Noncommutative geometry & supersymmetry

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Title

Name

Affiliation

Date

Outline: NCG & Physics

Riemannian Geometry $\longrightarrow$ GRT

Noncommutative geometry $\longrightarrow$ GRT + Standard Model (Connes & coworkers)

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Noncommutative geometry $\longrightarrow$ GRT + Standard Model (Connes & coworkers)

+ supersymmetry?

A spectral triple

Definition:

For an involutive, unital algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$ and an operator $D : \mathcal{H} \to \mathcal{H}$ we call the data $(\mathcal{A}, \mathcal{H}, D)$ a spectral triple when

- $\mathcal{A}$ is represented as bounded operators on $\mathcal{H}$
- D is self-adjoint and has compact resolvent
- for any $a \in \mathcal{A}$, $[D, a]$ is a bounded operator on $\mathcal{H}$

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Roughly:

$\mathcal{A}$ encodes a space / space(-time)

$\mathcal{H}$ accounts for the (physical) states

D takes care of the dynamics

Prime examples

Example: the triple

$$\mathcal{A}_F = \bigoplus_i M_{n_i}(\mathbb{K}_i) \quad \mathbb{K}_i = \mathbb{R}, \mathbb{C} \text{ or } \mathbb{H}$$

$\mathcal{H}_F$ a finite dimensional representation of $\mathcal{A}_F$

D_F a matrix on $\mathcal{H}_F$

is called a finite spectral triple

Prime examples

Example: finite spectral triple

Example: let M be a compact Riemannian manifold (w/o boundary)

- $C^\infty(M)$: smooth complex-valued functions
- $L^2(M, S)$: square integrable sections of the spinor bundle
- $i\gamma^\mu(\partial_\mu + \omega_\mu)$, locally,

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then $\left(C^\infty(M), L^2(M,S), i\gamma^\mu(\partial_\mu + \omega_\mu) \right)$ is a spectral triple (the canonical spectral triple):

- $C^\infty(M)$ acts on $L^2(M,S)$ by $(f\psi)(x) := f(x)\psi(x)$
- $i\gamma^\mu(\partial_\mu + \omega_\mu)$ is self-adjoint
- $[D,f] = i\gamma^\mu\partial_\mu(f)$ is bounded

Chirality & Antiparticles

- Chirality:

  A grading operator $\gamma : \mathcal{H}_F \to \mathcal{H}_F$ (satisfying: $\gamma^* = \gamma, \gamma^2 = 1$)

  with $D\gamma = -\gamma D$

  and $\gamma a = a\gamma$ for all $a \in \mathcal{A}$

  i.e. $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ (e.g. Weyl spinors)

  Notation: $(\mathcal{A}, \mathcal{H}, D; \gamma)$ (even spectral triple)

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- Charge conjugation:

  A real structure: an anti-isometry $J : \mathcal{H} \to \mathcal{H}$ satisfying:

$$J^2 = \pm 1 \qquad JD = \pm DJ \qquad J\gamma = \pm \gamma J$$

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- Compatibility conditions:

$$[a, Jb^* J^*] = 0 \qquad [[D, a], Jb^* J^*] = 0 \quad \forall a, b \in \mathcal{A}$$

(first order condition)

Gauge interactions: inner fluctuations

Definition: algebra $\mathcal{B}$ is said to be Morita equivalent to $\mathcal{A}$ iff $\exists\ \mathcal{E}$ such that $\mathcal{B} \simeq \mathrm{End}^0_A(\mathcal{E})$

Question: $\exists\ (\mathcal{B}, \mathcal{H}', D'; J')$ that implements the Mor. equiv.?

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$$J' = J$$

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$$D' = D + A \pm JAJ^* =: D_A$$

with $A \in \Omega^1_D(\mathcal{A}) = \left\{ \sum_i a_i[D, b_i],\ \ a_i, b_i \in \mathcal{A} \right\}$

self-adjoint

$\Rightarrow$ Inner fluctuations of D

Gauge group: unitarily equivalence

Definition: Two spectral triples

$$(\mathcal{A}, \mathcal{H}, D; J, \gamma) \quad \text{and} \quad (\mathcal{A}, \mathcal{H}, D'; J', \gamma')$$

are called unitarily equivalent if there exists unitary $U : \mathcal{H} \to \mathcal{H}$ such that

- $U\pi(a)U^* = \pi(\sigma(a))$
- $\gamma' = U\gamma U^*$
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Example: Take $U = uJuJ^*$, for any unitary element $u \in U(\mathcal{A})$, then

- $\sigma(a) = uau^*$
- $J' = J$
- $\gamma' = \gamma$
- $D' = D + u[D, u^*] \pm Ju[D, u^*]J^*$

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Note: for a fluctuated Dirac operator:

$$D_A \to D_{A^u} \quad \text{with:} \quad A^u = uAu^* + u[D, u^*]$$

Action: the spectral action

Spectral action

$$\mathrm{Tr}[f(D_A/\Lambda)]$$

f some even function $\quad$ Λ a cutoff scale

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If for a compact Riemannian manifold M of dimension m and a vector bundle V, D_V on $L^2(M, S \otimes V)$ is such that $D_V^2 = \nabla^*\nabla - E$ for some endomorphism E, then we have

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$$\mathrm{Tr}[f(D_V/\Lambda)] \sim f(0)a_m(D_V^2) + \sum_{n\geq 0, n\neq m} \Lambda^{m-n} \frac{2f_{m-n}}{\Gamma((m-n)/2)} a_n(D_V^2)$$

with $a_n(D_V^2) = \int_M a_n(D_V^2, x)\sqrt{g}\mathrm{d}^m x$.

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f some even function $\quad \Lambda$ a cutoff scale

For $m = 4$ we the first contributions are:

$$\mathrm{Tr}[f(D_V/\Lambda)] \sim f(0)a_4(D_V^2) + 2f_2\Lambda^2 a_2(D_A^2) + 2f_4\Lambda^4 a_0(D_A^2) + \mathcal{O}(\Lambda^{-2})$$

where

- $a_0(D_V^2, x) = (4\pi)^{-m/2}\mathrm{Tr}(\mathrm{id})$
- $a_2(D_V^2, x) = (4\pi)^{-m/2}\mathrm{Tr}(-\frac{1}{6}R + E)$
- $a_4(D_V^2, x) = (4\pi)^{-m/2}\mathrm{Tr}(-12R_{;\mu}{}^{\mu} + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 60RE + 60E_{;\mu}{}^{\mu} + 180E^2 + 30\Omega_{\mu\nu}\Omega^{\mu\nu})$

Accounts for: gauge kinetic terms, scalar kinetic terms, scalar masses, scalar self-interactions.

Application: the commutative case

Take again ($C^\infty(M), L^2(M,S), i\gamma^\mu(\partial_\mu + \omega_\mu)$) with:

- $\gamma := \gamma^5$;
- $(J\psi)(x) := C(\psi(x))$.

M 4-d: $\quad J^2 = -1 \qquad J\gamma = \gamma J \qquad J\partial\!\!\!/ = \partial\!\!\!/ J$

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Results:

- No inner fluctuations / gauge interactions;
- Spectral action:

$$\mathrm{Tr}[f(D/\Lambda)] = \Lambda^4 \frac{f_4}{4\pi^2}\int_M \sqrt{g}\,\mathrm{d}^4x + \Lambda^2 \frac{f_2}{48\pi^2}\int_M R\sqrt{g}\,\mathrm{d}^4x + \mathcal{O}(\Lambda^0)$$

Action: the inner product

$\langle .,. \rangle$ Inner product on $\mathcal{H}$, $D_A \equiv D + \mathbb{A}$

Inner product

$$\langle \psi, D_A \psi \rangle \qquad \psi \in \mathcal{H}$$

(Accounts for: fermion kinetic terms, minimal coupling, Yukawa-coupling)

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Solution:

The new inner product

$\frac{1}{2}\langle J\psi, D_A \psi \rangle \qquad \psi \in \mathcal{H}_+$

Works: since $J\gamma = -\gamma J$

The Einstein-Yang-Mills system (1/2)

Take the spectral triple:

$$\mathcal{A} = C^\infty(M) \otimes M_N(\mathbb{C})$$
$$\mathcal{H} = L^2(M,S) \otimes M_N(\mathbb{C})$$
$$D = \partial\!\!\!/ \otimes 1_N$$

Make it real and even with:

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Results:

Inner fluctuations (gauge interactions) $i\mathbb{A} \equiv A + JA^*J$

(with $A = \sum_k \gamma^\mu a_k \partial_\mu(b_k) \in \Omega^1_D(\mathcal{A})$ self-adjoint):

- act in the adjoint representation: $i\mathbb{A} = -i\gamma^\mu \mathrm{ad} A_\mu$
- transform as a $\mathfrak{su}(N)$ gauge field

The Einstein-Yang-Mills system (2/2)

Inner product:

$$\langle \psi, D_A \psi \rangle = \int_M \psi^*(x)\gamma^\mu(\partial_\mu + \mathbb{A}_\mu)\psi(x)\mathrm{dvol}(M)$$

Spectral action:

$$\begin{aligned}\mathrm{Tr}[f(D_A/\Lambda)] \sim -\frac{f(0)}{24\pi^2}\int_M \mathrm{Tr}(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu})\mathrm{dvol}(M) \\ + f(0)\frac{N^2}{1440}\int_M \left[5R^2 - 8R_{\mu\nu}R^{\mu\nu} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right]\mathrm{dvol}(M) \\ + 2f_4\Lambda^4 N^2 \mathrm{vol}(M)\end{aligned}$$

Result: SU(N) Yang-Mills theory (with fermions in the adjoint representation), coupled to gravity.

Supersymmetry in EYM? (See also [1])

Question: Does the Einstein-Yang-Mills system exhibit supersymmetry?

[1] A.H. Chamseddine, Connection between space-time supersymmetry and non-commutative geometry, Phys. Lett. B332 (1994) 349-357

Supersymmetry in EYM? (See also [1])

Question: Does the Einstein-Yang-Mills system exhibit supersymmetry?

To what extent is the action of the Einstein-Yang-Mills system invariant under supersymmetry?

i.e. define $\delta\psi, \delta\mathbb{A}$ and determine

$$\delta S_\Lambda[\psi, \mathbb{A}] = \frac{\mathrm{d}}{\mathrm{d}t} S_\Lambda[\psi + t\delta\psi, \mathbb{A} + t\delta\mathbb{A}]$$

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SUSY in the EYM system (1/2)

Problem: Fermionic & Bosonic DOF don't match.

Degrees of freedom:	Continuous	Finite
Bosonic:	4	$N^2 - 1$
Fermionic:	8	$2N^2$

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1 Continuous part

$$J^2 = -1 \quad J\gamma = \gamma J$$

No Majoranas

No reduction trough
$$\langle \psi, D_A\psi\rangle \to \tfrac{1}{2}\langle J\psi, D_A\psi\rangle \quad \psi \in \mathcal{H}^+$$

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- Minkowski: $(\psi, \overline{\psi})$ with $\gamma_M^5 \psi = \psi \quad \gamma_M^5 \overline{\psi} = \overline{\psi}$

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- Relax hermiticity of action: $(\psi, \overline{\chi})$
- Wick rotate: $(\psi_E, \overline{\chi_E})$ with $\gamma^5_E \psi_E = \psi_E$

  but $\gamma^5_E \overline{\chi_E} = -\overline{\chi_E}$

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  but $\gamma_E^5 \overline{\chi_E} = -\overline{\chi_E}$
- Take as inner product $\langle \chi, D_A \psi \rangle, \psi \in \mathcal{H}^+, \chi \in \mathcal{H}^-$

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2 **Finite part**

- Use $L^2(M, S) \otimes_{\mathbb{C}} M_N(\mathbb{C}) \simeq L^2(M, S) \otimes_{\mathbb{R}} \mathfrak{u}(N)$
- and $\mathfrak{u}(N) \simeq \mathfrak{u}(1) \oplus \mathfrak{su}(N)$: write $\psi = \operatorname{Tr} \widetilde{\psi} + \psi$

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- and $\mathfrak{u}(N) \simeq \mathfrak{u}(1) \oplus \mathfrak{su}(N)$: write $\psi = \mathrm{Tr}\, \widetilde{\psi} + \psi$
- Result: action decouples and $\mathrm{Tr}\, \widetilde{\psi}$ lacks gauge interactions:

$$S[\widetilde{\psi}, \mathbb{A}] = S_1[\mathrm{Tr}\, \widetilde{\psi}] + S_2[\psi, \mathbb{A}]$$

- Discard $\mathrm{Tr}\, \widetilde{\psi}$

SUSY in the EYM system (2/2)

Let $\epsilon_\pm = \pm\gamma^5\epsilon_\pm$ be two constant spinors

Define:

$$\delta\psi = F\epsilon_+ \in \mathcal{H}^+ \qquad F = \gamma^\mu\gamma^\nu\left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right)$$

$$\delta\chi = F\epsilon_- \in \mathcal{H}^-$$

$$\delta\mathbb{A} = c\gamma^\mu[\mathrm{ad}(\epsilon_-, \gamma_\mu\psi) + \mathrm{ad}(\chi, \gamma^\mu\epsilon_+)] \in \mathcal{B}(\mathcal{H})$$

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Then:

Lemma:

The action of the Einstein-Yang-Mills system (after reducing DOF) is SUSY invariant for at least all positive powers of Λ —i.e. for $a_0(D_A^2)$, $a_2(D_A^2)$ and $a_4(D_A^2)$ — upon taking:

$$c = \frac{-3i\pi}{2f(0)N}$$

Supersymmetry

Big question: Can a spectral triple be defined that yields the action of the minmal supersymmetric standard model (MSSM)?

- Superpartners (‘sfermions’, ‘gauginos’) of SM particles
- Undetected yet
- Action invariant under ‘supersymmetry’
- Peculiar Higgs sector

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- Peculiar Higgs sector

Small question: Can a spectral triple be defined that yields the action of super-QCD?

Super-QCD; the spectral triple

Start with the spectral triple of the EYM-system. Now:

- The algebra stays intact

$$\mathcal{A} = C^\infty(M) \otimes M_3(\mathbb{C})$$

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- As for the Hilbert space

$$\mathcal{H} = L^2(M,S) \otimes M_3(\mathbb{C}) \longrightarrow L^2(M,S) \otimes \left(\mathbb{C}^3 \oplus M_3(\mathbb{C}) \oplus \overline{\mathbb{C}}^3\right)$$

with $\pi(a)(q,g,q') = (aq, ag, q')$

$\pi^o(a)(q,g,q') \equiv J\pi^*(a)J^*(q,g,q') = (q, ga, a^t q')$

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$$\mathcal{H} = L^2(M,S) \otimes M_3(\mathbb{C}) \longrightarrow L^2(M,S) \otimes \left(\overset{\text{‘Quark’}}{\mathbb{C}^3} \oplus \overset{\text{Gluino}}{M_3(\mathbb{C})} \oplus \overset{\text{‘Antiquark’}}{\overline{\mathbb{C}}^3}\right)$$

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- Real structure/ charge conjugation

$$J = J_M \otimes {}^* \longrightarrow J_M \otimes J_F, \quad J_F(q,g,\overline{q'}) = (q', g^*, \overline{q})$$

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- Real structure/ charge conjugation

$$J = J_M \otimes {}^* \longrightarrow J_M \otimes J_F, \quad J_F(q,g,\overline{q'}) = (q', g^*, \overline{q})$$

- Grading stays intact

$$\gamma = \gamma^5 \otimes 1_N$$

Super-QCD; the spectral triple

Start with the spectral triple of the EYM-system. Now:

- The algebra stays intact

$$\mathcal{A} = C^\infty(M) \otimes M_3(\mathbb{C})$$

- As for the Hilbert space

'Quark' Gluino 'Antiquark'

$$\mathcal{H} = L^2(M,S) \otimes M_3(\mathbb{C}) \longrightarrow L^2(M,S) \otimes \left(\mathbb{C}^3 \oplus M_3(\mathbb{C}) \oplus \overline{\mathbb{C}}^3\right)$$

with $\pi(a)(q,g,q') = (aq, ag, q')$

$$\pi^o(a)(q,g,q') \equiv J\pi^*(a)J^*(q,g,q') = (q, ga, a^t q')$$

- Real structure/ charge conjugation

$$J = J_M \otimes {}^* \longrightarrow J_M \otimes J_F, \quad J_F(q,g,\overline{q'}) = (q', g^*, \overline{q})$$

- Grading stays intact

$$\gamma = \gamma^5 \otimes 1_N$$

- Dirac operator:

$$D \longrightarrow \not{\partial} \otimes 1_N + \gamma^5 \otimes D_F \qquad D_F : \mathbb{C}^3 \oplus M_3(\mathbb{C}) \oplus \overline{\mathbb{C}}^3 \circlearrowleft$$

Super-QCD; the Dirac operator

A Dirac operator that is self-adjoint, fulfills $D_F J_F = J_F D_F$ and is order one ($[[D, a], b^o]] \;\forall\; a, b \in \mathcal{A}$) is of the form:

$$\begin{pmatrix} D_{11} & D_{12} & 0 \\ D_{12}^* & D_{22} & D_{32}^* \\ 0 & D_{32} & D_{33} \end{pmatrix} \quad \text{with} \quad \begin{aligned} &\overline{D_{12} g} = D_{32} g^* \\ &[D_{21}, a] = 0 \\ &[D_{23}, a^o] = 0 \quad \forall\; a \in M_3(\mathbb{C}) \end{aligned}$$

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- Put $D_{11} = D_{22} = D_{33} = 0$
- D_{12}, D_{32} parametrized by $v \in \mathbb{C}^3$
- Write

$$D_F = \begin{pmatrix} 0 & d_v & 0 \\ d_v^* & 0 & e_v^* \\ 0 & e_v & 0 \end{pmatrix} \qquad \begin{aligned} d_v(g) &= g(v) \in \mathbb{C}^3 \\ e_v(g) &= g^t(\overline{v}) \in \overline{\mathbb{C}}^3 \end{aligned}$$

Super-QCD; inner fluctuations

Inner fluctuations of D_F :

For $A^{(0,1)} := \sum_i a_i[D_F, b_i] \quad a_i, b_i \in M_3(\mathbb{C})$ self-adjoint:

$$\big(D_F + A^{(0,1)} + J(A^{(0,1)})^* J^*\big)(q_1, g, q_2)$$
$$= (g\widetilde{q}, q_1\overline{\widetilde{q}}^t + \widetilde{q}q_2^t, g^t\overline{\widetilde{q}}) \in \mathbb{C}^3 \oplus M_3(\mathbb{C}) \oplus \overline{\mathbb{C}}^3$$

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with $\widetilde{q} = \sum_i [1 + m_i(1 - n_i)]v = \sum_i n_i^* v$

Write $D_{\widetilde{q}} := D_F + A^{(0,1)} + J(A^{(0,1)})^* J^*$

Super-QCD; gauge group

Recall: action of the gauge group by $uJuJ^*$:

Gauge group: gluino

For $U := uJuJ^*$, $u \in U(M_3(\mathbb{C}))$ we find:

$$M_3(\mathbb{C}) \ni g \to UgU* = ugu^*$$

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And:

Gauge group: squark

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$$D_{\widetilde{q}} \to UD_{\widetilde{q}}U^* = D_{u\widetilde{q}}$$

Super-QCD; the action (1/2)

Adding D_F results in:

$$\Omega_{\mu\nu} \to \Omega_{\mu\nu}$$

$$E \to E + i\gamma^5\gamma^\mu[\partial_\mu + \mathbb{A}_\mu, 1 \otimes D_{\widetilde{q}}] - 1 \otimes D_{\widetilde{q}}^2$$

(Was: $E = \frac{1}{4}R \otimes \mathrm{id} - \frac{1}{2}\gamma^\mu\gamma^\nu\mathbb{F}_{\mu\nu}$)

the latter appearing as:

$$\mathcal{O}(\Lambda^2): \quad \mathrm{Tr}(E)$$

$$\mathcal{O}(\Lambda^0): \quad \mathrm{Tr}(RE), \mathrm{Tr}(E^2)$$

Super-QCD; the action (1/2)

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$$\mathrm{Tr} D_{\widetilde{q}}^4 = 16|\widetilde{q}|^4$$

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we have

Theorem

Adding the finite part of the algebra to the finite Hilbert space of QCD and introducing D_F as given before, the spectral action gives the following extra terms (on the orders of Λ^4, Λ^2, Λ^0):

$$\int_M \Bigg[- \left(\frac{6 f_2}{\pi^2} - \frac{f(0)}{4\pi^2 \Lambda^2} R \right) \Lambda^2 |\widetilde{q}|^2 + \frac{f(0)}{4\pi^2} \left(8|\widetilde{q}|^4 + 6|(\partial_\mu + A_\mu)\widetilde{q}|^2 \right) \Bigg] \mathrm{dvol}(M)$$

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squark self-interaction (a 4 squark vertex)

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gluon-squark-squark interaction

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gluon-gluon-squark-squark term

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Super-QCD; the action (2/2)

Theorem

Adding the finite part of the algebra to the finite Hilbert space of QCD and introducing D_F as given before, the inner product gives the following extra terms:

$$\langle \chi_g, (D + \mathbb{A})\psi_g \rangle = \int_M (\chi_g, (\partial\!\!\!/ + \mathbb{A})\psi_g)\mathrm{dvol}(M)$$

$$\langle \chi, (\gamma^5 \otimes D_{\widetilde{q}})\psi \rangle = \int_M [(\chi_q^i, \gamma^5 \psi_g^a)\widetilde{q}^k - (\chi_g^a, \gamma^5 \psi_q^k)\overline{\widetilde{q}}^i](T_a)_{ik}\mathrm{dvol}(M)$$

gluino-gluino-gluon

Super-QCD; the action (2/2)

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gluino-gluino-gluon

squark-quark-gluino

Summary

Big question: Can a spectral triple be defined that yields the action of the minimal supersymmetric standard model (MSSM)?

Small answer: Supersymmetric QCD has been done [1]:

[1] TvdB, Walter D. van Suijlekom, Supersymmetric QCD and noncommutative geometry, 1003.3788

Summary

Big question: Can a spectral triple be defined that yields the action of the minimal supersymmetric standard model (MSSM)?

Small answer: Supersymmetric QCD has been done [1]:

Next step(s):

- ...
- full MSSM
- Predictions?

Plus: Conceptual understanding of SUSY within NCG is desired.

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Small answer: Supersymmetric QCD has been done [1]:

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- ...
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And: Thanks for the attention and now off to lunch!

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