\Xi_A + \epsilon(\overline{Y}_\mu)\Xi_D + \overline{\Xi_C}\epsilon(X_\mu) + \tfrac{3}{2}(\overline{Y}_\mu(\Xi_c)^T - \tfrac{1}{3}\,\mathrm{tr}(\overline{Y}_\mu\Xi_c^T)\mathbb{1}_3)\big) , \\
&D_\mu\Xi_F^0 := \partial_\mu\Xi_F^0 + \mathrm{i}\tfrac{g_0}{2}\big([\mathbf{G}_\mu,\Xi_F^0] + (\tfrac{3}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu + W_\mu^3)\Xi_F^0 - (W_\mu^1 - \mathrm{i}W_\mu^2)\Xi_E^0 \\
&\qquad +\epsilon(\overline{Y}_\mu)\Xi_B + \epsilon(\overline{X}_\mu)\Xi_D + \overline{\Xi_C}\epsilon(Y_\mu) - \tfrac{3}{2}(\overline{X}_\mu\Xi_c^T - \tfrac{1}{3}\,\mathrm{tr}(\overline{X}_\mu\Xi_c^T)\mathbb{1}_3)\big) , \\
&D_\mu\Xi_a := \partial_\mu\Xi_a + \mathrm{i}\tfrac{g_0}{2}\big(\mathbf{G}_\mu\Xi_a + (-\tfrac{7}{\sqrt{15}}A'_\mu - \sqrt{\tfrac{2}{5}}\tilde{A}_\mu + W_\mu^3)\Xi_a \\
&\qquad +(W_\mu^1 - \mathrm{i}W_\mu^2)\Xi_b + \overline{\Xi_D}\overline{X}_\mu - \overline{\Xi_A}\overline{Y}_\mu + (\Xi_0+1)Y_\mu + \tfrac{3}{2}\epsilon(\overline{X}_\mu)\overline{\Xi_c}\big) , \\
&D_\mu\Xi_b := \partial_\mu\Xi_b + \mathrm{i}\tfrac{g_0}{2}\big(\mathbf{G}_\mu\Xi_b + (-\tfrac{7}{\sqrt{15}}A'_\mu - \sqrt{\tfrac{2}{5}}\tilde{A}_\mu - W_\mu^3)\Xi_b \\
&\qquad +(W_\mu^1 + \mathrm{i}W_\mu^2)\Xi_a - \overline{\Xi_D}\overline{Y}_\mu + \overline{\Xi_B}\overline{X}_\mu - (\Xi_0+1)X_\mu + \tfrac{3}{2}\epsilon(\overline{Y}_\mu)\overline{\Xi_c}\big) , \\
&D_\mu\Xi_c := \partial_\mu\Xi_c + \mathrm{i}\tfrac{g_0}{2}\big(\mathbf{G}_\mu\Xi_c + (\tfrac{2}{\sqrt{15}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu)\Xi_c \\
&\qquad +\Xi_E^0 Y_\mu - \Xi_F^0 X_\mu + \epsilon(\overline{X}_\mu)\overline{\Xi_a} + \epsilon(\overline{Y}_\mu)\overline{\Xi_b}\big) , \\
&D_\mu\Xi_0 := \partial_\mu\Xi_0 + \mathrm{i}\tfrac{g_0}{2}\big((-\tfrac{12}{\sqrt{15}}A'_\mu - \sqrt{\tfrac{2}{5}}\tilde{A}_\mu)(\Xi_0+1) + 2Y_\mu^*\Xi_a - 2X_\mu^*\Xi_b\big) .
\end{aligned}
$$

This gives

$$
\begin{aligned}
&\tfrac{\mu_3}{g_0^2}\,\mathrm{tr}_c\big((\mathbf{d}\Xi + \breve{A}(\Xi+\mathbf{m}') + (\Xi+\mathbf{m}')\breve{A}^T - A''(\Xi+\mathbf{m}'))^*\times \\
&\qquad\qquad \times(\mathbf{d}\Xi + \breve{A}(\Xi+\mathbf{m}') + (\Xi+\mathbf{m}')\breve{A}^T - A''(\Xi+\mathbf{m}'))\big) \qquad (6.38)
\end{aligned}
$$

$$
\begin{aligned}
&= \tfrac{4\mu_3}{g_0^2}\delta^{\mu\nu}\{\mathrm{tr}((D_\mu\Xi_A)^*(D_\nu\Xi_A)) + \mathrm{tr}((D_\mu\Xi_B)^*(D_\nu\Xi_B)) + \mathrm{tr}((D_\mu\Xi_C)^*(D_\nu\Xi_C)) \\
&+ 2\,\mathrm{tr}((D_\mu\Xi_D)^*(D_\nu\Xi_D)) + 2\,\mathrm{tr}((D_\mu\Xi_E^0)^*(D_\nu\Xi_E^0)) + 2\,\mathrm{tr}((D_\mu\Xi_F^0)^*(D_\nu\Xi_F^0)) \\
&+ 2(D_\mu\Xi_a)^*(D_\nu\Xi_a) + 2(D_\mu\Xi_b)^*(D_\nu\Xi_b) + 3(D_\mu\Xi_c)^*(D_\nu\Xi_c) + (D_\mu\Xi_0)^*(D_\nu\Xi_0)\}\ .
\end{aligned}
$$

The Lagrangian $\mathscr{L}_1$ is the sum of formulae (6.32), (6.34), (6.36) and (6.38).

6.3.5 Summary

Here we collect the results of Sections 6.3.3 and 6.3.4. As already mentioned we must restrict ourselves to the quadratic terms in $\mathscr{L}_0$. This means that we pick up only the mass terms and neglect the interactions of the Higgs fields. In the same way, let us pick up only the "interesting part" of $\mathscr{L}_2$ and $\mathscr{L}_1$ and denote the rest by I.T ("interaction terms"):

$$
\begin{aligned}
\mathscr{L}_2 &= \tfrac{1}{4}\delta^{\kappa\mu}\delta^{\lambda\nu}\big(\textstyle\sum_{a=1}^8 G^a_{\kappa\lambda}G^a_{\mu\nu} + \sum_{a=1}^3 W^a_{\kappa\lambda}W^a_{\mu\nu} + A'_{\kappa\lambda}A'_{\mu\nu} + \tilde{A}_{\kappa\lambda}\tilde{A}_{\mu\nu} \\
&\quad + \textstyle\sum_{a=1}^6 \partial_{[\kappa}X^a_{\lambda]}\,\partial_{[\mu}X^a_{\nu]} + \sum_{a=1}^6 \partial_{[\kappa}Y^a_{\lambda]}\,\partial_{[\mu}Y^a_{\nu]}\big) + I.T\ , \qquad (6.39a)\\
\mathscr{L}_1 &= \tfrac{8\mu_0}{g_0^2}\delta^{\mu\nu}\big(\textstyle\sum_{a=0}^8 \partial_\mu\Psi_a\,\partial_\nu\Psi_a + \sum_{a=1}^3 \partial_\mu\Psi'_a\,\partial_\nu\Psi'_a\big) + \tfrac{4\mu_1}{g_0^2}\delta^{\mu\nu}\ \textstyle\sum_{a=0}^6 \partial_\mu\Phi_a\,\partial_\nu\Phi_a \\
&\quad + \tfrac{8\mu_2}{g_0^2}\delta^{\mu\nu}\ \textstyle\sum_{i=0}^{89} \partial_\mu\Upsilon_i\,\partial_\nu\Upsilon_i + \tfrac{4\mu_3}{g_0^2}\delta^{\mu\nu}\ \sum_{i=0}^{98} \partial_\mu\Xi_i\,\partial_\nu\Xi_i \\
&\quad + (\mu_1+12\mu_2)\delta^{\mu\nu}\big(W^1_\mu W^1_\nu + W^2_\mu W^2_\nu + \qquad (6.39b)\\
&\qquad\qquad + (W^3_\mu - \sqrt{\tfrac{3}{5}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu)(W^3_\nu - \sqrt{\tfrac{3}{5}}A'_\nu + \sqrt{\tfrac{2}{5}}\tilde{A}_\nu)\big) \\
&\quad + \mu_3\delta^{\mu\nu}(4\sqrt{\tfrac{3}{5}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu)(4\sqrt{\tfrac{3}{5}}A'_\nu + \sqrt{\tfrac{2}{5}}\tilde{A}_\nu) \\
&\quad + \delta^{\mu\nu}\textstyle\sum_{a=1}^6 \big((2\mu_0+\mu_1+12\mu_2+2\mu_3)X^a_\mu X^a_\nu + (2\mu_0+32\mu_2+2\mu_3)Y^a_\mu Y^a_\nu\big) + I.T\ .
\end{aligned}
$$

6.3.6 The Sector of Neutral Gauge Fields

Here we diagonalize the mass matrix of the three completely neutral gauge fields $W^3, A', \tilde{A}$. One linear combination of these three fields, the photon P, is massless. The other two orthogonal linear combinations Z, Z' are massive.

We perform the orthogonal transformation by Euler angles

$$
\begin{pmatrix} Z_\mu \\ Z'_\mu \\ P_\mu \end{pmatrix} = \begin{pmatrix} \cos\phi_E & -\sin\phi_E & 0 \\ \sin\phi_E & \cos\phi_E & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_E & -\sin\theta_E \\ 0 & \sin\theta_E & \cos\theta_E \end{pmatrix} \begin{pmatrix} \cos\psi_E & -\sin\psi_E & 0 \\ \sin\psi_E & \cos\psi_E & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} W^3_\mu \\ A'_\mu \\ \tilde{A}_\mu \end{pmatrix}. \tag{6.40a}
$$

The photon P_μ is the massless linear combination, which is perpendicular to the plane spanned by $(W^3_\mu - \sqrt{\frac{3}{5}}A'_\mu + \sqrt{\frac{2}{5}}\tilde{A}_\mu)$ and $(4\sqrt{\frac{3}{5}}A'_\mu + \sqrt{\frac{2}{5}}\tilde{A}_\mu)$, see (6.39b). Calculating the vector product yields immediately

$$
P_\mu = \sqrt{\tfrac{3}{8}}W^3_\mu + \sqrt{\tfrac{1}{40}}A'_\mu - \sqrt{\tfrac{3}{5}}\tilde{A}_\mu\ , \tag{6.40b}
$$

which implies

$$\cos\theta_E = -\sqrt{\tfrac{3}{5}}\,, \qquad \sin\theta_E = \sqrt{\tfrac{2}{5}}\,, \qquad \cos\psi_E = \tfrac{1}{4}\,, \qquad \sin\psi_E = \sqrt{\tfrac{15}{16}}\,. \tag{6.40c}$$

The Euler angle ϕ_E is determined by the diagonalization of the mass matrix. The result is

$$\tan 2\phi_E = -\tfrac{3}{4} + \tfrac{25}{4}\lambda_4\,, \qquad \lambda_4 := \tfrac{(\mu_1+12\mu_2)}{25\mu_3}\,. \tag{6.40d}$$

We choose $\cos\phi_E < 0$ and $\sin\phi_E > 0$. Then, the inverse transformation is for $\lambda_4 \ll 1$ given by

$$\begin{aligned}
W^3_\mu &= \sqrt{\tfrac{5}{8}}Z_\mu + \sqrt{\tfrac{3}{8}}P_\mu - \sqrt{\tfrac{5}{2}}\lambda_4 Z'_\mu\,,\\
A'_\mu &= -\sqrt{\tfrac{3}{200}}(1-16\lambda_4)Z_\mu + \sqrt{\tfrac{1}{40}}P_\mu + \sqrt{\tfrac{24}{25}}(1+\tfrac{1}{4}\lambda_4)Z'_\mu\,,\\
\tilde{A}_\mu &= \tfrac{3}{5}(1+\tfrac{2}{3}\lambda_4)Z_\mu - \sqrt{\tfrac{3}{5}}P_\mu + \tfrac{1}{5}(1-6\lambda_4)Z'_\mu\,.
\end{aligned} \tag{6.40e}$$

6.3.7 The Canonical Form of the Bosonic Lagrangian

Here we reparametrize the Higgs fields in order to give the canonic form to the bosonic Lagrangian. Moreover, we read off the masses of the gauge fields in terms of the so far undetermined parameters μ_i.

The Lagrangian (6.39b) requires to perform the reparametrizations

$$\begin{aligned}
&\Psi_i = \tfrac{g_0}{\sqrt{16\mu_0}}\psi_i\,, && i=0,\dots,8\,, && \Psi'_i = \tfrac{g_0}{\sqrt{16\mu_0}}\psi'_i\,, && i=1,\dots,3\,,\\
&\Phi_i = \tfrac{g_0}{\sqrt{8\mu_1}}\phi_i\,, && i=0,\dots,6\,, && && \\
&\Upsilon_i = \tfrac{g_0}{\sqrt{16\mu_2}}\upsilon_i\,, && i=0,\dots,89\,, && \Xi_i = \tfrac{g_0}{\sqrt{8\mu_3}}\xi_i\,, && i=0,\dots,98\,.
\end{aligned} \tag{6.41}$$

We also perform a Wick rotation from the Riemannian manifold X to the Minkowski manifold X_M by introduction of a global minus sign in the action and by replacing[16]

$$\delta^{\mu\nu} \mapsto -g^{\mu\nu}\,, \qquad g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)\,. \tag{6.42}$$

We define $P_{\mu\nu} := \partial_{[\mu}P_{\nu]}$ and

$$\begin{aligned}
&m_W^2 = (2\mu_1 + 24\mu_2)\,, && m_Z^2 = \tfrac{1}{\cos^2(\theta_W - \theta'_W)}m_W^2\,,\\
&m_{Z'}^2 = 32\mu_3\cos^2(\theta_W - \theta'_W)\,, && \\
&m_X^2 = (4\mu_0 + 2\mu_1 + 24\mu_2 + 4\mu_3)\,, && m_Y^2 = (4\mu_0 + 64\mu_2 + 4\mu_3)\,,
\end{aligned} \tag{6.43a}$$

where

$$\sin\theta_W := \sqrt{\tfrac{3}{8}}\,, \qquad \theta'_W := \tfrac{1}{2}\sqrt{\tfrac{5}{3}}\lambda_4\,. \tag{6.43b}$$

[16] The minus sign in $\delta^{\mu\nu} \mapsto -g^{\mu\nu}$ is due to $(\hat{\gamma}^5)^* = -\hat{\gamma}^5$ on the Minkowski space.

Now we can write down the final formula for the bosonic Lagrangian:

$$\begin{aligned}
\mathscr{L} = &-\tfrac{1}{4}g^{\kappa\mu}g^{\lambda\nu}\big(\textstyle\sum_{a=1}^{8}(G^a_{\kappa\lambda}G^a_{\mu\nu}) + P_{\kappa\lambda}P_{\mu\nu}\big)\\
&+\textstyle\sum_{a=1}^{2}\big(-\tfrac{1}{4}g^{\kappa\mu}g^{\lambda\nu}\,\partial_{[\kappa}W^a_{\lambda]}\,\partial_{[\mu}W^a_{\nu]} + \tfrac{1}{2}g^{\mu\nu}m_W^2W^a_\mu W^a_\nu\big)\\
&+\big(-\tfrac{1}{4}g^{\kappa\mu}g^{\lambda\nu}\,\partial_{[\kappa}Z_{\lambda]}\,\partial_{[\mu}Z_{\nu]} + \tfrac{1}{2}g^{\mu\nu}m_Z^2Z_\mu Z_\nu\big)\\
&+\big(-\tfrac{1}{4}g^{\kappa\mu}g^{\lambda\nu}\,\partial_{[\kappa}Z'_{\lambda]}\,\partial_{[\mu}Z'_{\nu]} + \tfrac{1}{2}g^{\mu\nu}m_{Z'}^2Z'_\mu Z'_\nu\big) + \mathscr{L}_{ew}(P,W,Z,Z')\\
&+\textstyle\sum_{a=1}^{6}\big(-\tfrac{1}{4}g^{\kappa\mu}g^{\lambda\nu}\,\partial_{[\kappa}X^a_{\lambda]}\,\partial_{[\mu}X^a_{\nu]} + \tfrac{1}{2}g^{\mu\nu}m_X^2X^a_\mu X^a_\nu\big)\\
&+\textstyle\sum_{a=1}^{6}\big(-\tfrac{1}{4}g^{\kappa\mu}g^{\lambda\nu}\,\partial_{[\kappa}Y^a_{\lambda]}\,\partial_{[\mu}Y^a_{\nu]} + \tfrac{1}{2}g^{\mu\nu}m_Y^2Y^a_\mu Y^a_\nu\big) + \mathscr{L}_H + I.T\;,
\end{aligned} \tag{6.44a}$$

where

$$\begin{aligned}
&\mathscr{L}_{ew}(P,W,Z,Z')\\
&\quad= g_0\,g^{\kappa\mu}g^{\lambda\nu}\big(\partial_{[\kappa}W^1_{\lambda]}\,W^2_\mu W^3_\nu + \partial_{[\kappa}W^2_{\lambda]}\,W^3_\mu W^1_\nu + \partial_{[\kappa}W^3_{\lambda]}\,W^1_\mu W^2_\nu\big)\\
&\quad-\tfrac{1}{2}g_0^2(g^{\kappa\mu}g^{\lambda\nu} - g^{\kappa\nu}g^{\lambda\mu})(W^1_\kappa W^1_\mu W^2_\lambda W^2_\nu + W^1_\kappa W^1_\mu W^3_\lambda W^3_\nu + W^2_\kappa W^2_\mu W^3_\lambda W^3_\nu)\;,
\end{aligned} \tag{6.44b}$$

$$\begin{aligned}
\mathscr{L}_H = \tfrac{1}{2}g^{\mu\nu}\big(&\textstyle\sum_{i=0}^{8}\partial_\mu\psi_i\,\partial_\nu\psi_i + \sum_{i=1}^{3}\partial_\mu\psi'_i\,\partial_\nu\psi'_i\\
&+\textstyle\sum_{i=0}^{6}\partial_\mu\phi_i\,\partial_\nu\phi_i + \sum_{i=0}^{98}\partial_\mu\xi_i\,\partial_\nu\xi_i + \sum_{i=0}^{89}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i\big) - \mathscr{L}_0\;.
\end{aligned} \tag{6.44c}$$

This is precisely the bosonic Lagrangian of the flipped SU(5)×U(1)*–model, where the masses of the gauge bosons are given in* (6.43). *The parameters* μ_1,μ_2,μ_3 *and the Weinberg angle* θ_W *will be determined in Section 6.4 when discussing the fermionic action. Within our framework there is no possibility to determine* μ_0. *However, we will find in Section 6.4 that the X and Y bosons lead to proton decay. In order to suppress the proton decay sufficiently we need* $\mu_0 \gg \max(\mu_1,\mu_2)$. *Then, it remains to extract the quadratic terms from the Lagrangian* $\mathscr{L}_0$ *in order to obtain the masses of the Higgs fields, which will be done in Section 7.*

6.4 The Fermionic Action

In this subsection we investigate the fermionic action of the flipped SU(5)×U(1)*–Grand Unification model. In Section 6.4.1 we recall the structure of the operator* $\mathsf{D}+\mathrm{i}\rho$ *in Euclidian space and pass by a Wick rotation to Minkowski space. In Section 6.4.2 we impose a chirality condition to the fermions. However, in contrast to what one could expect, we do not employ the grading operator* Γ *of the L–cycle. Instead, we modify the signs and use a new non–canonical operator* $\tilde{\Gamma}$. *The purpose is to project away the Grand Unification part* $\mathcal{M}_i$ *of the mass matrix, see* (5.25) *on page 103. This is necessary in order to avoid ultrahigh masses for the fermions. Then, the chiral theory has two further symmetries: a trivial one and the charge conjugation. The three symmetries allow us to reduce the number of independent fermionic degrees of freedom by a factor 8, and the formula for the fermionic action simplifies considerably. The matrix decomposition of the fermionic Lagrangian is presented in Section 6.4.3. We recover the fermionic sector of the standard model as a part of that Lagrangian. Finally, we formally derive the* SU(5)*–Grand Unification model in Section 6.4.4.*

6.4.1 Introduction

Now we write down the fermionic action

$$S_F = \tfrac{1}{4}\langle\psi,(D+\mathrm{i}\rho)\psi\rangle_h\;, \qquad \psi\in h\;,\;\; \rho\in\mathcal{H}^1\mathfrak{g}\;. \tag{6.45}$$

Please do not confuse $\psi\in h$ with the Higgs fields $\psi_0,\dots,\psi_8$. The factor $\frac{1}{4}$ additional to (3.128b) occurs because we are going to impose constraints on ψ, which require precisely the form (6.45) for the action, see below. More explicitly, inserting (6.1) and (6.2) and using (5.25) we have

$$D+\mathrm{i}\rho = \tag{6.46}$$
$$\begin{pmatrix} \mathsf{D}+\mathrm{i}\tilde\pi(A{+}A'') & -\gamma^5\tilde\pi(\tilde{\mathbf{\Psi}}) & -\gamma^5\tilde\pi(\tilde{\mathbf{\Phi}}{+}\tilde{\Xi}{+}\tilde{\mathbf{\Upsilon}}) & 0 \\ -(\gamma^5\tilde\pi(\tilde{\mathbf{\Psi}}))^* & \mathsf{D}+\mathrm{i}\tilde\pi(A{+}A'') & 0 & -\gamma^5\tilde\pi(\tilde{\mathbf{\Phi}}{+}\tilde{\Xi}{+}\tilde{\mathbf{\Upsilon}}) \\ -(\gamma^5\tilde\pi(\tilde{\mathbf{\Phi}}{+}\tilde{\Xi}{+}\tilde{\mathbf{\Upsilon}}))^* & 0 & \mathsf{D}-\gamma_C\overline{(\mathrm{i}\tilde\pi(A{+}A''))}\gamma_C & -\overline{\gamma^5\tilde\pi(\tilde{\mathbf{\Psi}})} \\ 0 & -(\gamma^5\tilde\pi(\tilde{\mathbf{\Phi}}{+}\tilde{\Xi}{+}\tilde{\mathbf{\Upsilon}}))^* & -\overline{(\gamma^5\tilde\pi(\tilde{\mathbf{\Psi}}))^*} & \mathsf{D}-\gamma_C\overline{(\mathrm{i}\tilde\pi(A{+}A''))}\gamma_C \end{pmatrix},$$

where

$$\begin{aligned} \tilde\pi(A{+}A'') &:= \mathrm{diag}((\pi_{10}(A)-\tfrac{1}{2}A''\mathbb{1}_{10})\otimes\mathbb{1}_3\;,\; \gamma_C\overline{(\pi_5(A)-\tfrac{3}{2}A''\mathbb{1}_5)}\gamma_C\otimes\mathbb{1}_3\;,\; -\tfrac{5}{2}A''\otimes\mathbb{1}_3)\;, \\ \tilde\pi(\tilde{\mathbf{\Psi}}) &:= \mathrm{diag}\big((\check{\mathbf{\Psi}}+\check{\mathbf{m}})\otimes M_{10}\;,\; -\overline{(\mathbf{\Psi}+\mathbf{m})\otimes M_5}\;,\; 0_{3\times3}\big)\;, \end{aligned} \tag{6.47}$$

$$\tilde\pi(\tilde{\mathbf{\Phi}}{+}\tilde{\Xi}{+}\tilde{\mathbf{\Upsilon}}) := \begin{pmatrix} \begin{bmatrix} (\hat{\mathbf{\Phi}}+\hat{\mathbf{n}})\otimes M_d \\ +(\Xi+\mathbf{m}')\otimes M_N \end{bmatrix} & \begin{bmatrix} (\check{\mathbf{\Phi}}+\check{\mathbf{n}})\otimes M_{\tilde u} \\ +(\mathbf{\Upsilon}+\mathbf{n}')\otimes M_{\tilde n} \end{bmatrix} & 0 \\ \begin{bmatrix} (\check{\mathbf{\Phi}}+\check{\mathbf{n}})^T\otimes M_{\tilde u}^T \\ +(\mathbf{\Upsilon}+\mathbf{n}')^T\otimes M_{\tilde n}^T \end{bmatrix} & 0 & \overline{(\mathbf{\Phi}+\mathbf{n})}\otimes M_e \\ 0 & (\mathbf{\Phi}+\mathbf{n})^*\otimes M_e^T & 0 \end{pmatrix}.$$

Here, γ_C denotes the complex conjugation matrix, fulfilling $\gamma_C\overline{\gamma^\mu}\gamma_C=\gamma^\mu$. Therefore, we have $[\mathsf{D},\bar f]=-\gamma_C\overline{[\mathsf{D},f]}\gamma_C$, which is the reason that γ_C occurs in (6.46). Within our conventions (C.4) we have $\gamma_C=\gamma^2\gamma^4$, up to the sign. We recall that the Hilbert space is $h=L^2(X,S)\otimes\mathbb{C}^{192}$, where elements $\psi\in h$ have in terms of the decomposition (6.46) the form

$$\psi=(\psi_1,\psi_2,\psi_3,\psi_4)^T\;, \quad \psi_i\in L^2(X,S)\otimes\mathbb{C}^{48}\;. \tag{6.48}$$

It is necessary to perform a Wick rotation to Minkowski space X_M. We must replace h by $h_M=L^2(X_M,S)\otimes\mathbb{C}^{192}$ and the Euclidian gamma matrices γ^μ by Minkowskian gamma matrices $\hat\gamma^\mu$, see Appendix C. Moreover, the scalar product (6.45) must be replaced by the invariant product

$$S_F=\tfrac{1}{4}\int_{X_M} \mathrm{v}_M\,\psi^*\hat\gamma^0(D+\mathrm{i}\rho_M)\psi\;, \tag{6.49}$$

where v_M is the volume form on Minkowski space. Now, the complex conjugation matrix is in our conventions (C.7) $\hat{\gamma}_C = \hat{\gamma}^2$. *It is extremely important to be aware of the identity* $\hat{\gamma}^5 = -(\hat{\gamma}^5)^*$ *on Minkowski space*[17]. Therefore, we have in Minkowski space

$$
D + \mathrm{i}\rho_M = \tag{6.50}
$$
$$
\begin{pmatrix}
\mathsf{D} + \mathrm{i}\tilde{\pi}(A{+}A'') & -\hat{\gamma}^5\tilde{\pi}(\boldsymbol{\Psi}+\mathbf{m}) & -\hat{\gamma}^5\tilde{\pi}(\tilde{\boldsymbol{\Phi}}{+}\tilde{\Xi}{+}\tilde{\boldsymbol{\Upsilon}}) & 0 \\
\hat{\gamma}^5\tilde{\pi}(\tilde{\boldsymbol{\Psi}})^* & \mathsf{D} + \mathrm{i}\tilde{\pi}(A{+}A'') & 0 & -\hat{\gamma}^5\tilde{\pi}(\tilde{\boldsymbol{\Phi}}{+}\tilde{\Xi}{+}\tilde{\boldsymbol{\Upsilon}}) \\
\hat{\gamma}^5\tilde{\pi}(\tilde{\boldsymbol{\Phi}}{+}\tilde{\Xi}{+}\tilde{\boldsymbol{\Upsilon}})^* & 0 & \mathsf{D} - \hat{\gamma}^2\overline{(\mathrm{i}\tilde{\pi}(A{+}A''))}\hat{\gamma}^2 & -\hat{\gamma}^5\overline{\tilde{\pi}(\tilde{\boldsymbol{\Psi}})} \\
0 & \hat{\gamma}^5\tilde{\pi}(\tilde{\boldsymbol{\Phi}}{+}\tilde{\Xi}{+}\tilde{\boldsymbol{\Upsilon}})^* & \hat{\gamma}^5\tilde{\pi}(\tilde{\boldsymbol{\Psi}})^T & \mathsf{D} - \hat{\gamma}^2\overline{(\mathrm{i}\tilde{\pi}(A{+}A''))}\hat{\gamma}^2
\end{pmatrix}.
$$

6.4.2 Restrictions on the Fermions

Let us impose a chirality condition on physical fermions. The natural candidate would be $\Gamma\boldsymbol{\psi} = \boldsymbol{\psi}$, however, we have the freedom to choose a different one,

$$
\tilde{\Gamma}\boldsymbol{\psi} = \boldsymbol{\psi}\,, \quad \tilde{\Gamma} := \operatorname{diag}\big(-\hat{\gamma}^5 \otimes \mathbb{1}_{48}\,,\ -\hat{\gamma}^5 \otimes \mathbb{1}_{48}\,,\ \hat{\gamma}^5 \otimes \mathbb{1}_{48}\,,\ \hat{\gamma}^5 \otimes \mathbb{1}_{48}\big)\,. \tag{6.51}
$$

Of course, this choice breaks the structure of the model, which is precisely our intention. Since $\tilde{\Gamma}$ commutes with $\pi(a)$, the gauge invariance is not destroyed. But $\tilde{\Gamma}$ no longer anticommutes with the whole D. More precisely, if we apply $D + \mathrm{i}\rho_M$ to chiral fermions (6.51), we see that only the part

$$
(D + \mathrm{i}\rho_M)' := \tfrac{1}{2}(\mathrm{id}_h - \tilde{\Gamma})(D + \mathrm{i}\rho')\tfrac{1}{2}(\mathrm{id}_h + \tilde{\Gamma}) \tag{6.52}
$$

of $D + \mathrm{i}\rho_M$ survives. That part (6.52) differs from the matrix (6.50) by the absence of $\hat{\gamma}^5\tilde{\pi}(\tilde{\boldsymbol{\Psi}})$. *This was our motivation for* (6.51). *The presence of* $\hat{\gamma}^5\tilde{\pi}(\tilde{\boldsymbol{\Psi}})$ *in the fermionic action would be a desaster, because the fermions would get ultrahigh masses.*

Now, the chiral theory has two further symmetries: We define an operation $\mathscr{C} : \tilde{\Gamma}h_M \to \tilde{\Gamma}h_M$, the charge conjugation, by

$$
\mathscr{C} := \begin{pmatrix}
0 & 0 & -\hat{\gamma}^2 \otimes \mathbb{1}_{48} & 0 \\
0 & 0 & 0 & -\hat{\gamma}^2 \otimes \mathbb{1}_{48} \\
-\hat{\gamma}^2 \otimes \mathbb{1}_{48} & 0 & 0 & 0 \\
0 & -\hat{\gamma}^2 \otimes \mathbb{1}_{48} & 0 & 0
\end{pmatrix} \circ \text{complex conjugation}\,. \tag{6.53}
$$

It is straightforward to check

$$
\mathscr{C}^2 = \mathrm{id}_{h_M}\,, \quad \mathscr{C}(D + \mathrm{i}\rho_M)'\mathscr{C} = (D + \mathrm{i}\rho_M)'\,, \quad \mathscr{C}\pi(a)\mathscr{C} = \pi(a)\,, \quad \mathscr{C}\tilde{\Gamma}\mathscr{C} = \tilde{\Gamma}\,, \tag{6.54}
$$

[17] *A careless look at the matrix structure could lead to the conclusion* $\hat{\gamma}^5 = (\hat{\gamma}^5)^*$*, but this is wrong! In the past, this missing sign has been a source of misunderstandings in certain applications of non–commutative geometry.*

where one has to use (5.25) and

$$\hat{\gamma}^2\overline{\hat{\gamma}^\mu}\overline{\hat{\gamma}^2} = -\hat{\gamma}^\mu\,, \quad \hat{\gamma}^2\overline{\hat{\gamma}^5}\overline{\hat{\gamma}^2} = -\hat{\gamma}^5\,, \quad \hat{\gamma}^2\overline{\hat{\gamma}^2} = 1\,, \tag{6.55}$$

see Appendix C. Thus, it is natural to demand that also $\tilde{\Gamma}h_M$ is invariant under the charge conjugation $\mathscr{C}$.

The second symmetry $\mathcal{S} : \tilde{\Gamma}h_M \to \tilde{\Gamma}h_M$ of the chiral theory is

$$\mathcal{S} := \begin{pmatrix} 0 & \mathbb{1}_{48} & 0 & 0 \\ \mathbb{1}_{48} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{48} \\ 0 & 0 & \mathbb{1}_{48} & 0 \end{pmatrix}. \tag{6.56}$$

Obviously, we have

$$\mathcal{S}^2 = \mathrm{id}_{h_M}\,, \qquad \mathcal{S}(D+\mathrm{i}\rho_M)'\mathcal{S} = (D+\mathrm{i}\rho_M)'\,, \qquad \mathcal{S}\pi(a)\mathcal{S} = \pi(a)\,,$$
$$\mathcal{S}\tilde{\Gamma}\mathcal{S} = \tilde{\Gamma}\,, \qquad \mathcal{S}\mathscr{C}\mathcal{S} = \mathscr{C}\,.$$

Again, we restrict[18] ourselves to elements of h_M invariant under $\mathcal{S}$:

$$\psi = \hat{\Gamma}\psi = \mathscr{C}\psi = \mathcal{S}\psi = \begin{pmatrix} \frac{1}{2}(1-\hat{\gamma}^5)\psi_1 \\ \frac{1}{2}(1-\hat{\gamma}^5)\psi_1 \\ -\frac{1}{2}(1+\hat{\gamma}^5)\hat{\gamma}^2\overline{\psi_1} \\ -\frac{1}{2}(1+\hat{\gamma}^5)\hat{\gamma}^2\overline{\psi_1} \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{2}(1-\hat{\gamma}^5)\psi_1 \\ \frac{1}{2}(1-\hat{\gamma}^5)\psi_1 \\ -\hat{\gamma}^2\overline{\frac{1}{2}(1-\hat{\gamma}^5)\psi_1} \\ -\hat{\gamma}^2\overline{\frac{1}{2}(1-\hat{\gamma}^5)\psi_1} \end{pmatrix}. \tag{6.57}$$

Within our conventions one has the block structure

$$\tfrac{1}{2}(1-\hat{\gamma}^5)\psi_1 = \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix}, \qquad \psi_0 \in L^2(X_M)\otimes\mathbb{C}^2\otimes\mathbb{C}^{48}\,, \tag{6.58}$$

where $L^2(X_M)$ denotes the space of square integrable functions on the Minkowski space.

In local bases we have

$$\mathsf{D} = \mathrm{i}\hat{\gamma}^\mu\partial_\mu\,, \qquad A = A_\mu\hat{\gamma}^\mu\,, \qquad A'' = A''_\mu\hat{\gamma}^\mu\,. \tag{6.59}$$

We define $\sigma^0 = \tilde{\sigma}^0 = \mathbb{1}_2$ and $\tilde{\sigma}^a = -\sigma^a$, $a = 1,2,3$, or in a symbolic notation

$$\sigma^\mu = \{\mathbb{1}_2, \sigma^a\}\,, \quad \tilde{\sigma}^\mu = \{\mathbb{1}_2, -\sigma^a\}\,, \quad \mu = 0,1,2,3\,, \quad a = 1,2,3\,.$$

[18] It was pointed out in [47] that in non–commutative geometry there is a fundamental discrepancy between the bosonic theory and the fermionic theory. The bosonic action is built using the full Hilbert space h, whereas for the fermionic action one uses a projected Hilbert space such as (6.57). The same projection in the bosonic action leads to a wrong result. The criticism [47] applies literally to our model, and I do not know a solution.

Then, from (6.49), (6.50), (6.57) and (C.7) we get

$$S_F = \frac{1}{2}\int_{X_M} \mathrm{v}_M \left(\psi_0^*, \psi_0^T\sigma^2\right)\begin{pmatrix} \mathrm{i}\tilde{\sigma}^\mu(\partial_\mu + \tilde{\pi}(A_\mu + A''_\mu))\,; & -\tilde{\pi}(\tilde{\mathbf{\Phi}}+\tilde{\mathbf{\Xi}}+\tilde{\mathbf{\Upsilon}}) \\ -\tilde{\pi}(\tilde{\mathbf{\Phi}}+\tilde{\mathbf{\Xi}}+\tilde{\mathbf{\Upsilon}})^*\,; & \mathrm{i}\sigma^\mu(\partial_\mu + \overline{\tilde{\pi}(A_\mu + A''_\mu)}) \end{pmatrix}\begin{pmatrix} \psi_0 \\ \sigma^2\overline{\psi_0} \end{pmatrix}. \quad (6.60)$$

This formula can be further simplified, because we have

$$\begin{aligned} \int_{X_M} \mathrm{v}_M\ \psi_0^T\sigma^2\mathrm{i}\sigma^\mu(\partial_\mu + \overline{\tilde{\pi}(A_\mu{+}A''_\mu)})\sigma^2\overline{\psi_0} &= \int_{X_M} \mathrm{v}_M\, \psi_0^T\mathrm{i}(\tilde{\sigma}^\mu)^T(\partial_\mu + \overline{\tilde{\pi}(A_\mu{+}A''_\mu)})\overline{\psi_0} \\ &= \int_{X_M} \mathrm{v}_M \left((-\mathrm{i}\partial_\mu\psi_0^T)(\tilde{\sigma}^\mu)^T\overline{\psi_0} + \psi_0^T(\tilde{\sigma}^\mu)^T(-\mathrm{i}\tilde{\pi}(A_\mu{+}A''_\mu))^T\overline{\psi_0}\right) \quad (6.61) \\ &= \int_{X_M} \mathrm{v}_M\, \psi_0^*\mathrm{i}\tilde{\sigma}^\mu(\partial_\mu + \tilde{\pi}(A_\mu{+}A''_\mu))\psi_0\ . \end{aligned}$$

Here, we have integrated by parts and made use of $\tilde{\pi}(A_\mu{+}A''_\mu) = -\tilde{\pi}(A_\mu{+}A''_\mu)^*$. In the last step we took into account that in quantum mechanics the fields ψ_0 are annihilation operators and the fields $\overline{\psi_0}$ creation operators. In normal ordered products, all creation operators must stand on the left of all annihilation operators. This means that in (6.61) we have to exchange ψ_0 and $\overline{\psi_0}$. But since they represent fermions, which anticommute, this change of order gives a minus sign. Now, (6.60) takes the form

$$S_F = \int_{X_M} \mathrm{v}_M \left(\psi_0^*\mathrm{i}\tilde{\sigma}^\mu(\partial_\mu + \tilde{\pi}(A_\mu{+}A''_\mu))\psi_0 - \tfrac{1}{2}(\psi_0^*\tilde{\pi}(\tilde{\mathbf{\Phi}}{+}\tilde{\mathbf{\Xi}}{+}\tilde{\mathbf{\Upsilon}})\sigma^2\overline{\psi_0} + \mathrm{h.c})\right), \quad (6.62)$$

where h.c denotes the Hermitian conjugate of the preceding term, without change of signs when exchanging fermion fields.

For $\psi_0 \in L^2(X_M)\otimes\mathbb{C}^2\otimes\mathbb{C}^{48}$ we choose the following parametrization:

$$\psi_0 = \quad (6.63a)$$
$$\left(u_L^r, u_L^b, u_L^g, d_L^r, d_L^b, d_L^g, \sigma^2\overline{d_R^r}, \sigma^2\overline{d_R^b}, \sigma^2\overline{d_R^g}, \sigma^2\overline{\nu_R}, -\sigma^2\overline{u_R^r}, -\sigma^2\overline{u_R^b}, -\sigma^2\overline{u_R^g}, -e_L, \nu_L, \sigma^2\overline{e_R}\right)^t,$$
$$\sigma^2\overline{\psi_0} = \quad (6.63b)$$
$$\left(\sigma^2\overline{u_L^r}, \sigma^2\overline{u_L^b}, \sigma^2\overline{u_L^g}, \sigma^2\overline{d_L^r}, \sigma^2\overline{d_L^b}, \sigma^2\overline{d_L^g}, -d_R^r, -d_R^b, -d_R^g, -\nu_R, u_R^r, u_R^b, u_R^g, -\sigma^2\overline{e_L}, \sigma^2\overline{\nu_L}, -e_R\right)^t,$$

where $u_L^r, \dots, e_R \in L^2(X_M)\otimes\mathbb{C}^2\otimes\mathbb{C}^3$ and ^{t} means transposition only of the row, without transposing the matrix elements.

6.4.3 The Fermionic Lagrangian

Respecting (5.7) and (5.11) and inserting the matrix structures of (6.20), (6.23a) and (6.24) for the gauge and Higgs fields, it is not difficult to write down the explicit formula for the fermionic action. However, the transformation (6.40e) requires some care. Let us derive the coefficients of P, Z, Z' corresponding to the u_L–quarks. From (6.47), (5.7) and (6.24a) we find with (6.43b) for $\lambda_4 \ll 1$ in good approximation

$$\begin{aligned} \pi_{u_L}(A_\mu{+}\,A''_\mu) &\to \mathrm{i}\tfrac{g_0}{2}(W_\mu^3 - \tfrac{1}{3}\sqrt{\tfrac{3}{5}}A' - \tfrac{1}{2}\sqrt{\tfrac{2}{5}}\tilde{A}_\mu) \\ &= \mathrm{i}g_0\big(\tfrac{1}{2\cos(\theta_W - 2\theta'_W)}(Z_\mu - \tfrac{1}{2}(1{+}2\sqrt{15}\theta'_W)Z'_\mu) - \tfrac{1}{3}(\cos\theta_W Z'_\mu + 4\theta'_W\sin\theta_W Z_\mu) \\ &\quad + \tfrac{2}{3}\sin\theta_W(P_\mu - \tan(\theta_W{-}2\theta'_W)Z_\mu + (\tfrac{4}{\sqrt{15}} + \tfrac{12}{5}\theta'_W)Z'_\mu)\big)\ . \end{aligned} \quad (6.64)$$

Thus, it is convenient to define

$$
\begin{aligned}
\tilde{P}_\mu &:= P_\mu - \tan(\theta_W - 2\theta'_W) Z_\mu + \big(\tfrac{4}{\sqrt{15}} + \tfrac{12}{5}\theta'_W\big) Z'_\mu\,, \\
\tilde{Z}_\mu &:= Z_\mu - \tfrac{1}{2}(1 + 2\sqrt{15}\theta'_W) Z'_\mu\,, \\
\tilde{Z}'_\mu &:= Z'_\mu + 4\theta'_W \tan\theta_W Z_\mu\,, \\
e &:= \sin\theta_W g_0\,, \qquad \tilde{e} := \cos\theta_W g_0\,.
\end{aligned}
\tag{6.65}
$$

Moreover, we express $\Phi_0, \Phi_g, \Xi_A, \dots, \Xi_c, \Upsilon_A, \dots, \Upsilon_g$ in terms of the physical Higgs bosons $\phi_0, \phi_g, \xi_A, \dots, \xi_c, \upsilon_A, \dots, \upsilon_g$, see (6.20), (6.23a) and (6.41). Then we arrive at the following formula for the fermionic Lagrangian:

$$
S_F = \int_{X_M} \mathrm{v}_M \,(\mathscr{L}_q + \mathscr{L}_\ell + \mathscr{L}_m + \mathscr{L}_x + \mathscr{L}_h + \mathscr{L}'_h + \mathscr{L}''_h)\,, \qquad \text{where} \tag{6.66a}
$$

$$
\begin{aligned}
\mathscr{L}_q = \big(\boldsymbol{u}_L^*, \boldsymbol{d}_L^*\big)\Big(\tilde{\sigma}^\mu &\begin{pmatrix} \begin{bmatrix} \mathrm{i}\partial_\mu - \frac{g_0}{2}\mathbf{G}_\mu - \big(\frac{g_0}{2\cos(\theta_W - 2\theta'_W)}\tilde{Z}_\mu \\ +\frac{2}{3}e\tilde{P}_\mu - \frac{1}{3}\tilde{e}\tilde{Z}'_\mu\big)\mathbb{1}_3 \end{bmatrix} & -\frac{g_0}{2}(W^1_\mu - \mathrm{i}W^2_\mu)\mathbb{1}_3 \\ -\frac{g_0}{2}(W^1_\mu + \mathrm{i}W^2_\mu)\mathbb{1}_3 & \begin{bmatrix} \mathrm{i}\partial_\mu - \frac{g_0}{2}\mathbf{G}_\mu - \big(-\frac{g_0}{2\cos(\theta_W - 2\theta'_W)}\tilde{Z}_\mu \\ -\frac{1}{3}e\tilde{P}_\mu - \frac{1}{3}\tilde{e}\tilde{Z}'_\mu\big)\mathbb{1}_3 \end{bmatrix} \end{pmatrix} \otimes \mathbb{1}_3\Big) \begin{pmatrix} \boldsymbol{u}_L \\ \boldsymbol{d}_L \end{pmatrix} \\
&+ \boldsymbol{u}_R^*\big(\sigma^\mu(\mathrm{i}\partial_\mu - \tfrac{g_0}{2}\mathbf{G}_\mu - (\tfrac{2}{3}e\tilde{P}_\mu - \tfrac{1}{3}\tilde{e}\tilde{Z}'_\mu)\mathbb{1}_3) \otimes \mathbb{1}_3\big)\boldsymbol{u}_R \\
&+ \boldsymbol{d}_R^*\big(\sigma^\mu(\mathrm{i}\partial_\mu - \tfrac{g_0}{2}\mathbf{G}_\mu - (-\tfrac{1}{3}e\tilde{P}_\mu - \tfrac{1}{3}\tilde{e}\tilde{Z}'_\mu)\mathbb{1}_3) \otimes \mathbb{1}_3\big)\boldsymbol{d}_R\,,
\end{aligned}
\tag{6.66b}
$$

$$
\begin{aligned}
\mathscr{L}_\ell = \big(\nu_L^*, e_L^*\big)\Big(\tilde{\sigma}^\mu &\begin{pmatrix} \mathrm{i}\partial_\mu - \big(\frac{g_0}{2\cos(\theta_W - 2\theta'_W)}\tilde{Z}_\mu + \tilde{e}\tilde{Z}'_\mu\big) & -\frac{g_0}{2}(W^1_\mu - \mathrm{i}W^2_\mu) \\ -\frac{g_0}{2}(W^1_\mu + \mathrm{i}W^2_\mu) & \begin{bmatrix} \mathrm{i}\partial_\mu - \big(-\frac{g_0}{2\cos(\theta_W - 2\theta'_W)}\tilde{Z}_\mu \\ -e\tilde{P}_\mu + \tilde{e}\tilde{Z}'_\mu\big) \end{bmatrix} \end{pmatrix} \otimes \mathbb{1}_3\Big) \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\
&+ \nu_R^*\big(\sigma^\mu(\mathrm{i}\partial_\mu - \tilde{e}\tilde{Z}'_\mu) \otimes \mathbb{1}_3\big)\nu_R \\
&+ e_R^*\big(\sigma^\mu(\mathrm{i}\partial_\mu - (-e\tilde{P}_\mu + \tilde{e}\tilde{Z}'_\mu)) \otimes \mathbb{1}_3\big)e_R\,,
\end{aligned}
\tag{6.66c}
$$

$$
\begin{aligned}
\mathscr{L}_m = \Big(&-\boldsymbol{d}_L^*(\mathbb{1}_3 \otimes (M_d + \tfrac{g_0}{\sqrt{8\mu_1}}\phi_0 M_d) - \tfrac{g_0}{\sqrt{8\mu_3}}(\xi_F^0)^* \otimes M_N)\boldsymbol{d}_R - e_L^*(M_e + \tfrac{g_0}{\sqrt{8\mu_1}}\phi_0 M_e)e_R \\
&- \boldsymbol{u}_L^*(\mathbb{1}_3 \otimes (M_u + \tfrac{g_0}{\sqrt{8\mu_1}}\phi_0 M_{\tilde{u}}) + \tfrac{g_0}{4\sqrt{\mu_2}}\upsilon_A \otimes M_{\tilde{n}})\boldsymbol{u}_R \\
&- \nu_L^*(M_n^T + \tfrac{g_0}{\sqrt{8\mu_1}}\phi_0 M_{\tilde{u}}^T - \tfrac{3g_0}{4\sqrt{6\mu_2}}(\upsilon_0 + \mathrm{i}\upsilon_{45})M_{\tilde{n}}^T)\nu_R \\
&- \boldsymbol{d}_L^*(\tfrac{g_0}{4\sqrt{\mu_2}}\upsilon_B \otimes M_{\tilde{n}})\boldsymbol{u}_R - e_L^*(\tfrac{3g_0}{4\sqrt{6\mu_2}}(\upsilon_{18} + \mathrm{i}\upsilon_{63}) \otimes M_{\tilde{n}}^T)\nu_R \\
&+ \boldsymbol{u}_L^*((\xi_E^0)^* \otimes M_N)\boldsymbol{d}_R - \tfrac{1}{2}\nu_R^T\sigma_2(M_N + \tfrac{g_0}{\sqrt{8\mu_3}}\xi_0 M_N)\nu_R\Big) + \text{h.c}\,,
\end{aligned}
\tag{6.66d}
$$

$$
\begin{aligned}
\mathscr{L}_x = \tfrac{g_0}{2}\Big(&-\boldsymbol{u}_L^*(\tilde{\sigma}^\mu\sigma^2\epsilon(\overline{X_\mu}) \otimes \mathbb{1}_3)\overline{\boldsymbol{d}}_R - \boldsymbol{d}_L^*(\tilde{\sigma}^\mu\sigma^2\epsilon(\overline{Y_\mu}) \otimes \mathbb{1}_3)\overline{\boldsymbol{d}}_R \\
&+ \boldsymbol{u}_L^*(\tilde{\sigma}^\mu\sigma^2 Y_\mu \otimes \mathbb{1}_3)\overline{\nu}_R - \boldsymbol{d}_L^*(\tilde{\sigma}^\mu\sigma^2 X_\mu \otimes \mathbb{1}_3)\overline{\nu}_R \\
&- \nu_L^*(\tilde{\sigma}^\mu\sigma^2 Y_\mu^T \otimes \mathbb{1}_3)\overline{\boldsymbol{u}}_R + e_L^*(\tilde{\sigma}^\mu\sigma^2 X_\mu^T \otimes \mathbb{1}_3)\overline{\boldsymbol{u}}_R\Big) + \text{h.c}\,,
\end{aligned}
\tag{6.66e}
$$

$$
\begin{aligned}
\mathscr{L}_h = \tfrac{g_0}{\sqrt{8\mu_1}}\Big(&-\boldsymbol{u}_L^*\{\sigma^2\epsilon(\overline{\phi_g}) \otimes M_d\}\overline{\boldsymbol{d}}_L + \nu_R^T\{\sigma^2\phi_g^* \otimes M_d\}\boldsymbol{d}_R - \boldsymbol{u}_L^*\{\sigma^2\phi_g \otimes M_{\tilde{u}}\}\overline{e}_L \\
&+ \boldsymbol{d}_L^*\{\sigma^2\phi_g \otimes M_{\tilde{u}}\}\overline{\nu}_L + \boldsymbol{d}_R^T\{\sigma^2\epsilon(\phi_g) \otimes M_{\tilde{u}}\}\boldsymbol{u}_R - \boldsymbol{u}_R^T\{\sigma^2\overline{\phi_g} \otimes M_e\}e_R\Big) + \text{h.c}\,,
\end{aligned}
\tag{6.66f}
$$

$$
\begin{aligned}
\mathscr{L}_h' = \tfrac{g_0}{4\sqrt{\mu_2}}\big(&\boldsymbol{u}_L^*(\sigma^2(\upsilon_a+\upsilon_b)\otimes M_{\tilde{n}})\overline{e_L} - \boldsymbol{d}_L^*(\sigma^2(\upsilon_a-\upsilon_b)\otimes M_{\tilde{n}})\overline{\nu_L} \\
&- \boldsymbol{d}_R^T(\sigma^2(\upsilon_C-\epsilon(\upsilon_a))\otimes M_{\tilde{n}})\boldsymbol{u}_R - \boldsymbol{u}_L^*(\sigma^2\upsilon_c\otimes M_{\tilde{n}})\overline{\nu_L} + \boldsymbol{d}_L^*(\sigma^2\upsilon_d\otimes M_{\tilde{n}})\overline{e_L} \\
&- e_L^*(\upsilon_e^*\otimes M_{\tilde{n}}^T)\boldsymbol{d}_R + \nu_L^*(\upsilon_f^*\otimes M_{\tilde{n}}^T)\boldsymbol{d}_R - \nu_R^T(\sigma^2\upsilon_g^*\otimes M_{\tilde{n}})\boldsymbol{u}_R\big) + \text{h.c}, \qquad (6.66g)
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{L}_h'' = \tfrac{g_0}{\sqrt{8\mu_3}}\big(&-\tfrac{1}{2}\boldsymbol{u}_L^*(\sigma^2\overline{\xi_A}\otimes M_N)\overline{\boldsymbol{u}}_L - \boldsymbol{u}_L^*(\sigma^2(\overline{\xi_D}-\tfrac{1}{2}\varepsilon(\overline{\xi_c}))\otimes M_N)\overline{\boldsymbol{d}}_L \\
&+ \boldsymbol{u}_L^*(\xi_a\otimes M_N)\nu_R - \tfrac{1}{2}\boldsymbol{d}_L^*(\sigma^2\overline{\xi_B}\otimes M_N)\overline{\boldsymbol{d}}_L + \boldsymbol{d}_L^*(\xi_b\otimes M_N)\nu_R \\
&+ \tfrac{1}{2}\boldsymbol{d}_R^T(\sigma_2\xi_C\otimes M_N)\boldsymbol{d}_R + \boldsymbol{d}_R^T(\sigma_2\overline{\xi_c}\otimes M_N)\nu_R\big) + \text{h.c}. \qquad (6.66h)
\end{aligned}
$$

The Lagrangian $\mathscr{L}_q$ contains the kinetic terms and the strong and electroweak interactions of quarks. The Lagrangian $\mathscr{L}_\ell$ contains the kinetic terms and electroweak interactions of leptons. The Lagrangian $\mathscr{L}_m$ contains the mass terms of the fundamental fermions and their interactions with the Higgs fields $\phi_0, \xi_E^0, \xi_F^0, \xi_0, \upsilon_A$ and υ_B. The masses of the u,c,t–quarks, the d,s,b–quarks and the e,μ,τ–leptons are the eigenvalues of M_u, M_d and M_e. The mass Lagrangian of the neutrino sector is given by

$$
-\tfrac{1}{2}\big(-\nu_L^*,\, \nu_R^T\sigma_2\big)\begin{pmatrix} 0 & -M_n \\ -M_n & M_N \end{pmatrix}\begin{pmatrix} \sigma_2\overline{\nu_L} \\ \nu_R \end{pmatrix} + \text{h.c}\,. \qquad (6.67)
$$

The diagonalization of the mass matrix occurring in (6.67) yields the masses of the neutrinos. The mixing angles are small for $\|M_N\| \gg \|M_n\|$. In this case, the left–handed neutrinos receive a mass of the order $\frac{\|M_n\|^2}{2\|M_N\|}$ and the right–handed neutrinos a mass of the order $\frac{1}{2}\|M_N\|$. Thus, for $\|M_n\|$ being of the order of the mass of the top quark and $\|M_N\|$ being of the order of the unification scale, we obtain very low masses for the left–handed neutrinos, which is compatible with experiments (seesaw mechanisms). Moreover, the matrices M_u, M_d, M_e, M_n and M_N contain mixing angles between the fermions, which constitute generalized Kobayashi–Maskawa matrices. Finally, the Lagrangians $\mathscr{L}_x, \mathscr{L}_h, \mathscr{L}_h'$ and $\mathscr{L}_h''$ describe the coupling of the fundamental fermions to the X and Y leptoquarks, the Higgs bosons ϕ_g and the remaining Higgs bosons υ_i and ξ_i, respectively. All terms of these Lagrangians contribute to the proton decay.

Observe that the Lagrangians $\mathscr{L}_q$ and $\mathscr{L}_\ell$ differ from the corresponding Lagrangians of the standard model in two aspects: First, there occurs the massive gauge field Z', which of course is not a terrible problem if its mass is sufficiently large. Second, the universal Weinberg angle θ_W of the standard model is modified by angles of the order θ_W'. However, this angle θ_W' is extremely small if $m_{Z'}$ is very large against m_Z. This means that experiments will certainly not detect θ_W'.

6.4.4 The SU(5)–Model

Formally, we can derive the SU(5)–model from our flipped SU(5)×U(1)–model discussed so far by the following restrictions and replacements: We put $\tilde{A}_\mu \equiv 0$ ad hoc. Strictly speaking, this step is not allowed within our theory. However, one could imagine a formalism that does not use $(\Lambda^1\otimes \mathbb{r}^0\mathfrak{a})\oplus(\Lambda^0\gamma\otimes\mathbb{r}^1\mathfrak{a})$–valued connection forms but $\pi(\Omega^1\mathfrak{g})$–valued connection forms. Now, taking the same L–cycle as before, however with $M_N \equiv 0$, we obtain indeed a SU(5)–Grand Unification model. The calculation is the same as before. However, since the graded centralizer of $\pi(\Omega^*\mathfrak{g})$ is not relevant in such a model, we must put $J_3 = 0$ and $\zeta_{A,B,U,V} = 0$ in the factorization procedure of Section 5.6. This leads instead of (5.81) to the equations

$$\begin{aligned}
&\hat{M}^{10}_{aa} = \tfrac{3}{10} M'_{10}{}^2 - \tfrac{1}{220}\,\mathrm{tr}(9M'_{10}{}^2 + 2M'_5{}^2)\mathbb{1}_6\,, \qquad \hat{M}^{10}_{nn} = \tfrac{1}{10} M'_{\tilde{n}} M'_{\tilde{n}}{}^* - \tfrac{1}{44}\,\mathrm{tr}(M'_{\tilde{n}} M'_{\tilde{n}}{}^*)\mathbb{1}_6\,,\\
&\hat{M}^{10}_{bb} = \tfrac{2}{5} M'_{\tilde{u}} M'_{\tilde{u}}{}^* + \tfrac{3}{5} M'_d M'_d{}^* - \mathrm{tr}(\tfrac{1}{11} M'_{\tilde{u}} M'_{\tilde{u}}{}^* + \tfrac{9}{110} M'_d M'_d{}^* + \tfrac{1}{110} M'_e M'_e{}^*)\mathbb{1}_6\,,\\
&\hat{M}^{5}_{aa} = \tfrac{1}{5} M'_5{}^2 - \tfrac{1}{330}\,\mathrm{tr}(9M'_{10}{}^2 + 2M'_5{}^2)\mathbb{1}_6\,, \qquad \hat{M}^{5}_{nn} = \tfrac{1}{5} M'_{\tilde{n}}{}^* M'_{\tilde{n}} - \tfrac{1}{66}\,\mathrm{tr}(M'_{\tilde{n}} M'_{\tilde{n}}{}^*)\mathbb{1}_6\,,\\
&\hat{M}^{5}_{bb} = \tfrac{4}{5} M'_{\tilde{u}}{}^* M'_{\tilde{u}} + \tfrac{1}{5} \overline{M}'_e M'_e{}^T - \mathrm{tr}(\tfrac{2}{33} M'_{\tilde{u}} M'_{\tilde{u}}{}^* + \tfrac{3}{55} M'_d M'_d{}^* + \tfrac{1}{165} M'_e M'_e{}^*)\mathbb{1}_6\,,\\
&\hat{M}^{1}_{aa} = \hat{M}^{1}_{nn} = \hat{M}^{1}_{bb} = 0\,.
\end{aligned} \tag{6.68}$$

Moreover, instead of (6.40a) we perform the orthogonal transformation

$$-\begin{pmatrix} Z_\mu \\ P_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W^3_\mu \\ A'_\mu \end{pmatrix}, \qquad \sin\theta_W = \sqrt{\tfrac{3}{8}}\,. \tag{6.69}$$

If we compute the electric charges we find that the labels are unconvenient now, because u describes the d quarks (and vice versa) and ν the electrons (and vice versa). Thus, we must permute the labels $u \leftrightarrow d$ and $\nu \leftrightarrow e$. Then we obtain almost the same form (6.66) for the fermionic action, with the following modifications:

1) We have $M_N \equiv 0$, in particular, the Lagrangian $\mathscr{L}''_h$ and the fields ξ^0_E, ξ^0_F and ξ_0 are absent.
2) In the Lagrangians $\mathscr{L}_q$ and $\mathscr{L}_\ell$ we have $Z'_\mu \equiv 0$, $\tilde{Z}'_\mu \equiv 0$ and $\theta'_W \equiv 0$.
3) In the Lagrangians $\mathscr{L}_x$, $\mathscr{L}_h$ and $\mathscr{L}'_h$, the fermion labels $\boldsymbol{u}$ and $\boldsymbol{d}$ are exchanged and e and ν are exchanged.
4) The same exchanges occurs for the mass matrices: $M_u \leftrightarrow M_d$, $M_{\tilde{u}} \leftrightarrow M_{\tilde{d}}$, $M_e \leftrightarrow M_n$, $M_{\tilde{n}} \leftrightarrow M_{\tilde{e}}$.

7 Higgs Potential and Masses of Higgs Fields

There is a last step to go: we must evaluate the Higgs potential $\mathscr{L}_0$ given in (6.10d). *However, since this Higgs potential is so complicated, we must be satisfied with the terms quadratically in the fields. Even this had been an unsolvable problem few decades ago. Fortunately, there exist computers and sophisticated computer algebra packages. Once the matrices for the Higgs fields given in Section 6.3.1 and their relevant products are typed in, it is a matter of only few minutes to select the quadratic terms. We present the results of a Mathematica*^{TM} *calculation in Appendix B. One has to take into consideration that each gauge invariant term of the Higgs potential is the trace of a product of two terms occurring in $\mathfrak{e}(\theta)$, see* (6.7)*. The terms in* (6.7) *vanish for vanishing physical Higgs fields (those without the tilde). Therefore, we must multiply the parts of* (6.7) *which are linear in the Higgs fields in order to get the quadratic terms in the Higgs potential. It remains to multiply the results by the parameters $\mu^i, \tilde{\mu}^i, \breve{\mu}^i, \hat{\mu}^i$. We use a convenient ordering: The outcome of equations* (B.1.i), (B.2.i), (B.3.i) *and* (B.4.i) *is multiplied by $\mu^i, \tilde{\mu}^i, \breve{\mu}^i$ and $\hat{\mu}^i$, in this order. The result given in Section 7.1 is still very long. Then we compute the parameters $\mu^i, \tilde{\mu}^i, \breve{\mu}^i, \hat{\mu}^i$ in Section 7.2 under certain assumptions for the mass matrices $M_{u,d,e,n,N}$ and $M_{10,5}$. This computation is very boring and we use again computer algebra. Inserting this result into the Higgs potential, we get the masses of our Higgs fields in Section 7.3. By a simple reparametrization we can derive in Section 7.4 the masses of the Higgs fields in the* SU(5)*–model.*

7.1 The Higgs Potential

We compute the quadratic terms of the Higgs potential (6.10d) in Appendix B. Here, we present the summary of the results obtained in (B.1), (B.2), (B.3) and (B.4), after performing the reparametrizations (6.41). Moreover, we perform an orthogonal transformation in the ϕ_0–υ_0–sector:

$$\begin{pmatrix}\phi_0\\ \upsilon_0\end{pmatrix} = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\phi_0'\\ \upsilon_0'\end{pmatrix}, \qquad \tan\alpha = \sqrt{\tfrac{12\mu_2}{\mu_1}}\,. \tag{7.1}$$

The motivation for this transformation is that the linear combination ϕ_0' receives a much smaller mass than all other Higgs fields, see below. The quadratic terms of the Higgs potential take the following form:

$$\begin{aligned}
&\mathscr{L}_0 \to \tfrac{1}{384}\Big\{ \\
&\tfrac{1}{12\mu_2+\mu_1}{\phi_0'}^2\big(8\mu^{\mathrm{b}} + 1152\mu^{\mathrm{c}} + 96\mu^{\mathrm{f}} + \tfrac{3072}{5}\tilde{\mu}^{\mathrm{b}} + \tfrac{1024}{15}\tilde{\mu}^{\mathrm{c}} + \tfrac{3072}{5}\tilde{\mu}^{\mathrm{d}} + \tfrac{1024}{5}\tilde{\mu}^{\mathrm{j}} - \tfrac{3072}{5}\tilde{\mu}^{\mathrm{k}} - \tfrac{1024}{5}\tilde{\mu}^{\mathrm{m}} \\
&\qquad\qquad + 256\tilde{\mu}^{\mathrm{p}} + 256\tilde{\mu}^{\mathrm{q}} - 256\tilde{\mu}^{\mathrm{s}}\big) \\
&+\sqrt{\tfrac{6}{5}}\tfrac{1}{\sqrt{\mu_0(12\mu_2+\mu_1)}}\psi_0\phi_0'\big(8\mu^{\mathrm{d}} + 96\mu^{\mathrm{e}} - \tfrac{32}{3}\hat{\mu}^{\mathrm{c}} - \tfrac{32}{9}\hat{\mu}^{\mathrm{d}} + \tfrac{32}{3}\hat{\mu}^{\mathrm{e}} - \tfrac{80}{3}\hat{\mu}^{\mathrm{m}} + \tfrac{80}{3}\hat{\mu}^{\mathrm{n}} \\
&\qquad\qquad -512\tilde{\mu}^{\mathrm{b}} - \tfrac{512}{9}\tilde{\mu}^{\mathrm{c}} - 512\tilde{\mu}^{\mathrm{d}} + \tfrac{32}{5}\tilde{\mu}^{\mathrm{f}} + \tfrac{32}{15}\tilde{\mu}^{\mathrm{g}} - \tfrac{32}{5}\tilde{\mu}^{\mathrm{h}} - \tfrac{512}{3}\tilde{\mu}^{\mathrm{j}} + 512\tilde{\mu}^{\mathrm{k}} \\
&\qquad\qquad + \tfrac{512}{3}\tilde{\mu}^{\mathrm{m}} - \tfrac{1280}{3}\tilde{\mu}^{\mathrm{p}} - \tfrac{1280}{3}\tilde{\mu}^{\mathrm{q}} + \tfrac{1280}{3}\tilde{\mu}^{\mathrm{s}}\big)
\end{aligned} \tag{7.2}$$

$$
\begin{aligned}
&+\sqrt{3}\tfrac{1}{12\mu_2+\mu_1}\phi_0'\upsilon_0'\Big(\sqrt{\tfrac{\mu_1}{\mu_2}}(384\mu^{\rm c}+16\mu^{\rm f}+\tfrac{448}{5}\tilde{\mu}^{\rm b}+\tfrac{1216}{45}\tilde{\mu}^{\rm c}+64\tilde{\mu}^{\rm d}+\tfrac{832}{15}\tilde{\mu}^{\rm j}-\tfrac{384}{5}\tilde{\mu}^{\rm k}-\tfrac{256}{5}\tilde{\mu}^{\rm m}\\
&\qquad\qquad+\tfrac{256}{3}\tilde{\mu}^{\rm p}+\tfrac{128}{3}\tilde{\mu}^{\rm q}-64\tilde{\mu}^{\rm s})\\
&\qquad\qquad+\sqrt{\tfrac{\mu_2}{\mu_1}}(-32\mu^{\rm b}-192\mu^{\rm f}-\tfrac{6912}{5}\tilde{\mu}^{\rm b}+\tfrac{256}{5}\tilde{\mu}^{\rm c}-\tfrac{8448}{5}\tilde{\mu}^{\rm d}-\tfrac{768}{5}\tilde{\mu}^{\rm j}+1536\tilde{\mu}^{\rm k}\\
&\qquad\qquad+\tfrac{1024}{5}\tilde{\mu}^{\rm m}-512\tilde{\mu}^{\rm q}+256\tilde{\mu}^{\rm s})\Big)\\
&+\sqrt{3}\tfrac{1}{\sqrt{\mu_2(12\mu_2+\mu_1)}}\phi_0'\upsilon_{45}(-32\tilde{\mu}^{\rm l}-\tfrac{32}{3}\tilde{\mu}^{\rm n}+32\tilde{\mu}^{\rm o}-\tfrac{64}{3}\tilde{\mu}^{\rm t}+\tfrac{64}{3}\tilde{\mu}^{\rm u})\\
&+\sqrt{2}\tfrac{\sqrt{2}}{\sqrt{\mu_3(12\mu_2+\mu_1)}}\phi_0'\xi_0(4\breve{\mu}^{\rm c}+48\breve{\mu}^{\rm d}+\tfrac{64}{5}\hat{\mu}^{\rm c}+\tfrac{64}{15}\hat{\mu}^{\rm d}-\tfrac{64}{5}\hat{\mu}^{\rm e}+16\hat{\mu}^{\rm m}-16\hat{\mu}^{\rm n})\\
&+\tfrac{1}{12\mu_2+\mu_1}{\upsilon_0'}^2\Big((\tfrac{9}{4}\breve{\mu}^{\rm l}-96\mu^{\rm f}+24\mu^{\rm h}+3\mu^{\rm i}+3\mu^{\rm j}+9\mu^{\rm k}+9\mu^{\rm l}-3\mu^{\rm m}-9\mu^{\rm t}+\tfrac{1728}{5}\tilde{\mu}^{\rm b}+\tfrac{192}{5}\tilde{\mu}^{\rm c}\\
&\qquad\qquad-192\tilde{\mu}^{\rm d}-192\tilde{\mu}^{\rm j}+\tfrac{384}{5}\tilde{\mu}^{\rm k}+\tfrac{896}{5}\tilde{\mu}^{\rm m}-128\tilde{\mu}^{\rm q}+128\tilde{\mu}^{\rm s})\\
&\qquad\qquad+\tfrac{\mu_1}{\mu_2}(96\mu^{\rm c}+\tfrac{3}{32}\breve{\mu}^{\rm l}+\mu^{\rm h}+\tfrac{1}{8}\mu^{\rm i}+\tfrac{1}{8}\mu^{\rm j}+\tfrac{3}{8}\mu^{\rm k}+\tfrac{3}{8}\mu^{\rm l}-\tfrac{1}{8}\mu^{\rm m}-\tfrac{3}{8}\mu^{\rm t}\\
&\qquad\qquad+\tfrac{184}{5}\tilde{\mu}^{\rm b}+\tfrac{376}{45}\tilde{\mu}^{\rm c}+8\tilde{\mu}^{\rm d}+\tfrac{88}{15}\tilde{\mu}^{\rm j}-16\tilde{\mu}^{\rm k}-\tfrac{16}{3}\tilde{\mu}^{\rm m}+\tfrac{64}{3}\tilde{\mu}^{\rm p}+\tfrac{16}{3}\tilde{\mu}^{\rm q}-\tfrac{32}{3}\tilde{\mu}^{\rm s})\\
&\qquad\qquad+\tfrac{\mu_2}{\mu_1}(96\mu^{\rm b}+\tfrac{27}{2}\breve{\mu}^{\rm l}+144\mu^{\rm h}+18\mu^{\rm i}+18\mu^{\rm j}+54\mu^{\rm k}+54\mu^{\rm l}-18\mu^{\rm m}-54\mu^{\rm t}\\
&\qquad\qquad+\tfrac{31104}{5}\tilde{\mu}^{\rm b}+\tfrac{384}{5}\tilde{\mu}^{\rm c}+\tfrac{19584}{5}\tilde{\mu}^{\rm d}-\tfrac{3456}{5}\tilde{\mu}^{\rm j}-\tfrac{20736}{5}\tilde{\mu}^{\rm k}+\tfrac{2304}{5}\tilde{\mu}^{\rm m}+768\tilde{\mu}^{\rm q})\Big)\\
&+\sqrt{6}\tfrac{1}{\sqrt{\mu_0(12\mu_2+\mu_1)}}\psi_3'\upsilon_0'(\sqrt{\tfrac{\mu_1}{\mu_2}}+12\sqrt{\tfrac{\mu_2}{\mu_1}})(2\mu^{\rm h}+\tfrac{1}{4}\mu^{\rm i}+\tfrac{1}{4}\mu^{\rm j}-\tfrac{3}{4}\mu^{\rm k}-\tfrac{3}{4}\mu^{\rm l}-\tfrac{1}{4}\mu^{\rm m}-\tfrac{1}{4}\mu^{\rm n}\\
&\qquad\qquad+\tfrac{1}{2}\mu^{\rm p}-\tfrac{1}{4}\mu^{\rm r}+\tfrac{3}{4}\mu^{\rm t}+48\tilde{\mu}^{\rm b}-\tfrac{16}{9}\tilde{\mu}^{\rm c}+16\tilde{\mu}^{\rm d}-3\tilde{\mu}^{\rm f}+\tfrac{1}{3}\tilde{\mu}^{\rm g}+\tilde{\mu}^{\rm h}\\
&\qquad\qquad+\tfrac{16}{3}\tilde{\mu}^{\rm j}-32\tilde{\mu}^{\rm k}+\hat{\mu}^{\rm c}-\tfrac{1}{9}\hat{\mu}^{\rm d}-\tfrac{1}{3}\hat{\mu}^{\rm e})\\
&+\tfrac{1}{\sqrt{\mu_2(12\mu_2+\mu_1)}}\upsilon_0'\upsilon_{45}\Big(\sqrt{\tfrac{\mu_1}{\mu_2}}(-16\tilde{\mu}^{\rm l}-\tfrac{16}{3}\tilde{\mu}^{\rm n}+8\tilde{\mu}^{\rm o}-\tfrac{32}{3}\tilde{\mu}^{\rm t}+\tfrac{16}{3}\tilde{\mu}^{\rm u})+\sqrt{\tfrac{\mu_2}{\mu_1}}(-96\tilde{\mu}^{\rm o}-64\tilde{\mu}^{\rm u})\Big)\\
&+\sqrt{\tfrac{2}{5}}\tfrac{1}{\sqrt{\mu_0(12\mu_2+\mu_1)}}\psi_0\upsilon_0'\Big(\sqrt{\tfrac{\mu_1}{\mu_2}}(48\mu^{\rm e}-10\mu^{\rm h}-\tfrac{5}{4}\mu^{\rm i}-\tfrac{5}{4}\mu^{\rm j}+\tfrac{15}{4}\mu^{\rm k}+\tfrac{15}{4}\mu^{\rm l}+\tfrac{5}{4}\mu^{\rm m}+\tfrac{5}{4}\mu^{\rm n}-\tfrac{5}{2}\mu^{\rm p}\\
&\qquad\qquad+\tfrac{5}{4}\mu^{\rm r}-\tfrac{15}{4}\mu^{\rm t}-112\tilde{\mu}^{\rm b}-\tfrac{304}{9}\tilde{\mu}^{\rm c}-80\tilde{\mu}^{\rm d}+\tfrac{7}{5}\tilde{\mu}^{\rm f}+\tfrac{19}{15}\tilde{\mu}^{\rm g}-\tilde{\mu}^{\rm h}-\tfrac{208}{3}\tilde{\mu}^{\rm j}\\
&\qquad\qquad+96\tilde{\mu}^{\rm k}+64\tilde{\mu}^{\rm m}-\tfrac{640}{3}\tilde{\mu}^{\rm p}-\tfrac{320}{3}\tilde{\mu}^{\rm q}+160\tilde{\mu}^{\rm s}\\
&\qquad\qquad-\tfrac{7}{3}\hat{\mu}^{\rm c}-\tfrac{19}{9}\hat{\mu}^{\rm d}+\tfrac{5}{3}\hat{\mu}^{\rm e}-\tfrac{40}{3}\hat{\mu}^{\rm m}+\tfrac{20}{3}\hat{\mu}^{\rm n})\\
&\qquad\qquad+\sqrt{\tfrac{\mu_1}{\mu_2}}(-48\mu^{\rm d}-120\mu^{\rm h}-15\mu^{\rm i}-15\mu^{\rm j}+45\mu^{\rm k}+45\mu^{\rm l}+15\mu^{\rm m}+15\mu^{\rm n}\\
&\qquad\qquad-30\mu^{\rm p}+15\mu^{\rm r}-45\mu^{\rm t}+1728\tilde{\mu}^{\rm b}-64\tilde{\mu}^{\rm c}+2112\tilde{\mu}^{\rm d}-\tfrac{108}{5}\tilde{\mu}^{\rm f}+\tfrac{12}{5}\tilde{\mu}^{\rm g}\\
&\qquad\qquad+\tfrac{132}{5}\tilde{\mu}^{\rm h}+192\tilde{\mu}^{\rm j}-1920\tilde{\mu}^{\rm k}-256\tilde{\mu}^{\rm m}+1280\tilde{\mu}^{\rm q}-640\tilde{\mu}^{\rm s}\\
&\qquad\qquad+36\hat{\mu}^{\rm c}-4\hat{\mu}^{\rm d}-44\hat{\mu}^{\rm e}-80\hat{\mu}^{\rm n})\Big)\\
&+\sqrt{6}\tfrac{\sqrt{2}}{\sqrt{\mu_3(12\mu_2+\mu_1)}}\xi_0\upsilon_0'\Big(\sqrt{\tfrac{\mu_1}{\mu_2}}(8\breve{\mu}^{\rm d}+\tfrac{14}{15}\hat{\mu}^{\rm c}+\tfrac{38}{45}\hat{\mu}^{\rm d}-\tfrac{2}{3}\hat{\mu}^{\rm e}+\tfrac{8}{3}\hat{\mu}^{\rm m}-\tfrac{4}{3}\hat{\mu}^{\rm n})\\
&\qquad\qquad+\sqrt{\tfrac{\mu_2}{\mu_1}}(-8\breve{\mu}^{\rm c}-\tfrac{72}{5}\hat{\mu}^{\rm c}+\tfrac{8}{5}\hat{\mu}^{\rm d}+\tfrac{88}{5}\hat{\mu}^{\rm e}+16\hat{\mu}^{\rm n})\Big)\\
&+\tfrac{1}{\mu_2}\upsilon_{45}^2(\tfrac{3}{32}\breve{\mu}^{\rm l}+\mu^{\rm h}+\tfrac{1}{8}\mu^{\rm i}+\tfrac{1}{8}\mu^{\rm j}+\tfrac{3}{8}\mu^{\rm k}+\tfrac{3}{8}\mu^{\rm l}-\tfrac{1}{8}\mu^{\rm m}-\tfrac{3}{8}\mu^{\rm t}+8\tilde{\mu}^{\rm e}+\tfrac{16}{3}\tilde{\mu}^{\rm r})\\
&+\tfrac{1}{\sqrt{\mu_0\mu_2}}\sqrt{\tfrac{2}{5}}\psi_0\upsilon_{45}(-\tfrac{7}{2}\mu^{\rm o}+4\mu^{\rm q}-\tfrac{1}{2}\mu^{\rm s}-\tilde{\mu}^{\rm i}+40\tilde{\mu}^{\rm l}+\tfrac{40}{3}\tilde{\mu}^{\rm n}-40\tilde{\mu}^{\rm o}+\tfrac{160}{3}\tilde{\mu}^{\rm t}-\tfrac{160}{3}\tilde{\mu}^{\rm u}+\tfrac{5}{3}\hat{\mu}^{\rm f}+\tfrac{20}{3}\hat{\mu}^{\rm o})\\
&+\sqrt{6}\tfrac{1}{\sqrt{\mu_0\mu_2}}\psi_3'\upsilon_{45}(-\tfrac{1}{2}\mu^{\rm o}+\tfrac{1}{2}\mu^{\rm s}+\tilde{\mu}^{\rm i}-8\tilde{\mu}^{\rm l}-\tfrac{8}{3}\tilde{\mu}^{\rm n}+8\tilde{\mu}^{\rm o}-\tfrac{1}{3}\hat{\mu}^{\rm f})\\
&+\sqrt{6}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\xi_0\upsilon_{45}(-\tfrac{2}{3}\hat{\mu}^{\rm f}-\tfrac{4}{3}\hat{\mu}^{\rm o})\\
&+\tfrac{1}{\mu_0}\psi_0^2(\tfrac{48}{5}\mu^{\rm a}+\tfrac{3}{5}\mu^{\rm g}+10\mu^{\rm h}+\tfrac{13}{5}\mu^{\rm i}+\tfrac{7}{5}\mu^{\rm j}+\tfrac{109}{5}\mu^{\rm k}+\tfrac{31}{5}\mu^{\rm l}-\tfrac{4}{5}\mu^{\rm m}-7\mu^{\rm n}+8\mu^{\rm p}-\mu^{\rm r}-\tfrac{52}{5}\mu^{\rm t}+\tfrac{2}{25}\tilde{\mu}^{\rm a}\\
&\qquad+128\tilde{\mu}^{\rm b}+\tfrac{128}{9}\tilde{\mu}^{\rm c}+128\tilde{\mu}^{\rm d}-\tfrac{16}{5}\tilde{\mu}^{\rm f}-\tfrac{16}{15}\tilde{\mu}^{\rm g}+\tfrac{16}{5}\tilde{\mu}^{\rm h}+\tfrac{128}{3}\tilde{\mu}^{\rm j}-128\tilde{\mu}^{\rm k}-\tfrac{128}{3}\tilde{\mu}^{\rm m}+\tfrac{640}{3}\tilde{\mu}^{\rm p}\\
&\qquad+\tfrac{640}{3}\tilde{\mu}^{\rm q}-\tfrac{640}{3}\tilde{\mu}^{\rm s}+\tfrac{48}{5}\breve{\mu}^{\rm e}+\tfrac{2}{9}\hat{\mu}^{\rm a}-\tfrac{2}{15}\hat{\mu}^{\rm b}+\tfrac{16}{3}\hat{\mu}^{\rm c}+\tfrac{16}{9}\hat{\mu}^{\rm d}-\tfrac{16}{3}\hat{\mu}^{\rm e}+\tfrac{10}{3}\hat{\mu}^{\rm g}+\tfrac{80}{3}\hat{\mu}^{\rm m}-\tfrac{80}{3}\hat{\mu}^{\rm n})
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{1}{\mu_0}\textstyle\sum_{i=1}^{3}\psi_i'^2(\mu^{\mathrm{g}}+6\mu^{\mathrm{h}}+3\mu^{\mathrm{i}}+\mu^{\mathrm{j}}+3\mu^{\mathrm{k}}+9\mu^{\mathrm{l}}+3\mu^{\mathrm{n}}-3\mu^{\mathrm{r}}+2\tilde{\mu}^{\mathrm{a}}+128\tilde{\mu}^{\mathrm{b}}+\tfrac{128}{9}\tilde{\mu}^{\mathrm{c}}+128\tilde{\mu}^{\mathrm{d}}\\
&\qquad-16\tilde{\mu}^{\mathrm{f}}-\tfrac{16}{3}\tilde{\mu}^{\mathrm{g}}+16\tilde{\mu}^{\mathrm{h}}+\tfrac{128}{3}\tilde{\mu}^{\mathrm{j}}-128\tilde{\mu}^{\mathrm{k}}-\tfrac{128}{3}\tilde{\mu}^{\mathrm{m}}+\tfrac{2}{9}\hat{\mu}^{\mathrm{a}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{b}}+\tfrac{16}{3}\hat{\mu}^{\mathrm{c}}+\tfrac{16}{9}\hat{\mu}^{\mathrm{d}}-\tfrac{16}{3}\hat{\mu}^{\mathrm{e}})\\
&+\tfrac{1}{\mu_0}\textstyle\sum_{i=1}^{8}\psi_i^2(16\mu^{\mathrm{h}}+2\mu^{\mathrm{i}}+2\mu^{\mathrm{j}}+2\mu^{\mathrm{k}}+2\mu^{\mathrm{l}}-2\mu^{\mathrm{m}}+2\mu^{\mathrm{n}}-4\mu^{\mathrm{p}}+2\mu^{\mathrm{r}}-2\mu^{\mathrm{t}}+2\tilde{\mu}^{\mathrm{a}}+128\tilde{\mu}^{\mathrm{b}}\\
&\qquad+\tfrac{128}{9}\tilde{\mu}^{\mathrm{c}}+128\tilde{\mu}^{\mathrm{d}}+16\tilde{\mu}^{\mathrm{f}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{g}}-16\tilde{\mu}^{\mathrm{h}}+\tfrac{128}{3}\tilde{\mu}^{\mathrm{j}}-128\tilde{\mu}^{\mathrm{k}}-\tfrac{128}{3}\tilde{\mu}^{\mathrm{m}}+\tfrac{256}{3}\tilde{\mu}^{\mathrm{p}}\\
&\qquad+\tfrac{256}{3}\tilde{\mu}^{\mathrm{q}}-\tfrac{256}{3}\tilde{\mu}^{\mathrm{s}}+\tfrac{2}{9}\hat{\mu}^{\mathrm{a}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{b}}+\tfrac{16}{3}\hat{\mu}^{\mathrm{c}}+\tfrac{16}{9}\hat{\mu}^{\mathrm{d}}-\tfrac{16}{3}\hat{\mu}^{\mathrm{e}}+\tfrac{4}{3}\hat{\mu}^{\mathrm{g}}+\tfrac{32}{3}\hat{\mu}^{\mathrm{m}}-\tfrac{32}{3}\hat{\mu}^{\mathrm{n}})\\
&+\sqrt{\tfrac{3}{5}}\psi_0\psi_3'(2\mu^{\mathrm{g}}-20\mu^{\mathrm{h}}+2\mu^{\mathrm{i}}-2\mu^{\mathrm{j}}+2\mu^{\mathrm{k}}-18\mu^{\mathrm{l}}+4\mu^{\mathrm{m}}+2\mu^{\mathrm{n}}-8\mu^{\mathrm{p}}+6\mu^{\mathrm{r}}+20\mu^{\mathrm{t}})\\
&+\sqrt{\tfrac{2}{\mu_0\mu_3}}\sqrt{\tfrac{3}{5}}\psi_0\xi_0(8\breve{\mu}^{\mathrm{b}}-\tfrac{8}{9}\hat{\mu}^{\mathrm{a}}+\tfrac{4}{15}\hat{\mu}^{\mathrm{b}}-\tfrac{32}{3}\hat{\mu}^{\mathrm{c}}-\tfrac{32}{9}\hat{\mu}^{\mathrm{d}}+\tfrac{32}{3}\hat{\mu}^{\mathrm{e}}-\tfrac{20}{3}\hat{\mu}^{\mathrm{g}}-\tfrac{80}{3}\hat{\mu}^{\mathrm{m}}+\tfrac{80}{3}\hat{\mu}^{\mathrm{n}})\\
&+\tfrac{2}{\mu_3}\xi_0^2(4\breve{\mu}^{\mathrm{a}}+\tfrac{8}{15}\hat{\mu}^{\mathrm{a}}+2\hat{\mu}^{\mathrm{g}})\\
&+\tfrac{1}{\sqrt{\mu_0\mu_2}}\sqrt{6}(\psi_1'\upsilon_{18}+\psi_2'\upsilon_{63})(2\mu^{\mathrm{h}}+\tfrac{1}{4}\mu^{\mathrm{i}}+\tfrac{1}{4}\mu^{\mathrm{j}}-\tfrac{3}{4}\mu^{\mathrm{k}}-\tfrac{3}{4}\mu^{\mathrm{l}}-\tfrac{1}{4}\mu^{\mathrm{m}}-\tfrac{1}{4}\mu^{\mathrm{n}}+\tfrac{1}{2}\mu^{\mathrm{p}}-\tfrac{1}{4}\mu^{\mathrm{r}}+\tfrac{3}{4}\mu^{\mathrm{t}}\\
&\qquad+48\tilde{\mu}^{\mathrm{b}}-\tfrac{16}{9}\tilde{\mu}^{\mathrm{c}}+16\tilde{\mu}^{\mathrm{d}}-3\tilde{\mu}^{\mathrm{f}}+\tfrac{1}{3}\tilde{\mu}^{\mathrm{g}}+\tilde{\mu}^{\mathrm{h}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{j}}-32\tilde{\mu}^{\mathrm{k}}\\
&\qquad+\hat{\mu}^{\mathrm{c}}-\tfrac{1}{9}\hat{\mu}^{\mathrm{d}}-\tfrac{1}{3}\hat{\mu}^{\mathrm{e}})\\
&+\tfrac{1}{\sqrt{\mu_0\mu_2}}\sqrt{6}(\psi_1'\upsilon_{63}-\psi_2'\upsilon_{18})(-\tfrac{1}{2}\mu^{\mathrm{o}}+\tfrac{1}{2}\mu^{\mathrm{s}}+\tilde{\mu}^{\mathrm{i}}-8\tilde{\mu}^{\mathrm{l}}-\tfrac{8}{3}\tilde{\mu}^{\mathrm{n}}+8\tilde{\mu}^{\mathrm{o}}-\tfrac{1}{3}\hat{\mu}^{\mathrm{f}})\\
&+\tfrac{1}{\sqrt{\mu_0\mu_2}}\textstyle\sum_{i=1}^{8}\psi_i\upsilon_i(8\mu^{\mathrm{h}}+\mu^{\mathrm{i}}+\mu^{\mathrm{j}}-3\mu^{\mathrm{k}}-3\mu^{\mathrm{l}}-\mu^{\mathrm{m}}-\mu^{\mathrm{n}}+2\mu^{\mathrm{p}}-\mu^{\mathrm{r}}+3\mu^{\mathrm{t}}-64\tilde{\mu}^{\mathrm{b}}-\tfrac{64}{9}\tilde{\mu}^{\mathrm{c}}+64\tilde{\mu}^{\mathrm{d}}\\
&\qquad-4\tilde{\mu}^{\mathrm{f}}-\tfrac{4}{3}\tilde{\mu}^{\mathrm{g}}-4\tilde{\mu}^{\mathrm{h}}-\tfrac{64}{3}\tilde{\mu}^{\mathrm{j}}-\tfrac{64}{3}\tilde{\mu}^{\mathrm{p}}+\tfrac{64}{3}\tilde{\mu}^{\mathrm{q}}-\tfrac{4}{3}\hat{\mu}^{\mathrm{c}}-\tfrac{4}{9}\hat{\mu}^{\mathrm{d}}-\tfrac{4}{3}\hat{\mu}^{\mathrm{e}}-\tfrac{4}{3}\hat{\mu}^{\mathrm{m}}-\tfrac{4}{3}\hat{\mu}^{\mathrm{n}})\\
&+\tfrac{1}{\sqrt{\mu_0\mu_2}}\textstyle\sum_{i=1}^{8}\psi_i\upsilon_{i+45}(-2\mu^{\mathrm{o}}+4\mu^{\mathrm{q}}-2\mu^{\mathrm{s}}-4\tilde{\mu}^{\mathrm{i}}-32\tilde{\mu}^{\mathrm{l}}-\tfrac{32}{3}\tilde{\mu}^{\mathrm{n}}+32\tilde{\mu}^{\mathrm{o}}-\tfrac{32}{3}\tilde{\mu}^{\mathrm{t}}+\tfrac{32}{3}\tilde{\mu}^{\mathrm{u}}-\tfrac{4}{3}\hat{\mu}^{\mathrm{f}}-\tfrac{4}{3}\hat{\mu}^{\mathrm{o}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_0\mu_3}}\textstyle\sum_{i=1}^{8}\psi_i\xi_{i+32}(-4\breve{\mu}^{\mathrm{g}}+2\hat{\mu}^{\mathrm{j}}+16\hat{\mu}^{\mathrm{p}}-16\hat{\mu}^{\mathrm{q}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_0\mu_3}}\textstyle\sum_{i=1}^{8}\psi_i\xi_{i+81}(-4\breve{\mu}^{\mathrm{h}}+2\hat{\mu}^{\mathrm{k}}+16\hat{\mu}^{\mathrm{s}}-16\hat{\mu}^{\mathrm{t}})\\
&+\tfrac{2}{\mu_1}\textstyle\sum_{i=1}^{6}\phi_i^2(\mu^{\mathrm{g}}+8\mu^{\mathrm{h}}+\tfrac{13}{4}\mu^{\mathrm{i}}+\tfrac{5}{4}\mu^{\mathrm{j}}+\tfrac{9}{4}\mu^{\mathrm{k}}+\tfrac{9}{4}\mu^{\mathrm{l}}-\tfrac{1}{4}\mu^{\mathrm{m}}-\tfrac{9}{4}\mu^{\mathrm{n}}\\
&\qquad+\tfrac{3}{2}\mu^{\mathrm{p}}+\tfrac{3}{4}\mu^{\mathrm{r}}-\tfrac{9}{4}\mu^{\mathrm{t}}+\mu^{\mathrm{u}}+\tfrac{1}{3}\mu^{\mathrm{v}}+\tfrac{32}{3}\mu^{\mathrm{w}}+162\tilde{\mu}^{\mathrm{b}}+2\tilde{\mu}^{\mathrm{c}}\\
&\qquad+98\tilde{\mu}^{\mathrm{d}}+2\tilde{\mu}^{\mathrm{e}}-18\tilde{\mu}^{\mathrm{j}}-126\tilde{\mu}^{\mathrm{k}}+14\tilde{\mu}^{\mathrm{m}}+\tfrac{64}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{64}{3}\tilde{\mu}^{\mathrm{r}}+2\hat{\mu}^{\mathrm{h}}+2\hat{\mu}^{\mathrm{i}})\\
&+\sqrt{2}\tfrac{2}{\sqrt{\mu_1\mu_3}}(\phi_1\xi_{44}+\phi_2\xi_{45}+\phi_3\xi_{46}+\phi_4\xi_{93}+\phi_5\xi_{94}+\phi_6\xi_{95})\\
&\qquad(-3\hat{\mu}^{\mathrm{c}}+\tfrac{1}{3}\hat{\mu}^{\mathrm{d}}+\tfrac{7}{3}\hat{\mu}^{\mathrm{e}}+2\hat{\mu}^{\mathrm{h}}+2\hat{\mu}^{\mathrm{i}}+\tfrac{8}{3}\hat{\mu}^{\mathrm{n}})\\
&+\sqrt{2}\tfrac{2}{\sqrt{\mu_1\mu_3}}(\phi_1\xi_{93}+\phi_2\xi_{94}+\phi_3\xi_{95}-\phi_4\xi_{44}-\phi_5\xi_{45}-\phi_6\xi_{46})(\tfrac{1}{3}\hat{\mu}^{\mathrm{f}}+\tfrac{8}{3}\hat{\mu}^{\mathrm{o}})\\
&+\sqrt{3}\tfrac{2}{\sqrt{\mu_1\mu_3}}(\phi_1\xi_{47}+\phi_2\xi_{48}+\phi_3\xi_{49}+\phi_4\xi_{96}+\phi_5\xi_{97}+\phi_6\xi_{98})\\
&\qquad(\tfrac{8}{3}\breve{\mu}^{\mathrm{g}}+\tfrac{4}{9}\breve{\mu}^{\mathrm{j}}-\tfrac{4}{9}\breve{\mu}^{\mathrm{m}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{j}}+\tfrac{8}{3}\hat{\mu}^{\mathrm{q}}-\tfrac{8}{3}\hat{\mu}^{\mathrm{u}})\\
&+\sqrt{3}\tfrac{2}{\sqrt{\mu_1\mu_3}}(\phi_1\xi_{96}+\phi_2\xi_{97}+\phi_3\xi_{98}-\phi_4\xi_{47}-\phi_5\xi_{48}-\phi_6\xi_{49})\\
&\qquad(\tfrac{8}{3}\breve{\mu}^{\mathrm{h}}+\tfrac{4}{9}\breve{\mu}^{\mathrm{k}}+\tfrac{4}{9}\breve{\mu}^{\mathrm{n}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{k}}+\tfrac{8}{3}\hat{\mu}^{\mathrm{r}}+\tfrac{8}{3}\hat{\mu}^{\mathrm{t}})\\
&+\sqrt{2}\tfrac{\sqrt{2}}{\sqrt{\mu_1\mu_2}}(\phi_1\upsilon_9+\phi_2\upsilon_{10}+\phi_3\upsilon_{11}+\phi_4\upsilon_{54}+\phi_5\upsilon_{55}+\phi_6\upsilon_{56})\\
&\qquad(-4\mu^{\mathrm{h}}-\tfrac{1}{2}\mu^{\mathrm{i}}-\tfrac{1}{2}\mu^{\mathrm{j}}-\tfrac{3}{2}\mu^{\mathrm{k}}-\tfrac{3}{2}\mu^{\mathrm{l}}+\tfrac{1}{2}\mu^{\mathrm{m}}+2\mu^{\mathrm{n}}-2\mu^{\mathrm{p}}+\tfrac{3}{2}\mu^{\mathrm{t}}+\mu^{\mathrm{u}}+\tfrac{1}{3}\mu^{\mathrm{v}}-\tfrac{16}{3}\mu^{\mathrm{w}}\\
&\qquad+18\tilde{\mu}^{\mathrm{b}}-\tfrac{10}{3}\tilde{\mu}^{\mathrm{c}}-14\tilde{\mu}^{\mathrm{d}}+2\tilde{\mu}^{\mathrm{e}}+14\tilde{\mu}^{\mathrm{j}}+2\tilde{\mu}^{\mathrm{k}}-\tfrac{38}{3}\tilde{\mu}^{\mathrm{m}}+\tfrac{32}{3}\tilde{\mu}^{\mathrm{q}}-\tfrac{32}{3}\tilde{\mu}^{\mathrm{r}}-\tfrac{16}{3}\tilde{\mu}^{\mathrm{s}})\\
&+\sqrt{2}\tfrac{\sqrt{2}}{\sqrt{\mu_1\mu_2}}(\phi_1\upsilon_{54}+\phi_2\upsilon_{55}+\phi_3\upsilon_{56}-\phi_4\upsilon_9-\phi_5\upsilon_{10}-\phi_6\upsilon_{11})\\
&\qquad(2\mu^{\mathrm{o}}-2\mu^{\mathrm{q}}+8\tilde{\mu}^{\mathrm{l}}-\tfrac{8}{3}\tilde{\mu}^{\mathrm{n}}-8\tilde{\mu}^{\mathrm{o}}-\tfrac{16}{3}\tilde{\mu}^{\mathrm{t}}+\tfrac{32}{3}\tilde{\mu}^{\mathrm{u}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_1\mu_2}}(\phi_1\upsilon_{12}+\phi_2\upsilon_{13}+\phi_3\upsilon_{14}+\phi_4\upsilon_{57}+\phi_5\upsilon_{58}+\phi_6\upsilon_{59})\\
&\qquad(2\mu^{\mathrm{u}}+\tfrac{2}{3}\mu^{\mathrm{v}}+\tfrac{16}{3}\mu^{\mathrm{w}}+36\tilde{\mu}^{\mathrm{b}}+4\tilde{\mu}^{\mathrm{c}}-28\tilde{\mu}^{\mathrm{d}}+4\tilde{\mu}^{\mathrm{e}}-20\tilde{\mu}^{\mathrm{j}}+4\tilde{\mu}^{\mathrm{k}}+12\tilde{\mu}^{\mathrm{m}}-\tfrac{32}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{32}{3}\tilde{\mu}^{\mathrm{r}}+16\tilde{\mu}^{\mathrm{s}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_1\mu_2}}(\phi_1\upsilon_{57}+\phi_2\upsilon_{58}+\phi_3\upsilon_{59}-\phi_4\upsilon_{12}-\phi_5\upsilon_{13}-\phi_6\upsilon_{14})(16\tilde{\mu}^{\mathrm{l}}-16\tilde{\mu}^{\mathrm{o}}+16\tilde{\mu}^{\mathrm{t}}-\tfrac{32}{3}\tilde{\mu}^{\mathrm{u}})
\end{aligned}
$$

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\begin{aligned}
&+\sqrt{2}\tfrac{\sqrt{2}}{\sqrt{\mu_1\mu_2}}(\phi_1\upsilon_{39}+\phi_2\upsilon_{40}+\phi_3\upsilon_{41}+\phi_4\upsilon_{84}+\phi_5\upsilon_{85}+\phi_6\upsilon_{86})(-6\hat{\mu}^{\mathrm{p}}+2\hat{\mu}^{\mathrm{q}}+2\hat{\mu}^{\mathrm{u}})\\
&+\sqrt{2}\tfrac{\sqrt{2}}{\sqrt{\mu_1\mu_2}}(\phi_1\upsilon_{84}+\phi_2\upsilon_{85}+\phi_3\upsilon_{86}-\phi_4\upsilon_{39}-\phi_5\upsilon_{40}-\phi_6\upsilon_{41})(-2\hat{\mu}^{\mathrm{r}}-6\hat{\mu}^{\mathrm{s}}+2\hat{\mu}^{\mathrm{t}})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=1}^{6}\xi_i^2+\sum_{i=50}^{55}\xi_i^2)(4\breve{\mu}^{\mathrm{e}}+\breve{\mu}^{\mathrm{l}})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=7}^{12}\xi_i^2+\sum_{i=56}^{61}\xi_i^2)(4\breve{\mu}^{\mathrm{e}}+2\hat{\mu}^{\mathrm{h}}+2\hat{\mu}^{\mathrm{i}})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=13}^{18}\xi_i^2+\sum_{i=62}^{67}\xi_i^2)(16\breve{\mu}^{\mathrm{e}}+2\hat{\mu}^{\mathrm{h}}+2\hat{\mu}^{\mathrm{i}})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=19}^{24}\xi_i^2+\sum_{i=68}^{73}\xi_i^2)(4\breve{\mu}^{\mathrm{e}}+\tfrac{1}{2}\breve{\mu}^{\mathrm{l}}+\hat{\mu}^{\mathrm{h}}+\hat{\mu}^{\mathrm{i}})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=25}^{32}\xi_i^2+\sum_{i=74}^{81}\xi_i^2)(9\breve{\mu}^{\mathrm{e}}+\breve{\mu}^{\mathrm{f}}+\tfrac{1}{2}\breve{\mu}^{\mathrm{l}}+\hat{\mu}^{\mathrm{h}}+\hat{\mu}^{\mathrm{i}})\\
&+\tfrac{2}{\mu_3}\textstyle\sum_{i=33}^{40}\xi_i^2(9\breve{\mu}^{\mathrm{e}}+\breve{\mu}^{\mathrm{f}}+4\hat{\mu}^{\mathrm{h}})+\tfrac{2}{\mu_3}\sum_{i=82}^{89}\xi_i^2(9\breve{\mu}^{\mathrm{e}}+\breve{\mu}^{\mathrm{f}}+4\hat{\mu}^{\mathrm{i}})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=41}^{43}\xi_i^2+\sum_{i=90}^{92}\xi_i^2)(\breve{\mu}^{\mathrm{e}}+\breve{\mu}^{\mathrm{f}}+\tfrac{8}{3}\breve{\mu}^{\mathrm{i}}+\tfrac{1}{3}\breve{\mu}^{\mathrm{l}}+\tfrac{1}{9}\hat{\mu}^{\mathrm{a}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{g}})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=44}^{46}\xi_i^2+\sum_{i=93}^{95}\xi_i^2)(\breve{\mu}^{\mathrm{e}}+\breve{\mu}^{\mathrm{f}}+\tfrac{1}{9}\hat{\mu}^{\mathrm{a}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{g}}+\hat{\mu}^{\mathrm{h}}+\hat{\mu}^{\mathrm{i}})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=47}^{49}\xi_i^2+\sum_{i=96}^{98}\xi_i^2)(4\breve{\mu}^{\mathrm{e}}+\tfrac{8}{3}\breve{\mu}^{\mathrm{f}}+\tfrac{16}{9}\breve{\mu}^{\mathrm{i}}+\tfrac{1}{18}\breve{\mu}^{\mathrm{l}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{g}}+\hat{\mu}^{\mathrm{h}}+\hat{\mu}^{\mathrm{i}})\\
&+\tfrac{2}{\mu_3}\textstyle\sum_{i=1}^{6}(\xi_{i+6}\xi_{i+12}+\xi_{i+55}\xi_{i+61})(4\hat{\mu}^{\mathrm{h}}-4\hat{\mu}^{\mathrm{i}})\\
&+\tfrac{2}{\mu_3}\textstyle\sum_{i=1}^{6}(\xi_{i+6}\xi_{i+61}-\xi_{i+55}\xi_{i+12})(-4\hat{\mu}^{\mathrm{l}})\\
&+\tfrac{2}{\mu_3}\textstyle\sum_{i=1}^{8}\xi_{i+32}\xi_{i+81}(4\hat{\mu}^{\mathrm{l}})\\
&+\sqrt{6}\tfrac{2}{\mu_3}(\xi_{44}\xi_{47}+\xi_{45}\xi_{48}+\xi_{46}\xi_{49}+\xi_{93}\xi_{96}+\xi_{94}\xi_{97}+\xi_{95}\xi_{98})(\tfrac{2}{3}\hat{\mu}^{\mathrm{j}})\\
&+\sqrt{6}\tfrac{2}{\mu_3}(\xi_{44}\xi_{96}+\xi_{45}\xi_{97}+\xi_{46}\xi_{98}-\xi_{93}\xi_{47}-\xi_{94}\xi_{48}-\xi_{95}\xi_{49})(\tfrac{2}{3}\hat{\mu}^{\mathrm{k}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{6}(\xi_{i+18}\upsilon_{i+29}+\xi_{i+67}\upsilon_{i+74})(-\breve{\mu}^{\mathrm{m}}-2\hat{\mu}^{\mathrm{p}}-2\hat{\mu}^{\mathrm{q}}-2\hat{\mu}^{\mathrm{u}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{6}(\xi_{i+18}\upsilon_{i+74}-\xi_{i+67}\upsilon_{i+29})(-\breve{\mu}^{\mathrm{n}}-2\hat{\mu}^{\mathrm{r}}+2\hat{\mu}^{\mathrm{s}}+2\hat{\mu}^{\mathrm{t}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{8}(\xi_{i+24}\upsilon_{i+18}+\xi_{i+73}\upsilon_{i+63})(\breve{\mu}^{\mathrm{g}}-\breve{\mu}^{\mathrm{m}}-2\hat{\mu}^{\mathrm{p}}-2\hat{\mu}^{\mathrm{q}}-2\hat{\mu}^{\mathrm{u}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{8}(\xi_{i+24}\upsilon_{i+63}-\xi_{i+73}\upsilon_{i+18})(-\breve{\mu}^{\mathrm{h}}-\breve{\mu}^{\mathrm{n}}-2\hat{\mu}^{\mathrm{r}}+2\hat{\mu}^{\mathrm{s}}+2\hat{\mu}^{\mathrm{t}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{8}(\xi_{i+32}\upsilon_{i}+\xi_{i+81}\upsilon_{i+45})(-\breve{\mu}^{\mathrm{g}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{8}(\xi_{i+32}\upsilon_{i+45}-\xi_{i+81}\upsilon_{i})(\breve{\mu}^{\mathrm{h}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{41}\upsilon_{15}+\xi_{42}\upsilon_{16}+\xi_{43}\upsilon_{17}+\xi_{90}\upsilon_{60}+\xi_{91}\upsilon_{61}+\xi_{92}\upsilon_{62})(\tfrac{2}{3}\hat{\mu}^{\mathrm{c}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{d}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{e}}-4\hat{\mu}^{\mathrm{m}}+\tfrac{4}{3}\hat{\mu}^{\mathrm{n}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{41}\upsilon_{60}+\xi_{42}\upsilon_{61}+\xi_{43}\upsilon_{62}-\xi_{90}\upsilon_{15}-\xi_{91}\upsilon_{16}-\xi_{92}\upsilon_{17})(\tfrac{2}{3}\hat{\mu}^{\mathrm{f}}+\tfrac{4}{3}\hat{\mu}^{\mathrm{o}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{41}\upsilon_{36}+\xi_{42}\upsilon_{37}+\xi_{43}\upsilon_{38}+\xi_{90}\upsilon_{81}+\xi_{91}\upsilon_{82}+\xi_{92}\upsilon_{83})(-\breve{\mu}^{\mathrm{g}}+\tfrac{4}{3}\breve{\mu}^{\mathrm{j}}-\tfrac{1}{3}\breve{\mu}^{\mathrm{m}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{41}\upsilon_{81}+\xi_{42}\upsilon_{82}+\xi_{43}\upsilon_{83}-\xi_{90}\upsilon_{36}-\xi_{91}\upsilon_{37}-\xi_{92}\upsilon_{38})(-\breve{\mu}^{\mathrm{h}}+\tfrac{4}{3}\breve{\mu}^{\mathrm{k}}+\tfrac{1}{3}\breve{\mu}^{\mathrm{n}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{41}\upsilon_{42}+\xi_{42}\upsilon_{43}+\xi_{43}\upsilon_{44}+\xi_{90}\upsilon_{87}+\xi_{91}\upsilon_{88}+\xi_{92}\upsilon_{89})\\
&\qquad(-\tfrac{8}{3}\breve{\mu}^{\mathrm{i}}-\tfrac{1}{3}\breve{\mu}^{\mathrm{l}}+2\hat{\mu}^{\mathrm{c}}-\tfrac{2}{9}\hat{\mu}^{\mathrm{d}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{e}}-\tfrac{4}{3}\hat{\mu}^{\mathrm{m}}-\tfrac{4}{3}\hat{\mu}^{\mathrm{n}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{41}\upsilon_{87}+\xi_{42}\upsilon_{88}+\xi_{43}\upsilon_{89}-\xi_{90}\upsilon_{42}-\xi_{91}\upsilon_{43}-\xi_{92}\upsilon_{44})(\tfrac{2}{3}\hat{\mu}^{\mathrm{f}}+\tfrac{4}{3}\hat{\mu}^{\mathrm{o}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{44}\upsilon_{9}+\xi_{45}\upsilon_{10}+\xi_{46}\upsilon_{11}+\xi_{93}\upsilon_{54}+\xi_{94}\upsilon_{55}+\xi_{95}\upsilon_{56})(-\tfrac{1}{3}\hat{\mu}^{\mathrm{c}}-\tfrac{5}{9}\hat{\mu}^{\mathrm{d}}-\tfrac{1}{3}\hat{\mu}^{\mathrm{e}}-\tfrac{4}{3}\hat{\mu}^{\mathrm{m}}+\tfrac{4}{3}\hat{\mu}^{\mathrm{n}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{44}\upsilon_{54}+\xi_{45}\upsilon_{55}+\xi_{46}\upsilon_{56}-\xi_{93}\upsilon_{9}-\xi_{94}\upsilon_{10}-\xi_{95}\upsilon_{11})(-\tfrac{1}{3}\hat{\mu}^{\mathrm{f}}+\tfrac{4}{3}\hat{\mu}^{\mathrm{o}})\\
&+\sqrt{2}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{44}\upsilon_{12}+\xi_{45}\upsilon_{13}+\xi_{46}\upsilon_{14}+\xi_{93}\upsilon_{57}+\xi_{94}\upsilon_{58}+\xi_{95}\upsilon_{59})(-\tfrac{1}{3}\hat{\mu}^{\mathrm{c}}+\tfrac{1}{3}\hat{\mu}^{\mathrm{d}}-\tfrac{1}{3}\hat{\mu}^{\mathrm{e}}+2\hat{\mu}^{\mathrm{m}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{n}})\\
&+\sqrt{2}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{44}\upsilon_{57}+\xi_{45}\upsilon_{58}+\xi_{46}\upsilon_{59}-\xi_{93}\upsilon_{12}-\xi_{94}\upsilon_{13}-\xi_{95}\upsilon_{14})(-\tfrac{1}{3}\hat{\mu}^{\mathrm{f}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{o}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{44}\upsilon_{39}+\xi_{45}\upsilon_{40}+\xi_{46}\upsilon_{41}+\xi_{93}\upsilon_{84}+\xi_{94}\upsilon_{85}+\xi_{95}\upsilon_{86})(-\breve{\mu}^{\mathrm{g}}-6\hat{\mu}^{\mathrm{p}}+2\hat{\mu}^{\mathrm{q}}+2\hat{\mu}^{\mathrm{u}})
\end{aligned}
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\begin{aligned}
&+\frac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{44}\upsilon_{84}+\xi_{45}\upsilon_{85}+\xi_{46}\upsilon_{86}-\xi_{93}\upsilon_{39}-\xi_{94}\upsilon_{40}-\xi_{95}\upsilon_{41})(-\breve{\mu}^{\mathrm{h}}-2\hat{\mu}^{\mathrm{r}}-6\hat{\mu}^{\mathrm{s}}+2\hat{\mu}^{\mathrm{t}})\\
&+\sqrt{6}\frac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{47}\upsilon_{9}+\xi_{48}\upsilon_{10}+\xi_{49}\upsilon_{11}+\xi_{96}\upsilon_{54}+\xi_{97}\upsilon_{55}+\xi_{98}\upsilon_{56})\\
&\qquad(-\tfrac{2}{3}\breve{\mu}^{\mathrm{g}}+\tfrac{2}{9}\breve{\mu}^{\mathrm{j}}+\tfrac{1}{9}\breve{\mu}^{\mathrm{m}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{p}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{q}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{u}})\\
&+\sqrt{6}\frac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{47}\upsilon_{54}+\xi_{48}\upsilon_{55}+\xi_{49}\upsilon_{56}-\xi_{96}\upsilon_{9}-\xi_{97}\upsilon_{10}-\xi_{98}\upsilon_{11})\\
&\qquad(\tfrac{2}{3}\breve{\mu}^{\mathrm{h}}-\tfrac{2}{9}\breve{\mu}^{\mathrm{k}}+\tfrac{1}{9}\breve{\mu}^{\mathrm{n}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{r}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{s}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{t}})\\
&+\sqrt{3}\frac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{47}\upsilon_{12}+\xi_{48}\upsilon_{13}+\xi_{49}\upsilon_{14}+\xi_{96}\upsilon_{57}+\xi_{97}\upsilon_{58}+\xi_{98}\upsilon_{59})(\tfrac{4}{9}\breve{\mu}^{\mathrm{j}}-\tfrac{1}{9}\breve{\mu}^{\mathrm{m}}+2\hat{\mu}^{\mathrm{p}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{q}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{u}})\\
&+\sqrt{3}\frac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{47}\upsilon_{57}+\xi_{48}\upsilon_{58}+\xi_{49}\upsilon_{59}-\xi_{96}\upsilon_{12}-\xi_{97}\upsilon_{13}-\xi_{98}\upsilon_{14})(-\tfrac{4}{9}\breve{\mu}^{\mathrm{k}}-\tfrac{1}{9}\breve{\mu}^{\mathrm{n}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{r}}-2\hat{\mu}^{\mathrm{s}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{t}})\\
&+\sqrt{6}\frac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{47}\upsilon_{39}+\xi_{48}\upsilon_{40}+\xi_{49}\upsilon_{41}+\xi_{96}\upsilon_{84}+\xi_{97}\upsilon_{85}+\xi_{98}\upsilon_{86})(-2\hat{\mu}^{\mathrm{m}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{n}})\\
&+\sqrt{6}\frac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{47}\upsilon_{84}+\xi_{48}\upsilon_{85}+\xi_{49}\upsilon_{86}-\xi_{96}\upsilon_{39}-\xi_{97}\upsilon_{40}-\xi_{98}\upsilon_{41})(-\tfrac{2}{3}\hat{\mu}^{\mathrm{o}})\\
&+\frac{1}{\mu_2}\textstyle\sum_{i=1}^{8}\upsilon_i^2(\mu^{\mathrm{h}}+\frac{1}{8}\mu^{\mathrm{i}}+\frac{1}{8}\mu^{\mathrm{j}}+\frac{9}{8}\mu^{\mathrm{k}}+\frac{9}{8}\mu^{\mathrm{l}}-\frac{1}{8}\mu^{\mathrm{m}}-\frac{3}{8}\mu^{\mathrm{n}}+\frac{3}{4}\mu^{\mathrm{p}}-\frac{3}{8}\mu^{\mathrm{r}}-\frac{9}{8}\mu^{\mathrm{t}}\\
&\qquad+8\tilde{\mu}^{\mathrm{b}}+\tfrac{8}{9}\tilde{\mu}^{\mathrm{c}}+8\tilde{\mu}^{\mathrm{d}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{j}}+8\tilde{\mu}^{\mathrm{k}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{m}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{p}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{s}})\\
&+\frac{1}{\mu_2}\textstyle\sum_{i=46}^{53}\upsilon_i^2(\mu^{\mathrm{h}}+\frac{1}{8}\mu^{\mathrm{i}}+\frac{1}{8}\mu^{\mathrm{j}}+\frac{9}{8}\mu^{\mathrm{k}}+\frac{9}{8}\mu^{\mathrm{l}}-\frac{1}{8}\mu^{\mathrm{m}}-\frac{3}{8}\mu^{\mathrm{n}}+\frac{3}{4}\mu^{\mathrm{p}}-\frac{3}{8}\mu^{\mathrm{r}}-\frac{9}{8}\mu^{\mathrm{t}}+8\tilde{\mu}^{\mathrm{e}}+\frac{16}{3}\tilde{\mu}^{\mathrm{r}})\\
&+\frac{1}{\mu_2}\textstyle\sum_{i=1}^{8}\upsilon_i\upsilon_{i+45}(8\tilde{\mu}^{\mathrm{l}}+\frac{8}{3}\tilde{\mu}^{\mathrm{n}}+8\tilde{\mu}^{\mathrm{o}}+\frac{16}{3}\tilde{\mu}^{\mathrm{t}}+\frac{16}{3}\tilde{\mu}^{\mathrm{u}})\\
&+\frac{1}{\mu_2}(\textstyle\sum_{i=9}^{11}\upsilon_i^2+\sum_{i=54}^{56}\upsilon_i^2)(\mu^{\mathrm{h}}+\frac{1}{8}\mu^{\mathrm{i}}+\frac{1}{8}\mu^{\mathrm{j}}+\frac{29}{8}\mu^{\mathrm{k}}+\frac{5}{8}\mu^{\mathrm{l}}-\frac{1}{8}\mu^{\mathrm{m}}-\frac{5}{8}\mu^{\mathrm{n}}+\frac{3}{4}\mu^{\mathrm{p}}-\frac{1}{8}\mu^{\mathrm{r}}-\frac{9}{8}\mu^{\mathrm{t}}\\
&\qquad+\tfrac{1}{2}\mu^{\mathrm{u}}+\tfrac{1}{6}\mu^{\mathrm{v}}+\tfrac{4}{3}\mu^{\mathrm{w}}+\tilde{\mu}^{\mathrm{b}}+\tfrac{25}{9}\tilde{\mu}^{\mathrm{c}}+\tilde{\mu}^{\mathrm{d}}+\tilde{\mu}^{\mathrm{e}}+\tfrac{5}{3}\tilde{\mu}^{\mathrm{j}}+\tilde{\mu}^{\mathrm{k}}+\tfrac{5}{3}\tilde{\mu}^{\mathrm{m}}\\
&\qquad+\tfrac{8}{3}\tilde{\mu}^{\mathrm{p}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{r}}-\tfrac{8}{3}\tilde{\mu}^{\mathrm{s}})\\
&+\frac{1}{\mu_2}(\textstyle\sum_{i=12}^{14}\upsilon_i^2+\sum_{i=57}^{59}\upsilon_i^2)(\mu^{\mathrm{h}}+\frac{1}{8}\mu^{\mathrm{i}}+\frac{1}{8}\mu^{\mathrm{j}}+\frac{9}{8}\mu^{\mathrm{k}}+\frac{1}{8}\mu^{\mathrm{l}}-\frac{1}{8}\mu^{\mathrm{m}}-\frac{3}{8}\mu^{\mathrm{n}}+\frac{1}{4}\mu^{\mathrm{p}}+\frac{1}{8}\mu^{\mathrm{r}}+\frac{3}{8}\mu^{\mathrm{t}}+\mu^{\mathrm{u}}\\
&\qquad+\tfrac{1}{3}\mu^{\mathrm{v}}+\tfrac{2}{3}\mu^{\mathrm{w}}+2\tilde{\mu}^{\mathrm{b}}+2\tilde{\mu}^{\mathrm{c}}+2\tilde{\mu}^{\mathrm{d}}+2\tilde{\mu}^{\mathrm{e}}-2\tilde{\mu}^{\mathrm{j}}+2\tilde{\mu}^{\mathrm{k}}-2\tilde{\mu}^{\mathrm{m}}\\
&\qquad+12\tilde{\mu}^{\mathrm{p}}+\tfrac{4}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{4}{3}\tilde{\mu}^{\mathrm{r}}-4\tilde{\mu}^{\mathrm{s}})\\
&+\frac{1}{\mu_2}(\textstyle\sum_{i=15}^{17}\upsilon_i^2+\sum_{i=60}^{62}\upsilon_i^2)(\mu^{\mathrm{h}}+\frac{1}{8}\mu^{\mathrm{i}}+\frac{1}{8}\mu^{\mathrm{j}}+\frac{9}{8}\mu^{\mathrm{k}}+\frac{1}{8}\mu^{\mathrm{l}}-\frac{1}{8}\mu^{\mathrm{m}}-\frac{3}{8}\mu^{\mathrm{n}}+\frac{1}{4}\mu^{\mathrm{p}}+\frac{1}{8}\mu^{\mathrm{r}}+\frac{3}{8}\mu^{\mathrm{t}}\\
&\qquad+4\tilde{\mu}^{\mathrm{b}}+4\tilde{\mu}^{\mathrm{c}}+4\tilde{\mu}^{\mathrm{d}}+4\tilde{\mu}^{\mathrm{e}}-4\tilde{\mu}^{\mathrm{j}}+4\tilde{\mu}^{\mathrm{k}}-4\tilde{\mu}^{\mathrm{m}}+24\tilde{\mu}^{\mathrm{p}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{r}}-8\tilde{\mu}^{\mathrm{s}})\\
&+\frac{1}{\mu_2}(\upsilon_{18}^2+\upsilon_{63}^2)(\tfrac{3}{32}\breve{\mu}^{\mathrm{l}}+\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{3}{8}\mu^{\mathrm{k}}+\tfrac{3}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}-\tfrac{3}{8}\mu^{\mathrm{t}}+\tfrac{3}{2}\mu^{\mathrm{u}}+\tfrac{1}{2}\mu^{\mathrm{v}}\\
&\qquad+27\tilde{\mu}^{\mathrm{b}}+\tfrac{1}{3}\tilde{\mu}^{\mathrm{c}}+3\tilde{\mu}^{\mathrm{d}}+3\tilde{\mu}^{\mathrm{e}}-3\tilde{\mu}^{\mathrm{j}}-9\tilde{\mu}^{\mathrm{k}}+\tilde{\mu}^{\mathrm{m}})\\
&+\frac{1}{\mu_2}(\textstyle\sum_{i=19}^{26}\upsilon_i^2+\sum_{i=64}^{71}\upsilon_i^2)(\mu^{\mathrm{h}}+\frac{1}{8}\mu^{\mathrm{i}}+\frac{1}{8}\mu^{\mathrm{j}}+\frac{9}{8}\mu^{\mathrm{k}}+\frac{9}{8}\mu^{\mathrm{l}}-\frac{1}{8}\mu^{\mathrm{m}}-\frac{3}{8}\mu^{\mathrm{n}}+\frac{3}{4}\mu^{\mathrm{p}}-\frac{3}{8}\mu^{\mathrm{r}}-\frac{9}{8}\mu^{\mathrm{t}}+2\mu^{\mathrm{w}}\\
&\qquad+4\tilde{\mu}^{\mathrm{p}}+4\tilde{\mu}^{\mathrm{q}}+4\tilde{\mu}^{\mathrm{r}}+4\tilde{\mu}^{\mathrm{s}})\\
&+\frac{1}{\mu_2}(\textstyle\sum_{i=27}^{29}\upsilon_i^2+\sum_{i=72}^{74}\upsilon_i^2)(\mu^{\mathrm{h}}+\frac{1}{8}\mu^{\mathrm{i}}+\frac{1}{8}\mu^{\mathrm{j}}+\frac{9}{8}\mu^{\mathrm{k}}+\frac{1}{8}\mu^{\mathrm{l}}-\frac{1}{8}\mu^{\mathrm{m}}-\frac{3}{8}\mu^{\mathrm{n}}+\frac{1}{4}\mu^{\mathrm{p}}\\
&\qquad+\tfrac{1}{8}\mu^{\mathrm{r}}+\tfrac{3}{8}\mu^{\mathrm{t}}+2\mu^{\mathrm{u}}+\tfrac{2}{3}\mu^{\mathrm{v}}+\tfrac{4}{3}\mu^{\mathrm{w}})\\
&+\frac{1}{\mu_2}(\textstyle\sum_{i=30}^{35}\upsilon_i^2+\sum_{i=75}^{80}\upsilon_i^2)(\mu^{\mathrm{h}}+\frac{1}{8}\mu^{\mathrm{i}}+\frac{1}{8}\mu^{\mathrm{j}}+\frac{49}{8}\mu^{\mathrm{k}}+\frac{9}{8}\mu^{\mathrm{l}}-\frac{1}{8}\mu^{\mathrm{m}}-\frac{7}{8}\mu^{\mathrm{n}}+\frac{5}{4}\mu^{\mathrm{p}}-\frac{3}{8}\mu^{\mathrm{r}}-\frac{21}{8}\mu^{\mathrm{t}}\\
&\qquad+2\mu^{\mathrm{w}}+4\tilde{\mu}^{\mathrm{p}}+4\tilde{\mu}^{\mathrm{q}}+4\tilde{\mu}^{\mathrm{r}}+4\tilde{\mu}^{\mathrm{s}})\\
&+\frac{1}{\mu_2}(\textstyle\sum_{i=36}^{38}\upsilon_i^2+\sum_{i=82}^{84}\upsilon_i^2)(\mu^{\mathrm{h}}+\frac{1}{8}\mu^{\mathrm{i}}+\frac{1}{8}\mu^{\mathrm{j}}+\frac{49}{8}\mu^{\mathrm{k}}+\frac{1}{8}\mu^{\mathrm{l}}-\frac{1}{8}\mu^{\mathrm{m}}-\frac{7}{8}\mu^{\mathrm{n}}\\
&\qquad+\tfrac{3}{4}\mu^{\mathrm{p}}+\tfrac{1}{8}\mu^{\mathrm{r}}+\tfrac{7}{8}\mu^{\mathrm{t}}+2\mu^{\mathrm{u}}+\tfrac{2}{3}\mu^{\mathrm{v}}+\tfrac{4}{3}\mu^{\mathrm{w}})\\
&+\frac{1}{\mu_2}(\textstyle\sum_{i=39}^{41}\upsilon_i^2+\sum_{i=84}^{86}\upsilon_i^2)(\mu^{\mathrm{h}}+\frac{1}{8}\mu^{\mathrm{i}}+\frac{1}{8}\mu^{\mathrm{j}}+\frac{49}{8}\mu^{\mathrm{k}}+\frac{1}{8}\mu^{\mathrm{l}}-\frac{1}{8}\mu^{\mathrm{m}}-\frac{7}{8}\mu^{\mathrm{n}}+\frac{3}{4}\mu^{\mathrm{p}}+\frac{1}{8}\mu^{\mathrm{r}}+\frac{7}{8}\mu^{\mathrm{t}}\\
&\qquad+2\mu^{\mathrm{w}}+36\tilde{\mu}^{\mathrm{p}}+4\tilde{\mu}^{\mathrm{q}}+4\tilde{\mu}^{\mathrm{r}}-12\tilde{\mu}^{\mathrm{s}})
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{1}{\mu_2}(\textstyle\sum_{i=42}^{44} \upsilon_i^2+\sum_{i=87}^{89} \upsilon_i^2)(\tfrac{2}{3}\breve{\mu}^{\mathrm{i}}+\tfrac{1}{12}\breve{\mu}^{\mathrm{l}}+\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{1}{8}\mu^{\mathrm{k}}+\tfrac{9}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}+\tfrac{1}{8}\mu^{\mathrm{n}}\\
&\qquad\qquad+\tfrac{1}{4}\mu^{\mathrm{p}}-\tfrac{3}{8}\mu^{\mathrm{r}}+\tfrac{3}{8}\mu^{\mathrm{t}}+36\tilde{\mu}^{\mathrm{b}}+\tfrac{4}{9}\tilde{\mu}^{\mathrm{c}}+4\tilde{\mu}^{\mathrm{d}}+4\tilde{\mu}^{\mathrm{e}}-4\tilde{\mu}^{\mathrm{j}}\\
&\qquad\qquad-12\tilde{\mu}^{\mathrm{k}}+\tfrac{4}{3}\tilde{\mu}^{\mathrm{m}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{p}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{r}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{s}})\\
&+\sqrt{2}\tfrac{1}{\mu_2}(\upsilon_9\upsilon_{12}+\upsilon_{10}\upsilon_{13}+\upsilon_{11}\upsilon_{14}+\upsilon_{54}\upsilon_{57}+\upsilon_{55}\upsilon_{58}+\upsilon_{56}\upsilon_{59})\\
&\qquad(\mu^{\mathrm{u}}+\tfrac{1}{3}\mu^{\mathrm{v}}-\tfrac{4}{3}\mu^{\mathrm{w}}+2\tilde{\mu}^{\mathrm{b}}-\tfrac{10}{3}\tilde{\mu}^{\mathrm{c}}+2\tilde{\mu}^{\mathrm{d}}+2\tilde{\mu}^{\mathrm{e}}+\tfrac{2}{3}\tilde{\mu}^{\mathrm{j}}+2\tilde{\mu}^{\mathrm{k}}+\tfrac{2}{3}\tilde{\mu}^{\mathrm{m}}-8\tilde{\mu}^{\mathrm{p}}-\tfrac{8}{3}\tilde{\mu}^{\mathrm{q}}-\tfrac{8}{3}\tilde{\mu}^{\mathrm{r}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{s}})\\
&+\sqrt{2}\tfrac{1}{\mu_2}(\upsilon_9\upsilon_{57}+\upsilon_{10}\upsilon_{58}+\upsilon_{11}\upsilon_{59}-\upsilon_{12}\upsilon_{54}-\upsilon_{13}\upsilon_{55}-\upsilon_{14}\upsilon_{56})(\tfrac{8}{3}\tilde{\mu}^{\mathrm{n}}-\tfrac{8}{3}\tilde{\mu}^{\mathrm{t}})\\
&+\tfrac{1}{\mu_2}(\upsilon_{15}\upsilon_{42}+\upsilon_{16}\upsilon_{43}+\upsilon_{17}\upsilon_{54}+\upsilon_{60}\upsilon_{87}+\upsilon_{61}\upsilon_{88}+\upsilon_{62}\upsilon_{89})\\
&\qquad(24\tilde{\mu}^{\mathrm{b}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{c}}-8\tilde{\mu}^{\mathrm{d}}+8\tilde{\mu}^{\mathrm{e}}-\tfrac{40}{3}\tilde{\mu}^{\mathrm{j}}+8\tilde{\mu}^{\mathrm{k}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{m}}+16\tilde{\mu}^{\mathrm{p}}-\tfrac{16}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{r}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{s}})\\
&+\tfrac{1}{\mu_2}(\upsilon_{15}\upsilon_{87}+\upsilon_{16}\upsilon_{88}+\upsilon_{17}\upsilon_{89}-\upsilon_{60}\upsilon_{42}-\upsilon_{61}\upsilon_{43}-\upsilon_{62}\upsilon_{44})\\
&\qquad(-8\tilde{\mu}^{\mathrm{l}}-\tfrac{8}{3}\tilde{\mu}^{\mathrm{n}}+8\tilde{\mu}^{\mathrm{o}}-\tfrac{16}{3}\tilde{\mu}^{\mathrm{t}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{u}})\\
&+\tfrac{1}{\mu_2}(\upsilon_{36}\upsilon_{42}+\upsilon_{37}\upsilon_{43}+\upsilon_{38}\upsilon_{54}+\upsilon_{81}\upsilon_{87}+\upsilon_{82}\upsilon_{88}+\upsilon_{83}\upsilon_{89})(-\tfrac{2}{3}\breve{\mu}^{\mathrm{j}}+\tfrac{1}{6}\breve{\mu}^{\mathrm{m}})\\
&+\tfrac{1}{\mu_2}(\upsilon_{36}\upsilon_{87}+\upsilon_{37}\upsilon_{88}+\upsilon_{38}\upsilon_{89}-\upsilon_{81}\upsilon_{42}-\upsilon_{82}\upsilon_{43}-\upsilon_{83}\upsilon_{44})(\tfrac{2}{3}\breve{\mu}^{\mathrm{k}}+\tfrac{1}{6}\breve{\mu}^{\mathrm{n}})\Big\}\;.
\end{aligned}
$$

7.2 The Parameters Occurring in the Higgs Potential

In this subsection we will compute the parameters $\mu^i, \tilde{\mu}^i, \breve{\mu}^i$ and $\hat{\mu}^i$ of the Higgs potential, which depend according to (6.10f), (6.10h), (6.10g) *and* (7.2) *on the mass matrices occurring in the generalized Dirac operator $\mathcal{M}$.*

7.2.1 The Mass Matrices M_u, M_d, M_n

We have found in Section 6.4 that the eigenvalues

$$
\begin{aligned}
&\text{of } M_u M_u^* \text{ are } m_u^2, m_c^2, m_t^2\,, \qquad\qquad \text{of } M_d M_d^* \text{ are } m_d^2, m_s^2, m_b^2\,, \\
&\text{of } M_e M_e^* \text{ are } m_e^2, m_\mu^2, m_\tau^2\,,
\end{aligned}
\tag{7.3}
$$

referring to the usual names of the fermions. By unitary transformations we can achieve that M_u is diagonal,

$$M_u = \mathrm{diag}(m_u, m_c, m_t)\;. \tag{7.4a}$$

It is necessary to make several assumptions to simplify the calculation: Since the Kobayashi–Maskawa matrix between M_u and M_d is approximately the identity matrix, let us assume[19]

$$M_d = \mathrm{diag}(m_d, m_s, m_b)\;. \tag{7.4b}$$

[19]For the determination of the structure of the connection form we have used that the Kobayashi–Maskawa matrix is non–trivial. Now, for the computation of traces, it is indeed possible to neglect small matrix elements.

The experimental data show that m_t is much bigger than all other eigenvalues. Among the remaining eigenvalues we neglect all but m_b^2 and m_τ^2. For simplicity we also neglect m_τ^2 against m_b^2, although this is not completely justified. Unfortunately, there are no experimental values for the matrix M_n. Therefore, we can only estimate its contribution: We assume that in the case (7.4a) we have

$$M_n = \mathrm{diag}(0,0,\mathrm{e}^{\mathrm{i}\chi} m_n) \,. \tag{7.4c}$$

Quantum corrections suggest that m_n is of the order m_t. Thus, we get from (5.44b)

$$\begin{aligned}
&M'_{\tilde u} M'^{\,*}_{\tilde u} = M'^{\,*}_{\tilde u} M'_{\tilde u} = \tfrac{1}{16}(9m_t^2 + 6m_t m_n \cos\chi + m_n^2)\,\mathrm{diag}(0,0,1,0,0,1)\,,\\
&M'_{\tilde n} M'^{\,*}_{\tilde n} = M'^{\,*}_{\tilde n} M'_{\tilde n} = \tfrac{1}{16}(m_t^2 - 2m_t m_n \cos\chi + m_n^2)\,\mathrm{diag}(0,0,1,0,0,1)\,,\\
&\tfrac{1}{2}(M'_{\tilde u} M'^{\,*}_{\tilde n} + M'_{\tilde n} M'^{\,*}_{\tilde u}) = \tfrac{1}{2}(M'^{\,*}_{\tilde n} M'_{\tilde u} + M'^{\,*}_{\tilde u} M'_{\tilde n})\\
&\qquad\qquad\qquad\qquad\quad = \tfrac{1}{16}(3m_t^2 - 2m_t m_n \cos\chi - m_n^2)\,\mathrm{diag}(0,0,1,0,0,1)\,,\\
&\tfrac{1}{2\mathrm{i}}(M'_{\tilde u} M'^{\,*}_{\tilde n} - M'_{\tilde n} M'^{\,*}_{\tilde u}) = \tfrac{1}{2\mathrm{i}}(M'^{\,*}_{\tilde n} M'_{\tilde u} - M'^{\,*}_{\tilde u} M'_{\tilde n}) = \tfrac{1}{4} m_t m_n \sin\chi\,\mathrm{diag}(0,0,1,0,0,1)\,.
\end{aligned} \tag{7.5}$$

Inserting these matrices into formulae (6.10e) we get approximately

$$\begin{aligned}
\mu_1 &= \tfrac{1}{8} m_b^2 + \tfrac{1}{96}(9m_t^2 + 6m_t m_n \cos\chi + m_n^2) + \tfrac{1}{24} m_\tau^2\,,\\
\mu_2 &= \tfrac{1}{384}(m_t^2 - 2m_t m_n \cos\chi + m_n^2)\,,
\end{aligned} \tag{7.6}$$

which yields according to (6.43a) for the mass m_W of the W boson

$$m_W^2 = \tfrac{1}{4}(m_t^2 + m_b^2 + \tfrac{1}{3} m_n^2 + \tfrac{1}{3} m_\tau^2)\,. \tag{7.7}$$

The comparison with the experimental values for m_t and m_W requires that m_n is small against m_t. Thus, we shall neglect m_n against m_t whenever this is possible.

7.2.2 The Mass Matrices M_N, M_{10}, M_5

Since at energies accessible at present the standard model is in excellent agreement with experiments, the parameter $\mu_3 \sim \mathrm{tr}(M_N M_N^*)$ must be very large, see Sections 6.3 and 6.4. We choose the parametrization

$$M_N = m_N U\,\mathrm{diag}(\sin\theta_N \cos\phi_N\,,\ \sin\theta_N \sin\phi_N\,,\ \cos\theta_N) U^T\,, \qquad U \in \mathrm{U}(3)\,, \tag{7.8}$$

where $m_N \gg m_t$ determines the mass scale.

The mass of the X and Y bosons must be very large in order to suppress the proton decay. This could be achieved by a sufficiently large μ_3, however, there are also Higgs bosons which induce an insufficient lifetime for the proton if μ_0 is too small. Therefore, we must demand

$$\max(\mathrm{tr}(M_{10} M_{10}^*)\,,\ \mathrm{tr}(M_5 M_5^*)) \gg \mathrm{tr}(M_u M_u^*)\,. \tag{7.9}$$

We put[20]

$$M_{10} = M\mathbb{1}_3 + m_{10}\ , \quad M_5 = M\mathbb{1}_3 + m_5\ , \quad M \in \mathbb{R}\ , \tag{7.10}$$

where $m_{10}, m_5 \in \mathrm{M}_3\mathbb{C}$ are perturbations, which we consider for the time being as small against $M\mathbb{1}_3$. Thus, we obtain for (6.10e) approximately

$$\mu_0 = \tfrac{1}{4}M^2\ , \qquad \mu_1 = \tfrac{3}{32}m_t^2\ , \qquad \mu_2 = \tfrac{1}{384}m_t^2\ , \qquad \mu_3 = \tfrac{1}{48}m_N^2\ . \tag{7.11}$$

The parameters μ^{g} to μ^{t} take in leading order the form

$$\begin{aligned}
&\mu^{\mathrm{g}} \to 12M^2m_b^2\ , && \breve{\mu}^{\mathrm{e}} \to 2M^2m_N^2\ ,\\
&\mu^{\mathrm{i}} = \mu^{\mathrm{j}} = \tfrac{1}{2}\mu^{\mathrm{m}} \to \tfrac{9}{4}M^2m_t^2\ , && \mu^{\mathrm{k}} = \mu^{\mathrm{l}} = \tfrac{1}{2}\mu^{\mathrm{t}} \to \tfrac{1}{4}M^2m_t^2\ ,\\
&\mu^{\mathrm{n}} = \mu^{\mathrm{p}} = \mu^{\mathrm{r}} \to \tfrac{3}{2}M^2m_t^2\ , && \mu^{\mathrm{o}} = \mu^{\mathrm{q}} = \mu^{\mathrm{s}} \to 2M^2m_tm_n\sin\chi\ .
\end{aligned} \tag{7.12}$$

Inserting this leading approximation into the quadratic terms (7.2) of the Higgs potential, we can distinguish two groups. Terms of the first group receive a contribution from (7.12), whereas terms of second group do not. One finds that the terms of the second group contain the following combinations of μ^{i} to μ^{t} (or linear combinations of (7.13a)):

$$\begin{aligned}
&\tfrac{1}{4}\mu^{\mathrm{i}}+\tfrac{1}{4}\mu^{\mathrm{j}}+\tfrac{1}{4}\mu^{\mathrm{k}}+\tfrac{1}{4}\mu^{\mathrm{l}}-\tfrac{1}{4}\mu^{\mathrm{m}}+\tfrac{1}{4}\mu^{\mathrm{n}}-\tfrac{1}{2}\mu^{\mathrm{p}}+\tfrac{1}{4}\mu^{\mathrm{r}}-\tfrac{1}{4}\mu^{\mathrm{t}} = \tfrac{1}{2}\,\mathrm{tr}(\tilde{M}_u\tilde{M}_u^*+\hat{M}_u\hat{M}_u^*) =: \tilde{\lambda}_1^2m_t^4\ ,\\
&\tfrac{1}{4}\mu^{\mathrm{i}}+\tfrac{1}{4}\mu^{\mathrm{j}}+\tfrac{9}{4}\mu^{\mathrm{k}}+\tfrac{9}{4}\mu^{\mathrm{l}}-\tfrac{1}{4}\mu^{\mathrm{m}}-\tfrac{3}{4}\mu^{\mathrm{n}}+\tfrac{3}{2}\mu^{\mathrm{p}}-\tfrac{3}{4}\mu^{\mathrm{r}}-\tfrac{9}{4}\mu^{\mathrm{t}} = \tfrac{1}{2}\,\mathrm{tr}(\tilde{M}_n\tilde{M}_n^*+\hat{M}_n\hat{M}_n^*) =: \tilde{\lambda}_2^2m_t^2m_n^2\ ,\\
&\tfrac{1}{2}\mu^{\mathrm{i}}+\tfrac{1}{2}\mu^{\mathrm{j}}-\tfrac{3}{2}\mu^{\mathrm{k}}-\tfrac{3}{2}\mu^{\mathrm{l}}-\tfrac{1}{2}\mu^{\mathrm{m}}-\tfrac{1}{2}\mu^{\mathrm{n}}+\mu^{\mathrm{p}}-\tfrac{1}{2}\mu^{\mathrm{r}}+\tfrac{3}{2}\mu^{\mathrm{t}}\\
&\qquad = \tfrac{1}{2}\,\mathrm{tr}(\tilde{M}_u\tilde{M}_n^*+\hat{M}_u\hat{M}_n^*+\tilde{M}_n\tilde{M}_u^*+\hat{M}_n\hat{M}_u^*) =: 2\tilde{\lambda}_3^2m_t^3m_n\cos\chi\ ,\\
&\mu^{\mathrm{o}}-2\mu^{\mathrm{q}}+\mu^{\mathrm{s}} = \tfrac{1}{2}\mathrm{i}\,\mathrm{tr}(\tilde{M}_u\tilde{M}_n^*+\hat{M}_u\hat{M}_n^*-\tilde{M}_n\tilde{M}_u^*-\hat{M}_n\hat{M}_u^*) =: 2\tilde{\lambda}_4^2m_t^3m_n\sin\chi\ ,
\end{aligned} \tag{7.13a}$$

where

$$\begin{aligned}
&\tilde{M}_u = m_{10}M_u - M_um_5\ , && \hat{M}_u = m_{10}^*M_u - M_um_5^*\ ,\\
&\tilde{M}_n = m_{10}M_n - M_nm_5\ , && \hat{M}_n = m_{10}^*M_n - M_nm_5^*\ ,
\end{aligned} \tag{7.13b}$$

see (5.44a) and (6.10f). Within our assumptions (7.4) we have

$$\tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_3 = \tilde{\lambda}_4 \equiv \lambda\ . \tag{7.13c}$$

In any case, the Higgs fields of the second group receive from the terms μ^{i} to μ^{t} a contribution to their mass not bigger than $\tilde{\lambda}_1^2m_t^2 + \tilde{\lambda}_2^2m_n^2$, up to a factor of the order one. We also need the following linear combinations:

$$\begin{aligned}
|\mu^{\mathrm{n}}-\mu^{\mathrm{p}}| &= 4|\mathrm{Re}(\mathrm{tr}((M'_{10}M'_{\tilde{u}}-M'_{\tilde{u}}M'_5)M'^{\,*}_{\tilde{n}}M'_{10}))|\\
&= 2M|\,\mathrm{tr}((m_{10}+m_{10}^*)(M_{\tilde{u}}M_{\tilde{n}}^*+M_{\tilde{n}}M_{\tilde{u}}^*)-(M_{\tilde{n}}^*M_{\tilde{u}}+M_{\tilde{u}}^*M_{\tilde{n}})(m_5+m_5^*))|\\
&= \tfrac{3}{4}Mm_t|\,\mathrm{tr}((m_{10}+m_{10}^*)M_u-M_u(m_5+m_5^*))|\\
&\le \tfrac{3}{4}\sqrt{3}Mm_t\Big(\sqrt{\mathrm{tr}(\tilde{M}_u\tilde{M}_u^*)}+\sqrt{\mathrm{tr}(\hat{M}_u\hat{M}_u^*)}\Big) \le \tfrac{3}{4}\sqrt{6}Mm_t\sqrt{\mathrm{tr}(\tilde{M}_u\tilde{M}_u^*+\hat{M}_u\hat{M}_u^*)}\\
&= \tfrac{3}{2}\sqrt{3}\lambda Mm_t^3\ ,
\end{aligned}$$

[20] The choice $M_{10} = (M\mathbb{1}_3 + m_{10})\ ,\ \ M_5 = \mathrm{e}^{\mathrm{i}\hat{\chi}}(M\mathbb{1}_3 + m_5)$ yields the same results.

$$\begin{aligned}|\mu^{\mathrm{o}}-\mu^{\mathrm{q}}| &= 4|\mathrm{Im}(\mathrm{tr}((M'_{10}M'_{\tilde{u}}-M'_{\tilde{u}}M'_5)M'^{\,*}_{\tilde{n}}M'_{10}))| \\ &= 2M|\,\mathrm{tr}((m_{10}+m_{10}^*)(M_{\tilde{u}}M_{\tilde{n}}^*-M_{\tilde{n}}M_{\tilde{u}}^*)-(M_{\tilde{n}}^*M_{\tilde{u}}-M_{\tilde{u}}^*M_{\tilde{n}})(m_5+m_5^*))| \\ &= Mm_n|\sin\chi|\,|\mathrm{tr}((m_{10}+m_{10}^*)M_u-M_u(m_5+m_5^*))| \\ &= \tfrac{4m_n}{3m_t}|\sin\chi|\,|\mu^{\mathrm{n}}-\mu^{\mathrm{p}}| \le 2\sqrt{3}\lambda Mm_t^2m_n|\sin\chi|\ .\end{aligned}$$

The approximation is well up to terms of the order $\max(\|m_{10}\|,\|m_5\|)/M$.

The matrices M'_{10} and M'_5 enter the matrices (5.78) only quadratically. Neglecting quadratic terms in m_{10} and m_5 we have

$$M_i^2 = \mathrm{diag}(M^2\mathbb{1}_3+M(m_i+m_i^*),\, M^2\mathbb{1}_3+M(m_i+m_i^*))\ , \quad i=10,\,5\ .$$

Thus, we may assume $m_{10}=m_{10}^*$ and $m_5=m_5^*$. Moreover, we may assume $\mathrm{tr}(m_{10})=0$, because the transformation $m_5\mapsto m_5+\nu\,\mathbb{1}_3$ and $m_{10}\mapsto m_{10}+\nu\,\mathbb{1}_3$, for $\nu\in\mathbb{R}$, leaves the matrices M^i_{aa} and $\hat{M}^i_{aa}$ invariant. Therefore, we make the ansatz ($v_i^j\in\mathbb{R}$, $j\in\{10,5\}$)

$$m_j = \begin{pmatrix} \sqrt{\tfrac13}v_0^j+v_3^j+\sqrt{\tfrac13}v_8^j & v_1^j-\mathrm{i}v_2^j & v_4^j-\mathrm{i}v_5^j \\ v_1^j+\mathrm{i}v_2^j & \sqrt{\tfrac13}v_0^j-v_3^j+\sqrt{\tfrac13}v_8^j & v_6^j-\mathrm{i}v_7^j \\ v_4^j+\mathrm{i}v_5^j & v_6^j+\mathrm{i}v_7^j & \sqrt{\tfrac13}v_0^j-\sqrt{\tfrac43}v_8^j \end{pmatrix},\quad j=10,5\ ,\quad v_0^{10}\equiv 0\ . \tag{7.14}$$

7.2.3 Some Abbreviations

We introduce the abbreviations

$$\begin{aligned}&\cos^4\theta_N+\sin^4\theta_N(\cos^4\phi_N+\sin^4\phi_N)\equiv\tfrac13(1+2\cos^2\chi_N)\ ,\\ &v_{10}^2=2\textstyle\sum_{i=1}^8(v_i^{10})^2\ , \qquad v_5^2=2\textstyle\sum_{i=1}^8(v_i^5)^2\ ,\\ &v_8^{10}=\tfrac{1}{\sqrt2}v_{10}\cos\tilde\chi\ , \qquad v_3^{10}=\tfrac{1}{\sqrt2}v_{10}\sin\tilde\chi\cos\tilde\chi'\ ,\\ &(v_1^{10})^2+(v_2^{10})^2=\tfrac{1}{\sqrt2}v_{10}\sin\tilde\chi\sin\tilde\chi'\cos\tilde\chi''\ .\end{aligned} \tag{7.15a}$$

Then, the inequality $|\mathrm{tr}(AB)|^2\le\mathrm{tr}(A^*A)\,\mathrm{tr}(B^*B)$, for any matrices A,B, yields

$$\begin{aligned}&|\mathrm{tr}(m_{10}M_NM_N^*)|\le\sqrt{\mathrm{tr}(m_{10}^2)\,\mathrm{tr}((M_NM_N^*)^2)}\le m_N^2v_{10}\sqrt{\tfrac13(1+2\cos^2\chi_N)}\ ,\\ &0\le\mathrm{tr}(\mathrm{diag}(0,0,1)M_NM_N^*)\le m_N^2\sqrt{\tfrac13(1+2\cos^2\chi_N)}\ .\end{aligned}$$

To deal with equalities we put

$$\mathrm{tr}(m_{10}M_NM_N^*)=m_N^2v_{10}\sqrt{\tfrac13(1+2\cos^2\chi_N)}\cos\chi'_N\ , \tag{7.15b}$$

$$\mathrm{tr}(\mathrm{diag}(0,0,1)M_NM_N^*)=m_N^2\sqrt{\tfrac13(1+2\cos^2\chi_N)}\cos^2\chi''_N\ . \tag{7.15c}$$

We also need $\mathrm{tr}(\mathrm{diag}(0,0,1)M_N)$, for which we get

$$|\mathrm{tr}(\mathrm{diag}(0,0,1)M_N)|^2\le m_N^2\sqrt{\tfrac13(1+2\cos^2\chi_N)}\cos^2\chi''_N\ ,$$

or in terms of an equality,

$$\mathrm{tr}(\mathrm{diag}(0,0,1)M_N) = m_N \sqrt[4]{\tfrac{1}{3}(1+2\cos^2\chi_N)}\cos\chi''_N \cos\theta'_N \mathrm{e}^{\mathrm{i}\phi'_N} \ . \tag{7.15d}$$

These definitions enable us to compute the matrices $M'_{d\tilde{n}} \equiv \mathrm{diag}(M_{d\tilde{n}}, M_{d\tilde{n}})$ and $M'_{Nu} \equiv \mathrm{diag}(M_{Nu}, M_{Nu})$, see (5.83b):

$$\begin{aligned}
M_{d\tilde{n}} &= \tfrac{1}{4} m_b (m_t - \mathrm{e}^{-\mathrm{i}\chi} m_n) \Big(\mathrm{diag}(0,0,1) - \frac{\cos\theta'_N \mathrm{e}^{-\mathrm{i}\phi'_N}}{\sqrt[4]{\tfrac{1}{3}(1+2\cos^2\chi_N)}\cos\chi''_N} (\mathrm{diag}(0,0,1)\frac{M_N}{m_N}) \Big) \ , \\
M_{Nu} &= \mathrm{diag}(0,0,0) \ .
\end{aligned}$$

7.2.4 The Parameters $\beta_i, \gamma_i, \delta_i, \epsilon_i$

From (5.57a) we obtain approximately

$$\begin{aligned}
M^2_{ud} &\to M'_u M'^*_u - M'_d M'^*_d - \tfrac{1}{12}(m_t^2 - m_b^2)\mathbb{1}_6 \ , \\
M^2_{en} &\to \qquad\qquad\qquad\quad\ - \tfrac{1}{4}(m_t^2 - m_b^2)\mathbb{1}_6 \ , \\
&\Rightarrow \delta_{tb} := \mathrm{tr}(3(M^2_{ud})^2 + (M^2_{en})^2) \to \tfrac{11}{2}(m_t^2 - m_b^2)^2 \ .
\end{aligned} \tag{7.16}$$

This yields for (5.80)

$$\begin{aligned}
\beta_A &= -M^2 - \tfrac{\sqrt{3}}{2} v_0^5 M \ , & \delta_A &= \tfrac{1}{\sqrt{3}\delta_{tb}} M(3v_0^5 + 8v_8^{10})(m_t^2 - m_b^2) \ , \\
\check{\beta}_V &= -\tfrac{1}{64} m_t^2 \ , & \check{\delta}_V &= \tfrac{1}{32\,\delta_{tb}}(m_t^2 - m_b^2)m_t^2 \ , \\
\beta_V &= -\tfrac{1}{64} m_t^2 \ , & \delta_V &= -\tfrac{11}{32\,\delta_{tb}}(m_t^2 - m_b^2)m_t^2 \ , \\
\beta_W &= -\tfrac{1}{16} m_t^2 \ , & \delta_W &= -\tfrac{1}{4\,\delta_{tb}}(m_t^2 - m_b^2)m_t^2 \ , \\
\beta'_W &= -\tfrac{1}{12} m_t m_n \sin\chi \ , & \delta'_W &= -\tfrac{1}{3\,\delta_{tb}}(m_t^2 - m_b^2)m_n m_t \sin\chi \ , \\
\beta_U &= -\tfrac{1}{4} m_N^2 \ , & \delta_U &= \tfrac{1}{2\,\delta_{tb}} m_N^2 (m_t^2 - m_b^2)(1 - 12\sqrt{\tfrac{1}{3}(1+2\cos\chi_N)}\cos^2\chi''_N) \ , \\
\gamma_V &= -\tfrac{1}{48} m_t^2 \ , & \epsilon_V &= -\tfrac{1}{16}\frac{2}{\sqrt{3\,\mathrm{tr}((\tilde{M}_V^2)^2)}}\cos\hat{\chi} m_t^2 \ , \\
\gamma_W &= -\tfrac{1}{16} m_t^2 \ , & \epsilon_W &= -\tfrac{3}{16}\frac{2}{\sqrt{3\,\mathrm{tr}((\tilde{M}_V^2)^2)}}\cos\hat{\chi} m_t^2 \ , \\
\gamma'_W &= -\tfrac{1}{12} m_t m_n \sin^2\chi \ , & \epsilon'_W &= -\tfrac{1}{4}\frac{2}{\sqrt{3\,\mathrm{tr}((\tilde{M}_V^2)^2)}}\cos\hat{\chi} m_t m_n \sin\chi \ , \\
\gamma_U &= -\tfrac{1}{3} m_N^2 \ , & \epsilon_U &= -\frac{2}{\sqrt{3\,\mathrm{tr}((\tilde{M}_V^2)^2)}} m_N^2 \cos\chi_N \cos\hat{\chi}_U \ ,
\end{aligned}$$
$$\begin{aligned}
\tilde{\gamma}_U &= -\tfrac{1}{3} m_N m_b \sqrt[4]{\tfrac{1}{3}(1+2\cos^2\chi_N)}\cos\chi''_N \cos\theta'_N \cos\phi'_N \ , \\
\tilde{\epsilon}_U &= \frac{m_b m_N}{\sqrt{\mathrm{tr}((\tilde{M}_V^2)^2)}} \sqrt[4]{\tfrac{1}{3}(1+2\cos^2\chi_N)}\cos\chi''_N \sqrt{\sin^2\theta'_N + \cos^2\theta'_N \cos^2\phi'_N}\cos\hat{\chi}' \ , \\
\tilde{\gamma}'_U &= -\tfrac{1}{3} m_N m_b \sqrt[4]{\tfrac{1}{3}(1+2\cos^2\chi_N)}\cos\chi''_N \cos\theta'_N \sin\phi'_N \ , \\
\tilde{\epsilon}'_U &= \frac{m_b m_N}{\sqrt{\mathrm{tr}((\tilde{M}_V^2)^2)}} \sqrt[4]{\tfrac{1}{3}(1+2\cos^2\chi_N)}\cos\chi''_N \sqrt{\sin^2\theta'_N + \cos^2\theta'_N \sin^2\phi'_N}\cos\hat{\chi}'' \ .
\end{aligned} \tag{7.17}$$

The result (7.17) is independent of the special form of $\tilde{M}_V^2$. We have in good approximation

$$\tilde{M}_V^2 = \mathrm{diag}\left(2Mm_{10} + \tfrac{1}{2}(M_N M_N^* - \tfrac{1}{3}m_N^2 \mathbb{1}_3), 2Mm_{10} + \tfrac{1}{2}(M_N M_N^* - \tfrac{1}{3}m_N^2 \mathbb{1}_3)\right),$$

see (5.57b) and (5.48a). If we neglect $2Mm_{10}$ against $\frac{1}{2}(M_N M_N^* - \frac{1}{3}m_N^2 \mathbb{1}_3)$ then we get $\epsilon_U = -2$. Moreover, a direct calculation of ϵ_V yields

$$\sqrt{\tfrac{1}{3}(1+2\cos^2\chi_N)}\cos^2\chi_N'' = \tfrac{1}{3}|1+2\cos\chi_N\cos\hat{\chi}| \,. \tag{7.18}$$

7.2.5 The Parameters $\mu^i, \breve{\mu}^i, \tilde{\mu}^i, \hat{\mu}^i$

Now we obtain from (6.10f), (6.10g), (6.10h) and (7.2) the following result:

$$\begin{aligned}
&\mu^{\mathrm{a}} \to \tfrac{8}{5}M^2(\tfrac{9}{2}v_{10}^2 + \tfrac{9}{16}(v_0^5)^2 + v_5^2) =: \tilde{\lambda}^2 M^2 m_t^2\,, \\
&\mu^{\mathrm{b}} \to \tfrac{351}{160}m_t^4 + \tfrac{63}{20}m_b^2 m_t^2 + \tfrac{51}{10}m_b^4\,, \qquad \mu^{\mathrm{c}} \to \tfrac{13}{7680}m_t^4\,, \\
&\mu^{\mathrm{d}} \to -\tfrac{3\sqrt{3}}{5}M((v_0^5 + 16v_8^{10})m_b^2 + (-\tfrac{3}{4}v_0^5 + 4v_8^5 + 6v_8^{10})m_t^2)\,, \\
&\mu^{\mathrm{e}} \to -\tfrac{\sqrt{3}}{80}M(-v_0^5 + \tfrac{16}{3}v_8^5 + 8v_8^{10})m_t^2\,, \qquad \mu^{\mathrm{f}} \to \tfrac{39}{320}m_t^4 + \tfrac{7}{80}m_b^2 m_t^2\,, \\
&\mu^{\mathrm{h}} \to \tfrac{1}{2}m_b^2 v_{10}^2 \sin^2\tilde{\chi}\sin^2\tilde{\chi}'\sin^2\tilde{\chi}''\,, \qquad \mu^{\mathrm{u}} \le \tfrac{1}{4}m_\tau^2 m_t^2\,, \\
&\mu^{\mathrm{v}} \to \tfrac{1}{4}m_b^2 m_t^2\,, \qquad \mu^{\mathrm{w}} \to \tfrac{1}{4}m_b^2 m_t^2 \sin^2\theta_N'\,,
\end{aligned} \tag{7.19a}$$

$$\begin{aligned}
&\breve{\mu}^{\mathrm{a}} \to \tfrac{1}{15}m_N^4((1+2\cos^2\chi_N) - \tfrac{7}{8})\,, \\
&\breve{\mu}^{\mathrm{b}} \to \tfrac{12}{5}Mm_N^2\left(v_{10}\sqrt{\tfrac{1}{3}(1+2\cos^2\chi_N)}\cos\chi_N' - \tfrac{\sqrt{3}}{24}v_0^5\right), \\
&\breve{\mu}^{\mathrm{c}} \to \tfrac{1}{10}m_N^2\left(3m_t^2(|1+2\cos\chi_N\cos\hat{\chi}| - \tfrac{5}{4}) + 8m_b^2(|1+2\cos\chi_N\cos\hat{\chi}| - \tfrac{7}{8})\right), \\
&\breve{\mu}^{\mathrm{d}} \to \tfrac{1}{120}m_N^2 m_t^2(|1+2\cos\chi_N\cos\hat{\chi}| - \tfrac{5}{4})\,, \\
&\breve{\mu}^{\mathrm{f}} \to m_N^2 v_{10}^2 \sin^2\tilde{\chi}\cos^2\chi_\nu\,, \\
&\breve{\mu}^{\mathrm{g}} \to \sqrt{2}m_b m_N v_{10}^2 \sin^2\tilde{\chi}\sin\tilde{\chi}'\sin\tilde{\chi}''\cos\chi_\nu\cos\theta_\nu\cos\phi_\nu\,, \\
&\breve{\mu}^{\mathrm{h}} \to \sqrt{2}m_b m_N v_{10}^2 \sin^2\tilde{\chi}\sin\tilde{\chi}'\sin\tilde{\chi}''\cos\chi_\nu\cos\theta_\nu\sin\phi_\nu\,, \\
&\breve{\mu}^{\mathrm{i}} \to \tfrac{1}{12}m_N^2 m_t^2|1+2\cos\chi_N\cos\hat{\chi}|\,, \\
&\breve{\mu}^{\mathrm{j}} \to \tfrac{1}{2}m_b m_N m_t^2\sqrt{\tfrac{1}{3}|1+2\cos\chi_N\cos\hat{\chi}|}\cos\theta_N'\cos\phi_N'\,, \\
&\breve{\mu}^{\mathrm{k}} \to \tfrac{1}{2}m_b m_N m_t^2\sqrt{\tfrac{1}{3}|1+2\cos\chi_N\cos\hat{\chi}|}\cos\theta_N'\sin\phi_N'\,, \\
&\breve{\mu}^{\mathrm{l}} \to O(m_N^2 m_c^2)\,, \qquad \breve{\mu}^{\mathrm{m}} \to O(m_N m_b m_t m_c)\,, \qquad \breve{\mu}^{\mathrm{n}} \to O(m_N m_b m_t m_c)\,,
\end{aligned} \tag{7.19b}$$

$$\begin{aligned}
&\tilde{\mu}^{\mathrm{a}} \to 8M^2(v_5^2 + \tfrac{1}{3}v_{10}^2\sin^2\tilde{\chi} + \tfrac{2}{11}(v_0^5 - v_8^{10})^2) =: 5\mu^{\mathrm{a}}\sin^2\chi_0\,, \\
&\tilde{\mu}^{\mathrm{b}} \to \tfrac{1}{176}m_t^4\,, \qquad \tilde{\mu}^{\mathrm{c}} \to \tfrac{1}{176}m_n^4\tfrac{m_t^4}{(m_t^2 - m_b^2)^2}\,, \\
&\tilde{\mu}^{\mathrm{d}} \to \tfrac{9}{176}m_t^4\,, \qquad \tilde{\mu}^{\mathrm{e}} \to \tfrac{1}{11}m_t^2 m_n^2\sin^2\chi\,, \\
&\tilde{\mu}^{\mathrm{f}} \to \tfrac{\sqrt{3}}{33}Mm_t^2(v_0^5 - 11v_8^5 - v_8^{10})\,, \qquad \tilde{\mu}^{\mathrm{g}} \to \tfrac{\sqrt{3}}{33}Mm_n^2(v_0^5 - 11v_8^5 - v_8^{10})\tfrac{m_t^2}{m_t^2 - m_b^2}\,, \\
&\tilde{\mu}^{\mathrm{h}} \to \tfrac{\sqrt{3}}{11}Mm_t^2(v_0^5 - 11v_8^5 - v_8^{10})\,, \qquad \tilde{\mu}^{\mathrm{i}} \to \tfrac{4\sqrt{3}}{33}M(v_0^5 - 11v_8^5 - v_8^{10})m_t m_n\sin\chi\,,
\end{aligned}$$

$$
\begin{aligned}
&\tilde{\mu}^{\mathrm{j}} \to \tfrac{1}{88} m_n^2 m_t^2 \tfrac{m_t^2}{m_t^2-m_b^2}\,, && \tilde{\mu}^{\mathrm{k}} \to \tfrac{3}{88} m_t^4\,,\\
&\tilde{\mu}^{\mathrm{l}} \to \tfrac{1}{22} m_t^3 m_n \sin\chi\,, && \tilde{\mu}^{\mathrm{m}} \to \tfrac{3}{88} m_t^2 m_n^2 \tfrac{m_t^2}{m_t^2-m_b^2}\,,\\
&\tilde{\mu}^{\mathrm{n}} \to \tfrac{1}{22} m_t m_n^3 \sin\chi \tfrac{m_t^2}{m_t^2-m_b^2}\,, && \tilde{\mu}^{\mathrm{o}} \to \tfrac{3}{22} m_t^3 m_n \sin\chi\,,\\
&\tilde{\mu}^{\mathrm{p}} \to \tfrac{1}{192} m_t^4 \sin^2\hat{\chi}\,, && \tilde{\mu}^{\mathrm{q}} \to \tfrac{3}{64} m_t^4 \sin^2\hat{\chi}\,,\\
&\tilde{\mu}^{\mathrm{r}} \to \tfrac{1}{12} m_t^2 m_n^2 \sin^2\chi \sin^2\hat{\chi}\,, && \tilde{\mu}^{\mathrm{s}} \to \tfrac{1}{32} m_t^4 \sin^2\hat{\chi}\,,\\
&\tilde{\mu}^{\mathrm{t}} \to \tfrac{1}{24} m_t^3 m_n \sin\chi \sin^2\hat{\chi}\,, && \tilde{\mu}^{\mathrm{u}} \to \tfrac{1}{8} m_t^3 m_n \sin\chi \sin^2\hat{\chi}\,,
\end{aligned}
\tag{7.19c}
$$

$$
\begin{aligned}
\hat{\mu}^{\mathrm{a}} &= 2m_N^4((1+2\cos^2\chi_N)-\tfrac{1}{4})-\delta_U^2\delta_{tb} =: 2m_N^4((1+2\cos^2\chi_N)-\tfrac{1}{4})\cos^2\hat{\chi}_a\,,\\
\hat{\mu}^{\mathrm{b}} &= 4Mm_N^2\big(v_{10}\sqrt{\tfrac{1}{3}(1+2\cos^2\chi_N)}\cos\chi'_N-\tfrac{\sqrt{3}}{33}(9v_0^5+2v_8^{10})\\
&\qquad+\tfrac{\sqrt{1}}{11}|1+2\cos\chi_N\cos\hat{\chi}|(3v_0^5+8v_8^{10})\big)\,,\\
\hat{\mu}^{\mathrm{c}} &= \tfrac{1}{88} m_N^2 m_t^2(|1+2\cos\chi_N\cos\hat{\chi}|-1)\,,\\
\hat{\mu}^{\mathrm{d}} &= \tfrac{1}{88} m_N^2 m_n^2(|1+2\cos\chi_N\cos\hat{\chi}|-1)\tfrac{m_t^2}{m_t^2-m_b^2}\,,\\
\hat{\mu}^{\mathrm{e}} &= \tfrac{3}{88} m_N^2 m_t^2(|1+2\cos\chi_N\cos\hat{\chi}|-1)\,,\\
\hat{\mu}^{\mathrm{f}} &= \tfrac{1}{22} m_N^2 m_t m_n \sin\chi(|1+2\cos\chi_N\cos\hat{\chi}|-1)\,,\\
\hat{\mu}^{\mathrm{g}} &= O(m_N^2 M v_{10})\,,\\
\hat{\mu}^{\mathrm{h}} &= \tfrac{1}{3} m_N^2 m_b^2|1+2\cos\chi_N\cos\hat{\chi}|(\sin^2\theta'_N+\tfrac{4}{3}\cos^2\theta'_N\cos^2\phi'_N)\sin^2\hat{\chi}'\,,\\
\hat{\mu}^{\mathrm{i}} &= \tfrac{1}{3} m_N^2 m_b^2|1+2\cos\chi_N\cos\hat{\chi}|(\sin^2\theta'_N+\tfrac{4}{3}\cos^2\theta'_N\sin^2\phi'_N)\sin^2\hat{\chi}''\,,\\
\hat{\mu}^{\mathrm{j}} &= 2\sqrt{\hat{\mu}^{\mathrm{g}}\hat{\mu}^{\mathrm{h}}}\cos\hat{\chi}_j\,, \qquad \hat{\mu}^{\mathrm{k}} = 2\sqrt{\hat{\mu}^{\mathrm{g}}\hat{\mu}^{\mathrm{i}}}\cos\hat{\chi}_k\,,\\
\hat{\mu}^{\mathrm{l}} &= \tfrac{4}{9} m_N^2 m_b^2|1+2\cos\chi_N\cos\hat{\chi}|\big(\cos^2\theta'_N\sin(2\phi'_N)-\\
&\qquad-\sqrt{\tfrac{9}{4}\sin^4\theta'_N+3\cos^2\theta'_N\sin^2\theta'_N+\cos^4\theta'_N\sin^2(2\phi'_N)}\cos\hat{\chi}'\cos\hat{\chi}''\big)\,,\\
\hat{\mu}^{\mathrm{m}} &= 2\sqrt{\hat{\mu}^{\mathrm{g}}\tilde{\mu}^{\mathrm{p}}} = O(\sqrt{Mv_{10}}\,m_N m_t^2)\,, \qquad \hat{\mu}^{\mathrm{n}} = 2\sqrt{\hat{\mu}^{\mathrm{g}}\tilde{\mu}^{\mathrm{q}}} = O(\sqrt{Mv_{10}}\,m_N m_t^2)\,,\\
\hat{\mu}^{\mathrm{o}} &= 2\sqrt{\hat{\mu}^{\mathrm{g}}\tilde{\mu}^{\mathrm{r}}} = O(\sqrt{Mv_{10}}\,m_N m_t m_n\sin\chi)\,,\\
\hat{\mu}^{\mathrm{p}} &= \tfrac{1}{6} m_N m_t^2 m_b\sqrt{\tfrac{1}{3}|1+2\cos\chi_N\cos\hat{\chi}|}\times\\
&\qquad\times\big(\cos\theta'_N\cos\phi'_N-\sqrt{\tfrac{3}{4}\sin^2\theta'_N+\cos^2\theta'_N\cos^2\phi'_N}\cos\hat{\chi}\cos\hat{\chi}'\big)\,,\\
\hat{\mu}^{\mathrm{q}} &= \tfrac{1}{2} m_N m_t^2 m_b\sqrt{\tfrac{1}{3}|1+2\cos\chi_N\cos\hat{\chi}|}\times\\
&\qquad\times\big(\cos\theta'_N\cos\phi'_N-\sqrt{\tfrac{3}{4}\sin^2\theta'_N+\cos^2\theta'_N\cos^2\phi'_N}\cos\hat{\chi}\cos\hat{\chi}'\big)\,,\\
\hat{\mu}^{\mathrm{r}} &= \tfrac{2}{3} m_N m_t m_b m_n\sin\chi\sqrt{\tfrac{1}{3}|1+2\cos\chi_N\cos\hat{\chi}|}\times\\
&\qquad\times\big(\cos\theta'_N\cos\phi'_N-\sqrt{\tfrac{3}{4}\sin^2\theta'_N+\cos^2\theta'_N\cos^2\phi'_N}\cos\hat{\chi}\cos\hat{\chi}'\big)\,,\\
\hat{\mu}^{\mathrm{s}} &= \tfrac{1}{6} m_N m_t^2 m_b\sqrt{\tfrac{1}{3}|1+2\cos\chi_N\cos\hat{\chi}|}\times\\
&\qquad\times\big(\cos\theta'_N\sin\phi'_N-\sqrt{\tfrac{3}{4}\sin^2\theta'_N+\cos^2\theta'_N\sin^2\phi'_N}\cos\hat{\chi}\cos\hat{\chi}''\big)\,,
\end{aligned}
\tag{7.19d}
$$

$$\begin{aligned}
\hat{\mu}^{\mathrm{t}} &= \tfrac{1}{2} m_N m_t^2 m_b \sqrt{\tfrac{1}{3}|1+2\cos\chi_N\cos\hat{\chi}|}\times \\
&\quad \times\big(\cos\theta_N'\sin\phi_N' - \sqrt{\tfrac{3}{4}\sin^2\theta_N' + \cos^2\theta_N'\sin^2\phi_N'}\cos\hat{\chi}\cos\hat{\chi}''\big) , \\
\hat{\mu}^{\mathrm{u}} &= \tfrac{2}{3} m_N m_t m_b m_n \sin\chi \sqrt{\tfrac{1}{3}|1+2\cos\chi_N\cos\hat{\chi}|}\times \\
&\quad \times\big(\cos\theta_N'\sin\phi_N' - \sqrt{\tfrac{3}{4}\sin^2\theta_N' + \cos^2\theta_N'\sin^2\phi_N'}\cos\hat{\chi}\cos\hat{\chi}''\big) .
\end{aligned}$$

The parameters $\tilde{\lambda}$ and λ are not correlated. One can easily imagine choices of m_{10} and m_5 where $\tilde{\lambda}$ is much smaller than λ and vice versa. However, if we do not insist on such a fine tuning then $\tilde{\lambda}$ and λ have approximately the same order of magnitude.

7.3 The Masses of the Higgs Fields

7.3.1 The Higgs Field Lagrangian

We insert the parameters (7.11), (7.12), (7.13) *and* (7.19) *into the Higgs potential* (7.2). *We write down the result under the assumption*

$$M, m_N \gg \lambda m_t, \tilde{\lambda} m_t \gg m_t \gg m_b, m_n, m_\tau , \tag{7.20}$$

where we can consider $1 \gg \frac{1}{10}$ *in this formula. We also neglect mixing terms if the square of the coefficient of* $2AB$ *is small against the product of the coefficients of* A^2 *and* B^2 *. The tree–level predictions for the Higgs masses are obtained by a diagonalization of the mass matrices. The result is presented in Table 2.*

We find for (6.44c)

$$\begin{aligned}
\mathscr{L}_H = {} & \tfrac{1}{2} g^{\mu\nu}(\partial_\mu\phi_0'\,\partial_\nu\phi_0' + \partial_\mu\upsilon_0'\,\partial_\nu\upsilon_0' + \partial_\mu\upsilon_{45}\,\partial_\nu\upsilon_{45} \\
& \quad +\partial_\mu\psi_0\,\partial_\nu\psi_0 + \partial_\mu\psi_3'\,\partial_\nu\psi_3' + \partial_\mu\xi_0\,\partial_\nu\xi_0) \\
& \quad -\tfrac{1}{2}\big(\lambda^2 m_t^2 \upsilon_0^2 + \tfrac{3}{4}\lambda^2 m_t^2 \upsilon_{45}^2 + \tfrac{m_N^4}{M^2}(\tfrac{1}{144}\cos^2\hat{\chi}_a + \tfrac{1}{54}\cos^2\chi_N\cos^2\hat{\chi}_a)\psi_3'^{\,2} \\
& \quad +\tfrac{1}{48M^2}(\tfrac{48}{5}\breve{\mu}^{\mathrm{e}} + \tfrac{2}{9}\hat{\mu}^{\mathrm{a}})\psi_0^2 + \tfrac{1}{2m_N^2}(4\breve{\mu}^{\mathrm{a}} + \tfrac{8}{15}\hat{\mu}^{\mathrm{a}})\xi_0^2 \\
& \quad +(\tfrac{207}{110} + \tfrac{2}{9}\sin^2\hat{\chi})m_t^2\phi_0'^{\,2} + \tfrac{1}{8\sqrt{15}Mm_t}(-\tfrac{32}{3}\hat{\mu}^{\mathrm{c}} + \tfrac{32}{3}\hat{\mu}^{\mathrm{e}})\psi_0\phi_0' \\
& \quad +\tfrac{1}{\sqrt{24}m_N m_t}(4\breve{\mu}^{\mathrm{c}} + 48\breve{\mu}^{\mathrm{d}} + \tfrac{64}{5}\hat{\mu}^{\mathrm{c}} - \tfrac{64}{5}\hat{\mu}^{\mathrm{e}})\phi_0'\xi_0 - \tfrac{2}{9\sqrt{10}Mm_N}\hat{\mu}^{\mathrm{a}}\psi_0\xi_0\big) \\
& + \tfrac{1}{2} g^{\mu\nu}(\partial_\mu\psi_1'\,\partial_\nu\psi_1' + \partial_\mu\psi_2'\,\partial_\nu\psi_2' + \partial_\mu\upsilon_{18}\,\partial_\nu\upsilon_{18} + \partial_\mu\upsilon_{63}\,\partial_\nu\upsilon_{63}) \\
& \quad -\tfrac{1}{2}\big(\tfrac{m_N^4}{M^2}(\tfrac{1}{144}\cos^2\hat{\chi}_a + \tfrac{1}{54}\cos^2\chi_N\cos^2\hat{\chi}_a)(\psi_1'^{\,2} + \psi_2'^{\,2}) + \tfrac{3}{4}\lambda^2 m_t^2(\upsilon_{18}^2 + \upsilon_{63}^2)\big) \\
& + \tfrac{1}{2} g^{\mu\nu}(\textstyle\sum_{i=1}^{8}\partial_\mu\psi_i\,\partial_\nu\psi_i + \sum_{i=1}^{8}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i + \sum_{i=46}^{53}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i \\
& \quad +\textstyle\sum_{i=33}^{40}\partial_\mu\xi_i\,\partial_\nu\xi_i + \sum_{i=82}^{89}\partial_\mu\xi_i\,\partial_\nu\xi_i) \\
& \quad -\tfrac{1}{2}\big(\tfrac{m_N^4}{M^2}(\tfrac{1}{144}\cos^2\hat{\chi}_a + \tfrac{1}{54}\cos^2\chi_N\cos^2\hat{\chi}_a)\textstyle\sum_{i=1}^{8}\psi_i^2 \\
& \quad +(\lambda^2 m_n^2 + \tfrac{m_b^2 v_{10}^2}{m_t^2}\sin^2\tilde{\chi}\sin^2\tilde{\chi}'\sin^2\tilde{\chi}'')(\textstyle\sum_{i=1}^{8}\upsilon_i^2 + \sum_{i=46}^{53}\upsilon_i^2) \\
& \quad +9M^2(\textstyle\sum_{i=33}^{40}\xi_i^2 + \sum_{i=82}^{89}\xi_i^2)\big)
\end{aligned} \tag{7.21}$$

$$
\begin{aligned}
&+\tfrac{1}{2}g^{\mu\nu}\big(\textstyle\sum_{i=19}^{26}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=64}^{71}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=25}^{32}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=74}^{81}\partial_\mu\xi_i\,\partial_\nu\xi_i\big)\\
&\quad-\tfrac{1}{2}\Big(\big(\lambda^2m_n^2+\tfrac{m_b^2v_{10}^2}{m_t^2}\sin^2\tilde\chi\sin^2\tilde\chi'\sin^2\tilde\chi''\big)\big(\textstyle\sum_{i=19}^{26}\upsilon_i^2+\sum_{i=64}^{71}\upsilon_i^2\big)\\
&\quad+9M^2\big(\textstyle\sum_{i=25}^{32}\xi_i^2+\sum_{i=74}^{81}\xi_i^2\big)\Big)\\
&+\tfrac{1}{2}g^{\mu\nu}\big(\textstyle\sum_{i=1}^{6}\partial_\mu\phi_i\,\partial_\nu\phi_i+\sum_{i=9}^{14}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=54}^{59}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i\\
&\quad+\textstyle\sum_{i=30}^{35}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=75}^{80}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=39}^{41}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=84}^{86}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i\\
&\quad+\textstyle\sum_{i=19}^{24}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=68}^{73}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=44}^{46}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=93}^{95}\partial_\mu\xi_i\,\partial_\nu\xi_i\\
&\quad+\textstyle\sum_{i=47}^{49}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=96}^{98}\partial_\mu\xi_i\,\partial_\nu\xi_i\big)\\
&\quad-\tfrac{1}{2}\Big(M^2\big(\textstyle\sum_{i=1}^{6}\phi_i^2+\sum_{i=9}^{14}\upsilon_i^2+\sum_{i=54}^{59}\upsilon_i^2+\sum_{i=30}^{35}\upsilon_i^2+\sum_{i=75}^{80}\upsilon_i^2\big)\\
&\quad+4M^2\big(\textstyle\sum_{i=39}^{41}\upsilon_i^2+\sum_{i=84}^{86}\upsilon_i^2+\sum_{i=19}^{24}\xi_i^2+\sum_{i=68}^{73}\xi_i^2+\sum_{i=47}^{49}\xi_i^2+\sum_{i=96}^{98}\xi_i^2\big)\\
&\quad+\big(M^2+m_N^2(\tfrac{1}{12}\cos^2\hat\chi_a+\tfrac{2}{9}\cos^2\chi_N\cos^2\hat\chi_a)\big)\big(\textstyle\sum_{i=44}^{46}\xi_i^2+\sum_{i=93}^{95}\xi_i^2\big)\Big)\\
&+\tfrac{1}{2}g^{\mu\nu}\big(\textstyle\sum_{i=7}^{18}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=56}^{67}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=41}^{43}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=90}^{92}\partial_\mu\xi_i\,\partial_\nu\xi_i\\
&\quad+\textstyle\sum_{i=15}^{17}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=60}^{62}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=42}^{44}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=87}^{89}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i\\
&\quad+\textstyle\sum_{i=36}^{38}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=81}^{83}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i\big)\\
&\quad-\tfrac{1}{2}\Big(M^2\big(\textstyle\sum_{i=15}^{17}\upsilon_i^2+\sum_{i=60}^{62}\upsilon_i^2+\sum_{i=42}^{44}\upsilon_i^2+\sum_{i=87}^{89}\upsilon_i^2\big)\\
&\quad+4M^2\big(\textstyle\sum_{i=36}^{38}\upsilon_i^2+\sum_{i=81}^{83}\upsilon_i^2+\sum_{i=7}^{12}\xi_i^2+\sum_{i=56}^{61}\xi_i^2\big)+16M^2\big(\sum_{i=13}^{18}\xi_i^2+\sum_{i=62}^{69}\xi_i^2\big)\\
&\quad+\big(M^2+m_N^2(\tfrac{1}{12}\cos^2\hat\chi_a+\tfrac{2}{9}\cos^2\chi_N\cos^2\hat\chi_a)\big)\big(\textstyle\sum_{i=41}^{43}\xi_i^2+\sum_{i=90}^{92}\xi_i^2\big)\Big)\\
&+\tfrac{1}{2}g^{\mu\nu}\big(\textstyle\sum_{i=27}^{29}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=72}^{74}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=1}^{6}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=50}^{55}\partial_\mu\xi_i\,\partial_\nu\xi_i\big)\\
&\quad-\tfrac{1}{2}\Big(4M^2\big(\textstyle\sum_{i=1}^{6}\xi_i^2+\sum_{i=50}^{55}\xi_i^2\big)+M^2\big(\sum_{i=27}^{29}\upsilon_i^2+\sum_{i=72}^{74}\upsilon_i^2\big)\Big)\;.
\end{aligned}
$$

It remains to find the eigenvalues of the quadratic form[21] determined by the ϕ_0'–ψ_0–ξ_0 sector in (7.21). We use the general result that the smallest (largest) eigenvalue is smaller (larger) than the smallest (largest) diagonal matrix element. This means that the mass of the ϕ_0' Higgs field is smaller than $\sqrt{\frac{2083}{990}}\,m_t \approx 1.45\,m_t$. We assume $\frac{48}{5}\check\mu^{\rm e} \gg \frac{2}{9}\hat\mu^{\rm a}$, or $M^2 \gg \frac{55}{864}m_N^2$. Then, the large parameter $\check\mu^{\rm e}$ occurring in the coefficient of ψ_0^2 stabilizes the other two eigenvalues near the diagonal matrix elements $\frac{1}{5M^2}\check\mu^{\rm e}$ and $\frac{1}{m_N^2}(2\check\mu^{\rm a}+\frac{4}{15}\hat\mu^{\rm a})$, respectively.

7.3.2 Summary of the Higgs Field Masses

For convenience we list in Table 2 the masses of the Higgs fields and the masses of the gauge fields derived in Section 6.3. We recall that m_t is the mass of the top quark, m_N the mass scale of the right neutrinos and M the Grand Unification scale, where we have

[21] The corresponding matrix is positive definite by construction. This is not apparent when inserting (7.19), because there are complicated relations between $\chi_N, \hat\chi_a, \hat\chi$.

Particle	Mass	Particle	Mass
1. The completely neutral Higgs fields:			
ϕ_0'	$(0\ldots 1.45)\,m_t$	ξ_0	$(\sqrt{\frac{1}{60}}\ldots\sqrt{\frac{7}{4}})m_N$
υ_0'	λm_t	υ_{45}	$\frac{1}{2}\sqrt{3}\lambda m_t$
ψ_0	$\sqrt{\frac{2}{5}}m_N$	ψ_3'	$(0\ldots\frac{1}{12}\sqrt{\frac{11}{3}})\frac{m_N^2}{M}$
2. The colour–neutral Higgs fields of charge ∓ 1:			
$\frac{1}{\sqrt{2}}(\upsilon_{18}\pm \mathrm{i}\upsilon_{63})$	$\frac{1}{2}\sqrt{3}\lambda m_t$	$\frac{1}{\sqrt{2}}(\psi_1\pm \mathrm{i}\psi_2)$	$(0\ldots\frac{1}{12}\sqrt{\frac{11}{3}})\frac{m_N^2}{M}$
3. The neutral Higgs fields, for $i=0,\ldots,7$:			
ψ_{1+i}	$(0\ldots\frac{1}{12}\sqrt{\frac{11}{3}})\frac{m_N^2}{M}$		
υ_{1+i}	$(\lambda\ldots\lambda{+}\check{\lambda})m_n$	υ_{46+i}	$(\lambda\ldots\lambda{+}\check{\lambda})m_n$
ξ_{33+i}	$3M$	ξ_{82+i}	$3M$
4. The Higgs fields of charge ∓ 1, for $i=0\ldots 7$:			
$\frac{1}{\sqrt{2}}(\upsilon_{19+i}\pm \mathrm{i}\upsilon_{64+i})$	$(\lambda\ldots\lambda{+}\check{\lambda})m_n$	$\frac{1}{\sqrt{2}}(\xi_{25+i}\pm \mathrm{i}\xi_{74+i})$	$3M$
5. The Higgs fields of charge $\mp\frac{1}{3}$, for $i=0,1,2$ and $j=0,\ldots,5$:			
$\frac{1}{\sqrt{2}}(\phi_{1+i}\pm \mathrm{i}\phi_{4+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{9+i}\pm \mathrm{i}\upsilon_{54+i})$	M
$\frac{1}{\sqrt{2}}(\upsilon_{12+i}\pm \mathrm{i}\upsilon_{57+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{39+i}\pm \mathrm{i}\upsilon_{84+i})$	$2M$
$\frac{1}{\sqrt{2}}(\xi_{44+i}\pm \mathrm{i}\xi_{93+i})$	M	$\frac{1}{\sqrt{2}}(\xi_{47+i}\pm \mathrm{i}\xi_{96+i})$	$2M$
$\frac{1}{\sqrt{2}}(\xi_{19+j}\pm \mathrm{i}\xi_{68+j})$	$2M$	$\frac{1}{\sqrt{2}}(\upsilon_{30+j}\pm \mathrm{i}\upsilon_{75+j})$	M
6. The Higgs fields of charge $\pm\frac{2}{3}$, for $i=0,1,2$ and $j=0,\ldots,5$:			
$\frac{1}{\sqrt{2}}(\upsilon_{15+i}\pm \mathrm{i}\upsilon_{60+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{36+i}\pm \mathrm{i}\upsilon_{81+i})$	$2M$
$\frac{1}{\sqrt{2}}(\upsilon_{42+i}\pm \mathrm{i}\upsilon_{87+i})$	M	$\frac{1}{\sqrt{2}}(\xi_{41+i}\pm \mathrm{i}\xi_{90+i})$	M
$\frac{1}{\sqrt{2}}(\xi_{7+j}\pm \mathrm{i}\xi_{56+j})$	$2M$	$\frac{1}{\sqrt{2}}(\xi_{13+j}\pm \mathrm{i}\xi_{62+j})$	$4M$
7. The Higgs fields of charge $\mp\frac{4}{3}$, for $i=0,1,2$ and $j=0,\ldots,5$:			
$\frac{1}{\sqrt{2}}(\upsilon_{27+i}\pm \mathrm{i}\upsilon_{72+i})$	M	$\frac{1}{\sqrt{2}}(\xi_{1+j}\pm \mathrm{i}\xi_{50+j})$	$2M$
8. The neutral massive gauge fields:			
Z	$\sqrt{\frac{2}{5}}\,m_t$	Z'	$\frac{1}{2}\sqrt{\frac{5}{3}}m_N$
9. The massive gauge fields of charge ± 1:			
$\frac{1}{\sqrt{2}}(W_1\mp \mathrm{i}W_2)$	$\frac{1}{2}m_t$	Weinberg angle: $\sin^2\theta_W=\frac{3}{8}$	
10. The leptoquarks leading to proton decay, for $i=0,1,2$:			
$\frac{1}{\sqrt{2}}(X_{1+i}\mp \mathrm{i}X_{4+i})$	M	charge: $\mp\frac{1}{3}$	
$\frac{1}{\sqrt{2}}(Y_{1+i}\mp \mathrm{i}Y_{4+i})$	M	charge: $\pm\frac{2}{3}$	

Table 2: The particle masses for the SU(5)×U(1)–model

assumed $m_N, M \gg m_t$. Moreover, we have introduced the abbreviation

$$\breve{\lambda} = \sqrt{\lambda^2 + \tfrac{m_b^2 v_{10}^2}{m_t^2 m_n^2}} - \lambda \geq 0 \;.$$

It is interesting to perform the transformation (7.1) in the Yukawa Larangian $\mathscr{L}_m$ of the fermionic action (6.66). The contribution of the coupling of the ϕ_0' Higgs field to the fermions takes the form

$$\begin{aligned}
\mathscr{L}_{\phi_0'} &= \big(-\boldsymbol{d}_L^*(\mathbb{1}_3 \otimes (M_d + \tfrac{g_0}{\sqrt{8(\mu_1+12\mu_2)}}\phi_0' M_d))\boldsymbol{d}_R - e_L^*(M_e + \tfrac{g_0}{\sqrt{8(\mu_1+12\mu_2)}}\phi_0' M_e) e_R \\
&\quad - \boldsymbol{u}_L^*(\mathbb{1}_3 \otimes (M_u + \tfrac{g_0}{\sqrt{8(\mu_1+12\mu_2)}}\phi_0' M_u))\boldsymbol{u}_R - \nu_L^*(M_n^T + \tfrac{g_0}{\sqrt{8(\mu_1+12\mu_2)}}\phi_0' M_n^T)\nu_R\big) + \text{h.c} \\
&= \big(-\boldsymbol{d}_L^*(\mathbb{1}_3 \otimes (1 + \tfrac{g_0}{m_t}\phi_0')M_d)\boldsymbol{d}_R - e_L^*((1 + \tfrac{g_0}{m_t}\phi_0')M_e) e_R \\
&\quad - \boldsymbol{u}_L^*(\mathbb{1}_3 \otimes (1 + \tfrac{g_0}{m_t}\phi_0')M_u))\boldsymbol{u}_R - \nu_L^*((1 + \tfrac{g_0}{m_t}\phi_0')M_n^T)\nu_R\big) + \text{h.c} \;. \qquad (7.22)
\end{aligned}$$

Thus, the Higgs field ϕ_0' has the same properties as the standard model Higgs field.

All other Higgs fields are too massive to observe. All Higgs and gauge fields with fractional–valued charge lead to proton decay. Without exception they receive a mass of the order of the grand unification scale M, which must be chosen sufficiently large to ensure the observed stability of matter. The mass of the remaining Higgs fields with integer–valued charge is of the order $M, \lambda m_t, \lambda m_n, m_N$ or $\frac{m_N^2}{M}$. These mass scales are situated somewhere between m_t and M. By assumption, m_N and $\frac{m_N^2}{M}$ are very close to M. Moreover, for generic choices of the mass matrices M_{10} and M_5, also λm_t and λm_n are close to M.

The results presented in Table 2 require some comments. Of course, all mass predictions are only valid on tree–level. There are few chances that in near future one gets estimations for the quantum corrections. Therefore, let us employ the following naïve consideration: We assume that all contributions of higher order perturbation theory only yield terms of the Lagrangian which are already present at tree–level. Thus, we assume that only the coefficients of the vertices are modified and that no further observable vertices emerge. Then, the renormalized bosonic vertices are no longer related to the fermionic mass parameters (unless there occur reductions of couplings). This means that the predictions for the masses of the W and Z bosons and for the standard model Higgs field ϕ_0' will be changed. But as these fields do not feel the GUT–sector in leading approximation, the mass modification will be of the order of the fermion masses. Thus, we can be sure that the mass of all standard model bosons is of the order m_t. On the other hand, the GUT–scale M is not related to the fermions. Thus, we can adopt the point of view that the GUT–matrices M_{10} and M_5 are already those of the effective theory. Hence, the mass predictions for the heavy bosons are exact under the technical assumptions we made to simplify the calculation. The outcome shows that the masses of all particles with fractional–valued charge are stable under small variations of the GUT–matrices. Therefore, we can be sure that the masses of those particles do not migrate significantly below the GUT–scale. This result is indirectly confirmed by *the large lifetime of the proton. For certain Higgs fields with integer–valued charge we get no prediction at all. This is because these masses react very sensibly to variations of the GUT–matrices. We cannot exclude the possibility that some of them are observable in the near future.*

7.4 The SU(5)–Model

7.4.1 The Higgs Field Lagrangian

The SU(5)–model can be derived from the SU(5)×U(1)–model studied so far by the replacements discussed in Section 6.4.4. In particular, we must replace in (7.19)

$$m_t \leftrightarrow m_b\,, \qquad\qquad m_\tau \leftrightarrow m_n\,, \tag{7.23}$$

which leads to different numerical relations. With these replacements, formulae (7.19) are correct for the parameters μ^i, $i = \mathrm{g},\dots,\mathrm{v}$, and all $\tilde{\mu}^i$. We clearly have $\mu^{\mathrm{w}} \equiv \mu^{\mathrm{v}} = \frac{1}{4}m_b^2 m_t^2$. In the SU(5)–model, the Higgs fields ξ_i are absent and we have $m_N = 0$. Therefore, all parameters $\hat{\mu}^i$ and $\check{\mu}^i$ vanish. Thus, it remains to compute the parameters μ^i, $i = \mathrm{a},\dots,\mathrm{f}$, using the matrices (6.68) instead of (5.81). We find in leading approximation

$$\begin{aligned}
&\mu^{\mathrm{a}} \to \tfrac{8}{5}M^2(\tfrac{9}{2}v_{10}^2 + \tfrac{9}{11}(v_0^5)^2 + v_5^2) =: \tilde{\lambda}^2 M^2 m_b^2\,, && \mu^{\mathrm{b}} \to \tfrac{288}{55}m_t^4\,,\\
&\mu^{\mathrm{c}} \to \tfrac{37}{21120}m_b^4\,, && \mu^{\mathrm{d}} \to -\tfrac{48\sqrt{3}}{45}Mm_t^2(v_0^5 + 11v_8^{10})\,,\\
&\mu^{\mathrm{e}} \to -\tfrac{\sqrt{3}}{55}Mm_b^2(-v_0^5 + \tfrac{11}{3}v_8^5 + \tfrac{11}{2}v_8^{10})\,, && \mu^{\mathrm{f}} \to \tfrac{9}{110}m_b^2 m_t^2\,.
\end{aligned} \tag{7.24}$$

The parameters (6.10e) take the form

$$\mu_0 = \tfrac{1}{4}M^2\,, \qquad \mu_1 = \tfrac{1}{8}m_t^2\,, \qquad \mu_2 = \tfrac{1}{384}m_b^2\,. \tag{7.25}$$

The relations (6.43a) remain unchanged. We get for the bosonic Lagrangian the same formula (6.44a), however with $Z'_\mu \equiv 0$, and with a modified Higgs potential $\mathscr{L}_H$. Again, we only consider the leading terms under the assumption

$$M \gg \lambda m_b, \tilde{\lambda} m_b \gg m_t \gg m_b, m_\tau, m_n\,, \tag{7.26}$$

and neglect mixing terms if the square of the coefficient of $2AB$ is small against the product of the coefficients of A^2 and B^2. Now, replacing $v_{10} \mapsto \frac{m_b}{m_t}v_{10}$, we find for (6.44c)

$$\begin{aligned}
\mathscr{L}_H = {}& \tfrac{1}{2}g^{\mu\nu}(\partial_\mu\psi''_3\,\partial_\nu\psi''_3 + \partial_\mu\psi''_0\,\partial_\nu\psi''_0 + \partial_\mu\phi'_0\,\partial_\nu\phi'_0 + \partial_\mu\upsilon'_0\,\partial_\nu\upsilon'_0 + \partial_\mu\upsilon_{45}\,\partial_\nu\upsilon_{45})\\
&-\tfrac{1}{2}\big(\tfrac{5}{24}\tilde{\lambda}^2 m_b^2 \sin^2\chi_0 {\psi'_3}^2 + (\tfrac{3}{4}\lambda^2 m_b^2 + v_{10}^2\sin^2\tilde{\chi}\sin^2\tilde{\chi}'\sin^2\tilde{\chi}'')({\upsilon'_0}^2 + \upsilon_{45}^2)\\
&\quad + \tfrac{1}{5M^2}(\mu^{\mathrm{a}} + \tfrac{1}{120}\tilde{\mu}^{\mathrm{a}})\psi_0^2 + \tfrac{1}{3m_t^2}\mu^{\mathrm{b}}{\phi'_0}^2 + \tfrac{1}{\sqrt{15}Mm_t}\mu^{\mathrm{d}}\phi'_0\psi_0\big)\\
&+\tfrac{1}{2}g^{\mu\nu}(\partial_\mu\psi'_1\,\partial_\nu\psi'_1 + \partial_\mu\psi'_2\,\partial_\nu\psi'_2 + \partial_\mu\upsilon_{18}\,\partial_\nu\upsilon_{18} + \partial_\mu\upsilon_{63}\,\partial_\nu\upsilon_{63})\\
&-\tfrac{1}{2}\big(\tfrac{5}{24}\tilde{\lambda}^2 m_b^2\sin^2\chi_0({\psi'_1}^2 + {\psi'_2}^2)\\
&\quad + (\tfrac{3}{4}\lambda^2 m_b^2 + v_{10}^2\sin^2\tilde{\chi}\sin^2\tilde{\chi}'\sin^2\tilde{\chi}'')(\upsilon_{18}^2 + \upsilon_{63}^2)\big)
\end{aligned} \tag{7.27}$$

$$
\begin{aligned}
&+\tfrac{1}{2}g^{\mu\nu}(\textstyle\sum_{i=1}^{8}\partial_\mu\psi_i\,\partial_\nu\psi_i+\sum_{i=1}^{8}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=46}^{53}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i)\\
&\quad-\tfrac{1}{2}\big((\lambda^2m_\tau^2+v_{10}^2\sin^2\tilde{\chi}\sin^2\tilde{\chi}'\sin^2\tilde{\chi}'')(\textstyle\sum_{i=1}^{8}\upsilon_i^2+\sum_{i=46}^{53}\upsilon_i^2)\\
&\quad+\tfrac{5}{24}\tilde{\lambda}^2m_b^2\sin^2\chi_0\textstyle\sum_{i=1}^{8}\psi_i^2\big)\\
&+\tfrac{1}{2}g^{\mu\nu}(\textstyle\sum_{i=19}^{26}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=64}^{71}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i)\\
&\quad-\tfrac{1}{2}(\lambda^2m_\tau^2+v_{10}^2\sin^2\tilde{\chi}\sin^2\tilde{\chi}'\sin^2\tilde{\chi}'')(\textstyle\sum_{i=19}^{26}\upsilon_i^2+\sum_{i=64}^{71}\upsilon_i^2)\\
&+\tfrac{1}{2}g^{\mu\nu}(\textstyle\sum_{i=1}^{6}\partial_\mu\phi_i\,\partial_\nu\phi_i+\sum_{i=9}^{14}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=54}^{59}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i\\
&\quad+\textstyle\sum_{i=30}^{35}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=75}^{80}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i)\\
&\quad-\tfrac{1}{2}M^2(\textstyle\sum_{i=1}^{6}\phi_i^2+\sum_{i=9}^{14}\upsilon_i^2+\sum_{i=54}^{59}\upsilon_i^2+\sum_{i=30}^{35}\upsilon_i^2+\sum_{i=75}^{80}\upsilon_i^2)\\
&+\tfrac{1}{2}g^{\mu\nu}(\textstyle\sum_{i=15}^{17}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=60}^{62}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=42}^{44}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=87}^{89}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i)\\
&\quad-\tfrac{1}{2}M^2(\textstyle\sum_{i=15}^{17}\upsilon_i^2+\sum_{i=60}^{62}\upsilon_i^2+\sum_{i=42}^{44}\upsilon_i^2+\sum_{i=87}^{89}\upsilon_i^2)\\
&+\tfrac{1}{2}g^{\mu\nu}(\textstyle\sum_{i=27}^{29}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=72}^{74}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=36}^{38}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=81}^{83}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i)\\
&\quad-\tfrac{1}{2}\big(M^2(\textstyle\sum_{i=27}^{29}\upsilon_i^2+\sum_{i=72}^{74}\upsilon_i^2)+4M^2(\sum_{i=36}^{38}\upsilon_i^2+\sum_{i=81}^{83}\upsilon_i^2)\big)\\
&+\tfrac{1}{2}g^{\mu\nu}(\textstyle\sum_{i=39}^{41}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i+\sum_{i=84}^{86}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i)+4M^2(\sum_{i=39}^{41}\upsilon_i^2+\sum_{i=84}^{86}\upsilon_i^2)\;.
\end{aligned}
$$

Again, we must find the eigenvalues of the quadratic form determined by the ϕ_0'–ψ_0 sector in (7.27). The smallest of the two eigenvalues is approximately $\frac{1}{3m_t^2}(\mu^{\mathrm{b}}-\frac{(\mu^{\mathrm{d}})^2}{4\mu^{\mathrm{a}}})$, which is smaller than $\frac{96}{55}m_t^2$.

7.4.2 Summary of the Higgs Field Masses

We list in Table 3 the masses of the Higgs fields and the masses of the gauge fields for the SU(5)–model. We have introduced the abbreviation

$$\breve{\lambda}=\sqrt{\lambda^2+\frac{4v_{10}^2}{3m_b^2}}-\lambda\geq 0\;,\qquad \breve{\lambda}'=\sqrt{\lambda^2+\frac{v_{10}^2}{m_\tau^2}}-\lambda\geq 0\;.$$

The transformation (7.1) in the Yukawa Larangian $\mathscr{L}_m$ of the fermionic action (6.66) leads to the same contribution (7.22) of the coupling of the ϕ_0' Higgs field to the fermions. Thus, the Higgs field ϕ_0' has the same properties as the standard model Higgs field in the SU(5)–model, too. Again, all Higgs and gauge fields with fractional–valued charge lead to proton decay and receive a mass of the order of the grand unification scale M. The mass of the remaining Higgs fields with integer–valued charge is of the order λm_b and $\tilde{\lambda}m_b$. These mass scales are situated somewhere between m_t and M (for generic choices of the mass matrices M_{10} and M_5 close to M). The remarks at the end of Section 7.3 on the validity of the predictions hold for the SU(5)–model, too.

Particle	Mass	Particle	Mass

1. The completely neutral Higgs fields:

ϕ_0'	$(0\ldots 1.32)\,m_t$		
υ_0'	$\frac{\sqrt{3}}{2}\lambda m_b$	υ_{45}	$\frac{\sqrt{3}}{2}(\lambda\ldots\lambda+\check{\lambda})m_b$
ψ_0	$(\sqrt{\frac{1}{5}}\ldots\sqrt{\frac{5}{24}})\tilde{\lambda}m_b$	ψ_3'	$(0\ldots\sqrt{\frac{5}{24}}\tilde{\lambda})m_b$

2. The colour–neutral Higgs fields of charge ∓ 1:

$\frac{1}{\sqrt{2}}(\upsilon_{18}\pm \mathrm{i}\upsilon_{63})$	$\frac{\sqrt{3}}{2}(\lambda\ldots\lambda+\check{\lambda})m_b$	$\frac{1}{\sqrt{2}}(\psi_1\pm \mathrm{i}\psi_2)$	$(0\ldots\sqrt{\frac{5}{24}}\tilde{\lambda})m_b$

3. The neutral Higgs fields, for $i=0,\ldots,7$:

ψ_{1+i}	$(0\ldots\sqrt{\frac{5}{24}}\tilde{\lambda})m_b$		
υ_{1+i}	$(\lambda\ldots\lambda+\check{\lambda}')m_\tau$	υ_{46+i}	$(\lambda\ldots\lambda+\check{\lambda}')m_\tau$

4. The Higgs fields of charge ∓ 1, for $i=0\ldots 7$:

$\frac{1}{\sqrt{2}}(\upsilon_{19+i}\pm \mathrm{i}\upsilon_{64+i})$	$(\lambda\ldots\lambda+\check{\lambda}')m_t$

5. The Higgs fields of charge $\mp\frac{1}{3}$, for $i=0,1,2$ and $j=0,\ldots,5$:

$\frac{1}{\sqrt{2}}(\phi_{1+i}\pm \mathrm{i}\phi_{4+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{9+i}\pm \mathrm{i}\upsilon_{54+i})$	M
$\frac{1}{\sqrt{2}}(\upsilon_{12+i}\pm \mathrm{i}\upsilon_{57+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{30+j}\pm \mathrm{i}\upsilon_{75+j})$	M

6. The Higgs fields of charge $\pm\frac{2}{3}$, for $i=0,1,2$:

$\frac{1}{\sqrt{2}}(\upsilon_{27+i}\pm \mathrm{i}\upsilon_{72+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{36+i}\pm \mathrm{i}\upsilon_{81+i})$	$2M$

7. The Higgs fields of charge $\mp\frac{4}{3}$, for $i=0,1,2$:

$\frac{1}{\sqrt{2}}(\upsilon_{15+i}\pm \mathrm{i}\upsilon_{60+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{42+i}\pm \mathrm{i}\upsilon_{87+i})$	M

8. The Higgs fields of charge $\pm\frac{5}{3}$, for $i=0,1,2$:

$\frac{1}{\sqrt{2}}(\upsilon_{39+i}\pm \mathrm{i}\upsilon_{84+i})$	$2M$

9. The massive electroweak gauge fields:

$\frac{1}{\sqrt{2}}(W_1\mp \mathrm{i}W_2)$	$\frac{1}{2}m_t$	Z	$\sqrt{\frac{2}{5}}\,m_t$
Weinberg angle:		$\sin^2\theta_W=\frac{3}{8}$	

10. The leptoquarks leading to proton decay, for $i=0,1,2$:

$\frac{1}{\sqrt{2}}(X_{1+i}\mp \mathrm{i}X_{4+i})$	M	charge: $\mp\frac{1}{3}$	
$\frac{1}{\sqrt{2}}(Y_{1+i}\mp \mathrm{i}Y_{4+i})$	M	charge: $\mp\frac{4}{3}$	

Table 3: The particle masses for the SU(5)–model

8 Conclusion

1) We have invented a new branch of non–commutative geometry, which we baptized "non–associative geometry". This branch is based upon Lie algebras and generalized Dirac operators acting on Hilbert spaces – i.e. the minimal set of information to be specified in gauge field theories. We have developed the general scheme of the new approach and have applied it successfully to the standard model and the flipped $\mathrm{SU}(5)\times\mathrm{U}(1)$–Grand Unification model. For certain modifications of the calculus we were able to derive the SU(5)–model from the flipped $\mathrm{SU}(5)\times\mathrm{U}(1)$–model. In all of these models we have found interesting tree–level relations between fermionic and bosonic parameters: Given the fermionic parameters (fermion masses and Kobayashi–Maskawa mixing angles) and – in the Grand Unification case – two 3×3–matrices determining the unification scale as input, we were able to compute all bosonic quantities on tree–level:

   - the occurring multiplets of Higgs fields,
   - the spontaneous symmetry breaking pattern,
   - the masses of all Higgs fields,
   - the masses of all Yang–Mills fields,
   - the Weinberg angle.

   However, since in the Grand Unification models not all input parameters are known, we were forced to be satisfied with estimations for some of the masses.

2) The representation of the U(1)–part of the $\mathrm{SU}(5)\times\mathrm{U}(1)$–model is not an input but an algebraic consequence of the theory. This U(1)–representation is unique and realized in nature.

3) In the standard model there occurs only the usual complex doublet of Higgs fields. Of these four component there survives one Higgs field the spontaneous symmetry breaking, and three gauge fields (W^+, W^-, Z) become massive.

   In the SU(5)–model there occur Higgs fields in complex $\underline{5}$–, complex $\underline{45}$– and real $\underline{24}$–plets. After the spontaneous symmetry breaking, there survive 12 Higgs fields of the $\underline{24}$–representation, 7 Higgs fields of the $\underline{5}$–representation and 90 Higgs fields of the $\underline{45}$–representation, and 15 gauge fields become massive.

   In the $\mathrm{SU}(5)\times\mathrm{U}(1)$–model there occurs additionally a complex $\underline{50}$–plet of Higgs bosons. Of this multiplet there survive 99 Higgs fields the spontaneous symmetry breaking, and an additional massive neutral gauge field emerges.

4) There occur three mass scales in the unification models:

   - The lowest mass scale is the scale of the fermion masses reaching from the neutrino masses to the mass of the top quark. Moreover, also the electroweak gauge fields Z, W^+, W^- belong to this scale, and – remarkably – one Higgs field as well.

- In the two Grand Unification models, the mass of all fields leading to proton decay is of the order of the Grand Unification scale M.
- The masses of Higgs fields which do not lead to proton decay lie between the fermions scale and the Grand Unification scale, generically close to M.

5) In both the SU(5)–model and the SU(5)×U(1)–model there exists precisely one light Higgs field ϕ_0', which has exactly the same properties as the standard model Higgs field. It couples to a fermion of the mass m_f with the coupling constant $g_0 m_f/m_t$. Moreover, it has the same couplings with the intermediate vector bosons Z, W^+, W^- as the standard model Higgs field. The Higgs field ϕ_0' is a certain linear combination of the $\underline{5}$–representation and the $\underline{45}$–representation. This linear combination is the only one which corresponds to a zero mode of the Grand Unification sector. That the mass of ϕ_0' is generically different from zero is due to the fermion masses. Therefore, the Higgs field ϕ_0' receives a mass of the order of the mass m_t of the top quark: For $m_t = 176\,\text{GeV}$ we have in tree–level approximation

$$\begin{aligned} m_{\phi_0'} &\leq 255\,\text{GeV} && \text{for the SU(5)}\times\text{U(1)–model}\,, \\ m_{\phi_0'} &\leq 232\,\text{GeV} && \text{for the SU(5)–model}\,. \end{aligned}$$

   The reason that only an upper bound can be given is the incomplete knowledge of the input parameters. The upper bound is independent of any parameters related to Grand Unification.

6) The standard model is in perfect agreement with experiment. However, we have shown that the low energy sector of both the SU(5)×U(1)– and SU(5)–Grand Unification models is identical with the standard model. This means that it is not possible to decide by means of present energy experiments which of the three models is correct. One essential advantage of the Grand Unification models is that they explain why proton and electron have up to the sign the same electric charge. On the other hand, the proton is not a stable particle in Grand Unification models. Concerning this question, the SU(5)×U(1)–model is favoured, because it yields a larger lifetime for the proton [24]. The insufficient lifetime of the proton predicted within the SU(5)–model was the reason that the SU(5)–model in its simplest form had to be rejected. The SU(5)×U(1)–model is preferred, because the additional U(1)–group prevents the existence of magnetic monopoles, which may catalyse the proton decay [24].

These remarks demonstrate that non–associative geometry is a powerful tool to study (classical) gauge field theories.

A Supplements to Section 3.2

Lemma 13. *For $c_i^1 \in C^1$ we have*

$$\tfrac{1}{2}(c_0^1(c_1^1\wedge\ldots\wedge c_n^1)+(-1)^n(c_1^1\wedge\ldots\wedge c_n^1)c_0^1) = c_0^1\wedge c_1^1\wedge c_2^1\wedge\ldots\wedge c_n^1\,, \tag{A.1}$$
$$\tfrac{1}{2}(c_0^1(c_1^1\wedge\ldots\wedge c_n^1)-(-1)^n(c_1^1\wedge\ldots\wedge c_n^1)c_0^1) = c_0^1\,\lrcorner\,(c_1^1\wedge c_2^1\wedge\ldots\wedge c_n^1)\,. \tag{A.2}$$

Proof.

$$\begin{aligned}
&c_0^1\, c^1_{\pi(1)}c^1_{\pi(2)}\cdots c^1_{\pi(n)}\\
&\quad= \tfrac{1}{n+1}c_0^1c^1_{\pi(1)}c^1_{\pi(2)}\cdots c^1_{\pi(n)}\\
&\quad+\tfrac{n}{n+1}(c_0^1c^1_{\pi(1)}+c^1_{\pi(1)}c_0^1)c^1_{\pi(2)}\cdots c^1_{\pi(n)}-\tfrac{n}{n+1}c^1_{\pi(1)}c_0^1c^1_{\pi(2)}\cdots c^1_{\pi(n)}\\
&\quad=\cdots=\tfrac{1}{n+1}\textstyle\sum_{j=0}^n(-1)^j c^1_{\pi(1)}c^1_{\pi(2)}\cdots c^1_{\pi(j)}c_0^1c^1_{\pi(j+1)}\cdots c^1_{\pi(n)}\\
&\quad-\textstyle\sum_{j=1}^n(-1)^j\,\tfrac{n+1-j}{n+1}c^1_{\pi(1)}c^1_{\pi(2)}\cdots c^1_{\pi(j-1)}\{c_0^1,c^1_{\pi(j)}\}c^1_{\pi(j+1)}\cdots c^1_{\pi(n)}\,,\\
&c^1_{\pi(1)}\cdots c^1_{\pi(n-1)}c^1_{\pi(n)}c_0^1\\
&\quad=\tfrac{1}{n+1}c^1_{\pi(1)}\cdots c^1_{\pi(n-1)}c^1_{\pi(n)}c_0^1\\
&\quad+\tfrac{n}{n+1}c^1_{\pi(1)}\cdots c^1_{\pi(n-1)}(c_0^1c^1_{\pi(n)}+c^1_{\pi(n)}c_0^1)-\tfrac{n}{n+1}c^1_{\pi(1)}\cdots c^1_{\pi(n-1)}c_0^1c^1_{\pi(n)}\\
&\quad=\cdots=\tfrac{1}{n+1}\textstyle\sum_{j=0}^n(-1)^{n-j}c^1_{\pi(1)}c^1_{\pi(2)}\cdots c^1_{\pi(j)}c_0^1c^1_{\pi(j+1)}\cdots c^1_{\pi(n)}\\
&\quad+\textstyle\sum_{j=1}^n(-1)^{n-j}\,\tfrac{j}{n+1}c^1_{\pi(1)}c^1_{\pi(2)}\cdots c^1_{\pi(j-1)}\{c_0^1,c^1_{\pi(j)}\}c^1_{\pi(j+1)}\cdots c^1_{\pi(n)}\,.
\end{aligned}$$

This gives

$$\begin{aligned}
&\tfrac{1}{2}\big(\,c_0^1(c^1_{\pi(1)}c^1_{\pi(2)}\cdots c^1_{\pi(n)})+(-1)^n(c^1_{\pi(1)}c^1_{\pi(2)}\cdots c^1_{\pi(n)})c_0^1\big)\\
&=\tfrac{1}{n+1}\textstyle\sum_{j=0}^n(-1)^j c^1_{\pi(1)}c^1_{\pi(2)}\cdots c^1_{\pi(j)}c_0^1c^1_{\pi(j+1)}\cdots c^1_{\pi(n)}\\
&\quad+\tfrac{1}{n+1}\textstyle\sum_{j=1}^n(-1)^j(j-\tfrac{n+1}{2})c^1_{\pi(1)}c^1_{\pi(2)}\cdots c^1_{\pi(j-1)}\{c_0^1,c^1_{\pi(j)}\}c^1_{\pi(j+1)}\cdots c^1_{\pi(n)}\,.
\end{aligned}\tag{A.3}$$

Now we take in the last formula the sum over all permutations of the numbers $1,\ldots,n$, respecting the sign. In the second sum on the r.h.s. of (A.3) we collect all terms having the same anticommutator with c_0^1. Denoting the set of permutations of the numbers $1,\ldots j-1, j+1,\ldots,n$ by P_j^{n-1} we obtain for the second sum

$$\begin{aligned}
&\tfrac{1}{n+1}\textstyle\sum_{\pi\in P^n}\sum_{j=1}^n(-1)^{j+\mathrm{sign}(\pi)}(j-\tfrac{n+1}{2})c^1_{\pi(1)}c^1_{\pi(2)}\cdots c^1_{\pi(j-1)}\{c_0^1,c^1_{\pi(j)}\}c^1_{\pi(j+1)}\cdots c^1_{\pi(n)}\\
&=\tfrac{1}{n+1}\textstyle\sum_{j=1}^n\sum_{\pi'\in P_j^{n-1}}\sum_{k=1}^n(-1)^{j+\mathrm{sign}(\pi')}\{c_0^1,c_j^1\}(k-\tfrac{n+1}{2})c^1_{\pi'(1)}\overset{\overset{j}{\vee}}{\cdots}c^1_{\pi'(n)}=0\,,
\end{aligned}$$

due to $\sum_{k=1}^n(k-\frac{n+1}{2})=0$. The sum over the permutations of the first sum of the r.h.s. of (A.3), together with the factor $\frac{1}{n!}$, yields the assertion (A.1). On the other hand,

$$\begin{aligned}
&\tfrac{1}{2}\big(\,c_0^1(c^1_{\pi(1)}c^1_{\pi(2)}\cdots c^1_{\pi(n)})-(-1)^n(c^1_{\pi(1)}c^1_{\pi(2)}\cdots c^1_{\pi(n)})c_0^1\big)\\
&\quad=\tfrac{1}{2}\textstyle\sum_{j=1}^n(-1)^{j+1}c^1_{\pi(1)}c^1_{\pi(2)}\cdots c^1_{\pi(j-1)}\{c_0^1,c^1_{\pi(j)}\}c^1_{\pi(j+1)}\cdots c^1_{\pi(n)}\,.
\end{aligned}$$

Taking the sum over all permutations we find precisely (A.2). □

Next, we check that the exterior differential

$$\mathbf{d}c^k := \sum_{j=1}^N c(e^j)\wedge\nabla_{e_j}(c^k)\,, \quad c^k\in\Lambda^k\,,$$

has the desired properties. First, for $k=0$ and $c^0\equiv f\in C^\infty(X)$ we have $\mathbf{d}f = \sum_{j=1}^N c(e^j)e_j(f) = c(\sum_{j=1}^N e_j(f)e^j) = c(\mathsf{d}f)$, where $\mathsf{d}: C^\infty(X)\to\Gamma^\infty(T^*X)$ is the usual exterior differential, fulfilling

$$\langle \mathsf{d}f, v\rangle = \langle \sum_{j=1}^N e^j e_j(f), \sum_{i=1}^N v^i e_i\rangle = \sum_{j=1}^N v^j e_j(f) = v(f)\,,$$

for any vector field $v=\sum_{i=1}^N v^i e_i\in\Gamma^\infty(T_*X)$. Obviously, for $c^k\in\Lambda^k$ and $\tilde c^l\in\Lambda^l$ we have

$$\begin{aligned}\mathbf{d}(c^k\wedge\tilde c^l) &= \sum_{j=1}^N c(e^j)\wedge\nabla_{e_j}(c^k\wedge\tilde c^l) = \sum_{j=1}^N (c(e^j)\wedge\nabla_{e_j}(c^k)\wedge\tilde c^l + c(e^j)\wedge c^k\wedge\nabla_{e_j}(\tilde c^l))\\ &= \sum_{j=1}^N (c(e^j)\wedge\nabla_{e_j}(c^k)\wedge\tilde c^l + (-1)^k c^k\wedge c(e^j)\wedge\nabla_{e_j}(\tilde c^l))\\ &= (\mathbf{d}c^k)\wedge\tilde c^l + (-1)^k c^k\wedge(\mathbf{d}\tilde c^l)\,.\end{aligned}$$

Thus, $\mathbf{d}$ is a graded derivation. Moreover, we have $\mathbf{d}^2\equiv 0$ on $C^\infty(X)$, see [7]:

$$\mathbf{d}^2 f = \sum_{i,j=1}^N c(e^i)\wedge\nabla_{e_i}(c(e^j\,e_j(f))) = \sum_{i,j=1}^N c(e^i\wedge\nabla_{e_i}(e^j)\,e_j(f) + e^i\wedge e^j\,e_ie_j(f))$$

Using

$$\nabla_{e_i}(e^j) = \sum_{k=1}^N \Gamma^j_{ik}e^k = \sum_{k=1}^N\langle\nabla_{e_i}(e^j), e_k\rangle e^k = -\sum_{k=1}^N\langle e^j,\nabla_{e_i}(e_k)\rangle e^k \tag{A.4}$$

and $[e_i,e_k] = \sum_{j=1}^N c^j_{ik}e_j = \sum_{j=1}^N\langle e^j,[e_i,e_k]\rangle e_j$ we get

$$\begin{aligned}\mathbf{d}^2 f &= \sum_{i,j,k=1}^N c(-e^i\wedge e^k\langle e^j,\nabla_{e_i}(e_k)\rangle e_j(f) + e^i\wedge e^k\langle e^j,\tfrac12[e_i,e_k]\rangle e_j(f))\\ &= \sum_{i,j,k=1}^N c(-\tfrac12 e^i\wedge e^k\langle e^j,\nabla_{e_i}(e_k)-\nabla_{e_k}(e_i)-[e_i,e_k]\rangle e_j(f))\equiv 0\,,\end{aligned}$$

because the Levi–Civita connection ∇_v has vanishing torsion, $T(v,w)=\nabla_v(w)-\nabla_w(v)-[v,w]=0$, for any $v,w\in\Gamma^\infty(T_*X)$. Since Λ^* is generated by $C^\infty(X)$ and $\mathbf{d}C^\infty(X)$ we get that $\mathbf{d}$ is indeed a graded differential on Λ^*.

Lemma 14. *Within our conventions one has the representation*

$$\mathbf{d}^*c^k = -\sum_{j=1}^N c(e^j)\,\lrcorner\,\nabla_{e_j}(c^k)\,. \tag{A.5}$$

Proof. For $c^k\in\Lambda^k$ and $\tilde c^{k+1}\in\Lambda^{k+1}$ we have

$$\begin{aligned}(\mathbf{d}c^k,\tilde c^{k+1})_{\Lambda^*} &\equiv \int_X \mathrm{v}_g\ \mathrm{tr}_c\left(\sum_{j=1}^N (c(e^j)\wedge\nabla_{e_j}(c^k))^*\tilde c^{k+1}\right)\\ &= \int_X \mathrm{v}_g\ \mathrm{tr}_c\left(\sum_{j=1}^N ((\nabla_{e_j}c^k)^*\wedge c(e^j))\tilde c^{k+1}\right)\\ &= \int_X \mathrm{v}_g\ \mathrm{tr}_c\left(\sum_{j=1}^N \tfrac12((\nabla_{e_j}c^k)^*c(e^j) + (-1)^k c(e^j)(\nabla_{e_j}c^k)^*)\tilde c^{k+1}\right)\\ &= \int_X \mathrm{v}_g\ \sum_{j=1}^N \mathrm{tr}_c\left(\tfrac12 c(e^j)(\tilde c^{k+1}(\nabla_{e_j}c^k)^* + (-1)^k(\nabla_{e_j}c^k)^*\tilde c^{k+1})\right)\end{aligned}$$

$$= \int_X \mathrm{v}_g \, \sum_{j=1}^N \mathrm{tr}_c \left(\tfrac{1}{2} c(e^j) \nabla_{e_j} (\tilde{c}^{k+1} c^{k^*} + (-1)^k c^{k^*} \tilde{c}^{k+1}) \right) \qquad \text{(A.6)}$$
$$- \int_X \mathrm{v}_g \, \sum_{j=1}^N \mathrm{tr}_c \left(\tfrac{1}{2} c(e^j) (\nabla_{e_j} (\tilde{c}^{k+1}) c^{k^*} + (-1)^k c^{k^*} \nabla_{e_j} (\tilde{c}^{k+1})) \right) . \qquad \text{(A.7)}$$

Here we have used that the sections e_j and e^j are selfadjoint. The integral (A.7) takes the form

$$\begin{aligned}
\text{(A.7)} &= - \int_X \mathrm{v}_g \, \sum_{j=1}^N \mathrm{tr}_c \left(\tfrac{1}{2} c(e^j) (\nabla_{e_j} (\tilde{c}^{k+1}) c^{k^*} + (-1)^k c^{k^*} \nabla_{e_j} (\tilde{c}^{k+1})) \right) \\
&= - \int_X \mathrm{v}_g \, \mathrm{tr}_c \left(c^{k^*} \sum_{j=1}^N \tfrac{1}{2} (c(e^j) \nabla_{e_j} (\tilde{c}^{k+1}) + (-1)^k \nabla_{e_j} (\tilde{c}^{k+1}) c(e^j)) \right) \\
&= \int_X \mathrm{v}_g \, \mathrm{tr}_c \left(c^{k^*} (- \textstyle\sum_{j=1}^N c(e^j) \lrcorner \nabla_{e_j} (\tilde{c}^{k+1})) \right) .
\end{aligned}$$

Thus, we arrive at (A.5) provided that the integral (A.6) vanishes, which is indeed the case. To see this, observe that there can only be a contribution from the Λ^1–component of $\frac{1}{2}(\tilde{c}^{k+1} c^{k^*} + (-1)^k c^{k^*} \tilde{c}^{k+1})$, which we denote by $\hat{c}^1$. Thus, we must show that

$$\int_X \mathrm{v}_g \, \sum_{j=1}^N \mathrm{tr}_c \left(c(e^j) \nabla_{e_j} (\hat{c}^1) \right) \equiv 0 ,$$

for all $\hat{c}^1 \in \Lambda^1$. We use the decompositions $\nabla_{e_j}(\hat{c}^1) \equiv \sum_{i=1}^N A_{ji} c(e^i)$ and $c(\mathrm{v}_g) = f c(e^1) \wedge \ldots \wedge c(e^N)$, where the precise form $A_{ji}, f \in C^\infty(X)$ is not important in the moment. Then,

$$\sum_{j=1}^N \mathrm{v}_g \, \mathrm{tr}_c \left(c(e^j) \nabla_{e_j} (\hat{c}^1) \right) = \sum_{i,j=1}^N c^{-1} \left(c(\mathrm{v}_g) A_{ji} \, \mathrm{tr}_c (\tfrac{1}{2} \{ c(e^j), c(e^i) \}) \right) .$$

On the other hand,

$$\begin{aligned}
&\sum_{j=1}^N c^{-1} \left(c(e^j) \wedge (\nabla_{e_j} (\hat{c}^1) \lrcorner c(\mathrm{v}_g)) \right) \\
&\quad = \sum_{i,j,k=1}^N c^{-1} \left(c(e^j) \wedge (A_{ji} f (-1)^{k+1} \tfrac{1}{2} \{ c(e^i), c(e^k) \} c(e^1) \wedge \overset{k}{\overset{\vee}{\ldots}} \wedge c(e^N)) \right) \\
&\quad = \sum_{i,j=1}^N c^{-1} \left(A_{ji} \tfrac{1}{2} \{ c(e^i), c(e^j) \} c(\mathrm{v}_g) \right) .
\end{aligned}$$

Since $\{c(e^i), c(e^j)\}$ is proportional to the identity operator we obtain

$$\begin{aligned}
\text{(A.6)} &= \mathrm{tr}_c(1) \int_X \sum_{j=1}^N c^{-1} \left(c(e^j) \wedge (\nabla_{e_j} (\hat{c}^1) \lrcorner c(\mathrm{v}_g)) \right) \\
&= \mathrm{tr}_c(1) \int_X \sum_{j=1}^N c^{-1} \left(c(e^j) \wedge \nabla_{e_j} (\hat{c}^1 \lrcorner c(\mathrm{v}_g)) \right) \\
&= \mathrm{tr}_c(1) \int_X c^{-1} \left(\mathbf{d} (\hat{c}^1 \lrcorner c(\mathrm{v}_g)) \right) \equiv \mathrm{tr}_c(1) \int_X \mathsf{d} (c^{-1} \left(\hat{c}^1 \lrcorner c(\mathrm{v}_g) \right)) \equiv 0 ,
\end{aligned}$$

where we have used Stokes' theorem and the fact that $\nabla_w \mathrm{v}_g \equiv 0$. □

Finally, using (3.21) we calculate for $f \in C^\infty(X)$

$$[\mathsf{D}^2, f] = -((-\mathrm{i}\mathsf{D})[-\mathrm{i}\mathsf{D}, f] + [-\mathrm{i}\mathsf{D}, f](-\mathrm{i}\mathsf{D})) = \mathbf{d}^* \mathbf{d} f - 2 \nabla^S_{g^{-1}(c^{-1}(\mathbf{d} f))} \equiv \Delta f - 2 \nabla^S_{\mathrm{grad}\, f} ,$$

where $\mathrm{grad}\, f := g^{-1}(c^{-1}(\mathbf{d} f)) = g^{-1}(\mathsf{d} f)$ and $\Delta f \equiv \mathbf{d}^* \mathbf{d} f$. In terms of local coordinates, the scalar Laplacian Δ takes the form

$$\begin{aligned}
\Delta f \equiv \mathbf{d}^* \mathbf{d} f &= - \sum_{i,j=1}^N c(e^i) \lrcorner \nabla_{e_i} (c(e^j) \nabla_{e_j} (f)) \\
&= - \sum_{i,j=1}^N (c(e^i) \lrcorner c(e^j) \nabla_{e_i} \nabla_{e_j} (f) + c(e^i) \lrcorner c(\nabla_{e_i} (e^j)) \nabla_{e_j} (f)) ,
\end{aligned}$$

see (3.12) and (3.15). Using (A.4) and $c(e^i) \lrcorner c(e^j) = \frac{1}{2}\{c(e^i), c(e^j)\} = g^{-1}(e^i, e^j)1$ we obtain

$$\begin{aligned}\Delta f &= -\textstyle\sum_{i,j=1}^N g^{-1}(e^i,e^j)(\nabla_{e_i}\nabla_{e_j}(f) - \sum_{k=1}^N \langle e^k, \nabla_{e_i}(e_j)\rangle \nabla_{e_k}(f)) \\ &= -\textstyle\sum_{i,j=1}^N g^{-1}(e^i,e^j)(\nabla_{e_i}\nabla_{e_j} - \nabla_{\nabla_{e_i}e_j})(f)\,.\end{aligned}$$

B The Quadratic Terms of the Higgs Potential

Here we compute the quadratic terms in the Higgs potential $\mathscr{L}_0$, see (6.10d). We find:

$$(\mathrm{tr}((\boldsymbol{\Psi}+\mathbf{m})^2) - \tfrac{6}{5})^2 \to 4(\mathrm{tr}(\boldsymbol{\Psi}\mathbf{m}))^2 = \tfrac{48}{5}\Psi_0^2\,, \tag{B.1a}$$

$$((\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n}) - 1)^2 \to (\mathbf{n}^*\boldsymbol{\Phi} + \boldsymbol{\Phi}^*\mathbf{n})^2 = 4\Phi_0^2\,, \tag{B.1b}$$

$$(\mathrm{tr}((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\boldsymbol{\Upsilon}+\mathbf{n}')) - 12)^2 \to (\mathrm{tr}(\mathbf{n}'^*\boldsymbol{\Upsilon} + \boldsymbol{\Upsilon}^*\mathbf{n}'))^2 = 96\Upsilon_0^2\,, \tag{B.1c}$$

$$(\mathrm{tr}((\boldsymbol{\Psi}+\mathbf{m})^2) - \tfrac{6}{5})((\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n}) - 1) \to 2\,\mathrm{tr}(\mathbf{m}\boldsymbol{\Psi})(\mathbf{n}^*\boldsymbol{\Phi} + \boldsymbol{\Phi}^*\mathbf{n}) = 8\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0\,, \tag{B.1d}$$

$$\begin{aligned}&(\mathrm{tr}((\boldsymbol{\Psi}+\mathbf{m})^2) - \tfrac{6}{5})(\mathrm{tr}((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\boldsymbol{\Upsilon}+\mathbf{n}')) - 12) \\ &\to 2\,\mathrm{tr}(\mathbf{m}\boldsymbol{\Psi})(\mathrm{tr}(\mathbf{n}'^*\boldsymbol{\Upsilon} + \boldsymbol{\Upsilon}^*\mathbf{n}')) = 48\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0\,,\end{aligned} \tag{B.1e}$$

$$\begin{aligned}&((\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n}) - 1)(\mathrm{tr}((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\boldsymbol{\Upsilon}+\mathbf{n}')) - 12) \\ &\to (\boldsymbol{\Phi}^*\mathbf{n} + \mathbf{n}^*\boldsymbol{\Phi})\,\mathrm{tr}(\mathbf{n}'^*\boldsymbol{\Upsilon} + \boldsymbol{\Upsilon}^*\mathbf{n}') = 8\sqrt{6}\Phi_0\Upsilon_0\,,\end{aligned} \tag{B.1f}$$

$$\begin{aligned}&((\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Psi}+\mathbf{m}) - \tfrac{3}{5}(\boldsymbol{\Phi}+\mathbf{n})^*)((\boldsymbol{\Psi}+\mathbf{m})(\boldsymbol{\Phi}+\mathbf{n}) - \tfrac{3}{5}(\boldsymbol{\Phi}+\mathbf{n})) \\ &\to (\boldsymbol{\Phi}^*\mathbf{m} + \mathbf{n}^*\boldsymbol{\Psi} - \tfrac{3}{5}\boldsymbol{\Phi}^*)(\mathbf{m}\boldsymbol{\Phi} + \boldsymbol{\Psi}\mathbf{n} - \tfrac{3}{5}\boldsymbol{\Phi}) = \tfrac{3}{5}\Psi_0^2 + \textstyle\sum_{i=1}^3 \Psi_i'^2 + \sum_{i=1}^6 \Phi_i^2 + 2\sqrt{\tfrac{3}{5}}\Psi_0\Psi_3'\,,\end{aligned} \tag{B.1g}$$

$$\begin{aligned}&\mathrm{tr}(((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}}) - (\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^T + \tfrac{1}{2}(\hat{\boldsymbol{\Upsilon}}+\hat{\mathbf{n}}')) \\ &\qquad\times((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}}) - (\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^T + \tfrac{1}{2}(\hat{\boldsymbol{\Upsilon}}+\hat{\mathbf{n}}'))^*) \\ &\to \mathrm{tr}((\check{\boldsymbol{\Psi}}\hat{\mathbf{n}} + \check{\mathbf{m}}\hat{\boldsymbol{\Phi}} - \hat{\mathbf{n}}\check{\boldsymbol{\Psi}}^T - \hat{\boldsymbol{\Phi}}\check{\mathbf{m}}^T + \tfrac{1}{2}\hat{\boldsymbol{\Upsilon}})(\check{\boldsymbol{\Psi}}\hat{\mathbf{n}} + \check{\mathbf{m}}\hat{\boldsymbol{\Phi}} - \hat{\mathbf{n}}\check{\boldsymbol{\Psi}}^T - \hat{\boldsymbol{\Phi}}\check{\mathbf{m}}^T + \tfrac{1}{2}\hat{\boldsymbol{\Upsilon}})^*) \\ &= 10\Psi_0^2 + 16\textstyle\sum_{i=1}^8 \Psi_i^2 + 6\sum_{i=1}^8 \Psi_i'^2 + 6\Phi_0^2 + 8\sum_{i=1}^6 \Phi_i^2 - 20\sqrt{\tfrac{3}{5}}\Psi_0\Psi_3' + 20\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0 \\ &\quad -12\Psi_3'\Phi_0 - 10\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0 - 2\sqrt{6}\Phi_0\Upsilon_0 + 2\sqrt{6}(\Psi_1'\Upsilon_{18} + \Psi_2'\Upsilon_{63} + \Psi_3'\Upsilon_0) + 8\textstyle\sum_{i=1}^8 \Psi_i\Upsilon_i \\ &\quad -4\sqrt{2}(\Phi_1\Upsilon_9 + \Phi_2\Upsilon_{10} + \Phi_3\Upsilon_{11} + \Phi_4\Upsilon_{54} + \Phi_5\Upsilon_{55} + \Phi_6\Upsilon_{56}) + \textstyle\sum_{i=0}^{89}\Upsilon_i^2\,,\end{aligned} \tag{B.1h}$$

$$\begin{aligned}&\mathrm{tr}(((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) - \tfrac{9}{20}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) + \tfrac{1}{4}(\boldsymbol{\Upsilon}+\mathbf{n}'))^*((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) - \tfrac{9}{20}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) + \tfrac{1}{4}(\boldsymbol{\Upsilon}+\mathbf{n}'))) \\ &\to \mathrm{tr}((\check{\boldsymbol{\Phi}}^*\check{\mathbf{m}} + \check{\mathbf{n}}^*\check{\boldsymbol{\Psi}} - \tfrac{9}{20}\check{\boldsymbol{\Phi}}^* + \tfrac{1}{4}\boldsymbol{\Upsilon}^*)(\check{\boldsymbol{\Psi}}\check{\mathbf{n}} + \check{\mathbf{m}}\check{\boldsymbol{\Phi}} - \tfrac{9}{20}\check{\boldsymbol{\Phi}} + \tfrac{1}{4}\boldsymbol{\Upsilon})) \\ &= \tfrac{13}{5}\Psi_0^2 + 2\textstyle\sum_{i=1}^8 \Psi_i^2 + 3\sum_{i=1}^3 \Psi_i'^2 + \tfrac{13}{4}\sum_{i=1}^6 \Phi_i^2 + 2\sqrt{\tfrac{3}{5}}\Psi_0\Psi_3' + \tfrac{5}{2}\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0 - \tfrac{3}{2}\Psi_3'\Phi_0 + \tfrac{3}{4}\Phi_0^2 \\ &\quad -\tfrac{5}{4}\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0 + \tfrac{1}{4}\sqrt{6}(\Psi_1'\Upsilon_{18} + \Psi_2'\Upsilon_{63} + \Psi_3'\Upsilon_0) - \tfrac{1}{4}\sqrt{6}\Phi_0\Upsilon_0 + \textstyle\sum_{i=1}^8 \Psi_i\Upsilon_i \\ &\quad -\tfrac{1}{2}\sqrt{2}(\Phi_1\Upsilon_9 + \Phi_2\Upsilon_{10} + \Phi_3\Upsilon_{11} + \Phi_4\Upsilon_{54} + \Phi_5\Upsilon_{55} + \Phi_6\Upsilon_{56}) + \tfrac{1}{8}\textstyle\sum_{i=0}^{89}\Upsilon_i^2\,,\end{aligned} \tag{B.1i}$$

$$\begin{aligned}&\mathrm{tr}(((\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})(\boldsymbol{\Psi}+\mathbf{m}) + \tfrac{3}{20}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) + \tfrac{1}{4}(\boldsymbol{\Upsilon}+\mathbf{n}'))^*((\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})(\boldsymbol{\Psi}+\mathbf{m}) + \tfrac{3}{20}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) + \tfrac{1}{4}(\boldsymbol{\Upsilon}+\mathbf{n}'))) \\ &\to \mathrm{tr}((\mathbf{m}\check{\boldsymbol{\Phi}}^* + \boldsymbol{\Psi}\check{\mathbf{n}}^* + \tfrac{3}{20}\check{\boldsymbol{\Phi}}^* + \tfrac{1}{4}\boldsymbol{\Upsilon}^*)(\check{\mathbf{n}}\boldsymbol{\Psi} + \check{\boldsymbol{\Phi}}\mathbf{m} + \tfrac{3}{20}\check{\boldsymbol{\Phi}} + \tfrac{1}{4}\boldsymbol{\Upsilon}))\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{7}{5}\Psi_0^2 + 2\textstyle\sum_{i=1}^{8}\Psi_i^2 + \sum_{i=1}^{3}\Psi_i'^2 + \tfrac{5}{4}\sum_{i=1}^{6}\Phi_i^2 - 2\sqrt{\tfrac{3}{5}}\Psi_0\Psi_3' + \tfrac{5}{2}\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0 - \tfrac{3}{2}\Psi_3'\Phi_0 + \tfrac{3}{4}\Phi_0^2 \\
&\quad -\tfrac{5}{4}\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0 + \tfrac{1}{4}\sqrt{6}(\Psi_1'\Upsilon_{18} + \Psi_2'\Upsilon_{63} + \Psi_3'\Upsilon_0) - \tfrac{1}{4}\sqrt{6}\Phi_0\Upsilon_0 + \textstyle\sum_{i=1}^{8}\Psi_i\Upsilon_i \\
&\quad -\tfrac{1}{2}\sqrt{2}(\Phi_1\Upsilon_9 + \Phi_2\Upsilon_{10} + \Phi_3\Upsilon_{11} + \Phi_4\Upsilon_{54} + \Phi_5\Upsilon_{55} + \Phi_6\Upsilon_{56}) + \tfrac{1}{8}\textstyle\sum_{i=0}^{89}\Upsilon_i^2\,,
\end{aligned}
\tag{B.1j}
$$

$$
\begin{aligned}
&\mathrm{tr}(((\check{\boldsymbol{\Psi}} + \check{\mathbf{m}})(\boldsymbol{\Upsilon} + \mathbf{n}') + \tfrac{3}{4}(\check{\boldsymbol{\Phi}} + \check{\mathbf{n}}) - \tfrac{19}{20}(\boldsymbol{\Upsilon} + \mathbf{n}'))^*((\check{\boldsymbol{\Psi}} + \check{\mathbf{m}})(\boldsymbol{\Upsilon} + \mathbf{n}') + \tfrac{3}{4}(\check{\boldsymbol{\Phi}} + \check{\mathbf{n}}) - \tfrac{19}{20}(\boldsymbol{\Upsilon} + \mathbf{n}'))) \\
&\to \mathrm{tr}((\boldsymbol{\Upsilon}^*\check{\mathbf{m}} + \mathbf{n}'^*\check{\boldsymbol{\Psi}} + \tfrac{3}{4}\check{\boldsymbol{\Phi}}^* - \tfrac{19}{20}\boldsymbol{\Upsilon}^*)(\check{\boldsymbol{\Psi}}\mathbf{n}' + \check{\mathbf{m}}\boldsymbol{\Upsilon} + \tfrac{3}{4}\check{\boldsymbol{\Phi}} - \tfrac{19}{20}\boldsymbol{\Upsilon})) \\
&= \tfrac{109}{5}\Psi_0^2 + 2\textstyle\sum_{i=1}^{8}\Psi_i^2 + 3\sum_{i=1}^{3}\Psi_i'^2 + \tfrac{9}{4}\sum_{i=1}^{6}\Phi_i^2 + 2\sqrt{\tfrac{3}{5}}\Psi_0\Psi_3' - \tfrac{15}{2}\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0 + \tfrac{9}{2}\Psi_3'\Phi_0 + \tfrac{9}{4}\Phi_0^2 \\
&\quad +\tfrac{15}{4}\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0 - \tfrac{3}{4}\sqrt{6}(\Psi_1'\Upsilon_{18} + \Psi_2'\Upsilon_{63} + \Psi_3'\Upsilon_0) - \tfrac{3}{4}\sqrt{6}\Phi_0\Upsilon_0 + \tfrac{3}{8}\Upsilon_0^2 + \tfrac{3}{8}\Upsilon_{45}^2 - 3\textstyle\sum_{i=1}^{8}\Psi_i\Upsilon_i \\
&\quad -\tfrac{3}{2}\sqrt{2}(\Phi_1\Upsilon_9 + \Phi_2\Upsilon_{10} + \Phi_3\Upsilon_{11} + \Phi_4\Upsilon_{54} + \Phi_5\Upsilon_{55} + \Phi_6\Upsilon_{56}) \\
&\quad +\tfrac{9}{8}\textstyle\sum_{i=1}^{8}\Upsilon_i^2 + \tfrac{29}{8}\sum_{i=9}^{11}\Upsilon_i^2 + \tfrac{9}{8}\sum_{i=12}^{17}\Upsilon_i^2 + \tfrac{3}{8}\Upsilon_{18}^2 + \tfrac{9}{8}\sum_{i=19}^{29}\Upsilon_i^2 + \tfrac{49}{8}\sum_{i=30}^{41}\Upsilon_i^2 + \tfrac{1}{8}\sum_{i=42}^{44}\Upsilon_i^2 \\
&\quad +\tfrac{9}{8}\textstyle\sum_{i=46}^{53}\Upsilon_i^2 + \tfrac{29}{8}\sum_{i=54}^{56}\Upsilon_i^2 + \tfrac{9}{8}\sum_{i=57}^{62}\Upsilon_i^2 + \tfrac{3}{8}\Upsilon_{63}^2 + \tfrac{9}{8}\sum_{i=64}^{74}\Upsilon_i^2 + \tfrac{49}{8}\sum_{i=75}^{86}\Upsilon_i^2 + \tfrac{1}{8}\sum_{i=87}^{89}\Upsilon_i^2\,,
\end{aligned}
\tag{B.1k}
$$

$$
\begin{aligned}
&\mathrm{tr}(((\boldsymbol{\Upsilon} + \mathbf{n}')(\boldsymbol{\Psi} + \mathbf{m}) + \tfrac{3}{4}(\check{\boldsymbol{\Phi}} + \check{\mathbf{n}}) - \tfrac{7}{20}(\boldsymbol{\Upsilon} + \mathbf{n}'))^*((\boldsymbol{\Upsilon} + \mathbf{n}')(\boldsymbol{\Psi} + \mathbf{m}) + \tfrac{3}{4}(\check{\boldsymbol{\Phi}} + \check{\mathbf{n}}) - \tfrac{7}{20}(\boldsymbol{\Upsilon} + \mathbf{n}'))) \\
&\to \mathrm{tr}((\mathbf{m}\boldsymbol{\Upsilon}^* + \boldsymbol{\Psi}\mathbf{n}'^* + \tfrac{3}{4}\check{\boldsymbol{\Phi}}^* - \tfrac{7}{20}\boldsymbol{\Upsilon}^*)(\mathbf{n}'\boldsymbol{\Psi} + \boldsymbol{\Upsilon}\mathbf{m} + \tfrac{3}{4}\check{\boldsymbol{\Phi}} - \tfrac{7}{20}\boldsymbol{\Upsilon})) \\
&= \tfrac{31}{5}\Psi_0^2 + 2\textstyle\sum_{i=1}^{8}\Psi_i^2 + 9\sum_{i=1}^{3}\Psi_i'^2 + \tfrac{9}{4}\sum_{i=1}^{6}\Phi_i^2 - 18\sqrt{\tfrac{3}{5}}\Psi_0\Psi_3' - \tfrac{15}{2}\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0 + \tfrac{9}{2}\Psi_3'\Phi_0 + \tfrac{9}{4}\Phi_0^2 \\
&\quad +\tfrac{15}{4}\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0 - \tfrac{3}{4}\sqrt{6}(\Psi_1'\Upsilon_{18} + \Psi_2'\Upsilon_{63} + \Psi_3'\Upsilon_0) - \tfrac{3}{4}\sqrt{6}\Phi_0\Upsilon_0 + \tfrac{3}{8}\Upsilon_0^2 + \tfrac{3}{8}\Upsilon_{45}^2 - 3\textstyle\sum_{i=1}^{8}\Psi_i\Upsilon_i \\
&\quad -\tfrac{3}{2}\sqrt{2}(\Phi_1\Upsilon_9 + \Phi_2\Upsilon_{10} + \Phi_3\Upsilon_{11} + \Phi_4\Upsilon_{54} + \Phi_5\Upsilon_{55} + \Phi_6\Upsilon_{56}) \\
&\quad +\tfrac{9}{8}\textstyle\sum_{i=1}^{8}\Upsilon_i^2 + \tfrac{5}{8}\sum_{i=9}^{11}\Upsilon_i^2 + \tfrac{1}{8}\sum_{i=12}^{17}\Upsilon_i^2 + \tfrac{3}{8}\Upsilon_{18}^2 + \tfrac{9}{8}\sum_{i=19}^{26}\Upsilon_i^2 + \tfrac{1}{8}\sum_{i=27}^{29}\Upsilon_i^2 + \tfrac{9}{8}\sum_{i=30}^{35}\Upsilon_i^2 \\
&\quad +\tfrac{1}{8}\textstyle\sum_{i=36}^{41}\Upsilon_i^2 + \tfrac{9}{8}\sum_{i=42}^{44}\Upsilon_i^2 + \tfrac{9}{8}\sum_{i=46}^{53}\Upsilon_i^2 + \tfrac{5}{8}\sum_{i=54}^{56}\Upsilon_i^2 + \tfrac{1}{8}\sum_{i=57}^{62}\Upsilon_i^2 + \tfrac{3}{8}\Upsilon_{63}^2 \\
&\quad +\tfrac{9}{8}\textstyle\sum_{i=64}^{71}\Upsilon_i^2 + \tfrac{1}{8}\sum_{i=72}^{74}\Upsilon_i^2 + \tfrac{9}{8}\sum_{i=75}^{80}\Upsilon_i^2 + \tfrac{1}{8}\sum_{i=81}^{86}\Upsilon_i^2 + \tfrac{9}{8}\sum_{i=87}^{89}\Upsilon_i^2\,,
\end{aligned}
\tag{B.1l}
$$

$$
\begin{aligned}
&\mathrm{Re}(-\mathrm{tr}(((\check{\boldsymbol{\Psi}} + \check{\mathbf{m}})(\check{\boldsymbol{\Phi}} + \check{\mathbf{n}}) - \tfrac{9}{20}(\check{\boldsymbol{\Phi}} + \check{\mathbf{n}}) + \tfrac{1}{4}(\boldsymbol{\Upsilon} + \mathbf{n}'))^*((\check{\boldsymbol{\Phi}} + \check{\mathbf{n}})(\boldsymbol{\Psi} + \mathbf{m}) + \tfrac{3}{20}(\check{\boldsymbol{\Phi}} + \check{\mathbf{n}}) + \tfrac{1}{4}(\boldsymbol{\Upsilon} + \mathbf{n}')))) \\
&\to \mathrm{Re}(-\mathrm{tr}((\check{\boldsymbol{\Phi}}^*\check{\mathbf{m}} + \check{\mathbf{n}}^*\check{\boldsymbol{\Psi}} - \tfrac{9}{20}\check{\boldsymbol{\Phi}}^* + \tfrac{1}{4}\boldsymbol{\Upsilon}^*)(\check{\mathbf{n}}\boldsymbol{\Psi} + \check{\boldsymbol{\Phi}}\mathbf{m} + \tfrac{3}{20}\check{\boldsymbol{\Phi}} + \tfrac{1}{4}\boldsymbol{\Upsilon}))) \\
&= -\tfrac{4}{5}\Psi_0^2 - 2\textstyle\sum_{i=1}^{8}\Psi_i^2 - \tfrac{1}{4}\sum_{i=1}^{6}\Phi_i^2 + 4\sqrt{\tfrac{3}{5}}\Psi_0\Psi_3' - \tfrac{5}{2}\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0 + \tfrac{3}{2}\Psi_3'\Phi_0 - \tfrac{3}{4}\Phi_0^2 \\
&\quad +\tfrac{5}{4}\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0 - \tfrac{1}{4}\sqrt{6}(\Psi_1'\Upsilon_{18} + \Psi_2'\Upsilon_{63} + \Psi_3'\Upsilon_0) + \tfrac{1}{4}\sqrt{6}\Phi_0\Upsilon_0 - \textstyle\sum_{i=1}^{8}\Psi_i\Upsilon_i \\
&\quad +\tfrac{1}{2}\sqrt{2}(\Phi_1\Upsilon_9 + \Phi_2\Upsilon_{10} + \Phi_3\Upsilon_{11} + \Phi_4\Upsilon_{54} + \Phi_5\Upsilon_{55} + \Phi_6\Upsilon_{56}) - \tfrac{1}{8}\textstyle\sum_{i=1}^{90}\Upsilon_i^2\,,
\end{aligned}
\tag{B.1m}
$$

$$
\begin{aligned}
&\mathrm{Re}(\mathrm{tr}(((\check{\boldsymbol{\Psi}} + \check{\mathbf{m}})(\check{\boldsymbol{\Phi}} + \check{\mathbf{n}}) - \tfrac{9}{20}(\check{\boldsymbol{\Phi}} + \check{\mathbf{n}}) + \tfrac{1}{4}(\boldsymbol{\Upsilon} + \mathbf{n}'))^*((\check{\boldsymbol{\Psi}} + \check{\mathbf{m}})(\boldsymbol{\Upsilon} + \mathbf{n}') + \tfrac{3}{4}(\check{\boldsymbol{\Phi}} + \check{\mathbf{n}}) - \tfrac{19}{20}(\boldsymbol{\Upsilon} + \mathbf{n}')))) \\
&\to \mathrm{Re}(\mathrm{tr}((\check{\boldsymbol{\Phi}}^*\check{\mathbf{m}} + \check{\mathbf{n}}^*\check{\boldsymbol{\Psi}} - \tfrac{9}{20}\check{\boldsymbol{\Phi}}^* + \tfrac{1}{4}\boldsymbol{\Upsilon}^*)(\check{\boldsymbol{\Psi}}\mathbf{n}' + \check{\mathbf{m}}\boldsymbol{\Upsilon} + \tfrac{3}{4}\check{\boldsymbol{\Phi}} - \tfrac{19}{20}\boldsymbol{\Upsilon}))) \\
&= -7\Psi_0^2 + 2\textstyle\sum_{i=1}^{8}\Psi_i^2 + 3\sum_{i=1}^{3}\Psi_i'^2 - \tfrac{9}{4}\sum_{i=1}^{6}\Phi_i^2 + 2\sqrt{\tfrac{3}{5}}\Psi_0\Psi_3' - \tfrac{5}{2}\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0 + \tfrac{3}{2}\Psi_3'\Phi_0 \\
&\quad +\tfrac{5}{4}\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0 - \tfrac{1}{4}\sqrt{6}(\Psi_1'\Upsilon_{18} + \Psi_2'\Upsilon_{63} + \Psi_3'\Upsilon_0) - \textstyle\sum_{i=1}^{8}\Psi_i\Upsilon_i \\
&\quad +2\sqrt{2}(\Phi_1\Upsilon_9 + \Phi_2\Upsilon_{10} + \Phi_3\Upsilon_{11} + \Phi_4\Upsilon_{54} + \Phi_5\Upsilon_{55} + \Phi_6\Upsilon_{56}) \\
&\quad -\tfrac{3}{8}\textstyle\sum_{i=1}^{8}\Upsilon_i^2 - \tfrac{5}{8}\sum_{i=9}^{11}\Upsilon_i^2 - \tfrac{3}{8}\sum_{i=12}^{17}\Upsilon_i^2 - \tfrac{3}{8}\sum_{i=19}^{29}\Upsilon_i^2 - \tfrac{7}{8}\sum_{i=30}^{41}\Upsilon_i^2 + \tfrac{1}{8}\sum_{i=42}^{44}\Upsilon_i^2 \\
&\quad -\tfrac{3}{8}\textstyle\sum_{i=46}^{53}\Upsilon_i^2 - \tfrac{5}{8}\sum_{i=54}^{56}\Upsilon_i^2 - \tfrac{3}{8}\sum_{i=57}^{62}\Upsilon_i^2 - \tfrac{3}{8}\sum_{i=64}^{74}\Upsilon_i^2 - \tfrac{7}{8}\sum_{i=75}^{86}\Upsilon_i^2 + \tfrac{1}{8}\sum_{i=87}^{89}\Upsilon_i^2\,,
\end{aligned}
\tag{B.1n}
$$

$$
\begin{aligned}
&\mathrm{Im}(\mathrm{tr}(((\check{\boldsymbol{\Psi}} + \check{\mathbf{m}})(\check{\boldsymbol{\Phi}} + \check{\mathbf{n}}) - \tfrac{9}{20}(\check{\boldsymbol{\Phi}} + \check{\mathbf{n}}) + \tfrac{1}{4}(\boldsymbol{\Upsilon} + \mathbf{n}'))^*((\check{\boldsymbol{\Psi}} + \check{\mathbf{m}})(\boldsymbol{\Upsilon} + \mathbf{n}') + \tfrac{3}{4}(\check{\boldsymbol{\Phi}} + \check{\mathbf{n}}) - \tfrac{19}{20}(\boldsymbol{\Upsilon} + \mathbf{n}')))) \\
&\to \mathrm{Im}(\mathrm{tr}((\check{\boldsymbol{\Phi}}^*\check{\mathbf{m}} + \check{\mathbf{n}}^*\check{\boldsymbol{\Psi}} - \tfrac{9}{20}\check{\boldsymbol{\Phi}}^* + \tfrac{1}{4}\boldsymbol{\Upsilon}^*)(\check{\boldsymbol{\Psi}}\mathbf{n}' + \check{\mathbf{m}}\boldsymbol{\Upsilon} + \tfrac{3}{4}\check{\boldsymbol{\Phi}} - \tfrac{19}{20}\boldsymbol{\Upsilon})))
\end{aligned}
$$

$$
\begin{aligned}
&= -\tfrac{7}{2}\sqrt{\tfrac{2}{5}}\varPsi_0\varUpsilon_{45} - \tfrac{1}{2}\sqrt{6}(\varPsi'_1\varUpsilon_{63} - \varPsi'_2\varUpsilon_{18} + \varPsi'_3\varUpsilon_{45}) - 2\textstyle\sum_{i=1}^{8}\varPsi_i\varUpsilon_{i+45} \\
&\quad + 2\sqrt{2}(\varPhi_1\varUpsilon_{54} + \varPhi_2\varUpsilon_{55} + \varPhi_3\varUpsilon_{56} - \varPhi_4\varUpsilon_9 - \varPhi_5\varUpsilon_{10} - \varPhi_6\varUpsilon_{11})\,,
\end{aligned}
\tag{B.1o}
$$

$$
\begin{aligned}
&\mathrm{Re}(-\mathrm{tr}(((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) - \tfrac{9}{20}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) + \tfrac{1}{4}(\boldsymbol{\Upsilon}+\mathbf{n}'))^*((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Psi}+\mathbf{m}) + \tfrac{3}{4}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) - \tfrac{7}{20}(\boldsymbol{\Upsilon}+\mathbf{n}')) \\
&\quad + ((\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})(\boldsymbol{\Psi}+\mathbf{m}) + \tfrac{3}{20}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) + \tfrac{1}{4}(\boldsymbol{\Upsilon}+\mathbf{n}'))^*((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\boldsymbol{\Upsilon}+\mathbf{n}') + \tfrac{3}{4}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) - \tfrac{19}{20}(\boldsymbol{\Upsilon}+\mathbf{n}'))))) \\
&\to \mathrm{Re}(-\mathrm{tr}((\check{\boldsymbol{\Phi}}^*\check{\mathbf{m}} + \check{\mathbf{n}}^*\check{\boldsymbol{\Psi}} - \tfrac{9}{20}\check{\boldsymbol{\Phi}}^* + \tfrac{1}{4}\boldsymbol{\Upsilon}^*)(\mathbf{n}'\boldsymbol{\Psi} + \boldsymbol{\Upsilon}\mathbf{m} + \tfrac{3}{4}\check{\boldsymbol{\Phi}} - \tfrac{7}{20}\boldsymbol{\Upsilon}) \\
&\quad + (\mathbf{m}\check{\boldsymbol{\Phi}}^* + \boldsymbol{\Psi}\check{\mathbf{n}}^* + \tfrac{3}{20}\check{\boldsymbol{\Phi}}^* + \tfrac{1}{4}\boldsymbol{\Upsilon}^*)(\check{\boldsymbol{\Psi}}\mathbf{n}' + \check{\mathbf{m}}\boldsymbol{\Upsilon} + \tfrac{3}{4}\check{\boldsymbol{\Phi}} - \tfrac{19}{20}\boldsymbol{\Upsilon}))) \\
&= 8\varPsi_0^2 - 4\textstyle\sum_{i=1}^{8}\varPsi_i^2 + \tfrac{3}{2}\sum_{i=1}^{6}\varPhi_i^2 - 8\sqrt{\tfrac{3}{5}}\varPsi_0\varPsi'_3 + 5\sqrt{\tfrac{3}{5}}\varPsi_0\varPhi_0 - 3\varPsi'_3\varPhi_0 \\
&\quad - \tfrac{5}{2}\sqrt{\tfrac{2}{5}}\varPsi_0\varUpsilon_0 + \tfrac{1}{2}\sqrt{6}(\varPsi'_1\varUpsilon_{18} + \varPsi'_2\varUpsilon_{63} + \varPsi'_3\varUpsilon_0) + 2\textstyle\sum_{i=1}^{8}\varPsi_i\varUpsilon_i \\
&\quad - 2\sqrt{2}(\varPhi_1\varUpsilon_9 + \varPhi_2\varUpsilon_{10} + \varPhi_3\varUpsilon_{11} + \varPhi_4\varUpsilon_{54} + \varPhi_5\varUpsilon_{55} + \varPhi_6\varUpsilon_{56}) \\
&\quad + \tfrac{3}{4}\textstyle\sum_{i=1}^{11}\varUpsilon_i^2 + \tfrac{1}{4}\sum_{i=12}^{17}\varUpsilon_i^2 + \tfrac{3}{4}\sum_{i=19}^{26}\varUpsilon_i^2 + \tfrac{1}{4}\sum_{i=27}^{29}\varUpsilon_i^2 + \tfrac{5}{4}\sum_{i=30}^{35}\varUpsilon_i^2 + \tfrac{3}{4}\sum_{i=36}^{41}\varUpsilon_i^2 \\
&\quad + \tfrac{1}{4}\textstyle\sum_{i=42}^{44}\varUpsilon_i^2 + \tfrac{3}{4}\sum_{i=46}^{56}\varUpsilon_i^2 + \tfrac{1}{4}\sum_{i=57}^{62}\varUpsilon_i^2 + \tfrac{3}{4}\sum_{i=64}^{71}\varUpsilon_i^2 + \tfrac{1}{4}\sum_{i=72}^{74}\varUpsilon_i^2 + \tfrac{5}{4}\sum_{i=75}^{80}\varUpsilon_i^2 \\
&\quad + \tfrac{3}{4}\textstyle\sum_{i=81}^{86}\varUpsilon_i^2 + \tfrac{1}{4}\sum_{i=87}^{89}\varUpsilon_i^2\,,
\end{aligned}
\tag{B.1p}
$$

$$
\begin{aligned}
&\mathrm{Im}(-\mathrm{tr}(((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) - \tfrac{9}{20}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) + \tfrac{1}{4}(\boldsymbol{\Upsilon}+\mathbf{n}'))^*((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Psi}+\mathbf{m}) + \tfrac{3}{4}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) - \tfrac{7}{20}(\boldsymbol{\Upsilon}+\mathbf{n}')) \\
&\quad + ((\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})(\boldsymbol{\Psi}+\mathbf{m}) + \tfrac{3}{20}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) + \tfrac{1}{4}(\boldsymbol{\Upsilon}+\mathbf{n}'))^*((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\boldsymbol{\Upsilon}+\mathbf{n}') + \tfrac{3}{4}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) - \tfrac{19}{20}(\boldsymbol{\Upsilon}+\mathbf{n}'))))) \\
&\to \mathrm{Im}(-\mathrm{tr}((\check{\boldsymbol{\Phi}}^*\check{\mathbf{m}} + \check{\mathbf{n}}^*\check{\boldsymbol{\Psi}} - \tfrac{9}{20}\check{\boldsymbol{\Phi}}^* + \tfrac{1}{4}\boldsymbol{\Upsilon}^*)(\mathbf{n}'\boldsymbol{\Psi} + \boldsymbol{\Upsilon}\mathbf{m} + \tfrac{3}{4}\check{\boldsymbol{\Phi}} - \tfrac{7}{20}\boldsymbol{\Upsilon}) \\
&\quad + (\mathbf{m}\check{\boldsymbol{\Phi}}^* + \boldsymbol{\Psi}\check{\mathbf{n}}^* + \tfrac{3}{20}\check{\boldsymbol{\Phi}}^* + \tfrac{1}{4}\boldsymbol{\Upsilon}^*)(\check{\boldsymbol{\Psi}}\mathbf{n}' + \check{\mathbf{m}}\boldsymbol{\Upsilon} + \tfrac{3}{4}\check{\boldsymbol{\Phi}} - \tfrac{19}{20}\boldsymbol{\Upsilon}))) \\
&= 4\sqrt{\tfrac{2}{5}}\varPsi_0\varUpsilon_{45} + 4\textstyle\sum_{i=1}^{8}\varPsi_i\varUpsilon_{i+45} \\
&\quad - 2\sqrt{2}(\varPhi_1\varUpsilon_{54} + \varPhi_2\varUpsilon_{55} + \varPhi_3\varUpsilon_{56} - \varPhi_4\varUpsilon_9 - \varPhi_5\varUpsilon_{10} - \varPhi_6\varUpsilon_{11})\,,
\end{aligned}
\tag{B.1q}
$$

$$
\begin{aligned}
&\mathrm{Re}(\mathrm{tr}(((\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})(\boldsymbol{\Psi}+\mathbf{m}) + \tfrac{3}{20}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) + \tfrac{1}{4}(\boldsymbol{\Upsilon}+\mathbf{n}'))^*((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Psi}+\mathbf{m}) + \tfrac{3}{4}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) - \tfrac{7}{20}(\boldsymbol{\Upsilon}+\mathbf{n}')))) \\
&\to \mathrm{Re}(\mathrm{tr}((\mathbf{m}\check{\boldsymbol{\Phi}}^* + \boldsymbol{\Psi}\check{\mathbf{n}}^* + \tfrac{3}{20}\check{\boldsymbol{\Phi}}^* + \tfrac{1}{4}\boldsymbol{\Upsilon}^*)(\mathbf{n}'\boldsymbol{\Psi} + \boldsymbol{\Upsilon}\mathbf{m} + \tfrac{3}{4}\check{\boldsymbol{\Phi}} - \tfrac{7}{20}\boldsymbol{\Upsilon}))) \\
&= -\varPsi_0^2 + 2\textstyle\sum_{i=1}^{8}\varPsi_i^2 - 3\sum_{i=1}^{3}\varPsi_i'^2 + \tfrac{3}{4}\sum_{i=1}^{6}\varPhi_i^2 + 6\sqrt{\tfrac{3}{5}}\varPsi_0\varPsi'_3 - \tfrac{5}{2}\sqrt{\tfrac{3}{5}}\varPsi_0\varPhi_0 + \tfrac{3}{2}\varPsi'_3\varPhi_0 \\
&\quad + \tfrac{5}{4}\sqrt{\tfrac{2}{5}}\varPsi_0\varUpsilon_0 - \tfrac{1}{4}\sqrt{6}(\varPsi'_1\varUpsilon_{18} + \varPsi'_2\varUpsilon_{63} + \varPsi'_3\varUpsilon_0) - \textstyle\sum_{i=1}^{8}\varPsi_i\varUpsilon_i \\
&\quad - \tfrac{3}{8}\textstyle\sum_{i=1}^{8}\varUpsilon_i^2 - \tfrac{1}{8}\sum_{i=9}^{11}\varUpsilon_i^2 + \tfrac{1}{8}\sum_{i=12}^{17}\varUpsilon_i^2 - \tfrac{3}{8}\sum_{i=19}^{26}\varUpsilon_i^2 + \tfrac{1}{8}\sum_{i=27}^{29}\varUpsilon_i^2 - \tfrac{3}{8}\sum_{i=30}^{35}\varUpsilon_i^2 \\
&\quad + \tfrac{1}{8}\textstyle\sum_{i=36}^{41}\varUpsilon_i^2 - \tfrac{3}{8}\sum_{i=42}^{44}\varUpsilon_i^2 - \tfrac{3}{8}\sum_{i=46}^{53}\varUpsilon_i^2 - \tfrac{1}{8}\sum_{i=54}^{56}\varUpsilon_i^2 + \tfrac{1}{8}\sum_{i=57}^{62}\varUpsilon_i^2 - \tfrac{3}{8}\sum_{i=64}^{71}\varUpsilon_i^2 \\
&\quad + \tfrac{1}{8}\textstyle\sum_{i=72}^{74}\varUpsilon_i^2 - \tfrac{3}{8}\sum_{i=75}^{80}\varUpsilon_i^2 + \tfrac{1}{8}\sum_{i=81}^{86}\varUpsilon_i^2 - \tfrac{3}{8}\sum_{i=87}^{89}\varUpsilon_i^2\,,
\end{aligned}
\tag{B.1r}
$$

$$
\begin{aligned}
&\mathrm{Im}(\mathrm{tr}(((\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})(\boldsymbol{\Psi}+\mathbf{m}) + \tfrac{3}{20}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) + \tfrac{1}{4}(\boldsymbol{\Upsilon}+\mathbf{n}'))^*((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Psi}+\mathbf{m}) + \tfrac{3}{4}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) - \tfrac{7}{20}(\boldsymbol{\Upsilon}+\mathbf{n}')))) \\
&\to \mathrm{Im}(\mathrm{tr}((\mathbf{m}\check{\boldsymbol{\Phi}}^* + \boldsymbol{\Psi}\check{\mathbf{n}}^* + \tfrac{3}{20}\check{\boldsymbol{\Phi}}^* + \tfrac{1}{4}\boldsymbol{\Upsilon}^*)(\mathbf{n}'\boldsymbol{\Psi} + \boldsymbol{\Upsilon}\mathbf{m} + \tfrac{3}{4}\check{\boldsymbol{\Phi}} - \tfrac{7}{20}\boldsymbol{\Upsilon}))) \\
&= -\tfrac{1}{2}\sqrt{\tfrac{2}{5}}\varPsi_0\varUpsilon_{45} + \tfrac{1}{2}\sqrt{6}(\varPsi'_1\varUpsilon_{63} - \varPsi'_2\varUpsilon_{18} + \varPsi'_3\varUpsilon_{45}) - 2\textstyle\sum_{i=1}^{8}\varPsi_i\varUpsilon_{i+45}\,,
\end{aligned}
\tag{B.1s}
$$

$$
\begin{aligned}
&\mathrm{Re}(-\mathrm{tr}(((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\boldsymbol{\Upsilon}+\mathbf{n}') + \tfrac{3}{4}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) - \tfrac{19}{20}(\boldsymbol{\Upsilon}+\mathbf{n}'))^*((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Psi}+\mathbf{m}) + \tfrac{3}{4}(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) - \tfrac{7}{20}(\boldsymbol{\Upsilon}+\mathbf{n}')))) \\
&\to \mathrm{Re}(-\mathrm{tr}((\boldsymbol{\Upsilon}^*\check{\mathbf{m}} + \mathbf{n}'^*\check{\boldsymbol{\Psi}} + \tfrac{3}{4}\check{\boldsymbol{\Phi}}^* - \tfrac{19}{20}\boldsymbol{\Upsilon}^*)(\mathbf{n}'\boldsymbol{\Psi} + \boldsymbol{\Upsilon}\mathbf{m} + \tfrac{3}{4}\check{\boldsymbol{\Phi}} - \tfrac{7}{20}\boldsymbol{\Upsilon}))) \\
&= -\tfrac{52}{5}\varPsi_0^2 - 2\textstyle\sum_{i=1}^{8}\varPsi_i^2 - \tfrac{9}{4}\sum_{i=1}^{6}\varPhi_i^2 + 20\sqrt{\tfrac{3}{5}}\varPsi_0\varPsi'_3 + \tfrac{15}{2}\sqrt{\tfrac{3}{5}}\varPsi_0\varPhi_0 - \tfrac{9}{2}\varPsi'_3\varPhi_0 - \tfrac{9}{4}\varPhi_0^2 \\
&\quad - \tfrac{15}{4}\sqrt{\tfrac{2}{5}}\varPsi_0\varUpsilon_0 + \tfrac{3}{4}\sqrt{6}(\varPsi'_1\varUpsilon_{18} + \varPsi'_2\varUpsilon_{63} + \varPsi'_3\varUpsilon_0) + \tfrac{3}{4}\sqrt{6}\varPhi_0\varUpsilon_0 - \tfrac{3}{8}\varUpsilon_0^2 - \tfrac{3}{8}\varUpsilon_{45}^2 + 3\textstyle\sum_{i=1}^{8}\varPsi_i\varUpsilon_i \\
&\quad + \tfrac{3}{2}\sqrt{2}(\varPhi_1\varUpsilon_9 + \varPhi_2\varUpsilon_{10} + \varPhi_3\varUpsilon_{11} + \varPhi_4\varUpsilon_{54} + \varPhi_5\varUpsilon_{55} + \varPhi_6\varUpsilon_{56})
\end{aligned}
$$

$$
\begin{aligned}
&-\tfrac{9}{8}\textstyle\sum_{i=1}^{11}\Upsilon_i^2+\tfrac{3}{8}\sum_{i=12}^{17}\Upsilon_i^2-\tfrac{3}{8}\Upsilon_{18}^2-\tfrac{9}{8}\sum_{i=19}^{26}\Upsilon_i^2+\tfrac{3}{8}\sum_{i=27}^{29}\Upsilon_i^2\\
&-\tfrac{21}{8}\textstyle\sum_{i=30}^{35}\Upsilon_i^2+\tfrac{7}{8}\sum_{i=36}^{41}\Upsilon_i^2+\tfrac{3}{8}\sum_{i=42}^{44}\Upsilon_i^2-\tfrac{9}{8}\sum_{i=46}^{56}\Upsilon_i^2+\tfrac{3}{8}\sum_{i=57}^{62}\Upsilon_i^2-\tfrac{3}{8}\Upsilon_{63}^2\\
&-\tfrac{9}{8}\textstyle\sum_{i=64}^{71}\Upsilon_i^2+\tfrac{3}{8}\sum_{i=72}^{74}\Upsilon_i^2-\tfrac{21}{8}\sum_{i=75}^{80}\Upsilon_i^2+\tfrac{7}{8}\sum_{i=81}^{86}\Upsilon_i^2+\tfrac{3}{8}\sum_{i=87}^{89}\Upsilon_i^2\,, \qquad \text{(B.1t)}
\end{aligned}
$$

$$
\begin{aligned}
&(\mathbf{\Phi}+\mathbf{n})^*(\mathbf{\Upsilon}+\mathbf{n}')^*(\mathbf{\Upsilon}+\mathbf{n}')(\mathbf{\Phi}+\mathbf{n})\to(\mathbf{\Phi}^*\mathbf{n}'^*+\mathbf{n}^*\mathbf{\Upsilon}^*)(\mathbf{\Upsilon}\mathbf{n}+\mathbf{n}'\mathbf{\Phi})\\
&=\textstyle\sum_{i=1}^{6}\Phi_i^2+\sqrt{2}(\Phi_1\Upsilon_9+\Phi_2\Upsilon_{10}+\Phi_3\Upsilon_{11}+\Phi_4\Upsilon_{54}+\Phi_5\Upsilon_{55}+\Phi_6\Upsilon_{56})\\
&\quad+2(\Phi_1\Upsilon_{12}+\Phi_2\Upsilon_{13}+\Phi_3\Upsilon_{14}+\Phi_4\Upsilon_{57}+\Phi_5\Upsilon_{58}+\Phi_6\Upsilon_{59})\\
&\quad+\tfrac{1}{2}\textstyle\sum_{i=9}^{11}\Upsilon_i^2+\sum_{i=12}^{14}\Upsilon_i^2+\tfrac{3}{2}\Upsilon_{18}^2+2\sum_{i=27}^{29}\Upsilon_i^2+2\sum_{i=36}^{38}\Upsilon_i^2\\
&\quad+\tfrac{1}{2}\textstyle\sum_{i=54}^{56}\Upsilon_i^2+\sum_{i=57}^{59}\Upsilon_i^2+\tfrac{3}{2}\Upsilon_{63}^2+2\sum_{i=72}^{74}\Upsilon_i^2+2\sum_{i=82}^{84}\Upsilon_i^2\\
&\quad+\sqrt{2}(\Upsilon_9\Upsilon_{12}+\Upsilon_{10}\Upsilon_{13}+\Upsilon_{11}\Upsilon_{14}+\Upsilon_{54}\Upsilon_{57}+\Upsilon_{55}\Upsilon_{58}+\Upsilon_{56}\Upsilon_{59})\,, \qquad \text{(B.1u)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{tr}(((\mathbf{\Upsilon}+\mathbf{n}')^*(\hat{\mathbf{\Phi}}+\hat{\mathbf{n}}))_{\underline{10}}((\hat{\mathbf{\Phi}}+\hat{\mathbf{n}})^*(\mathbf{\Upsilon}+\mathbf{n}'))_{\underline{10}})\to\mathrm{tr}((\mathbf{\Upsilon}^*\hat{\mathbf{n}}+\mathbf{n}'^*\hat{\mathbf{\Phi}})_{\underline{10}}(\hat{\mathbf{\Phi}}^*\mathbf{n}'+\hat{\mathbf{n}}^*\mathbf{\Upsilon}))_{\underline{10}}\\
&=\tfrac{1}{3}\textstyle\sum_{i=1}^{6}\Phi_i^2+\tfrac{1}{3}\sqrt{2}(\Phi_1\Upsilon_9+\Phi_2\Upsilon_{10}+\Phi_3\Upsilon_{11}+\Phi_4\Upsilon_{54}+\Phi_5\Upsilon_{55}+\Phi_6\Upsilon_{56})\\
&\quad+\tfrac{2}{3}(\Phi_1\Upsilon_{12}+\Phi_2\Upsilon_{13}+\Phi_3\Upsilon_{14}+\Phi_4\Upsilon_{57}+\Phi_5\Upsilon_{58}+\Phi_6\Upsilon_{59})\\
&\quad+\tfrac{1}{6}\textstyle\sum_{i=9}^{11}\Upsilon_i^2+\tfrac{1}{3}\sum_{i=12}^{14}\Upsilon_i^2+\tfrac{1}{2}\Upsilon_{18}^2+\tfrac{2}{3}\sum_{i=27}^{29}\Upsilon_i^2+\tfrac{2}{3}\sum_{i=36}^{38}\Upsilon_i^2\\
&\quad+\tfrac{1}{6}\textstyle\sum_{i=54}^{56}\Upsilon_i^2+\tfrac{1}{3}\sum_{i=57}^{59}\Upsilon_i^2+\tfrac{1}{2}\Upsilon_{63}^2+\tfrac{2}{3}\sum_{i=72}^{74}\Upsilon_i^2+\tfrac{2}{3}\sum_{i=81}^{83}\Upsilon_i^2\\
&\quad+\tfrac{1}{3}\sqrt{2}(\Upsilon_9\Upsilon_{12}+\Upsilon_{10}\Upsilon_{13}+\Upsilon_{11}\Upsilon_{14}+\Upsilon_{54}\Upsilon_{57}+\Upsilon_{55}\Upsilon_{58}+\Upsilon_{56}\Upsilon_{59})\,, \qquad \text{(B.1v)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{tr}(((\mathbf{\Upsilon}+\mathbf{n}')^*(\hat{\mathbf{\Phi}}+\hat{\mathbf{n}}))_{\underline{40}}((\hat{\mathbf{\Phi}}+\hat{\mathbf{n}})^*(\mathbf{\Upsilon}+\mathbf{n}'))_{\underline{40}})\to\mathrm{tr}((\mathbf{\Upsilon}^*\hat{\mathbf{n}}+\mathbf{n}'^*\hat{\mathbf{\Phi}})_{\underline{40}}(\hat{\mathbf{\Phi}}^*\mathbf{n}'+\hat{\mathbf{n}}^*\mathbf{\Upsilon}))_{\underline{40}}\\
&=\tfrac{32}{3}\textstyle\sum_{i=1}^{6}\Phi_i^2-\tfrac{16}{3}\sqrt{2}(\Phi_1\Upsilon_9+\Phi_2\Upsilon_{10}+\Phi_3\Upsilon_{11}+\Phi_4\Upsilon_{54}+\Phi_5\Upsilon_{55}+\Phi_6\Upsilon_{56})\\
&\quad+\tfrac{16}{3}(\Phi_1\Upsilon_{12}+\Phi_2\Upsilon_{13}+\Phi_3\Upsilon_{14}+\Phi_4\Upsilon_{57}+\Phi_5\Upsilon_{58}+\Phi_6\Upsilon_{59})\\
&\quad+\tfrac{4}{3}\textstyle\sum_{i=9}^{11}\Upsilon_i^2+\tfrac{2}{3}\sum_{i=12}^{14}\Upsilon_i^2+2\sum_{i=19}^{26}\Upsilon_i^2+\tfrac{4}{3}\sum_{i=27}^{30}\Upsilon_i^2+2\sum_{i=30}^{35}\Upsilon_i^2+\tfrac{4}{3}\sum_{i=36}^{38}\Upsilon_i^2+2\sum_{i=39}^{41}\Upsilon_i^2\\
&\quad+\tfrac{4}{3}\textstyle\sum_{i=54}^{56}\Upsilon_i^2+\tfrac{2}{3}\sum_{i=57}^{59}\Upsilon_i^2+2\sum_{i=64}^{71}\Upsilon_i^2+\tfrac{4}{3}\sum_{i=72}^{74}\Upsilon_i^2+2\sum_{i=75}^{80}\Upsilon_i^2+\tfrac{4}{3}\sum_{i=81}^{83}\Upsilon_i^2+2\sum_{i=84}^{86}\Upsilon_i^2\\
&\quad-\tfrac{4}{3}\sqrt{2}(\Upsilon_9\Upsilon_{12}+\Upsilon_{10}\Upsilon_{13}+\Upsilon_{11}\Upsilon_{14}+\Upsilon_{54}\Upsilon_{57}+\Upsilon_{55}\Upsilon_{58}+\Upsilon_{56}\Upsilon_{59})\,, \qquad \text{(B.1w)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{tr}(((\mathbf{\Psi}+\mathbf{m})^2-\tfrac{1}{5}\mathrm{tr}((\mathbf{\Psi}+\mathbf{m})^2)\mathbb{1}_5-\tfrac{1}{5}(\mathbf{\Psi}+\mathbf{m}))^2)\\
&\to\mathrm{tr}((\mathbf{\Psi}\mathbf{m}+\mathbf{m}\mathbf{\Psi}-\tfrac{2}{5}\mathrm{tr}(\mathbf{\Psi}\mathbf{m})\mathbb{1}_5-\tfrac{1}{5}\mathbf{\Psi})^2)=\tfrac{2}{25}\Psi_0^2+2\textstyle\sum_{i=1}^{8}\Psi_i^2+2\sum_{i=1}^{3}\Psi_i'^2\,, \qquad \text{(B.2a)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{tr}(((\mathbf{\Upsilon}+\mathbf{n}')^*(\mathbf{\Upsilon}+\mathbf{n}')-\tfrac{1}{5}\mathrm{tr}((\mathbf{\Upsilon}+\mathbf{n}')^*(\mathbf{\Upsilon}+\mathbf{n}'))\mathbb{1}_5-8(\mathbf{\Psi}+\mathbf{m})+9(\mathbf{\Phi}+\mathbf{n})(\mathbf{\Phi}+\mathbf{n})^*\\
&\qquad-\tfrac{9}{5}(\mathbf{\Phi}+\mathbf{n})^*(\mathbf{\Phi}+\mathbf{n})\mathbb{1}_5)^2)\\
&\to\mathrm{tr}((\mathbf{\Upsilon}^*\mathbf{n}'+\mathbf{n}'^*\mathbf{\Upsilon}-\tfrac{1}{5}\mathrm{tr}(\mathbf{\Upsilon}^*\mathbf{n}'+\mathbf{n}'^*\mathbf{\Upsilon})\mathbb{1}_5-8\mathbf{\Psi}+9(\mathbf{\Phi}\mathbf{n}^*+\mathbf{n}\mathbf{\Phi}^*)-\tfrac{9}{5}\mathrm{tr}(\mathbf{\Phi}\mathbf{n}^*+\mathbf{n}\mathbf{\Phi}^*)\mathbb{1}_5)^2)\\
&=128\textstyle\sum_{i=0}^{8}\Psi_i^2+128\sum_{i=1}^{3}\Psi_i'^2+162\sum_{i=1}^{6}\Phi_i^2+\tfrac{1296}{5}\Phi_0^2-288\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0\\
&\quad+48\sqrt{6}(\Psi_1'\Upsilon_{18}+\Psi_2'\Upsilon_{63}+\Psi_3'\Upsilon_0)-288\Psi_3'\Phi_0-112\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0-\tfrac{144}{5}\sqrt{6}\Phi_0\Upsilon_0+\tfrac{184}{5}\Upsilon_0^2\\
&\quad-64\textstyle\sum_{i=1}^{8}\Psi_i\Upsilon_i+36(\Phi_1\Upsilon_{12}+\Phi_2\Upsilon_{13}+\Phi_3\Upsilon_{14}+\Phi_4\Upsilon_{57}+\Phi_5\Upsilon_{58}+\Phi_6\Upsilon_{59})\\
&\quad+18\sqrt{2}(\Phi_1\Upsilon_9+\Phi_2\Upsilon_{10}+\Phi_3\Upsilon_{11}+\Phi_4\Upsilon_{54}+\Phi_5\Upsilon_{55}+\Phi_6\Upsilon_{56})\\
&\quad+8\textstyle\sum_{i=1}^{8}\Upsilon_i^2+\sum_{i=9}^{11}\Upsilon_i^2+2\sum_{i=12}^{14}\Upsilon_i^2+4\sum_{i=15}^{17}\Upsilon_i^2+27\Upsilon_{18}^2+36\sum_{i=42}^{44}\Upsilon_i^2\\
&\quad+\textstyle\sum_{i=54}^{56}\Upsilon_i^2+2\sum_{i=57}^{59}\Upsilon_i^2+4\sum_{i=60}^{62}\Upsilon_i^2+27\Upsilon_{63}^2+36\sum_{i=87}^{89}\Upsilon_i^2\\
&\quad+24(\Upsilon_{15}\Upsilon_{42}+\Upsilon_{16}\Upsilon_{43}+\Upsilon_{17}\Upsilon_{44}+\Upsilon_{60}\Upsilon_{87}+\Upsilon_{61}\Upsilon_{88}+\Upsilon_{62}\Upsilon_{89})\\
&\quad+2\sqrt{2}(\Upsilon_9\Upsilon_{12}+\Upsilon_{10}\Upsilon_{13}+\Upsilon_{11}\Upsilon_{14}+\Upsilon_{54}\Upsilon_{57}+\Upsilon_{55}\Upsilon_{58}+\Upsilon_{56}\Upsilon_{59})\,, \qquad \text{(B.2b)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{tr}((((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Upsilon}+\mathbf{n}')^*)'-\tfrac{8}{3}(\boldsymbol{\Psi}+\mathbf{m})-(\boldsymbol{\Phi}+\mathbf{n})(\boldsymbol{\Phi}+\mathbf{n})^*+\tfrac{1}{5}(\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n})\mathbb{1}_5)^2)\\
&\to \mathrm{tr}(((\boldsymbol{\Upsilon}\mathbf{n}'^*+\mathbf{n}'\boldsymbol{\Upsilon}^*)'-\tfrac{8}{3}\boldsymbol{\Psi}-(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)+\tfrac{1}{5}\,\mathrm{tr}(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)\mathbb{1}_5)^2)\\
&=\tfrac{128}{9}\textstyle\sum_{i=0}^{8}\Psi_i^2+\tfrac{128}{9}\sum_{i=1}^{3}\Psi_i'^2+2\sum_{i=1}^{6}\Phi_i^2+\tfrac{16}{5}\Phi_0^2+\tfrac{32}{3}\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0+\tfrac{32}{3}\Psi_3'\Phi_0\\
&\quad-\tfrac{304}{9}\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0-\tfrac{16}{9}\sqrt{6}(\Psi_1'\Upsilon_{18}+\Psi_2'\Upsilon_{63}+\Psi_3'\Upsilon_0)-\tfrac{16}{5}\sqrt{6}\Phi_0\Upsilon_0+\tfrac{376}{45}\Upsilon_0^2-\tfrac{64}{9}\textstyle\sum_{i=1}^{8}\Psi_i\Upsilon_i\\
&\quad+4(\Phi_1\Upsilon_{12}+\Phi_2\Upsilon_{13}+\Phi_3\Upsilon_{14}+\Phi_4\Upsilon_{57}+\Phi_5\Upsilon_{58}+\Phi_6\Upsilon_{59})\\
&\quad-\tfrac{10}{3}\sqrt{2}(\Phi_1\Upsilon_9+\Phi_2\Upsilon_{10}+\Phi_3\Upsilon_{11}+\Phi_4\Upsilon_{54}+\Phi_5\Upsilon_{55}+\Phi_6\Upsilon_{56})\\
&\quad+\tfrac{8}{9}\textstyle\sum_{i=1}^{8}\Upsilon_i^2+\tfrac{25}{9}\sum_{i=9}^{11}\Upsilon_i^2+2\sum_{i=12}^{14}\Upsilon_i^2+4\sum_{i=15}^{17}\Upsilon_i^2+\tfrac{1}{3}\Upsilon_{18}^2+\tfrac{4}{9}\sum_{i=42}^{44}\Upsilon_i^2\\
&\quad+\tfrac{25}{9}\textstyle\sum_{i=54}^{56}\Upsilon_i^2+2\sum_{i=57}^{59}\Upsilon_i^2+4\sum_{i=60}^{62}\Upsilon_i^2+\tfrac{1}{3}\Upsilon_{63}^2+\tfrac{4}{9}\sum_{i=87}^{89}\Upsilon_i^2\\
&\quad+\tfrac{8}{3}(\Upsilon_{15}\Upsilon_{42}+\Upsilon_{16}\Upsilon_{43}+\Upsilon_{17}\Upsilon_{44}+\Upsilon_{60}\Upsilon_{87}+\Upsilon_{61}\Upsilon_{88}+\Upsilon_{62}\Upsilon_{89})\\
&\quad-\tfrac{10}{3}\sqrt{2}(\Upsilon_9\Upsilon_{12}+\Upsilon_{10}\Upsilon_{13}+\Upsilon_{11}\Upsilon_{14}+\Upsilon_{54}\Upsilon_{57}+\Upsilon_{55}\Upsilon_{58}+\Upsilon_{56}\Upsilon_{59})\,,
\end{aligned}
\tag{B.2c}
$$

$$
\begin{aligned}
&\mathrm{tr}(((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})+(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*(\boldsymbol{\Upsilon}+\mathbf{n}')+8(\boldsymbol{\Psi}+\mathbf{m})-6(\boldsymbol{\Phi}+\mathbf{n})(\boldsymbol{\Phi}+\mathbf{n})^*\\
&\qquad+\tfrac{6}{5}(\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n})\mathbb{1}_5)^2)\\
&\to \mathrm{tr}((\boldsymbol{\Upsilon}^*\check{\mathbf{n}}+\mathbf{n}'^*\check{\boldsymbol{\Phi}}+\check{\boldsymbol{\Phi}}^*\mathbf{n}'+\check{\mathbf{n}}^*\boldsymbol{\Upsilon}+8\boldsymbol{\Psi}-6(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)+\tfrac{6}{5}\,\mathrm{tr}(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)\mathbb{1}_5)^2)\\
&=128\textstyle\sum_{i=0}^{8}\Psi_i^2+128\sum_{i=1}^{3}\Psi_i'^2+98\sum_{i=1}^{6}\Phi_i^2+\tfrac{816}{5}\Phi_0^2-352\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0\\
&\quad-96\Psi_3'\Phi_0-80\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0+16\sqrt{6}\Phi_0\Upsilon_0+16\sqrt{6}(\Psi_1'\Upsilon_{18}+\Psi_2'\Upsilon_{63}+\Psi_3'\Upsilon_0)\\
&\quad-14\sqrt{2}(\Phi_1\Upsilon_9+\Phi_2\Upsilon_{10}+\Phi_3\Upsilon_{11}+\Phi_4\Upsilon_{54}+\Phi_5\Upsilon_{55}+\Phi_6\Upsilon_{56})\\
&\quad-28(\Phi_1\Upsilon_{12}+\Phi_2\Upsilon_{13}+\Phi_3\Upsilon_{14}+\Phi_4\Upsilon_{57}+\Phi_5\Upsilon_{58}+\Phi_6\Upsilon_{59})+64\textstyle\sum_{i=1}^{8}\Psi_i\Upsilon_i\\
&\quad+8\textstyle\sum_{i=0}^{8}\Upsilon_i^2+\sum_{i=9}^{11}\Upsilon_i^2+2\sum_{i=12}^{14}\Upsilon_i^2+4\sum_{i=15}^{17}\Upsilon_i^2+3\Upsilon_{18}^2+4\sum_{i=42}^{44}\Upsilon_i^2\\
&\quad+\textstyle\sum_{i=54}^{56}\Upsilon_i^2+2\sum_{i=57}^{59}\Upsilon_i^2+4\sum_{i=60}^{62}\Upsilon_i^2+3\Upsilon_{63}^2+4\sum_{i=87}^{89}\Upsilon_i^2\\
&\quad-8(\Upsilon_{15}\Upsilon_{42}+\Upsilon_{16}\Upsilon_{43}+\Upsilon_{17}\Upsilon_{44}+\Upsilon_{60}\Upsilon_{87}+\Upsilon_{61}\Upsilon_{88}+\Upsilon_{62}\Upsilon_{89})\\
&\quad+2\sqrt{2}(\Upsilon_9\Upsilon_{12}+\Upsilon_{10}\Upsilon_{13}+\Upsilon_{11}\Upsilon_{14}+\Upsilon_{54}\Upsilon_{57}+\Upsilon_{55}\Upsilon_{58}+\Upsilon_{56}\Upsilon_{59})\,,
\end{aligned}
\tag{B.2d}
$$

$$
\begin{aligned}
&-\mathrm{tr}(((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})-(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*(\boldsymbol{\Upsilon}+\mathbf{n}'))^2)\to-\mathrm{tr}((\boldsymbol{\Upsilon}^*\check{\mathbf{n}}+\mathbf{n}'^*\check{\boldsymbol{\Phi}}-\check{\boldsymbol{\Phi}}^*\mathbf{n}'-\check{\mathbf{n}}^*\boldsymbol{\Upsilon})^2)\\
&=2\textstyle\sum_{i=1}^{6}\Phi_i^2+2\sqrt{2}(\Phi_1\Upsilon_9+\Phi_2\Upsilon_{10}+\Phi_3\Upsilon_{11}+\Phi_4\Upsilon_{54}+\Phi_5\Upsilon_{55}+\Phi_6\Upsilon_{56})\\
&\quad+4(\Phi_1\Upsilon_{12}+\Phi_2\Upsilon_{13}+\Phi_3\Upsilon_{14}+\Phi_4\Upsilon_{57}+\Phi_5\Upsilon_{58}+\Phi_6\Upsilon_{59})\\
&\quad+\textstyle\sum_{i=9}^{11}\Upsilon_i^2+2\sum_{i=12}^{14}\Upsilon_i^2+4\sum_{i=15}^{17}\Upsilon_i^2+3\Upsilon_{18}^2+4\sum_{i=42}^{44}\Upsilon_i^2\\
&\quad+8\textstyle\sum_{i=45}^{53}\Upsilon_i^2+\sum_{i=54}^{56}\Upsilon_i^2+2\sum_{i=57}^{59}\Upsilon_i^2+4\sum_{i=60}^{62}\Upsilon_i^2+3\Upsilon_{63}^2+4\sum_{i=87}^{89}\Upsilon_i^2\\
&\quad+2\sqrt{2}(\Upsilon_9\Upsilon_{12}+\Upsilon_{10}\Upsilon_{13}+\Upsilon_{11}\Upsilon_{14}+\Upsilon_{54}\Upsilon_{57}+\Upsilon_{55}\Upsilon_{58}+\Upsilon_{56}\Upsilon_{59})\\
&\quad+8(\Upsilon_{15}\Upsilon_{42}+\Upsilon_{16}\Upsilon_{43}+\Upsilon_{17}\Upsilon_{44}+\Upsilon_{60}\Upsilon_{87}+\Upsilon_{61}\Upsilon_{88}+\Upsilon_{62}\Upsilon_{89})\,,
\end{aligned}
\tag{B.2e}
$$

$$
\begin{aligned}
&\mathrm{tr}(((\boldsymbol{\Psi}+\mathbf{m})^2-\tfrac{1}{5}\,\mathrm{tr}((\boldsymbol{\Psi}+\mathbf{m})^2)\mathbb{1}_5-\tfrac{1}{5}(\boldsymbol{\Psi}+\mathbf{m}))((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\boldsymbol{\Upsilon}+\mathbf{n}')-\tfrac{1}{5}\,\mathrm{tr}((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\boldsymbol{\Upsilon}+\mathbf{n}'))\mathbb{1}_5\\
&\qquad-8(\boldsymbol{\Psi}+\mathbf{m})+9(\boldsymbol{\Phi}+\mathbf{n})(\boldsymbol{\Phi}+\mathbf{n})^*-\tfrac{9}{5}(\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n})\mathbb{1}_5))\\
&\to \mathrm{tr}((\boldsymbol{\Psi}\mathbf{m}+\mathbf{m}\boldsymbol{\Psi}-\tfrac{2}{5}\,\mathrm{tr}(\boldsymbol{\Psi}\mathbf{m})\mathbb{1}_5-\tfrac{1}{5}\boldsymbol{\Psi})(\boldsymbol{\Upsilon}^*\mathbf{n}'+\mathbf{n}'^*\boldsymbol{\Upsilon}-\tfrac{1}{5}\,\mathrm{tr}(\boldsymbol{\Upsilon}^*\mathbf{n}'+\mathbf{n}'^*\boldsymbol{\Upsilon})\mathbb{1}_5-8\boldsymbol{\Psi}\\
&\quad+9(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)-\tfrac{9}{5}\,\mathrm{tr}(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)\mathbb{1}_5))\\
&=-\tfrac{16}{5}\Psi_0^2+16\textstyle\sum_{i=1}^{8}\Psi_i^2-16\sum_{i=1}^{3}\Psi_i'^2+\tfrac{18}{5}\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0+18\Psi_3'\Phi_0-3\sqrt{6}(\Psi_3'\Upsilon_0+\Psi_1'\Upsilon_{18}+\Psi_2'\Upsilon_{63})\\
&\quad+\tfrac{7}{5}\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0-4\textstyle\sum_{i=1}^{8}\Psi_i\Upsilon_i\,,
\end{aligned}
\tag{B.2f}
$$

$$\begin{aligned}
&\mathrm{tr}(((\boldsymbol{\Psi}+\mathbf{m})^2-\tfrac{1}{5}\,\mathrm{tr}((\boldsymbol{\Psi}+\mathbf{m})^2)\mathbb{1}_5-\tfrac{1}{5}(\boldsymbol{\Psi}+\mathbf{m}))(((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Upsilon}+\mathbf{n}')^*)'-\tfrac{8}{3}(\boldsymbol{\Psi}+\mathbf{m})\\
&\qquad-(\boldsymbol{\Phi}+\mathbf{n})(\boldsymbol{\Phi}+\mathbf{n})^*+\tfrac{1}{5}(\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n})\mathbb{1}_5))\\
&\to \mathrm{tr}((\boldsymbol{\Psi}\mathbf{m}+\mathbf{m}\boldsymbol{\Psi}-\tfrac{2}{5}\,\mathrm{tr}(\boldsymbol{\Psi}\mathbf{m})\mathbb{1}_5-\tfrac{1}{5}\boldsymbol{\Psi})((\boldsymbol{\Upsilon}\mathbf{n}'^*+\mathbf{n}'\boldsymbol{\Upsilon}^*)'-\tfrac{8}{3}\boldsymbol{\Psi}-(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)\\
&\quad+\tfrac{1}{5}\,\mathrm{tr}(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)\mathbb{1}_5))\\
&=-\tfrac{16}{15}\Psi_0^2+\tfrac{16}{3}\textstyle\sum_{i=1}^8\Psi_i^2-\tfrac{16}{3}\sum_{i=1}^3\Psi_i'^2-\tfrac{2}{5}\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0-2\Psi_3'\Phi_0+\tfrac{1}{3}\sqrt{6}(\Psi_3'\Upsilon_0+\Psi_1'\Upsilon_{18}+\Psi_2'\Upsilon_{63})\\
&\quad+\tfrac{19}{15}\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0-\tfrac{4}{3}\textstyle\sum_{i=1}^8\Psi_i\Upsilon_i\,,
\end{aligned}\tag{B.2g}$$

$$\begin{aligned}
&\mathrm{tr}(((\boldsymbol{\Psi}+\mathbf{m})^2-\tfrac{1}{5}\,\mathrm{tr}((\boldsymbol{\Psi}+\mathbf{m})^2)\mathbb{1}_5-\tfrac{1}{5}(\boldsymbol{\Psi}+\mathbf{m}))((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})+(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*(\boldsymbol{\Upsilon}+\mathbf{n}')\\
&\qquad+8(\boldsymbol{\Psi}+\mathbf{m})-6(\boldsymbol{\Phi}+\mathbf{n})(\boldsymbol{\Phi}+\mathbf{n})^*+\tfrac{6}{5}(\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n})\mathbb{1}_5))\\
&\to \mathrm{tr}((\boldsymbol{\Psi}\mathbf{m}+\mathbf{m}\boldsymbol{\Psi}-\tfrac{2}{5}\,\mathrm{tr}(\boldsymbol{\Psi}\mathbf{m})\mathbb{1}_5-\tfrac{1}{5}\boldsymbol{\Psi})(\boldsymbol{\Upsilon}^*\check{\mathbf{n}}+\mathbf{n}'^*\check{\boldsymbol{\Phi}}+\check{\boldsymbol{\Phi}}^*\mathbf{n}'+\check{\mathbf{n}}^*\boldsymbol{\Upsilon}+8\boldsymbol{\Psi}\\
&\quad-6(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)+\tfrac{6}{5}\,\mathrm{tr}(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)\mathbb{1}_5))\\
&=\tfrac{16}{5}\Psi_0^2-16\textstyle\sum_{i=1}^8\Psi_i^2+16\sum_{i=1}^3\Psi_i'^2-\tfrac{22}{5}\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0-6\Psi_3'\Phi_0+\sqrt{6}(\Psi_1'\Upsilon_{18}+\Psi_2'\Upsilon_{63}+\Psi_3'\Upsilon_0)\\
&\quad-\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0-4\textstyle\sum_{i=1}^8\Psi_i\Upsilon_i\,,
\end{aligned}\tag{B.2h}$$

$$\begin{aligned}
&\mathrm{i}\,\mathrm{tr}(((\boldsymbol{\Psi}+\mathbf{m})^2-\tfrac{1}{5}\,\mathrm{tr}((\boldsymbol{\Psi}+\mathbf{m})^2)\mathbb{1}_5-\tfrac{1}{5}(\boldsymbol{\Psi}+\mathbf{m}))((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})-(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*(\boldsymbol{\Upsilon}+\mathbf{n}')))\\
&\to \mathrm{i}\,\mathrm{tr}((\boldsymbol{\Psi}\mathbf{m}+\mathbf{m}\boldsymbol{\Psi}-\tfrac{2}{5}\,\mathrm{tr}(\boldsymbol{\Psi}\mathbf{m})\mathbb{1}_5-\tfrac{1}{5}\boldsymbol{\Psi})(\boldsymbol{\Upsilon}^*\check{\mathbf{n}}+\mathbf{n}'^*\check{\boldsymbol{\Phi}}-\check{\boldsymbol{\Phi}}^*\mathbf{n}'-\check{\mathbf{n}}^*\boldsymbol{\Upsilon}))\\
&=-\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_{45}+\sqrt{6}(\Psi_1'\Upsilon_{63}-\Psi_2'\Upsilon_{18}+\Psi_3'\Upsilon_{45})-4\textstyle\sum_{i=1}^8\Psi_i\Upsilon_{i+45}\,,
\end{aligned}\tag{B.2i}$$

$$\begin{aligned}
&\mathrm{tr}(((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\boldsymbol{\Upsilon}+\mathbf{n}')-\tfrac{1}{5}\,\mathrm{tr}((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\boldsymbol{\Upsilon}+\mathbf{n}'))\mathbb{1}_5-8(\boldsymbol{\Psi}+\mathbf{m})+9(\boldsymbol{\Phi}+\mathbf{n})(\boldsymbol{\Phi}+\mathbf{n})^*\\
&\qquad-\tfrac{9}{5}(\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n})\mathbb{1}_5)(((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Upsilon}+\mathbf{n}')^*)'-\tfrac{8}{3}(\boldsymbol{\Psi}+\mathbf{m})-(\boldsymbol{\Phi}+\mathbf{n})(\boldsymbol{\Phi}+\mathbf{n})^*\\
&\qquad+\tfrac{1}{5}(\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n})\mathbb{1}_5))\\
&\to \mathrm{tr}((\boldsymbol{\Upsilon}^*\mathbf{n}'+\mathbf{n}'^*\boldsymbol{\Upsilon}-\tfrac{1}{5}\,\mathrm{tr}(\boldsymbol{\Upsilon}^*\mathbf{n}'+\mathbf{n}'^*\boldsymbol{\Upsilon})\mathbb{1}_5-8\boldsymbol{\Psi}+9(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)-\tfrac{9}{5}\,\mathrm{tr}(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)\mathbb{1}_5)\\
&\quad((\boldsymbol{\Upsilon}\mathbf{n}'^*+\mathbf{n}'\boldsymbol{\Upsilon}^*)'-\tfrac{8}{3}\boldsymbol{\Psi}-(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)+\tfrac{1}{5}\,\mathrm{tr}(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)\mathbb{1}_5))\\
&=\tfrac{128}{3}\textstyle\sum_{i=0}^8\Psi_i^2+\tfrac{128}{3}\sum_{i=1}^3\Psi_i'^2-18\sum_{i=1}^6\Phi_i^2-32\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0-32\Psi_3'\Phi_0\\
&\quad-\tfrac{144}{5}\Phi_0^2-\tfrac{208}{3}\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0+16\sqrt{6}\Phi_0\Upsilon_0+\tfrac{88}{15}\Upsilon_0^2+\tfrac{16}{3}\sqrt{6}(\Psi_1'\Upsilon_{18}+\Psi_2'\Upsilon_{63}+\Psi_3'\Upsilon_0)\\
&\quad-\tfrac{64}{3}\textstyle\sum_{i=1}^8\Psi_i\Upsilon_i+14\sqrt{2}(\Phi_1\Upsilon_9+\Phi_2\Upsilon_{10}+\Phi_3\Upsilon_{11}+\Phi_4\Upsilon_{54}+\Phi_5\Upsilon_{55}+\Phi_6\Upsilon_{56})\\
&\quad-20(\Phi_1\Upsilon_{12}+\Phi_2\Upsilon_{13}+\Phi_3\Upsilon_{14}+\Phi_4\Upsilon_{57}+\Phi_5\Upsilon_{58}+\Phi_6\Upsilon_{59})\\
&\quad+\tfrac{8}{3}\textstyle\sum_{i=1}^8\Upsilon_i^2+\tfrac{5}{3}\sum_{i=9}^{11}\Upsilon_i^2-2\sum_{i=12}^{14}\Upsilon_i^2-4\sum_{i=15}^{17}\Upsilon_i^2-3\Upsilon_{18}^2-4\sum_{i=42}^{44}\Upsilon_i^2\\
&\quad+\tfrac{5}{3}\textstyle\sum_{i=54}^{56}\Upsilon_i^2-2\sum_{i=57}^{59}\Upsilon_i^2-4\sum_{i=60}^{62}\Upsilon_i^2-3\Upsilon_{63}^2-4\sum_{i=87}^{89}\Upsilon_i^2\\
&\quad-\tfrac{40}{3}(\Upsilon_{15}\Upsilon_{42}+\Upsilon_{16}\Upsilon_{43}+\Upsilon_{17}\Upsilon_{44}+\Upsilon_{60}\Upsilon_{87}+\Upsilon_{61}\Upsilon_{88}+\Upsilon_{62}\Upsilon_{89})\\
&\quad+\tfrac{2}{3}\sqrt{2}(\Upsilon_9\Upsilon_{12}+\Upsilon_{10}\Upsilon_{13}+\Upsilon_{11}\Upsilon_{14}+\Upsilon_{54}\Upsilon_{57}+\Upsilon_{55}\Upsilon_{58}+\Upsilon_{56}\Upsilon_{59})\,,
\end{aligned}\tag{B.2j}$$

$$\begin{aligned}
&\mathrm{tr}(((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\boldsymbol{\Upsilon}+\mathbf{n}')-\tfrac{1}{5}\,\mathrm{tr}((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\boldsymbol{\Upsilon}+\mathbf{n}'))\mathbb{1}_5-8(\boldsymbol{\Psi}+\mathbf{m})+9(\boldsymbol{\Phi}+\mathbf{n})(\boldsymbol{\Phi}+\mathbf{n})^*\\
&\qquad-\tfrac{9}{5}(\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n})\mathbb{1}_5)((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})+(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*(\boldsymbol{\Upsilon}+\mathbf{n}')+8(\boldsymbol{\Psi}+\mathbf{m})\\
&\qquad-6(\boldsymbol{\Phi}+\mathbf{n})(\boldsymbol{\Phi}+\mathbf{n})^*+\tfrac{6}{5}(\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n})\mathbb{1}_5))\\
&\to \mathrm{tr}((\boldsymbol{\Upsilon}^*\mathbf{n}'+\mathbf{n}'^*\boldsymbol{\Upsilon}-\tfrac{1}{5}\,\mathrm{tr}(\boldsymbol{\Upsilon}^*\mathbf{n}'+\mathbf{n}'^*\boldsymbol{\Upsilon})\mathbb{1}_5-8\boldsymbol{\Psi}+9(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)-\tfrac{9}{5}\,\mathrm{tr}(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)\mathbb{1}_5)\times\\
&\quad\times(\boldsymbol{\Upsilon}^*\check{\mathbf{n}}+\mathbf{n}'^*\check{\boldsymbol{\Phi}}+\check{\boldsymbol{\Phi}}^*\mathbf{n}'+\check{\mathbf{n}}^*\boldsymbol{\Upsilon}+8\boldsymbol{\Psi}-6(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)+\tfrac{6}{5}\,\mathrm{tr}(\boldsymbol{\Phi}\mathbf{n}^*+\mathbf{n}\boldsymbol{\Phi}^*)\mathbb{1}_5))
\end{aligned}$$

$$
\begin{aligned}
&= -128\textstyle\sum_{i=0}^{8}\mathit{\Psi}_i^2 - 128\sum_{i=1}^{3}\mathit{\Psi}_i'^2 - 126\sum_{i=1}^{6}\mathit{\Phi}_i^2 + 320\sqrt{\tfrac{3}{5}}\mathit{\Psi}_0\mathit{\Phi}_0 + 192\mathit{\Psi}_3'\mathit{\Phi}_0\\
&\quad -\tfrac{864}{5}\mathit{\Phi}_0^2 + 96\sqrt{\tfrac{2}{5}}\mathit{\Psi}_0\mathit{\Upsilon}_0 - \tfrac{32}{5}\sqrt{6}\mathit{\Phi}_0\mathit{\Upsilon}_0 - 16\mathit{\Upsilon}_0^2 - 32\sqrt{6}(\mathit{\Psi}_3'\mathit{\Upsilon}_0 + \mathit{\Psi}_1'\mathit{\Upsilon}_{18} + \mathit{\Psi}_2'\mathit{\Upsilon}_{63})\\
&\quad +2\sqrt{2}(\mathit{\Phi}_1\mathit{\Upsilon}_9 + \mathit{\Phi}_2\mathit{\Upsilon}_{10} + \mathit{\Phi}_3\mathit{\Upsilon}_{11} + \mathit{\Phi}_4\mathit{\Upsilon}_{54} + \mathit{\Phi}_5\mathit{\Upsilon}_{55} + \mathit{\Phi}_6\mathit{\Upsilon}_{56})\\
&\quad +4(\mathit{\Phi}_1\mathit{\Upsilon}_{12} + \mathit{\Phi}_2\mathit{\Upsilon}_{13} + \mathit{\Phi}_4\mathit{\Upsilon}_{14} + \mathit{\Phi}_4\mathit{\Upsilon}_{57} + \mathit{\Phi}_5\mathit{\Upsilon}_{58} + \mathit{\Phi}_6\mathit{\Upsilon}_{59})\\
&\quad +8\textstyle\sum_{i=1}^{8}\mathit{\Upsilon}_i^2 + \sum_{i=9}^{11}\mathit{\Upsilon}_i^2 + 2\sum_{i=12}^{14}\mathit{\Upsilon}_i^2 + 4\sum_{i=15}^{17}\mathit{\Upsilon}_i^2 - 9\mathit{\Upsilon}_{18}^2 - 12\sum_{i=42}^{44}\mathit{\Upsilon}_i^2\\
&\quad +\textstyle\sum_{i=54}^{56}\mathit{\Upsilon}_i^2 + 2\sum_{i=57}^{59}\mathit{\Upsilon}_i^2 + 4\sum_{i=60}^{62}\mathit{\Upsilon}_i^2 - 9\mathit{\Upsilon}_{63}^2 - 12\sum_{i=87}^{89}\mathit{\Upsilon}_i^2\\
&\quad +8(\mathit{\Upsilon}_{15}\mathit{\Upsilon}_{42} + \mathit{\Upsilon}_{16}\mathit{\Upsilon}_{43} + \mathit{\Upsilon}_{17}\mathit{\Upsilon}_{44} + \mathit{\Upsilon}_{60}\mathit{\Upsilon}_{87} + \mathit{\Upsilon}_{61}\mathit{\Upsilon}_{88} + \mathit{\Upsilon}_{62}\mathit{\Upsilon}_{89})\\
&\quad +2\sqrt{2}(\mathit{\Upsilon}_9\mathit{\Upsilon}_{12} + \mathit{\Upsilon}_{10}\mathit{\Upsilon}_{13} + \mathit{\Upsilon}_{11}\mathit{\Upsilon}_{14} + \mathit{\Upsilon}_{54}\mathit{\Upsilon}_{57} + \mathit{\Upsilon}_{55}\mathit{\Upsilon}_{58} + \mathit{\Upsilon}_{56}\mathit{\Upsilon}_{59})\,,
\end{aligned}
\tag{B.2k}
$$

$$
\begin{aligned}
&\mathrm{i}\,\mathrm{tr}(((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\boldsymbol{\Upsilon}+\mathbf{n}') - \tfrac{1}{5}\mathrm{tr}((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\boldsymbol{\Upsilon}+\mathbf{n}'))\mathbb{1}_5 - 8(\boldsymbol{\Psi}+\mathbf{m}) + 9(\boldsymbol{\Phi}+\mathbf{n})(\boldsymbol{\Phi}+\mathbf{n})^*\\
&\qquad -\tfrac{9}{5}(\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n})\mathbb{1}_5)((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) - (\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*(\boldsymbol{\Upsilon}+\mathbf{n}')))\\
&\to \mathrm{i}\,\mathrm{tr}((\boldsymbol{\Upsilon}^*\mathbf{n}' + \mathbf{n}'^*\boldsymbol{\Upsilon} - \tfrac{1}{5}\mathrm{tr}(\boldsymbol{\Upsilon}^*\mathbf{n}' + \mathbf{n}'^*\boldsymbol{\Upsilon})\mathbb{1}_5 - 8\boldsymbol{\Psi} + 9(\boldsymbol{\Phi}\mathbf{n}^* + \mathbf{n}\boldsymbol{\Phi}^*) - \tfrac{9}{5}\mathrm{tr}(\boldsymbol{\Phi}\mathbf{n}^* + \mathbf{n}\boldsymbol{\Phi}^*)\mathbb{1}_5)\times\\
&\quad \times(\boldsymbol{\Upsilon}^*\check{\mathbf{n}} + \mathbf{n}'^*\check{\boldsymbol{\Phi}} - \check{\boldsymbol{\Phi}}^*\mathbf{n}' - \check{\mathbf{n}}^*\boldsymbol{\Upsilon}))\\
&= 40\sqrt{\tfrac{2}{5}}\mathit{\Psi}_0\mathit{\Upsilon}_{45} - 8\sqrt{6}(\mathit{\Psi}_1'\mathit{\Upsilon}_{63} - \mathit{\Psi}_2'\mathit{\Upsilon}_{18} + \mathit{\Psi}_3'\mathit{\Upsilon}_{45}) - 16\mathit{\Upsilon}_0\mathit{\Upsilon}_{45} - 32\textstyle\sum_{i=1}^{8}\mathit{\Psi}_i\mathit{\Upsilon}_{i+45} + 8\sum_{i=1}^{8}\mathit{\Upsilon}_i\mathit{\Upsilon}_{i+45}\\
&\quad +8\sqrt{2}(\mathit{\Phi}_1\mathit{\Upsilon}_{54} + \mathit{\Phi}_2\mathit{\Upsilon}_{55} + \mathit{\Phi}_3\mathit{\Upsilon}_{56} - \mathit{\Phi}_4\mathit{\Upsilon}_9 - \mathit{\Phi}_5\mathit{\Upsilon}_{10} - \mathit{\Phi}_6\mathit{\Upsilon}_{11})\\
&\quad +16(\mathit{\Phi}_1\mathit{\Upsilon}_{57} + \mathit{\Phi}_2\mathit{\Upsilon}_{58} + \mathit{\Phi}_3\mathit{\Upsilon}_{59} - \mathit{\Phi}_4\mathit{\Upsilon}_{12} - \mathit{\Phi}_5\mathit{\Upsilon}_{13} - \mathit{\Phi}_6\mathit{\Upsilon}_{14})\\
&\quad -8(\mathit{\Upsilon}_{15}\mathit{\Upsilon}_{87} + \mathit{\Upsilon}_{16}\mathit{\Upsilon}_{88} + \mathit{\Upsilon}_{17}\mathit{\Upsilon}_{89} - \mathit{\Upsilon}_{42}\mathit{\Upsilon}_{60} - \mathit{\Upsilon}_{43}\mathit{\Upsilon}_{61} - \mathit{\Upsilon}_{44}\mathit{\Upsilon}_{62})\,,
\end{aligned}
\tag{B.2l}
$$

$$
\begin{aligned}
&\mathrm{tr}((((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Upsilon}+\mathbf{n}')^*)' - \tfrac{8}{3}(\boldsymbol{\Psi}+\mathbf{m}) - (\boldsymbol{\Phi}+\mathbf{n})(\boldsymbol{\Phi}+\mathbf{n})^* + \tfrac{1}{5}(\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n})\mathbb{1}_5)\\
&\qquad ((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) + (\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*(\boldsymbol{\Upsilon}+\mathbf{n}') + 8(\boldsymbol{\Psi}+\mathbf{m}) - 6(\boldsymbol{\Phi}+\mathbf{n})(\boldsymbol{\Phi}+\mathbf{n})^*\\
&\qquad +\tfrac{6}{5}(\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n})\mathbb{1}_5))\\
&\to \mathrm{tr}(((\boldsymbol{\Upsilon}\mathbf{n}'^* + \mathbf{n}'\boldsymbol{\Upsilon}^*)' - \tfrac{8}{3}\boldsymbol{\Psi} - (\boldsymbol{\Phi}\mathbf{n}^* + \mathbf{n}\boldsymbol{\Phi}^*) + \tfrac{1}{5}\mathrm{tr}(\boldsymbol{\Phi}\mathbf{n}^* + \mathbf{n}\boldsymbol{\Phi}^*)\mathbb{1}_5)\times\\
&\quad \times(\boldsymbol{\Upsilon}^*\check{\mathbf{n}} + \mathbf{n}'^*\check{\boldsymbol{\Phi}} + \check{\boldsymbol{\Phi}}^*\mathbf{n}' + \check{\mathbf{n}}^*\boldsymbol{\Upsilon} + 8\boldsymbol{\Psi} - 6(\boldsymbol{\Phi}\mathbf{n}^* + \mathbf{n}\boldsymbol{\Phi}^*) + \tfrac{6}{5}\mathrm{tr}(\boldsymbol{\Phi}\mathbf{n}^* + \mathbf{n}\boldsymbol{\Phi}^*)\mathbb{1}_5))\\
&= -\tfrac{128}{3}\textstyle\sum_{i=0}^{8}\mathit{\Psi}_i^2 - \tfrac{128}{3}\sum_{i=1}^{3}\mathit{\Psi}_i'^2 + 14\sum_{i=1}^{6}\mathit{\Phi}_i^2 + \tfrac{128}{3}\sqrt{\tfrac{3}{5}}\mathit{\Psi}_0\mathit{\Phi}_0 + \tfrac{96}{5}\mathit{\Phi}_0^2 + 64\sqrt{\tfrac{2}{5}}\mathit{\Psi}_0\mathit{\Upsilon}_0\\
&\quad -\tfrac{224}{15}\sqrt{6}\mathit{\Phi}_0\mathit{\Upsilon}_0 - \tfrac{16}{3}\mathit{\Upsilon}_0^2 - \tfrac{38}{3}\sqrt{2}(\mathit{\Phi}_1\mathit{\Upsilon}_9 + \mathit{\Phi}_2\mathit{\Upsilon}_{10} + \mathit{\Phi}_3\mathit{\Upsilon}_{11} + \mathit{\Phi}_4\mathit{\Upsilon}_{54} + \mathit{\Phi}_5\mathit{\Upsilon}_{55} + \mathit{\Phi}_6\mathit{\Upsilon}_{56})\\
&\quad +12(\mathit{\Phi}_1\mathit{\Upsilon}_{12} + \mathit{\Phi}_2\mathit{\Upsilon}_{13} + \mathit{\Phi}_4\mathit{\Upsilon}_{14} + \mathit{\Phi}_4\mathit{\Upsilon}_{57} + \mathit{\Phi}_5\mathit{\Upsilon}_{58} + \mathit{\Phi}_6\mathit{\Upsilon}_{59})\\
&\quad +\tfrac{8}{3}\textstyle\sum_{i=1}^{8}\mathit{\Upsilon}_i^2 + \tfrac{5}{3}\sum_{i=9}^{11}\mathit{\Upsilon}_i^2 - 2\sum_{i=12}^{14}\mathit{\Upsilon}_i^2 - 4\sum_{i=15}^{17}\mathit{\Upsilon}_i^2 + \mathit{\Upsilon}_{18}^2 + \tfrac{4}{3}\sum_{i=42}^{44}\mathit{\Upsilon}_i^2\\
&\quad +\tfrac{5}{3}\textstyle\sum_{i=54}^{56}\mathit{\Upsilon}_i^2 - 2\sum_{i=57}^{59}\mathit{\Upsilon}_i^2 - 4\sum_{i=60}^{62}\mathit{\Upsilon}_i^2 + \mathit{\Upsilon}_{63}^2 + \tfrac{4}{3}\sum_{i=87}^{89}\mathit{\Upsilon}_i^2\\
&\quad +\tfrac{8}{3}(\mathit{\Upsilon}_{15}\mathit{\Upsilon}_{42} + \mathit{\Upsilon}_{16}\mathit{\Upsilon}_{43} + \mathit{\Upsilon}_{17}\mathit{\Upsilon}_{44} + \mathit{\Upsilon}_{60}\mathit{\Upsilon}_{87} + \mathit{\Upsilon}_{61}\mathit{\Upsilon}_{88} + \mathit{\Upsilon}_{62}\mathit{\Upsilon}_{89})\\
&\quad +\tfrac{2}{3}\sqrt{2}(\mathit{\Upsilon}_9\mathit{\Upsilon}_{12} + \mathit{\Upsilon}_{10}\mathit{\Upsilon}_{13} + \mathit{\Upsilon}_{11}\mathit{\Upsilon}_{14} + \mathit{\Upsilon}_{54}\mathit{\Upsilon}_{57} + \mathit{\Upsilon}_{55}\mathit{\Upsilon}_{58} + \mathit{\Upsilon}_{56}\mathit{\Upsilon}_{59})\,,
\end{aligned}
\tag{B.2m}
$$

$$
\begin{aligned}
&\mathrm{i}\,\mathrm{tr}((((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Upsilon}+\mathbf{n}')^*)' - \tfrac{8}{3}(\boldsymbol{\Psi}+\mathbf{m}) - (\boldsymbol{\Phi}+\mathbf{n})(\boldsymbol{\Phi}+\mathbf{n})^* + \tfrac{1}{5}(\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n})\mathbb{1}_5)\\
&\qquad ((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) - (\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*(\boldsymbol{\Upsilon}+\mathbf{n}')))\\
&\to \mathrm{i}\,\mathrm{tr}(((\boldsymbol{\Upsilon}\mathbf{n}'^* + \mathbf{n}'\boldsymbol{\Upsilon}^*)' - \tfrac{8}{3}\boldsymbol{\Psi} - (\boldsymbol{\Phi}\mathbf{n}^* + \mathbf{n}\boldsymbol{\Phi}^*) + \tfrac{1}{5}\mathrm{tr}(\boldsymbol{\Phi}\mathbf{n}^* + \mathbf{n}\boldsymbol{\Phi}^*)\mathbb{1}_5)\times\\
&\quad \times(\boldsymbol{\Upsilon}^*\check{\mathbf{n}} + \mathbf{n}'^*\check{\boldsymbol{\Phi}} - \check{\boldsymbol{\Phi}}^*\mathbf{n}' - \check{\mathbf{n}}^*\boldsymbol{\Upsilon}))\\
&= \tfrac{40}{3}\sqrt{\tfrac{2}{5}}\mathit{\Psi}_0\mathit{\Upsilon}_{45} - \tfrac{8}{3}\sqrt{6}(\mathit{\Psi}_1'\mathit{\Upsilon}_{63} - \mathit{\Psi}_2'\mathit{\Upsilon}_{18} + \mathit{\Psi}_3'\mathit{\Upsilon}_{45}) - \tfrac{16}{3}\mathit{\Upsilon}_0\mathit{\Upsilon}_{45} - \tfrac{32}{3}\textstyle\sum_{i=1}^{8}\mathit{\Psi}_i\mathit{\Upsilon}_{i+45}\\
&\quad -\tfrac{8}{3}\sqrt{2}(\mathit{\Phi}_1\mathit{\Upsilon}_{54} + \mathit{\Phi}_2\mathit{\Upsilon}_{55} + \mathit{\Phi}_3\mathit{\Upsilon}_{56} - \mathit{\Phi}_4\mathit{\Upsilon}_9 - \mathit{\Phi}_5\mathit{\Upsilon}_{10} - \mathit{\Phi}_6\mathit{\Upsilon}_{11}) + \tfrac{8}{3}\textstyle\sum_{i=1}^{8}\mathit{\Upsilon}_i\mathit{\Upsilon}_{i+45}\\
&\quad -\tfrac{8}{3}(\mathit{\Upsilon}_{15}\mathit{\Upsilon}_{87} + \mathit{\Upsilon}_{16}\mathit{\Upsilon}_{88} + \mathit{\Upsilon}_{17}\mathit{\Upsilon}_{89} - \mathit{\Upsilon}_{42}\mathit{\Upsilon}_{60} - \mathit{\Upsilon}_{43}\mathit{\Upsilon}_{61} - \mathit{\Upsilon}_{44}\mathit{\Upsilon}_{62})\\
&\quad +\tfrac{8}{3}\sqrt{2}(\mathit{\Upsilon}_9\mathit{\Upsilon}_{57} + \mathit{\Upsilon}_{10}\mathit{\Upsilon}_{58} + \mathit{\Upsilon}_{11}\mathit{\Upsilon}_{59} - \mathit{\Upsilon}_{12}\mathit{\Upsilon}_{54} - \mathit{\Upsilon}_{13}\mathit{\Upsilon}_{55} - \mathit{\Upsilon}_{14}\mathit{\Upsilon}_{56})\,,
\end{aligned}
\tag{B.2n}
$$

$$
\begin{aligned}
&\mathrm{i}\,\mathrm{tr}(((\mathbf{\Upsilon}+\mathbf{n}')^*(\check{\mathbf{\Phi}}+\check{\mathbf{n}})+(\check{\mathbf{\Phi}}+\check{\mathbf{n}})^*(\mathbf{\Upsilon}+\mathbf{n}')+8(\mathbf{\Psi}+\mathbf{m})-6(\mathbf{\Phi}+\mathbf{n})(\mathbf{\Phi}+\mathbf{n})^* \\
&\qquad+\tfrac{6}{5}(\mathbf{\Phi}+\mathbf{n})^*(\mathbf{\Phi}+\mathbf{n})\mathbb{1}_5)((\mathbf{\Upsilon}+\mathbf{n}')^*(\check{\mathbf{\Phi}}+\check{\mathbf{n}})+(\check{\mathbf{\Phi}}+\check{\mathbf{n}})^*(\mathbf{\Upsilon}+\mathbf{n}'))) \\
&\to=\mathrm{i}\,\mathrm{tr}((\mathbf{\Upsilon}^*\check{\mathbf{n}}+\mathbf{n}'^*\check{\mathbf{\Phi}}+\check{\mathbf{\Phi}}^*\mathbf{n}'+\check{\mathbf{n}}^*\mathbf{\Upsilon}+8\mathbf{\Psi}-6(\mathbf{\Phi}\mathbf{n}^*+\mathbf{n}\mathbf{\Phi}^*)+\tfrac{6}{5}\,\mathrm{tr}(\mathbf{\Phi}\mathbf{n}^*+\mathbf{n}\mathbf{\Phi}^*)\mathbb{1}_5) \\
&\quad(\mathbf{\Upsilon}^*\check{\mathbf{n}}+\mathbf{n}'^*\check{\mathbf{\Phi}}-\check{\mathbf{\Phi}}^*\mathbf{n}'-\check{\mathbf{n}}^*\mathbf{\Upsilon})) \\
&=8\sqrt{6}\mathit{\Upsilon}_{45}\mathit{\Phi}_0+8\mathit{\Upsilon}_0\mathit{\Upsilon}_{45}-40\sqrt{\tfrac{2}{5}}\mathit{\Psi}_0\mathit{\Upsilon}_{45}+8\sqrt{6}(\mathit{\Psi}'_1\mathit{\Upsilon}_{63}-\mathit{\Psi}'_2\mathit{\Upsilon}_{18}+\mathit{\Psi}'_3\mathit{\Upsilon}_{45})+32\textstyle\sum_{i=1}^{8}\mathit{\Psi}_i\mathit{\Upsilon}_{i+45} \\
&\quad-8\sqrt{2}(\mathit{\Phi}_1\mathit{\Upsilon}_{54}+\mathit{\Phi}_2\mathit{\Upsilon}_{55}+\mathit{\Phi}_3\mathit{\Upsilon}_{56}-\mathit{\Phi}_4\mathit{\Upsilon}_9-\mathit{\Phi}_5\mathit{\Upsilon}_{10}-\mathit{\Phi}_6\mathit{\Upsilon}_{11})+8\textstyle\sum_{i=1}^{8}\mathit{\Upsilon}_i\mathit{\Upsilon}_{i+45} \\
&\quad-16(\mathit{\Phi}_1\mathit{\Upsilon}_{57}+\mathit{\Phi}_2\mathit{\Upsilon}_{58}+\mathit{\Phi}_3\mathit{\Upsilon}_{59}-\mathit{\Phi}_4\mathit{\Upsilon}_{12}-\mathit{\Phi}_5\mathit{\Upsilon}_{13}-\mathit{\Phi}_6\mathit{\Upsilon}_{14}) \\
&\quad+8(\mathit{\Upsilon}_{15}\mathit{\Upsilon}_{87}+\mathit{\Upsilon}_{16}\mathit{\Upsilon}_{88}+\mathit{\Upsilon}_{17}\mathit{\Upsilon}_{89}-\mathit{\Upsilon}_{42}\mathit{\Upsilon}_{60}-\mathit{\Upsilon}_{43}\mathit{\Upsilon}_{61}-\mathit{\Upsilon}_{44}\mathit{\Upsilon}_{62})\,,
\end{aligned}
\tag{B.2o}
$$

$$
\begin{aligned}
&\big(\mathrm{tr}(((\mathbf{\Upsilon}+\mathbf{n}')(\mathbf{\Upsilon}+\mathbf{n}')^*-4(\check{\mathbf{\Psi}}+\check{\mathbf{m}})^2)^2)-\tfrac{1}{10}(\mathrm{tr}((\mathbf{\Upsilon}+\mathbf{n}')^*(\mathbf{\Upsilon}+\mathbf{n}')-12(\mathbf{\Psi}+\mathbf{m})^2))^2 \\
&\qquad-3\,\mathrm{tr}((((\mathbf{\Upsilon}+\mathbf{n}')(\mathbf{\Upsilon}+\mathbf{n}')^*)'-\tfrac{4}{3}(\mathbf{\Psi}+\mathbf{m})^2+\tfrac{4}{15}\,\mathrm{tr}((\mathbf{\Psi}+\mathbf{m})^2)\mathbb{1}_5)^2)\big) \\
&\to\mathrm{tr}((\mathbf{\Upsilon}\mathbf{n}'^*+\mathbf{n}'\mathbf{\Upsilon}^*-4\check{\mathbf{\Psi}}\check{\mathbf{m}}-4\check{\mathbf{m}}\check{\mathbf{\Psi}})^2)-\tfrac{1}{10}(\mathrm{tr}(\mathbf{\Upsilon}^*\mathbf{n}'+\mathbf{n}'^*\mathbf{\Upsilon}-24\mathbf{m}\mathbf{\Psi}))^2 \\
&\quad-3\,\mathrm{tr}(((\mathbf{\Upsilon}\mathbf{n}'^*+\mathbf{n}'\mathbf{\Upsilon}^*)'-\tfrac{4}{3}(\mathbf{\Psi}\mathbf{m}+\mathbf{m}\mathbf{\Psi})+\tfrac{8}{15}\,\mathrm{tr}(\mathbf{m}\mathbf{\Psi})\mathbb{1}_5)^2) \\
&=\tfrac{640}{3}\mathit{\Psi}_0^2+\tfrac{256}{3}\textstyle\sum_{i=1}^{8}\mathit{\Psi}_i^2-\tfrac{640}{3}\sqrt{\tfrac{2}{5}}\mathit{\Psi}_0\mathit{\Upsilon}_0+\tfrac{64}{3}\mathit{\Upsilon}_0^2-\tfrac{64}{3}\textstyle\sum_{i=1}^{8}\mathit{\Psi}_i\mathit{\Upsilon}_i \\
&\quad+\tfrac{16}{3}\textstyle\sum_{i=1}^{8}\mathit{\Upsilon}_i^2+\tfrac{8}{3}\sum_{i=9}^{11}\mathit{\Upsilon}_i^2+12\sum_{i=12}^{14}\mathit{\Upsilon}_i^2+24\sum_{i=15}^{17}\mathit{\Upsilon}_i^2+4\sum_{i=19}^{26}\mathit{\Upsilon}_i^2 \\
&\quad+4\textstyle\sum_{i=30}^{35}\mathit{\Upsilon}_i^2+36\sum_{i=39}^{41}\mathit{\Upsilon}_i^2+\tfrac{8}{3}\sum_{i=42}^{44}\mathit{\Upsilon}_i^2+\tfrac{8}{3}\sum_{i=54}^{56}\mathit{\Upsilon}_i^2+12\sum_{i=57}^{59}\mathit{\Upsilon}_i^2 \\
&\quad+24\textstyle\sum_{i=60}^{62}\mathit{\Upsilon}_i^2+4\sum_{i=64}^{71}\mathit{\Upsilon}_i^2+4\sum_{i=75}^{80}\mathit{\Upsilon}_i^2+36\sum_{i=84}^{86}\mathit{\Upsilon}_i^2+\tfrac{8}{3}\sum_{i=87}^{89}\mathit{\Upsilon}_i^2 \\
&\quad+16(\mathit{\Upsilon}_{15}\mathit{\Upsilon}_{42}+\mathit{\Upsilon}_{16}\mathit{\Upsilon}_{43}+\mathit{\Upsilon}_{17}\mathit{\Upsilon}_{44}+\mathit{\Upsilon}_{60}\mathit{\Upsilon}_{87}+\mathit{\Upsilon}_{61}\mathit{\Upsilon}_{88}+\mathit{\Upsilon}_{62}\mathit{\Upsilon}_{89}) \\
&\quad-8\sqrt{2}(\mathit{\Upsilon}_9\mathit{\Upsilon}_{12}+\mathit{\Upsilon}_{10}\mathit{\Upsilon}_{13}+\mathit{\Upsilon}_{11}\mathit{\Upsilon}_{14}+\mathit{\Upsilon}_{54}\mathit{\Upsilon}_{57}+\mathit{\Upsilon}_{55}\mathit{\Upsilon}_{58}+\mathit{\Upsilon}_{56}\mathit{\Upsilon}_{59})\,,
\end{aligned}
\tag{B.2p}
$$

$$
\begin{aligned}
&\big(\mathrm{tr}(((\check{\mathbf{\Phi}}+\check{\mathbf{n}})(\mathbf{\Upsilon}+\mathbf{n}')^*+(\mathbf{\Upsilon}+\mathbf{n}')(\check{\mathbf{\Phi}}+\check{\mathbf{n}})^*+4(\check{\mathbf{\Psi}}+\check{\mathbf{m}})^2)^2)-\tfrac{72}{5}(\mathrm{tr}((\mathbf{\Psi}+\mathbf{m})^2))^2 \\
&\qquad-\tfrac{1}{3}\,\mathrm{tr}((((\mathbf{\Upsilon}+\mathbf{n}')^*(\check{\mathbf{\Phi}}+\check{\mathbf{n}})+(\check{\mathbf{\Phi}}+\check{\mathbf{n}})^*(\mathbf{\Upsilon}+\mathbf{n}')+4(\mathbf{\Psi}+\mathbf{m})^2-\tfrac{4}{5}\,\mathrm{tr}((\mathbf{\Psi}+\mathbf{m})^2)\mathbb{1}_5)^2)\big) \\
&\to\mathrm{tr}((\check{\mathbf{n}}\mathbf{\Upsilon}^*+\check{\mathbf{\Phi}}\mathbf{n}'^*+\mathbf{\Upsilon}\check{\mathbf{n}}^*+\mathbf{n}'\check{\mathbf{\Phi}}^*+4\check{\mathbf{\Psi}}\check{\mathbf{m}}+4\check{\mathbf{m}}\check{\mathbf{\Psi}})^2)-\tfrac{288}{5}(\mathrm{tr}(\mathbf{m}\mathbf{\Psi}))^2 \\
&\quad-\tfrac{1}{3}\,\mathrm{tr}((\mathbf{\Upsilon}^*\check{\mathbf{n}}+\mathbf{n}'^*\check{\mathbf{\Phi}}+\check{\mathbf{n}}^*\mathbf{\Upsilon}+\check{\mathbf{\Phi}}^*\mathbf{n}'+4\mathbf{\Psi}\mathbf{m}+4\mathbf{m}\mathbf{\Psi}-\tfrac{8}{5}\,\mathrm{tr}(\mathbf{m}\mathbf{\Psi})\mathbb{1}_5)^2) \\
&=\tfrac{640}{3}\mathit{\Psi}_0^2+\tfrac{256}{3}\textstyle\sum_{i=1}^{8}\mathit{\Psi}_i^2+\tfrac{64}{3}\sum_{i=1}^{6}\mathit{\Phi}_i^2+32\mathit{\Phi}_0^2-\tfrac{640}{3}\sqrt{\tfrac{3}{5}}\mathit{\Psi}_0\mathit{\Phi}_0 \\
&\quad-\tfrac{320}{3}\sqrt{\tfrac{2}{5}}\mathit{\Psi}_0\mathit{\Upsilon}_0+\tfrac{32}{3}\sqrt{6}\mathit{\Phi}_0\mathit{\Upsilon}_0+\tfrac{32}{3}\sqrt{2}(\mathit{\Phi}_1\mathit{\Upsilon}_9+\mathit{\Phi}_2\mathit{\Upsilon}_{10}+\mathit{\Phi}_3\mathit{\Upsilon}_{11}+\mathit{\Phi}_4\mathit{\Upsilon}_{54}+\mathit{\Phi}_5\mathit{\Upsilon}_{55}+\mathit{\Phi}_6\mathit{\Upsilon}_{56}) \\
&\quad-\tfrac{32}{3}(\mathit{\Phi}_1\mathit{\Upsilon}_{12}+\mathit{\Phi}_2\mathit{\Upsilon}_{13}+\mathit{\Phi}_3\mathit{\Upsilon}_{14}+\mathit{\Phi}_4\mathit{\Upsilon}_{57}+\mathit{\Phi}_5\mathit{\Upsilon}_{58}+\mathit{\Phi}_6\mathit{\Upsilon}_{59})+\tfrac{64}{3}\textstyle\sum_{i=1}^{8}\mathit{\Psi}_i\mathit{\Upsilon}_i \\
&\quad+\tfrac{16}{3}\textstyle\sum_{i=0}^{8}\mathit{\Upsilon}_i^2+\tfrac{8}{3}\sum_{i=9}^{11}\mathit{\Upsilon}_i^2+\tfrac{4}{3}\sum_{i=12}^{14}\mathit{\Upsilon}_i^2+\tfrac{8}{3}\sum_{i=15}^{17}\mathit{\Upsilon}_i^2+4\sum_{i=19}^{26}\mathit{\Upsilon}_i^2 \\
&\quad+4\textstyle\sum_{i=30}^{35}\mathit{\Upsilon}_i^2+4\sum_{i=39}^{41}\mathit{\Upsilon}_i^2+\tfrac{8}{3}\sum_{i=42}^{44}\mathit{\Upsilon}_i^2+\tfrac{8}{3}\sum_{i=54}^{56}\mathit{\Upsilon}_i^2+\tfrac{4}{3}\sum_{i=57}^{59}\mathit{\Upsilon}_i^2 \\
&\quad+\tfrac{8}{3}\textstyle\sum_{i=60}^{62}\mathit{\Upsilon}_i^2+4\sum_{i=64}^{71}\mathit{\Upsilon}_i^2+4\sum_{i=75}^{80}\mathit{\Upsilon}_i^2+4\sum_{i=84}^{86}\mathit{\Upsilon}_i^2+\tfrac{8}{3}\sum_{i=87}^{89}\mathit{\Upsilon}_i^2 \\
&\quad-\tfrac{16}{3}(\mathit{\Upsilon}_{15}\mathit{\Upsilon}_{42}+\mathit{\Upsilon}_{16}\mathit{\Upsilon}_{43}+\mathit{\Upsilon}_{17}\mathit{\Upsilon}_{44}+\mathit{\Upsilon}_{60}\mathit{\Upsilon}_{87}+\mathit{\Upsilon}_{61}\mathit{\Upsilon}_{88}+\mathit{\Upsilon}_{62}\mathit{\Upsilon}_{89}) \\
&\quad-\tfrac{8}{3}\sqrt{2}(\mathit{\Upsilon}_9\mathit{\Upsilon}_{12}+\mathit{\Upsilon}_{10}\mathit{\Upsilon}_{13}+\mathit{\Upsilon}_{11}\mathit{\Upsilon}_{14}+\mathit{\Upsilon}_{54}\mathit{\Upsilon}_{57}+\mathit{\Upsilon}_{55}\mathit{\Upsilon}_{58}+\mathit{\Upsilon}_{56}\mathit{\Upsilon}_{59})\,,
\end{aligned}
\tag{B.2q}
$$

$$
\begin{aligned}
&\mathrm{tr}(-((\check{\mathbf{\Phi}}+\check{\mathbf{n}})(\mathbf{\Upsilon}+\mathbf{n}')^*-(\mathbf{\Upsilon}+\mathbf{n}')(\check{\mathbf{\Phi}}+\check{\mathbf{n}})^*)^2)+\tfrac{1}{3}\,\mathrm{tr}(((\mathbf{\Upsilon}+\mathbf{n}')^*(\check{\mathbf{\Phi}}+\check{\mathbf{n}})-(\check{\mathbf{\Phi}}+\check{\mathbf{n}})^*(\mathbf{\Upsilon}+\mathbf{n}'))^2) \\
&\to\mathrm{tr}(-(\check{\mathbf{n}}\mathbf{\Upsilon}^*+\check{\mathbf{\Phi}}\mathbf{n}'^*-\mathbf{\Upsilon}\check{\mathbf{n}}^*-\mathbf{n}'\check{\mathbf{\Phi}}^*)^2)+\tfrac{1}{3}\,\mathrm{tr}((\mathbf{\Upsilon}^*\check{\mathbf{n}}+\mathbf{n}'^*\check{\mathbf{\Phi}}-\check{\mathbf{n}}^*\mathbf{\Upsilon}-\check{\mathbf{\Phi}}^*\mathbf{n}')^2) \\
&=\tfrac{64}{3}\textstyle\sum_{i=1}^{6}\mathit{\Phi}_i^2-\tfrac{32}{3}\sqrt{2}(\mathit{\Phi}_1\mathit{\Upsilon}_9+\mathit{\Phi}_2\mathit{\Upsilon}_{10}+\mathit{\Phi}_3\mathit{\Upsilon}_{11}+\mathit{\Phi}_4\mathit{\Upsilon}_{54}+\mathit{\Phi}_5\mathit{\Upsilon}_{55}+\mathit{\Phi}_6\mathit{\Upsilon}_{56}) \\
&\quad+\tfrac{32}{3}(\mathit{\Phi}_1\mathit{\Upsilon}_{12}+\mathit{\Phi}_2\mathit{\Upsilon}_{13}+\mathit{\Phi}_3\mathit{\Upsilon}_{14}+\mathit{\Phi}_4\mathit{\Upsilon}_{57}+\mathit{\Phi}_5\mathit{\Upsilon}_{58}+\mathit{\Phi}_6\mathit{\Upsilon}_{59})
\end{aligned}
$$

$$\begin{aligned}
&+\tfrac{8}{3}\textstyle\sum_{i=9}^{11}\Upsilon_i^2+\tfrac{4}{3}\sum_{i=12}^{14}\Upsilon_i^2+\tfrac{8}{3}\sum_{i=15}^{17}\Upsilon_i^2+4\sum_{i=19}^{26}\Upsilon_i^2+4\sum_{i=30}^{35}\Upsilon_i^2\\
&+4\textstyle\sum_{i=39}^{41}\Upsilon_i^2+\tfrac{8}{3}\sum_{i=42}^{44}\Upsilon_i^2+\tfrac{16}{3}\sum_{i=45}^{53}\Upsilon_i^2+\tfrac{8}{3}\sum_{i=54}^{56}\Upsilon_i^2+\tfrac{4}{3}\sum_{i=57}^{59}\Upsilon_i^2\\
&+\tfrac{8}{3}\textstyle\sum_{i=60}^{62}\Upsilon_i^2+4\sum_{i=64}^{71}\Upsilon_i^2+4\sum_{i=75}^{80}\Upsilon_i^2+4\sum_{i=84}^{86}\Upsilon_i^2+\tfrac{8}{3}\sum_{i=87}^{89}\Upsilon_i^2\\
&-\tfrac{8}{3}\sqrt{2}(\Upsilon_9\Upsilon_{12}+\Upsilon_{10}\Upsilon_{13}+\Upsilon_{11}\Upsilon_{14}+\Upsilon_{54}\Upsilon_{57}+\Upsilon_{55}\Upsilon_{58}+\Upsilon_{56}\Upsilon_{59})\\
&+\tfrac{16}{3}(\Upsilon_{15}\Upsilon_{42}+\Upsilon_{16}\Upsilon_{43}+\Upsilon_{17}\Upsilon_{44}+\Upsilon_{60}\Upsilon_{87}+\Upsilon_{61}\Upsilon_{88}+\Upsilon_{62}\Upsilon_{89})\,,
\end{aligned}\tag{B.2r}$$

$$\begin{aligned}
&\big(\operatorname{tr}(((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Upsilon}+\mathbf{n}')^*-4(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^2)((\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})(\boldsymbol{\Upsilon}+\mathbf{n}')^*+(\boldsymbol{\Upsilon}+\mathbf{n}')(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*+4(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^2))\\
&\qquad-\tfrac{6}{5}\operatorname{tr}((\boldsymbol{\Psi}+\mathbf{m})^2)\operatorname{tr}((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\boldsymbol{\Upsilon}+\mathbf{n}')-12(\boldsymbol{\Psi}+\mathbf{m})^2)\\
&\qquad-\operatorname{tr}((((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Upsilon}+\mathbf{n}')^*)'-\tfrac{4}{3}(\boldsymbol{\Psi}+\mathbf{m})^2+\tfrac{4}{15}\operatorname{tr}((\boldsymbol{\Psi}+\mathbf{m})^2)\mathbb{1}_5)\times\\
&\qquad\times((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})+(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*(\boldsymbol{\Upsilon}+\mathbf{n}')+4(\boldsymbol{\Psi}+\mathbf{m})^2-\tfrac{4}{5}\operatorname{tr}((\boldsymbol{\Psi}+\mathbf{m})^2)\mathbb{1}_5))\big)\\
&\to\operatorname{tr}((\boldsymbol{\Upsilon}\mathbf{n}'^*+\mathbf{n}'\boldsymbol{\Upsilon}^*-4\check{\mathbf{m}}\check{\boldsymbol{\Psi}}-4\check{\boldsymbol{\Psi}}\check{\mathbf{m}})(\check{\boldsymbol{\Phi}}\mathbf{n}'^*+\check{\mathbf{n}}\boldsymbol{\Upsilon}^*+\boldsymbol{\Upsilon}\check{\mathbf{n}}^*+\mathbf{n}'\check{\boldsymbol{\Phi}}^*+4\check{\mathbf{m}}\check{\boldsymbol{\Psi}}+4\check{\boldsymbol{\Psi}}\check{\mathbf{m}}))\\
&\quad-\tfrac{12}{5}\operatorname{tr}(\boldsymbol{\Psi}\mathbf{m})\operatorname{tr}(\boldsymbol{\Upsilon}^*\mathbf{n}'+\mathbf{n}'^*\boldsymbol{\Upsilon}-24\boldsymbol{\Psi}\mathbf{m})-\operatorname{tr}(((\boldsymbol{\Upsilon}\mathbf{n}'^*+\mathbf{n}'\boldsymbol{\Upsilon}^*)'-\tfrac{4}{3}(\mathbf{m}\boldsymbol{\Psi}+\boldsymbol{\Psi}\mathbf{m})+\tfrac{8}{15}\operatorname{tr}(\mathbf{m}\boldsymbol{\Psi})\mathbb{1}_5)\times\\
&\quad\times(\boldsymbol{\Upsilon}^*\check{\mathbf{n}}+\mathbf{n}'^*\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}^*\boldsymbol{\Upsilon}+\check{\boldsymbol{\Phi}}^*\mathbf{n}'+4\boldsymbol{\Psi}\mathbf{m}+4\mathbf{m}\boldsymbol{\Psi}-\tfrac{8}{5}\operatorname{tr}(\mathbf{m}\boldsymbol{\Psi})\mathbb{1}_5))\\
&=-\tfrac{640}{3}\Psi_0^2-\tfrac{256}{3}\textstyle\sum_{i=1}^{8}\Psi_i^2+\tfrac{320}{3}\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0+160\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0\\
&\quad-\tfrac{32}{3}\sqrt{6}\Phi_0\Upsilon_0-\tfrac{32}{3}\Upsilon_0^2-\tfrac{16}{3}\sqrt{2}(\Phi_1\Upsilon_9+\Phi_2\Upsilon_{10}+\Phi_3\Upsilon_{11}+\Phi_4\Upsilon_{54}+\Phi_5\Upsilon_{55}+\Phi_6\Upsilon_{56})\\
&\quad+16(\Phi_1\Upsilon_{12}+\Phi_2\Upsilon_{13}+\Phi_4\Upsilon_{14}+\Phi_4\Upsilon_{57}+\Phi_5\Upsilon_{58}+\Phi_6\Upsilon_{59})\\
&\quad+\tfrac{16}{3}\textstyle\sum_{i=1}^{8}\Upsilon_i^2-\tfrac{8}{3}\sum_{i=9}^{11}\Upsilon_i^2-4\sum_{i=12}^{14}\Upsilon_i^2-8\sum_{i=15}^{17}\Upsilon_i^2+4\sum_{i=19}^{26}\Upsilon_i^2\\
&\quad+4\textstyle\sum_{i=30}^{35}\Upsilon_i^2-12\sum_{i=39}^{41}\Upsilon_i^2+\tfrac{8}{3}\sum_{i=42}^{44}\Upsilon_i^2-\tfrac{8}{3}\sum_{i=54}^{56}\Upsilon_i^2-4\sum_{i=57}^{59}\Upsilon_i^2\\
&\quad-8\textstyle\sum_{i=60}^{62}\Upsilon_i^2+4\sum_{i=64}^{71}\Upsilon_i^2+4\sum_{i=75}^{80}\Upsilon_i^2-12\sum_{i=84}^{86}\Upsilon_i^2+\tfrac{8}{3}\sum_{i=87}^{89}\Upsilon_i^2\\
&\quad+\tfrac{16}{3}(\Upsilon_{15}\Upsilon_{42}+\Upsilon_{16}\Upsilon_{43}+\Upsilon_{17}\Upsilon_{44}+\Upsilon_{60}\Upsilon_{87}+\Upsilon_{61}\Upsilon_{88}+\Upsilon_{62}\Upsilon_{89})\\
&\quad+\tfrac{16}{3}\sqrt{2}(\Upsilon_9\Upsilon_{12}+\Upsilon_{10}\Upsilon_{13}+\Upsilon_{11}\Upsilon_{14}+\Upsilon_{54}\Upsilon_{57}+\Upsilon_{55}\Upsilon_{58}+\Upsilon_{56}\Upsilon_{59})\,,
\end{aligned}\tag{B.2s}$$

$$\begin{aligned}
&\mathrm{i}\big(\operatorname{tr}(((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Upsilon}+\mathbf{n}')^*-4(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^2)((\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})(\boldsymbol{\Upsilon}+\mathbf{n}')^*-(\boldsymbol{\Upsilon}+\mathbf{n}')(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*))-\\
&\quad-\tfrac{1}{3}\operatorname{tr}((((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Upsilon}+\mathbf{n}')^*)'-\tfrac{4}{3}(\boldsymbol{\Psi}+\mathbf{m})^2)((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})-(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*(\boldsymbol{\Upsilon}+\mathbf{n}')))\big)\\
&\to\mathrm{i}\big(\operatorname{tr}((\boldsymbol{\Upsilon}\mathbf{n}'^*+\mathbf{n}'\boldsymbol{\Upsilon}^*-4\check{\mathbf{m}}\check{\boldsymbol{\Psi}}-4\check{\boldsymbol{\Psi}}\check{\mathbf{m}})(\check{\boldsymbol{\Phi}}\mathbf{n}'^*+\check{\mathbf{n}}\boldsymbol{\Upsilon}^*-\boldsymbol{\Upsilon}\check{\mathbf{n}}^*-\mathbf{n}'\check{\boldsymbol{\Phi}}^*))-\\
&\quad-\operatorname{tr}(((\boldsymbol{\Upsilon}\mathbf{n}'^*+\mathbf{n}'\boldsymbol{\Upsilon}^*)'-\tfrac{4}{3}(\mathbf{m}\boldsymbol{\Psi}+\boldsymbol{\Psi}\mathbf{m}))(\boldsymbol{\Upsilon}^*\check{\mathbf{n}}+\mathbf{n}'^*\check{\boldsymbol{\Phi}}-\check{\boldsymbol{\Phi}}^*\mathbf{n}'-\check{\mathbf{n}}^*\boldsymbol{\Upsilon}))\big)\\
&=\tfrac{160}{3}\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_{45}-\tfrac{32}{3}\Upsilon_0\Upsilon_{45}-\tfrac{32}{3}\textstyle\sum_{i=1}^{8}\Psi_i\Upsilon_{i+45}+\tfrac{16}{3}\sum_{i=1}^{8}\Upsilon_i\Upsilon_{i+45}\\
&\quad-\tfrac{16}{3}\sqrt{2}(\Phi_1\Upsilon_{54}+\Phi_2\Upsilon_{55}+\Phi_3\Upsilon_{56}-\Phi_4\Upsilon_9-\Phi_5\Upsilon_{10}-\Phi_6\Upsilon_{11})\\
&\quad+16(\Phi_1\Upsilon_{57}+\Phi_2\Upsilon_{58}+\Phi_3\Upsilon_{59}-\Phi_4\Upsilon_{12}-\Phi_5\Upsilon_{13}-\Phi_6\Upsilon_{14})\\
&\quad-\tfrac{8}{3}\sqrt{2}(\Upsilon_9\Upsilon_{57}+\Upsilon_{10}\Upsilon_{58}+\Upsilon_{11}\Upsilon_{59}-\Upsilon_{12}\Upsilon_{54}-\Upsilon_{13}\Upsilon_{55}-\Upsilon_{14}\Upsilon_{56})\\
&\quad-\tfrac{16}{3}(\Upsilon_{15}\Upsilon_{87}+\Upsilon_{16}\Upsilon_{88}+\Upsilon_{17}\Upsilon_{89}-\Upsilon_{42}\Upsilon_{60}-\Upsilon_{43}\Upsilon_{61}-\Upsilon_{44}\Upsilon_{62})\,,
\end{aligned}\tag{B.2t}$$

$$\begin{aligned}
&\mathrm{i}\big(\operatorname{tr}(((\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})(\boldsymbol{\Upsilon}+\mathbf{n}')^*+(\boldsymbol{\Upsilon}+\mathbf{n}')(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*+4(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^2)\times\\
&\qquad\qquad\times((\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})(\boldsymbol{\Upsilon}+\mathbf{n}')^*-(\boldsymbol{\Upsilon}+\mathbf{n}')(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*))-\\
&\quad-\tfrac{1}{3}\operatorname{tr}(((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})+(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*(\boldsymbol{\Upsilon}+\mathbf{n}')+4(\boldsymbol{\Psi}+\mathbf{m})^2)\times\\
&\qquad\qquad\times((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})-(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*(\boldsymbol{\Upsilon}+\mathbf{n}')))\big)\\
&\to\mathrm{i}\big(\operatorname{tr}((\check{\boldsymbol{\Phi}}\mathbf{n}'^*+\check{\mathbf{n}}\boldsymbol{\Upsilon}^*+\boldsymbol{\Upsilon}\check{\mathbf{n}}^*+\mathbf{n}'\check{\boldsymbol{\Phi}}^*+4\check{\boldsymbol{\Psi}}\check{\mathbf{m}}+4\check{\mathbf{m}}\check{\boldsymbol{\Psi}})(\check{\boldsymbol{\Phi}}\mathbf{n}'^*+\check{\mathbf{n}}\boldsymbol{\Upsilon}^*-\boldsymbol{\Upsilon}\check{\mathbf{n}}^*-\mathbf{n}'\check{\boldsymbol{\Phi}}^*))-\\
&\quad-\tfrac{1}{3}\operatorname{tr}((\boldsymbol{\Upsilon}^*\check{\mathbf{n}}+\mathbf{n}'^*\check{\boldsymbol{\Phi}}+\check{\boldsymbol{\Phi}}^*\mathbf{n}'+\check{\mathbf{n}}^*\boldsymbol{\Upsilon}+4\boldsymbol{\Psi}\mathbf{m}+4\mathbf{m}\boldsymbol{\Psi})(\boldsymbol{\Upsilon}^*\check{\mathbf{n}}+\mathbf{n}'^*\check{\boldsymbol{\Phi}}-\check{\boldsymbol{\Phi}}^*\mathbf{n}'-\check{\mathbf{n}}^*\boldsymbol{\Upsilon}))\big)
\end{aligned}$$

$$\begin{aligned}
&= -\tfrac{160}{3}\sqrt{\tfrac{2}{5}}\,\Psi_0\Upsilon_{45} + \tfrac{16}{3}\Upsilon_0\Upsilon_{45} + \tfrac{16}{3}\sqrt{6}\Phi_0\Upsilon_{45} + \tfrac{32}{3}\textstyle\sum_{i=1}^{8}\Psi_i\Upsilon_{i+45} + \tfrac{16}{3}\textstyle\sum_{i=1}^{8}\Upsilon_i\Upsilon_{i+45}\\
&\quad + \tfrac{32}{3}\sqrt{2}(\Phi_1\Upsilon_{54} + \Phi_2\Upsilon_{55} + \Phi_3\Upsilon_{56} - \Phi_4\Upsilon_9 - \Phi_5\Upsilon_{10} - \Phi_6\Upsilon_{11})\\
&\quad - \tfrac{32}{3}(\Phi_1\Upsilon_{57} + \Phi_2\Upsilon_{58} + \Phi_3\Upsilon_{59} - \Phi_4\Upsilon_{12} - \Phi_5\Upsilon_{13} - \Phi_6\Upsilon_{14})\\
&\quad + \tfrac{16}{3}(\Upsilon_{15}\Upsilon_{87} + \Upsilon_{16}\Upsilon_{88} + \Upsilon_{17}\Upsilon_{89} - \Upsilon_{42}\Upsilon_{60} - \Upsilon_{43}\Upsilon_{61} - \Upsilon_{44}\Upsilon_{62})\ ,
\end{aligned} \tag{B.2u}$$

$$(\mathrm{tr}((\boldsymbol{\Xi}+\mathbf{m}')(\boldsymbol{\Xi}+\mathbf{m}')^*)-1)^2 \to (\mathrm{tr}(\boldsymbol{\Xi}\mathbf{m}'^* + \mathbf{m}'\boldsymbol{\Xi}^*))^2 = 4\Xi_0^2\ , \tag{B.3a}$$

$$\begin{aligned}
&(\mathrm{tr}((\boldsymbol{\Xi}+\mathbf{m}')(\boldsymbol{\Xi}+\mathbf{m}')^*)-1)(\mathrm{tr}((\boldsymbol{\Psi}+\mathbf{m})^2)-\tfrac{6}{5})\\
&\to 2\,\mathrm{tr}(\boldsymbol{\Xi}\mathbf{m}'^* + \mathbf{m}'\boldsymbol{\Xi}^*)\,\mathrm{tr}(\boldsymbol{\Psi}\mathbf{m}) = 8\sqrt{\tfrac{3}{5}}\Psi_0\Xi_0\ ,
\end{aligned} \tag{B.3b}$$

$$\begin{aligned}
&(\mathrm{tr}((\boldsymbol{\Xi}+\mathbf{m}')(\boldsymbol{\Xi}+\mathbf{m}')^*)-1)((\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n})-1)\\
&\to \mathrm{tr}(\boldsymbol{\Xi}\mathbf{m}'^* + \mathbf{m}'\boldsymbol{\Xi}^*)(\mathbf{n}^*\boldsymbol{\Phi} + \boldsymbol{\Phi}^*\mathbf{n}) = 4\Phi_0\Xi_0\ ,
\end{aligned} \tag{B.3c}$$

$$\begin{aligned}
&(\mathrm{tr}((\boldsymbol{\Xi}+\mathbf{m}')(\boldsymbol{\Xi}+\mathbf{m}')^*)-1)(\mathrm{tr}((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\boldsymbol{\Upsilon}+\mathbf{n}'))-12)\\
&\to \mathrm{tr}(\boldsymbol{\Xi}\mathbf{m}'^* + \mathbf{m}'\boldsymbol{\Xi}^*)\,\mathrm{tr}(\mathbf{n}'^*\boldsymbol{\Upsilon} + \boldsymbol{\Upsilon}^*\mathbf{n}') = 8\sqrt{6}\Xi_0\Upsilon_0\ ,
\end{aligned} \tag{B.3d}$$

$$\begin{aligned}
&\mathrm{tr}(((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\boldsymbol{\Xi}+\mathbf{m}') + (\boldsymbol{\Xi}+\mathbf{m}')(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^T - \tfrac{12}{5}(\boldsymbol{\Xi}+\mathbf{m}'))\times\\
&\qquad \times((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\boldsymbol{\Xi}+\mathbf{m}') + (\boldsymbol{\Xi}+\mathbf{m}')(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^T - \tfrac{12}{5}(\boldsymbol{\Xi}+\mathbf{m}'))^*)\\
&\to \mathrm{tr}((\check{\boldsymbol{\Psi}}\mathbf{m}' + \check{\mathbf{m}}\boldsymbol{\Xi} + \mathbf{m}'\check{\boldsymbol{\Psi}}^T + \boldsymbol{\Xi}\check{\mathbf{m}}^T - \tfrac{12}{5}\boldsymbol{\Xi})(\check{\boldsymbol{\Psi}}\mathbf{m}' + \check{\mathbf{m}}\boldsymbol{\Xi} + \mathbf{m}'\check{\boldsymbol{\Psi}}^T + \boldsymbol{\Xi}\check{\mathbf{m}}^T - \tfrac{12}{5}\boldsymbol{\Xi})^*)\\
&= \tfrac{48}{5}\Psi_0^2 + 4\textstyle\sum_{i=1}^{12}\Xi_i^2 + 16\sum_{i=13}^{18}\Xi_i^2 + 4\sum_{i=19}^{24}\Xi_i^2 + 9\sum_{i=25}^{40}\Xi_i^2 + \sum_{i=41}^{46}\Xi_i^2 + 4\sum_{i=47}^{49}\Xi_i^2\\
&\quad + 4\textstyle\sum_{i=50}^{61}\Xi_i^2 + 16\sum_{i=62}^{67}\Xi_i^2 + 4\sum_{i=68}^{73}\Xi_i^2 + 9\sum_{i=74}^{89}\Xi_i^2 + \sum_{i=90}^{95}\Xi_i^2 + 4\sum_{i=96}^{98}\Xi_i^2\ ,
\end{aligned} \tag{B.3e}$$

$$\begin{aligned}
&\mathrm{tr}(((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\boldsymbol{\Xi}+\mathbf{m}') - (\boldsymbol{\Xi}+\mathbf{m}')(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^T)((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\boldsymbol{\Xi}+\mathbf{m}') - (\boldsymbol{\Xi}+\mathbf{m}')(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^T)^*)\\
&\to \mathrm{tr}((\check{\boldsymbol{\Psi}}\mathbf{m}' + \check{\mathbf{m}}\boldsymbol{\Xi} - \mathbf{m}'\check{\boldsymbol{\Psi}}^T - \boldsymbol{\Xi}\check{\mathbf{m}}^T)(\check{\boldsymbol{\Psi}}\mathbf{m}' + \check{\mathbf{m}}\boldsymbol{\Xi} - \mathbf{m}'\check{\boldsymbol{\Psi}}^T - \boldsymbol{\Xi}\check{\mathbf{m}}^T)^*)\\
&= \textstyle\sum_{i=25}^{46}\Xi_i^2 + \tfrac{8}{3}\sum_{i=47}^{49}\Xi_i^2 + \sum_{i=74}^{95}\Xi_i^2 + \tfrac{8}{3}\sum_{i=96}^{98}\Xi_i^2\ ,
\end{aligned} \tag{B.3f}$$

$$\begin{aligned}
&\mathrm{Re}(\mathrm{tr}(((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}}) - (\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^T + \tfrac{1}{2}(\hat{\boldsymbol{\Upsilon}}+\hat{\mathbf{n}}'))\times\\
&\qquad \times((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\boldsymbol{\Xi}+\mathbf{m}') - (\boldsymbol{\Xi}+\mathbf{m}')(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^T)^*))\\
&\to \mathrm{Re}(\mathrm{tr}((\check{\boldsymbol{\Psi}}\hat{\mathbf{n}} + \check{\mathbf{m}}\hat{\boldsymbol{\Phi}} - \hat{\mathbf{n}}\check{\boldsymbol{\Psi}}^T - \hat{\boldsymbol{\Phi}}\check{\mathbf{m}}^T + \tfrac{1}{2}\hat{\boldsymbol{\Upsilon}})(\check{\boldsymbol{\Psi}}\mathbf{m}' + \check{\mathbf{m}}\boldsymbol{\Xi} - \mathbf{m}'\check{\boldsymbol{\Psi}}^T - \boldsymbol{\Xi}\check{\mathbf{m}}^T)^*))\\
&= -4\textstyle\sum_{i=1}^{8}\Psi_i\Xi_{i+32} - \sum_{i=1}^{8}\Xi_{i+32}\Upsilon_i - \sum_{i=1}^{8}\Xi_{i+81}\Upsilon_{i+45} + \sum_{i=1}^{8}\Xi_{i+24}\Upsilon_{i+18} + \sum_{i=1}^{8}\Xi_{i+73}\Upsilon_{i+63}\\
&\quad + \tfrac{8}{3}\sqrt{3}(\Phi_1\Xi_{47} + \Phi_2\Xi_{48} + \Phi_3\Xi_{49} + \Phi_4\Xi_{96} + \Phi_5\Xi_{97} + \Phi_6\Xi_{98})\\
&\quad - 2\sqrt{\tfrac{2}{3}}(\Xi_{47}\Upsilon_9 + \Xi_{48}\Upsilon_{10} + \Xi_{49}\Upsilon_{11} + \Xi_{96}\Upsilon_{54} + \Xi_{97}\Upsilon_{55} + \Xi_{98}\Upsilon_{56})\\
&\quad - (\Xi_{41}\Upsilon_{36} + \Xi_{42}\Upsilon_{37} + \Xi_{43}\Upsilon_{38} + \Xi_{90}\Upsilon_{81} + \Xi_{91}\Upsilon_{82} + \Xi_{92}\Upsilon_{83})\\
&\quad - (\Xi_{44}\Upsilon_{39} + \Xi_{45}\Upsilon_{40} + \Xi_{46}\Upsilon_{41} + \Xi_{93}\Upsilon_{84} + \Xi_{94}\Upsilon_{85} + \Xi_{95}\Upsilon_{86})\ ,
\end{aligned} \tag{B.3g}$$

$$\begin{aligned}
&\mathrm{Im}(\mathrm{tr}(((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})(\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}}) - (\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^T + \tfrac{1}{2}(\hat{\boldsymbol{\Upsilon}}+\hat{\mathbf{n}}'))((\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})\times\\
&\qquad \times(\boldsymbol{\Xi}+\mathbf{m}') - (\boldsymbol{\Xi}+\mathbf{m}')(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^T)^*))\\
&\to \mathrm{Im}(\mathrm{tr}((\check{\boldsymbol{\Psi}}\hat{\mathbf{n}} + \check{\mathbf{m}}\hat{\boldsymbol{\Phi}} - \hat{\mathbf{n}}\check{\boldsymbol{\Psi}}^T - \hat{\boldsymbol{\Phi}}\check{\mathbf{m}}^T + \tfrac{1}{2}\hat{\boldsymbol{\Upsilon}})(\check{\boldsymbol{\Psi}}\mathbf{m}' + \check{\mathbf{m}}\boldsymbol{\Xi} - \mathbf{m}'\check{\boldsymbol{\Psi}}^T - \boldsymbol{\Xi}\check{\mathbf{m}}^T)^*))
\end{aligned}$$

$$
\begin{aligned}
&= -4\textstyle\sum_{i=1}^{8} \Psi_i \Xi_{i+81} + \sum_{i=1}^{8} \Xi_{i+32} \Upsilon_{i+45} - \sum_{i=1}^{8} \Xi_{i+81} \Upsilon_i - \sum_{i=1}^{8} \Xi_{i+24} \Upsilon_{i+63} + \sum_{i=1}^{8} \Xi_{i+73} \Upsilon_{i+18} \\
&\quad + \tfrac{8}{3}\sqrt{3}(\Phi_1 \Xi_{96} + \Phi_2 \Xi_{97} + \Phi_3 \Xi_{98} - \Phi_4 \Xi_{47} - \Phi_5 \Xi_{48} - \Phi_6 \Xi_{49}) \\
&\quad + 2\sqrt{\tfrac{2}{3}}(\Xi_{47}\Upsilon_{54} + \Xi_{48}\Upsilon_{55} + \Xi_{49}\Upsilon_{56} - \Xi_{96}\Upsilon_9 - \Xi_{97}\Upsilon_{10} - \Xi_{98}\Upsilon_{11}) \\
&\quad - (\Xi_{41}\Upsilon_{81} + \Xi_{42}\Upsilon_{82} + \Xi_{43}\Upsilon_{83} - \Xi_{90}\Upsilon_{36} - \Xi_{91}\Upsilon_{37} - \Xi_{92}\Upsilon_{38}) \\
&\quad - (\Xi_{44}\Upsilon_{84} + \Xi_{45}\Upsilon_{85} + \Xi_{46}\Upsilon_{86} - \Xi_{93}\Upsilon_{39} - \Xi_{94}\Upsilon_{40} - \Xi_{95}\Upsilon_{41})\,,
\end{aligned}
\tag{B.3h}
$$

$$
\begin{aligned}
&\mathrm{tr}(((\mathbf{\Upsilon} + \mathbf{n}')^*(\mathbf{\Xi} + \mathbf{m}'))_{\underline{10}}((\mathbf{\Upsilon} + \mathbf{n}')^*(\mathbf{\Xi} + \mathbf{m}'))^*_{\underline{10}}) \rightarrow \mathrm{tr}((\mathbf{\Upsilon}^*\mathbf{m}' + \mathbf{n}'^*\mathbf{\Xi})_{\underline{10}}(\mathbf{\Upsilon}^*\mathbf{m}' + \mathbf{n}'^*\mathbf{\Xi})^*_{\underline{10}}) \\
&= \tfrac{8}{3}\textstyle\sum_{i=41}^{43} \Xi_i^2 + \tfrac{16}{9}\sum_{i=47}^{49} \Xi_i^2 + \tfrac{8}{3}\sum_{i=90}^{92} \Xi_i^2 + \tfrac{16}{9}\sum_{i=96}^{98} \Xi_i^2 + \tfrac{2}{3}\sum_{i=42}^{44} \Upsilon_i^2 + \tfrac{2}{3}\sum_{i=87}^{89} \Upsilon_i^2 \\
&\quad - \tfrac{8}{3}(\Xi_{41}\Upsilon_{42} + \Xi_{42}\Upsilon_{43} + \Xi_{43}\Upsilon_{44} + \Xi_{90}\Upsilon_{87} + \Xi_{91}\Upsilon_{88} + \Xi_{92}\Upsilon_{89})\,,
\end{aligned}
\tag{B.3i}
$$

$$
\begin{aligned}
&\mathrm{Re}\,(\mathrm{tr}(((\mathbf{\Upsilon} + \mathbf{n}')^*(\hat{\mathbf{\Phi}} + \hat{\mathbf{n}}))_{\underline{10}}((\mathbf{\Upsilon} + \mathbf{n}')^*(\mathbf{\Xi} + \mathbf{m}'))^*_{\underline{10}})) \\
&\rightarrow \mathrm{Re}\,(\mathrm{tr}((\mathbf{\Upsilon}^*\hat{\mathbf{n}} + \mathbf{n}'^*\hat{\mathbf{\Phi}})_{\underline{10}}(\mathbf{\Upsilon}^*\mathbf{m}' + \mathbf{n}'^*\mathbf{\Xi})^*_{\underline{10}})) \\
&= \tfrac{4}{9}\sqrt{3}(\Phi_1\Xi_{47} + \Phi_2\Xi_{48} + \Phi_3\Xi_{49} + \Phi_4\Xi_{96} + \Phi_5\Xi_{97} + \Phi_6\Xi_{98}) \\
&\quad + \tfrac{2}{3}\sqrt{\tfrac{2}{3}}(\Xi_{47}\Upsilon_9 + \Xi_{48}\Upsilon_{10} + \Xi_{49}\Upsilon_{11} + \Xi_{96}\Upsilon_{54} + \Xi_{97}\Upsilon_{55} + \Xi_{98}\Upsilon_{56}) \\
&\quad + \tfrac{4}{9}\sqrt{3}(\Xi_{47}\Upsilon_{12} + \Xi_{48}\Upsilon_{13} + \Xi_{49}\Upsilon_{14} + \Xi_{96}\Upsilon_{57} + \Xi_{97}\Upsilon_{58} + \Xi_{98}\Upsilon_{59}) \\
&\quad + \tfrac{4}{3}(\Xi_{41}\Upsilon_{36} + \Xi_{42}\Upsilon_{37} + \Xi_{43}\Upsilon_{38} + \Xi_{90}\Upsilon_{81} + \Xi_{91}\Upsilon_{82} + \Xi_{92}\Upsilon_{83}) \\
&\quad - \tfrac{2}{3}(\Upsilon_{36}\Upsilon_{42} + \Upsilon_{37}\Upsilon_{43} + \Upsilon_{38}\Upsilon_{44} + \Upsilon_{81}\Upsilon_{87} + \Upsilon_{82}\Upsilon_{88} + \Upsilon_{83}\Upsilon_{89})\,,
\end{aligned}
\tag{B.3j}
$$

$$
\begin{aligned}
&\mathrm{Im}\,(\mathrm{tr}(((\mathbf{\Upsilon} + \mathbf{n}')^*(\hat{\mathbf{\Phi}} + \hat{\mathbf{n}}))_{\underline{10}}((\mathbf{\Upsilon} + \mathbf{n}')^*(\mathbf{\Xi} + \mathbf{m}'))^*_{\underline{10}})) \\
&\rightarrow \mathrm{Im}\,(\mathrm{tr}((\mathbf{\Upsilon}^*\hat{\mathbf{n}} + \mathbf{n}'^*\hat{\mathbf{\Phi}})_{\underline{10}}(\mathbf{\Upsilon}^*\mathbf{m}' + \mathbf{n}'^*\mathbf{\Xi})^*_{\underline{10}})) \\
&= -\tfrac{4}{9}\sqrt{3}(\Phi_1\Xi_{96} + \Phi_2\Xi_{97} + \Phi_3\Xi_{98} - \Phi_4\Xi_{47} - \Phi_5\Xi_{48} - \Phi_6\Xi_{49}) \\
&\quad - \tfrac{2}{3}\sqrt{\tfrac{2}{3}}(\Xi_{47}\Upsilon_{54} + \Xi_{48}\Upsilon_{55} + \Xi_{49}\Upsilon_{56} - \Xi_{96}\Upsilon_9 - \Xi_{97}\Upsilon_{10} - \Xi_{98}\Upsilon_{11}) \\
&\quad - \tfrac{4}{9}\sqrt{3}(\Xi_{47}\Upsilon_{57} + \Xi_{48}\Upsilon_{58} + \Xi_{49}\Upsilon_{59} - \Xi_{96}\Upsilon_{12} - \Xi_{97}\Upsilon_{13} - \Xi_{98}\Upsilon_{14}) \\
&\quad + \tfrac{4}{3}(\Xi_{41}\Upsilon_{81} + \Xi_{42}\Upsilon_{82} + \Xi_{43}\Upsilon_{83} - \Xi_{90}\Upsilon_{36} - \Xi_{91}\Upsilon_{37} - \Xi_{92}\Upsilon_{38}) \\
&\quad + \tfrac{2}{3}(\Upsilon_{36}\Upsilon_{87} + \Upsilon_{37}\Upsilon_{88} + \Upsilon_{38}\Upsilon_{89} - \Upsilon_{81}\Upsilon_{42} - \Upsilon_{82}\Upsilon_{43} - \Upsilon_{83}\Upsilon_{44})\,,
\end{aligned}
\tag{B.3k}
$$

$$
\begin{aligned}
&\mathrm{tr}((\tfrac{1}{4}((\mathbf{\Upsilon} + \mathbf{n}')^*(\mathbf{\Xi} + \mathbf{m}'))_{\underline{40}} + \tfrac{3}{4}(\check{\mathbf{\Phi}} + \check{\mathbf{n}})^*(\mathbf{\Xi} + \mathbf{m}'))\times \\
&\qquad \times(\tfrac{1}{4}((\mathbf{\Upsilon} + \mathbf{n}')^*(\mathbf{\Xi} + \mathbf{m}'))_{\underline{40}} + \tfrac{3}{4}(\check{\mathbf{\Phi}} + \check{\mathbf{n}})^*(\mathbf{\Xi} + \mathbf{m}'))^*) \\
&\rightarrow \mathrm{tr}((\tfrac{1}{4}(\mathbf{\Upsilon}^*\mathbf{m}' + \mathbf{n}'^*\mathbf{\Xi})_{\underline{40}} + \tfrac{3}{4}(\check{\mathbf{\Phi}}^*\mathbf{m}' + \check{\mathbf{n}}^*\mathbf{\Xi}))(\tfrac{1}{4}(\mathbf{\Upsilon}^*\mathbf{m}' + \mathbf{n}'^*\mathbf{\Xi})_{\underline{40}} + \tfrac{3}{4}(\check{\mathbf{\Phi}}^*\mathbf{m}' + \check{\mathbf{n}}^*\mathbf{\Xi}))^*) \\
&= \tfrac{9}{16}\Phi_0^2 - \tfrac{3}{16}\sqrt{6}\Phi_0\Upsilon_0 + \tfrac{3}{32}(\Upsilon_0^2 + \Upsilon_{18} + \Upsilon_{45}^2 + \Upsilon_{63}^2) \\
&\quad + \textstyle\sum_{i=1}^{6} \Xi_i^2 + \tfrac{1}{2}\sum_{i=19}^{32} \Xi_i^2 + \tfrac{1}{3}\sum_{i=41}^{43} \Xi_i^2 + \tfrac{1}{18}\sum_{i=47}^{49} \Xi_i^2 + \sum_{i=50}^{55} \Xi_i^2 \\
&\quad + \tfrac{1}{2}\textstyle\sum_{i=68}^{81} \Xi_i^2 + \tfrac{1}{3}\sum_{i=90}^{92} \Xi_i^2 + \tfrac{1}{18}\sum_{i=96}^{98} \Xi_i^2 + \tfrac{1}{12}\sum_{i=42}^{44} \Upsilon_i^2 + \tfrac{1}{12}\sum_{i=87}^{89} \Upsilon_i^2 \\
&\quad - \tfrac{1}{3}(\Xi_{41}\Upsilon_{42} + \Xi_{42}\Upsilon_{43} + \Xi_{43}\Upsilon_{44} + \Xi_{90}\Upsilon_{87} + \Xi_{91}\Upsilon_{88} + \Xi_{92}\Upsilon_{89})\,,
\end{aligned}
\tag{B.3l}
$$

$$
\begin{aligned}
&\mathrm{Re}\,(\mathrm{tr}((\tfrac{1}{4}((\mathbf{\Upsilon} + \mathbf{n}')^*(\mathbf{\Xi} + \mathbf{m}'))_{\underline{40}} + \tfrac{3}{4}(\check{\mathbf{\Phi}} + \check{\mathbf{n}})^*(\mathbf{\Xi} + \mathbf{m}'))((\mathbf{\Upsilon} + \mathbf{n}')^*(\hat{\mathbf{\Phi}} + \hat{\mathbf{n}}))_{\underline{40}})) \\
&\rightarrow \mathrm{Re}(\mathrm{tr}((\tfrac{1}{4}(\mathbf{\Upsilon}^*\mathbf{m}' + \mathbf{n}'^*\mathbf{\Xi})_{\underline{40}} + \tfrac{3}{4}(\check{\mathbf{\Phi}}^*\mathbf{m}' + \check{\mathbf{n}}^*\mathbf{\Xi}))(\mathbf{\Upsilon}^*\hat{\mathbf{n}} + \mathbf{n}'^*\hat{\mathbf{\Phi}})_{\underline{40}})) \\
&= -\tfrac{4}{9}\sqrt{3}(\Phi_1\Xi_{47} + \Phi_2\Xi_{48} + \Phi_3\Xi_{49} + \Phi_4\Xi_{96} + \Phi_5\Xi_{97} + \Phi_6\Xi_{98})
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{1}{3}\sqrt{\tfrac{2}{3}}(\Xi_{47}\Upsilon_{9}+\Xi_{48}\Upsilon_{10}+\Xi_{49}\Upsilon_{11}+\Xi_{96}\Upsilon_{54}+\Xi_{97}\Upsilon_{55}+\Xi_{98}\Upsilon_{56})\\
&-\tfrac{1}{9}\sqrt{3}(\Xi_{47}\Upsilon_{12}+\Xi_{48}\Upsilon_{13}+\Xi_{49}\Upsilon_{14}+\Xi_{96}\Upsilon_{57}+\Xi_{97}\Upsilon_{58}+\Xi_{98}\Upsilon_{59})\\
&-\textstyle\sum_{i=1}^{8}\Xi_{i+24}\Upsilon_{i+18}-\sum_{i=1}^{8}\Xi_{i+73}\Upsilon_{i+63}-\sum_{i=1}^{6}\Xi_{i+18}\Upsilon_{i+29}-\sum_{i=1}^{6}\Xi_{i+67}\Upsilon_{i+74}\\
&-\tfrac{1}{3}(\Xi_{41}\Upsilon_{36}+\Xi_{42}\Upsilon_{37}+\Xi_{43}\Upsilon_{38}+\Xi_{90}\Upsilon_{81}+\Xi_{91}\Upsilon_{82}+\Xi_{92}\Upsilon_{83})\\
&+\tfrac{1}{6}(\Upsilon_{36}\Upsilon_{42}+\Upsilon_{37}\Upsilon_{43}+\Upsilon_{38}\Upsilon_{44}+\Upsilon_{81}\Upsilon_{87}+\Upsilon_{82}\Upsilon_{88}+\Upsilon_{83}\Upsilon_{89})\,,
\end{aligned}
\tag{B.3m}
$$

$$
\begin{aligned}
&\mathrm{Im}\,(\mathrm{tr}((\tfrac{1}{4}((\mathbf{\Upsilon}+\mathbf{n}')^*(\mathbf{\Xi}+\mathbf{m}'))_{\underline{40}}+\tfrac{3}{4}(\check{\mathbf{\Phi}}+\check{\mathbf{n}})^*(\mathbf{\Xi}+\mathbf{m}'))((\mathbf{\Upsilon}+\mathbf{n}')^*(\hat{\mathbf{\Phi}}+\hat{\mathbf{n}}))_{\underline{40}}))\\
&\to \mathrm{Im}(\mathrm{tr}((\tfrac{1}{4}(\mathbf{\Upsilon}^*\mathbf{m}'+\mathbf{n}'^*\mathbf{\Xi})_{\underline{40}}+\tfrac{3}{4}(\check{\mathbf{\Phi}}^*\mathbf{m}'+\check{\mathbf{n}}^*\mathbf{\Xi}))(\mathbf{\Upsilon}^*\hat{\mathbf{n}}+\mathbf{n}'^*\hat{\mathbf{\Phi}})_{\underline{40}}))\\
&=\tfrac{4}{9}\sqrt{3}(\Phi_1\Xi_{96}+\Phi_2\Xi_{97}+\Phi_3\Xi_{98}-\Phi_4\Xi_{47}-\Phi_5\Xi_{48}-\Phi_6\Xi_{49})\\
&\quad+\tfrac{1}{3}\sqrt{\tfrac{2}{3}}(\Xi_{47}\Upsilon_{54}+\Xi_{48}\Upsilon_{55}+\Xi_{49}\Upsilon_{56}-\Xi_{96}\Upsilon_{9}-\Xi_{97}\Upsilon_{10}-\Xi_{98}\Upsilon_{11})\\
&\quad-\tfrac{1}{9}\sqrt{3}(\Xi_{47}\Upsilon_{57}+\Xi_{48}\Upsilon_{58}+\Xi_{49}\Upsilon_{59}-\Xi_{96}\Upsilon_{12}-\Xi_{97}\Upsilon_{13}-\Xi_{98}\Upsilon_{14})\\
&\quad-\textstyle\sum_{i=1}^{8}\Xi_{i+24}\Upsilon_{i+63}+\sum_{i=1}^{8}\Xi_{i+73}\Upsilon_{i+18}-\sum_{i=1}^{6}\Xi_{i+18}\Upsilon_{i+74}+\sum_{i=1}^{8}\Xi_{i+67}\Upsilon_{i+29}\\
&\quad+\tfrac{1}{3}(\Xi_{41}\Upsilon_{81}+\Xi_{42}\Upsilon_{82}+\Xi_{43}\Upsilon_{83}-\Xi_{90}\Upsilon_{36}-\Xi_{91}\Upsilon_{37}-\Xi_{92}\Upsilon_{38})\\
&\quad+\tfrac{1}{6}(\Upsilon_{36}\Upsilon_{87}+\Upsilon_{37}\Upsilon_{88}+\Upsilon_{38}\Upsilon_{89}-\Upsilon_{81}\Upsilon_{42}-\Upsilon_{82}\Upsilon_{43}-\Upsilon_{83}\Upsilon_{44})\,,
\end{aligned}
\tag{B.3n}
$$

$$
\begin{aligned}
&\mathrm{tr}((((\mathbf{\Xi}+\mathbf{m}')(\mathbf{\Xi}+\mathbf{m}'))'-\tfrac{1}{3}(\mathbf{\Psi}+\mathbf{m}))^2)\to\mathrm{tr}(((\mathbf{\Xi}\mathbf{m}'+\mathbf{m}'\mathbf{\Xi})'-\tfrac{1}{3}\mathbf{\Psi})^2)\\
&=\tfrac{2}{9}\textstyle\sum_{i=0}^{8}\Psi_i^2+\tfrac{2}{9}\sum_{i=1}^{3}\Psi_i'^2-\tfrac{8}{9}\sqrt{\tfrac{3}{5}}\Psi_0\Xi_0+\tfrac{8}{15}\Xi_0^2+\tfrac{1}{9}\sum_{i=41}^{46}\Xi_i^2+\tfrac{1}{9}\sum_{i=90}^{95}\Xi_i^2\,,
\end{aligned}
\tag{B.4a}
$$

$$
\begin{aligned}
&\mathrm{tr}((((\mathbf{\Xi}+\mathbf{m}')(\mathbf{\Xi}+\mathbf{m}'))'-\tfrac{1}{3}(\mathbf{\Psi}+\mathbf{m}))((\mathbf{\Psi}+\mathbf{m})^2-\tfrac{1}{5}\mathrm{tr}((\mathbf{\Psi}+\mathbf{m})^2)\mathbb{1}_5-\tfrac{1}{5}(\mathbf{\Psi}+\mathbf{m})))\\
&\to\mathrm{tr}(((\mathbf{\Xi}\mathbf{m}'+\mathbf{m}'\mathbf{\Xi})'-\tfrac{1}{3}\mathbf{\Psi})(\mathbf{\Psi}\mathbf{m}+\mathbf{m}\mathbf{\Psi}-\tfrac{2}{5}\mathrm{tr}(\mathbf{\Psi}\mathbf{m})\mathbb{1}_5-\tfrac{1}{5}\mathbf{\Psi}))\\
&=-\tfrac{2}{15}\Psi_0^2+\tfrac{2}{3}\textstyle\sum_{i=0}^{8}\Psi_i^2-\tfrac{2}{3}\sum_{i=1}^{3}\Psi_i'^2+\tfrac{4}{15}\sqrt{\tfrac{3}{5}}\Psi_0\Xi_0\,,
\end{aligned}
\tag{B.4b}
$$

$$
\begin{aligned}
&\mathrm{tr}((((\mathbf{\Xi}+\mathbf{m}')(\mathbf{\Xi}+\mathbf{m}'))'-\tfrac{1}{3}(\mathbf{\Psi}+\mathbf{m}))((\mathbf{\Upsilon}+\mathbf{n}')^*(\mathbf{\Upsilon}+\mathbf{n}')-\tfrac{1}{5}\mathrm{tr}((\mathbf{\Upsilon}+\mathbf{n}')^*(\mathbf{\Upsilon}+\mathbf{n}'))\mathbb{1}_5-\\
&\qquad-8(\mathbf{\Psi}+\mathbf{m})+9(\mathbf{\Phi}+\mathbf{n})(\mathbf{\Phi}+\mathbf{n})^*-\tfrac{9}{5}(\mathbf{\Phi}+\mathbf{n})^*(\mathbf{\Phi}+\mathbf{n})\mathbb{1}_5))\\
&\to\mathrm{tr}(((\mathbf{\Xi}\mathbf{m}'+\mathbf{m}'\mathbf{\Xi})'-\tfrac{1}{3}\mathbf{\Psi})(\mathbf{\Upsilon}^*\mathbf{n}'+\mathbf{n}'^*\mathbf{\Upsilon}-\tfrac{1}{5}\mathrm{tr}(\mathbf{\Upsilon}^*\mathbf{n}'+\mathbf{n}'^*\mathbf{\Upsilon})\mathbb{1}_5\\
&\quad-8\mathbf{\Psi}+9(\mathbf{\Phi}\mathbf{n}^*+\mathbf{n}\mathbf{\Phi}^*)-\tfrac{9}{5}\mathrm{tr}(\mathbf{\Phi}\mathbf{n}^*+\mathbf{n}\mathbf{\Phi}^*)\mathbb{1}_5))\\
&=\tfrac{16}{3}\textstyle\sum_{i=0}^{8}\Psi_i^2+\tfrac{16}{3}\sum_{i=1}^{3}\Psi_i'^2-6\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0-\tfrac{7}{3}\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0-6\Psi_3'\Phi_0-\tfrac{32}{3}\sqrt{\tfrac{3}{5}}\Psi_0\Xi_0\\
&\quad+\tfrac{36}{5}\Phi_0\Xi_0-\tfrac{4}{3}\textstyle\sum_{i=1}^{8}\Psi_i\Upsilon_i+\sqrt{6}(\Psi_1'\Upsilon_{18}+\Psi_2'\Upsilon_{63}+\Psi_3'\Upsilon_0)+\tfrac{14}{5}\sqrt{\tfrac{2}{3}}\Xi_0\Upsilon_0\\
&\quad-3\sqrt{2}(\Phi_1\Xi_{44}+\Phi_2\Xi_{45}+\Phi_3\Xi_{45}+\Phi_4\Xi_{93}+\Phi_5\Xi_{94}+\Phi_6\Xi_{95})\\
&\quad-\tfrac{1}{3}(\Xi_{44}\Upsilon_{9}+\Xi_{45}\Upsilon_{10}+\Xi_{46}\Upsilon_{11}+\Xi_{93}\Upsilon_{54}+\Xi_{94}\Upsilon_{55}+\Xi_{95}\Upsilon_{56})\\
&\quad-\tfrac{1}{3}\sqrt{2}(\Xi_{44}\Upsilon_{12}+\Xi_{45}\Upsilon_{13}+\Xi_{46}\Upsilon_{14}+\Xi_{93}\Upsilon_{57}+\Xi_{94}\Upsilon_{58}+\Xi_{95}\Upsilon_{59})\\
&\quad+\tfrac{2}{3}(\Xi_{41}\Upsilon_{15}+\Xi_{42}\Upsilon_{16}+\Xi_{43}\Upsilon_{17}+\Xi_{90}\Upsilon_{60}+\Xi_{91}\Upsilon_{61}+\Xi_{92}\Upsilon_{62})\\
&\quad+2(\Xi_{41}\Upsilon_{42}+\Xi_{42}\Upsilon_{43}+\Xi_{43}\Upsilon_{44}+\Xi_{90}\Upsilon_{87}+\Xi_{91}\Upsilon_{88}+\Xi_{92}\Upsilon_{89})\,,
\end{aligned}
\tag{B.4c}
$$

$$
\begin{aligned}
&\mathrm{tr}((((\mathbf{\Xi}+\mathbf{m}')(\mathbf{\Xi}+\mathbf{m}'))'-\tfrac{1}{3}(\mathbf{\Psi}+\mathbf{m}))(((\mathbf{\Upsilon}+\mathbf{n}')(\mathbf{\Upsilon}+\mathbf{n}')^*)'-\tfrac{8}{3}(\mathbf{\Psi}+\mathbf{m})-\\
&\qquad-(\mathbf{\Phi}+\mathbf{n})(\mathbf{\Phi}+\mathbf{n})^*+\tfrac{1}{5}(\mathbf{\Phi}+\mathbf{n})^*(\mathbf{\Phi}+\mathbf{n})\mathbb{1}_5))\\
&\to\mathrm{tr}(((\mathbf{\Xi}\mathbf{m}'+\mathbf{m}'\mathbf{\Xi})'-\tfrac{1}{3}\mathbf{\Psi})((\mathbf{\Upsilon}\mathbf{n}'^*+\mathbf{n}'\mathbf{\Upsilon}^*)'-\tfrac{8}{3}\mathbf{\Psi}-(\mathbf{\Phi}\mathbf{n}^*+\mathbf{n}\mathbf{\Phi}^*)+\tfrac{1}{5}\mathrm{tr}(\mathbf{\Phi}\mathbf{n}^*+\mathbf{n}\mathbf{\Phi}^*)\mathbb{1}_5))
\end{aligned}
$$

$$
\begin{aligned}
&= \tfrac{16}{9}\textstyle\sum_{i=0}^{8}\Psi_i^2 + \tfrac{16}{9}\sum_{i=1}^{3}\Psi_i'^2 + \tfrac{2}{3}\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0 - \tfrac{19}{9}\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0 + \tfrac{2}{3}\Psi_3'\Phi_0 - \tfrac{4}{5}\Phi_0\Xi_0\\
&\quad -\tfrac{32}{9}\sqrt{\tfrac{3}{5}}\Psi_0\Xi_0 + \tfrac{38}{15}\sqrt{\tfrac{2}{3}}\Xi_0\Upsilon_0 - \tfrac{4}{9}\textstyle\sum_{i=1}^{8}\Psi_i\Upsilon_i - \tfrac{1}{9}\sqrt{6}(\Psi_1'\Upsilon_{18} + \Psi_2'\Upsilon_{63} + \Psi_3'\Upsilon_0)\\
&\quad +\tfrac{1}{3}\sqrt{2}(\Phi_1\Xi_{44} + \Phi_2\Xi_{45} + \Phi_3\Xi_{45} + \Phi_4\Xi_{93} + \Phi_5\Xi_{94} + \Phi_6\Xi_{95})\\
&\quad -\tfrac{5}{9}(\Xi_{44}\Upsilon_9 + \Xi_{45}\Upsilon_{10} + \Xi_{46}\Upsilon_{11} + \Xi_{93}\Upsilon_{54} + \Xi_{94}\Upsilon_{55} + \Xi_{95}\Upsilon_{56})\\
&\quad +\tfrac{1}{3}\sqrt{2}(\Xi_{44}\Upsilon_{12} + \Xi_{45}\Upsilon_{13} + \Xi_{46}\Upsilon_{14} + \Xi_{93}\Upsilon_{57} + \Xi_{94}\Upsilon_{58} + \Xi_{95}\Upsilon_{59})\\
&\quad -\tfrac{2}{3}(\Xi_{41}\Upsilon_{15} + \Xi_{42}\Upsilon_{16} + \Xi_{43}\Upsilon_{17} + \Xi_{90}\Upsilon_{60} + \Xi_{91}\Upsilon_{61} + \Xi_{92}\Upsilon_{62})\\
&\quad -\tfrac{2}{9}(\Xi_{41}\Upsilon_{42} + \Xi_{42}\Upsilon_{43} + \Xi_{43}\Upsilon_{44} + \Xi_{90}\Upsilon_{87} + \Xi_{91}\Upsilon_{88} + \Xi_{92}\Upsilon_{89}) \,, \qquad \text{(B.4d)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{tr}((((\boldsymbol{\Xi}+\mathbf{m}')(\boldsymbol{\Xi}+\mathbf{m}'))' - \tfrac{1}{3}(\boldsymbol{\Psi}+\mathbf{m}))((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) + (\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*(\boldsymbol{\Upsilon}+\mathbf{n}') +\\
&\qquad +8(\boldsymbol{\Psi}+\mathbf{m}) - 6(\boldsymbol{\Phi}+\mathbf{n})(\boldsymbol{\Phi}+\mathbf{n})^* + \tfrac{6}{5}(\boldsymbol{\Phi}+\mathbf{n})^*(\boldsymbol{\Phi}+\mathbf{n})\mathbb{1}_5))\\
&\to \mathrm{tr}(((\boldsymbol{\Xi}\mathbf{m}' + \mathbf{m}'\boldsymbol{\Xi})' - \tfrac{1}{3}\boldsymbol{\Psi})(\boldsymbol{\Upsilon}^*\check{\mathbf{n}} + \mathbf{n}'^*\check{\boldsymbol{\Phi}} + \check{\boldsymbol{\Phi}}^*\mathbf{n}' + \check{\mathbf{n}}^*\boldsymbol{\Upsilon} + 8\boldsymbol{\Psi}\\
&\qquad -6(\boldsymbol{\Phi}\mathbf{n}^* + \mathbf{n}\boldsymbol{\Phi}^*) + \tfrac{6}{5}\,\mathrm{tr}(\boldsymbol{\Phi}\mathbf{n}^* + \mathbf{n}\boldsymbol{\Phi}^*)\mathbb{1}_5))\\
&= -\tfrac{16}{3}\textstyle\sum_{i=0}^{8}\Psi_i^2 - \tfrac{16}{3}\sum_{i=1}^{3}\Psi_i'^2 + \tfrac{22}{3}\sqrt{\tfrac{3}{5}}\Psi_0\Phi_0 + \tfrac{5}{3}\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_0 + 2\Phi_0\Upsilon_3' - \tfrac{44}{5}\Phi_0\Xi_0\\
&\quad +\tfrac{32}{3}\sqrt{\tfrac{3}{5}}\Psi_0\Xi_0 - 2\sqrt{\tfrac{2}{3}}\Xi_0\Upsilon_0 - \tfrac{4}{3}\textstyle\sum_{i=1}^{8}\Psi_i\Upsilon_i - \tfrac{1}{3}\sqrt{6}(\Psi_1'\Upsilon_{18} + \Psi_2'\Upsilon_{63} + \Psi_3'\Upsilon_0)\\
&\quad +\tfrac{7}{3}\sqrt{2}(\Phi_1\Xi_{44} + \Phi_2\Xi_{45} + \Phi_3\Xi_{45} + \Phi_4\Xi_{93} + \Phi_5\Xi_{94} + \Phi_6\Xi_{95})\\
&\quad -\tfrac{1}{3}(\Xi_{44}\Upsilon_9 + \Xi_{45}\Upsilon_{10} + \Xi_{46}\Upsilon_{11} + \Xi_{93}\Upsilon_{54} + \Xi_{94}\Upsilon_{55} + \Xi_{95}\Upsilon_{56})\\
&\quad -\tfrac{1}{3}\sqrt{2}(\Xi_{44}\Upsilon_{12} + \Xi_{45}\Upsilon_{13} + \Xi_{46}\Upsilon_{14} + \Xi_{93}\Upsilon_{57} + \Xi_{94}\Upsilon_{58} + \Xi_{95}\Upsilon_{59})\\
&\quad +\tfrac{2}{3}(\Xi_{41}\Upsilon_{15} + \Xi_{42}\Upsilon_{16} + \Xi_{43}\Upsilon_{17} + \Xi_{90}\Upsilon_{60} + \Xi_{91}\Upsilon_{61} + \Xi_{92}\Upsilon_{62})\\
&\quad -\tfrac{2}{3}(\Xi_{41}\Upsilon_{42} + \Xi_{42}\Upsilon_{43} + \Xi_{43}\Upsilon_{44} + \Xi_{90}\Upsilon_{87} + \Xi_{91}\Upsilon_{88} + \Xi_{92}\Upsilon_{89}) \,, \qquad \text{(B.4e)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{i}\,\mathrm{tr}((((\boldsymbol{\Xi}+\mathbf{m}')(\boldsymbol{\Xi}+\mathbf{m}'))' - \tfrac{1}{3}(\boldsymbol{\Psi}+\mathbf{m}))((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\check{\boldsymbol{\Phi}}+\check{\mathbf{n}}) - (\check{\boldsymbol{\Phi}}+\check{\mathbf{n}})^*(\boldsymbol{\Upsilon}+\mathbf{n}')))\\
&\to \mathrm{i}\,\mathrm{tr}(((\boldsymbol{\Xi}\mathbf{m}' + \mathbf{m}'\boldsymbol{\Xi})' - \tfrac{1}{3}\boldsymbol{\Psi})(\boldsymbol{\Upsilon}^*\check{\mathbf{n}} + \mathbf{n}'^*\check{\boldsymbol{\Phi}} - \check{\boldsymbol{\Phi}}^*\mathbf{n}' - \check{\mathbf{n}}^*\boldsymbol{\Upsilon}))\\
&= \tfrac{5}{3}\sqrt{\tfrac{2}{5}}\Psi_0\Upsilon_{45} - 2\sqrt{\tfrac{2}{3}}\Xi_0\Upsilon_{45} - \tfrac{4}{3}\textstyle\sum_{i=1}^{8}\Psi_i\Upsilon_{i+45} - \tfrac{1}{3}\sqrt{6}(\Psi_1'\Upsilon_{63} - \Psi_2'\Upsilon_{18} + \Psi_3'\Upsilon_{45})\\
&\quad +\tfrac{1}{3}\sqrt{2}(\Phi_1\Xi_{93} + \Phi_2\Xi_{94} + \Phi_3\Xi_{95} - \Phi_4\Xi_{44} - \Phi_5\Xi_{45} - \Phi_6\Xi_{46})\\
&\quad -\tfrac{1}{3}(\Xi_{44}\Upsilon_{54} + \Xi_{45}\Upsilon_{55} + \Xi_{46}\Upsilon_{56} - \Xi_{93}\Upsilon_9 - \Xi_{94}\Upsilon_{10} - \Xi_{95}\Upsilon_{11})\\
&\quad -\tfrac{1}{3}\sqrt{2}(\Xi_{44}\Upsilon_{57} + \Xi_{45}\Upsilon_{58} + \Xi_{46}\Upsilon_{59} - \Xi_{93}\Upsilon_{12} - \Xi_{94}\Upsilon_{13} - \Xi_{95}\Upsilon_{14})\\
&\quad +\tfrac{2}{3}(\Xi_{41}\Upsilon_{60} + \Xi_{42}\Upsilon_{61} + \Xi_{43}\Upsilon_{62} - \Xi_{90}\Upsilon_{15} - \Xi_{91}\Upsilon_{16} - \Xi_{92}\Upsilon_{17})\\
&\quad +\tfrac{2}{3}(\Xi_{41}\Upsilon_{87} + \Xi_{42}\Upsilon_{88} + \Xi_{43}\Upsilon_{89} - \Xi_{90}\Upsilon_{42} - \Xi_{91}\Upsilon_{43} - \Xi_{92}\Upsilon_{44}) \,, \qquad \text{(B.4f)}
\end{aligned}
$$

$$
\begin{aligned}
&\big(\mathrm{tr}(((\boldsymbol{\Xi}+\mathbf{m}')(\boldsymbol{\Xi}+\mathbf{m}')^* - \tfrac{1}{2}(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^2)^2) - \tfrac{1}{10}(\mathrm{tr}((\boldsymbol{\Xi}+\mathbf{m}')(\boldsymbol{\Xi}+\mathbf{m}')^*) - \tfrac{3}{2}\,\mathrm{tr}((\boldsymbol{\Psi}+\mathbf{m})^2))^2 -\\
&\qquad -3\,\mathrm{tr}((((\boldsymbol{\Xi}+\mathbf{m}')(\boldsymbol{\Xi}+\mathbf{m}')^*)' - \tfrac{1}{6}(\boldsymbol{\Psi}+\mathbf{m})^2 + \tfrac{1}{30}\,\mathrm{tr}((\boldsymbol{\Psi}+\mathbf{m})^2)\mathbb{1}_5)^2)\big)\\
&\to \big(\mathrm{tr}((\boldsymbol{\Xi}\mathbf{m}'^* + \mathbf{m}'\boldsymbol{\Xi}^* - \tfrac{1}{2}(\check{\boldsymbol{\Psi}}\check{\mathbf{m}} + \check{\mathbf{m}}\check{\boldsymbol{\Psi}}))^2) - \tfrac{1}{10}(\mathrm{tr}(\boldsymbol{\Xi}\mathbf{m}'^* + \mathbf{m}'\boldsymbol{\Xi}^*) - 3\,\mathrm{tr}(\boldsymbol{\Psi}\mathbf{m}))^2 -\\
&\qquad -3\,\mathrm{tr}(((\boldsymbol{\Xi}\mathbf{m}'^* + \mathbf{m}'\boldsymbol{\Xi}^*)' - \tfrac{1}{6}(\boldsymbol{\Psi}\mathbf{m} + \mathbf{m}\boldsymbol{\Psi}) + \tfrac{1}{15}\,\mathrm{tr}(\boldsymbol{\Psi}\mathbf{m})\mathbb{1}_5)^2)\big)\\
&= \tfrac{10}{3}\Psi_0^2 + \tfrac{4}{3}\textstyle\sum_{i=1}^{8}\Psi_i^2 - \tfrac{20}{3}\sqrt{\tfrac{3}{5}}\Psi_0\Xi_0 + 2\Xi_0^2 + \tfrac{2}{3}\sum_{i=41}^{49}\Xi_i^2 + \tfrac{2}{3}\sum_{i=90}^{98}\Xi_i^2 \,, \qquad \text{(B.4g)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{tr}(((\boldsymbol{\Xi}+\mathbf{m}')(\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})^* + (\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})(\boldsymbol{\Xi}+\mathbf{m}')^*)^2) \to \mathrm{tr}((\boldsymbol{\Xi}\hat{\mathbf{n}}^* + \mathbf{m}'\hat{\boldsymbol{\Phi}}^* + \hat{\boldsymbol{\Phi}}\mathbf{m}'^* + \hat{\mathbf{n}}\boldsymbol{\Xi}^*)^2)\\
&= 2\textstyle\sum_{i=1}^{6}\Phi_i^2 + 2\sum_{i=7}^{18}\Xi_i^2 + \sum_{i=19}^{32}\Xi_i^2 + 4\sum_{i=33}^{40}\Xi_i^2 + \sum_{i=44}^{49}\Xi_i^2\\
&\quad +2\textstyle\sum_{i=56}^{67}\Xi_i^2 + \sum_{i=68}^{81}\Xi_i^2 + \sum_{i=93}^{98}\Xi_i^2\\
&\quad +2\sqrt{2}(\Phi_1\Xi_{44} + \Phi_2\Xi_{45} + \Phi_3\Xi_{45} + \Phi_4\Xi_{93} + \Phi_5\Xi_{94} + \Phi_6\Xi_{95})
\end{aligned}
$$

$$
\begin{aligned}
&+4(\Xi_7\Xi_{13}+\Xi_8\Xi_{14}+\Xi_9\Xi_{15}+\Xi_{10}\Xi_{16}+\Xi_{11}\Xi_{17}+\Xi_{12}\Xi_{18})\\
&+4(\Xi_{56}\Xi_{62}+\Xi_{57}\Xi_{63}+\Xi_{58}\Xi_{64}+\Xi_{59}\Xi_{65}+\Xi_{60}\Xi_{66}+\Xi_{61}\Xi_{67})\,, \qquad \text{(B.4h)}
\end{aligned}
$$

$$
\begin{aligned}
&-\mathrm{tr}(((\boldsymbol{\Xi}+\mathbf{m}')(\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})^*-(\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})(\boldsymbol{\Xi}+\mathbf{m}')^*)^2)\to-\mathrm{tr}((\boldsymbol{\Xi}\hat{\mathbf{n}}^*+\mathbf{m}'\hat{\boldsymbol{\Phi}}^*-\hat{\boldsymbol{\Phi}}\mathbf{m}'^*-\hat{\mathbf{n}}\boldsymbol{\Xi}^*)^2)\\
&=2\textstyle\sum_{i=1}^6\Phi_i^2+2\sum_{i=7}^{18}\Xi_i^2+\sum_{i=19}^{32}\Xi_i^2+\sum_{i=44}^{49}\Xi_i^2\\
&\quad+2\textstyle\sum_{i=56}^{67}\Xi_i^2+\sum_{i=68}^{81}\Xi_i^2+4\sum_{i=82}^{89}\Xi_i^2+\sum_{i=93}^{98}\Xi_i^2\\
&\quad+2\sqrt{2}(\Phi_1\Xi_{44}+\Phi_2\Xi_{45}+\Phi_3\Xi_{45}+\Phi_4\Xi_{93}+\Phi_5\Xi_{94}+\Phi_6\Xi_{95})\\
&\quad-4(\Xi_7\Xi_{13}+\Xi_8\Xi_{14}+\Xi_9\Xi_{15}+\Xi_{10}\Xi_{16}+\Xi_{11}\Xi_{17}+\Xi_{12}\Xi_{18})\\
&\quad-4(\Xi_{56}\Xi_{62}+\Xi_{57}\Xi_{63}+\Xi_{58}\Xi_{64}+\Xi_{59}\Xi_{65}+\Xi_{60}\Xi_{66}+\Xi_{61}\Xi_{67})\,, \qquad \text{(B.4i)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{tr}(((\boldsymbol{\Xi}+\mathbf{m}')(\boldsymbol{\Xi}+\mathbf{m}')^*-\tfrac{1}{2}(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^2)((\boldsymbol{\Xi}+\mathbf{m}')(\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})^*+(\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})(\boldsymbol{\Xi}+\mathbf{m}')^*))\\
&\to\mathrm{tr}((\boldsymbol{\Xi}\mathbf{m}'^*+\mathbf{m}'\boldsymbol{\Xi}^*-\tfrac{1}{2}(\check{\boldsymbol{\Psi}}\check{\mathbf{m}}+\check{\mathbf{m}}\check{\boldsymbol{\Psi}}))(\boldsymbol{\Xi}\hat{\mathbf{n}}^*+\mathbf{m}'\hat{\boldsymbol{\Phi}}^*+\hat{\boldsymbol{\Phi}}\mathbf{m}'^*+\hat{\mathbf{n}}\boldsymbol{\Xi}^*))\\
&=2\textstyle\sum_{i=1}^8\Psi_i\Xi_{i+32}+\frac{2}{3}\sqrt{3}(\Phi_1\Xi_{47}+\Phi_2\Xi_{48}+\Phi_3\Xi_{49}+\Phi_4\Xi_{96}+\Phi_5\Xi_{97}+\Phi_6\Xi_{98})\\
&\quad+2\sqrt{\tfrac{2}{3}}(\Xi_{44}\Xi_{47}+\Xi_{45}\Xi_{48}+\Xi_{46}\Xi_{49}+\Xi_{93}\Xi_{96}+\Xi_{94}\Xi_{97}+\Xi_{95}\Xi_{98})\,, \qquad \text{(B.4j)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{i}\,\mathrm{tr}(((\boldsymbol{\Xi}+\mathbf{m}')(\boldsymbol{\Xi}+\mathbf{m}')^*-\tfrac{1}{2}(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^2)((\boldsymbol{\Xi}+\mathbf{m}')(\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})^*-(\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})(\boldsymbol{\Xi}+\mathbf{m}')^*))\\
&\to\mathrm{i}\,\mathrm{tr}((\boldsymbol{\Xi}\mathbf{m}'^*+\mathbf{m}'\boldsymbol{\Xi}^*-\tfrac{1}{2}(\check{\boldsymbol{\Psi}}\check{\mathbf{m}}+\check{\mathbf{m}}\check{\boldsymbol{\Psi}}))(\boldsymbol{\Xi}\hat{\mathbf{n}}^*+\mathbf{m}'\hat{\boldsymbol{\Phi}}^*-\hat{\boldsymbol{\Phi}}\mathbf{m}'^*-\hat{\mathbf{n}}\boldsymbol{\Xi}^*))\\
&=2\textstyle\sum_{i=1}^8\Psi_i\Xi_{i+81}+\frac{2}{3}\sqrt{3}(\Phi_1\Xi_{96}+\Phi_2\Xi_{97}+\Phi_3\Xi_{98}-\Phi_4\Xi_{47}-\Phi_5\Xi_{48}-\Phi_6\Xi_{49})\\
&\quad+2\sqrt{\tfrac{2}{3}}(\Xi_{44}\Xi_{96}+\Xi_{45}\Xi_{97}+\Xi_{46}\Xi_{98}-\Xi_{93}\Xi_{47}-\Xi_{94}\Xi_{48}-\Xi_{95}\Xi_{49})\,, \qquad \text{(B.4k)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{i}\,\mathrm{tr}(((\boldsymbol{\Xi}+\mathbf{m}')(\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})^*+(\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})(\boldsymbol{\Xi}+\mathbf{m}')^*)((\boldsymbol{\Xi}+\mathbf{m}')(\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})^*-(\hat{\boldsymbol{\Phi}}+\hat{\mathbf{n}})(\boldsymbol{\Xi}+\mathbf{m}')^*))\\
&\to\mathrm{i}\,\mathrm{tr}((\boldsymbol{\Xi}\hat{\mathbf{n}}^*+\mathbf{m}'\hat{\boldsymbol{\Phi}}^*+\hat{\boldsymbol{\Phi}}\mathbf{m}'^*+\hat{\mathbf{n}}\boldsymbol{\Xi}^*)(\boldsymbol{\Xi}\hat{\mathbf{n}}^*+\mathbf{m}'\hat{\boldsymbol{\Phi}}^*-\hat{\boldsymbol{\Phi}}\mathbf{m}'^*-\hat{\mathbf{n}}\boldsymbol{\Xi}^*))\\
&=4\textstyle\sum_{i=1}^8\Xi_{i+32}\Xi_{i+81}-4(\Xi_7\Xi_{62}+\Xi_8\Xi_{63}+\Xi_9\Xi_{64}+\Xi_{10}\Xi_{65}+\Xi_{11}\Xi_{66}+\Xi_{12}\Xi_{67})\\
&\quad+4(\Xi_{13}\Xi_{56}+\Xi_{14}\Xi_{57}+\Xi_{15}\Xi_{58}+\Xi_{16}\Xi_{59}+\Xi_{17}\Xi_{60}+\Xi_{18}\Xi_{61})\,, \qquad \text{(B.4l)}
\end{aligned}
$$

$$
\begin{aligned}
&\big(\mathrm{tr}(((\boldsymbol{\Xi}+\mathbf{m}')(\boldsymbol{\Xi}+\mathbf{m}')^*-\tfrac{1}{2}(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^2)((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Upsilon}+\mathbf{n}')^*-4(\check{\boldsymbol{\Psi}}+\check{\mathbf{m}})^2))-\\
&\qquad-\tfrac{1}{10}(\mathrm{tr}((\boldsymbol{\Xi}+\mathbf{m}')(\boldsymbol{\Xi}+\mathbf{m}')^*)-\tfrac{3}{2}\mathrm{tr}((\boldsymbol{\Psi}+\mathbf{m})^2))\,\mathrm{tr}((\boldsymbol{\Upsilon}+\mathbf{n}')^*(\boldsymbol{\Upsilon}+\mathbf{n}')-12(\boldsymbol{\Psi}+\mathbf{m})^2)-\\
&\qquad-3\,\mathrm{tr}((((\boldsymbol{\Xi}+\mathbf{m}')(\boldsymbol{\Xi}+\mathbf{m}')^*)'-\tfrac{1}{6}(\boldsymbol{\Psi}+\mathbf{m})^2+\tfrac{1}{30}\mathrm{tr}((\boldsymbol{\Psi}+\mathbf{m})^2)\mathbb{1}_5)\times\\
&\qquad\quad\times(((\boldsymbol{\Upsilon}+\mathbf{n}')(\boldsymbol{\Upsilon}+\mathbf{n}')^*)'-\tfrac{4}{3}(\boldsymbol{\Psi}+\mathbf{m})^2+\tfrac{4}{15}\mathrm{tr}((\boldsymbol{\Psi}+\mathbf{m})^2)\mathbb{1}_5))\big)\\
&\to\big(\mathrm{tr}((\boldsymbol{\Xi}\mathbf{m}'^*+\mathbf{m}'\boldsymbol{\Xi}^*-\tfrac{1}{2}(\check{\boldsymbol{\Psi}}\check{\mathbf{m}}+\check{\mathbf{m}}\check{\boldsymbol{\Psi}}))(\boldsymbol{\Upsilon}\mathbf{n}'^*+\mathbf{n}'\boldsymbol{\Upsilon}^*-4\check{\boldsymbol{\Psi}}\check{\mathbf{m}}-4\check{\mathbf{m}}\check{\boldsymbol{\Psi}}))-\\
&\qquad-\tfrac{1}{10}(\mathrm{tr}(\boldsymbol{\Xi}\mathbf{m}'^*+\mathbf{m}'\boldsymbol{\Xi}^*)-3\,\mathrm{tr}(\boldsymbol{\Psi}\mathbf{m}))\,\mathrm{tr}(\boldsymbol{\Upsilon}^*\mathbf{n}'+\mathbf{n}'^*\boldsymbol{\Upsilon}-24\mathbf{m}\boldsymbol{\Psi})-\\
&\qquad-3\,\mathrm{tr}(((\boldsymbol{\Xi}\mathbf{m}'^*+\mathbf{m}'\boldsymbol{\Xi}^*)'-\tfrac{1}{6}(\boldsymbol{\Psi}\mathbf{m}+\mathbf{m}\boldsymbol{\Psi})+\tfrac{1}{15}\mathrm{tr}(\boldsymbol{\Psi}\mathbf{m})\mathbb{1}_5)\times\\
&\qquad\quad\times((\boldsymbol{\Upsilon}\mathbf{n}'^*+\mathbf{n}'\boldsymbol{\Upsilon}^*)'-\tfrac{4}{3}(\boldsymbol{\Psi}\mathbf{m}+\mathbf{m}\boldsymbol{\Psi})+\tfrac{8}{15}\mathrm{tr}(\mathbf{m}\boldsymbol{\Psi})\mathbb{1}_5))\big)\\
&=\tfrac{80}{3}\Psi_0^2+\tfrac{32}{3}\textstyle\sum_{i=1}^8\Psi_i^2-\frac{40}{3}\sqrt{\frac{2}{5}}\Psi_0\Upsilon_0-\frac{80}{3}\sqrt{\frac{3}{5}}\Psi_0\Xi_0+8\sqrt{\frac{2}{3}}\Xi_0\Upsilon_0-\frac{4}{3}\sum_{i=1}^8\Psi_i\Upsilon_i\\
&\quad-\tfrac{4}{3}(\Xi_{44}\Upsilon_9+\Xi_{45}\Upsilon_{10}+\Xi_{46}\Upsilon_{11}+\Xi_{93}\Upsilon_{54}+\Xi_{94}\Upsilon_{55}+\Xi_{95}\Upsilon_{56})\\
&\quad+2\sqrt{2}(\Xi_{44}\Upsilon_{12}+\Xi_{45}\Upsilon_{13}+\Xi_{46}\Upsilon_{14}+\Xi_{93}\Upsilon_{57}+\Xi_{94}\Upsilon_{58}+\Xi_{95}\Upsilon_{59})\\
&\quad-4(\Xi_{41}\Upsilon_{15}+\Xi_{42}\Upsilon_{16}+\Xi_{43}\Upsilon_{17}+\Xi_{90}\Upsilon_{60}+\Xi_{91}\Upsilon_{61}+\Xi_{92}\Upsilon_{62})\\
&\quad-\tfrac{4}{3}(\Xi_{41}\Upsilon_{42}+\Xi_{42}\Upsilon_{43}+\Xi_{43}\Upsilon_{44}+\Xi_{90}\Upsilon_{87}+\Xi_{91}\Upsilon_{88}+\Xi_{92}\Upsilon_{89})\\
&\quad-2\sqrt{6}(\Xi_{47}\Upsilon_{39}+\Xi_{48}\Upsilon_{40}+\Xi_{49}\Upsilon_{41}+\Xi_{96}\Upsilon_{84}+\Xi_{97}\Upsilon_{85}+\Xi_{98}\Upsilon_{86})\,, \qquad \text{(B.4m)}
\end{aligned}
$$

$$\Big(\operatorname{tr}(((\mathbf{\Xi}+\mathbf{m}')(\mathbf{\Xi}+\mathbf{m}')^*-\tfrac{1}{2}(\check{\mathbf{\Psi}}+\check{\mathbf{m}})^2)((\check{\mathbf{\Phi}}+\check{\mathbf{n}})(\mathbf{\Upsilon}+\mathbf{n}')^*+(\mathbf{\Upsilon}+\mathbf{n}')(\check{\mathbf{\Phi}}+\check{\mathbf{n}})^*+4(\check{\mathbf{\Psi}}+\check{\mathbf{m}})^2))-$$
$$-\tfrac{12}{10}(\operatorname{tr}((\mathbf{\Xi}+\mathbf{m}')(\mathbf{\Xi}+\mathbf{m}')^*)-\tfrac{3}{2}\operatorname{tr}((\mathbf{\Psi}+\mathbf{m})^2))\operatorname{tr}((\mathbf{\Psi}+\mathbf{m})^2)-$$
$$-\operatorname{tr}((((\mathbf{\Xi}+\mathbf{m}')(\mathbf{\Xi}+\mathbf{m}')^*)'-\tfrac{1}{6}(\mathbf{\Psi}+\mathbf{m})^2+\tfrac{1}{30}\operatorname{tr}((\mathbf{\Psi}+\mathbf{m})^2)\mathbb{1}_5)\times$$
$$\times((\mathbf{\Upsilon}+\mathbf{n}')^*(\check{\mathbf{\Phi}}+\check{\mathbf{n}})+(\check{\mathbf{\Phi}}+\check{\mathbf{n}})^*(\mathbf{\Upsilon}+\mathbf{n}')+4(\mathbf{\Psi}+\mathbf{m})^2-\tfrac{4}{5}\operatorname{tr}((\mathbf{\Psi}+\mathbf{m})^2)\mathbb{1}_5))\Big)$$
$$\to\Big(\operatorname{tr}((\mathbf{\Xi}\mathbf{m}'^*+\mathbf{m}'\mathbf{\Xi}^*-\tfrac{1}{2}(\check{\mathbf{\Psi}}\check{\mathbf{m}}+\check{\mathbf{m}}\check{\mathbf{\Psi}}))(\check{\mathbf{n}}\mathbf{\Upsilon}^*+\check{\mathbf{\Phi}}\mathbf{n}'^*+\mathbf{\Upsilon}\check{\mathbf{n}}^*+\mathbf{n}'\check{\mathbf{\Phi}}^*+4\check{\mathbf{\Psi}}\check{\mathbf{m}}+4\check{\mathbf{m}}\check{\mathbf{\Psi}}))-$$
$$-\tfrac{24}{10}(\operatorname{tr}(\mathbf{\Xi}\mathbf{m}'^*+\mathbf{m}'\mathbf{\Xi}^*)-3\operatorname{tr}(\mathbf{\Psi}\mathbf{m}))\operatorname{tr}(\mathbf{m}\mathbf{\Psi})-$$
$$-\operatorname{tr}(((\mathbf{\Xi}\mathbf{m}'^*+\mathbf{m}'\mathbf{\Xi}^*)'-\tfrac{1}{6}(\mathbf{\Psi}\mathbf{m}+\mathbf{m}\mathbf{\Psi})+\tfrac{1}{15}\operatorname{tr}(\mathbf{\Psi}\mathbf{m})\mathbb{1}_5)\times$$
$$\times(\mathbf{\Upsilon}^*\check{\mathbf{n}}+\mathbf{n}'^*\check{\mathbf{\Phi}}+\check{\mathbf{n}}^*\mathbf{\Upsilon}+\check{\mathbf{\Phi}}^*\mathbf{n}'+4\mathbf{\Psi}\mathbf{m}+4\mathbf{m}\mathbf{\Psi}-\tfrac{8}{5}\operatorname{tr}(\mathbf{m}\mathbf{\Psi})\mathbb{1}_5))\Big)$$
$$=-\tfrac{80}{3}\mathit{\Psi}_0^2-\tfrac{32}{3}\textstyle\sum_{i=1}^8\mathit{\Psi}_i^2+\tfrac{20}{3}\sqrt{\tfrac{2}{5}}\mathit{\Psi}_0\mathit{\Upsilon}_0+\tfrac{80}{3}\sqrt{\tfrac{3}{5}}\mathit{\Psi}_0\mathit{\Xi}_0-4\sqrt{\tfrac{2}{3}}\mathit{\Xi}_0\mathit{\Upsilon}_0-8\mathit{\Phi}_0\mathit{\Xi}_0$$
$$+\tfrac{40}{3}\sqrt{\tfrac{3}{5}}\mathit{\Psi}_0\mathit{\Phi}-\tfrac{4}{3}\textstyle\sum_{i=1}^8\mathit{\Psi}_i\mathit{\Upsilon}_i+\tfrac{8}{3}\sqrt{2}(\mathit{\Phi}_1\mathit{\Xi}_{44}+\mathit{\Phi}_2\mathit{\Xi}_{45}+\mathit{\Phi}_3\mathit{\Xi}_{45}+\mathit{\Phi}_4\mathit{\Xi}_{93}+\mathit{\Phi}_5\mathit{\Xi}_{94}+\mathit{\Phi}_6\mathit{\Xi}_{95})$$
$$+\tfrac{4}{3}(\mathit{\Xi}_{44}\mathit{\Upsilon}_9+\mathit{\Xi}_{45}\mathit{\Upsilon}_{10}+\mathit{\Xi}_{46}\mathit{\Upsilon}_{11}+\mathit{\Xi}_{93}\mathit{\Upsilon}_{54}+\mathit{\Xi}_{94}\mathit{\Upsilon}_{55}+\mathit{\Xi}_{95}\mathit{\Upsilon}_{56})$$
$$-\tfrac{2}{3}\sqrt{2}(\mathit{\Xi}_{44}\mathit{\Upsilon}_{12}+\mathit{\Xi}_{45}\mathit{\Upsilon}_{13}+\mathit{\Xi}_{46}\mathit{\Upsilon}_{14}+\mathit{\Xi}_{93}\mathit{\Upsilon}_{57}+\mathit{\Xi}_{94}\mathit{\Upsilon}_{58}+\mathit{\Xi}_{95}\mathit{\Upsilon}_{59})$$
$$+\tfrac{4}{3}(\mathit{\Xi}_{41}\mathit{\Upsilon}_{15}+\mathit{\Xi}_{42}\mathit{\Upsilon}_{16}+\mathit{\Xi}_{43}\mathit{\Upsilon}_{17}+\mathit{\Xi}_{90}\mathit{\Upsilon}_{60}+\mathit{\Xi}_{91}\mathit{\Upsilon}_{61}+\mathit{\Xi}_{92}\mathit{\Upsilon}_{62})$$
$$-\tfrac{4}{3}(\mathit{\Xi}_{41}\mathit{\Upsilon}_{42}+\mathit{\Xi}_{42}\mathit{\Upsilon}_{43}+\mathit{\Xi}_{43}\mathit{\Upsilon}_{44}+\mathit{\Xi}_{90}\mathit{\Upsilon}_{87}+\mathit{\Xi}_{91}\mathit{\Upsilon}_{88}+\mathit{\Xi}_{92}\mathit{\Upsilon}_{89})$$
$$+\tfrac{2}{3}\sqrt{6}(\mathit{\Xi}_{47}\mathit{\Upsilon}_{39}+\mathit{\Xi}_{48}\mathit{\Upsilon}_{40}+\mathit{\Xi}_{49}\mathit{\Upsilon}_{41}+\mathit{\Xi}_{96}\mathit{\Upsilon}_{84}+\mathit{\Xi}_{97}\mathit{\Upsilon}_{85}+\mathit{\Xi}_{98}\mathit{\Upsilon}_{86})\,, \tag{B.4n}$$

$$\Big(\mathrm{i}\operatorname{tr}(((\mathbf{\Xi}+\mathbf{m}')(\mathbf{\Xi}+\mathbf{m}')^*-\tfrac{1}{2}(\check{\mathbf{\Psi}}+\check{\mathbf{m}})^2)((\check{\mathbf{\Phi}}+\check{\mathbf{n}})(\mathbf{\Upsilon}+\mathbf{n}')^*-(\mathbf{\Upsilon}+\mathbf{n}')(\check{\mathbf{\Phi}}+\check{\mathbf{n}})^*))-$$
$$-\mathrm{i}\operatorname{tr}((((\mathbf{\Xi}+\mathbf{m}')(\mathbf{\Xi}+\mathbf{m}')^*)'-\tfrac{1}{6}(\mathbf{\Psi}+\mathbf{m})^2+\tfrac{1}{30}\operatorname{tr}((\mathbf{\Psi}+\mathbf{m})^2)\mathbb{1}_5)\times$$
$$\times((\mathbf{\Upsilon}+\mathbf{n}')^*(\check{\mathbf{\Phi}}+\check{\mathbf{n}})-(\check{\mathbf{\Phi}}+\check{\mathbf{n}})^*(\mathbf{\Upsilon}+\mathbf{n}')))\Big)$$
$$\to\Big(\mathrm{i}\operatorname{tr}((\mathbf{\Xi}\mathbf{m}'^*+\mathbf{m}'\mathbf{\Xi}^*-\tfrac{1}{2}(\check{\mathbf{\Psi}}\check{\mathbf{m}}+\check{\mathbf{m}}\check{\mathbf{\Psi}}))(\check{\mathbf{n}}\mathbf{\Upsilon}^*+\check{\mathbf{\Phi}}\mathbf{n}'^*-\mathbf{\Upsilon}\check{\mathbf{n}}^*-\mathbf{n}'\check{\mathbf{\Phi}}^*))-$$
$$-\mathrm{i}\operatorname{tr}(((\mathbf{\Xi}\mathbf{m}'^*+\mathbf{m}'\mathbf{\Xi}^*)'-\tfrac{1}{6}(\mathbf{\Psi}\mathbf{m}+\mathbf{m}\mathbf{\Psi})+\tfrac{1}{15}\operatorname{tr}(\mathbf{\Psi}\mathbf{m})\mathbb{1}_5)(\mathbf{\Upsilon}^*\check{\mathbf{n}}+\mathbf{n}'^*\check{\mathbf{\Phi}}-\check{\mathbf{n}}^*\mathbf{\Upsilon}-\check{\mathbf{\Phi}}^*\mathbf{n}'))\Big)$$
$$=\tfrac{20}{3}\sqrt{\tfrac{2}{5}}\mathit{\Psi}_0\mathit{\Upsilon}_{45}-4\sqrt{\tfrac{2}{3}}\mathit{\Xi}_0\mathit{\Upsilon}_{45}-\tfrac{4}{3}\textstyle\sum_{i=1}^8\mathit{\Psi}_i\mathit{\Upsilon}_{i+45}$$
$$+\tfrac{8}{3}\sqrt{2}(\mathit{\Phi}_1\mathit{\Xi}_{93}+\mathit{\Phi}_2\mathit{\Xi}_{94}+\mathit{\Phi}_3\mathit{\Xi}_{95}-\mathit{\Phi}_4\mathit{\Xi}_{44}-\mathit{\Phi}_5\mathit{\Xi}_{45}-\mathit{\Phi}_6\mathit{\Xi}_{46})$$
$$+\tfrac{4}{3}(\mathit{\Xi}_{44}\mathit{\Upsilon}_{54}+\mathit{\Xi}_{45}\mathit{\Upsilon}_{55}+\mathit{\Xi}_{46}\mathit{\Upsilon}_{56}-\mathit{\Xi}_{93}\mathit{\Upsilon}_9-\mathit{\Xi}_{94}\mathit{\Upsilon}_{10}-\mathit{\Xi}_{95}\mathit{\Upsilon}_{11})$$
$$-\tfrac{2}{3}\sqrt{2}(\mathit{\Xi}_{44}\mathit{\Upsilon}_{57}+\mathit{\Xi}_{45}\mathit{\Upsilon}_{58}+\mathit{\Xi}_{46}\mathit{\Upsilon}_{59}-\mathit{\Xi}_{93}\mathit{\Upsilon}_{12}+\mathit{\Xi}_{94}\mathit{\Upsilon}_{13}-\mathit{\Xi}_{95}\mathit{\Upsilon}_{14})$$
$$+\tfrac{4}{3}(\mathit{\Xi}_{41}\mathit{\Upsilon}_{60}+\mathit{\Xi}_{42}\mathit{\Upsilon}_{61}+\mathit{\Xi}_{43}\mathit{\Upsilon}_{62}-\mathit{\Xi}_{90}\mathit{\Upsilon}_{15}-\mathit{\Xi}_{91}\mathit{\Upsilon}_{16}-\mathit{\Xi}_{92}\mathit{\Upsilon}_{17})$$
$$+\tfrac{4}{3}(\mathit{\Xi}_{41}\mathit{\Upsilon}_{87}+\mathit{\Xi}_{42}\mathit{\Upsilon}_{88}+\mathit{\Xi}_{43}\mathit{\Upsilon}_{89}-\mathit{\Xi}_{90}\mathit{\Upsilon}_{42}+\mathit{\Xi}_{91}\mathit{\Upsilon}_{43}+\mathit{\Xi}_{92}\mathit{\Upsilon}_{44})$$
$$-\tfrac{2}{3}\sqrt{6}(\mathit{\Xi}_{47}\mathit{\Upsilon}_{84}+\mathit{\Xi}_{48}\mathit{\Upsilon}_{85}+\mathit{\Xi}_{49}\mathit{\Upsilon}_{86}-\mathit{\Xi}_{96}\mathit{\Upsilon}_{39}+\mathit{\Xi}_{97}\mathit{\Upsilon}_{40}+\mathit{\Xi}_{98}\mathit{\Upsilon}_{41})\,, \tag{B.4o}$$

$$\operatorname{tr}(((\mathbf{\Xi}+\mathbf{m}')(\hat{\mathbf{\Phi}}+\hat{\mathbf{n}})^*+(\hat{\hat{\mathbf{\Phi}}}+\hat{\mathbf{n}})(\mathbf{\Xi}+\mathbf{m}')^*)((\mathbf{\Upsilon}+\mathbf{n}')(\mathbf{\Upsilon}+\mathbf{n}')^*-4(\check{\mathbf{\Psi}}+\check{\mathbf{m}})^2))$$
$$\to\operatorname{tr}((\mathbf{\Xi}\hat{\mathbf{n}}^*+\mathbf{m}'\hat{\mathbf{\Phi}}^*+\hat{\hat{\mathbf{\Phi}}}\mathbf{m}'^*+\hat{\mathbf{n}}\mathbf{\Xi}^*)(\mathbf{\Upsilon}\mathbf{n}'^*+\mathbf{n}'\mathbf{\Upsilon}^*-4(\check{\mathbf{\Psi}}\check{\mathbf{m}}+\check{\mathbf{m}}\check{\mathbf{\Psi}})))$$
$$=16\textstyle\sum_{i=1}^8\mathit{\Psi}_i\mathit{\Xi}_{i+32}-\tfrac{2}{3}\sqrt{6}(\mathit{\Xi}_{47}\mathit{\Upsilon}_9+\mathit{\Xi}_{48}\mathit{\Upsilon}_{10}+\mathit{\Xi}_{49}\mathit{\Upsilon}_{11}+\mathit{\Xi}_{96}\mathit{\Upsilon}_{54}+\mathit{\Xi}_{97}\mathit{\Upsilon}_{55}+\mathit{\Xi}_{98}\mathit{\Upsilon}_{56})$$
$$+2\sqrt{3}(\mathit{\Xi}_{47}\mathit{\Upsilon}_{12}+\mathit{\Xi}_{48}\mathit{\Upsilon}_{13}+\mathit{\Xi}_{49}\mathit{\Upsilon}_{14}+\mathit{\Xi}_{96}\mathit{\Upsilon}_{57}+\mathit{\Xi}_{97}\mathit{\Upsilon}_{58}+\mathit{\Xi}_{98}\mathit{\Upsilon}_{59})$$
$$-2\textstyle\sum_{i=1}^8\mathit{\Xi}_{i+24}\mathit{\Upsilon}_{i+18}-2\sum_{i=1}^8\mathit{\Xi}_{i+73}\mathit{\Upsilon}_{i+63}-2\sum_{i=1}^6\mathit{\Xi}_{i+18}\mathit{\Upsilon}_{i+29}-2\sum_{i=1}^6\mathit{\Xi}_{i+67}\mathit{\Upsilon}_{i+74}$$
$$-6\sqrt{2}(\mathit{\Phi}_1\mathit{\Xi}_{39}+\mathit{\Phi}_2\mathit{\Xi}_{40}+\mathit{\Phi}_3\mathit{\Xi}_{41}-\mathit{\Phi}_4\mathit{\Xi}_{84}-\mathit{\Phi}_5\mathit{\Xi}_{85}-\mathit{\Phi}_6\mathit{\Xi}_{86})$$
$$-6(\mathit{\Xi}_{44}\mathit{\Upsilon}_{39}+\mathit{\Xi}_{45}\mathit{\Upsilon}_{40}+\mathit{\Xi}_{46}\mathit{\Upsilon}_{41}+\mathit{\Xi}_{93}\mathit{\Upsilon}_{84}+\mathit{\Xi}_{94}\mathit{\Upsilon}_{85}+\mathit{\Xi}_{95}\mathit{\Upsilon}_{86})\,, \tag{B.4p}$$

$$
\begin{aligned}
&\mathrm{tr}(((\mathbf{\Xi}+\mathbf{m}')(\hat{\mathbf{\Phi}}+\hat{\mathbf{n}})^* + (\hat{\bar{\mathbf{\Phi}}}+\hat{\mathbf{n}})(\mathbf{\Xi}+\mathbf{m}')^*)((\check{\mathbf{\Phi}}+\check{\mathbf{n}})(\mathbf{\Upsilon}+\mathbf{n}')^* + (\mathbf{\Upsilon}+\mathbf{n}')(\check{\mathbf{\Phi}}+\check{\mathbf{n}})^* + 4(\check{\mathbf{\Psi}}+\check{\mathbf{m}})^2))\\
&\to \mathrm{tr}((\mathbf{\Xi}\hat{\mathbf{n}}^* + \mathbf{m}'\hat{\mathbf{\Phi}}^* + \hat{\bar{\mathbf{\Phi}}}\mathbf{m}'^* + \hat{\mathbf{n}}\mathbf{\Xi}^*)(\check{\mathbf{n}}\mathbf{\Upsilon}^* + \check{\mathbf{\Phi}}\mathbf{n}'^* + \mathbf{\Upsilon}\check{\mathbf{n}}^* + \mathbf{n}'\check{\mathbf{\Phi}}^* + 4\check{\mathbf{\Psi}}\check{\mathbf{m}} + 4\check{\mathbf{m}}\check{\mathbf{\Psi}}))\\
&= -16\textstyle\sum_{i=1}^{8} \varPsi_i \varXi_{i+32} + \frac{2}{3}\sqrt{6}(\varXi_{47}\varUpsilon_9 + \varXi_{48}\varUpsilon_{10} + \varXi_{49}\varUpsilon_{11} + \varXi_{96}\varUpsilon_{54} + \varXi_{97}\varUpsilon_{55} + \varXi_{98}\varUpsilon_{56})\\
&\quad -\tfrac{2}{3}\sqrt{3}(\varXi_{47}\varUpsilon_{12} + \varXi_{48}\varUpsilon_{13} + \varXi_{49}\varUpsilon_{14} + \varXi_{96}\varUpsilon_{57} + \varXi_{97}\varUpsilon_{58} + \varXi_{98}\varUpsilon_{59})\\
&\quad -2\textstyle\sum_{i=1}^{8} \varXi_{i+24}\varUpsilon_{i+18} - 2\sum_{i=1}^{8} \varXi_{i+73}\varUpsilon_{i+63} - 2\sum_{i=1}^{6} \varXi_{i+18}\varUpsilon_{i+29} + 2\sum_{i=1}^{6} \varXi_{i+67}\varUpsilon_{i+74}\\
&\quad +\tfrac{8}{3}\sqrt{3}(\varPhi_1\varXi_{47} + \varPhi_2\varXi_{48} + \varPhi_3\varXi_{49} + \varPhi_4\varXi_{96} + \varPhi_5\varXi_{97} + \varPhi_6\varXi_{98})\\
&\quad +2\sqrt{2}(\varPhi_1\varUpsilon_{39} + \varPhi_2\varUpsilon_{40} + \varPhi_3\varUpsilon_{41} + \varPhi_4\varUpsilon_{84} + \varPhi_5\varUpsilon_{85} + \varPhi_6\varUpsilon_{86})\\
&\quad +2(\varXi_{44}\varUpsilon_{39} + \varXi_{45}\varUpsilon_{40} + \varXi_{46}\varUpsilon_{41} + \varXi_{93}\varUpsilon_{84} + \varXi_{94}\varUpsilon_{85} + \varXi_{95}\varUpsilon_{86})\,,
\end{aligned}
\tag{B.4q}
$$

$$
\begin{aligned}
&\mathrm{i}\,\mathrm{tr}(((\mathbf{\Xi}+\mathbf{m}')(\hat{\mathbf{\Phi}}+\hat{\mathbf{n}})^* + (\hat{\bar{\mathbf{\Phi}}}+\hat{\mathbf{n}})(\mathbf{\Xi}+\mathbf{m}')^*)((\check{\mathbf{\Phi}}+\check{\mathbf{n}})(\mathbf{\Upsilon}+\mathbf{n}')^* - (\mathbf{\Upsilon}+\mathbf{n}')(\check{\mathbf{\Phi}}+\check{\mathbf{n}})^*))\\
&\to \mathrm{i}\,\mathrm{tr}((\mathbf{\Xi}\hat{\mathbf{n}}^* + \mathbf{m}'\hat{\mathbf{\Phi}}^* + \hat{\bar{\mathbf{\Phi}}}\mathbf{m}'^* + \hat{\mathbf{n}}\mathbf{\Xi}^*)(\check{\mathbf{n}}\mathbf{\Upsilon}^* + \check{\mathbf{\Phi}}\mathbf{n}'^* - \mathbf{\Upsilon}\check{\mathbf{n}}^* - \mathbf{n}'\check{\mathbf{\Phi}}^*))\\
&= \tfrac{2}{3}\sqrt{6}(\varXi_{47}\varUpsilon_{54} + \varXi_{48}\varUpsilon_{55} + \varXi_{49}\varUpsilon_{56} - \varXi_{96}\varUpsilon_9 - \varXi_{97}\varUpsilon_{10} - \varXi_{98}\varUpsilon_{11})\\
&\quad -\tfrac{2}{3}\sqrt{3}(\varXi_{47}\varUpsilon_{57} + \varXi_{48}\varUpsilon_{58} + \varXi_{49}\varUpsilon_{59} - \varXi_{96}\varUpsilon_{12} - \varXi_{97}\varUpsilon_{13} - \varXi_{98}\varUpsilon_{14})\\
&\quad -2\textstyle\sum_{i=1}^{8} \varXi_{i+24}\varUpsilon_{i+63} + 2\sum_{i=1}^{8} \varXi_{i+73}\varUpsilon_{i+18} - 2\sum_{i=1}^{6} \varXi_{i+18}\varUpsilon_{i+74} + 2\sum_{i=1}^{6} \varXi_{i+67}\varUpsilon_{i+29}\\
&\quad +\tfrac{8}{3}\sqrt{3}(\varPhi_1\varXi_{96} + \varPhi_2\varXi_{97} + \varPhi_3\varXi_{98} - \varPhi_4\varXi_{47} - \varPhi_5\varXi_{48} - \varPhi_6\varXi_{49})\\
&\quad -2\sqrt{2}(\varPhi_1\varUpsilon_{84} + \varPhi_2\varUpsilon_{85} + \varPhi_3\varUpsilon_{86} - \varPhi_4\varUpsilon_{39} - \varPhi_5\varUpsilon_{40} - \varPhi_6\varUpsilon_{41})\\
&\quad -2(\varXi_{44}\varUpsilon_{84} + \varXi_{45}\varUpsilon_{85} + \varXi_{46}\varUpsilon_{86} - \varXi_{93}\varUpsilon_{39} - \varXi_{94}\varUpsilon_{40} - \varXi_{95}\varUpsilon_{41})\,,
\end{aligned}
\tag{B.4r}
$$

$$
\begin{aligned}
&\mathrm{i}\,\mathrm{tr}(((\mathbf{\Xi}+\mathbf{m}')(\hat{\mathbf{\Phi}}+\hat{\mathbf{n}})^* - (\hat{\bar{\mathbf{\Phi}}}+\hat{\mathbf{n}})(\mathbf{\Xi}+\mathbf{m}')^*)((\mathbf{\Upsilon}+\mathbf{n}')(\mathbf{\Upsilon}+\mathbf{n}')^* - 4(\check{\mathbf{\Psi}}+\check{\mathbf{m}})^2))\\
&\to \mathrm{i}\,\mathrm{tr}((\mathbf{\Xi}\hat{\mathbf{n}}^* + \mathbf{m}'\hat{\mathbf{\Phi}}^* - \hat{\bar{\mathbf{\Phi}}}\mathbf{m}'^* - \hat{\mathbf{n}}\mathbf{\Xi}^*)(\mathbf{\Upsilon}\mathbf{n}'^* + \mathbf{n}'\mathbf{\Upsilon}^* - 4(\check{\mathbf{\Psi}}\check{\mathbf{m}} + \check{\mathbf{m}}\check{\mathbf{\Psi}})))\\
&= 16\textstyle\sum_{i=1}^{8} \varPsi_i \varXi_{i+81} + \frac{2}{3}\sqrt{6}(\varXi_{47}\varUpsilon_{54} + \varXi_{48}\varUpsilon_{55} + \varXi_{49}\varUpsilon_{56} - \varXi_{96}\varUpsilon_9 - \varXi_{97}\varUpsilon_{10} - \varXi_{98}\varUpsilon_{11})\\
&\quad -2\sqrt{3}(\varXi_{47}\varUpsilon_{57} + \varXi_{48}\varUpsilon_{58} + \varXi_{49}\varUpsilon_{59} - \varXi_{96}\varUpsilon_{12} - \varXi_{97}\varUpsilon_{13} - \varXi_{98}\varUpsilon_{14})\\
&\quad +2\textstyle\sum_{i=1}^{8} \varXi_{i+24}\varUpsilon_{i+63} - 2\sum_{i=1}^{8} \varXi_{i+73}\varUpsilon_{i+18} + 2\sum_{i=1}^{6} \varXi_{i+18}\varUpsilon_{i+74} - 2\sum_{i=1}^{6} \varXi_{i+67}\varUpsilon_{i+29}\\
&\quad -6\sqrt{2}(\varPhi_1\varUpsilon_{84} + \varPhi_2\varUpsilon_{85} + \varPhi_3\varUpsilon_{86} - \varPhi_4\varUpsilon_{39} - \varPhi_5\varUpsilon_{40} - \varPhi_6\varUpsilon_{41})\\
&\quad -6(\varXi_{44}\varUpsilon_{84} + \varXi_{45}\varUpsilon_{85} + \varXi_{46}\varUpsilon_{86} - \varXi_{93}\varUpsilon_{39} - \varXi_{94}\varUpsilon_{40} - \varXi_{95}\varUpsilon_{41})\,,
\end{aligned}
\tag{B.4s}
$$

$$
\begin{aligned}
&\mathrm{i}\,\mathrm{tr}(((\mathbf{\Xi}+\mathbf{m}')(\hat{\mathbf{\Phi}}+\hat{\mathbf{n}})^* - (\hat{\bar{\mathbf{\Phi}}}+\hat{\mathbf{n}})(\mathbf{\Xi}+\mathbf{m}')^*)((\check{\mathbf{\Phi}}+\check{\mathbf{n}})(\mathbf{\Upsilon}+\mathbf{n}')^* + (\mathbf{\Upsilon}+\mathbf{n}')(\check{\mathbf{\Phi}}+\check{\mathbf{n}})^* + 4(\check{\mathbf{\Psi}}+\check{\mathbf{m}})^2))\\
&\to \mathrm{i}\,\mathrm{tr}((\mathbf{\Xi}\hat{\mathbf{n}}^* + \mathbf{m}'\hat{\mathbf{\Phi}}^* - \hat{\bar{\mathbf{\Phi}}}\mathbf{m}'^* - \hat{\mathbf{n}}\mathbf{\Xi}^*)(\check{\mathbf{n}}\mathbf{\Upsilon}^* + \check{\mathbf{\Phi}}\mathbf{n}'^* + \mathbf{\Upsilon}\check{\mathbf{n}}^* + \mathbf{n}'\check{\mathbf{\Phi}}^* + 4\check{\mathbf{\Psi}}\check{\mathbf{m}} + 4\check{\mathbf{m}}\check{\mathbf{\Psi}}))\\
&= -16\textstyle\sum_{i=1}^{8} \varPsi_i \varXi_{i+81} - \frac{2}{3}\sqrt{6}(\varXi_{47}\varUpsilon_{54} + \varXi_{48}\varUpsilon_{55} + \varXi_{49}\varUpsilon_{56} - \varXi_{96}\varUpsilon_9 - \varXi_{97}\varUpsilon_{10} - \varXi_{98}\varUpsilon_{11})\\
&\quad +\tfrac{2}{3}\sqrt{3}(\varXi_{47}\varUpsilon_{57} + \varXi_{48}\varUpsilon_{58} + \varXi_{49}\varUpsilon_{59} - \varXi_{96}\varUpsilon_{12} - \varXi_{97}\varUpsilon_{13} - \varXi_{98}\varUpsilon_{14})\\
&\quad +2\textstyle\sum_{i=1}^{8} \varXi_{i+24}\varUpsilon_{i+63} - 2\sum_{i=1}^{8} \varXi_{i+73}\varUpsilon_{i+18} + 2\sum_{i=1}^{6} \varXi_{i+18}\varUpsilon_{i+74} - 2\sum_{i=1}^{6} \varXi_{i+67}\varUpsilon_{i+29}\\
&\quad +2\sqrt{2}(\varPhi_1\varUpsilon_{84} + \varPhi_2\varUpsilon_{85} + \varPhi_3\varUpsilon_{86} - \varPhi_4\varUpsilon_{39} - \varPhi_5\varUpsilon_{40} - \varPhi_6\varUpsilon_{41})\\
&\quad +2(\varXi_{44}\varUpsilon_{84} + \varXi_{45}\varUpsilon_{85} + \varXi_{46}\varUpsilon_{86} - \varXi_{93}\varUpsilon_{39} - \varXi_{94}\varUpsilon_{40} - \varXi_{95}\varUpsilon_{41})\\
&\quad +\tfrac{8}{3}\sqrt{3}(\varPhi_1\varXi_{96} + \varPhi_2\varXi_{97} + \varPhi_3\varXi_{98} - \varPhi_4\varXi_{47} - \varPhi_5\varXi_{48} - \varPhi_6\varXi_{49})\,,
\end{aligned}
\tag{B.4t}
$$

$$
\begin{aligned}
&-\mathrm{tr}(((\mathbf{\Xi}+\mathbf{m}')(\hat{\mathbf{\Phi}}+\hat{\mathbf{n}})^* - (\hat{\bar{\mathbf{\Phi}}}+\hat{\mathbf{n}})(\mathbf{\Xi}+\mathbf{m}')^*)((\check{\mathbf{\Phi}}+\check{\mathbf{n}})(\mathbf{\Upsilon}+\mathbf{n}')^* - (\mathbf{\Upsilon}+\mathbf{n}')(\check{\mathbf{\Phi}}+\check{\mathbf{n}})^*))\\
&\to -\mathrm{tr}((\mathbf{\Xi}\hat{\mathbf{n}}^* + \mathbf{m}'\hat{\mathbf{\Phi}}^* - \hat{\bar{\mathbf{\Phi}}}\mathbf{m}'^* - \hat{\mathbf{n}}\mathbf{\Xi}^*)(\check{\mathbf{n}}\mathbf{\Upsilon}^* + \check{\mathbf{\Phi}}\mathbf{n}'^* - \mathbf{\Upsilon}\check{\mathbf{n}}^* - \mathbf{n}'\check{\mathbf{\Phi}}^*))
\end{aligned}
$$

$$
\begin{aligned}
&= \tfrac{2}{3}\sqrt{6}(\Xi_{47}\Upsilon_9 + \Xi_{48}\Upsilon_{10} + \Xi_{49}\Upsilon_{11} + \Xi_{96}\Upsilon_{54} + \Xi_{97}\Upsilon_{55} + \Xi_{98}\Upsilon_{56}) \\
&\quad -\tfrac{2}{3}\sqrt{3}(\Xi_{47}\Upsilon_{12} + \Xi_{48}\Upsilon_{13} + \Xi_{49}\Upsilon_{14} + \Xi_{96}\Upsilon_{57} + \Xi_{97}\Upsilon_{58} + \Xi_{98}\Upsilon_{59}) \\
&\quad -2\textstyle\sum_{i=1}^{8}\Xi_{i+24}\Upsilon_{i+18} - 2\sum_{i=1}^{8}\Xi_{i+73}\Upsilon_{i+63} - 2\sum_{i=1}^{6}\Xi_{i+18}\Upsilon_{i+29} - 2\sum_{i=1}^{6}\Xi_{i+67}\Upsilon_{i+74} \\
&\quad -\tfrac{8}{3}\sqrt{3}(\Phi_1\Xi_{47} + \Phi_2\Xi_{48} + \Phi_3\Xi_{49} + \Phi_4\Xi_{96} + \Phi_5\Xi_{97} + \Phi_6\Xi_{98}) \\
&\quad +2\sqrt{2}(\Phi_1\Upsilon_{39} + \Phi_2\Upsilon_{40} + \Phi_3\Upsilon_{41} + \Phi_4\Upsilon_{84} + \Phi_5\Upsilon_{85} + \Phi_6\Upsilon_{86}) \\
&\quad +2(\Xi_{44}\Upsilon_{39} + \Xi_{45}\Upsilon_{40} + \Xi_{46}\Upsilon_{41} + \Xi_{93}\Upsilon_{84} + \Xi_{94}\Upsilon_{85} + \Xi_{95}\Upsilon_{86}) \,. \qquad \text{(B.4u)}
\end{aligned}
$$

C Conventions

The Pauli matrices σ^a, $a = 1, 2, 3$, are

$$
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad \text{(C.1)}
$$

They obey

$$
[\sigma^a, \sigma^b] = 2\mathrm{i}\textstyle\sum_{c=1}^{3}\varepsilon^{abc}\sigma^c\,, \quad \varepsilon^{123} = -\varepsilon^{321} = 1 \quad \text{plus cyclic permutations}. \qquad \text{(C.2)}
$$

All remaining ε^{abc} are equal to zero. Moreover, we have

$$
\mathrm{tr}(\sigma^a\sigma^b) = 2\,\delta^{ab}\,. \qquad \text{(C.3)}
$$

We use the following convention for the gamma matrices $\{\gamma^\mu\}_{\mu=1,2,3,4}$ in Euclidian space:

$$
\gamma^1 = \begin{pmatrix} 0 & -\mathrm{i}\sigma^1 \\ \mathrm{i}\sigma^1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -\mathrm{i}\sigma^2 \\ \mathrm{i}\sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & -\mathrm{i}\sigma^3 \\ \mathrm{i}\sigma^3 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \qquad \text{(C.4)}
$$

This gives

$$
\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,\delta^{\mu\nu}\,\mathbb{1}_4\,, \qquad \gamma^\mu = \gamma^{\mu *}\,, \qquad \text{(C.5)}
$$

$$
\begin{aligned}
&\mathrm{tr}_c((\gamma^\kappa \wedge \gamma^\lambda)(\gamma^\mu \wedge \gamma^\nu)) = 4(\delta^{\lambda\mu}\delta^{\kappa\nu} - \delta^{\kappa\mu}\delta^{\lambda\nu})\,, \\
&\mathrm{tr}_c(\gamma^\mu\gamma^\nu) = 4\delta^{\mu\nu}\,, \qquad \mathrm{tr}_c(1) = 4\,.
\end{aligned} \qquad \text{(C.6)}
$$

We use the following convention for the gamma matrices $\{\hat{\gamma}^\mu\}_{\mu=0,1,2,3}$ in Minkowski space:

$$
\begin{aligned}
&\hat{\gamma}^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \hat{\gamma}^1 = \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad \hat{\gamma}^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \\
&\hat{\gamma}^3 = \begin{pmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \qquad \hat{\gamma}^5 = \mathrm{i}\hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}.
\end{aligned} \qquad \text{(C.7)}
$$

This gives

$$
\begin{aligned}
&\hat{\gamma}^{\mu}\hat{\gamma}^{\nu}+\hat{\gamma}^{\nu}\hat{\gamma}^{\mu}=2g^{\mu\nu}\,\mathbb{1}_4\,, \qquad g^{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)\,,\\
&\hat{\gamma}^{0}=\overline{\hat{\gamma}^{0}}\,,\quad \hat{\gamma}^{1}=\overline{\hat{\gamma}^{1}}\,,\quad \hat{\gamma}^{2}=-\overline{\hat{\gamma}^{2}}\,,\quad \hat{\gamma}^{3}=\overline{\hat{\gamma}^{3}}\,,\quad \hat{\gamma}^{5}=\overline{\hat{\gamma}^{5}}\,.
\end{aligned}
\tag{C.8}
$$

The Gell–Mann matrices $\{\lambda^a\}_{a=1,\dots 8}$ are

$$
\begin{aligned}
&\lambda^1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\ \lambda^2=\begin{pmatrix}0&-\mathrm{i}&0\\\mathrm{i}&0&0\\0&0&0\end{pmatrix},\ \lambda^3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\\
&\lambda^4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\ \lambda^5=\begin{pmatrix}0&0&-\mathrm{i}\\0&0&0\\\mathrm{i}&0&0\end{pmatrix},\\
&\lambda^6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\ \lambda^7=\begin{pmatrix}0&0&0\\0&0&-\mathrm{i}\\0&\mathrm{i}&0\end{pmatrix},\ \lambda^8=\begin{pmatrix}\sqrt{\frac{1}{3}}&0&0\\0&\sqrt{\frac{1}{3}}&0\\0&0&-\sqrt{\frac{4}{3}}\end{pmatrix},
\end{aligned}
\tag{C.9}
$$

The Gell–Mann matrices obey

$$
\begin{aligned}
&[\lambda^b,\lambda^c]=\textstyle\sum_{a=1}^{8}2\mathrm{i}\,f_{abc}\lambda^a\,, \qquad (C.10)\\
&f^{147}=-f^{741}=f^{246}=-f^{642}=f^{257}=-f^{752}=f^{345}=-f^{543}=\tfrac{1}{2}\,,\\
&f^{123}=-f^{321}=1\,,\ f^{458}=-f^{854}=f^{678}=-f^{876}=\tfrac{1}{2}\sqrt{3}\quad \text{plus cyclic permutations}\,.
\end{aligned}
$$

All remaining f^{abc} are equal to zero. Moreover, we have

$$
\mathrm{tr}(\lambda^a\lambda^b)=2\,\delta^{ab}\,. \tag{C.11}
$$

D List of Symbols

Symbol	Explanation	Page
General Symbols		
i	imaginary unit	
$\mathbb{R}$	real numbers	
$\mathbb{C}$	complex numbers	
$\mathbb{Z}$	integers	
$\mathbb{N}$	integers not smaller than zero	
$\mathrm{M}_F\mathbb{C}$	complex $F \times F$–matrices	
1_n	$n \times n$–unity matrix	
$0_{n\times m}$	$n \times m$–zero matrix	
$\mathrm{diag}(\dots)$	diagonal matrix	
$\mathrm{grad}(\,.\,)$	gradient	
$\deg(\,.\,)$	degree	
$\mathrm{id}_{.}$	identity operator	
$\mathrm{End}(\,.\,)$	endomorphism	
$\mathrm{Hom}(\,.\,,\,.\,)$	homomorphism	
$\mathrm{tr}(\,.\,)$	trace	
$\mathrm{tr}_c(\,.\,)$	trace which explicitly includes the trace over gamma matrices	
$\mathrm{Tr}_\omega(\,.\,)$	Dixmier trace	25
$[\,.\,,\,.\,]$	commutator	
$[\,.\,,\,.\,]_g$	graded commutator	32
$\{\,.\,,\,.\,\}$	anticommutator	
$\{\,.\,,\,.\,\}_g$	graded anticommutator	60
A^T	transposition of a matrix A	
$\overline{A}$	complex conjugation of a matrix A	
$A^* = \overline{A^T}$	Hermitian conjugation of a matrix A	
Exterior Differential Algebra		
X	compact Riemannian spin manifold	43
X_M	Minkowski space	146
N	dimension of X	43

Symbol	Explanation	Page
$C^\infty(X)$	algebra of real–valued smooth functions on X	43
v_g	volume form on X	46
v_M	volume form on X_M	148
T^*X	cotangent bundle of X	
T_*X	tangent bundle of X	
$\Gamma^\infty(C)$	smooth sections of the Clifford bundle over X	44
$\Gamma^\infty(T^*X)$	smooth sections of T^*X	44
$\Gamma^\infty(T_*X)$	smooth sections of T_*X	45
C^k	space of elements of $\Gamma^\infty(C)$ of degree not bigger than k and the same parity	44
Λ^k	space of differential k–forms, represented by gamma matrices	45
B^k	space of k–coboundaries	49
$\lrcorner$	interior product between differential forms	45
$\wedge$	exterior product between differential forms	45
$\tilde{\wedge}$	module multiplication in $\Lambda^* \times \Omega^*_D\mathfrak{g}$	77
$\mathbf{d}$	exterior differential	46
$\mathbf{d}^*$	codifferential	46
c^k	element of Λ^k	46
e^j	basis element of $\Gamma^\infty(T^*X)$	46
e_j	basis element of $\Gamma^\infty(T_*X)$	46
$g(.\,,.)$	Riemannian metric on X	45
$c(.)$	isomorphism from $\Gamma^\infty(T^*X)$ to Λ^1	44
∇_v	Levi–Civita covariant derivative with respect to the vector field v	46
∇^S_v	Spin covariant derivative with respect to the vector field v	47
∇_{c^n}	covariant derivative with respect to the differential form $c^n \in \Lambda^n$	65
∇_Ω	covariant derivative with respect to elements of $\pi(\Omega^*\mathfrak{g})$	64
$L^2(X,S)$	Hilbert space of square integrable sections of the spinor bundle over X	43
D	Dirac operator on $L^2(X,S)$	47
γ	$\mathbb{Z}_2$–grading operator on $L^2(X,S)$	47
Δ	scalar Laplacian	47
$\langle .\,,.\rangle_{\Lambda^*}$	scalar product on Λ^*	46

Symbol	Explanation	Page
L–Cycle		
$\mathfrak{g}$	skew–adjoint Lie algebra	24
$\mathfrak{a}$	skew–adjoint matrix Lie algebra	43
$\mathfrak{a}'$	semisimple matrix Lie algebra	48
$\mathfrak{a}''$	Abelian matrix Lie algebra	48
h	Hilbert space	24
$\mathscr{B}(h)$	linear bounded operators on h	24
$\mathscr{O}_0$	even linear (not necessarily bounded) operators on h	32
$\mathscr{O}_1$	odd linear (not necessarily bounded) operators on h	32
$\langle\cdot,\cdot\rangle_h$	scalar product on h	32
π	representation of $\mathfrak{g}$ on h,	24
	representation of $\Omega^*\mathfrak{g}$ on h	32
$\hat{\pi}$	representation of $\mathfrak{a}$ on $\mathbb{C}^F$,	43
	representation of $\Omega^*\mathfrak{a}$ on $\mathbb{C}^F$	50
D	generalized Dirac operator on h	24
$\mathcal{M}$	$F\times F$–mass matrix = generalized Dirac operator on $\mathbb{C}^F$	43
Γ	$\mathbb{Z}_2$–grading operator on h	24
$\hat{\Gamma}$	$\mathbb{Z}_2$–grading operator on $\mathbb{C}^F$	43
Graded Differential Lie Algebras		
d	universal differential	27
$V(\mathfrak{g})$	free vector space generated by $\mathfrak{g}$	25
$V(d\mathfrak{g})$	free vector space generated by $d\mathfrak{g}$	25
$T(\mathfrak{g})$	tensor algebra of $V(\mathfrak{g})\oplus V(d\mathfrak{g})$	26
δ_a	function on $\mathfrak{g}$ that equals 1 at $a\in\mathfrak{g}$ and 0 else	25
δ_{da}	function on $d\mathfrak{g}$ that equals 1 at $da\in d\mathfrak{g}$ and 0 else	25
$\tilde{\Omega}^*\mathfrak{g}$	graded Lie subalgebra of $T(\mathfrak{g})$	26
$I(\mathfrak{g})$	graded ideal of $\tilde{\Omega}^*\mathfrak{g}$	26
$\Omega^*\mathfrak{g}$	graded universal differential Lie algebra	27
$\imath(a)$	factorization mapping from $\mathfrak{g}$ to $\Omega^0\mathfrak{g}$, $a\in\mathfrak{g}$	30
$\imath(da)$	factorization mapping from $d\mathfrak{g}$ to $\Omega^1\mathfrak{g}$, $a\in\mathfrak{g}$	30

Symbol	Explanation	Page
$\jmath^*\mathfrak{g}$	graded differential ideal of $\Omega^*\mathfrak{g}$	33
$\tilde{\jmath}^j_n\mathfrak{a}$	matrix parts of $\pi(\jmath^*\mathfrak{g})$	75
$\Omega^*_D\mathfrak{g}$	graded differential Lie algebra	33
σ	linear mapping from $\Omega^*\mathfrak{g}$ to linear (not necessarily bounded) operators on h	35
$\hat{\sigma}$	linear mapping from $\Omega^*\mathfrak{a}$ to $\mathrm{M}_F\mathbb{C}$	53
$\hat{\sigma}_\mathfrak{g}$	linear mapping from $\Omega^*\mathfrak{g}$ to $\mathscr{B}(h)$	53
$\mathcal{H}^*\mathfrak{g}$	graded Lie homomorphisms of $\pi(\Omega^*\mathfrak{g})$	37
$\tilde{\mathbb{c}}^*\mathfrak{g}$	graded centre of $\pi(\Omega^*\mathfrak{g})$ in linear operators on h	37
$\mathbb{c}^*\mathfrak{a}$	matrix parts of $\tilde{\mathbb{c}}^*\mathfrak{g}$	79
$\mathbb{J}^*\mathfrak{g}$	ideal of $\mathcal{H}^*\mathfrak{g}$	38
$\mathbb{j}^*\mathfrak{a}$	matrix parts of $\mathbb{J}^*\mathfrak{g}$	80
$\hat{\mathcal{H}}^*\mathfrak{g}$	graded differential Lie algebra	38
$T^j_n\mathfrak{a}$	vector subspaces of $T(\mathfrak{g})$	50
$\tilde{K}^j_n\mathfrak{a}$	subspaces of $\mathrm{M}_F\mathbb{C}$ determining $\pi(\jmath^*\mathfrak{g})$	61
Ω^*X	universal graded differential algebra over $C^\infty(X)$	54
$\mathfrak{d}$	universal differential on Ω^*X	54
$\hat{T}^*(\mathfrak{a})$	universal graded differential enveloping algebra of $\mathfrak{a}$	54
$\hat{\otimes}$	skew–tensor product	54
$\hat{d}$	differential on $\Omega^*X\hat{\otimes}\hat{T}^*(\mathfrak{a})$	55
$\mathbf{i}$	homomorphism of $\Omega^*\mathfrak{g}$ to $\Omega^*X\hat{\otimes}\hat{T}^*(\mathfrak{a})$	55
t	ternary number	57
$\#_i(\mathrm{t})$	number of digits i in t	57
$\Omega_\mathrm{t}X$	subspace of Ω^*X	57
$\hat{T}_\mathrm{t}(\mathfrak{a})$	subspace of $\hat{T}^*(\mathfrak{a})$	57
$\mathrm{f}_{\alpha,A}$, $\tilde{\mathrm{f}}_{\alpha,A}$	very useful functions	62

Connections and Gauge Transformations

$\mathbb{u}(\mathfrak{g})$	infinitesimal gauge transformations	38
$\mathbb{u}_0(\mathfrak{g})$	local infinitesimal gauge transformations	80
exp	exponential mapping	38
Ad	adjoint representation of $\exp(\mathbb{u})$ in $\Omega^*_D\mathfrak{g}$	39

Symbol	Explanation	Page
∇_h	covariant derivative on h	40
∇	connection on $\Omega_D^*\mathfrak{g}$	40
ρ	connection form on h	40
$\mathbb{r}^*\mathfrak{a}$	matrix parts of the connection form	78
$\hat{\rho}$	connection form on $\Omega_D^*\mathfrak{g}$	40
θ	curvature form	40
$\mathfrak{e}(\theta)$	representative of θ orthogonal to $\mathbb{J}^2\mathfrak{g}$	42
$\mathcal{U}(\mathfrak{g})$	gauge group	41
$\mathcal{U}_0(\mathfrak{g})$	local gauge group	80
γ_u	gauge transformation by $u \in \mathcal{U}(\mathfrak{g})$	41
$\langle .\,,.\rangle_{\hat{\mathcal{H}}^2\mathfrak{g}}$	scalar product in $\hat{\mathcal{H}}^2\mathfrak{g}$	42
S_B	bosonic action	42
S_F	fermionic action	42
Physical Models		
SU(n), U(1)	unitary matrix Lie groups	
su(n), u(1)	skew–adjoint matrix Lie algebras	
$\underline{n}$	representation of su(5)	96
$a \in \mathfrak{a}$	elements of $\underline{24}$	97
$b \in \mathfrak{b}$	elements of $\underline{5}$	98
$c \in \mathfrak{c}$	elements of $\underline{50}$	101
$v \in \mathfrak{v}$	elements of $\underline{75}$	98
$w \in \mathfrak{w}$	elements of $\underline{45}^*$	100
$(b,b)'$	$\underline{24}$–part of bb^*, $b \in \underline{5}$	99
M_i	$i = u,d,e,n,N$ – 3×3–fermionic mass matrices	103
M_i'	$i = u,d,e,n,N$ – 6×6–fermionic mass matrices	103
M_5, M_{10}	3×3–Grand Unification mass matrices	103
M_5', M_{10}'	6×6–Grand Unification mass matrices	103
$M_{\tilde{u}}, M_{\tilde{n}}$	linear combinations of M_u, M_n	110
$M_{\tilde{u}}', M_{\tilde{n}}'$	linear combinations of M_u', M_n'	110
m, n, m', n'	special elements of $\underline{24}, \underline{5}, \underline{50}, \underline{45}^*$ that enter the mass matrix $\mathcal{M}$	102

Symbol	Explanation	Page
π_{10}, π_5	$\underline{24}$–representations of su(5)	97
$\pi_{i,j}$	other representations of su(5)	98
J_i	parts of $\mathbb{j}^2\mathfrak{a}$	111
M^i_{jk}	6×6–matrices determined by the fermionic and Grand Unification mass matrices	120
$\mu^i, \hat{\mu}^i, \tilde{\mu}^i, \breve{\mu}^i$	constants (of dimension mass^4) determined by the fermionic and Grand Unification mass matrices	132
$\mathscr{L}$	Lagrangian	129
$A, \breve{A}$	Yang–Mills field in $\underline{24}$–representation	129
A''	u(1) Yang–Mills field	125
$W^a_\mu, G^a_\mu,$ $A'_\mu, \tilde{A}_\mu,$ P_μ, Z_μ, Z'_μ	components of gauge fields	138
θ_W	Weinberg angle	146
$\mathbf{\Psi}$	Higgs field in $\underline{24}$–representation	137
$\mathbf{\Phi}$	Higgs field in complex $\underline{5}$–representation	137
$\mathbf{\Upsilon}$	Higgs field in complex $\underline{45}^*$–representation	140
$\mathbf{\Xi}$	Higgs field in complex $\underline{50}$–representation	139
$\mathbf{m}, \mathbf{n}, \mathbf{n}', \mathbf{m}'$	vacuum configuration of the Higgs field	136
$\mathit{\Psi}_i, \mathit{\Phi}_i, \mathit{\Xi}_i, \mathit{\Upsilon}_i$	components of Higgs fields	146
$\psi_i, \phi_i, \xi_i, \upsilon_i$	reparametrized components of Higgs fields	146
$\delta^{\mu\nu}$	metric tensor in Euclidian space	146
$g^{\mu\nu}$	metric tensor in Minkowski space	146
γ^μ	Euclidian gamma matrices	195
$\hat{\gamma}^\mu$	Minkowskian gamma matrices	195
γ_C	charge conjugation gamma matrix	148
σ^a	Pauli matrices	195
λ^a	Gell–Mann matrices	196
M	Grand Unification scale	162
m_t, m_b, m_τ	mass of the top quark, the bottom quark, the tauon	160
m_n, M_N	Dirac and Majorana mass scale for the neutrinos	161

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Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Dissertation selbständig und ohne unerlaubte fremde Hilfe angefertigt zu haben. Ich habe keine anderen als die im Schriftenverzeichnis angeführten Quellen benutzt und sämtliche Textstellen, die wörtlich oder sinngemäß aus veröffentlichten oder unveröffentlichten Schriften entnommen wurden, und alle Angaben, die auf mündlichen Auskünften beruhen, als solche kenntlich gemacht. Ebenfalls sind alle von anderen Personen bereitgestellten Materialien oder erbrachten Dienstleistungen als solche gekennzeichnet.

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1. Einleitung

Eine der wichtigsten Anwendungen der nichtkommutativen Geometrie [1] in der Physik ist die unifizierte Beschreibung des Standardmodells. Es gibt zahlreiche "nichtkommutative" Formulierungen des Standardmodells, z.B. einen auf Arbeiten von Connes und Lott [1,3] beruhenden Zugang oder ein von Gruppen aus Mainz und Marseille [4,5] entwickeltes Verfahren. Allen Varianten ist gemeinsam, daß neben den üblichen Yang–Mills–Feldern auch die Higgs–Felder Teil eines verallgemeinerten Eichpotentials sind. Dies führt zu einer einheitlichen Betrachtung von zuvor als unabhängig angesehenen Teilen des Standardmodells. Die wahrscheinlich tiefliegendste Formulierung aus mathematischer Sicht ist eine Weiterentwicklung des Connes–Lott–Zugangs. Diese Formulierung beruht auf einem K–Zyklus (heute meistens Spektraltripel genannt) mit reeller Struktur [2]. Ein K–Zyklus $(\mathcal{A}, h, D)$ besteht aus einer assoziativen $*$–Algebra mit Eins, die auf einem Hilbertraum h wirkt, sowie einem verallgemeinerten Dirac–Operator D auf h. Diese drei Objekte müssen noch gewisse technische Bedingungen erfüllen.

Wir untersuchen die Anwendbarkeit der nichtkommutativen Geometrie zur Lösung der folgenden allgemeinen Problemstellung: Es ist der klassische Lagrangian einer Eichfeldtheorie über der Raum–Zeit–Mannigfaltigkeit X aus den unter 1)–4) angegebenen Informationen zu gewinnen.

1) Eine unitäre Matrizengruppe G und die assoziierte Eichgruppe $\mathcal{G} = C^\infty(X) \otimes G$. Dabei bezeichnet $C^\infty(X)$ die Algebra der reellwertigen glatten Funktionen über X.
2) Multipletts ψ chiraler Fermionen, die sich nach einer Darstellung $\tilde{\pi}_0$ von G transformieren. Die induzierte Darstellung der Eichgruppe $\mathcal{G}$ sei $\tilde{\pi} \equiv \mathrm{id} \otimes \tilde{\pi}_0$.
3) Die fermionische Massenmatrix $\widetilde{\mathcal{M}}$, die die Massen der Fermionen sowie verallgemeinerte Kobayashi–Maskawa Matrizen beinhaltet.
4) Für gewisse unifizierte Modelle: Zusätzliche Informationen über das Schema der spontanen Symmetriebrechung von G.

Die Verbindung dieser physikalischen Daten zur nichtkommutativen Geometrie (K–Zyklen in der Connes–Lott–Formulierung des Standardmodells) wird wie folgt hergestellt: Man wählt eine Matrizenalgebra $\mathcal{A}_M$ derart, daß die Gruppe der lokalen Eichtransformationen $\mathcal{G} = C^\infty(X) \otimes G$ isomorph zur Gruppe der unitären Elemente von $\mathcal{A} = C^\infty(X) \otimes \mathcal{A}_M$ ist. Ferner bestimmen die fermionischen Multipletts ψ den Hilbertraum h und die fermionische Massenmatrix $\widetilde{\mathcal{M}}$ den verallgemeinerten Dirac–Operator D.

Zwar steht das Standardmodell nicht im Widerspruch zu den bisherigen Experimenten, doch es gibt Gründe, weshalb man an einer Beschreibung der Physik mit sogenannten großen unifizierten Theorien (GUT) interessiert ist. Es zeigt sich jedoch [6], daß eine Behandlung solcher unifizierter Theorien im Rahmen der strengsten Version des Connes–Lott–Zugangs nicht möglich ist; man benötigt zusätzliche Strukturen oder schwächere Beziehungen zwischen K–Zyklus und Modell.

In der Dissertation wird eine neue Richtung der nichtkommutativen Geometrie entwickelt, die auch ohne die Einführung zusätzlicher Strukturen in der Lage ist, eine größere Klasse von physikalischen Modellen zu beschreiben.

2. Die Idee — oder: Was ist ein L–Zyklus?

Weshalb erlaubt die strengste Version [2] des Connes–Lott–Zugang nur die Formulierung des Standardmodells? Das Problem ist die Erweiterung der Darstellungen $\tilde{\pi} = \mathrm{id} \otimes \tilde{\pi}_0$ der Eichgruppe $\mathcal{G} = C^\infty(X) \otimes G$ zu Darstellungen $\pi = \mathrm{id} \otimes \pi_0$ der Algebra $\mathcal{A} = C^\infty(X) \otimes \mathcal{A}_M$. Diese Erweiterung muß verträglich mit linearen Operationen sein, und das ist im allgemeinen nicht möglich. Erweiterbare irreduzible Darstellungen von Matrizengruppen sind die Identität und die komplexe Konjugation. Während die im Standardmodell auftretenden Darstellungen genau von diesem Typ sind, sind für große unifizierte Theorien auch andere Gruppendarstellungen erforderlich.

Wie können wir diese Einschränkung überwinden? — Unsere Idee ist, die Lie–Algebra $\mathfrak{g} = C^\infty(X) \otimes \mathfrak{a}$ der Eichgruppe $\mathcal{G} = C^\infty(X) \otimes G$ in unserem mathematischen Kalkül zu benutzen, und nicht eine die Eichgruppe $\mathcal{G}$ erweiternde assoziative Algebra. Dabei ist $\mathfrak{a}$ die (Matrix–) Lie–Algebra von G.

Folglich ersetzen wir in der Definition des K–Zyklus [1, 3] die assoziative $*$–Algebra $\mathcal{A}$ durch eine schiefadjungierte Lie–Algebra $\mathfrak{g}$. Das Ergebnis nennen wir "L–Zyklus", wobei der Buchstabe L an Lie erinnert:

Definition: *Ein L–Zyklus $(\mathfrak{g}, h, D, \pi, \Gamma)$ besteht aus einer $*$–Darstellung π einer schiefadjungierten Lie–Algebra $\mathfrak{g}$ als beschränkte Operatoren auf einem Hilbertraum h, einem selbstadjungierten Operator D auf h mit kompakter Resolvente und einem selbstadjungierten Operator Γ auf h, $\Gamma^2 = \mathrm{id}_h$, der mit $\pi(\mathfrak{g})$ kommutiert und mit D antikommutiert. Der Operator D darf unbeschränkt sein, aber so, daß $[D, \pi(\mathfrak{g})]$ beschränkt bleibt.*

Zu einem gegebenen physikalischen Modell finden wir wie folgt den zugehörigen L–Zyklus $(\mathfrak{g}, h, D, \pi, \Gamma)$: Aus technischen Gründen müssen wir zunächst von der physikalischen Raum–Zeit–Mannigfaltigkeit zu einer kompakten Euklidischen Spinmannigfaltigkeit X übergehen. Dann konstituieren die euklidischen fermionischen Multipletts ψ den Hilbertraum h des L–Zyklus. Wir wählen $\mathfrak{g} = C^\infty(X) \otimes \mathfrak{a}$ als die Lie–Algebra der gegebenen Eichgruppe $\mathcal{G} = C^\infty(X) \otimes G$. Die Darstellung $\pi = \mathrm{id} \otimes \hat{\pi}$ von $\mathfrak{g}$ auf h ist einfach gegeben durch das Differential der Gruppendarstellung $\tilde{\pi}$. Weiterhin setzen wir $D = \mathsf{D} + \gamma^5 \mathcal{M}$, wobei D der Dirac–Operator zum Spin–Zusammenhang ist. Die Matrix $\mathcal{M}$ enthält die fermionischen Massenparameter und eventuell Beiträge, die das gewünschte Schema der spontanen Symmetriebrechung bewirken. Die Chiralitätseigenschaften der Fermionen werden im Graduierungsoperator Γ kodiert. Nach einer Wick–Rotation zum Minkowski–Raum verwenden wir Γ, um von h auf den Raum der chiralen Fermionen zu projizieren. Dann muß $\gamma^5 \mathcal{M}$, angewendet auf chirale Fermionen, mit der Massenmatrix $\widetilde{\mathcal{M}}$ zusammenfallen.

3. Mathematische Ergebnisse [7]

1) In Analogie zum Vorgehen in der nichtkommutativen Geometrie wird eine universelle graduierte Differential–Lie–Algebra $(\Omega^*\mathfrak{g}, d, [\ ,\ ])$ über der Lie–Algebra $\mathfrak{g}$ des L–Zyklus definiert. Universelle graduierte Differential–Lie–Algebren scheinen in der Literatur nur wenig bekannt zu sein, obwohl ihre Konstruktion völlig kanonisch ist. Man kann sich $\Omega^*\mathfrak{g}$ als die Menge aller graduierten Kommutatoren vorstellen, die man aus Elementen von $\mathfrak{g}$ und $d\mathfrak{g}$ bildet. Die graduierte Differential–Lie–Algebra $\Omega^*\mathfrak{g}$ ist universell in dem Sinne, daß jede graduierte Differential–Lie–Algebra, die durch $\pi(\mathfrak{g})$ und $\mathrm{d}\pi(\mathfrak{g})$ generiert wird, durch Faktorisierung von $\Omega^*\mathfrak{g}$ nach einem Differentialideal erhalten werden kann. Da ein L–Zyklus ein kanonisches Differentialideal liefert, erhalten wir somit auf natürliche Weise eine graduierte Differential–Lie–Algebra $\Omega^*_D\mathfrak{g}$.

2) Es wird eine Darstellung π der universellen Differential–Lie–Algebra $\Omega^*\mathfrak{g}$ auf dem Hilbertraum h definiert, die die im L–Zyklus gegebene Darstellung π erweitert. Es

wird gezeigt, daß $\mathcal{J}^*\mathfrak{g} := \ker\pi + d\ker\pi \subset \Omega^*\mathfrak{g}$ ein Differentialideal ist. Folglich ist

$$\Omega_D^*\mathfrak{g} = \bigoplus_{n=0}^{\infty} \Omega_D^n\mathfrak{g} \;, \qquad\qquad \Omega_D^n\mathfrak{g} := \frac{\Omega^n\mathfrak{g}}{\mathcal{J}^n\mathfrak{g}} \cong \frac{\pi(\Omega^n\mathfrak{g})}{\pi(\mathcal{J}^n\mathfrak{g})} \;,$$

eine graduierte Differential–Lie–Algebra. Die Berechnung des Differentials in $\Omega_D^*\mathfrak{g}$ erfordert die Kenntnis von $\pi(d\omega^k)$ zu gegebenen $\pi(\omega^k) \in \pi(\Omega^k\mathfrak{g})$. Wir finden die äußerst nützliche Beziehung

$$\pi(d\omega^k) = (-\mathrm{i}D)\circ\pi(\omega^k) - (-1)^k\pi(\omega^k)\circ(-\mathrm{i}D) + \sigma(\omega^k) \;,$$

wobei σ eine lineare Abbildung von $\Omega^*\mathfrak{g}$ in den Raum der linearen Operatoren auf h ist. Diese Beziehung ist der Schlüssel zur Berechnung des Differentialideals $\pi(\mathcal{J}^*\mathfrak{g})$. Dabei ist entscheidend, daß wir eine einfache rekursive Berechnungsvorschrift für $\sigma(\omega^k)$ erhalten.

3) Wir definieren Räume von graduierten Lie–Homomorphismen $\mathcal{H}^*\mathfrak{g}$ von $\pi(\Omega^*\mathfrak{g})$ und $\hat{\mathcal{H}}^*\mathfrak{g}$ von $\Omega_D^*\mathfrak{g}$, die den mathematischen Rahmen für die Definition von Zusammenhängen, Eichtransformationen und physikalischen Wirkungen bilden. Eine Zusammenhangsform $\hat{\rho}$ ist ein Element von $\hat{\mathcal{H}}^1\mathfrak{g}$ und deren zugehörige Krümmungsform θ ein Element von $\hat{\mathcal{H}}^2\mathfrak{g}$. Die durch die Exponentialabbildung von gewissen beschränkten Elementen von $\mathcal{H}^0\mathfrak{g}$ erhaltene unitäre Gruppe $\mathcal{U}(\mathfrak{g})$ spielt die Rolle einer Eichgruppe in unserem Kalkül. Die bosonische Wirkung wird als die Dixmier–Spur des Quadrats der Krümmungform und die fermionische Wirkung als Erwartungswert der kovarianten Ableitung definiert. Es zeigt sich, daß die Beziehungen zwischen allen so definierten geometrischen Objekten die gleiche Form wie in gewöhnlichen Eichtheorien haben. Die vorkommenden Räume sind jedoch komplizierter, was die Formulierung von unifizierten physikalischen Modellen erst ermöglicht.

4) Für physikalische Anwendungen ist man an der speziellen Situation interessiert, daß die Lie–Algebra $\mathfrak{g}$ des L–Zyklus das Tensorprodukt aus der Funktionenalgebra $C^\infty(X)$ über der Raum–Zeit–Mannigfaltigkeit X und einer Matrix–Lie–Algebra $\mathfrak{a}$ ist. Ebenso zerlegen sich der verallgemeinerte Dirac–Operator D, der Hilbertraum h und der Graduierungsoperator Γ in Raum–Zeit– und Matrix–Anteile. Es stellt sich nun die Frage, wie sich die zum L–Zyklus assoziierten Räume $\pi(\Omega^n\mathfrak{g})$, $\pi(\mathcal{J}^n\mathfrak{g})$ und $\Omega_D^n\mathfrak{g}$ in Raum–Zeit– und Matrix–Anteile aufspalten. Für physikalische Anwendungen ist die Kenntnis der Zerlegungen für $n \leq 2$ notwendig. Wir lösen ein allgemeineres Problem: Für beliebige Matrix–Lie–Algebren und beliebigen Grad n geben wir Rekursionsformeln zur Berechnung dieser Zerlegungen an. Die Ergebnisse werden für $\pi(\Omega^n\mathfrak{g})$ in Proposition 15, für $\pi(\mathcal{J}^n\mathfrak{g})$ in Theorem 22 und für $\Omega_D^n\mathfrak{g}$ in Corollary 24 angegeben. Außerdem erhalten wir explizite Formeln für den Kommutator und das Differential von Elementen aus $\Omega_D^*\mathfrak{g}$.

5) Im Hinblick auf physikalische Anwendungen wird der Begriff der lokalen Zusammenhänge eingeführt. In diesem Fall ergeben sich Beziehungen zwischen dem Matrix–Anteil der Zusammenhangsform und den Matrix–Anteilen von $\pi(\Omega^n\mathfrak{g})$ und $\pi(\jmath^n\mathfrak{g})$. Außerdem gewinnen wir eine einfachere Berechnungsvorschrift für die bosonische und fermionische Wirkung. Es zeigt sich, daß die so erhaltene Zusammenhangsform differentielle 1–Formen und 0–Formen enthält. Die 1–Formen werden als Yang–Mills–Felder interpretiert und die 0–Formen als Higgs–Felder. Die bosonische Wirkung enthält damit in unifizierter Form den Yang–Mills–Lagrangian, die kovariante Ableitung der Higgs–Felder und ein Polynom vierter Ordnung in den Higgs–Feldern, das Higgs–Potential. Die fermionische Wirkung umfaßt eine minimale Kopplung der Fermionen an die Yang–Mills–Felder und eine Yukawa–Kopplung zwischen Fermionen und Higgs–Feldern.

4. Physikalische Ergebnisse

Der entwickelte mathematische Kalkül der nichtassoziativen Geometrie wird zur vereinheitlichenden Beschreibung dreier physikalischer Modelle angewendet: des Standardmodells der Teilchenphysik, des geflippten SU(5)×U(1)–Modells der Großen Unifizierung und (als dessen Spezialfall) des SU(5)–Modells der Großen Unifizierung. Es ist auch sehr lehrreich, die Formulierung der chiralen U(1)–Spinor–Elektrodynamik anzugehen. Dabei zeigt sich, daß wir zwar den korrekten fermionischen Sektor dieses Modells erhalten, daß es aber nicht möglich ist, eine bosonische Wirkung zu bekommen. Der Grund ist, daß durch die Kommutativität der Lie–Algebra gewisse Entartungen auftreten, die keine nichtverschwindende Krümmung zulassen.

1) Der L–Zyklus für das Standardmodell ist das unmittelbare Abbild der physikalischen Situation: Wir verwenden die Lie–Algebra $C^\infty(X)\otimes(\mathrm{su}(3)\oplus\mathrm{su}(2)\oplus\mathrm{u}(1))$ und deren übliche Darstellung auf dem fermionischen Hilbertraum. Unser Formalismus erzeugt ein komplexes Higgs–Dublett und das bekannte quartische Higgs–Potential [8]. Die bosonische Wirkung enthält in unifizierter Form den Yang–Mills–Lagrangian, die kovariante Ableitung des Higgs–Dubletts und das Higgs–Potential. Die fermionische Wirkung vereinheitlicht die minimale Kopplung der Eichfelder mit der Yukawa–Kopplung des Higgs–Feldes. Es ergeben sich in klassischer Näherung Vorhersagen für alle bosonischen Massen: Falls rechte Neutrinos existieren, erhalten wir

$$m_W=\tfrac{1}{2}m_t\,,\qquad m_Z=m_W/\cos\theta_W\,,\qquad \sin^2\theta_W=\tfrac{3}{8}\,,\qquad m_H=\tfrac{3}{2}m_t\,.$$

Ohne rechte Neutrinos finden wir als einzige Modifikation $m_H=\sqrt{\frac{43}{20}}m_t$. Dabei bezeichnen m_t, m_W, m_Z und m_H die Massen des Top–Quarks, des W–Bosons, des Z–Bosons und des Higgs–Bosons. Es ist jedoch noch unklar, wie diese Massenbeziehungen durch Quantenkorrekturen modifiziert werden. Diese Frage wird Gegenstand zukünftiger Untersuchungen sein.

2) Bisherige Formulierungen des geflippten SU(5)×U(1)–Modells der Großen Unifizierung geben unter anderem die Darstellungen des U(1)–Anteils vor. Das ist bei unserer Behandlung dieses Modells nicht erforderlich, wir benötigen lediglich die Lie–Algebra $\mathfrak{g} = C^\infty(X) \otimes \mathrm{su}(5)$ sowie die su(5)–Darstellungen $\underline{10}, \underline{5}^*, \underline{1}$. Trotzdem erhalten wir ein u(1)–Eichfeld und U(1)–Eichtransformationen wegen der natürlichen Erweiterung von $\Omega^1_D\mathfrak{g}$ zu $\hat{\mathscr{H}}^1\mathfrak{g}$. Es ist bemerkenswert, daß die erhaltenen Darstellungen dieses u(1)–Eichfeldes auf dem fermionischen Hilbertraum erstens eindeutig und zweitens in der Natur realisiert sind. Die auftretenden Higgs–Multipletts sind durch den Formalismus eindeutig festgelegt, es sind $\underline{24}, \underline{5}, \underline{45}^*, \underline{50}$–Darstellungen. Folglich gibt es insgesamt 224 Higgs–Felder und 25 Eichfelder. Der bosonische Lagrangian enthält wieder den Yang–Mills–Lagrangian, die kovariante Ableitung der Higgs–Felder und das äußerst komplizierte Higgs–Potential. Deshalb muß die Auswertung dieses Higgs–Potentials mittels Computeralgebra durchgeführt werden. Die spontane Symmetriebrechung von $C^\infty(X) \otimes (\mathrm{su}(5) \oplus \mathrm{u}(1))$ zu $C^\infty(X) \otimes (\mathrm{su}(3)_C \oplus \mathrm{u}(1)_{EM})$ entfernt 16 Higgs–Komponenten und gibt 16 Eichfeldern eine Masse. Wir erhalten in klassischer Näherung die gleichen Vorhersagen für die Massen der W– und Z–Bosonen und für den Weinberg–Winkel wie im Standardmodell:

$$m_W = \tfrac{1}{2} m_t \;, \qquad m_Z = m_W / \cos\theta_W \;, \qquad \sin^2\theta_W = \tfrac{3}{8} \;.$$

Es zeigt sich, daß genau ein Higgs–Feld eine Masse der Größenordnung m_t erhält. Dieses Higgs–Feld ist eine gewisse Linearkombination von neutralen Higgs–Komponenten der $\underline{5}$– und $\underline{45}^*$–Darstellungen. Es hat die gleichen Eigenschaften wie das Higgs–Feld des Standardmodells, und wir finden in klassischer Näherung für seine Masse die obere Schranke $m_H = 1.45\, m_t$. Alle übrigen 207 Higgs–Felder und 13 Eichfelder sind zu massereich, um in heutigen Experimenten beobachtet zu werden. Alle Bosonen mit drittelzahliger elektrischer Ladung, die deshalb zum Protonenzerfall beitragen, erhalten eine Masse in der Gr ößenordnung der GUT–Skala. Für eine generische Wahl der freien Parameter liegt auch die Masse der nicht im Standardmodell vorkommenden Bosonen mit ganzzahliger elektrischer Ladung in der Nähe der GUT–Skala.

3) Durch ein geringfügiges Modifizieren des mathematischen Kalküls und das damit verbundende Weglassen des U(1)–Anteils können wir aus dem SU(5)×U(1)–Modell das SU(5)–Modell der Großen Unifizierung herleiten. Es sind nur gewisse Umbezeichnungen der Fermionen notwendig, jedoch keine größeren Rechnungen. Higgs–Felder in der $\underline{50}$–Darstellung treten in diesem Modell nicht auf, ebenso fehlt das zur U(1)–Gruppe gehörende Eichfeld. Für die Massen der übrigen Felder erhalten wir in klassischer Näherung quantitativ nahezu identische Resultate, z.B. als Obergrenze für die Masse des einzigen leichten Higgs–Feldes $m_H = 1.32\, m_t$.

Die Rechnungen beweisen die Anwendbarkeit des entwickelten mathematischen Kalküls auf Modelle der Großen Unifizierung. Damit ist das Ziel der Arbeit erreicht.

Wir gelangen zu dem physikalisch interessanten Resultat, daß der Niederenergie–Sektor der SU(5)×U(1)– und SU(5)–Modelle identisch mit dem Standardmodell ist. Das bedeutet, daß aus der hervorragenden experimentellen Bestätigung der Vorhersagen des Standardmodells nicht geschlossen werden darf, daß das Standardmodell auch bei höheren Energien gültig bleibt.

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