

Noncommutative geometry & supersymmetry

Title

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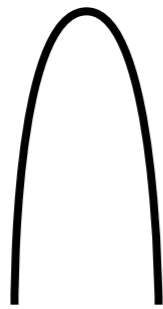
Affiliation

June 19th '10

Date

Outline: NCG & Physics

Riemannian Geometry



GRT

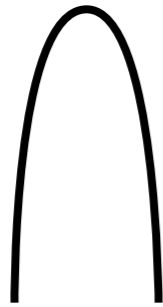
Noncommutative
geometry



GRT + Standard Model
(Connes & coworkers)

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Riemannian Geometry



GRT

Noncommutative
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GRT + Standard Model
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+ supersymmetry?

A spectral triple

Definition:

For an involutive, unital algebra \mathcal{A} , a Hilbert space \mathcal{H} and an operator $D : \mathcal{H} \rightarrow \mathcal{H}$ we call the data $(\mathcal{A}, \mathcal{H}, D)$ a **spectral triple** when

- \mathcal{A} is represented as bounded operators on \mathcal{H}
- D is self-adjoint and has compact resolvent
- for any $a \in \mathcal{A}$, $[D, a]$ is a bounded operator on \mathcal{H}

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Roughly:

\mathcal{A} encodes a space / space(-time)

\mathcal{H} accounts for the (physical) states

D takes care of the dynamics

Prime examples

Example: the triple

$$\mathcal{A}_F = \bigoplus_i M_{n_i}(\mathbb{K}_i) \quad \mathbb{K}_i = \mathbb{R}, \mathbb{C} \text{ or } \mathbb{H}$$

\mathcal{H}_F a finite dimensional representation of \mathcal{A}_F

D_F a matrix on \mathcal{H}_F

is called a **finite spectral triple**

Prime examples

Example: finite spectral triple

Example: let M be a compact Riemannian manifold (w/o boundary)

- $C^\infty(M)$: smooth complex-valued functions
- $L^2(M, S)$: square integrable sections of the spinor bundle
- $i\gamma^\mu(\partial_\mu + \omega_\mu)$, locally,

Prime examples

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- $L^2(M, S)$: square integrable sections of the spinor bundle
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then $(C^\infty(M), L^2(M, S), i\gamma^\mu(\partial_\mu + \omega_\mu))$

is a spectral triple (the **canonical spectral triple**):

- $C^\infty(M)$ acts on $L^2(M, S)$ by $(f\psi)(x) := f(x)\psi(x)$
- $i\gamma^\mu(\partial_\mu + \omega_\mu)$ is self-adjoint
- $[D, f] = i\gamma^\mu\partial_\mu(f)$ is bounded

Chirality & Antiparticles

- Chirality:

A **grading operator** $\gamma : \mathcal{H}_F \rightarrow \mathcal{H}_F$ (satisfying: $\gamma^* = \gamma, \gamma^2 = 1$)

with $D\gamma = -\gamma D$

and $\gamma a = a\gamma$ for all $a \in \mathcal{A}$

i.e. $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ (e.g. Weyl spinors)

Notation: $(\mathcal{A}, \mathcal{H}, D; \gamma)$ (**even spectral triple**)

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- Charge conjugation:

A **real structure**: an anti-isometry $J : \mathcal{H} \rightarrow \mathcal{H}$ satisfying:

$$J^2 = \pm 1 \quad JD = \pm DJ \quad J\gamma = \pm \gamma J$$

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- Compatibility conditions:

$$[a, Jb^* J^*] = 0 \quad [[D, a], Jb^* J^*] = 0 \quad \forall a, b \in \mathcal{A}$$

(**first order condition**)

Gauge interactions: inner fluctuations

Definition: algebra \mathcal{B} is said to be Morita equivalent to \mathcal{A} iff
 $\exists \mathcal{E}$ such that $\mathcal{B} \simeq \text{End}_A^0(\mathcal{E})$

Question: $\exists (\mathcal{B}, \mathcal{H}', D'; J')$ that implements the Mor. equiv.?

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$$J' = J$$

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$$D' = D + A \pm JAJ^* =: D_A$$

$$\text{with } A \in \Omega_D^1(\mathcal{A}) = \left\{ \sum_i a_i [D, b_i], a_i, b_i \in \mathcal{A} \right\}$$

self-adjoint

\Rightarrow **Inner fluctuations** of D

Gauge group: unitarily equivalence

Definition: Two spectral triples

$$(\mathcal{A}, \mathcal{H}, D; J, \gamma) \quad \text{and} \quad (\mathcal{A}, \mathcal{H}, D'; J', \gamma')$$

are called **unitarily equivalent** if there exists unitary $U : \mathcal{H} \rightarrow \mathcal{H}$ such that

- $U\pi(a)U^* = \pi(\sigma(a))$
- $J' = UJU^*$
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Example: Take $U = uJuJ^*$, for any unitary element $u \in U(\mathcal{A})$, then

- $\sigma(a) = uau^*$
- $J' = J$
- $\gamma' = \gamma$
- $D' = D + u[D, u^*] \pm Ju[D, u^*]J^*$

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Note: for a fluctuated Dirac operator:

$$D_A \rightarrow D_{A^u} \quad \text{with:} \quad A^u = uAu^* + u[D, u^*]$$

Action: the spectral action

Spectral action

$$\text{Tr}[f(D_A/\Lambda)]$$

f some even function Λ a cutoff scale

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If for a compact Riemannian manifold M of dimension m and a vector bundle V , D_V on $L^2(M, S \otimes V)$ is such that $D_V^2 = \nabla^* \nabla - E$ for some endomorphism E , then we have

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$$\mathrm{Tr}[f(D_V/\Lambda)] \sim f(0)a_m(D_V^2) + \sum_{n \geq 0, n \neq m} \Lambda^{m-n} \frac{2f_{m-n}}{\Gamma((m-n)/2)} a_n(D_V^2)$$

with $a_n(D_V^2) = \int_M a_n(D_V^2, x) \sqrt{g} d^m x$.

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For $m = 4$ we the first contributions are:

$$\mathrm{Tr}[f(D_V/\Lambda)] \sim f(0)a_4(D_V^2) + 2f_2\Lambda^2 a_2(D_A^2) + 2f_4\Lambda^4 a_0(D_A^2) + \mathcal{O}(\Lambda^{-2})$$

where

- $a_0(D_V^2, x) = (4\pi)^{-m/2} \mathrm{Tr}(\mathrm{id})$
- $a_2(D_V^2, x) = (4\pi)^{-m/2} \mathrm{Tr}(-\frac{1}{6}R + E)$
- $a_4(D_V^2, x) = (4\pi)^{-m/2} \mathrm{Tr}(-12R_{;\mu}^{\mu} + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 60RE + 60E_{;\mu}^{\mu} + 180E^2 + 30\Omega_{\mu\nu}\Omega^{\mu\nu})$

Accounts for: gauge kinetic terms, scalar kinetic terms, scalar masses, scalar self-interactions.

Application: the commutative case

Take again $(C^\infty(M), L^2(M, S), i\gamma^\mu(\partial_\mu + \omega_\mu))$ with:

- $\gamma := \gamma^5$;
- $(J\psi)(x) := C(\psi(x))$.

$$M \text{ 4-d: } J^2 = -1 \quad J\gamma = \gamma J \quad J\not{\partial} = \not{\partial}J$$

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Results:

- No inner fluctuations / gauge interactions;
- Spectral action:

$$\begin{aligned} \text{Tr}[f(D/\Lambda)] &= \Lambda^4 \frac{f_4}{4\pi^2} \int_M \sqrt{g} d^4x + \\ &\quad \Lambda^2 \frac{f_2}{48\pi^2} \int_M R \sqrt{g} d^4x \\ &\quad + \mathcal{O}(\Lambda^0) \end{aligned}$$

Action: the inner product

$\langle \cdot, \cdot \rangle$ Inner product on \mathcal{H} , $D_A \equiv D + \mathbb{A}$

Inner product

$$\langle \psi, D_A \psi \rangle \quad \psi \in \mathcal{H}$$

(Accounts for: fermion kinetic terms, minimal coupling, Yukawa-coupling)

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Problem: in the SM four times too many fermionic DOF.

Solution:

The new inner product

$$\frac{1}{2} \langle J\psi, D_A \psi \rangle \quad \psi \in \mathcal{H}_+$$

Works: since $J\gamma = -\gamma J$

The Einstein-Yang-Mills system (1/2)

Take the spectral triple:

$$\mathcal{A} = C^\infty(M) \otimes M_N(\mathbb{C})$$

$$\mathcal{H} = L^2(M, S) \otimes M_N(\mathbb{C})$$

$$D = \not{D} \otimes 1_N$$

Make it real and even with:

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Results:

Inner fluctuations (gauge interactions) $i\mathbb{A} \equiv A + JA^*J$

(with $A = \sum_k \gamma^\mu a_k \partial_\mu(b_k) \in \Omega_D^1(\mathcal{A})$ self-adjoint):

- act in the adjoint representation: $i\mathbb{A} = -i\gamma^\mu \text{ad}A_\mu$
- transform as a $\mathfrak{su}(N)$ gauge field

The Einstein-Yang-Mills system (2/2)

Inner product:

$$\langle \psi, D_A \psi \rangle = \int_M \psi^*(x) \gamma^\mu (\partial_\mu + A_\mu) \psi(x) \text{dvol}(M)$$

Spectral action:

$$\begin{aligned} \text{Tr}[f(D_A/\Lambda)] &\sim -\frac{f(0)}{24\pi^2} \int_M \text{Tr}(\mathbb{F}_{\mu\nu} \mathbb{F}^{\mu\nu}) \text{dvol}(M) \\ &+ f(0) \frac{N^2}{1440} \int_M [5R^2 - 8R_{\mu\nu} R^{\mu\nu} - 7R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}] \text{dvol}(M) \\ &+ 2f_4 \Lambda^4 N^2 \text{vol}(M) \end{aligned}$$

Result: SU(N) Yang-Mills theory (with fermions in the adjoint representation), coupled to gravity.

Supersymmetry in EYM? (See also [1])

Question: Does the Einstein-Yang-Mills system exhibit supersymmetry?

[1] A.H. Chamseddine, Connection between space-time supersymmetry and non-commutative geometry, Phys. Lett. B332 (1994) 349-357

Supersymmetry in EYM? (See also [1])

Question: Does the Einstein-Yang-Mills system exhibit supersymmetry?

To what extent is the action of the Einstein-Yang-Mills system invariant under supersymmetry?

i.e. define $\delta\psi, \delta\mathbb{A}$ and determine

$$\delta S_{\Lambda}[\psi, \mathbb{A}] = \frac{d}{dt} S_{\Lambda}[\psi + t\delta\psi, \mathbb{A} + t\delta\mathbb{A}]$$

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SUSY in the EYM system (1/2)

Problem: Fermionic & Bosonic DOF don't match.

Degrees of freedom:	Continuous	Finite
Bosonic:	4	$N^2 - 1$
Fermionic:	8	$2N^2$

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1 Continuous part

$$J^2 = -1 \quad J\gamma = \gamma J$$

No Majoranas

No reduction trough

$$\langle \psi, D_A \psi \rangle \rightarrow \frac{1}{2} \langle J\psi, D_A \psi \rangle \quad \psi \in \mathcal{H}^+$$

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- Minkowski: $(\psi, \bar{\psi})$ with $\gamma_M^5 \psi = \psi$ $\gamma_M^5 \bar{\psi} = \bar{\psi}$

[1] P. van Nieuwenhuizen and A. Waldron. On Euclidean spinors and Wick rotations. Phys. Lett. B, 389:29 – 36

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but $\gamma_E^5 \bar{\chi}_E = -\bar{\chi}_E$

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but $\gamma_E^5 \bar{\chi}_E = -\bar{\chi}_E$
- Take as inner product $\langle \chi, D_A \psi \rangle, \psi \in \mathcal{H}^+, \chi \in \mathcal{H}^-$

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2 Finite part

- Use $L^2(M, S) \otimes_{\mathbb{C}} M_N(\mathbb{C}) \simeq L^2(M, S) \otimes_{\mathbb{R}} \mathfrak{u}(N)$
- and $\mathfrak{u}(N) \simeq \mathfrak{u}(1) \oplus \mathfrak{su}(N)$: write $\psi = \text{Tr } \tilde{\psi} + \psi$

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- and $\mathfrak{u}(N) \simeq \mathfrak{u}(1) \oplus \mathfrak{su}(N)$: write $\psi = \text{Tr } \tilde{\psi} + \psi$
- Result:** action decouples and $\text{Tr } \tilde{\psi}$ lacks gauge interactions:

$$S[\tilde{\psi}, A] = S_1[\text{Tr } \tilde{\psi}] + S_2[\psi, A]$$

- Discard $\text{Tr } \tilde{\psi}$

SUSY in the EYM system (2/2)

Let $\epsilon_{\pm} = \pm \gamma^5 \epsilon_{\pm}$ be two constant spinors

Define:

$$\delta\psi = F\epsilon_+ \in \mathcal{H}^+ \quad F = \gamma^\mu \gamma^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])$$

$$\delta\chi = F\epsilon_- \in \mathcal{H}^-$$

$$\delta A = c\gamma^\mu [\text{ad}(\epsilon_-, \gamma_\mu \psi) + \text{ad}(\chi, \gamma^\mu \epsilon_+)] \in \mathcal{B}(\mathcal{H})$$

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Then:

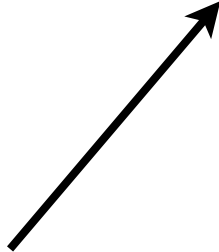
Lemma:

The action of the Einstein-Yang-Mills system (after reducing DOF) is SUSY invariant for at least all positive powers of Λ —i.e. for $a_0(D_A^2)$, $a_2(D_A^2)$ and $a_4(D_A^2)$ — upon taking:

$$c = \frac{-3i\pi}{2f(0)N}$$

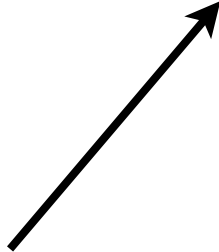
Supersymmetry

Big question: Can a spectral triple be defined that yields the action of the minimal supersymmetric standard model (MSSM)?

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- Superpartners ('sfermions', 'gauginos') of SM particles
 - Undetected yet
 - Action invariant under 'supersymmetry'
 - Peculiar Higgs sector

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Small question: Can a spectral triple be defined that yields the action of super-QCD?

Super-QCD; the spectral triple

Start with the spectral triple of the EYM-system. Now:

- The algebra stays intact

$$\mathcal{A} = C^\infty(M) \otimes M_3(\mathbb{C})$$

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- As for the Hilbert space

$$\mathcal{H} = L^2(M, S) \otimes M_3(\mathbb{C}) \longrightarrow L^2(M, S) \otimes (\mathbb{C}^3 \oplus M_3(\mathbb{C}) \oplus \overline{\mathbb{C}^3})$$

with $\pi(a)(q, g, q') = (aq, ag, q')$

$$\pi^o(a)(q, g, q') \equiv J\pi^*(a)J^*(q, g, q') = (q, ga, a^t q')$$

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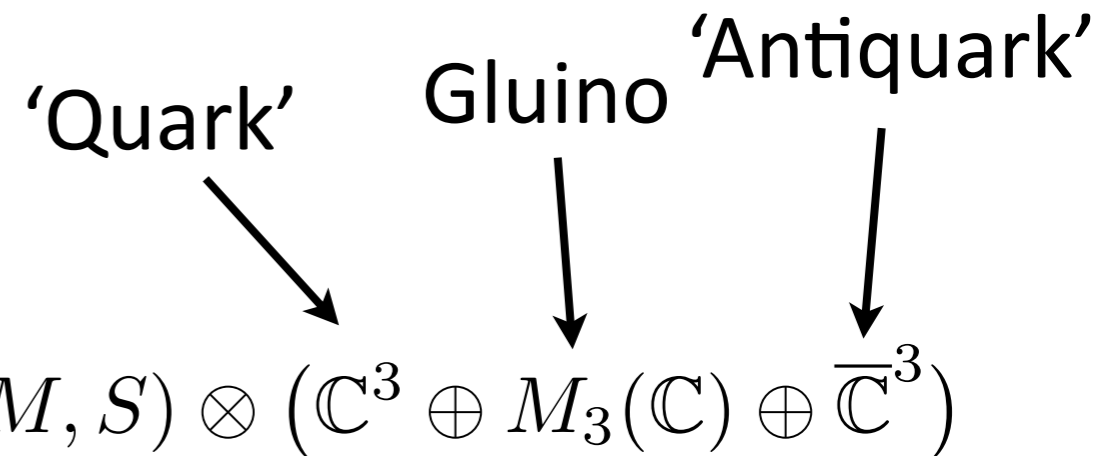
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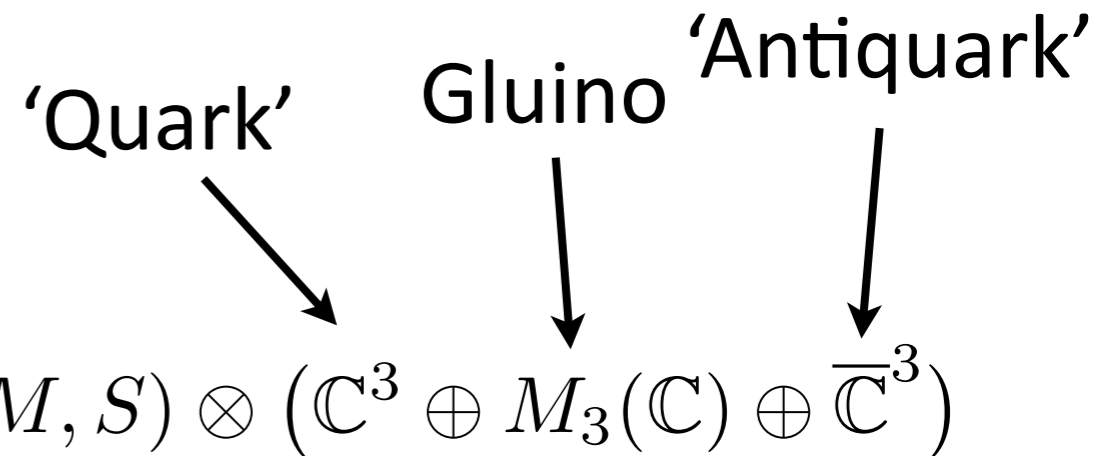
$$\mathcal{H} = L^2(M, S) \otimes M_3(\mathbb{C}) \longrightarrow L^2(M, S) \otimes (\mathbb{C}^3 \oplus M_3(\mathbb{C}) \oplus \overline{\mathbb{C}^3})$$

with $\pi(a)(q, g, q') = (aq, ag, q')$

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$$J = J_M \otimes * \longrightarrow J_M \otimes J_F, \quad J_F(q, g, \overline{q'}) = (q', g^*, \overline{q})$$



Super-QCD; the spectral triple

Start with the spectral triple of the EYM-system. Now:

- The algebra stays intact

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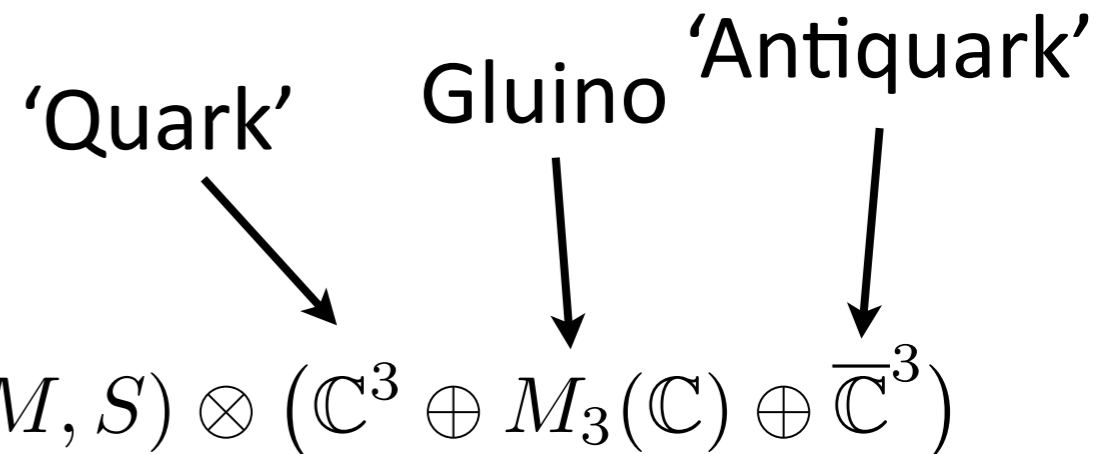
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- Dirac operator:

$$D \longrightarrow \not{D} \otimes 1_N + \gamma^5 \otimes D_F \quad D_F : \mathbb{C}^3 \oplus M_3(\mathbb{C}) \oplus \overline{\mathbb{C}^3} \curvearrowright$$

'Quark' Gluino 'Antiquark'

Super-QCD; the Dirac operator

A Dirac operator that is self-adjoint, fulfills $D_F J_F = J_F D_F$ and is order one ($[[D, a], b^o] \quad \forall a, b \in \mathcal{A}$) is of the form:

$$\begin{pmatrix} D_{11} & D_{12} & 0 \\ D_{12}^* & D_{22} & D_{32}^* \\ 0 & D_{32} & D_{33} \end{pmatrix} \quad \text{with} \quad \begin{aligned} \overline{D_{12}g} &= D_{32}g^* \\ [D_{21}, a] &= 0 \\ [D_{23}, a^o] &= 0 \quad \forall a \in M_3(\mathbb{C}) \end{aligned}$$

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- Put $D_{11} = D_{22} = D_{33} = 0$
- D_{12}, D_{32} parametrized by $v \in \mathbb{C}^3$
- Write

$$D_F = \begin{pmatrix} 0 & d_v & 0 \\ d_v^* & 0 & e_v^* \\ 0 & e_v & 0 \end{pmatrix} \quad \begin{aligned} d_v(g) &= g(v) \in \mathbb{C}^3 \\ e_v(g) &= g^t(\bar{v}) \in \overline{\mathbb{C}}^3 \end{aligned}$$

Super-QCD; inner fluctuations

Inner fluctuations of D_F :

For $A^{(0,1)} := \sum_i a_i [D_F, b_i]$ $a_i, b_i \in M_3(\mathbb{C})$ self-adjoint:

$$\begin{aligned} & (D_F + A^{(0,1)} + J(A^{(0,1)})^* J^*)(q_1, g, q_2) \\ &= (g\tilde{q}, q_1 \tilde{q}^t + \tilde{q} q_2^t, g^t \tilde{q}) \in \mathbb{C}^3 \oplus M_3(\mathbb{C}) \oplus \overline{\mathbb{C}}^3 \end{aligned}$$

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with $\tilde{q} = \sum_i [1 + m_i(1 - n_i)]v = \sum_i n_i^* v$

Write $D_{\tilde{q}} := D_F + A^{(0,1)} + J(A^{(0,1)})^* J^*$

Super-QCD; gauge group

Recall: action of the gauge group by $uJuJ^*$:

Gauge group: gluino

For $U := uJuJ^*$, $u \in U(M_3(\mathbb{C}))$ we find:

$$M_3(\mathbb{C}) \ni g \rightarrow Ugu^* = ugu^*$$

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And:

Gauge group: squark

For $U := uJ u J^*$, $u \in U(M_3(\mathbb{C}))$ we find:

$$D_{\tilde{q}} \rightarrow U D_{\tilde{q}} U^* = D_{u\tilde{q}}$$

Super-QCD; the action (1/2)

Adding D_F results in:

$$\Omega_{\mu\nu} \rightarrow \Omega_{\mu\nu}$$

$$E \rightarrow E + i\gamma^5 \gamma^\mu [\partial_\mu + \mathbb{A}_\mu, 1 \otimes D_{\tilde{q}}] - 1 \otimes D_{\tilde{q}}^2$$

$$\text{(Was: } E = \frac{1}{4} R \otimes \text{id} - \frac{1}{2} \gamma^\mu \gamma^\nu \mathbb{F}_{\mu\nu} \text{)}$$

the latter appearing as:

$$\mathcal{O}(\Lambda^2) : \quad \text{Tr}(E)$$

$$\mathcal{O}(\Lambda^0) : \quad \text{Tr}(RE), \text{Tr}(E^2)$$

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we have

Theorem

Adding the finite part of the algebra to the finite Hilbert space of QCD and introducing D_F as given before, the spectral action gives the following extra terms (on the orders of Λ^4 , Λ^2 , Λ^0):

$$\int_M \left[- \left(\frac{6f_2}{\pi^2} - \frac{f(0)}{4\pi^2\Lambda^2} R \right) \Lambda^2 |\tilde{q}|^2 + \frac{f(0)}{4\pi^2} \left(8|\tilde{q}|^4 + 6|(\partial_\mu + A_\mu)\tilde{q}|^2 \right) \right] \text{dvol}(M)$$

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squark self-interaction (a 4 squark vertex)



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gluon-squark-squark interaction

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gluon-gluon-squark-squark term

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Super-QCD; the action (2/2)

Theorem

Adding the finite part of the algebra to the finite Hilbert space of QCD and introducing D_F as given before, the inner product gives the following extra terms:

$$\langle \chi_g, (D + \mathbb{A})\psi_g \rangle = \int_M (\chi_g, (\not{\partial} + \mathbb{A})\psi_g) \text{dvol}(M)$$

$$\langle \chi, (\gamma^5 \otimes D_{\tilde{q}})\psi \rangle = \int_M [(\chi_q^i, \gamma^5 \psi_g^a) \tilde{q}^k - (\chi_g^a, \gamma^5 \psi_q^k) \tilde{q}^i] (T_a)_{ik} \text{dvol}(M)$$

gluino-gluino-gluon

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gluino-gluino-gluon

squark-quark-gluino

Summary

Big question: Can a spectral triple be defined that yields the action of the minimal supersymmetric standard model (MSSM)?

Small answer: Supersymmetric QCD has been done [1]:

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Next step(s):

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And: Thanks for the attention and now off to lunch!

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